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A study on Extended Newton-type Methods for Variational Inclusions

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A STUDY ON EXTENDED NEWTON-TYPE METHODS FOR VARIATIONAL INCLUSIONS

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TO

MY PARENTS

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(**Mst. Zamilla Khaton**)

June, 2021

Certificate from Supervisor

This is to certify that the Ph.D thesis entitled "**A STUDY ON EXTENDED NEWTON-TYPE METHODS FOR VARIATIONAL INCLUSIONS** " has been prepared by Mst. Zamilla Khaton under my supervision and guidance for submission to the Department of Mathematics, University of Rajshahi, Bangladesh, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Computational Mathematics. I am fully convinced that the results embodied in the thesis are new and this thesis has not been previously submitted to any other university or institute for any diploma or degree.

To the best of my knowledge Mst. Zamilla Khaton bears a good moral character and is mentally and physically fit to get the degree. I wish her a bright future and every success in her life.

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Abstract

In this work, we deal with the two types of variational inclusions. Firstly, we consider the variational inclusion problem of the form

$$0 \in \zeta(\bar{s}) + g(\bar{s}) + \xi(\bar{s}), \qquad (\mathcal{A})$$

where $\mathcal{S}$ and $\mathcal{T}$ are Banach spaces, $\zeta : \mathcal{S} \to \mathcal{T}$ is differentiable in a neighborhood $\Upsilon \subseteq \mathcal{S}$ of a solution s^* of $(\mathcal{A})$, $g : \mathcal{S} \to \mathcal{T}$ is differentiable at s^* but may not differentiable in Υ and $\xi \colon \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ is a set-valued mapping with closed graph. This work consists three parts and the main works we have done in this dissertation that are organized as follows.

In the first part, particularly in Chapter 3, we study the Newton-type method for solving the variational inclusion problem $(\mathcal{A})$ which is introduced in [2]. Under some suitable assumptions on the Fréchet derivative of the differentiable function and divided difference admissible function, we establish the existence of any sequence generated by the Newton-type method and prove that the sequence generated by the method (3.1.3) converges linearly, quadratically and superlinearly to a solution of the variational inclusion $(\mathcal{A})$. Specifically, when the Fréchet derivative of the differentiable function is continuous, Lipschitz continuous and Hölder continuous, divided difference admissible function admits first order divided difference and the set-valued mapping is pseudo-Lipschitz continuous, we show the linear, quadratic and superlinear convergence by the method (3.1.3).

In Chapter 4, we introduce and study the extended Newton-type method for solving the variational inclusion $(\mathcal{A})$. We establish the convergence criteria of the extended Newton-type method, which guarantees the existence and the convergence of any sequence under the conditions that $\eta > 1$, $\nabla\zeta$ is continuous, Lipschitz continuous and Hölder continuous, g admits first order divided difference as well as $(\zeta + g + \xi)^{-1}$ is Lipschitz-like. To validate our theoretical result we have presented numerical experiments and these works extend and improve the result corresponding to [13, 62, 103, 105]. More precisely, semilocal and local convergence of the extended Newton-type method are analyzed.

Next, when $g = 0$ in $(\mathcal{A})$, we are motivated to study the special type of nonsmooth

variational inclusion of the following form:

$$0 \in \zeta(\bar{s}) + \xi(\bar{s}), \qquad (\mathcal{B})$$

where $\zeta\colon \Upsilon \subseteq \mathcal{S} \to \mathcal{T}$ be a nonsmooth single-valued function that admits (n,α)-point-based approximation A on Υ with a constant $L > 0$ and $\xi\colon \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ be a set-valued mapping with closed graph.

In the second part, especially in Chapter 5, we introduce and study an extended Newton-type method for solving the nonsmooth variational inclusion $(\mathcal{B})$ and analyze its semilocal and local convergence under the conditions that $(\zeta+\xi)^{-1}$ is Lipschitz-like and ζ admits a (n,α)-point-based approximation. For smooth functions in the cases $n = 1$ and $n = 2$ as well as for normal maps, we provide applications of (n,α)-point-based approximation, that is, $(1,\alpha)$-point-based approximation and $(2,\alpha)$-point-based approximation are provided for the smooth functions and we construct a (n,α)-point-based approximation for the normal maps $\zeta_{\mathcal{C}} + \xi$ when ζ has a (n,α)-point-based approximation. At the end we have given a numerical experiment to illustrates our theoretical result.

Keywords: Set-valued mappings, pseudo-Lipschitz continuity, Lipschitz-like mappings, variational inclusions, extended Newton-type method, local convergence, semilocal convergence, (n,α)-point-based approximation.

(2000) AMS (MOS) Subject Classification: 49J53, 47H04, 65K10, 90C30.

Contents

List of Abbreviations

NM = Newton Method

N-type = Newton-type

EN = Extended Newton

ENM = Extended Newton-type Method

PBA = Point-based Approximation

FODD = First Order Divided Deference

SODD = Second Order Divided Deference

GN = Gauss-Newton

VI = Variational Inclusion

Chapter 1

Introduction

Robinson [113, 114] introduced variational inclusion as an abstract model for various problems and it has been explored as a general tool for solving, analyzing and describing different problems in a unified manner. These type of inclusion problems have been studied extensively; see for examples [31, 40–42, 49, 53, 55, 71, 76, 91]. It has been well recognized that this model provide a convenient framework for the unified study of optimal solutions in many optimization-related are as including variational inequalities, mathematical programming, optimal control, systems of inequalities, linear and nonlinear complementarity problems, systems of nonlinear equations, equilibrium problems, game theory, etc. also have a lot of applications in engineering (traffic equilibrium problems, analysis of elastoplastic structures etc.) and economics (Nash equilibrium, Walrasian equilibrium etc.). For more details on these applications and others we have not mention here, one can read one can refer to [39, 75, 113–115].

Let $\mathcal{S}$ and $\mathcal{T}$ be two Banach spaces and Υ be an open subset of $\mathcal{S}$. Suppose that $\zeta : \Upsilon \to \mathcal{T}$ is a function, which is Fréchet differentiable and the derivative of this function is denoted by $\nabla\zeta$, the linear function $g : \Upsilon \to \mathcal{T}$ is differentiable at s^* but may not differentiable in a neighborhood Υ and its FODD on the points s and t is denoted by $[s, t; g]$ and $\xi \colon \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ be a set-valued mapping with closed graph.

By smooth variational inclusion we mean a variational inclusion involving a smooth single-valued function, while by nonsmooth variational inclusion we mean a variational inclusion involving a nonsmooth single-valued function.

Here we consider a variational inclusion problem to approximate a point $\bar{s} \in \Upsilon$ satisfying the following form:

$$0 \in \zeta(\bar{s}) + g(\bar{s}) + \xi(\bar{s}). \tag{1.0.1}$$

When $\xi = \{0\}$, (1.0.1) is reduced to the classical problem of solving systems of nonlinear equations: $0 \in \zeta(\bar{s}) + g(\bar{s})$. Cătinas [21] proposed the following method for solving $0 \in \zeta(\bar{s}) + g(\bar{s})$ by using the combination of Newton's method with the secants method when ζ is differentiable and g is a continuous function admitting first and second order divided differences:

$$0 \in \zeta(s_k) + g(s_k) + (\nabla\zeta(s_k) + [s_{k-1},\, s_k;\, g])(s_{k+1} - s_k), \qquad k = 1, 2, ...,$$

where the FODD of g is denoted by $[s,\, t;\, g]$ and the Fréchet derivative of ζ at s_k is denoted by $\nabla\zeta(s_k)$.

For solving (1.0.1), Jean-Alexis and Piétrus [2] presented the method (3.1.1). They proved that the sequence generated by the method (3.1.1) converges superlinearly by considering that $\nabla\zeta$ and the FODD of g are p-Hölder continuous around a solution s^* and that $(\zeta+g+\xi)^{-1}$ is pseudo-Lipschitz around $(0, s^*)$ with ξ having closed graph. In recent time, Rashid *et al.* [109] have been presented the improvement of the result corresponding one in Jean-Alexis and Piétrus [2] by fixing a gap and show that if $\nabla\zeta$ and the FODD of g are p-Hölder continuous at a solution s^*, then the method (3.1.1) converges superlinearly. A vast number of iterative procedures have been introduced and studied for solving (1.0.1); see for details in [9, 101–104, 110].

For solving (1.0.1) various iterative methods have been studied. To solve the problem (1.0.1), Geoffroy and Piétrus [43] associate the method (3.1.3). They studied this method by using the conditions that $\nabla\zeta$ and the SODD of g are Lipschitz continuous around a solution s^*. They proved that the sequence generated by (3.1.3) converges superlinearly.

Moreover, for solving (1.0.1), Hilout *et al.* [50] associate the following sequence:

$$\begin{cases} s_0 \text{ and } s_1 \text{ are given two starting points} & \\ t_k = \alpha s_k + (1-\alpha)s_{k-1}; & \text{when } \alpha \text{ is fixed in } (0,1) \\ 0 \in \zeta(s_k) + [t_k, s_k; \zeta](s_{k+1} - s_k) + \xi(s_{k+1}), & \end{cases} \tag{1.0.2}$$

where the FODD of the function ζ on the points t_k and s_k is $[t_k, s_k; /zeta]$. They have proven that the sequence generated by the method (1.0.2) converges superlinearly. Further, in the case when $\xi = \{0\}$, for solving (1.0.1), it should be noteded that Argyros [8] has studied local as well as semilocal convergence analysis for two-point Newton-type (N-type) methods in a Banach space setting under very general Lipschitz type conditions. An extensive study on these issues has been investigated by Rashid [100, 103, 104] and other researchers when $g = 0$. In the case when ξ is either zero mapping or nonzero mapping, a large number of N-type iterative methods have been studied and we are not mention here all in detail.

In the case when $g = 0$, Rashid *et al.* [110] introduced GN method to obtain the solution of the variational inclusion (1.0.1) and established its semilocal convergence. Moreover, in the same case, Rashid [105, 106, 108] introduced different kinds of methods for obtaining the solution of (1.0.1) and attained the local and semilocal convergence.

In the framework of the variational inclusion (1.0.1), we assume that the single-valued function ζ is smooth function, that is, ζ is Fréchet differentiable and it can be expressed as a classical linearization $\zeta(s) + \nabla\zeta(s)(\cdot - s)$ for given s.

When the single-valued functions involved in (1.0.1) are differentiable, N-type methods can be considered to solve this variational inclusion, such an approach has been used in many contributions to this subject; see for example [26, 27, 89, 90]. In particular, when ζ is smooth function, the classical method to find an approximate solution is the N-type method, which was introduced by Dontchev [27] and is defined by the method (3.1.3) (see subsection 3.1.1 in Chapter 3).

In other words, Dontchev in [27], applied the Newton method to the smooth part ζ of the variational inclusion only (or leaving the nonsmooth part) by keeping the set-valued map ξ unchanged and showed that the sequence constructed by the method (3.1.3) converges quadratically to a solution $\bar{s}$ of (1.0.1). Moreover, when $\nabla\zeta$ is Lipschitz on a neighborhood of $\bar{s}$, Dontchev[29], showed that the stability of this method and certain Lipschitz condition is satisfied.

It is pointed out that the method (3.1.3) viewed as a N-type method based on a partial linearization of ζ. When applying Newton's method, ζ is replaced by its linearization $\zeta(s) + \nabla\zeta(s)(\cdot - s)$ for given s. We still cover known methods for solving variational problems by leaving the set-valued map ξ unchanged. If $\xi = \{0\}$, then the method (3.1.3) becomes the classical Newton method which is widely used and well known for finding an approximate solution of (1.0.1) where ζ has Lipschitz continuous Fréchet derivatives. Semilocal and local convergence results for Newton method can be found in the survey [12, 45, 47, 61, 83, 94] and its references.

For solving (1.0.1) various iterative methods have been studied. Piétrus [90] showed that the sequence generated by the N-type method (3.1.3) converges superlinearly when $\nabla\zeta$ is Hölder continuous on a neighborhood of $\bar{s}$ and certain Lipschitz condition is satisfied, while in [89], he also showed the stability of this method under mild conditions. Furthermore, for analysing (1.0.1), Hilout *et al*. [50] considered the sequence (1.0.2), when ζ is only continuous and also differentiable at $\bar{s}$. They proved the sequence converges superlinearly which is generated by the method (3.1.3).

Usually, there are two types of convergence issues focus on about the EN-type method (Algorithm 2 or Algorithm 3). One of them is local convergence and another one is semi-local convergence analysis. Local convergence analysis is concerned with the convergence ball based on the information in a neighborhood of a solution of (1.0.1) and semi-local convergence analysis is concerned with the convergence criterion based on the information around initial point.

If $\xi = \{0\}$, Algorithm 2 reduces the famous GN method which is well recognized iterative procedure for solving nonlinear least squares (model fitting) problems. To see an extensive study on this subject one can refer to [24, 74, 131]. On the other hand, if $\xi = C$, where a closed convex cone is denoted by C, Algorithm 2 is turned to the EN-type method for analysing convex inclusion problem, which was introduced and studied by Robinson [116]. For solving convex composite optimization problems the GN method are studied in [20, 73] and its references.

In the case when $\xi = \{0\}$ and $g = 0$, a number of useful results have been invented on semilocal convergence analysis for the GN method. For the detail one can refer to Dedieu and Kim [24]; Dedieu and Shub [25]; He, Wang and Li [49]; Xu and Li [130] or in the case when $\xi = C$ and $g = 0$ we can also refer to Li and Ng [72] for more details. Nevertheless, to our best knowledge, there is no study on semilocal convergence analysis discovered for the general case, even for the N-type method (3.1.3) or for the Algorithm 2.

The first main study of this thesis we present in Chapter 3 and Chapter 4 are as follows:

In chapter 3 we analyze the local convergence for the N-type method, which is defined by the method (3.1.3) for finding the solution of (1.0.1). The main tool is the FODD of g and $\nabla\zeta$ is continuous, Lipschitz continuous and Hölder continuous for studying the method (3.1.3), Or, the reader could refer to [63] in our paper to see the achievement on this topic. Relevant research topic for smooth analysis, there have been studied by many mathematician; see for example [13, 30, 50, 55, 90] and the references therein.

In this study, particularly in chapter 4, Argyros and Hilout [13, Theorem 4.1] showed that, for any point in Υ, there exists a sequence which is constructed by Algorithm 1 is quadratically convergent by using some suitable assumptions around the solution s^* of the variational inclusion (1.0.1). This reflection we definitely understood that the convergence result guarantees the existence of a convergent sequence, which is mentioned in [13]. Consequently, for any initial point close to a solution, the sequences which is constructed by Algorithm 1, in the section 4.1 are not identically defined and not each constructed sequence is convergent. Therefore, from a numerical computational point of view this type of method is not convenient to apply in numerical practice. This difficulty inspired us to introduce a kind of method "so-called" extended Newton-type (EN-type) method which is employed in Algorithm 2. The reader could refer to Khaton *et al.* [62] to know on this issue for more detail.

In section 4.3, we provide the EN-type method, (see Algorithm 3 in Chapter 4), for solving the variational inclusion (1.0.1) by using the weaker conditions than that are used in Khaton *et al.* [62]. We analyze this method under the conditions that, the Fréchet derivative of ζ and the FODD of g are Hölder continuous on Υ. In fact, semilocal and local convergence analysis are presented for EN-type method for solving (1.0.1). The reader could refer to [64] to see the contribution on this issue. To validate our theoretical result we have presented numerical experiments.

For the second part of this work, established in Chapter 5, when $g = 0$, in (1.0.1) we consider the special type of nonsmooth variational inclusion for the following form:

$$0 \in \zeta(\bar{s}) + \xi(\bar{s}), \tag{1.0.3}$$

where $\zeta\colon \Upsilon \subseteq \mathcal{S} \to \mathcal{T}$ be a nonsmooth single-valued function that admits (n, α)-point-based approximation (in short PBA) A on Υ with a constant $L > 0$ and $\xi\colon \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ be a set-valued mapping which has closed graph.

Now in the framework of the variational inclusion (1.0.3), we assume that the single-valued function ζ is nonsmooth function, that is, ζ doesn't possess Fréchet derivative and its classical linearization is no longer available. Then no one can give the clear result that how one can give a design of the Newton algorithm. So that it needs to seek a replacement for such type of linearization. A lot of researchers have worked on this question and a number of methods have been introduced and justified in particular cases of its importance in applications. A number of papers have worked on the N-type methods for nonsmooth equations and variational inequalities; see for example [18, 66, 69, 77, 87, 118, 119, 123, 129] for inspiration and advanced works on these areas.

In particular, Wilson [128] proposed an idea for solving nonlinear programming problems by replacing the original problem with a sequence of quadratic programming problems whose data depended on the progress of the solution. In [117], Robinson established a local convergence theorem explaining the quadratic convergence observed in Wilson's method. Eaves [37] and Robinson [120] each suggested N-type linearization methods for solving nonlinear variational inequalities in finite-dimensional spaces. This approach was developed by Josephy [57] to extend Newton's method for solving variational inequalities and complementarity problems. He also extended his analysis to quasi-Newton methods [58], and applied it to a particular problem in energy modeling [59, 60].

Numerous other authors have investigated N-type methods for solving various problems with some types of nonsmoothness: see for example [19, 22, 47, 54, 67, 84–86, 92, 93, 120]. Also, methods of damping and other modifications have been proposed for ensuring convergence: see [46, 48, 97]. All the methods discussed above were developed for solving the nonsmooth variational inclusions (1.0.1) in case of $\xi = \{0\}$.

Moreover, Robinson introduced (see [115] and also [118] based on his earlier preprint) the concept of PBA and proposed a N-type method to solve nonsmooth generalized equations (1.0.3) when $\xi = \{0\}$. Further, he considered PBA in [118, Theorem 3.2] to show the Newton's method converges under Newton-Kantorovich-type hypothesis. In a recent work, Argyros [11] presented a semilocal convergence analysis of Newton's method based on a suitable PBA. More pricisely, in order to solve a more comprehensive problem than those discussed in [118], he was taken weaker conditions in PBA by considering it as Hölderian property rather than Lipschitzian property and therefore he showed the result of convergence for Newton's method.

In addition, for superlinear convergence Kummer [70] presented a necessary and adequate conditions of the Newton method and the conditions of a nonsmooth function was originally designed for derivative-type approximations around an isolated zero. Contextual results, for finding the solution of the nonsmooth variational inclusion (1.0.3) are given in [38, 65, 103].

In case of nonsmoothness of ζ, for solving (1.0.3), Dontchev [26] introduced the N-type method (5.1.2), (see subsection 5.1.1 in Chapter 5), which is a nonsmooth version of the method (3.1.3) and presented the nonsmooth analogue of the Kantorovich-type theorem for this procedure by assuming the Aubin continuity of the map $(A(s_0, \cdot) + \xi(\cdot))^{-1}$ at $(0, s_1)$. In [42], Geoffory and Piétrus presented a general iterative procedure (5.1.2) for solving variational inclusions in the nonsmooth frame-work (1.0.3) by considering a class of functions admitting a certain type of approximation and established a local convergence theorem. It is obvious that if $\xi = \{0\}$, the procedure (5.1.2) reduces to the N-type method which is proposed by Robinson [118].

Genearally, the method (5.1.2) guarantees the existence of a sequence and the sequence is a convergent. Therefore, for a starting point near to a solution, we know that, the sequences are not uniquely defined, which is constructed by the method (5.1.2). For example, Dontchev presented a convergence result which is established in [42, Theorem 3.3] and the result confirms the existence of a convergent sequence. Thus, in view of numerical computation, this kind of Newton-type methods are not convenient in practical application. This drawback allows us to propose the iterative procedure "so-called" extended Newton-type method (ENM) to solve the nonsmooth variational inclusion (1.0.3).

The second main purpose in this work, established in Chapter 5, is to study the semilocal and local convergence for the extended Newton-type (ENM) method defined by Algorithm 4 for solving the nonsmooth variational inclusion (1.0.3) using the notion of (n, α)-PBA introduced by Geoffroy and Piétrus [42] and Lipschitz-like property. The main results, established in section 5.3, are the convergence criterion, which based on the information around the initial point, provides convergence criteria for starting point to determine condition ensuring the convergence to a solution of any sequence which is constructed by Algorithm 4. As consequences, local convergence results for the ENM method are obtained.

Rashid *et al.* [110] presented a method which called the GN-type method. They replaced A by the classical linearization of ζ and then the Algorithm 4 is turned into the GN-type method. For obtaining the solution of (1.0.3) Rashid [103] presented and worked the same algorithm. When the involved single-valued function does not possess Fréchet derivatives, he studied this method under the condition that ζ has a PBA and ζ is Lipsctiz-like mapping and he presented local and semilocal convergence results. Furthermore, the single-valued function is smooth when it involved in (1.0.3). Many mathematician show their interset on semilocal and local convergence analysis with this method (see, for example, [103–105, 109, 110] and the references therein). Finally, we have given some applications of (n, α)-PBA for smooth functions in the case when $n = 1$, $n = 2$ and $0 < \alpha < 1$ and for normal maps $\zeta_{\mathcal{C}} + \xi$ which is reformulated by Rashid [103]. We have given a numerical experiment to illustrates the theoretical result.

The materials in this thesis are divided into six Chapters. The introduction is enclosed in the first Chapter. Chapter 2 contains a review of some basic definitions, notations and some preliminary results that are used in the subsequent Chapters. In Chapter 3, we study a N-type method for solving the variational inclusion defined by the sums of a Fréchet differentiable function, divided difference admissible function and a set-valued mapping with closed graph. Under some suitable assumptions on the Fréchet derivative of the differentiable function and divided difference admissible function, we establish the existence of any sequence constructed by the N-type method and prove that the sequence constructed by this method converges linearly, quadratically and superlinearly to a solution of the variational inclusion. Specifically, when the Fréchet derivative of the differentiable function is continuous, Lipschitz continuous and Hölder continuous, divided difference admissible function admits first order divided difference and the set- valued mapping is pseudo-Lipschitz continuous, we show the linear, quadratic and superlinear convergence respectively of the method.

In Chapter 4, specifically in section 4.1, the EN-type method, which is defined by Algorithm 2, is introduced for obtaining the solution of the variational inclusion (1.0.1). In the section 4.2, we show the existence of a sequence and establish the linear and quardatic convergence results of the sequence constructed by Algorithm 2 by using the conditions that $\nabla\zeta$ is continuous, Lipschitz continuous and g admits the FODD. The purpose of this section 4.2. is to analyze the semilocal convergence of the EN-type method which is defined by Algorithm 2. A detailed discussion on this topic, we have mentioned in our paper Khaton *et al*. [62]. The objective of the section 4.3. is to analyze the semilocal and local convergence for the EN-type method under the weaker conditions than [62], that is, $\nabla\zeta$ is (L, q)-Hölder continuous and g admits the FODD satisfying q-Hölderian condition. The main result of the section 4.3. is semilocal analysis for the EN-type method, that is, based on the information around the initial point, the main results are the convergence criteria, which provide few suitable conditions ensuring the convergence to a solution of any sequence constructed by Algorithm 3. Consequently, the results of the local convergence for the EN-type method are attained.

In Chapter 5, we introduce the EN-type method, which is defined by Algorithm 4, for solving the nonsmooth variational inclusion (1.0.3) under the conditions $\eta > 1$, $(\zeta + \xi)^{-1}$ is Lipschitz-like and the nonsmooth function ζ has a (n, α)-PBA and we prove the existence and establish the $(n + \alpha)$ order convergence results of the sequence which is constructed by Algorithm 4. Moreover, we have given the applications of (n, α)-PBA for smooth functions in the cases $n = 1$ and $n = 2$ with $0 < \alpha < 1$. In addition, we have given another application of (n, α)-PBA for normal maps $\zeta_{\mathcal{C}} + \xi$ which extends the concept of PBA reformulated by Rashid [103]. That is, we have shown that if ζ has a (n, α)-PBA, it is easy to construct a (n, α)-PBA for the $\zeta_{\mathcal{C}} + \xi$.

Finally, a summary of the main finding of this study is presented in Chapter 6.

Chapter 2

Notations and Preliminaries

Throughout the whole thesis, we assume that $\mathcal{S}$ and $\mathcal{T}$ are two real or complex Banach spaces and $\mathbb{N}$ is the set of all Natural numbers and $\mathbb{N}^* = \mathbb{N} - \{0\}$. Suppose that $\zeta : \Upsilon \to \mathcal{T}$ is a function, which is Fréchet differentiable, $\zeta : \mathcal{S} \to \mathcal{T}$ is a Fréchet differentiable function and $\xi\colon \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ is a set-valued map which has closed graph. Let $s \in \mathcal{S}$ and $\mathbb{B}_r(s) = \{u \in \mathcal{S} : \|u - s\| \leq r\}$ be denoted for the closed ball centered at s with radius $r > 0$. All the norms are denoted by $\|\cdot\|$, while $\mathcal{L}(\mathcal{S}, \mathcal{T})$ stands for the set of all bounded linear operators from $\mathcal{S}$ to $\mathcal{T}$.

The **domain** of ξ, denoted by $\operatorname{dom}\xi$, is defined by

$$\operatorname{dom}\xi := \{s \in \mathcal{S} : \ \xi(s) \neq \emptyset\}.$$

The **inverse** of ξ, denoted by ξ^{-1}, is defined by

$$\xi^{-1}(t) := \{s \in \mathcal{S} : t \in \xi(s)\} \quad \text{for each } t \in \mathcal{T}.$$

Let $D \subseteq \mathcal{S}$. The **distance** from a point s to a set D is defined by

$$\operatorname{dist}(s, D) := \inf\{\|s - a\| : a \in D\} \quad \text{for each } s \in \mathcal{S},$$

while the excess from the set D to the set $C \subseteq \mathcal{S}$ is defined by

$$e(C, D) = \sup\{\operatorname{dist}(s, D) : s \in C\}.$$

Definition 2.0.1. *A **sequence** is a function whose domain is the set of natural numbers* $\mathbb{N} = \{1, 2, 3, \cdots\}$. *A sequence* $\{s_n\}$ *in* $\mathcal{S}$ *is said to be convergent if* $\exists$*'s a point* s *in* $\mathcal{S}$ *such that for each* $\epsilon > 0$, $\exists$*'s a positive integer* (n_0) *such that* $n \geq n_0 \Rightarrow d(s_n, s) < \epsilon$.

Definition 2.0.2. *A sequence* $\{s_n\}$ *in* $(\mathcal{S}, d)$ *is said to be* ***Cauchy sequence*** *if for every* $\epsilon > 0$, *∃'s some* n_0 *such that* $d(s_n, s_m) < \epsilon$, *for all* $n, m \geq n_0$. *Again, a metric space* $(\mathcal{S}, d)$ *is complete if every Cauchy sequence in it converges.*

Definition 2.0.3. *Consider the set-valued mapping* $\xi : \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$. *Then the* ***graph of*** ξ *is defined by*

$$\operatorname{gph}\xi := \{(s,t) \in \mathcal{S} \times \mathcal{T} : t \in \xi(s)\}.$$

Definition 2.0.4. *A set-valued function* $\xi : \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ *is said to be a* ***closed graph*** *if the set* $\{(s,t) : t \in \xi(s)\}$ *is a closed subset of* $\mathcal{S} \times \mathcal{T}$ *in the product topology i.e. for all sequences* $\{s_k\}_{k\in \mathbb{N}}$ *and* $\{t_k\}_{k\in\mathbb{N}}$ *such that* $s_k \to s$ *and* $t_k \to t$ *and* $t_k \in \xi(s_k)$ *for all* n, *we have* $t \in \xi(s)$.

The following definitions of continuity, Lipschitz continuity and Hölder continuity are taken from the book [21].

Definition 2.0.5. *A map* $\zeta : \Upsilon \subseteq \mathcal{S} \to \mathcal{T}$ *is said to be* ***continuous*** *at* $\bar{s} \in \Upsilon$ *if for every* $\epsilon > 0$, *there exist a* $\delta > 0$ *such that*

$$\|\zeta(s) - \zeta(\bar{s})\| \leq \epsilon, \quad \text{for all } s \in \Upsilon, \quad \text{for which } \|s - \bar{s}\| < \delta.$$

Definition 2.0.6. *A map* $\zeta : \Upsilon \subseteq \mathcal{S} \to \mathcal{T}$ *is said to be* ***Lipschitz continuous*** *if there exist constant* $0 < c < 1$ *and such that*

$$\|\zeta(s) - \zeta(\bar{s})\| \leq c\|s - t\|, \quad \text{for all } s \in \Upsilon, \quad \text{for all } s \quad \text{and } t \quad \text{in the domain of } \zeta.$$

Definition 2.0.7. *A map* $\zeta : \Upsilon \subseteq \mathcal{S} \to \mathcal{T}$ *is said to be* ***Hölder continuous*** *if there exist a constant* $c > 0$ *and* $0 < p \leq 1$ *such that*

$$\|\zeta(s) - \zeta(\bar{s})\| \leq c\|s - t\|^p, \quad \text{for all } s \quad \text{and } t \quad \text{in the domain of } \zeta.$$

The following definitions of linear convergence, quadratic convergence and super linear convergence are taken from the book [71].

Definition 2.0.8. *Let $\{s_n\}$ be a sequence which converges to the number $\bar{s}$. Then the sequence $\{s_n\}$ is said to be **converges linearly** to $\bar{s}$, if there exists a number $0 < c < 1$ such that*

$$\|s_{n+1} - \bar{s}\| \leq c\|s_n - \bar{s}\|.$$

Definition 2.0.9. *Let $\{s_n\}$ be a sequence which converges to the number $\bar{s}$. Then the sequence $\{s_n\}$ is said to be **converges quadratically** to $\bar{s}$, if there exists a number $0 < c < 1$ such that*

$$\|s_{n+1} - \bar{s}\| \leq c\|s_n - \bar{s}\|^2.$$

Definition 2.0.10. *Let $\{s_n\}$ be a sequence which converges to the number $\bar{s}$. Then the sequence $\{s_n\}$ is said to be **converges super-linearly** to $\bar{s}$, if there exists a number $c > 1$ and $0 < p \leq 1$ such that*

$$\|s_{n+1} - \bar{s}\| \leq c\|s_n - \bar{s}\|^p.$$

Aubin [15, 16] introduced the notions of pseudo-Lipschitz and Lipchitz-like set-valued mappings and have been studied extensively. For more details one could refer to [1, 2, 8, 13, 27, 30, 35, 50, 55, 90, 127]. We recall the following notions from [110].

Definition 2.0.11. *Let $\xi : \mathcal{T} \rightrightarrows 2^{\mathcal{S}}$ be a set-valued mapping and let $(\bar{t}, \bar{s}) \in \operatorname{gph} \xi$. Let $r_{\bar{s}}$, $r_{\bar{t}}$ and M are positive constants. Then ξ is said to be*

(a) ***Lipchitz-like*** *on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M if the following inequality holds:*

$$e(\xi(t_1) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), \xi(t_2)) \leq M\|t_1 - t_2\|, \quad \text{for any } t_1, t_2 \in \mathbb{B}_{r_{\bar{t}}}(\bar{t}). \tag{2.0.1}$$

(b) ***pseudo-Lipschitz*** *around $(\bar{t}, \bar{s})$ if there exist constants $r'_{\bar{t}} > 0$, $r'_{\bar{s}} > 0$ and $M' > 0$ such that ξ is Lipchitz-like on $\mathbb{B}_{r'_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r'_{\bar{s}}}(\bar{s})$ with constant M'.*

The following lemma is due to Lemma 2.1 of Rashid, Yu, Li & Wu. This lemma is useful and it was proven by Rashid *et al.* in [110].

Lemma 2.0.1. *Let $(\bar{t}, \bar{s}) \in \operatorname{gph} \xi$ and let $\xi : \mathcal{T} \rightrightarrows 2^{\mathcal{S}}$ be a set-valued mapping. Suppose that ξ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M. Then*

$$\operatorname{dist}(s, \xi(t)) \leq M \operatorname{dist}(t, \xi^{-1}(s))$$

holds for each $s \in \mathbb{B}_{r_{\bar{s}}}(\bar{s})$ *and* $t \in \mathbb{B}_{\frac{r_{\bar{t}}}{3}}(\bar{t})$ *satisfying* $\operatorname{dist}(t, \xi^{-1}(s)) \leq \frac{r_{\bar{t}}}{3}$.

Remark 2.0.1. *The concept of pseudo-Lipschitz property of a set-valued mapping* ξ *is equivalent to the openness with linear rate of* ξ^{-1} *and to the metric regularity of* ξ^{-1} *(see [7, 15–17, 23, 32, 33, 36, 44, 52, 78, 80, 81, 124, 126] for more details).*

Remark 2.0.2. *Equivalently for the property* (a) *we can say that* ξ *is Lipschitz-like at* $(t_0, s_0) \in \operatorname{gph}\xi$ *on* $\mathbb{B}_{r_{t_0}}(t_0) \times \mathbb{B}_{r_{s_0}}(s_0)$ *with constant* M *if for each* $s_1, s_2 \in \mathbb{B}_{r_{s_0}}(s_0)$ *and for every* $s_1 \in \xi(t_1) \cap \mathbb{B}_{r_{s_0}}(s_0)$, $\exists$*'s* $s_2 \in \xi(t_2)$ *such that*

$$\|s_1 - s_2\| \leq M\|t_1 - t_2\|, \quad \textit{for every} \;\; t_1, t_2 \in \mathbb{B}_{r_{t_0}}(t_0).$$

The definition of the first and second order divided difference operators are collected from [43, 109]:

Definition 2.0.12. *Let* $g \in \mathcal{L}(\mathcal{S}, \mathcal{T})$. *Then* g *is said to have the* ***FODD*** *on the points* $s,\ t \in \mathcal{S}$ $(s \neq t)$ *if the following properties hold:*
(a) $[s,\ t;\ g](t - s) = g(t) - g(s)$ for $s \neq t$;
(b) If g *is Fréchet differentiable at* $s \in \mathcal{S}$ *then* $[s,\ s;\ g] = \nabla g(s)$.

Definition 2.0.13. *Let* $g \in \mathcal{L}(\mathcal{S}, \mathcal{T})$. *Then* g *is said to have the* ***SODD*** *on the points* $s, t, z \in \mathcal{S}$ $(s \neq t \neq z)$ *if the following properties hold:*
(a) $[s, t, z; g](z - s) = [t, z; g] - [s, t; g]$, *for the distinct points* s, t *and* z;
(b) If g *is twice differentiable at* $s \in \mathcal{S}$ *then* $[s, s, s; g] = \dfrac{\nabla^2 g(s)}{2}$.

The notion of **point based-approximation** (PBA) is given in [118] and studied many mathematicians; see for example [11, 12, 61] and the references therein. We employ the following concept of PBA which is introduced by Robinson in [118].

Definition 2.0.14. *Let* ζ *be a function from an open subset* Υ *of* $\mathcal{S}$ *to* $\mathcal{T}$. *Consider a scalar* κ *and a function* $A : \Upsilon \times \Upsilon \to \mathcal{T}$ *such that, for each* $p, q \in \Upsilon$, *the following assertions are hold:*

(a) $\|\zeta(q) - A(p, q)\| \leq \frac{1}{2}\kappa\|p - q\|^2$ *and*

(b) *The function* $A(p,\cdot) - A(q,\cdot)$ *is Lipschitz continuous on* Υ *which have a Lipschitz constant* $\kappa\|p-q\|$.

Then A *is said to be a PBA on* Υ *for* ζ *with modulus* κ.

In that time we say that ζ has a PBA on Υ with modulus κ.

Remark 2.0.3. *The definition of PBA actually captures some very familiar properties of linearizations. The easiest way to observe that a PBA of a function* ζ *which is Fréchet differentiable in* Υ *and the functions derivatives is Lipschitz continuous on* Υ *with modulus* κ*, is the function*

$$A : (p,q) \longmapsto \zeta(p) + \nabla\zeta(p)(q-p) \tag{2.0.2}$$

is a PBA for ζ *with modulus* κ *on* Υ.

Then we get from the part (a) of Definition (2.0.14) *that*

$$\|\zeta(q) - \zeta(p) - \nabla\zeta(p)(q-p)\| \leq \frac{1}{2}\kappa\|p-q\|^2.$$

Furthermore,

$$\begin{aligned} \|[A(p,s) - A(q,s)] - [A(p,t) - A(q,t)]\| &= \|(\nabla\zeta(p) - \nabla\zeta(q))(s-t)\| \\ &\leq \|\nabla\zeta(p) - \nabla\zeta(q)\|\|s-t\| \\ &\leq \kappa\|p-q\|\|s-t\|. \end{aligned}$$

Here we prove that the part (b) of Definition (2.0.14) *is equivalent to the Lipschitzian property of* $\nabla\zeta$ *with modulus* κ.

The following concept of (n,α)-PBA is extracted from Geoffroy and Piétrus [42].

Definition 2.0.15. *Let* $\zeta : \Upsilon \subseteq \mathcal{S} \to \mathcal{T}$ *be a function and* $n \in \mathbb{N}^*$, $\alpha > 0$. *Then a function* $A : \Upsilon \times \Upsilon \to \mathcal{T}$ *is said to be a* (n,α)***-PBA*** *on* Υ *for* ζ *with modulus* κ *if there exists a scalar* κ *such that, for each* $p, q \in \Upsilon$*, the following assertions are hold:*

(a) $\|\zeta(q) - A(p,q)\| \leq \dfrac{\kappa}{\pi_{n,\alpha}}\|p-q\|^{n+\alpha}$, *where*

$$\pi_{n,\alpha} = \Pi_{i=1}^{n}(\alpha + i); \tag{2.0.3}$$

(b) *The function* $A(p,\cdot) - A(q,\cdot)$ *is Lipschitz continuous on* Υ *with modulus* $\kappa\|p-q\|^{\alpha}$.

It is clear that when $n = 1$ and $\alpha = 1$, Definition 2.0.15 agrees with Robinson's definition of point-based approximation introduced in [118].

Recall the following definition of strict differentiability, which has been taken from [26].

Definition 2.0.16. *A function* $\zeta : \mathcal{S} \to \mathcal{T}$ *is said to be* ***strictly differentiable*** *at* s^* *with strict derivative* $\nabla\zeta(s^*)$ *if for every* $\varepsilon > 0$ $\exists$*'s* $\delta > 0$ *such that*

$$\|\zeta(s') - \zeta(s'') - \nabla\zeta(s^*)(s' - s'')\| \le \varepsilon\|s' - s''\|, \quad \textit{for} \ \ \textit{every}\ s', s'' \in \mathbb{B}_\delta(s^*).$$

The following result is a version of [26, Lemma 2]. The connection between the strict differentiability of ζ and (n,α)-PBA of a function ζ is established by this result.

Lemma 2.0.2. *Let* $s^* \in \Upsilon$ *and let* A *be a* (n,α)*-PBA of a function* ζ *in* Υ *with a scalar constant* κ*. Then the function* $A(s^*,\cdot) - \zeta(\cdot)$ *is strictly differentiable at the point* s^* *and its strict derivative at* s^* *is zero.*

We recall the following lemma from [31, Corollary 2].

Lemma 2.0.3. *Let* ζ, $g : \mathcal{S} \to \mathcal{T}$ *be two continuous functions and let* $\xi : \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ *be a set-valued mapping which has closed graph. Let* $(s^*, t^*) \in \mathrm{gph}\xi$, $\zeta(s^*) = g(s^*) = 0$ *and the difference* $\zeta - g$ *be strictly differentiable at the point* $s^* \in \mathcal{S}$ *with* $\nabla(\zeta - g)(s^*) = 0$*. Let* L *be a positive constant. Then both of the following are equivalent:*

(i) At (t^*, s^*) *the feature of the map* $(\zeta + \xi)^{-1}$ *is Lipschitz-like with modulus* $< L$*;*

(ii) At (t^*, s^*) *the feature of the map* $(g + \xi)^{-1}$ *is Lipschitz-like with modulus* $< L$*.*

Remark 2.0.4. *Combining Lemma 2.0.2 and Lemma 2.0.3, we can infer that if* A *is a* (n,α)*-PBA of a function* ζ *in an open neighborhood of some* $s^* \in (\zeta + \xi)^{-1}(t^*)$*, then* $(\zeta + \xi)^{-1}$ *is Lipschitz-like at* (t^*, s^*) *if and only if the map* $(A(s^*,\cdot) + \xi(\cdot))^{-1}$ *possesses the same property.*

The following theorem on the convergence of the Newton-type method is due to Dontchev; see [27, Theorem.]:

Theorem 2.0.1. *Let s^* be a solution of (1.0.3). Suppose that ζ is a Fréchet differentiable function on an open neighborhood Υ of s^* and the derivative of Fréchet differentiable function $\nabla\zeta$ is Lipschitz in Υ with constant L. Suppose that ξ has closed graph and the mapping $(\zeta(s^*) + \nabla\zeta(s^*)(\cdot - s^*) + \xi(\cdot))^{-1}$ is Aubin continuous at $(0, s^*)$ with modulus M. Then, for every $c > \dfrac{ML}{2}$, one can find $\delta > 0$ such that, for any starting point $s_0 \in \mathbb{B}_\delta(s^*)$, $\exists$'s a sequence $\{s_k\}$ generated by (3.1.1), which satisfies*

$$\|s_{k+1} - s^*\| \leq c\|s_k - s^*\|^2.$$

The following theorem on the convergence of the nonsmooth function using (n, α)-point-based approximation is due to Geoffroy and Piétrus; see [42, Theorem 3.3]:

Theorem 2.0.2. *Let the solution of (1.0.3) is s^*. Fix $n \in \mathbb{N}^*$ and $\alpha > 0$. Suppose that ξ has closed graph, ζ admits a (n, α)-PBA with modulus k which is denoted by A, on some open neighborhood Υ of s^* and the set-valued map $[A(s^*, \cdot) + \xi(\cdot)]^{-1}$ is M-pseudo-Lipschitz around $(0, s^*)$. Then for every $c > \dfrac{Mk}{\pi_{n,\alpha}}$, one can find $\delta > 0$ such that for every starting point $s_0 \in \mathbb{B}_\delta(s^*)$, $\exists$'s a sequence $\{s_k\}$ generated by (5.1.2), which satisfies*

$$\|s_{k+1} - s^*\| \leq c\|s_k - s^*\|^{n+\alpha}.$$

Dontchev and Hager [31] proved Banach fixed point theorem, which has been employing the standard iterative concept for contracting mapping. To prove the existence of the sequence generated by Algorithm 4, the following lemma will be played an important rule in this study.

Lemma 2.0.4. *Let $\Psi : \mathcal{S} \rightrightarrows 2^{\mathcal{S}}$ be a set-valued mapping. Let $s^* \in \mathcal{S}$, $0 < \lambda < 1$ and $r > 0$ be such that*

$$(a)\ \operatorname{dist}(s^*, \Psi(s^*)) < r(1 - \lambda) \tag{2.0.4}$$

and

$$(b)\ e(\Psi(s_1) \cap \mathbb{B}_r(s^*), \Psi(s_2)) \leq \lambda\|s_1 - s_2\| \quad \text{for all} \quad s_1, s_2 \in \mathbb{B}_r(s^*). \tag{2.0.5}$$

Then Ψ has a fixed point in $\mathbb{B}_r(s^*)$, that is, there exists $s \in \mathbb{B}_r(s^*)$ such that $s \in \Psi(s)$. Furthermore, if Ψ is single-valued, then there exists a fixed point $s \in \mathbb{B}_r(s^*)$ such that $s = \Psi(s)$.

The preceding lemma is a generalization of a fixed point theorem and it has been taken from [51], where in the second assertion the excess e is updated by Hausdorff distance.

Chapter 3

Local Convergence Analysis of a Newton-type Method for Solving Variational Inclusions

This Chapter consists three sections. Section 3.1 is dedicated to study the Newton-type method satisfying (3.1.3) for finding the approximate solution of the variational inclusions (1.0.1), while in Section 3.2 is devoted to study the linear, quadratic and superlinear convergence of the sequence generated by Newton-type method satisfying (3.1.3) for solving the variational inclusions (1.0.1).

3.1 Newton-type Method

In numerical analysis, Newton's method, also known as the Newton-Raphson method, named after Isaac Newton and Joseph Raphson, is a root finding algorithm which produces successively better approximation to the roots or zeroes of a real valued mapping. Newton's method is a classical numerical method to solve a system of linear equations. John Wallis [56] published Newton's method for the first time in 1685 in "A Treatise of Algebra both Historical and Practical". For solving general nonlinear equations Thomas Simpson described Newton's method as an iterative method using calculus in 1740 and also gives the generalization to systems of two equations and notes that by setting the gradient to zero Newton's method can be used for solving optimization problems .

The classical Newton method is very widely used and well known for finding zeros of functions having Lipschitz continuous Fréchet derivatives. For an excellent treatment of this method and many references, see the book of Ortega and Rheinboldt [82, 112]. However, when the functions being dealt with do not possess Fréchet derivatives, then no one can give the clear result that, how one can give a design of the Newton algorithm. In recent years A lot of researchers have worked on this question and a number of methods have been presented and justified in particular cases of its importance in applications.

In 1970, Robinson [117] established a local convergence theorem explaining the quadratic convergence observed in Wilson's method and Eaves [37] and Robinson [121] each suggested N-type linearization methods for solving nonlinear variational inequalities in finite-dimensional spaces.

3.1.1 Introduction

Let $\mathcal{S}$ and $\mathcal{T}$ be two Banach spaces and $\Upsilon \subseteq \mathcal{S}$. Suppose that $\zeta : \Upsilon \to \mathcal{T}$ is a function, which is Fréchet differentiable and the derivative of this function is denoted by $\nabla\zeta$, the linear function $g : \Upsilon \to \mathcal{T}$ is differentiable at s^*, but in a neighborhood Υ of s^* it may not be differentiable and its FODD on the points s and t is denoted by $[s, t; g]$ and $\xi : \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ is a set-valued mapping which has closed graph. We are concerned with the problem of finding a solution $s^* \in \Upsilon$ satisfying the variational inclusion (1.0.1) such as

$$0 \in \zeta(\bar{s}) + g(\bar{s}) + \xi(\bar{s}).$$

For obtaining the solution of (1.0.1), Jean-Alexis and Piétrus [2] introduced the method as follows:

$$0 \in \zeta(s_k) + g(s_k) + (\nabla\zeta(s_k) + [2s_{k+1} - s_k,\ s_k;\ g])(s_{k+1} - s_k) + \xi(s_{k+1}). \tag{3.1.1}$$

They proved that the sequence generated by the method (3.1.1) converges superlinearly by considering that $\nabla\zeta$ and the FODD of g are p-Hölder continuous around a solution s^* and that $(\zeta + g + \xi)^{-1}$ is pseudo-Lipschitz around $(0, s^*)$ with ξ having closed graph. In recent time, Rashid et al. [109] have been presented the improvement of the result corresponding one Jean-Alexis and Piétrus in [2] and show that if $\nabla\zeta$ and the FODD of g are p-Hölder continuous at a solution s^*, then the method (3.1.1) converges superlinearly.

when $g = 0$, the variational inclusion (1.0.1) turns into the following form:

$$0 \in \zeta(\bar{s}) + \xi(\bar{s}). \tag{3.1.2}$$

Several iterative methods have been studied for solving (3.1.2). Dontchev [27] established a quadratically convergent N-type method under a pseudo-Lipschitz property for set-valued mapping when $\nabla\zeta$ is Lipschitz on a neighborhood of a solution s^* of (3.1.2) and subsequently he [29] proved the stability of this method. When $\nabla\zeta$ is Hòlder on a neighborhood of s^*, Piétrus [89] obtained superlinear convergence by following Dontchev's method and later he [29] proved the stability of this method in this mild differentiability context. In the case g = 0, Geoffroy *et al.* [41] considered a second degree Taylor polynomial expansion of ζ under suitable first and second order differentiability assumptions and showed that the existence of a sequence cubically converging to the solution of (1.0.1). But we cannot apply the above methods, because of the lack of regularity of g, To carry out our objective, we propose a combination of Newton's method with the secant's one. When the single-valued functions involved in (1.0.1) is differentiable, N-type method can be considered to solve this variational inclusion, such an approach has been used in many contributions to this subject; see for example [2, 27, 28, 33, 43, 110]). To solve the problem (1.0.1), Geoffroy and Piétrus [43] associate in the following:

$$0 \in \zeta(s_k) + g(s_k) + (\nabla\zeta(s_k) + [s_{k-1},\, s_k;\, g])(s_{k+1} - s_k) + \xi(s_{k+1}). \tag{3.1.3}$$

They studied this method by using the assumptions that $\nabla\zeta$ and the SODD of g are Lipschitz continuous around a solution s^*. They proved that the sequence generated by (3.1.3) converges superlinearly.

The aim of this study is to extend the result given in [43] by using the concept of the FODD of g and $\nabla\zeta$ is continuous, Lipschitz continuous and Hölder continuous and then we prove the existence of a sequence generated by the method (3.1.3) and show the linear, quadratical and superlinear convergence of the method for solving the variational inclusion (1.0.1).

3.2 Convergence Analysis of Newton-type Method

This section is devoted to study the existence and the convergence of any sequence generated by the method (3.1.3) for the variational inclusion (1.0.1). Let $\zeta : \mathcal{S} \to \mathcal{T}$ be a single valued function, $g : \mathcal{S} \to \mathcal{T}$ admits FODD and $\xi : \mathcal{S} \rightrightarrows 2^T$ be a set-valued mapping. Let s^* be a solution of (1.0.1). Let $s \in \mathcal{S}$ and define a set valued mapping $R_{s^*} : \mathcal{S} \rightrightarrows 2^T$ by

$$R_{s^*}(\cdot) := \zeta(s^*) + g(\cdot) + \nabla\zeta(s^*)(\cdot - s^*) + \xi(\cdot). \tag{3.2.1}$$

Consider the following assumptions:

(A0) ξ has closed graph;

(A1) ζ is Fréchet differentiable in a neighborhood of s^*;

(A2) g is differentiable at s^*;

(A3) The set valued map $R_{s^*}^{-1}$ is M-pseudo-Lipschitz around $(0, s^*)$.

Define a single valued function $G_k : \mathcal{S} \to \mathcal{T}$, for $k \in \mathbb{N}$ and $s_k \in \mathcal{S}$, by

$$G_k(s)(=)\zeta(s^*)+g(s)+\nabla\zeta(s^*)(s-s^*)-\zeta(s_k)-g(s_k)-(\nabla\zeta(s_k)+[s_{k-1}, s_k; g])(s-s_k), \tag{3.2.2}$$

Also, define a set valued mapping $\Psi_k : \mathcal{S} \rightrightarrows 2^{\mathcal{S}}$ by

$$\Psi_k(s) = R_{s^*}^{-1}[G_k(s)]. \tag{3.2.3}$$

3.2.1 Linear Convergence

The subsection is devoted to study linear convergence result of the N-type method (3.1.3). To do this we will take the following assumptions:

(A4) $\nabla\zeta$ is continuous in a neighbourhood of s^* with constant $\epsilon > 0$ i.e, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|\nabla\zeta(s) - \nabla\zeta(t)\| < \epsilon, \quad \text{when } \|s - t\| \le \delta.$$

(A5) g admits FODD i.e, there exists $\kappa > 0$ such that, for all $s,\, t, s',\, t' \in \Upsilon$,

$$\|[s, t; g] - [s', t'; g] \le \kappa(\|s - s'\| + \|t - t'\|) \quad \text{with } s' \ne s,\, t' \ne t.$$

Let M, ϵ and κ be defined in (A3), (A4) and (A5) respectively satisfying the relation $14M(\epsilon+4\kappa) < 3$.

$$\text{Set } C = \frac{7M(\epsilon + 4\kappa)}{3}. \tag{3.2.4}$$

This together with above inequality implies that $C < \frac{1}{2}$.

Lemma 3.2.1. *Let s^* be a solution of (1.0.1). Suppose that assumptions (A0)-(A5) are hold. Let C be defined by (3.2.4). Then for every such C, there exists $\delta > 0$ such that for every distinct starting point $s_0, s_1 \in B_\delta(s^*)$, there exists a sequence $\{s_2\}$, defined by*

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0,\, s_1;\, g])(s_2 - s_1) + \xi(s_2) \tag{3.2.5}$$

and in $B_\delta(s^*)$ the map Ψ_1 has a fixed point s_2, which satisfies

$$\|s_2 - s^*\| \leq C\|s_1 - s^*\|. \tag{3.2.6}$$

Proof. The assumption (A3) implies that the mapping $R_{s^*}^{-1}$ is M-pseudo-Lipschitz around $(0, s^*)$. Hence there exists $r_{s^*} > 0$ and $r_0 > 0$ such that

$$e\big(R_{s^*}^{-1}(t_1) \cap \mathbb{B}_{r_{s^*}}(s^*), R_{s^*}^{-1}(t_2)\big) \leq M\|t_1 - t_2\| \;\text{ for any } t_1,\, t_2 \in \mathbb{B}_{r_0}(0). \tag{3.2.7}$$

Let $\delta > 0$ be such that

$$\delta \leq \max\Big\{r_{s^*},\, \frac{r_0}{3\epsilon + 8\kappa},\, \frac{4 - 7M\epsilon}{28M\kappa},\, 1\Big\}. \tag{3.2.8}$$

Fix $s_0, s_1 \in \mathbb{B}_\delta(s^*)$ such that $s_0 \neq s_1 \neq s^*$ and define

$$r_{s_2} = C\|s_1 - s^*\|.$$

Since $C < \frac{1}{2}$ from (3.2.4) and for $s_0,\, s_1 \in \mathbb{B}_\delta(s^*)$, we have

$$r_{s_2} = \; C\; \|s_1 - s^*\| \leq c.\delta \leq \frac{\delta}{2}.$$

This shows that $r_{s_2} \leq \; \delta \leq \; r_{s^*}$

Lemma 2.0.4 will be applied to the map Ψ_1 with $\eta_0 := s^*$, $r := r_{s_2}$ and $\lambda := \frac{4}{7}$ to conclude that there exists a fixed point $s_2 \in \mathbb{B}_{r_{s_2}}(s^*)$ such that $s_2 \in \Psi_1(s_2)$, that is, $G_1(s_2) \in R_{s^*}^{-1}(s_2)$, which implies that

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0,\, s_1;\, g])(s_2 - s_1) + \xi(s_2), \tag{3.2.9}$$

i.e. (3.2.5) holds. Furthermore, $s_2 \in \mathbb{B}_{r_{s_2}}(s^*) \subseteq \mathbb{B}_\delta(s^*)$ and so

$$\|s_2 - s^*\| \leq r_{s_2} = C\, \|s_1 - s^*\|,$$

i.e. (3.2.6) holds. Thus, to complete the proof, it is sufficient to show that Lemma 2.0.4 is applicable for the map Ψ_1 with $\eta_0 := s^*$, $r := r_{s_2}$ and $\lambda := \dfrac{4}{7}$. To do this, it remains to prove that both assertions (a) and (b) of Lemma 2.0.4 hold. It is obvious that $s^* \in R_{s^*}^{-1}(0) \cap \mathbb{B}_{r_{s_2}}(s^*)$. We get that from the definition of the excess e,

$$\operatorname{dist}\big(s^*, \Psi_1(s^*)\big) \leq e\Big(R_{s^*}^{-1}(0) \cap \mathbb{B}_{r_{s_2}}(s^*), \Psi_1(s^*)\Big). \tag{3.2.10}$$

In addition, for all $s_0,\ s_1 \in \mathbb{B}_{r_{s_2}}(s^*)$ such that s_0, s_1 and s^* are distinct, we have from (3.2.2) that

$$\begin{aligned}
\|G_1(s^*)\| &= \|\zeta(s^*) + g(s^*) - \zeta(s_1) - g(s_1) - \big(\nabla\zeta(s_1) + [s_0,\, s_1;\, g]\big)(s^* - s_1)\| \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|g(s^*) - g(s_1) - [s_0,\, s_1;\, g](s^* - s_1)\| \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|[s_1,\, s^*;\, g](s^* - s_1) - [s_0,\, s_1;\, g](s^* - s_1)\| \\
&\qquad \text{[By using definition 2.0.12].} \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|\big([s_1,\, s^*;\, g] - [s_0,\, s_1;\, g]\big)(s^* - s_1)\|
\end{aligned}$$

Since $\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1) = \int_0^1 [\nabla\zeta(s_1 + f(s^* - s_1)) - \nabla\zeta(s_1)](s^* - s_1)df$, we have that

$$\begin{aligned}
\|G_1(s^*)\| &= \int_0^1 \|[\nabla\zeta(s_1 + f(s^* - s_1)) - \nabla\zeta(s_1)](s^* - s_1)\|df \\
&\qquad + \|[s_1,\, s^*;\, g] - [s_0,\, s_1;\, g](s^* - s_1)\| \\
&= \epsilon \int_0^1 \|(s^* - s_1)df\| + \kappa\big(\|s_1 - s_0\| + \|s^* - s_1\|\big)\|s^* - s_1\| \\
&\qquad \text{[By using assumption (A4) and (A5)]} \\
&= \epsilon\|s^* - s_1\| \int_0^1 df + \kappa\big(\|s_1 - s_0\| + \|s^* - s_1\|\big)\|s^* - s_1\| \qquad (3.2.11) \\
&= \epsilon\|s^* - s_1\| + \kappa\big(\|s_1 - s^* + s^* - s_0\| + \|s^* - s_1\|\big)\|s^* - s_1\| \\
&= \epsilon\delta + \kappa(2\delta + \delta)\delta = \epsilon\delta + 3\kappa\delta^2 \\
&\leq \epsilon\delta + 3\kappa\delta = (\epsilon + 3\kappa)\delta \ < r_0 \qquad (\text{ by } 3.2.8).
\end{aligned}$$

This together with (3.2.7) and (3.2.10) (with $t_1 = 0$ and $t_2 = G_1(s^*)$) implies that

$$\begin{aligned}
\operatorname{dist}\big(s^*, \Psi_1(s^*)\big) &\leq M\|t_1 - t_2\| \;\leq M\|G_1(s^*)\| \\
&\leq M\Big(\epsilon\|s^* - s_1\| + \kappa\big(\|s_1 - s_0\| + \|s^* - s_0\|\big)\|s^* - s_1\|\Big) \quad (\text{ by using 3.2.11}) \\
&\leq M\big(\epsilon + 2\kappa\|s_1 - s_0\|\big)\|s_1 - s^*\| \\
&\leq M\big(\epsilon + 4\kappa\delta\big)\|s_1 - s^*\| \\
&\leq M\big(\epsilon + 4\kappa\big)\|s_1 - s^*\|, \quad \text{Since } \delta \leq 1 \quad (\text{by using 3.2.8}) \\
&\leq (1 - \frac{4}{7})r_{s_2} \;=\; r(1-\lambda).
\end{aligned}$$

Hence assertion (a) of Lemma 2.0.4 is satisfied.

Now, we show that assertion (b) of Lemma 2.0.4 is also satisfied. Let $s \in \mathbb{B}_\delta(s^*)$. Then

$$\begin{aligned}
\|G_1(s)\| &= \|\zeta(s^*) + g(s) - \nabla\zeta(s^*)(s^* - s) - \zeta(s_1) - g(s_1) - \big(\nabla\zeta(s_1) + [s_0,\, s_1;\, g]\big)(s^* - s_1)\| \\
&= \|\zeta(s^*) - \zeta(s) + \zeta(s) - \zeta(s_1) - \nabla\zeta(s^*)(s^* - s) + g(s) - g(s_1) \\
&\quad - (\nabla\zeta(s_1)(s - s_1) - [s_0,\, s_1;\, g])(s - s_1)\| \\
&\leq \|\zeta(s^*) - \zeta(s) - \nabla\zeta(s^*)(s^* - s)\| + \|\zeta(s) - \zeta(s_1) - \nabla\zeta(s_1)(s - s_1)\| \\
&\quad + \|g(s) - g(s_1) - [s_0,\, s_1;\, g](s - s_1)\| \\
&\leq \epsilon\|s - s^*\| + \epsilon\|s - s_1\| + \|[s_1,\, s;\, g](s - s_1) - [s_0,\, s_1;\, g](s - s_1)\| \\
&= \epsilon\|s - s^*\| + \epsilon\|s - s_1\| + \|[s_1,\, s;\, g] - [s_0,\, s_1;\, g]\|\|s - s_1\| \\
&\leq \epsilon\|s - s^*\| + \epsilon\|s - s_1\| + \kappa\big(\|s_1 - s_0\| + \|s - s_1\|\big)\|s - s_1\| \\
&\leq \epsilon\delta + 2\epsilon\delta + \kappa(2\delta + 2\delta)2\delta \\
&= 3\epsilon\delta + 8\kappa\delta^2 \;\leq\; 3\epsilon\delta + 8\kappa\delta, \qquad \text{Since } \delta \leq 1 \\
&= (3\epsilon + 8\kappa)\delta \;< r_0 \quad (\text{by 3.2.8}).
\end{aligned}$$

Hence we deduce that for all $s \in \mathbb{B}_\delta(s^*), G_1(s) \in \mathbb{B}_{r_0}(0)$. Let $s',\, s'' \in \mathbb{B}_\delta(s^*)$. This together with (3.2.7) (with $t_1 = G_1(s')$, and $t_2 = G_1(s'')$) we get

$$\begin{aligned}
e\Big(\Psi_1(s') \cap \mathbb{B}_{r_{s_2}}(s^*), \Psi_1(s'')\Big) &\leq e\Big(\Psi_1(s') \cap \mathbb{B}_\delta(s^*),\ \Psi_1(s'')\Big) \\
&= e\Big(R_{s^*}^{-1}[G_1(s')] \cap \mathbb{B}_\delta(s^*), R_{s^*}^{-1}[G_1(s'')]\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq M\|G_1(s') - G_1(s'')\| \\
&\leq M\|\big(\nabla\zeta(s^*) - \nabla\zeta(s_1)\big)(s' - s'') + M\|g(s') - g(s'') - [s_0,\, s_1;\, g](s' - s'')\| \\
&\leq M\epsilon\|s' - s''\| + M\|[s'' - s';\, g](s' - s'') - [s_0,\, s_1;\, g](s' - s'')\| \\
&\leq M\epsilon\|s' - s''\| + M\|\big([s'' - s';\, g] - [s_0,\, s_1;\, g]\big)(s' - s'')\| \\
&\leq M\epsilon\|s' - s''\| + M\|\kappa\big(\|s'' - s_0\| + \|s' - s_1\|\big)\|\|s' - s''\| \\
&\leq M\epsilon\|s' - s''\| + M\kappa(2\delta + 2\delta)\|s' - s''\| \\
&\leq M(\epsilon + 4\kappa\delta)\|s' - s''\| \qquad (3.2.12)
\end{aligned}
$$

Due to the relation $28M\kappa\delta \leq 4 - 7M\epsilon$ in (3.2.8), we obtain from (3.2.12) that

$$e\Big(\Psi_1(s') \cap \mathbb{B}_{r_{s_2}}(s^*),\, \Psi_1(s'')\Big) \leq \frac{4}{7}\|s' - s''\| = \lambda\|s' - s''\|.$$

Thus assertion (b) of Lemma 2.0.4 is satisfied. This completes the proof of the Lemma. □

Theorem 3.2.1. *Let s^* be a solution of (1.0.1). Suppose that assumptions (A0)-(A5) are satisfied. Let C be defined in (3.2.4). Then for every C, there exists $\delta > 0$ such that for every starting point $s_0,\, s_1 \in \mathbb{B}_\delta(s^*)$, there exists a sequence $\{s_k\}$ which is constructed by (3.1.3) with initial point $s_0,\, s_1$ which converges to s^* and satisfies the following inequality*

$$\|s_{k+1} - s^*\| \leq C\|s_k - s^*\| \quad \text{for each } k = 1,\, 2, ... \qquad (3.2.13)$$

Proof. By Lemma 3.2.1, for every C, there exists $\delta > 0$ such that for each $s_0,\, s_1 \in \mathbb{B}_\delta(s^*)$, there is $s_2 \in \mathbb{B}_\delta(s^*)$ such that (3.2.5) and (3.2.6) hold. Let $s_0,\, s_1 \in \mathbb{B}_\delta(s^*)$. It follows from Lemma 3.2.1 that there exists $s_2 \in \mathbb{B}_\delta(s^*)$ such that

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0,\, s_1;\, g])(s_2 - s_1) + \xi(s_2)$$

and

$$\|s_2 - s^*\| \leq r_{s_2} \leq C\|s_1 - s^*\|$$

and so (3.2.13) holds for $k = 1$. We will proceed by induction method. Now assume that $s_0,\, s_1,\, ..., s_k$ are generated by (3.1.3) satisfying (3.2.13). Then by Lemma 3.2.1, we can choose $s_{k+1} \in \mathbb{B}_\delta(s^*)$ such that

$$0 \in \zeta(s_k) + g(s_k) + (\nabla\zeta(s_k) + [s_{k-1},\, s_k;\, g])(s_{k+1} - s_k) + \xi(s_{k+1})$$

and

$$\|s_{k+1} - s^*\| \leq r_{s_2} \leq C\|s_k - s^*\|,$$

and so (3.2.13) holds for all $k \geq 1$. This completes the proof of the Theorem.

□

3.2.2 Quadratic Convergence

The subsection is devoted to study quadratic convergence result of the N-type method (3.1.3). To do this we will take the following assumptions:

(A6)$\nabla\zeta$ is Lipschitz continuous in a neighbourhood of s^* with constant L i.e, for every s, $t \in \Upsilon$, we have that,

$$\|\nabla\zeta(s) - \nabla\zeta(y)\| < L\|s - t\|.$$

(A7) g admits FODD i.e, there exists $\kappa > 0$ such that, for all s, t, s', $t' \in \Upsilon$,

$$\|[s, t; g] - [s', t'; g]\| \leq \kappa(\|s - s'\|^2 + \|t - t'\|^2) \quad \text{with } s' \neq s,\ t' \neq t.$$

Let M, L and κ be defined in (A3), (A6) and (A7) such that $3M(L + 8\kappa) < 1$. *Let*

$$\text{Set } \gamma = \frac{3M(L + 8\kappa)}{2}. \tag{3.2.14}$$

Then we obtain that $\gamma < \dfrac{1}{2}$.

Lemma 3.2.2. *Let s^* be a solution of the variational inclusion (1.0.1). Suppose that assumptions (A0)-(A3), (A6) and (A7) are hold. Let γ be defined by (3.2.14). Then for every such γ, there exists $\delta > 0$ such that for every distinct starting point $s_0, s_1 \in B_\delta(s^*)$, there exists a sequence $\{s_2\}$, defined by*

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0,\ s_1;\ g])(s_2 - s_1) + \xi(s_2) \tag{3.2.15}$$

and the map Ψ_1 has a fixed point s_2 in $B_\delta(s^*)$, which satisfies

$$\|s_2 - s^*\| \leq \gamma\ \|s_1 - s^*\|\max\{\|s_1 - s^*\|, \|s_1 - s_0\|\}. \tag{3.2.16}$$

Proof. The assumption (A3) implies that the mapping $R_{s^*}^{-1}$ is M-pseudo-Lipschitz around $(0, s^*)$. Hence there exists $r_{s^*} > 0$ and $r_0 > 0$ such that

$$e\big(R_{s^*}^{-1}(t_1) \cap \mathbb{B}_{r_{s^*}}(s^*), R_{s^*}^{-1}(t_2)\big) \leq M\|t_1 - t_2\| \text{ for any } t_1,\, t_2 \in \mathbb{B}_{r_0}(0). \tag{3.2.17}$$

Let $\delta > 0$ be such that

$$\delta \leq \max\Big\{r_{s^*}, \sqrt{\frac{2r_0}{5L+32\kappa}}, \frac{2}{3M(5L+8\kappa)}, 1\Big\}. \tag{3.2.18}$$

Fix $s_0, s_1 \in \mathbb{B}_\delta(s^*)$ such that $s_0 \neq s_1 \neq s^*$ and define

$$r_{s_2} = \gamma\ \|s_1 - s^*\|\max\{\|s_1 - s^*\|, \|s_0 - s^*\|\}.$$

This implies that $r_{s_2} \leq \gamma\delta.\delta\ \leq \gamma\delta\ \leq \delta$ and hence $r_{s_2}\ \leq \delta \leq r_{s^*}$.
Lemma 2.0.4 will be applied to the map Ψ_1 with $\eta_0 := s^*$ and $r := r_{s_2}$ and $\lambda := \dfrac{2}{3}$ to conclude that there exists a fixed point $s_2 \in \mathbb{B}_{r_{s_2}}(s^*)$ such that $s_2 \in \Psi_1(s_2)$, that is, $G_1(s_2) \in R_{s^*}^{-1}(s_2)$, which implies that

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0,\, s_1;\, g])(s_2 - s_1) + \xi(s_2),$$

Furthermore, $s_2 \in \mathbb{B}_{r_{s_2}}(s^*) \subseteq \mathbb{B}_\delta(s^*)$ and so

$$\|s_2 - s^*\| \leq r_{s_2} \leq \gamma\ \|s_1 - s^*\|\max\{\|s_1 - s^*\|, \|s_1 - s_0\|\}.$$

Thus, to complete the proof, it is sufficient to show that Lemma 2.0.4 is applicable for the map Ψ_1 with $\eta_0 := s^*$ and $r := r_{s_2}$ and $\lambda := \dfrac{2}{3}$. To do this, it remains to prove that both assertions (a) and (b) of Lemma 2.0.4 hold. It is obvious that $s^* \in R_{s^*}^{-1}(0) \cap \mathbb{B}_{r_{s_2}}(s^*)$. We get that from the definition of the excess e is as follows:

$$\operatorname{dist}\big(s^*, \Psi_1(s^*)\big) \ \leq\ e\Big(R_{s^*}^{-1}(0) \cap \mathbb{B}_{r_{s_2}}(s^*), \Psi_1(s^*)\Big). \tag{3.2.19}$$

In addition, for all $s_0,\ s_1 \in \mathbb{B}_{r_{s_2}}(s^*)$ such that s_0, s_1 and s^* are distinct, we have from (3.2.2) that

$$\begin{aligned}
\|G_1(s^*)\| &= \|\zeta(s^*) + g(s^*) - \zeta(s_1) - g(s_1) - \big(\nabla\zeta(s_1) + [s_0,\, s_1;\, g]\big)(s^* - s_1)\| \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|g(s^*) - g(s_1) - [s_0,\, s_1;\, g](s^* - s_1)\| \\
&\qquad \text{[By using definition 2.0.12].} \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|[s_1,\, s^*;\, g](s^* - s_1) - [s_0,\, s_1;\, g](s^* - s_1)\| \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|\big([s_1,\, s^*;\, g] - [s_0,\, s_1;\, g]\big)(s^* - s_1)\|
\end{aligned}$$

Since $\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1) = \int_0^1 [\nabla\zeta(s_1 + f(s^* - s_1)) - \nabla\zeta(s_1)](s^* - s_1)df$,

$$\begin{aligned}
\|G_1(s^*)\| &= \int_0^1 [\nabla\zeta(s_1 + f(s^* - s_1)) - \nabla\zeta(s_1)](s^* - s_1)df + \|[s_1, s^*; g] - [s_0, s_1; g]\|\|s^* - s_1\| \\
&= \int_0^1 L\|s_1 + f(s^* - s_1) - s_1\|\|s^* - s_1\|df + \kappa\big(\|s_1 - s_0\|^2 + \|s^* - s_1\|^2\big)\|s^* - s_1\| \\
&\quad \text{[By using assumption (A6) and (A7)]} \\
&= \int_0^1 L\|f(s^* - s_1)\|df\|s^* - s_1\| + \kappa\big(\|s_1 - s_0\|^2 + \|s^* - s_1\|^2\big)\|s^* - s_1\| \\
&\le \frac{L}{2}\|s^* - s_1\|^2 + 2\kappa\|s_1 - s_0\|^2\|s^* - s_1\| \\
&\le \frac{L}{2}\|s^* - s_1\|^2 + 2\kappa.2\delta\|s_1 - s_0\|\|s^* - s_1\| \\
&\le \frac{L}{2}\|s^* - s_1\|^2 + 4\kappa\|s_1 - s_0\|\|s^* - s_1\|, \text{ since } \delta \le 1 \qquad (3.2.20) \\
&\le \frac{L}{2}\delta^2 + 8\kappa.\delta.\delta \\
&\le \big(\frac{L}{2} + 8\kappa\big)\delta^2 \; < r_0 \qquad (\text{ by } 3.2.18).
\end{aligned}$$

This together with (3.2.17) and (3.2.19) (with $t_1 = 0$ and $t_2 = G_1(s^*)$) we get

$$\begin{aligned}
\text{dist}\Big(s^*, \Psi_1(s^*)\Big) &\le M\|t_1 - t_2\| \; \le M\|G_1(s^*)\| \\
&\le M\Big(\frac{L}{2}\|s^* - s_1\|^2 + 4\kappa\|s_1 - s_0\| \; \|s^* - s_1\|\Big) \qquad (\text{ by using } 3.2.20) \\
&\le M\Big(\frac{L}{2} + 4\kappa\Big)\|s_1 - s^*\|\max\Big\{\|s_1 - s^*\|, \|s_1 - s_0\|\Big\} \\
&\le \Big(1 - \frac{2}{3}\Big)\frac{3M(L + 4\kappa)}{2}\|s_1 - s^*\|\max\Big\{\|s_1 - s^*\|, \|s_1 - s_0\|\Big\} \\
&\le \Big(1 - \frac{2}{3}\Big)r_{s_2} \; = \; r(1 - \lambda).
\end{aligned}$$

So assertion (a) of Lemma 2.0.4 is satisfied.

Now, we show that assertion (b) of Lemma 2.0.4 is also satisfied. Let $s \in \mathbb{B}_{r_{s_2}}(s^*) \subseteq \mathbb{B}_\delta(s^*)$. Then

$$
\begin{aligned}
\|G_1(s)\| &= \|\zeta(s^*) + g(s) - \nabla\zeta(s^*)(s^* - s) - \zeta(s_1) - g(s_1) - \big(\nabla\zeta(s_1) + [s_0,\, s_1;\, g]\big)(s^* - s_1)\| \\
&= \|\zeta(s^*) - \zeta(s) + \zeta(s) - \zeta(s_1) - \nabla\zeta(s^*)(s^* - s) + g(s) - g(s_1) \\
&\quad - (\nabla\zeta(s_1)(s - s_1) - [s_0,\, s_1;\, g])(s - s_1)\| \\
&\leq \|\zeta(s^*) - \zeta(s) - \nabla\zeta(s^*)(s^* - s)\| + \|\zeta(s) - \zeta(s_1) - \nabla\zeta(s_1)(s - s_1)\| \\
&\quad + \|g(s) - g(s_1) - [s_0,\, s_1;\, g](s - s_1)\| \\
&\leq \frac{L}{2}\|s - s^*\|^2 + \frac{L}{2}\|s - s_1\|^2 + \|[s_1,\, s;\, g](s - s_1) - [s_0,\, s_1;\, g](s - s_1)\| \\
&= \frac{L}{2}\|s - s^*\|^2 + \frac{L}{2}\|s - s_1\|^2 + \|[s_1,\, s;\, g] - [s_0,\, s_1;\, g]\|\|s - s_1\| \\
&\leq \frac{L}{2}\|s - s^*\|^2 + \frac{L}{2}\|s - s_1\|^2 + \kappa\big(\|s_1 - s_0\|^2 + \|s - s_1\|^2\big)\|s - s_1\| \\
&\leq \frac{L}{2}\delta^2 + \frac{L}{2}(2\delta)^2 + \kappa\big((2\delta)^2 + (2\delta)^2\big)2\delta \\
&\leq \frac{L}{2}\delta^2 + 2L\delta^2 + 16\kappa\delta^3 \quad \text{Since } \delta \leq 1 \\
&\leq \frac{L}{2}\delta^2 + 2L\delta^2 + 16\kappa\delta^2 \\
&= \big(\frac{5L}{2} + 16\kappa\big)\delta^2 \; < r_0 \quad \text{(by 3.2.18)}.
\end{aligned}
$$

So we deduce that for all $s \in \mathbb{B}_\delta(s^*), G_1(s) \in \mathbb{B}_{r_0}(0)$. Let $s',\, s'' \in \mathbb{B}_{r_{s_2}}(s^*)$. This together with (3.2.17) $\big($with $t_1 = G_1(s')$, and $t_2 = G_1(s'')\big)$ we get

$$
\begin{aligned}
& e\Big(\Psi_1(s') \cap \mathbb{B}_{r_{s_2}}(s^*),\, \Psi_1(s'')\Big) \leq e\Big(\Psi_1(s') \cap \mathbb{B}_\delta(s^*),\, \Psi_1(s'')\Big) \\
&\quad = e\Big(R_{s^*}^{-1}[G_1(s')] \cap \mathbb{B}_\delta(s^*), R_{s^*}^{-1}[G_1(s'')]\Big) \\
&\quad \leq M\|G_1(s') - G_1(s'')\| \\
&\quad \leq M\|\big(\nabla\zeta(s^*) - \nabla\zeta(s_1)\big)(s' - s'')\| + M\|g(s') - g(s'') - [s_0,\, s_1;\, g](s' - s'')\| \\
&\quad \leq ML\|s^* - s_1\|\|s' - s''\| + M\|[s'' - s';\, g](s' - s'') - [s_0,\, s_1;\, g](s' - s'')\| \\
&\quad \leq ML\|s^* - s_1\|\|s' - s''\| + M\|\big([s'' - s';\, g] - [s_0,\, s_1;\, g]\big)(s' - s'')\| \\
&\quad \leq ML\|s^* - s_1\|\|s' - s''\| + M\big(\kappa\|s'' - s_0\|^2 + \|s' - s_1\|^2\big)\|s' - s''\| \\
&\quad \leq ML\delta\|s' - s''\| + M\kappa\big((2\delta)^2 + (2\delta)^2\big)\|s' - s''\| \\
&\quad \leq ML\delta\|s' - s''\| + M\kappa 8\delta^2\|s' - s''\| \\
&\quad \leq M(L + 8\kappa)\delta\|s' - s''\| \qquad (3.2.21)
\end{aligned}
$$

Now using the relation $3M(L+8\kappa)\delta \leq 2$ from (3.2.18) in (3.2.21) we have

$$e\Big(\Psi_1(s') \cap \mathbb{B}_{r_{s_2}}(s^*),\ \Psi_1(s'')\Big) \leq \frac{2}{3}\|s'-s''\| = \lambda\|s'-s''\|.$$

Thus assertion (b) of Lemma 2.0.4 is satisfied. This completes the proof of the Lemma. □

Theorem 3.2.2. *Let s^* be a solution of the variational inclusion (1.0.1). Suppose that assumptions (A0)-(A3), (A6) and (A7) are satisfied. Let γ be defined in (3.2.14). Then for every γ, there exists $\delta > 0$ such that for every starting point $s_0,\ s_1 \in \mathbb{B}_\delta(s^*)$, there exists a sequence $\{s_k\}$ which is constructed by the method (3.1.3) with two initial point $s_0,\ s_1$ which converges to s^* and satisfies that*

$$\|s_{k+1}-s^*\| \leq \gamma\ \|s_k - s^*\|\max\big\{\|s_k-s^*\|, \|s_k-s_{k-1}\|\big\} \quad \text{for each } k=1,\,2,\ldots \tag{3.2.22}$$

Proof. By Lemma 3.2.2, for every γ, there exists $\delta > 0$ such that for each $s_0,\ s_1 \in \mathbb{B}_\delta(s^*)$, there is $s_2 \in \mathbb{B}_\delta(s^*)$ such that (3.2.15) and (3.2.16) hold. Let $s_0,\ s_1 \in \mathbb{B}_\delta(s^*)$. It follows from Lemma 3.2.2 that there exists $s_2 \in \mathbb{B}_\delta(s^*)$ such that

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0,\ s_1;\ g])(s_2 - s_1) + \xi(s_2)$$

and

$$\|s_2 - s^*\| \leq r_{s_2} \leq \gamma\ \|s_1-s^*\|\max\big\{\|s_1-s^*\|, \|s_1-s_0\|\big\}$$

and so (3.2.22) holds for $k=1$. We will proceed by induction method. Now assume that $s_0,\ s_1,\ \ldots, s_k$ are generated by (3.1.3) satisfying (3.2.22). Then by Lemma 3.2.1, we can choose $s_{k+1} \in \mathbb{B}_\delta(s^*)$ such that

$$0 \in \zeta(s_k) + g(s_k) + (\nabla\zeta(s_k) + [s_{k-1},\ s_k;\ g])(s_{k+1} - s_k) + \xi(s_{k+1})$$

and

$$\|s_{k+1} - s^*\| \leq r_{s_{k+1}} \leq \gamma\ \|s_k-s^*\|\max\big\{\|s_k-s^*\|, \|s_k-s_{k-1}\|\big\}.$$

and so (3.2.22) holds for all $k \geq 1$. The Theorem is proved. □

3.2.3 Superlinear Convergence

The subsection is devoted to study the superlinear convergence result of the N-type method (3.1.3). To do this we will take the following assumptions:

(A8)$\nabla\zeta$ is Hölder continuous in a neighbourhood of s^* with constant L i.e, for every s, $t \in \Upsilon$, we have that,

$$\|\nabla\zeta(s) - \nabla\zeta(t)\| < L\|s - t\|^p.$$

(A9) g admits FODD i.e, there exists $\kappa > 0$ such that, for all s, t, s', $t' \in \Upsilon$,

$$\|[s, t; g] - [s', t'; g]\| \leq \kappa(\|s - s'\|^p + \|t - t'\|^p), \quad \text{with } s' \neq s,\ t' \neq t.$$

Let M, L and κ be defined in (A3),(A8) and (A9) such that $M(3p+5)\{L+8(p+1)\kappa\} < 1$.

$$\text{Then we obtain that } \sigma < \frac{M(3p+5)\{L+4(p+1)\kappa\}}{2(p+1)} \text{ and } q = (L + 2^{p+1}\kappa). \tag{3.2.23}$$

This together with the above inequality implies that $\sigma < \dfrac{1}{2(p+1)}$.

Lemma 3.2.3. *Let s^* be a solution of the variational inclusion (1.0.1). Suppose that assumptions (A0)-(A3), (A8) and (A9) are hold. Let σ be defined by the method (3.2.23). Then for every such σ, there exists $\delta > 0$ such that for every distinct starting point $s_0, s_1 \in B_\delta(s^*)$, there exists a sequence $\{s_2\}$, defined by*

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0,\ s_1;\ g])(s_2 - s_1) + \xi(s_2) \tag{3.2.24}$$

and the map Ψ_1 has a fixed point s_2 in $B_\delta(s^*)$, which satisfies

$$\|s_2 - s^*\| \leq \sigma\ \|s_1 - s^*\|\max\{\|s_1 - s^*\|^p, \|s_1 - s_0\|^p\}. \tag{3.2.25}$$

Proof. The assumption (A3) implies that the mapping $R_{s^*}^{-1}$ is M-pseudo-Lipschitz around $(0, s^*)$. Hence there exists $r_{s^*} > 0$ and $r_0 > 0$ such that

$$e\big(R_{s^*}^{-1}(t_1) \cap \mathbb{B}_{r_{s^*}}(s^*), R_{s^*}^{-1}(t_2)\big) \leq M\|t_1 - t_2\| \text{ for any } t_1,\ t_2 \in \mathbb{B}_{r_0}(0). \tag{3.2.26}$$

Let $\delta > 0$, such that

$$\delta \leq \max\Big\{r_{s^*},\ \Big(\frac{r_0(p+1)}{L(2^{p+1}+1)+(p+1)2^{p+2}\kappa}\Big)^{\frac{1}{p+1}},\ \Big(\frac{3(p+1)}{Mq(3p+5)}\Big)^{\frac{1}{p}},\ 1\Big\}. \tag{3.2.27}$$

Fix $s_0, s_1 \in \mathbb{B}_\delta(s^*)$ such that $s_0 \neq s_1 \neq s^*$ and define

$$r_{s_2} = \sigma \ \|s_1 - s^*\| \max\big\{\|s_1 - s^*\|^p, \|s_1 - s_0\|^p\big\}.$$

This implies that $r_{s_2} \leq \sigma\delta \ .\delta^p \ \leq \sigma\delta^{p+1} \ \leq \delta$ and hence $r_{s_2} \ \leq \delta \leq r_{s^*}$.
Lemma 2.0.4 will be applied to the map Ψ_1 with $\eta_0 := s^*$ and $r := r_{s_2}$ and $\lambda := \dfrac{3(p+1)}{3p+5}$ to conclude that there exists a fixed point $s_2 \in \mathbb{B}_{r_{s_2}}(s^*)$ such that $s_2 \in \Psi_1(s_2)$, that is, $G_1(s_2) \in R_{s^*}^{-1}(s_2)$, we get that

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0, \, s_1; \, g])(s_2 - s_1) + \xi(s_2),$$

Furthermore, $s_2 \in \mathbb{B}_{r_{s_2}}(s^*) \subseteq \mathbb{B}_\delta(s^*)$ and so

$$\|s_2 - s^*\| \leq r_{s_2} \leq \sigma \ \|s_1 - s^*\| \max\big\{\|s_1 - s^*\|^p, \|s_1 - s_0\|^p\big\}.$$

Thus, to complete the proof, it is sufficient to show that Lemma 2.0.4 is applicable for the map Ψ_1 with $\eta_0 := s^*$, $r := r_{s_2}$ and $\lambda := \dfrac{3(p+1)}{3p+5}$. To do this, it remains to prove that both assertions (a) and (b) of Lemma 2.0.4 hold. It is obvious that $s^* \in R_{s^*}^{-1}(0) \cap \mathbb{B}_{r_{s_2}}(s^*)$. We get that from the definition of the excess e is as follows:

$$\operatorname{dist}\big(s^*, \Psi_1(s^*)\big) \leq e\Big(R_{s^*}^{-1}(0) \cap \mathbb{B}_{r_{s_2}}(s^*), \Psi_1(s^*)\Big). \tag{3.2.28}$$

In addition, for all $s_0, \ s_1 \in \mathbb{B}_{r_{s_2}}(s^*)$ such that s_0, s_1 and s^* are distinct, we have from (3.2.2) that

$$\begin{aligned}
\|G_1(s^*)\| &= \|\zeta(s^*) + g(s^*) - \zeta(s_1) - g(s_1) - \big(\nabla\zeta(s_1) + [s_0, \, s_1; \, g]\big)(s^* - s_1)\| \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|g(s^*) - g(s_1) - [s_0, \, s_1; \, g](s^* - s_1)\| \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|[s_1, \, s^*; \, g](s^* - s_1) - [s_0, \, s_1; \, g](s^* - s_1)\| \\
&\qquad \text{[By using definition (2.0.12)].} \\
&\leq \|\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1)\| + \|\big([s_1, \, s^*; \, g] - [s_0, \, s_1; \, g]\big)(s^* - s_1)\|
\end{aligned}$$

Since $\zeta(s^*) - \zeta(s_1) - \nabla\zeta(s_1)(s^* - s_1) = \int_0^1 [\nabla\zeta(s_1 + f(s^* - s_1)) - \nabla\zeta(s_1)](s^* - s_1)df$,

$$
\begin{aligned}
&\|G_1(s^*)\| \\
= \;& \int_0^1 [\nabla\zeta(s_1 + f(s^* - s_1)) - \nabla\zeta(s_1)](s^* - s_1)df + \|[s_1,\, s^*;\, g] - [s_0,\, s_1;\, g]\|\; \|s^* - s_1\| \\
= \;& \int_0^1 L\|\big(s_1 + f(s^* - s_1) - s_1\big)\|^p\|s^* - s_1\|df + \kappa\big(\|s_1 - s_0\|^p + \|s^* - s_0\|^p\big)\|s^* - s_1\| \\
& \quad \text{[By using assumption (A8) and (A9)]} \\
= \;& \int_0^1 L\|f(s^* - s_1)\|^p df\|s^* - s_1\| + \kappa\big(\|s_1 - s_0\|^p + \|s^* - s_0\|^p\big)\|s^* - s_1\| \\
\leq \;& \frac{L}{p+1}\|s^* - s_1\|^{p+1} + 2\kappa\|s_1 - s_0\|^p\|s^* - s_1\| \qquad (3.2.29) \\
\leq \;& \frac{L}{p+1}\|s^* - s_1\|^{p+1} + 2\kappa(2\delta)^p.\delta \\
\leq \;& \frac{L}{p+1}\delta^{p+1} + 2^{p+1}\delta^{p+1}\kappa \\
\leq \;& \big(\frac{L}{p+1} + 2^{p+1}\kappa\big)\delta^{p+1} \;\; < r_0 \qquad (\text{ by } 3.2.27).
\end{aligned}
$$

This together with (3.2.26) and (3.2.28) (with $t_1 = 0$ and $t_2 = G_1(s^*)$)

$$
\begin{aligned}
\text{dist}\big(s^*, \Psi_1(s^*)\big) \;&\leq\; M\|t_1 - t_2\| \;\leq M\|G_1(s^*)\| \\
&\leq\; \frac{ML}{p+1}\|s^* - s_1\|^{p+1} + 2\kappa\|s_1 - s_0\|^p\|s^* - s_1\| \quad (\text{ by using } 3.2.29) \\
&\leq\; M\big(\frac{L}{p+1} + 4\kappa\big)\|s_1 - s^*\|\max\{\|s_1 - s^*\|^p, \|s_1 - s_0\|^p\} \\
&\leq\; \frac{M\big\{L + 4(p+1)\kappa\big\}}{p+1}\|s_1 - s^*\|\max\{\|s_1 - s^*\|^p, \|s_1 - s_0\|^p\} \\
&\leq\; \frac{2\sigma}{3p+5}\|s_1 - s^*\|\max\{\|s_1 - s^*\|^p, \|s_1 - s_0\|^p\} \text{ from(3.2.23)} \\
&\leq\; \big(1 - \frac{3(p+1)}{3p+5}\big)r_{s_2} \\
&\leq\; r(1 - \lambda).
\end{aligned}
$$

Hence assertion (a) of Lemma 2.0.4 is satisfied.

Now, we show that assertion (b) of Lemma 2.0.4 is also satisfied. Let $s \in \mathbb{B}_{r_{s_2}}(s^*) \subseteq \mathbb{B}_\delta(s^*)$. Then

$$
\begin{aligned}
\|G_1(s)\| &= \|\zeta(s^*) + g(s) - \nabla\zeta(s^*)(s^* - s) - \zeta(s_1) - g(s_1) - \big(\nabla\zeta(s_1) + [s_0,\, s_1;\, g]\big)(s^* - s_1)\| \\
&= \|\zeta(s^*) - \zeta(s) + \zeta(s) - \zeta(s_1) - \nabla\zeta(s^*)(s^* - s) + g(s) - g(s_1) \\
&\quad - (\nabla\zeta(s_1)(s - s_1) - [s_0,\, s_1;\, g])(s - s_1)\| \\
&\le \|\zeta(s^*) - \zeta(s) - \nabla\zeta(s^*)(s^* - s)\| + \|\zeta(s) - \zeta(s_1) - \nabla\zeta(s_1)(s - s_1)\| \\
&\quad + \|g(s) - g(s_1) - [s_0,\, s_1;\, g](s - s_1)\| \\
&\le \frac{L}{p+1}\|s - s^*\|^{p+1} + \frac{L}{p+1}\|s - s_1\|^{p+1} + \|[s_1,\, s;\, g](s - s_1) - [s_0,\, s_1;\, g](s - s_1)\| \\
&= \frac{L}{p+1}\|s - s^*\|^{p+1} + \frac{L}{p+1}\|s - s_1\|^{p+1} + \|[s_1,\, s;\, g] - [s_0,\, s_1;\, g]\|\|s - s_1\| \\
&\le \frac{L}{p+1}\|s - s^*\|^{p+1} + \frac{L}{p+1}\|s - s_1\|^{p+1} + \kappa\big(\|s_1 - s_0\|^p + \|s - s_1\|^p\big)\|s - s_1\| \\
&\le \frac{L}{p+1}\delta^{p+1} + \frac{L}{p+1}(2\delta)^{p+1} + \kappa((2\delta)^p + (2\delta)^p)2\delta \\
&\le \frac{L}{p+1}\delta^{p+1}(2^{p+1} + 1) + \kappa\delta^{p+1}2^{p+2} \\
&\le \big(\frac{L}{p+1}(2^{p+1} + 1) + \kappa 2^{p+2}\big)\delta^{p+1} \\
&\le \frac{L(2^{p+1} + 1) + (p+1)2^{p+2}\kappa}{p+1}\,\delta^{p+1} \;< r_0 \quad \text{(by 3.2.18).}
\end{aligned}
$$

so we deduce that for all $s \in \mathbb{B}_\delta(s^*), G_1(s) \in \mathbb{B}_{r_0}(0)$. Let $s',\, s'' \in \mathbb{B}_{r_{s_2}}(s^*)$. This together with (3.2.26) (with $t_1 = G_1(s')$, and $t_2 = G_1(s'')$) we get

$$
\begin{aligned}
e\Big(\Psi_1(s') \cap \mathbb{B}_{r_{s_2}}(s^*),\, \Psi_1(s'')\Big) &\le e\Big(\Psi_1(s') \cap \mathbb{B}_\delta(s^*),\, \Psi_1(s'')\Big) \\
&= e\Big(R_{s^*}^{-1}[G_1(s')] \cap \mathbb{B}_\delta(s^*), R_{s^*}^{-1}[G_1(s'')]\Big) \\
&\le M\|G_1(s') - G_1(s'')\| \\
&\le M\|\big(\nabla\zeta(s^*) - \nabla\zeta(s_1)\big)(s' - s'')\| + M\|g(s') - g(s'') - [s_0,\, s_1;\, g](s' - s'')\| \\
&\le ML\|s^* - s_1\|^p\|s' - s''\| + M\|[s'' - s';\, g](s' - s'') - [s_0,\, s_1;\, g](s' - s'')\| \\
&\le ML\|s^* - s_1\|^p\|s' - s''\| + M\|\big([s'' - s';\, g] - [s_0,\, s_1;\, g]\big)(s' - s'')\| \\
&\le ML\|s^* - s_1\|^p\|s' - s''\| + M\big(\kappa\|s'' - s_0\|^p + \|s' - s_1\|^p\big)\|s' - s''\| \\
&\le ML\delta^p\|s' - s''\| + 2M\kappa(2\delta)^p\|s' - s''\| \\
&\le ML\delta^p\|s' - s''\| + 2^{p+1}M\kappa\delta^p\|s' - s''\| \\
&\le M(L + 2^{p+1}\kappa)\delta^p\|s' - s''\|
\end{aligned}
$$

$$\leq Mq\delta^p \|s' - s''\| \quad \text{(from 3.2.27)}$$
$$\leq \frac{3(p+1)}{3p+5} \|s' - s''\| \quad \leq \lambda \|s' - s''\|$$

Thus assertion (b) of Lemma 2.0.4 is satisfied. This completes the proof of the Lemma. □

Theorem 3.2.3. *Let s^* be a solution of the variational inclusion (1.0.1). Suppose that assumptions (A0)-(A3),(A8) and (A9) are satisfied. Let σ be defined in (3.2.23). Then for every σ, there exists $\delta > 0$ such that for every starting point s_0, $s_1 \in \mathbb{B}_\delta(s^*)$, there exists a sequence $\{s_k\}$ generated by the method (3.1.3) with initial point s_0, s_1 which converges to s^* and satisfies that*

$$\|s_{k+1} - s^*\| \leq \sigma \ \|s_k - s^*\| \max\{\|s_k - s^*\|^p, \|s_k - s_{k-1}\|^p\}, \quad \text{for each } k = 1, 2, ... \quad (3.2.30)$$

Proof. By Lemma 3.2.3, for every σ, there exists $\delta > 0$ such that for each s_0, $s_1 \in \mathbb{B}_\delta(s^*)$, there is $s_2 \in \mathbb{B}_\delta(s^*)$ such that (3.2.24) and (3.2.25) hold. Let s_0, $s_1 \in \mathbb{B}_\delta(s^*)$. It follows from Lemma 3.2.3 that there exists $s_2 \in \mathbb{B}_\delta(s^*)$ such that

$$0 \in \zeta(s_1) + g(s_1) + (\nabla\zeta(s_1) + [s_0, s_1; g])(s_2 - s_1) + \xi(s_2)$$

and

$$\|s_2 - s^*\| \leq r_{s_2} \leq \sigma \ \|s_1 - s^*\| \max\{\|s_1 - s^*\|^p, \|s_1 - s_0\|^p\}$$

and so (3.2.30) holds for $k = 1$. We will proceed by induction method. Now assume that s_0, s_1, ..., s_k are generated by (3.1.3) satisfying (3.2.30). Then by Lemma 3.2.1, we can choose $s_{k+1} \in \mathbb{B}_\delta(s^*)$ such that

$$0 \in \zeta(s_k) + g(s_k) + (\nabla\zeta(s_k) + [s_{k-1}, s_k; g])(s_{k+1} - s_k) + \xi(s_{k+1})$$

and

$$\|s_{k+1} - s^*\| \leq r_{s_{k+1}} \leq \sigma \ \|s_k - s^*\| \max\{\|s_k - s^*\|^p, \|s_k - s_{k-1}\|^p\}.$$

and so (3.2.30) holds for all $k \geq 1$. The Theorem is proved. □

3.3 Concluding Remarks

We have established local convergence results of the Newton-type method for approximating the solution of variational inclusion (1.0.1) under the assumptions that $R_{s^*}^{-1}(\cdot)$ is pseudo-Lipschitz and $\nabla\zeta$ is continuous, Lipschitz continuous and Hölder continuous respectively and g is admissible for FODD. More clearly, we have shown that the N-type method defined by (3.1.3) converges linearly, quadratically and superlinearly to the solution of (1.0.1) if $\nabla\zeta$ is continuous and Lipschitz continuous and Hölder continuous respectively, together with a divided difference admissible function g. This study improves and extends the results corresponding to [43].

Chapter 4

Semilocal and Local Convergence Analysis of an Extended Newton-type Method for Solving Variational Inclusions

This Chapter is dedicated to study an extended Newton-type method for finding the solution of the variational inclusions (1.0.1). Specially the linear and quadratic convergence by an extended Newton-type method which is defined by the Algorithm 2 is presented in Section 4.2, while in Section 4.3, an extended Newton-type method with Hölderian assumption which is defined by the Algorithm 3 is presented for finding the solution of the variational inclusion (1.0.1).

4.1 Introduction

EN-type Method can provide an effective tool to select nearly minimal norm solution from the infinite ones in relatively short computation time. In this chapter we are concerned with the problem of finding a solution of the variational inclusion (1.0.1) and we present the semilocal and local convergence of the EN-type method.

Let $\mathcal{S}$ and $\mathcal{T}$ be two Banach spaces and $\Upsilon \subseteq \mathcal{S}$. suppose that $\zeta : \Upsilon \to \mathcal{T}$ is a function, which is Fréchet differentiable and the derivative of this function is denoted by $\nabla\zeta$, the linear function $g : \Upsilon \to \mathcal{T}$ is differentiable at s^*, which may not differentiable in a neighborhood Υ of $\bar{s}$ and the FODD of g on the points s and t is denoted by $[s, t; g]$ and $\xi : \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ is a set-valued mapping which has closed graph acting between two Banach spaces. Here we consider a variational inclusion problem (1.0.1) to finding a point $\bar{s} \in \Upsilon$.

For solving (1.0.1), Hilout *et al.* [50] considered the following sequence

$$\begin{cases} \quad s_0 \text{ and } s_1 \text{ are given two starting points} \\ t_k = \alpha s_k + (1-\alpha)s_{k-1}; \qquad\qquad \text{when } \alpha \in (0,1) \\ 0 \in \zeta(s_k) + [t_k, s_k; \zeta](s_{k+1} - s_k) + \xi(s_{k+1}), \end{cases}$$

and when the function ζ is only continuous and differentiable at s^*, then the authors verified the convergence is superlinear by using this method. In addition, for two-point Newton-type methods in a Banach space setting under very general Lipschitz type conditions, it should be mentioned that Argyros [8] has studied local as well as semilocal convergence analysis for finding the solution (1.0.1) when $\xi = \{0\}$. When $g = 0$, this study has been extended by Rashid [100, 103, 104].

Suppose that $s \in \mathcal{S}$ and $\mathcal{P}(s)$ is the subset of $\mathcal{S}$, which is defined as

$$\mathcal{P}(s) = \big\{d \in \mathcal{S} \colon 0 \in \zeta(s) + g(s) + (\nabla\zeta(s) + [s+d, s; g])d + \xi(s+d)\big\}. \tag{4.1.1}$$

Argyros and Hilout [13] associated the Newton-type (N-type) method mentioned in the Algorithm 1 for finding the solution of the variational inclusion (1.0.1), which is as follows:

Algorithm 1 (The N-type Algorithm)

Iter. 1. Select $s_0 \in \mathcal{S}$, and place $k := 0$.

Iter. 2. In case $0 \in \mathcal{P}(s_k)$, then stop; otherwise, go to the next Stair 3.

Iter. 3. In case $0 \notin \mathcal{P}(s_k)$, choose d_k such that $d_k \in \mathcal{P}(s_k)$.

Iter. 4. Set $s_{k+1} := s_k + d_k$.

Iter. 5. $k+1$ is replace by k and repeat this cycle from Iter 2.

Using some compatible assumptions in the region of the solution s^*, for the variational inclusion (1.0.1), Argyros and Hilout [13, Theorem 4.1] presented a method which is mentioned by Algorithm 1. For any point in Υ, they showed that there exists a sequence and the sequence is quadratically convergent. This reflection we definitely understood that the convergence result guarantees the existence of a convergent sequence, which is mention in [13]. Consequently, for any initial point close to a solution, the sequences which is constructed by Algorithm 1 are not identically defined and not each constructed sequence is convergent. Therefore, from a numerical computational point of view this type of method is not convenient to apply in numerical practice. This difficulty inspired us to introduce a kind of method "so-called" extended Newton-type (EN-type) method which is employed in Algorithm 2. In this way, we contemplate the following EN-type method:

Algorithm 2 (The EN-type Method))

Iter. 0. Pick $\eta \in [1, \infty)$, $s_0 \in \mathcal{S}$, and put $k := 0$.

Iter. 1. In case $0 \in \mathcal{P}(s_k)$, then stop; otherwise, go to the next Stair 3.

Iter. 2. In case $0 \notin \mathcal{P}(s_k)$, choose d_k such that $d_k \in \mathcal{P}(s_k)$ and

$$\|d_k\| \leq \eta \text{ dist } (0, \mathcal{P}(s_k)).$$

Iter. 3. Set $s_{k+1} := s_k + d_k$.

Iter. 4. $k+1$ is replaced by k and repeat this circle from Step 1.

The above two Algorithm differs in two features. The difference between two Algorithms is that, Algorithm 2 generates at least one sequence and the generated each sequence is convergent. Algorithm 1 generated sequence but each sequence does not converge. That's why the sequences which is constructed by Algorithm 1 are not uniquely defined. By comparison with this two algorithms we can assume that algorithm 2 is more explicit than Algorithm 1 in numerical computation.

When we replace the set $\mathcal{P}(s)$ by the set

$$D(s) := \left\{ d \in \mathcal{S} \colon 0 \in \zeta(s) + g(s) + (\nabla\zeta(s) + [2d + s, s; g])d + \xi(s + d) \right\},$$

then the Algorithm 2 is reduced and the reduced algorithm is just like the algorithm which was proposed by Rashid [110].

If $\xi = \{0\}$ and $g = 0$, many mathematician have invented a number of useful results on semilocal convergence analysis for the GN method. For the detail one can refer to Dedieu and Kim [24]; Dedieu and Shub [25]; Xu and Li [130] or in the case when $\xi = C$ and $g = 0$, we can also refer to Li and Ng [72] for more details. In the case when $g = 0$, Rashid *et al.* [110] introduced GN method to obtain the solution of the variational inclusion (1.0.1) and established its semilocal convergence. Moreover, in the same case, Rashid [105, 106, 108] introduced different kinds of methods for obtaining the solution of (1.0.1) and attained the semilocal and local convergence.

The purpose of this section is to evaluate the semilocal and local convergence of the EN-type method which is constructed by Algorithm 2. In this section we deal with the Lipschitz-like property of set-valued mappings as the main tool which was introduced by Aubin [15], in the context of nonsmooth analysis and studied by many mathematicians (see for example, [2, 13, 30, 50, 88, 90]) and the reference therein.

4.2 Convergence Analysis of Extended Newton-type Method

This section is dedicated to show the existence of a sequence which is constructed by the EN-type method, represented by the Algorithm 2.

Let $s \in \mathcal{S}$. Then for each $s \in \mathcal{S}$, we get

$$\begin{aligned} g(s) + [s+d, s; g]d &= g(s) - [s+d, s; g](s - (s+d)) \\ &= g(s) - (g(s) - g(s+d)) = g(s+d). \end{aligned} \tag{4.2.1}$$

Let $\mathcal{R}_s$ be a set-valued mapping, which is defined by

$$R_s(\cdot) := \zeta(s) + g(\cdot) + \nabla\zeta(s)(\cdot - s) + \xi(\cdot).$$

It holds, for the formation of $\mathcal{P}(s)$ and (4.2.1), that

$$\mathcal{P}(s) = \big\{d \in \mathcal{S} : 0 \in \mathcal{G}_x(s+d)\big\}.$$

In addition, for any $z \in \mathcal{S}$ and $t \in \mathcal{T}$, we get the following identity:

$$z \in R_s^{-1}(t) \text{ if and only if } t \in \zeta(s) + g(z) + \nabla\zeta(s)(z - s) + \xi(z). \tag{4.2.2}$$

Particularly, let $(\bar{s}, \bar{t}) \in \text{gph} R_{\bar{s}}$. Then, the definition of closed graphness of $R_{\bar{s}}$ signifies that

$$\bar{s} \in R_{\bar{s}}^{-1}(\bar{t}). \tag{4.2.3}$$

The following outcome constitutes the equivalence between $R_{\bar{s}}^{-1}$ and $(\zeta + g + \xi)^{-1}$. This result is the modification of [108].

Lemma 4.2.1. *Let $(\bar{s}, \bar{t}) \in$ gph $(\zeta+g+\xi)$. Suppose that ζ is a Fréchet differentiable function in an open neighborhood Υ at $\bar{s}$ and its derivative $\nabla\zeta$ is continuous around $\bar{s}$. Assume that g admits FODD and g is Fréchet differentiable at $\bar{s}$. Then the followings relation are equivalent:*

(i) *At the point $(\bar{t}, \bar{s})$ the nature of the mapping $(\zeta + g + \xi)^{-1}$ is pseudo-Lipschitz,*

(ii) *At the point $(\bar{t}, \bar{s})$ the nature of the mapping $R_{\bar{s}}^{-1}$ is pseudo-Lipschitz.*

Proof. The function $h\colon \mathcal{S} \to \mathcal{T}$ is defined by

$$h(s) := -\zeta(s) + \zeta(\bar{s}) + \nabla\zeta(\bar{s})(s - \bar{s})$$

The proof is similar to that of [108], because the proof does not depend on the property of g. □

For our suitability, let $r_{\bar{s}} > 0$, $r_{\bar{t}} > 0$ and $\mathbb{B}_{r_{\bar{s}}}(\bar{s}) \subseteq \Upsilon \cap \text{dom}\,\xi$. Suppose that $\nabla\zeta$ is Lipschitz continuous on $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$, i.e, ∃'s $L > 0$ such that

$$\|\nabla\zeta(s) - \nabla\zeta(s')\| \leq L\|s - s'\|, \quad q \in (0, 1], \quad \text{for any } s, s' \in \mathbb{B}_{r_{\bar{s}}}(\bar{s}), \tag{4.2.4}$$

g admits a FODD satisfying Lipschitz condition, that is, there exists $\nu > 0$ such that, ∀ $s, t, v, w \in \mathbb{B}_{r_{\bar{s}}}(\bar{s})$ $(v \neq w, s \neq t)$,

$$\|[s, t; g] - [v, w; g]\| \leq \nu(\|s - v\| + \|t - w\|), \tag{4.2.5}$$

and $R_{\bar{s}}^{-1}$ is Lipschitz-like on the ball $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ including constant M, that is,

$$e(R_{\bar{s}}^{-1}(y_1) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}(y_2)) \leq M\|t_1 - t_2\| \quad \text{for any } t_1,\, t_2 \in \mathbb{B}_{r_{\bar{t}}}(\bar{t}). \tag{4.2.6}$$

Further, for $\bar{t}$, the closed graph property of $R_{\bar{s}}$ implies that $\zeta + g + \xi$ is continuous at $\bar{s}$ i.e.

$$\lim_{s \to \bar{s}} \text{dist}\big(\bar{t}, \zeta(s) + g(s) + \xi(s)\big) = 0 \tag{4.2.7}$$

is hold.

Let $\varepsilon > 0$ and we write

$$\bar{r} := \min\Big\{r_{\bar{t}} - 2\varepsilon r_{\bar{s}}, \frac{r_{\bar{s}}(1 - M\varepsilon)}{4M}\Big\}. \tag{4.2.8}$$

Then

$$\bar{r} > 0 \text{ if and only if } \varepsilon < \min\Big\{\frac{r_{\bar{t}}}{2r_{\bar{s}}},\ \frac{1}{M}\Big\}. \tag{4.2.9}$$

The following lemma is extracted from [110, Lemma 3.1] and the Lemma plays a very important role for convergence analysis of the EN-type method.

Lemma 4.2.2. *Assume that $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M, i.e,*

$$\sup_{s', s'' \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})} \|\nabla\zeta(s') - \nabla\zeta(s'')\| \leq \varepsilon < \min\Big\{\frac{r_{\bar{t}}}{2r_{\bar{s}}},\ \frac{1}{M}\Big\}. \tag{4.2.10}$$

Let $s \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ and ε be defined by (4.2.9). *Suppose that $\nabla\zeta$ is continuous on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Let $\bar{r}$ be defined by* (4.2.8) *such that* (4.2.10) *is true. Then R_s^{-1} is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\frac{M}{1-M\varepsilon}$, i.e,*

$$e(R_s^{-1}(t_1) \cap \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}), R_s^{-1}(t_2)) \leq \frac{M}{1 - M\varepsilon}\|t_1 - t_2\| \quad \text{for any} \quad t_1,\ t_2 \in \mathbb{B}_{\bar{r}}(\bar{t}).$$

Proof. Let

$$t_1,\ t_2 \in \mathbb{B}_{\bar{r}}(\bar{t}) \quad \text{and} \quad s' \in R_s^{-1}(t_1) \cap \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}). \tag{4.2.11}$$

It is enough to prove that $\exists\ s'' \in R_s^{-1}(t_2)$ such that

$$\|s' - s''\| \leq \frac{M}{1 - M\varepsilon}\|t_1 - t_2\|.$$

To finish this, we will justify that $\exists$'s a sequence $\{s_n\} \subset \mathbb{B}_{r_{\bar{s}}}(\bar{s})$ such that

$$t_2 \in \zeta(s) + g(s_n) + \nabla\zeta(s)(s_{n-1} - s) + \nabla\zeta(\bar{s})(s_n - s_{n-1}) + \xi(s_n), \tag{4.2.12}$$

and

$$\|s_n - s_{n-1}\| \leq M\|t_1 - t_2\|(M\varepsilon)^{n-2} \tag{4.2.13}$$

for each $n = 2, 3, 4, \ldots.$ the inequality (4.2.13) is hold. We proceed by mathematical induction on n. Letting

$$u_i := t_i - \zeta(s) - \nabla\zeta(s)(s_1 - s) + \zeta(\bar{s}) + \nabla\zeta(\bar{s})(s_1 - \bar{s}) \quad \text{for each } i = 1, 2.$$

From (4.2.11) we get that

$$\|s-s'\| \leq \|s-\bar{s}\| + \|\bar{s}-s'\| \leq r_{\bar{s}}.$$

Since $\nabla\zeta$ is continuous around $\bar{s}$ with the constant ε, we have that

$$\begin{aligned}\|\zeta(s)-\zeta(\bar{s})-\nabla\zeta(\bar{s})(s-\bar{s})\| &= \|\int_0^1 [\nabla\zeta(\bar{s}+f(s-\bar{s}))-\nabla\zeta(\bar{s})](s-\bar{s})df\| \\ &\leq \int_0^1 \|\nabla\zeta(\bar{s}+f(s-\bar{s}))-\nabla\zeta(\bar{s})\|\|s-\bar{s}\|df \\ &\leq \varepsilon\|s-\bar{s}\|\int_0^1 df \\ &= \varepsilon\|s-\bar{s}\|(1-0)=\varepsilon\|s-\bar{s}\|,\end{aligned}$$

From (4.2.11) and the relation $\bar{r} \leq r_{\bar{t}} - 2\varepsilon r_{\bar{s}}$ by (4.2.8), it follows that

$$\begin{aligned}\|u_i-\bar{t}\| &\leq \|u_i-\bar{t}\| + \|(\nabla\zeta(s)-\nabla\zeta(\bar{s}))(s-s')\| + \|\zeta(s)-\zeta(\bar{s})-\nabla\zeta(\bar{s})(s-\bar{s})\| \\ &\leq \bar{r}+\varepsilon(\|s-s'\|+\|s-\bar{s}\|) \\ &\leq \bar{r}+\varepsilon(r_{\bar{s}}+\frac{r_{\bar{s}}}{2}) \leq r_{\bar{t}}.\end{aligned}$$

The preceding inequality implies that $u_i \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$ for every $i=1,2$. Now denote $s_1 := s'$. Then $s_1 \in R_s^{-1}(t_1)$ by (4.2.11) and it follows from (4.2.2) that

$$t_1 \in \zeta(s)+g(s_1)+\nabla\zeta(s)(s_1-s)+\xi(s_1).$$

The alternative form of the above inclusion is as follows:

$$t_1+\zeta(\bar{s})+\nabla\zeta(\bar{s})(s_1-\bar{s})) \in \zeta(\bar{s})+\nabla\zeta(\bar{s})(s_1-\bar{s})+g(s_1)+\zeta(s)+\nabla\zeta(s)(s_1-s)+\xi(s_1).$$

According to the definition of u_1, this yields that

$$u_1 \in \zeta(\bar{s})+g(s_1)+\nabla\zeta(\bar{s})(s_1-\bar{s})+\xi(s_1).$$

So $s_1 \in R_{\bar{s}}^{-1}(u_1)$ by (4.2.2). Then by (4.2.11), we have that

$$s_1 \in R_{\bar{s}}^{-1}(u_1) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}).$$

Since $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$, then for every $u_1, u_2 \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$, we have through (4.2.6) that $\exists$'s $s_2 \in R_{\bar{s}}^{-1}(u_2)$ such that

$$\|s_2-s_1\| \leq M\|u_1-u_2\| = M\|t_1-t_2\|.$$

In addition, by the construction of u_2 and $s_1 = s'$, we get that

$$s_2 \in R_{\bar{s}}^{-1}(u_2) = R_{\bar{s}}^{-1}\big(t_2 - \zeta(s) - \nabla\zeta(s)(s_1 - s) + \zeta(\bar{s}) + \nabla\zeta(\bar{s})(s_1 - \bar{s})\big).$$

This inequality with (4.2.2), gives us

$$t_2 \in \zeta(s) + g(s_2) + \nabla\zeta(\bar{s})(s_2 - s_1) + \nabla\zeta(s)(s_1 - s) + \xi(s_2).$$

This implies that (4.2.12) and (4.2.13) are true with the generated points s_1 and s_2.

Let the points $s_1, s_2, \ldots, s_k$ be generated, that's why (4.2.12) and (4.2.13) are true for $n = 2, 3, \ldots, k$. Now we have to generate the new point s_{k+1} such that (4.2.12) and (4.2.13) are also true for $n = k + 1$. For showing this, let, for each $i = 0, 1$,

$$u_i^k := t_2 - \zeta(s) - \nabla\zeta(s)(s_{k+i-1} - s) + \zeta(\bar{s}) + \nabla\zeta(\bar{s})(s_{k+i-1} - \bar{s}).$$

Then, form the above inductional assumption, we have that

$$\begin{aligned} \|u_0^k - u_1^k\| &= \|\big(\nabla\zeta(\bar{s}) - \nabla\zeta(s)\big)(s_k - s_{k-1})\| \\ &\leq \varepsilon\|s_k - s_{k-1}\| \leq \|t_1 - t_2\|(M\varepsilon)^{k-1}. \end{aligned} \tag{4.2.14}$$

We have from (4.2.11) that $\|s_1 - \bar{s}\| \leq \dfrac{r_{\bar{s}}}{2}$ and $\|t_1 - t_2\| \leq 2\bar{r}$. Thus, we have, from (4.2.13), that

$$\begin{aligned} \|s_k - \bar{s}\| &\leq \sum_{i=2}^{k} \|s_i - s_{i-1}\| + \|s_1 - \bar{s}\| \\ &\leq 2M\bar{r}\sum_{i=2}^{k}(M\varepsilon)^{i-2} + \frac{r_{\bar{s}}}{2} \\ &\leq \frac{2M\bar{r}}{1 - M\varepsilon} + \frac{r_{\bar{s}}}{2}. \end{aligned}$$

Note by (4.2.8) that $4M\bar{r} \leq r_{\bar{s}}(1 - M\varepsilon)$. Therefore, we have from the above inequality that

$$\|s_k - \bar{s}\| \leq r_{\bar{s}}. \tag{4.2.15}$$

Moreover, we attain that

$$\|s_k - s\| \leq \|s_k - \bar{s}\| + \|\bar{s} - s\| \leq \frac{3}{2}r_{\bar{s}}. \tag{4.2.16}$$

Furthermore, using (4.2.11) and (4.2.16), we get, for every $i = 0, 1$,

$$\begin{aligned}
&\|u_i^k - \bar{t}\| \\
\leq\ & \|t_2 - \bar{t}\| + \|\big(\nabla\zeta(s) - \nabla\zeta(\bar{s})\big)(s - s_{k+i-1})\| + \|\zeta(s) - \zeta(\bar{s}) - \nabla\zeta(\bar{s})(s - \bar{s})\| \\
\leq\ & \bar{r} + \varepsilon\big(\|s - s_{k+i-1}\| + \|s - \bar{s}\|\big) \leq \bar{r} + \varepsilon\big(\frac{3r_{\bar{s}}}{2} + \frac{r_{\bar{s}}}{2}\big) \\
=\ & \bar{r} + 2\varepsilon r_{\bar{s}}.
\end{aligned}$$

By the relation $\bar{r} \leq r_{\bar{t}} - 2\varepsilon r_{\bar{s}}$ in (4.2.8), it follows that $\|u_i^k - \bar{t}\| \leq r_{\bar{t}}$. This shows that $u_i^k \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$ for each $i = 0, 1$. By our condition (4.2.12) is true for $n = k$. Thus, we get that

$$t_2 \in \zeta(s) + g(s_k) + \nabla\zeta(s)(s_{k-1} - s) + \nabla\zeta(\bar{s})(s_k - s_{k-1}) + \xi(s_k).$$

We can write the above inequality as follows:

$$\begin{aligned}
t_2 + \zeta(\bar{s}) + \nabla\zeta(\bar{s})(s_{k-1} - \bar{s}) \ \in\ & \zeta(s) + \nabla\zeta(s)(s_{k-1} - x) + \zeta(\bar{s}) + g(s_k) \\
& + \nabla\zeta(\bar{s})(s_k - s_{k-1}) + \xi(s_k) + \nabla\zeta(\bar{s})(s_{k-1} - \bar{s}).
\end{aligned}$$

Then by the construction of u_0^k, we have that $u_0^k \in \zeta(\bar{s}) + g(s_k) + \nabla\zeta(\bar{s})(s_k - \bar{s}) + \xi(s_k)$. This together with (4.2.2) implies that $s_k \in R_{\bar{s}}^{-1}(u_0^k)$. It follows from (4.2.15) that

$$s_k \in R_{\bar{s}}^{-1}(u_0^k) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}).$$

By Lipschitz-like property of $R_{\bar{s}}^{-1}$, ∃'s an element $s_{k+1} \in R_{\bar{s}}^{-1}(u_1^k)$ such that

$$\|s_{k+1} - s_k\| \leq M\|u_0^k - u_1^k\|.$$

Then by (4.2.14), it follows that

$$\|s_{k+1} - s_k\| \leq M\|t_1 - t_2\|(M\varepsilon)^{k-1}. \tag{4.2.17}$$

By the construction of u_1^k, we have that

$$s_{k+1} \in R_{\bar{s}}^{-1}(u_1^k) = R_{\bar{s}}^{-1}(t_2 - \zeta(s) - \nabla\zeta(s)(s_k - s) + \zeta(\bar{s}) + \nabla\zeta(\bar{s})(s_k - \bar{s})).$$

This inequality with (4.2.2), implies that

$$t_2 \in \zeta(s) + g(s_{k+1}) + \nabla\zeta(s)(s_k - s) + \nabla\zeta(\bar{s})(s_{k+1} - s_k) + \xi(s_{k+1}).$$

The inequality (4.2.17) with the above inclusion completes the induction step and confirming the existence of a sequence $\{s_k\}$ which satisfies (4.2.12) and (4.2.13).

Whereas $M\varepsilon < 1$, than we get from (4.2.13) that $\{s_k\}$ is a Cauchy sequence and hence it is convergent, to say s'', that is, $s'' := \lim_{k\to\infty} s_k$. Note that ξ has closed graph. Then, taking limit in (4.2.12), we get $t_2 \in \zeta(s) + g(s'') + \nabla\zeta(s)(s'' - s) + \xi(s'')$, that is, $s'' \in R_s^{-1}(t_2)$.

Therefore, we obtain

$$\begin{aligned}\|s' - s''\| &\leq \limsup_{n\to\infty} \sum_{k=2}^{n} \|s_k - s_{k-1}\| \\ &\leq \limsup_{n\to\infty} \sum_{k=2}^{n} (M\varepsilon)^{k-2} M\|t_1 - t_2\| \\ &\leq \frac{M}{1 - M\varepsilon}\|t_1 - t_2\|.\end{aligned}$$

That is,

$$e(R_s^{-1}(t_1) \cap \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}), R_s^{-1}(t_2)) \leq \frac{M}{1 - M\varepsilon}\|t_1 - t_2\|.$$

The Lemma 4.2.2 is complectly proved. □

Before going to prove our main results, we would like to introduce some notations. For our convenience, first we let a mapping $I_s \colon \mathcal{S} \to \mathcal{T}$, for each $s \in \mathcal{S}$, which is defined by

$$I_s(\cdot) := \zeta(\bar{s}) + g(\cdot) + \nabla\zeta(\bar{s})(\cdot - \bar{s}) - \zeta(s) - g(s) - \big(\nabla\zeta(s) + [\cdot, s; g]\big)(\cdot - s).$$

and the set-valued mapping $\Psi_s \colon \mathcal{S} \rightrightarrows 2^{\mathcal{S}}$ is defined by

$$\Psi_s(\cdot) := R_{\bar{s}}^{-1}[I_s(\cdot)]. \tag{4.2.18}$$

For any point $s',\ s'' \in \mathcal{S}$, we get

$$\begin{aligned}\|I_s(s') - I_s(s'')\| &= \|g(s') - g(s'') - [s', s; g](s' - s) + [s'', s; g](s'' - s) \\ &\quad + (\nabla\zeta(\bar{s}) - \nabla\zeta(s))(s' - s'')\| \\ &\leq \|g(s') - g(s'') - [s'', s; g](s' - s'')\| + \|\big([s'', s; g] \\ &\quad - [s', s; g]\big)(s' - s)\| + \|\nabla\zeta(\bar{s}) - \nabla\zeta(s)\|\|s' - s''\| \\ &\leq \big(\|[s'', s'; g] - [s'', s; g]\| + \|\nabla\zeta(\bar{s}) - \nabla\zeta(s)\|\big)\|s' - s''\| \\ &\quad + \|[s'', s; g] - [s', s; g]\|\|s' - s\| \end{aligned} \tag{4.2.19}$$

4.2.1 Linear Convergence

The first main theorem of this study read as follows. This theorem gives some certain suitable assumptions confirming the convergence of the method called EN-type method with starting point s_0.

Theorem 4.2.1. *Assume that $\eta > 1$ and $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with a constant M. let $\bar{r}$ be defined in (4.2.8) and let $s \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Suppose that $\varepsilon > 0$ be such that (4.2.10) is hold and $\nabla\zeta$ is continuous on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant ε.*

Let $\nu > 0$ and $\delta > 0$ be such that

(a) $\delta \leq \min\left\{\frac{r_{\bar{s}}}{4}, \frac{r_{\bar{t}}}{7(\varepsilon + 3\nu)}, 1, \frac{3 - 5M\varepsilon}{30M\nu}, \frac{\bar{r}}{3(\varepsilon + 3\nu)}\right\}$,

(b) $6\eta M(\varepsilon + 3\nu) \leq 1 - M\varepsilon$,

(c) $\|\bar{t}\| < (\varepsilon + 3\nu)\delta$.

Suppose that $(\zeta + g + \xi)$ is continuous at $\bar{s}$ for $\bar{t}$ i.e. (4.2.7) is hold. Then $\exists$'s some $\hat{\delta} > 0$ such that any sequence $\{s_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}(\bar{s}, \hat{\delta})$ converges to a solution s^* of (1.0.1), that is, s^* satisfies $0 \in \zeta(s^*) + g(s^*) + \xi(s^*)$.

Proof. Setting that $q := \frac{\eta M(\varepsilon + 3\nu)}{1 - M\varepsilon}$. From the assumption (b) $6\eta M(\varepsilon + 3\nu) \leq 1 - M\varepsilon$, we find

$$q := \frac{\eta M(\varepsilon + 3\nu)}{1 - M\varepsilon} \leq \frac{1}{6}.$$

Pick up $0 < \hat{\delta} \leq \delta$ such that

$$\operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0)) \leq (\varepsilon + 3\nu)\delta, \quad \text{for each } s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s}) \tag{4.2.20}$$

(Mark that such $\hat{\delta}$ exists by (4.2.7) and assumption (c)). Let $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s})$. To prove that Algorithm 2 generates at least one sequence, we will proceed by mathematical induction. Again show that any sequence $\{s_n\}$ constructed by Algorithm 2, which is satisfies both of the following assertions:

$$\|s_n - \bar{s}\| \leq 2\delta \tag{4.2.21}$$

and

$$\|s_{n+1} - s_n\| \leq q^{n+1}\delta \tag{4.2.22}$$

the assertions hold for every $n = 0, 1, 2, ...$ For this objective, we will define

$$r_s := \frac{5}{2}\Big(M\big(\varepsilon + 3\nu\|s - \bar{s}\|\big)\|s - \bar{s}\| + M\|\bar{t}\|\Big), \quad \text{for each } s \in S. \tag{4.2.23}$$

Then, we get $6\eta M(\varepsilon + 3\nu) \leq 1 - M\varepsilon < 1$ from the assumption (b) and from assumption (c) we get $\|\bar{t}\| < (\varepsilon + 3\nu)\delta$. Whereas $\eta > 1$, (4.2.23) yields that

$$\begin{aligned} r_s &< 5M\big(\varepsilon + 6\nu\delta\big)\delta + M(\varepsilon + 3\nu)\delta < 5M(\varepsilon + 6\nu\delta) + M(\varepsilon + 3\nu)\delta \\ &= 6M\varepsilon\delta + 33M\nu\delta < 11M\varepsilon\delta + 33M\nu\delta = 11M(\varepsilon + 3\nu)\delta \leq \frac{11}{6\eta}\delta \\ &\leq 2\delta \quad \text{for each } s \in \mathbb{B}_{2\delta}(\bar{s}). \end{aligned} \tag{4.2.24}$$

Note that for $n = 0$ the assertion (4.2.21) is trivial. At first we need to show that s_1 exists. For that we have to show (4.2.22) are holds for $n = 0$. To complete this, we have to prove that $\mathcal{P}(s_0) \neq \emptyset$. For that we will apply Lemma 2.0.4 to the map Ψ_{s_0} with $\eta_0 = \bar{s}$. Let us verify that both assertions (2.0.4) and (2.0.5) of Lemma 2.0.4 hold with $r := r_{s_0}$ and $\lambda := \frac{3}{5}$. We will remark that $\bar{s} \in R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{2\delta}(\bar{s})$ by (4.2.3) and by the definition of the mapping Ψ_{s_0} and the excess e in (4.2.18), we get

$$\begin{aligned} \operatorname{dist}(\bar{s}, \Psi_{s_0}(\bar{s})) &\leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(\bar{s})) \leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{2\delta}(\bar{s}), \Psi_{s_0}(\bar{s})) \\ &\leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}[I_{s_0}(\bar{s})]) \end{aligned} \tag{4.2.25}$$

(we remark that $\mathbb{B}_{2\delta}(\bar{s}) \subseteq \mathbb{B}_{r_{\bar{s}}}(\bar{s})$). According to ε, we get

$$\begin{aligned} \|I_{s_0}(s) - \bar{t}\| &= \|\zeta(\bar{s}) + g(s) + \nabla\zeta(\bar{s})(s - \bar{s}) - \zeta(s_0) - g(s_0) \\ &\quad - (\nabla\zeta(s_0) + [s, s_0; g])(s - s_0) - \bar{t}\| \\ &\leq \|\zeta(\bar{s}) - \zeta(s_0) - \nabla\zeta(s_0)(\bar{s} - s_0)\| + \|\nabla\zeta(\bar{s} - \nabla\zeta(s_0))(s - \bar{s})\| \\ &\quad + \|g(s) - g(s_0) - [s, s_0; g](s - s_0)\| + \|\bar{t}\| \\ &\leq \varepsilon(\|\bar{s} - s_0\| + \|s - \bar{s}\|) + \|[s_0, s; g] - [s, s_0; g]\|\|s - s_0\| \\ &\quad + \|\bar{t}\| \\ &\leq \varepsilon(\|\bar{s} - s_0\| + \|s - \bar{s}\|) + \nu(\|s_0 - s\| + \|s - s_0\|)\|s - s_0\| \\ &\quad + \|\bar{t}\|. \end{aligned} \tag{4.2.26}$$

Remark that $\|s_0 - \bar{s}\| \leq \hat{\delta} \leq \delta$, $7(\varepsilon + 3\nu)\delta \leq r_{\bar{t}}$ by assumption (a) and $\|\bar{t}\| < (\varepsilon + 3\nu)\delta$ by assumption (c), it follows from (4.2.26) that, for each $s \in \mathbb{B}_{2\delta}(\bar{s})$,

$$\begin{aligned}\|I_{s_0}(s) - \bar{t}\| &\leq 3\varepsilon\delta + 18\nu\delta^2 + (\varepsilon + 3\nu)\delta < 3\varepsilon\delta + 18\nu\delta + (\varepsilon + 3\nu)\delta \\ &< 6\varepsilon\delta + 18\nu\delta + (\varepsilon + 3\nu)\delta = 7(\varepsilon + 3\nu)\delta \\ &\leq r_{\bar{t}}. \end{aligned} \tag{4.2.27}$$

This implies that for all $s \in \mathbb{B}_{2\delta}(\bar{s})$, $I_{s_0}(s) \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$. Particularly, let $s = \bar{s}$ in (4.2.26). Then we get that

$$\begin{aligned}\|I_{s_0}(\bar{s}) - \bar{t}\| &\leq \varepsilon\|\bar{s} - s_0\| + \nu\big(2\|s_0 - \bar{s}\| + \|\bar{s} - s_0\|\big)\|\bar{s} - s_0\| + \|\bar{t}\| \\ &= \big(\varepsilon + 3\nu\|\bar{s} - s_0\|\big)\|\bar{s} - s_0\| + \|\bar{t}\| \\ &\leq (\varepsilon + 3\nu\delta)\delta + \|\bar{t}\| < (\varepsilon + 3\nu)\delta + \|\bar{t}\| \\ &\leq 2(\varepsilon + 3\nu)\delta \leq r_{\bar{t}}; \end{aligned} \tag{4.2.28}$$

and so $I_{s_0}(\bar{s}) \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$.

Therefore, by (4.2.23), (4.2.25), (4.2.28) and assumed Lipschitz-like property, we have

$$\begin{aligned}\mathrm{dist}(\bar{s}, \Psi_{s_0}(\bar{s})) &\leq M\|\bar{t} - I_{s_0}(\bar{s})\| \\ &\leq M\big(\varepsilon + 3\nu\|\bar{s} - s_0\|\big)\|\bar{s} - s_0\| + M\|\bar{t}\| \\ &= \big(1 - \frac{3}{5}\big)r_{s_0} = \big(1 - \lambda\big)r; \end{aligned}$$

therefore, the first assertion (2.0.4) of Lemma 2.0.4 is satisfied.

Now, we will show that the second assertion (2.0.5) of Lemma 2.0.4 holds. To finish this, we assume $s', s'' \in \mathbb{B}_{r_{s_0}}(\bar{s})$. Then, it follows that $s', s'' \in \mathbb{B}_{r_{s_0}}(\bar{s}) \subseteq \mathbb{B}_{2\delta}(\bar{s}) \subseteq \mathbb{B}_{r_{\bar{s}}}(\bar{s})$ by (4.2.24) and assumption (a) and $I_{s_0}(s')$, $I_{s_0}(s'') \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$ by (4.2.27). This together with the assumed Lipschitz-like property implies that

$$\begin{aligned}e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(s'')) &\leq e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), \Psi_{s_0}(s'')) \\ &= e(R_{\bar{s}}^{-1}[I_{s_0}(s')] \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}[I_{s_0}(s'')]) \\ &\leq M\|I_{s_0}(s') - I_{s_0}(s'')\|. \end{aligned} \tag{4.2.29}$$

According to the choice of s_0 in (4.2.19), we get

$$\begin{aligned}\|I_{s_0}(s') - I_{s_0}(s'')\| &\leq \Big(\|[s'', s'; g] - [s'', s_0; g]\| + \|\nabla\zeta(\bar{s}) - \nabla\zeta(s_0)\|\Big)\|s' - s''\| \\ &\quad + \|[s'', s_0; g] - [s', s_0; g]\|\|s' - s_0\| \\ &\leq \Big(\nu\big(\|s' - s_0\| + \|s' - s_0\|\big) + \varepsilon\Big)\|s' - s''\| \\ &\leq (\varepsilon + 6\nu\delta)\|s' - s''\|. \end{aligned} \tag{4.2.30}$$

we get (4.2.29) and (4.2.30) together is as follows,

$$e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(s'')) \leq M(\varepsilon + 6\nu\delta)\|s' - s''\|.$$

The above inequality follows the assumption (a) $30M\nu\delta \leq 3 - 5M\varepsilon$, then we get that

$$e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(s'')) \leq \frac{3}{5}\|s' - s''\| = \lambda\|s' - s''\|.$$

This yields that the second assertion (2.0.5) of Lemma 2.0.4 is satisfied. Inasmuch as we have seen that both first and second assertions of Lemma 2.0.4 are fulfilled, we can say that the Lemma 2.0.4 is applicable and hence we can conclude that there exists $\hat{s}_1 \in \mathbb{B}_{r_{s_0}}(\bar{s})$ such that $\hat{s}_1 \in \Psi_{s_0}(\hat{s}_1)$. This yields that $0 \in \zeta(s_0) + g(s_0) + (\nabla\zeta(s_0) + [\hat{s}_1, s_0; g])(\hat{s}_1 - s_0) + \xi(\hat{s}_1)$ and thus we conclude that $\mathcal{P}(s_0) \neq \emptyset$. Since $\eta > 1$ and $\mathcal{P}(s_0) \neq \emptyset$, we can select $d_0 \in \mathcal{P}(s_0)$ such that

$$\|d_0\| \leq \eta \text{ dist}(0, \mathcal{P}(s_0)).$$

$s_1 := s_0 + d_0$ is defined for Algorithm 2. Moreover, according to the definition of $\mathcal{P}(s_0)$ and through (4.2.1), we get

$$\begin{aligned}\mathcal{P}(s_0) &:= \Big\{d_0 \in \mathcal{S} : 0 \in \zeta(s_0) + g(s_0) + (\nabla\zeta(s_0) + [d_0 + s_0, s_0; g])d_0 + \xi(s_0 + d_0)\Big\} \\ &= \Big\{d_0 \in \mathcal{S} : 0 \in \zeta(s_0) + g(s_0 + d_0) + \nabla\zeta(s_0)d_0 + \xi(s_0 + d_0)\Big\} \\ &= \Big\{d_0 \in \mathcal{S} : s_0 + d_0 \in R_{s_0}^{-1}(0)\Big\},\end{aligned}$$

and so

$$\text{dist}(0, \mathcal{P}(s_0)) = \text{dist}(s_0, R_{s_0}^{-1}(0)). \tag{4.2.31}$$

Now, we show that (4.2.22) holds also for $n = 0$. The continuity property of $\nabla\zeta$ implies that

$$\|\nabla\zeta(s) - \nabla\zeta(\bar{s})\| \leq \varepsilon, \text{ for all } s \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$$

and note that $\bar{r} > 0$ by assumption (a). Therefore, (4.2.10) satisfies (4.2.8). Since $R_{\bar{s}}^{-1}$ is Lipschitz-like, it follows from Lemma 4.2.2 that the mapping R_s^{-1} is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\dfrac{M}{1-M\varepsilon}$ for each $s \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Particularly, according to assumption (a) and the choice of $\hat{\delta}$, $R_{s_0}^{-1}$ is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\dfrac{M}{1-M\varepsilon}$ as $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s}) \subset \mathbb{B}_{\delta}(\bar{s}) \subset \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Moreover, by the relation $3(\varepsilon+3\nu)\delta \leq \bar{r}$ in assumption (a) and assumption(c) imply that

$$\|\bar{t}\| < (\varepsilon+3\nu)\delta \leq \frac{\bar{r}}{3} \tag{4.2.32}$$

and therefore (4.2.20) implies that

$$\begin{aligned} \operatorname{dist}(0, R_{s_0}(s_0)) &= \operatorname{dist}\big(0, \zeta(s_0)+g(s_0)+\xi(s_0)\big) \leq (\varepsilon+3\nu)\delta \\ &\leq \frac{\bar{r}}{3}. \end{aligned} \tag{4.2.33}$$

By (4.2.32), it is marked earlier that $s_0 \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ and $0 \in \mathbb{B}_{\frac{\bar{r}}{3}}(\bar{t})$. Thus, applying Lemma 2.0.1 it can be shown that

$$\operatorname{dist}\big(s_0, R_{s_0}^{-1}(0)\big) \leq \frac{M}{1-M\varepsilon} \operatorname{dist}\big(0, R_{s_0}(s_0)\big).$$

The above relation together with (4.2.31) yields that

$$\operatorname{dist}\big(0, \mathcal{P}(s_0)\big) = \operatorname{dist}\big(s_0, R_{s_0}^{-1}(0)\big) \leq \frac{M}{1-M\varepsilon} \operatorname{dist}\big(0, R_{s_0}(s_0)\big). \tag{4.2.34}$$

According to Algorithm 2 and using (4.2.33) and (4.2.34), we have

$$\begin{aligned} \|d_0\| &\leq \eta \operatorname{dist}\big(0, \mathcal{P}(s_0)\big) \leq \frac{\eta M}{1-M\varepsilon} \operatorname{dist}\big(0, R_{s_0}(s_0)\big) \\ &\leq \frac{\eta M(\varepsilon+3\nu)\delta}{1-M\varepsilon} \\ &= q\delta. \end{aligned} \tag{4.2.35}$$

This implies that

$$\|s_1 - s_0\| = \|d_0\| \leq q\delta$$

and therefore, (4.2.22) is hold for $n = 0$.

Assume that $s_1, s_2, \ldots, s_k$ are constructed. So that the inequality (4.2.21) and (4.2.22) are hold for $n = 0, 1, 2, \ldots, k-1$. Again we will verify that there exists s_{k+1} such that

(4.2.21) and (4.2.22) are also hold for $n = k$. Since (4.2.21) and (4.2.22) are true for each $n \leq k-1$, we get the inequality as follows:

$$\|s_k - \bar{s}\| \leq \sum_{i=0}^{k-1} \|d_i\| + \|s_0 - \bar{s}\| \leq \delta \sum_{i=0}^{k-1} q^{i+1} + \delta \leq \frac{\delta q}{1-q} + \delta \leq 2\delta.$$

This shows that (4.2.21) holds for $n = k$. Now we can also show that (4.2.22) hold for $n = k$, by the same argument as we did for the case when $n = 0$.

The proof is complete. □

When $\bar{t} = 0$, that is, $\bar{s}$ is a solution of (1.0.1), Theorem 4.2.1 is reduced to the following corollary, which gives the local convergent result for the extended Newton-type method.

Corollary 4.2.1. *Suppose that $\eta > 1$ and $\bar{s}$ is a solution of the variational inclusion (1.0.1). Let $R_{\bar{s}}^{-1}$ be pseudo-Lipschitz around $(0, \bar{s})$. Let $\tilde{r} > 0$, $\nu > 0$ and suppose that $\nabla\zeta$ is continuous on $\mathbb{B}_{\tilde{r}}(\bar{s})$ and that*

$$\lim_{s \to \bar{s}} \operatorname{dist}(0, \zeta(s) + g(s) + \xi(s)) = 0.$$

Then there exists some $\hat{\delta}$ such that any sequence $\{s_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}_{\hat{\delta}}(\bar{s})$ converges to a solution s^* of the variational inclusion (1.0.1).

Proof. Let $R_{\bar{s}}^{-1}$ is pseudo-Lipschitz around $(0, \bar{s})$. Then there exist constants r_0, $\hat{r}_{\bar{s}}$ and M satisfy the following condition:

$$e(R_{\bar{s}}^{-1}(t_1) \cap \mathbb{B}_{\hat{r}_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}(t_2)) \leq M\|t_1 - t_2\|, \quad \text{for every} \ \ t_1, t_2 \in \mathbb{B}_{r_0}(0). \tag{4.2.36}$$

Thus, according to the definition of Lipschitz-like property we can say that $Q_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_0}(0)$ relative to $\mathbb{B}_{\hat{r}_{\bar{s}}}(\bar{s})$ with constant M which satisfy (4.2.36). Then, for each $0 < \tilde{r} \leq \hat{r}_{\bar{s}}$, one has that

$$e(R_{\bar{s}}^{-1}(t_1) \cap \mathbb{B}_{\tilde{r}}(\bar{s}), R_{\bar{s}}^{-1}(t_2)) \leq M\|t_1 - t_2\|, \quad \text{for every} \ \ t_1, t_2 \in \mathbb{B}_{r_0}(0),$$

that is, $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_0}(0)$ relative to $\mathbb{B}_{\tilde{r}}(\bar{s})$ with constant M. Let $\varepsilon \in (0, 1)$ be such that $M((6\eta + 1)\varepsilon + 3\nu) \leq 1$. By the continuity of $\nabla\zeta$ we can choose $r_{\bar{s}} \in (0, \hat{r}_{\bar{s}})$ such that $\frac{r_{\bar{s}}}{2} \leq \tilde{r}$, $r_0 - 2\varepsilon r_{\bar{s}} > 0$ and

$$\|\nabla\zeta(s) - \nabla\zeta(s')\| \le \varepsilon, \text{ for each } s,\ s' \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}).$$

Then

$$\bar{r} = \min\left\{r_0 - 2\varepsilon r_{\bar{s}}, \frac{r_{\bar{s}}(1 - M\varepsilon)}{4M}\right\} > 0,$$

and

$$\min\left\{\frac{r_{\bar{s}}}{4}, \frac{\bar{r}}{3(\varepsilon + 3\nu)}, \frac{r_0}{7(\varepsilon + 3\nu)}, \frac{3 - 5M\varepsilon}{30M\nu}\right\} > 0. \tag{4.2.37}$$

By (4.2.37), we can choose $0 < \delta \le 1$ such that

$$\delta \le \min\left\{\frac{r_{\bar{s}}}{4}, \frac{\bar{r}}{3(\varepsilon + 3\nu)}, 1, \frac{r_0}{7(\varepsilon + 3\nu)}, \frac{3 - 5M\varepsilon}{30M\nu}\right\}.$$

Thus it is routine to check that inequalities (a) -(c) of Theorem 4.2.1 are satisfied. Therefore, Theorem 4.2.1 is applicable to complete the proof. □

4.2.2 Quadratic Convergence

In this section we consider $\nabla\zeta$ is Lipschitz continuous around $\bar{s}$ and we show that the sequence generated by Algorithm 2 converges quadratically.

Let $L > 0$ and define

$$r^* := \min\left\{r_{\bar{t}} - 2Lr_{\bar{s}}^2, \frac{r_{\bar{s}}(1 - MLr_{\bar{s}})}{4M}\right\}. \tag{4.2.38}$$

Now, we state our second main theorem as follows:

Theorem 4.2.2. *Suppose that $\eta > 1$ and let $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r^*}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M and that $\nabla\zeta$ is Lipschitz continuous on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with Lipschitz constant L. Let $\nu > 0$, $\delta > 0$ be such that*

(a) $\delta \le \min\left\{\frac{r_{\bar{s}}}{4}, \frac{10r^*}{3}, 1, \left(\frac{r_{\bar{t}}}{3(L + 4\nu)}\right)^{\frac{1}{2}}\right\}$,

(b) $(M + 1)(L + 4\nu)(\eta\delta + r_{\bar{s}}) \le 1$,

(c) $\|\bar{t}\| < \frac{(L + 4\nu)\delta^2}{2}$.

Suppose that

$$\lim_{s\to\bar{s}} \operatorname{dist}(\bar{t}, \zeta(s) + g(s) + \xi(s)) = 0. \tag{4.2.39}$$

Then there exist some $\hat{\delta} > 0$ such that any sequence $\{s_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}_{\hat{\delta}}(\bar{s})$ converges quadratically to a solution s^* of (1.0.1).

Proof. Setting

$$b := \frac{\eta M(L+4\nu)\delta}{1 - MLr_{\bar{s}}}. \tag{4.2.40}$$

Thanks to assumption (b). Since $\nu > 0$, it allows us to write the fact that

$$\begin{aligned} \eta M(L+4\nu)\delta + MLr_{\bar{s}} &< (M+1)(L+4\nu)\eta\delta + (M+1)(L+4\nu)r_{\bar{s}} \\ &= (M+1)(L+4\nu)(\eta\delta + r_{\bar{s}}) \leq 1. \end{aligned}$$

Thus, we have from (4.2.40) that

$$b := \frac{\eta M(L+4\nu)\delta}{1 - MLr_{\bar{s}}} \leq 1. \tag{4.2.41}$$

Pick $0 < \hat{\delta} \leq \delta$ be such that

$$\operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0)) \leq \frac{(L+4\nu)\delta^2}{2} \quad \text{for each } s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s}) \tag{4.2.42}$$

Since (4.2.39) is hold and assumption (c) is true, we assume that such $\hat{\delta}$ exists, which satisfies (4.2.42). Let $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s})$. Now we use the same argument whichever we used in Theorem 4.2.1 for complete the proof of the Theorem 4.2.2 We show that Algorithm 2 generates at least one sequence and such generated sequence $\{s_n\}$ satisfies the following assertions:

$$\|s_n - \bar{s}\| \leq 2\delta; \tag{4.2.43}$$

and

$$\|d_n\| \leq b\Big(\frac{1}{2}\Big)^{2^n}\delta. \tag{4.2.44}$$

hold for each $n = 0, 1, 2, ...$ Let

$$r_s := \frac{5M}{8}\Big((L+4\nu)\|s-\bar{s}\|^2 + 2\|\bar{t}\|\Big), \quad \text{for each } s \in X. \tag{4.2.45}$$

Owing to the fact $4\delta \leq r_{\bar{s}}$ in assumption (a) and $\eta > 1$, by assumption (b) we can write as follows

$$\begin{aligned}5(M+1)(L+4\nu)\delta &= (M+1)(L+4\nu)(\delta+4\delta)\\ &\leq (M+1)(L+4\nu)(\eta\delta+r_{\bar{s}})\\ &\leq 1.\end{aligned}$$

This gives

$$M(L+4\nu)\delta \leq \frac{1}{5} \quad \text{and} \quad (L+4\nu)\delta \leq \frac{1}{5}. \tag{4.2.46}$$

Hence by $3\delta \leq 5r^*$ in assumption (a) together with second inequality of (4.2.46), we get

$$\|\bar{t}\| < \frac{(L+4\nu)\delta^2}{2} \leq \frac{1}{5\cdot 2}\cdot\frac{10r^*}{3} = \frac{r^*}{3}. \tag{4.2.47}$$

Thanks to assumption (c). Utilizing the first inequality from (4.2.46) together with assumption (c), we obtain from (4.2.45) that

$$\begin{aligned}r_s &< \frac{5M}{8}\Big((L+4\nu)\delta^2+(L+4\nu)\delta^2\Big)\\ &= \frac{10M}{8}(L+4\nu)\delta^2 \leq \frac{10}{8\cdot 5}\delta\\ &= \frac{\delta}{4} < 2\delta, \quad \text{for each } s\in\mathbb{B}_{2\delta}(\bar{s}).\end{aligned} \tag{4.2.48}$$

Note that (4.2.43) is trivial for $n = 0$. In order to show that (4.2.44) is hold for $n = 0$, first we need to prove $\mathcal{P}(s_0) \neq \emptyset$. The nonemptyness of $\mathcal{P}(s_0)$ will ensure us to deduce the existence of the point s_1. To complete this, we will apply Lemma 2.0.4 to the map Ψ_{s_0} with $\eta_0 = \bar{s}$. Let us check that both assertions (2.0.4) and (2.0.5) of Lemma 2.0.4 hold with $r := r_{s_0}$ and $\lambda := \frac{1}{5}$. Here we note by (4.2.3) that $\bar{s} \in R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{2\delta}(\bar{s})$. Then, by the definition of the excess e and the mapping Ψ_{s_0} defined by (4.2.18), we have that

$$\begin{aligned}\operatorname{dist}(\bar{s},\Psi_{s_0}(\bar{s})) &\leq e(R_{\bar{s}}^{-1}(\bar{t})\cap\mathbb{B}_{r_{s_0}}(\bar{s}),\Psi_{s_0}(\bar{s})) \leq e(R_{\bar{s}}^{-1}(\bar{t})\cap\mathbb{B}_{2\delta}(\bar{s}),\Psi_{s_0}(\bar{s}))\\ &\leq e(R_{\bar{s}}^{-1}(\bar{t})\cap\mathbb{B}_{r_{\bar{s}}}(\bar{s}),R_{\bar{s}}^{-1}[I_{s_0}(\bar{s})]).\end{aligned} \tag{4.2.49}$$

For each $s \in \mathbb{B}_{2\delta}(\bar{s}) \subseteq \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ and Lipschitz continuous property of $\nabla\zeta$, we have that

$$\begin{aligned}
\|I_{s_0}(s)-\bar{t}\| &= \|\zeta(\bar{s})+g(s)+\nabla\zeta(\bar{s})(s-\bar{s})-\zeta(s_0)-g(s_0) \\
&\quad -(\nabla\zeta(s_0)+[s,s_0;g])(s-s_0)-\bar{t}\| \\
&\le \|\zeta(\bar{s})-\zeta(s_0)-\nabla\zeta(s_0)(\bar{s}-s_0)\|+\|\big(\nabla\zeta(s_0)-\nabla\zeta(\bar{s})\big)(\bar{s}-s)\| \\
&\quad +\|g(s)-g(s_0)-[s,s_0;g](s-s_0)\|+\|\bar{t}\| \\
&\le \frac{L}{2}\|\bar{s}-s_0\|^2+L\|s_0-\bar{s}\|\|\bar{s}-s\|+\|[s_0,s;g]-[s,s_0;g]\|\|s-s_0\| \\
&\quad +\|\bar{t}\| \\
&\le \frac{L}{2}\|\bar{s}-s_0\|^2+L\|s_0-\bar{s}\|\|\bar{s}-s\|+\nu\big(\|s_0-s\|+\|s-s_0\|\big)\|s-s_0\| \\
&\quad +\|\bar{t}\| \qquad (4.2.50) \\
&\le \frac{L}{2}(\delta^2+4\delta^2)+2\nu(2\delta)^2+\|\bar{t}\| = \frac{5L\delta^2}{2}+8\nu\delta^2+\|\bar{t}\| \\
&< \frac{5}{2}(L+4\nu)\delta^2+\|\bar{t}\|.
\end{aligned}$$

It follows, from the facts $3(L+4\nu)\delta^2 \le r_{\bar{t}}$ and $2\|\bar{t}\| < (L+4\nu)\delta^2$ respectively in assumptions (a) and (c), that

$$\begin{aligned}
\|I_{s_0}(s)-\bar{t}\| &\le \frac{5}{2}(L+4\nu)\delta^2+\frac{(L+4\nu)\delta^2}{2} \\
&= 3(L+4\nu)\delta^2 \le r_{\bar{t}}. \qquad (4.2.51)
\end{aligned}$$

This shows that $I_{s_0}(s) \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$. Particularly, let $s=\bar{s}$ in (4.2.50). Then it is easily shown that

$$I_{s_0}(\bar{s}) \in \mathbb{B}_{r_{\bar{t}}}(\bar{t}) \quad \text{and} \quad \|I_{s_0}(\bar{s})-\bar{t}\| \le \frac{(L+4\nu)}{2}\|\bar{s}-s_0\|^2+\|\bar{t}\|. \qquad (4.2.52)$$

Using the Lipschitz-like property of $R_{\bar{s}}^{-1}$ and (4.2.52) in (4.2.49), we have

$$\begin{aligned}
\operatorname{dist}(\bar{s},\Psi_{s_0}(\bar{s})) &\le M\|\bar{t}-I_{s_0}(\bar{s})\| \le \frac{M(L+4\nu)}{2}\|\bar{s}-s_0\|^2+M\|\bar{t}\| \\
&= (1-\frac{1}{5})r_{s_0} = (1-\lambda)r;
\end{aligned}$$

that is, the first assertion (2.0.4) of Lemma 2.0.4 is satisfied.

Now, we will show that the second assertion (2.0.5) of Lemma 2.0.4 holds. To finish this, we assume $s', s'' \in \mathbb{B}_{r_{s_0}}(\bar{s})$. Then it follows that $s', s'' \in \mathbb{B}_{r_{s_0}}(\bar{s}) \subseteq \mathbb{B}_{2\delta}(\bar{s}) \subseteq \mathbb{B}_{r_{\bar{s}}}(\bar{s})$ by

(4.2.48) and $I_{s_0}(s')$, $I_{s_0}(s'') \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$ by (4.2.51). This together with the assumed Lipschitz-like property of $R_{\bar{s}}^{-1}$ implies that

$$\begin{aligned} e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(s'')) &\leq e(\Psi_{s_0}(s') \cap \mathbb{B}_{2\delta}(\bar{s}), \Psi_{s_0}(s'')) \\ &\leq e(R_{\bar{s}}^{-1}[I_{s_0}(s')] \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}[I_{s_0}(s'')]) \\ &\leq M\|I_{s_0}(s') - I_{s_0}(s'')\|. \end{aligned} \tag{4.2.53}$$

According to the choice of s_0 in (4.2.19), we get

$$\begin{aligned} \|I_{s_0}(s') - I_{s_0}(s'')\| &\leq \big(\|[s'', s'; g] - [s'', s_0; g]\| + \|\nabla\zeta(\bar{s}) - \nabla\zeta(s_0)\|\big)\|s' - s''\| \\ &\qquad + \|[s'', s_0; g] - [s', s_0; g]\|\|s' - s_0\| \\ &\leq \Big(\nu\big(\|s_0 - s'\| + \|s' - s_0\|\big) + L\|\bar{s} - s_0\|\Big)\|s' - s''\| \\ &\leq (L + 4\nu)\delta\|s' - s''\|. \end{aligned} \tag{4.2.54}$$

The above two inequalities (4.2.53) and (4.2.54) together in (4.2.46) is as follows

$$\begin{aligned} e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(s'')) &\leq M(L + 4\nu)\delta\|s' - s''\| \\ &\leq \frac{1}{5}\|s' - s''\| = \lambda\|s' - s''\|. \end{aligned}$$

It seems that the second assertion (2.0.5) of Lemma 2.0.4 is also satisfied.

Thus, we have seen that both assertions (2.0.4) and (2.0.5) of Lemma 2.0.4 are fulfilled. So, we can conclude that Lemma 2.0.4 is applicable to deduce the existence of a point $\hat{s}_1 \in \mathbb{B}_{r_{s_0}}(\bar{s})$ such that $\hat{s}_1 \in \Psi_{s_0}(\hat{s}_1)$. This implies that $0 \in \zeta(s_0) + g(s_0) + (\nabla\zeta(s_0) + [\hat{s}_1, s_0; g])(\hat{s}_1 - s_0) + \xi(\hat{s}_1)$ and thus $\mathcal{P}(s_0) \neq \emptyset$. Since $\eta > 1$ and $\mathcal{P}(s_0) \neq \emptyset$, we can choose $d_0 \in \mathcal{P}(s_0)$ such that

$$\|d_0\| \leq \eta \ \mathrm{dist}(0, \mathcal{P}(s_0)).$$

By Algorithm 2, $s_1 := s_0 + d_0$ is defined. Furthermore, by the construction of $\mathcal{P}(s_0)$ and (4.2.1), we have that

$$\begin{aligned} \mathcal{P}(s_0) &:= \Big\{d_0 \in \mathcal{S} : 0 \in \zeta(s_0) + g(s_0) + (\nabla\zeta(s_0) + [d_0 + s_0, s_0; g])d_0 + \xi(s_0 + d_0)\Big\} \\ &= \Big\{d_0 \in \mathcal{S} : 0 \in \zeta(s_0) + g(s_0 + d_0) + \nabla\zeta(s_0)d_0 + \xi(s_0 + d_0)\Big\} \\ &= \Big\{d_0 \in \mathcal{S} : s_0 + d_0 \in R_{s_0}^{-1}(0)\Big\}, \end{aligned}$$

and so

$$\operatorname{dist}(0, \mathcal{P}(s_0)) = \operatorname{dist}(s_0, R_{s_0}^{-1}(0)). \tag{4.2.55}$$

Now, we are ready to show that (4.2.44) is hold for $n = 0$.

Note by assumption (a) that $r^* > 0$. Then, from (4.2.38) we conclude that

$$L < \{\frac{r_{\bar{t}}}{2r_{\bar{s}}^2}, \frac{1}{Mr_{\bar{s}}}\}.$$

Since $\nabla\zeta$ is Lipschitz continuous on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with Lipschitz constant L, we have for all $s', s'' \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$, that

$$\|\nabla\zeta(s') - \nabla\zeta(s'')\| \leq L\|s' - s''\| \leq Lr_{\bar{s}}.$$

This shows that Lemma 4.2.2 is applicable with $\varepsilon := Lr_{\bar{s}}$.

According to our assumption $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r^*}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$. Then, it follows from Lemma 4.2.2 that for each $s \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$, the mapping R_s^{-1} is Lipschitz-like on $\mathbb{B}_{r^*}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\frac{M}{1-MLr_{\bar{s}}}$. Specifically, $R_{s_0}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r^*}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\frac{M}{1-MLr_{\bar{s}}}$ as $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s}) \subseteq \mathbb{B}_{2\delta}(\bar{s}) \subseteq \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ by assumption (a). On the other hand, (4.2.42) implies that

$$\begin{aligned} \operatorname{dist}(0, R_{s_0}(s_0)) &= \operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0)) \\ &\leq \frac{r^*}{3}. \end{aligned}$$

We have shown by (4.2.47) that $0 \in \mathbb{B}_{\frac{r^*}{3}}(\bar{t})$ and it is noted earlier that $s_0 \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Thus by appying Lemma 2.0.1, we get the following inequality:

$$\operatorname{dist}(s_0, R_{s_0}^{-1}(0)) \leq \frac{M\ \operatorname{dist}(0, R_{s_0}(s_0))}{1 - MLr_{\bar{s}}} = \frac{M\ \operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0))}{1 - MLr_{\bar{s}}}.$$

But, by (4.2.55), we have that

$$\operatorname{dist}(0, \mathcal{P}(s_0)) = \operatorname{dist}(s_0, R_{s_0}^{-1}(0)) \leq \frac{M\ \operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0))}{1 - MLr_{\bar{s}}}. \tag{4.2.56}$$

According to Algorithm 2 and using (4.2.40), (4.2.42) and (4.2.56), we have

$$\begin{aligned} \|d_0\| &\leq \eta\ \operatorname{dist}(0, \mathcal{P}(s_0)) \\ &\leq \frac{\eta M\ \operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0))}{(1 - MLr_{\bar{s}})} \\ &\leq \frac{\eta M(L + 4\nu)\delta^2}{2(1 - MLr_{\bar{s}})} = b(\frac{1}{2})\delta. \end{aligned}$$

This means that

$$\|s_1 - s_0\| = \|d_0\| \leq b\big(\frac{1}{2}\big)\delta,$$

and therefore, (4.2.44) is true for $n = 0$.

Assume that $s_1, s_2, \ldots, s_k$ are constructed. The inequalities (4.2.43) and (4.2.44) are true for $n = 0, 1, 2, \ldots, k-1$. Again we will show that there exists s_{k+1} such that (4.2.43) and (4.2.44) are also hold for $n = k$. Since (4.2.43) and (4.2.44) are true for each $n \leq k-1$, we get the inequality as follows:

$$\|s_k - \bar{s}\| \leq \sum_{i=0}^{k-1} \|d_i\| + \|s_0 - \bar{s}\| \leq b\delta \sum_{i=0}^{k-1} \big(\frac{1}{2}\big)^{2^i} + \delta \leq 2\delta.$$

This shows that (4.2.43) holds for $n = k$.

Finally, we will show that the assertion (4.2.44) holds for $n = k$. For doing this, we will apply again the contraction mapping principle to Ψ_{s_k} with $r := r_{s_k}$ and $\lambda := \dfrac{1}{5}$. Then we can deduce the existence of a fixed point $\hat{s}_{k+1} \in \mathbb{B}_{r_{s_k}}(\bar{s})$ satisfying $\hat{s}_{k+1} \in \Psi_{s_k}(\hat{s}_{k+1})$, which translates to $I_{s_k}(\hat{s}_{k+1}) \in R_{\bar{s}}(\hat{s}_{k+1})$. This means that $0 \in \zeta(s_k) + g(s_k) + \big(\nabla\zeta(s_k) + [\hat{s}_{k+1}, s_k; g]\big)(\hat{s}_{k+1} - s_k) + \xi(\hat{s}_{k+1})$, that is, $\mathcal{P}(s_k) \neq \emptyset$. Choose $d_k \in \mathcal{P}(s_k)$ such that

$$\|d_k\| \leq \eta \ \mathrm{dist}(0, \mathcal{P}(s_k)).$$

Then by Algorithm 2, set $s_{k+1} := s_k + d_k$. Moreover, applying Lemma 4.2.2 we can infer that $R_{s_k}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r^*}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\frac{M}{1-MLr_{\bar{s}}}$. Therefore, we have that

$$\begin{aligned}
\|s_{k+1} - s_k\| &= \|d_k\| \leq \eta \ \mathrm{dist}(0, \mathcal{P}(s_k)) \\
&\leq \eta \ \mathrm{dist}(s_k, R_{s_k}^{-1}(0)) \\
&= \frac{\eta M}{1 - MLr_{\bar{s}}} \ \mathrm{dist}(0, \zeta(s_k) + g(s_k) + \xi(s_k)) \\
&\leq \frac{\eta M}{1 - MLr_{\bar{s}}} \|\zeta(s_k) + g(s_k) - \zeta(s_{k-1}) - g(s_{k-1}) \\
&\quad -\big(\nabla\zeta(s_{k-1}) + [s_k, s_{k-1}; g]\big)(s_k - s_{k-1})\|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\eta M}{1-MLr_{\bar{s}}}\big(\|\zeta(s_k)-\zeta(s_{k-1})-\nabla\zeta(s_{k-1})(s_k-s_{k-1})\| \\
&\quad +\|g(s_k)-g(s_{k-1})-[s_k,s_{k-1};g](s_k-s_{k-1})\|\big) \\
&\le \frac{\eta M}{2(1-MLr_{\bar{s}})}\big(L\|s_k-s_{k-1}\|^2+ \\
&\qquad 2\|[s_{k-1},s_k;g]-[s_k,s_{k-1};g]\|\|s_k-s_{k-1}\|\big) \\
&\le \frac{\eta M}{2(1-MLr_{\bar{s}})}\big(L\|s_k-s_{k-1}\|^2+ \\
&\qquad 2\nu(\|s_{k-1}-s_k\|+\|s_k-s_{k-1}\|)\|s_k-s_{k-1}\|\big) \\
&= \frac{\eta M(L+4\nu)}{2(1-MLr_{\bar{s}})}\|s_k-s_{k-1}\|^2 \\
&\le \frac{b}{2}\Big(b\big(\frac{1}{2}\big)^{2^{k-1}}\delta\Big)^2 \le b\big(\frac{1}{2}\big)^{2^k}\delta.
\end{aligned}
$$

This implies that (4.2.44) holds for $n=k$ and therefore the proof is completed.

□

Consider the special case when $\bar{s}$ is a solution of (1.0.1) (that is, $\bar{t}=0$) in Theorem 4.2.2. We have the following corollary, which gives the local quadratic convergence result for the EN-type method. The proof of this corollary is similar to that we did for Corollary 4.2.1.

Corollary 4.2.2. *Suppose that $\bar{s}$ is solution of the variational inclusion* (1.0.1) *and that $R_{\bar{s}}^{-1}$ is pseudo-Lipschitz around $(0,\bar{s})$. Let $\eta>1$, $\nu>0$, $\tilde{r}>0$ and suppose that $\nabla\zeta$ is Lipschitz continuous on $\mathbb{B}_{\tilde{r}}(\bar{s})$ with Lipschitz constant L. Suppose that*

$$\lim_{s\to\bar{s}} \operatorname{dist}(0,\zeta(s)+g(s)+\xi(s))=0.$$

Then there exist some $\hat{\delta}>0$ such that any sequence $\{s_n\}$ generated by Algorithm 2 with initial point in $\mathbb{B}_{\hat{\delta}}(\bar{s})$ converges quadratically to a solution s^* of the variational inclusion (1.0.1).

4.2.3 Numerical Experiment

To verify the semi-local convergence results of the EN-type method, a numerical example is presented in this section.

Example 4.2.1. *Let* $S = T = \mathbb{R}$, $s_0 = 0.01$, $\eta = 2$, $\nu = 0.3$, $M = 0.4$, and $\varepsilon = 0.1$. *Define a Fréchet differentiable function* ζ *on* $\mathbb{R}$ *by* $\zeta(s) = 2s^2$, *linear and divided difference admissible function* $g(s) = -\frac{1}{4}s$ *and a set-valued mapping* ξ *on* $\mathbb{R}$ *by* $\xi(s) = \{\frac{15s}{4} - 1, -\frac{17s}{4} + 1\}$. *Then* $\zeta + g + \xi$ *is a set-valued mapping on* $\mathbb{R}$ *defined by* $\zeta(s) + g(s) + \xi(s) = \{2s^2 + \frac{14s}{4} - 1, 2s^2 - \frac{18s}{4} + 1\}$. *Then Algorithm 2 generates a sequence which converges to* $s^* = 0.2500$.

Solution: Consider $\zeta(s) + g(s) + \xi(s) = 2s^2 + \frac{14s}{4} - 1$. It is manifest that $(\zeta + g + \xi)$ has a closed graph at $(-0.01, 1.002)$. In this way $(-0.01, 1.002) \in \text{gph}(\zeta + g + \xi)$. Then from the statement, it is clear that $(\zeta + g + \xi)^{-1}$ is Lipschitz-like at $(1.002, -0.01)$. Then from (4.1.1), we have that

$$\begin{aligned} P(s_k) &= \Big\{ d_k \in S : 0 \in \zeta(s_k) + g(s_k) + (\nabla\zeta(s_k) + [s_k + d_k, s_k; g]d_k + \xi(s_k + d_k) \Big\} \\ &= \Big\{ d_k \in S : 0 \in \zeta(s_k) + \nabla\zeta(s_k)d_k + g(s_k + d_k) + \xi(s_k + d_k) \Big\} \\ &= \Big\{ d_k \in \mathbb{R} : d_k = \frac{2 + 5s_k - 4s_k^2}{8s_k - 5} \Big\}. \end{aligned}$$

Otherwise, if $P(s_k) \neq \emptyset$, we obtain that

$$\begin{aligned} & 0 \in \zeta(s_k) + \nabla\zeta(s_k)(s_{k+1} - s_k) + g(s_{k+1}) + \xi(s_{k+1}) \\ & \Rightarrow s_{k+1} = \frac{1 + 2s_k^2}{8s_k - 5}. \end{aligned}$$

Thus from (4.2.35), we obtain that

$$\|d_k\| \leq \frac{\eta M(\varepsilon + 3\nu)]}{1 - M\varepsilon)} \|d_{k-1}\|.$$

We see that $\frac{\eta M(\varepsilon + 3\nu)]}{1 - M\varepsilon)} = .834 < 1$ for the values of η, M, L and ε. This shows that the sequence generated by Algorithm 2 converges linearly. Then the following Table 4.1, obtained by using Matlab code, indicates that the solution of the variational inclusion is 0 when $k = 4$.

Table 4.1 Numerical results for Example 4.2.1

iteration no.	s_k	$\zeta + \xi + g = 2s^2 + \frac{14s}{4} - 1$
1	0.0100	-0.9648
2	0.2825	0.1486
3	0.2505	0.0021
4	0.2500	0.0000
5	0.2500	0.0000

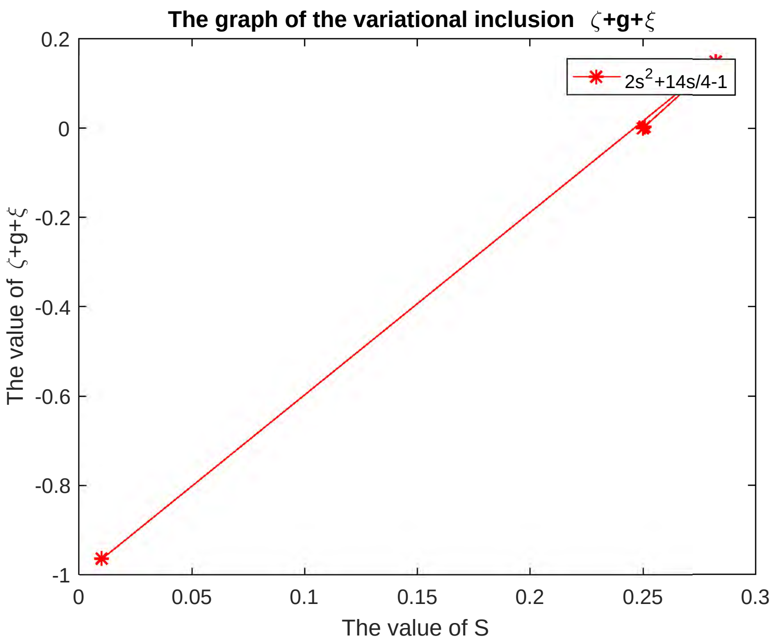

Figure 4.1: Finding a solution of variational inclusion

4.2.4 Concluding Remarks

The semilocal and local convergence results for the EN-type method are established under the conditions that $\eta > 1$, $\nabla\zeta$ is continuous and Lipschitz continuous, g admits first order divided difference as well as $R_{\bar{s}}^{-1}$ is Lipschitz-like. Finally to illustrates the theoretical result we have presented a numerical experiment. Therefore, this work extends and improves the result corresponding to [13, 105].

4.3 Convergence Analysis of an EN-type Method with Hölderian Assumptions

This section is organized as follows: In subsection 4.3.2, we consider the EN-type method defined by Algorithm 3 to approximate the solution of (1.0.1). Using the concept of Lipschitz-like property for the set-valued mapping, in this section we also establish the existence and superlinear convergence of the sequence generated by Algorithm 3 in both semilocal and local cases. At the end, we give a summary of the main results and present a comparison of this study with other known results.

In this section, we consider the variational inclusion $0 \in \zeta(\bar{s}) + g(\bar{s}) + \xi(\bar{s})$. Here we study the variational inclusion (1.0.1) with the help of EN-type method, introduced in Khaton *et al.* [62], under the weaker conditions than that are used in Khaton *et al.* [62]. Indeed, semilocal and local convergence analysis are provided for this method under some conditions that the Fréchet derivative of ζ and the FODD of g are Hölder continuous on Υ. In particular, we show this method converges superlinearly and these results extend and improve the corresponding results in Argyros [13] and Khaton *et al.* [62]).

4.3.1 Introduction

Let Υ be a subset of $\mathcal{S}$. Let $[s, t; g]$ denotes the FODD at the points s and t and ξ be a set-valued mapping from $\mathcal{S}$ to $\mathcal{T}$ which has closed graph. To find a point $\bar{s}$ in Υ, we consider the variational inclusion (1.0.1).

Suppose that $s \in \mathcal{S}$. $\mathcal{P}(s)$ is the subset of $\mathcal{S}$, which defined by

$$\mathcal{P}(s) = \left\{ d \in \mathcal{S} \colon 0 \in \zeta(s) + g(s) + (\nabla\zeta(s) + [s + d, s; g])d + \xi(s + d) \right\}.$$

Under some suitable conditions, Khaton *et al.* [62] introduced and studied extended Newton-type method, when $\nabla\zeta$ is continuous and Lipschitz continuous as well as g admits FODD satisfying Lipschitzian condition. Inspired by the work of in [13], Khaton *et al.* [62] considered the following, "so called" EN-type method (see Algorithm 3):

Algorithm 3 (The Extended Newton-type Method)

Iter. 0. Pick $\eta \in [1, \infty)$, $s_0 \in \mathcal{S}$, and put $k := 0$.

Iter. 1. In case $0 \in \mathcal{P}(s_k)$, then stop; otherwise, go to the next Stair 2.

Iter. 2. In case $0 \notin \mathcal{P}(s_k)$, choose $d_k \in \mathcal{P}(s_k)$ such that

$$\|d_k\| \leq \eta \text{ dist } (0, \mathcal{P}(s_k)).$$

Iter. 3. Set $s_{k+1} := s_k + d_k$.

Iter. 4. Replace k by $k+1$ and repeat this cycle from Iter. 1.

In contrast Algorithm 3 with the known results, we have the following conclusions:

When $\xi = \{0\}$ and $g = 0$, it is obvious that Algorithm 3 is turned into the known GN method which is a famous iterative technique for solving nonlinear least squares (model fitting) problems and has been studied widely; see for example [24, 25, 49, 74, 130, 131]. Within the case when $g = 0$, several kind of methods for solving (1.0.1) were established by Rashid [105, 106, 108] and also obtained their semilocal and local convergence.

The objective of this subsection is to continue to study the semilocal and local convergence for the EN-type method under the weaker conditions than [62], that is, $\nabla\zeta$ is (L, q)-Hölder continuous and g admits the FODD satisfying q-Hölderian condition. The Lipschitz-like property of set-valued mappings which is the main tool of this study whose concepts can be found in Aubin [15] in the context of non smooth analysis and it has been studied by a huge number of mathematicians [2, 13, 30, 50, 90]. The main result of this study is semilocal analysis for the extended Newton-type method, that is, based on the information around the initial point, the main results are the convergence criteria, which provide few suitable conditions ensuring the convergence to a solution of any sequence generated by Algorithm 3. Consequently, the results of the local convergence for the EN-type method are attained.

4.3.2 Convergence Analysis

This section is dedicated to prove the existence of a sequence generated by the EN-type method, represented by the Algorithm 3 and show the superlinear convergence of the sequence generated by this method.

For our suitability, let $r_{\bar{s}} > 0$, $r_{\bar{t}} > 0$ and $\mathbb{B}_{r_{\bar{s}}}(\bar{s}) \subseteq \Upsilon \cap \operatorname{dom}\xi$. Suppose that $\nabla\zeta$ is (L, q)-Hölder continuous on $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$, that is , there exists $L > 0$ such that

$$\|\nabla\zeta(s) - \nabla\zeta(s')\| \leq L\|s - s'\|^q, q \in (0, 1], \quad \text{for any } s, s' \in \mathbb{B}_{r_{\bar{s}}}(\bar{s}), \tag{4.3.1}$$

g admits a FODD satisfying q-Hölder condition, that is, there exists $\nu > 0$ such that, for all $s, t, v, w \in \mathbb{B}_{r_{\bar{s}}}(\bar{s})$ $(s \neq t, v \neq w)$,

$$\|[s, t; g] - [v, w; g]\| \leq \nu(\|s - v\|^q + \|t - w\|^q), \tag{4.3.2}$$

and the mapping $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M, that is,

$$e(R_{\bar{s}}^{-1}(t_1) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}(t_2)) \leq M\|t_1 - t_2\| \quad \text{for any } t_1,\ t_2 \in \mathbb{B}_{r_{\bar{t}}}(\bar{t}). \tag{4.3.3}$$

Further, for $\bar{t}$, the closed graph property of $R_{\bar{s}}$ implies that $\zeta + g + \xi$ is continuous at $\bar{s}$ i.e.

$$\lim_{s \to \bar{s}} \operatorname{dist}\big(\bar{t}, \zeta(s) + g(s) + \xi(s)\big) = 0 \tag{4.3.4}$$

is hold.

Let $\varepsilon_0 > 0$ and write

$$\bar{r} := \min\left\{r_{\bar{t}} - 2\varepsilon_0 r_{\bar{s}}, \frac{r_{\bar{s}}(1 - M\varepsilon_0)}{4M}\right\}. \tag{4.3.5}$$

Then

$$\bar{r} > 0 \text{ if and only if } \varepsilon_0 < \min\left\{\frac{r_{\bar{t}}}{2r_{\bar{s}}},\ \frac{1}{M}\right\}. \tag{4.3.6}$$

The following lemma is extracted from [110, Lemma 3.1] which plays a crucial role for convergence analysis of the extended Newton-type (EN-type) method.

Lemma 4.3.1. *Assume that $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M and that*

$$\sup_{s', s'' \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})} \|\nabla\zeta(s') - \nabla\zeta(s'')\| \leq \varepsilon_0 < \min\left\{\frac{r_{\bar{t}}}{2r_{\bar{s}}},\ \frac{1}{M}\right\}. \tag{4.3.7}$$

Let $s \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ and ε_0 be defined by (4.3.6). Suppose that $\nabla\zeta$ is continuous on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Let $\bar{r}$ be defined by (4.3.5) such that (4.3.7) is true. Then R_s^{-1} is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\frac{M}{1-M\varepsilon_0}$, that is,

$$e(R_s^{-1}(t_1) \cap \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}), R_s^{-1}(t_2)) \leq \frac{M}{1 - M\varepsilon_0}\|t_1 - t_2\| \quad \text{for any } \ t_1,\ t_2 \in \mathbb{B}_{\bar{r}}(\bar{t}).$$

Then for any point $s',\ s'' \in \mathcal{S}$, we have from (4.2.19)

$$\begin{aligned}\|I_s(s') - I_s(s'')\| &= \|g(s') - g(s'') - [s', s; g](s' - s) + [s'', s; g](s'' - s) \\ &\quad + (\nabla\zeta(\bar{s}) - \nabla\zeta(s))(s' - s'')\|. \end{aligned} \tag{4.3.8}$$

Furthermore, let $q \in (0, 1]$ and define

$$\hat{r} := \min\left\{r_{\bar{t}} - 2Lr_{\bar{s}}^{q+1}, \frac{r_{\bar{s}}(1 - MLr_{\bar{s}}^{q})}{4M}\right\}. \tag{4.3.9}$$

Then

$$\hat{r} > 0 \Leftrightarrow L < \min\left\{\frac{r_{\bar{t}}}{2r_{\bar{s}}^{q+1}}, \frac{1}{Mr_{\bar{s}}^{q}}\right\}. \tag{4.3.10}$$

4.3.3 Superlinear Convergence

In this section we will show that the sequence generated by Algorithm 2 converges superlinearly if $\nabla\zeta$ is (L, q)-Hölderian and g admits FODD satisfying (ν, q)-Hölder condition. In fact, the following theorem ensuring the convergence of the EN-type method with initial point s_0.

Theorem 4.3.1. *Let $\eta > 1$ and $q \in (0, 1]$. Assume that $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M and that $\nabla\zeta$ is (L, q)- Hölder continuous on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ and g admits FODD that satisfies* (4.3.2)*. Let $\hat{r}$ be defined by* (4.3.9) *so that* (4.3.10) *is satisfied. Let $\nu > 0$, $\delta > 0$ be such that*

(a) $\delta \le \min\left\{\frac{r_{\bar{s}}}{4},\ (q + 5)\hat{r},\ 1,\ \left(\frac{3(q + 1)r_{\bar{t}}}{[L(q + 2) + 2\nu(q + 1)](6.2^q + 1)}\right)^{\frac{1}{(q+1)}}\right\}$,

(b) $(2^q M + 1)[L(q + 2) + 2\nu(q + 1)]\left(\eta(q + 1)\delta^q + 4^{1-q} r_{\bar{s}}^{q}\right) \le (q + 1)$,

(c) $\|\bar{t}\| < \dfrac{[L(q + 2) + 2\nu(q + 1)]}{3(q + 1)}\delta^{q+1}$.

Suppose that

$$\lim_{s\to\bar{s}} \operatorname{dist}(\bar{t}, \zeta(s) + g(s) + \xi(s)) = 0. \tag{4.3.11}$$

Then there exist some $\hat{\delta} > 0$ such that any sequence $\{s_n\}$ generated by Algorithm 2 with initial point s_0 in $\mathbb{B}_{\hat{\delta}}(\bar{s})$ converges superlinearly to a solution s^* of (1.0.1).

Proof. According to the assumption (a) $4\delta \le r_{\bar{s}}$ and $\eta > 1$, by assumption (b) we can write the inequality as follows

$$\begin{aligned}
&(2^qM+1)(q+5)[L(q+2)+2\nu(q+1)]\delta^q \\
&\qquad = (2^qM+1)[L(q+2)+2\nu(q+1)]\big((q+1)\delta^q+4\delta^q\big) \\
&\qquad \le (2^qM+1)[L(q+2)+2\nu(q+1)]\big(\eta(q+1)\delta^q+4\delta^q\big) \\
&\qquad \le (2^qM+1)[L(q+2)+2\nu(q+1)]\big(\eta(q+1)\delta^q+4^{1-q}r_{\bar{s}}^q\big) \\
&\qquad \le (q+1). \qquad (4.3.12)
\end{aligned}$$

Furthermore, using assumption (a) $4\delta \le r_{\bar{s}}$ and assumption (b) we can reduce the inequality as follows:

$$\begin{aligned}
&\eta M[L(q+2)+2\nu(q+1)]\delta^q \\
&< \eta 2^qM[L(q+2)+2\nu(q+1)](q+5)\delta^q \\
&\le (2^qM+1)[L(q+2)+2\nu(q+1)](\eta(q+1)\delta^q+4\delta^q) - 2^qML4\delta^q \\
&\le (2^qM+1)[L(q+2)+2\nu(q+1)](\eta(q+1)\delta^q+4^{1-q}r_{\bar{s}}^q) - 2^qML4^{1-q}r_{\bar{s}}^q \\
&\le (q+1) - 2^qML4^{1-q}r_{\bar{s}}^q.
\end{aligned}$$

Since $q \in (0,1]$ then, we get $2^qML4^{1-q}r_{\bar{s}}^q \ge (q+1)MLr_{\bar{s}}^q$. Now using (4.3.12) in the above equation and it becomes

$$\eta M[L(q+2)+2\nu(q+1)]\delta^q \le (q+1) - (q+1)MLr_{\bar{s}}^q. \qquad (4.3.13)$$

Putting

$$b := \frac{\eta M[L(q+2)+2\nu(q+1)]\delta^q}{(q+1)(1-MLr_{\bar{s}}^q)}.$$

Then, from (4.3.13) we have that

$$b \le 1. \qquad (4.3.14)$$

Pick $0 < \hat{\delta} \le \delta$ such that, for each $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s})$,

$$\operatorname{dist}(0, \zeta(s_0)+g(s_0)+\xi(s_0)) \le \frac{[L(q+2)+2\nu(q+1)]}{3(q+1)}\delta^{q+1}. \qquad (4.3.15)$$

Note that since (4.3.11) holds and assumption (c) is true, we assume that such $\hat{\delta}$ exists, which satisfies (4.3.15). Let $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s})$. By induction we will show that Algorithm 3 generates at least one sequence and such generated sequence $\{s_n\}$ satisfies the statements as follows:

$$\|s_n - \bar{s}\| \le 2\delta \tag{4.3.16}$$

and

$$\|d_n\| \le b\Big(\frac{1}{3}\Big)^{(q+1)^n} \delta, \tag{4.3.17}$$

hold for every $n = 0, 1, 2, ...$

Define

$$r_s := \frac{(q+5)M}{4(q+1)}\Big([L(q+2) + 2\nu(q+1)]\|s - \bar{s}\|^{(q+1)} + (q+1)\|\bar{t}\|\Big) \quad \text{for each } s \in \mathcal{S}. \tag{4.3.18}$$

From (4.3.12) we get

$$2^q M[L(q+2) + 2\nu(q+1)]\delta^q \le \frac{q+1}{q+5}. \tag{4.3.19}$$

$$\text{and} \quad [L(q+2) + 2\nu(q+1)]\delta^q \le \frac{q+1}{q+5}. \tag{4.3.20}$$

Hence by the combination of $\delta \le (q+5)\hat{r}$ in assumption (a) and inequality (4.3.20), we get

$$\begin{aligned} \|\bar{t}\| &< \frac{[L(q+2) + 2\nu(q+1)]\delta^{q+1}}{3(q+1)} \\ &\le \frac{(q+1)}{(q+1)\cdot(q+5)} \cdot \frac{(q+5)\hat{r}}{3} = \frac{\hat{r}}{3}. \end{aligned} \tag{4.3.21}$$

Utilizing (4.3.19) and assumption (c) together with (4.3.20), we get from (4.3.18) that

$$\begin{aligned} r_s &\le \frac{(q+5)M}{4(q+1)}\Big([L(q+2) + 2\nu(q+1)]\|\bar{s} - s_0\|^{q+1} + \frac{[L(q+2) + 2\nu(q+1)]}{3}\delta^{q+1}\Big) \\ &< \frac{(q+5)M}{12(q+1)}\Big(3[L(q+2) + 2\nu(q+1)](2\delta)^{q+1} + 2^q[L(q+2) + 2\nu(q+1)]\delta^{q+1}\Big) \\ &= \frac{(q+5)M}{12(q+1)}[L(q+2) + 2\nu(q+1)]\delta^{q+1}(3.2.2^q + 2^q) \\ &= \frac{(q+5)(6\cdot 2^q + 2^q)M}{12(q+1)}[L(q+2) + 2\nu(q+1)]\delta^{q+1} \\ &= \frac{(q+5)7\cdot 2^q M}{12(q+1)}[L(q+2) + 2\nu(q+1)]\delta^{p+1} \\ &= \frac{7(q+5)}{12(q+1)} \cdot \frac{(q+1)}{(q+5)}\delta < \frac{7}{12}\delta < 2\delta, \quad \text{for each } \ s \in \mathbb{B}_{2\delta}(\bar{s}). \end{aligned} \tag{4.3.22}$$

Observe that (4.3.16) is trivial for $n = 0$.

At first, we need to prove $\mathcal{P}(s_0) \neq \emptyset$ to show that (4.3.17) holds for $n = 0$. The nonemptyness of $\mathcal{P}(s_0)$ will ensure us to deduce the existence of the point s_1. We will apply Lemma 2.0.4 to the map Ψ_{s_0} with $\eta_0 = \bar{s}$ for completing this. We have to show that Lemma 2.0.4 holds with $r := r_{s_0}$ and $\lambda := \dfrac{q+1}{q+5}$ satisfying both assertions (2.0.4) and (2.0.5). We get from (4.2.3) that $\bar{s} \in R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{2\delta}(\bar{s})$. By the definition of the excess e and (4.2.18), defined as the mapping of Ψ_{s_0}, we have that

$$\begin{aligned}
\operatorname{dist}(\bar{s}, \Psi_{s_0}(\bar{s})) &\leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(\bar{s})) \\
&\leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{2\delta}(\bar{s}), \Psi_{s_0}(\bar{s})) \\
&\leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}[I_{s_0}(\bar{s})]). \qquad (4.3.23)
\end{aligned}$$

Since $\nabla\zeta$ is (L, q)-Hölder continuous and g admits FODD satisfies Hölderian condition, for every $s \in \mathbb{B}_{2\delta}(\bar{s}) \subseteq \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$, we have that

$$\begin{aligned}
\|I_{s_0}(s) - \bar{t}\| &= \|\zeta(\bar{s}) + g(s) + \nabla\zeta(\bar{s})(s - \bar{s}) - \zeta(s_0) - g(s_0) \\
&\qquad - (\nabla\zeta(s_0) + [s, s_0; g])(s - s_0) - \bar{t}\| \\
&\leq \|\zeta(\bar{s}) - \zeta(s_0) - \nabla\zeta(s_0)(\bar{s} - s_0)\| + \|\big(\nabla\zeta(s_0) - \nabla\zeta(\bar{s})\big)(\bar{s} - s)\| \\
&\qquad + \|g(s) - g(s_0) - [s, s_0; g](s - s_0)\| + \|\bar{t}\| \\
&\leq \frac{L}{q+1}\|\bar{s} - s_0\|^{q+1} + L\|s_0 - \bar{s}\|^q\|\bar{s} - s\| \\
&\quad + \|[s_0, s; g] - [s, s_0; g]\|\|s - s_0\| + \|\bar{t}\| \qquad (4.3.24) \\
&\leq \frac{L}{q+1}\|\bar{s} - s_0\|^{q+1} + L\|s_0 - \bar{s}\|^q\|\bar{s} - s\| \\
&\quad + \nu\big(\|s_0 - x\|^q + \|s - s_0\|^q\big)\|s - s_0\| + \|\bar{t}\| \\
&\leq \frac{L}{q+1}(2\delta)^{q+1} + L(2\delta)^q \cdot 2\delta + \nu\big((2\delta)^q + (2\delta)^q\big) \cdot 2\delta + \|\bar{t}\| \\
&\leq \frac{L(q+2) + 2\nu(q+1)}{q+1}\delta^{q+1} \cdot 2^{q+1} + \|\bar{t}\|. \qquad (4.3.25)
\end{aligned}$$

Now through the assumptions (a) $\dfrac{[L(q+2) + 2\nu(q+1)](6 \cdot 2^q + 1)}{3(q+1)}\delta^{q+1} \leq r_{\bar{t}}$ and (c), (4.3.24) gives that

$$\begin{aligned}\|I_{s_0}(s)-\bar{t}\| &\leq \frac{[L(q+2)+2\nu(q+1)]}{q+1}2^{q+1}\delta^{q+1}+\frac{[L(q+2)+2\nu(q+1)]}{3(q+1)}\delta^{q+1}\\ &= \frac{[L(q+2)+2\nu(q+1)](3.2.2^q+1)}{3(q+1)}\delta^{q+1}\\ &< \frac{[L(q+2)+2\nu(q+1)](6\cdot 2^q+1)}{3(q+1)}\delta^{q+1}\leq r_{\bar{t}}. \qquad (4.3.26)\end{aligned}$$

This means that $I_{s_0}(s)\in\mathbb{B}_{r_{\bar{t}}}(\bar{t})$. Moreover, let $s=\bar{s}$ in (4.3.24). Then it is easily proved that

$$\begin{aligned}&I_{s_0}(\bar{s})\in\mathbb{B}_{r_{\bar{t}}}(\bar{t}) \qquad \text{and}\\ &\|I_{s_0}(\bar{s})-\bar{t}\|\leq\frac{[L+2\nu(q+1)]}{q+1}\|\bar{s}-s_0\|^{q+1}+\|\bar{t}\|. \qquad (4.3.27)\end{aligned}$$

By using the Lipschitz-like property of $R_{\bar{s}}^{-1}$ and (4.3.27) in (4.3.23), we obtain

$$\begin{aligned}\text{dist}(\bar{s},\Psi_{s_0}(\bar{s})) &\leq M\|\bar{t}-I_{s_0}(\bar{s})\|\\ &\leq \frac{M[L(q+2)+2\nu(q+1)]}{q+1}\|\bar{s}-s_0\|^{q+1}+M\|\bar{t}\|\\ &\leq \frac{4}{q+5}r_{s_0}=\big(1-\frac{q+1}{q+5}\big)r_{s_0}\\ &= (1-\lambda)r;\end{aligned}$$

i e, the statement (2.0.4) of Lemma 2.0.4 is hold. Now, it is evident to show that statement (2.0.5) of Lemma 2.0.4 holds. Let $s',s''\in\mathbb{B}_{r_{s_0}}(\bar{s})$. Then we get that $s',s''\in\mathbb{B}_{r_{s_0}}(\bar{s})\subseteq\mathbb{B}_{2\delta}(\bar{s})\subseteq\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ by (4.3.22) and $I_{s_0}(s')$, $I_{s_0}(s'')\in\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ by (4.3.26). This together with the assumed Lipschitz-like property of $R_{\bar{s}}^{-1}$ is as follows:

$$\begin{aligned}e(\Psi_{s_0}(s')\cap\mathbb{B}_{r_{s_0}}(\bar{s}),\Psi_{s_0}(s'')) &\leq e(\Psi_{s_0}(s')\cap\mathbb{B}_{2\delta}(\bar{s}),\Psi_{s_0}(s''))\\ &\leq e(R_{\bar{s}}^{-1}[I_{s_0}(s')]\cap\mathbb{B}_{r_{\bar{s}}}(\bar{s}),R_{\bar{s}}^{-1}[I_{s_0}(s'')])\\ &\leq M\|I_{s_0}(s')-I_{s_0}(s'')\|. \qquad (4.3.28)\end{aligned}$$

Now, using the definition of FODD of g in (4.3.8) we obtain

$$\begin{aligned}\|I_{s_0}(s')-I_{s_0}(s'')\| &= \|g(s')-g(s'')-[s',s_0;g](s'-s_0)+[s'',s_0;g](s''-s_0)\\ &\quad +(\nabla\zeta(\bar{s})-\nabla\zeta(s_0))(s'-s'')\|\\ &\leq \|g(s')-g(s'')+[s',s_0;g](s_0-s')-[s'',s_0;g](s_0-s'')\|\\ &\quad +\|\nabla\zeta(\bar{s})-\nabla\zeta(s_0)\|\|s'-s''\|\\ &\leq \|g(s')-g(s'')+g(s_0)-g(s')-g(s_0)+g(s'')\\ &\quad +\|\nabla\zeta(\bar{s})-\nabla\zeta(s_0)\|\|s'-s''\|\\ &\leq \|\nabla\zeta(\bar{s})-\nabla\zeta(s_0)\|\|s'-s''\| \leq L\|\bar{s}-s_0\|^q\|s'-s''\|\\ &\leq L.2^q\delta^q\|s'-s''\|. \end{aligned} \tag{4.3.29}$$

It follows from (4.3.28), that

$$e(\Psi_{s_0}(s')\cap\mathbb{B}_{r_{s_0}}(\bar{s}),\Psi_{s_0}(s'')) \leq ML.2^q\delta^q\|s'-s''\|.$$

Since $\nu, M, L>0$ and $q\in(0,1]$, then we can write $2^qML\delta^q < 2^qM[L(q+2)+2\nu(q+1)]\delta^p$ and hence the above inequality becomes

$$\begin{aligned} e(\Psi_{s_0}(s')\cap\mathbb{B}_{r_{s_0}}(\bar{s}),\Psi_{s_0}(s'')) &\leq 2^qM[L(q+2)+2\nu(q+1)]\delta^p\|s'-s''\|\\ &\leq \frac{q+1}{q+5}\|s'-s''\|\\ &= \lambda\|s'-s''\|. \end{aligned}$$

Thus the statement (2.0.5) of Lemma 2.0.4 is also hold. Hence, both statements (2.0.4) and (2.0.5) of Lemma 2.0.4 are accomplished. Finally, it shows that Lemma 2.0.4 is adequate to presume the position of a point $\hat{s}_1\in\mathbb{B}_{r_{s_0}}(\bar{s})$ such that $\hat{s}_1\in\Psi_{s_0}(\hat{s}_1)$ which implies that $0\in\zeta(s_0)+g(s_0)+(\nabla\zeta(s_0)+[\hat{s}_1,s_0;g])(\hat{s}_1-s_0)+\xi(\hat{s}_1)$ and hence $\mathcal{P}(s_0)\neq\emptyset$.

Next, it is sufficient to prove that (4.3.17) holds for $n=0$. As $\nabla\zeta$ is (L,q)- Hölder continuous on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$, we have for all $s',s''\in\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$, that

$$Lr_{\bar{s}}^q \geq \sup_{s',s''\in\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})}\|\nabla\zeta(s')-\nabla\zeta(s'')\|. \tag{4.3.30}$$

Observe the assumption (a) that $\hat{r}>0$. Therefore, from (4.3.9) and (4.3.30) imply that Lemma 4.3.1 is satisfied with $\varepsilon_0 := Lr_{\bar{s}}^p$. According to our assumption $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{y}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$. Then, it implies from Lemma 4.3.1 that, $R_{s_0}^{-1}$ is Lipschitz-like on $\mathbb{B}_{\hat{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\frac{M}{1-MLr_{\bar{s}}^q}$ as $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s}) \subseteq \mathbb{B}_{\delta}(\bar{s}) \subseteq \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ by assumption (a) and the choice of $\hat{\delta}$. On the other hand, (4.3.15) follows as

$$\begin{aligned} \operatorname{dist}(0, R_{s_0}(s_0)) &= \operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0)) \\ &\leq \frac{\hat{r}}{3}. \end{aligned}$$

Inequality (4.3.21) shows that $0 \in \mathbb{B}(\bar{t}, \frac{\hat{r}}{3})$ and observe before that $s_0 \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Hence using Lemma 2.0.1, we get

$$\begin{aligned} \operatorname{dist}(s_0, R_{s_0}^{-1}(0)) &\leq \frac{M}{1 - MLr_{\bar{s}}^q} \operatorname{dist}(0, R_{s_0}(s_0)) \\ &= \frac{M}{1 - MLr_{\bar{s}}^q} \operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0)). \end{aligned}$$

This together with (4.2.1), gives

$$\begin{aligned} \operatorname{dist}(0, \mathcal{P}(s_0)) &= \operatorname{dist}(s_0, R_{s_0}^{-1}(0)) \\ &\leq \frac{M}{1 - MLr_{\bar{s}}^q} \operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0)). \end{aligned} \tag{4.3.31}$$

According to Algorithm 2 and using (4.3.14), (4.2.42) and (4.3.31), we have

$$\begin{aligned} \|d_0\| &\leq \eta \operatorname{dist}(0, \mathcal{P}(s_0)) \\ &\leq \frac{\eta M}{(1 - MLr_{\bar{s}}^q)} \operatorname{dist}(0, \zeta(s_0) + g(s_0) + \xi(s_0)) \\ &\leq \frac{\eta M[L(q+2) + 2\nu(q+1)]\delta^{q+1}}{3(q+1)(1 - MLr_{\bar{s}}^q)} = b\Big(\frac{1}{3}\Big)\delta. \end{aligned}$$

This means that

$$\|s_1 - s_0\| = \|d_0\| \leq b\Big(\frac{1}{3}\Big)\delta,$$

and therefore, (4.3.17) is true for $n = 0$. Suppose $s_1, s_2, \ldots, s_k$ are formed. The inequalities (4.3.16) and (4.3.17) are hold for $n = 0, 1, 2, \ldots, k-1$. We show that there exists s_{k+1} such that (4.3.16) and (4.3.17) are also hold for $n = k$. Since (4.3.16) and (4.3.17) are true for each $n \leq k-1$, we get the following inequality:

$$\|s_k - \bar{s}\| \leq \sum_{i=0}^{k-1} \|d_i\| + \|s_0 - \bar{s}\| \leq b\delta \sum_{i=0}^{k-1} \Big(\frac{1}{3}\Big)^{(q+1)^i} + \delta \leq 2\delta.$$

This implies (4.3.16) holds for $n = k$. Now with all the same argument as we did for the case when $n = 0$, we can prove that $\mathcal{P}(s_k) \neq \emptyset$, that is, the point s_{k+1} exists and $R_{s_k}^{-1}$ is Lipschitz-like on $\mathbb{B}_{\hat{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\frac{M}{1-MLr_{\bar{s}}^q}$. Therefore, we have that

$$\begin{aligned}
\|s_{k+1} - s_k\| &= \|d_k\| \leq \eta \operatorname{dist}(0, \mathcal{P}(s_k)) \\
&\leq \eta \operatorname{dist}(s_k, R_{s_k}^{-1}(0)) \\
&= \frac{\eta M}{1 - MLr_{\bar{s}}^q} \operatorname{dist}(0, \zeta(s_k) + g(s_k) + \xi(s_k)) \\
&\leq \frac{\eta M}{1 - MLr_{\bar{s}}^q} \|\zeta(s_k) + g(s_k) - \zeta(s_{k-1}) - g(s_{k-1}) \\
&\quad - \big(\nabla\zeta(s_{k-1}) + [s_k, s_{k-1}; g]\big)(s_k - s_{k-1})\| \\
&\leq \frac{\eta M}{1 - MLr_{\bar{s}}^q} \big(\|\zeta(s_k) - \zeta(s_{k-1}) - \nabla\zeta(s_{k-1})(s_k - s_{k-1})\| \\
&\quad + \|g(s_k) - g(s_{k-1}) - [s_k, s_{k-1}; g](s_k - s_{k-1})\|\big) \\
&\leq \frac{\eta M}{(q+1)(1 - MLr_{\bar{s}}^q)} \big(L\|s_k - s_{k-1}\|^{q+1} + \\
&\quad (q+1)\|[s_{k-1}, s_k; g] - [s_k, s_{k-1}; g]\|\|s_k - s_{k-1}\|\big) \\
&\leq \frac{\eta M}{(q+1)(1 - MLr_{\bar{s}}^q)} \big(L\|s_k - s_{k-1}\|^{q+1} + \\
&\quad (q+1)\nu(\|s_{k-1} - s_k\|^q + \|s_k - s_{k-1}\|^q)\|s_k - s_{k-1}\|\big) \\
&\leq \frac{\eta M[L + 2\nu(q+1)]}{(q+1)(1 - MLr_{\bar{s}}^q)} \|d_{k-1}\|^{q+1} \\
&\leq \frac{\eta M[L(q+2) + 2\nu(q+1)]}{(q+1)(1 - MLr_{\bar{s}}^q)} \|d_{k-1}\|^{q+1} \qquad (4.3.32) \\
&\leq \frac{\eta M[L(q+2) + 2\nu(q+1)]}{(q+1)(1 - MLr_{\bar{s}}^q)} \big(b\big(\frac{1}{3}\big)^{(q+1)^{k-1}}\delta\big)^{q+1} \leq b\big(\frac{1}{3}\big)^{(q+1)^k}\delta.
\end{aligned}$$

This implies that (4.3.17) holds for $n = k$ and therefore the proof of the theorem is completed.

□

Consider the special case when $\bar{s}$ is a solution of (1.0.1)(that is, $\bar{t} = 0$) in Theorem 4.3.1. We have the following corollary, which describes the local superlinear convergence result for the EN-type method.

Corollary 4.3.1. *Suppose that $\bar{s}$ is a solution of the variational inclusion (1.0.1). Let $q \in (0, 1]$ and $\eta > 1$ and let $R_{\bar{s}}^{-1}$ be pseudo-Lipschitz around $(0, \bar{s})$. Let $\tilde{r} > o$ and suppose that $\nabla\zeta$ is (L, q)-Hölder continuous on $\mathbb{B}_{\tilde{r}}(\bar{s})$ and g admits FODD satisfying Hölderian condition* *on $\mathbb{B}_{\tilde{r}}(\bar{s})$. Assume that*

$$\lim_{s\to\bar{s}} \operatorname{dist}(0, R_s(s)) = 0. \tag{4.3.33}$$

Then, with an initial point s_0, there exists some $\hat{\delta} > 0$ such that any sequence $\{s_n\}$ generated by Algorithm 3 converges superlinearly to a solution s^* of the variational inclusion (1.0.1).

Proof. Suppose that $R_{\bar{s}}^{-1}$ is pseudo-Lipschitz around $(0, \bar{s})$. Then by definition of pseudo-Lipschitz continuty, there exist constants $M, \tilde{r}$ and r_0 such that $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{(\bar{t})}(r_0)$ relative to $\mathbb{B}_{\tilde{r}}(\bar{s})$ with constant M. Then, for each $0 < r_{\bar{s}} \leq \tilde{r}$, we have that

$$e(R_{\bar{s}}^{-1}(t_1) \cap \mathbb{B}(\bar{s}, r_{\bar{s}}), R_{\bar{s}}^{-1}(t_2) \leq M\|t_1 - t_2\| \ \text{ for any } \ t_1,\ t_2 \in \mathbb{B}_{r_0}(0),$$

that is, $R_{\bar{s}}^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_0}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M. Let $L \in (0, 1]$, $q \in (0, 1]$ and $\nu > 0$. By the (L, q)-Hölder continuty of $\nabla\zeta$ we can select $r_{\bar{s}} \in (0, \tilde{r})$ such that $\frac{r_{\bar{s}}}{2} \leq \tilde{r},\ r_0 - 2Lr_{\bar{s}}^{q+1} > 0,\ MLr_{\bar{s}}^q < 1$ and

$$Lr_{\bar{s}}^q \geq \sup_{s', s'' \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})} \|\nabla\zeta(s') - \nabla\zeta(s'')\|.$$

Then, define

$$\hat{r} := \min\left\{r_0 - 2Lr_{\bar{s}}^{q+1}, \frac{r_{\bar{s}}(1 - MLr_{\bar{s}}^q)}{4M}\right\} > 0.$$

and

$$\min\left\{\frac{r_{\bar{s}}}{4},\ (q+5)\hat{r},\ \frac{3(q+1)r_0}{[L(q+2) + 2\nu(q+1)](6.2^q + 1)}\right\} > 0$$

Thus, we can choose $0 < \delta \leq 1$ such that

$$\delta \leq \min\left\{\frac{r_{\bar{s}}}{4},\ (q+5)\hat{r},\ \frac{3(q+1)r_0}{[L(q+2) + 2\nu(q+1)](6.2^q + 1)}\right\}$$

and

$$(2^q M + 1)[L(q+2) + 2\nu(q+1)]\Big(\eta(q+1)\delta^q + 4^{1-q} r_{\bar{s}}^q\Big) \leq (q+1).$$

Now it is routine to check that conditions (a)-(c) of Theorem 4.3.1 are satisfied. Thus we can apply Theorem 4.3.1 to complete the proof. □

4.3.4 Numerical Experiment

To verify the semi-local convergence results of the EN-type method, a numerical example is presented in this section.

Example 4.3.1. *Let* $S = T = \mathbb{R}$, $s_0 = -0.2$, $\eta = 1.5$, $\nu = 0.4$, $M = 0.2$, $q = 0.9$, $r = 5$ and $L = 3$. *Define a Fréchet differentiable function* ζ *on* $\mathbb{R}$ *by* $\zeta(s) = 3s^2 + 1$, *linear and divided difference admissible function* $g(s) = -\frac{3s}{2}$ *and a set-valued mapping* ξ *on* $\mathbb{R}$ *by* $\xi(s) = \{-5s + 2, 2s - 2\}$. *Then* $\zeta + g + \xi$ *is a set-valued mapping on* $\mathbb{R}$ *defined by* $\zeta(s) + g(s) + \xi(s) = \{3s^2 - \frac{13s}{2} + 3, 3s^2 + \frac{s}{2} - 1\}$. *Then Algorithm 3 generates a sequence which converges to* $s^* = 0.666$.

Solution: Consider $\zeta(s) + g(s) + \xi(s) = 3s^2 - \frac{13s}{2} + 3$. It is manifest that $(\zeta + g + \xi)$ has a closed graph at $(-0.2, 4.42)$. In this way $(-0.2, 4.42) \in \mathrm{gph}(\zeta + g + \xi)$. Then from the statement, it is clear that $(\zeta + g + \xi)^{-1}$ is Lipschitz-like at $(4.42, -0.2)$. Then from (4.1.1), we have that

$$\begin{aligned} P(s_k) &= \Big\{d_k \in S : 0 \in \zeta(s_k) + g(s_k) + (\nabla\zeta(s_k) + [s_k + d_k, s_k; g]d_k + \xi(s_k + d_k)\Big\} \\ &= \Big\{d_k \in S : 0 \in \zeta(s_k) + \nabla\zeta(s_k)d_k + g(s_k + d_k) + \xi(s_k + d_k)\Big\} \\ &= \Big\{d_k \in \mathbb{R} : d_k = \frac{6s_k^2 - 13s_k + 6}{13 - 12s_k}\Big\}. \end{aligned}$$

Otherwise, if $P(s_k) \neq \emptyset$, we obtain that

$$\begin{aligned} & 0 \in \zeta(s_k) + \nabla\zeta(s_k)(s_{k+1} - s_k) + g(s_{k+1}) + \xi(s_{k+1}) \\ & \Rightarrow s_{k+1} = \frac{6 - 6s_k^2}{13 - 12s_k}. \end{aligned}$$

Thus from (4.3.32), we obtain that

$$\|d_k\| \leq \frac{\eta M[L + (q + 2) + 2\nu(q + 1)]}{(q + 1)(1 - MLr_{\bar{s}}^{q})}\|d_{k-1}\|^{1+q}.$$

Hereafter, for the given values of M, L, η, q, r and ν, we get that Algorithm 3 generates a superlinearly convergent sequence with initial point $s_0 = -0.2$ in a neighborhood of $\bar{s} = -0.19$. Then the following Table 4.3, obtained by using Matlab code, indicates that the solution of the variational inclusion is 0 when $k = 5$.

Table 4.3 Numerical results for Example 4.3.1

iteration no.	s_k	$\zeta + g + \xi = 3s^2 - \dfrac{13s}{2} + 3$
1	-0.2000	4.4200
2	0.3740	0.9885
3	0.6063	0.1619
4	0.6628	0.0096
5	0.6666	0.0000
6	0.6667	0.0000
7	0.6667	0

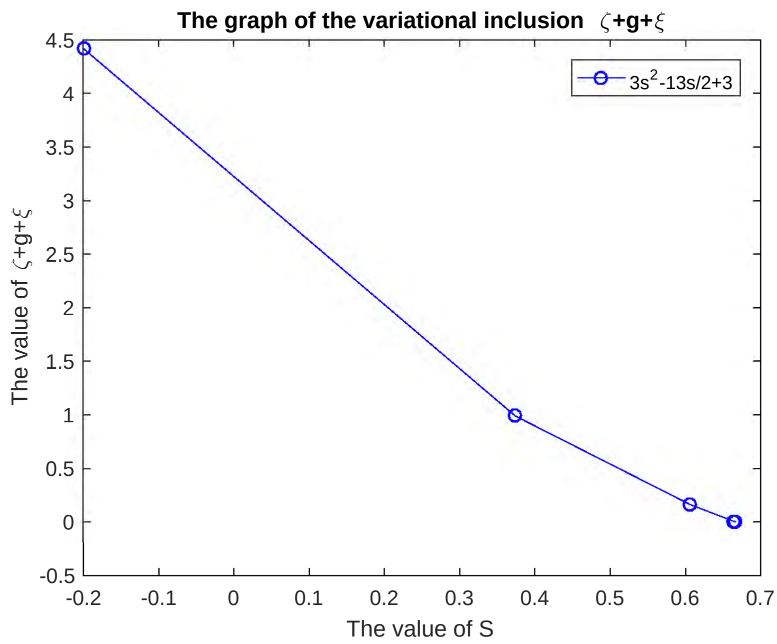

Figure 4.2: Finding a solution of variational inclusion

4.3.5 Concluding Remarks

The semilocal and local convergence results are presented for the EN-type method under the conditions that $\eta > 1$, $R_{\bar{s}}^{-1}$ is Lipschitz-like, $\nabla\zeta$ satisfies Hölderian condition and g admits FODD satisfying the Hölder condition defined by (4.3.2). In particular, we have presented semilocally superlinear convergence analysis for EN-type method in Theorem 4.3.1 while the locally superlinear convergence analysis for EN-type method is presented in Corollary 4.3.1. Here we have given a numerical experiment to illustrates the theoretical result. Therefore, this result extends and improves the corresponding ones [13, 62].

Moreover, according to our main results, we have the following conclusions:

(i) If we set $q = 0$ in Theorem 4.3.1, it gives the semilocal linear convergence result for the EN-type method and this result coincides with the result presented in [62, Theorem 3.1]. On the other hand, if we put $q = 0$ in Corollary 4.3.1, this result provides locally linear convergence result which is similar with the result presented in [62, Corollary 3.1].

(ii) If we put $q = 1$ in Theorem 4.3.1, it yields the semilocal quadratic convergence result for the EN-type method and this result is analogous to the outcome presented in [62, Theorem 3.2]. Furthermore, if we give $q = 1$ in Corollary 4.3.1, it gives the local quadratic convergence result for this method which is resembling the work presented in [62, Corollary 3.2].

Chapter 5

Semilocal and Local Convergence Analysis of an ENM for Nonsmooth Variational Inclusions

In this Chapter, we introduce an ENM for finding the solution of the nonsmooth variational inclusion (1.0.3) $0 \in \zeta(\bar{s}) + \xi(\bar{s})$ and analyze its semilocal and local convergence under the conditions that $(\zeta+\xi)^{-1}$ is Lipschitz-like and ζ admits a (n, α)-PBA. Applications of (n, α)-PBA are provided for smooth functions in the cases $n = 1$ and $n = 2$ as well as for normal maps. In particular, when $0 < \alpha < 1$ and the derivative of ζ, denoted $\nabla\zeta$, is (ℓ, α)-Hölder continuous, we have shown that ζ admits $(1, \alpha)$-PBA for $n = 1$ while ζ admits $(2, \alpha)$-PBA for $n = 2$, when $0 < \alpha < 1$ and the second derivative of ζ, denoted $\nabla^2\zeta$, is (K, α)-Hölder. Finally, we have constructed a (n, α)-PBA for the normal maps $\zeta_{\mathcal{C}} + \xi$ when ζ has a (n, α)-PBA.

5.1 ENM for Nonsmooth Variational Inclusions

5.1.1 Introduction

Let $\mathcal{S}$ and $\mathcal{T}$ be two Banach spaces, $\xi\colon \mathcal{S} \rightrightarrows 2^{\mathcal{T}}$ be a set-valued mapping which has closed graph and $\zeta\colon \Upsilon \subseteq \mathcal{S} \to \mathcal{T}$ be a nonsmooth single-valued function that admits (n, α)-PBA on

Υ with a constant $L > 0$. We are concerned with the problem of finding solution of the nonsmooth variational inclusion (1.0.3), which is as follows:

$$0 \in \zeta(\bar{s}) + \xi(\bar{s}). \tag{5.1.1}$$

The classical Newton method is very well known and extensively used to find solutions of (1.0.3) when $\xi = \{0\}$, where ζ has Lipschitz continuous Fréchet derivatives. Semilocal and local convergence results for Newton method can be found in the survey [12, 27, 43, 61] and its references. We assume that the single-valued function ζ is nonsmooth function, that is, ζ doesn't possess Fréchet derivative and its classical linearization is no longer available. Then no one can give the clear result that how one can give a design of the Newton algorithm. So that it needs to seek a replacement for such type of linearization. A lot of researchers have worked on this question and the applicants have presented different methods for a few things that are important in certain cases and have proved their justification. A lot of papers have worked on the Newton-type methods for solving the nonsmooth equations and variational inequalities; see for example [6, 10, 14, 34, 68, 69, 119, 123, 129] for inspiration and advanced works on these areas.

In the framework of nonsmooth variational inclusion (1.0.3), when the single-valued function is differentiable, several iterative methods have presented for solving this variational inclusion, such as N-type method, proximal point method, etc.; see for example [3–5, 25, 102, 105, 107, 110, 111]. The proximal point algorithm (PPA) is one of the most useful method for solving (1.0.3) in the case $\zeta = 0$ and $\mathcal{T} = \mathcal{S}$ a Hilbert space. About the root of PPA can be known in the works of Martinet [77] for variational inequalities. This PPA has been further polished and extended in [102, 125, 127] to a more general setting, including convex programs, convex-concave saddle point problems and variational inequality problems. In addition, Alom and Rashid [4] have been presented the Gauss-type proximal point method for solving (1.0.3) in the case of smooth function, that is, when ζ is Fréchet differentiable. A number of papers have appeared dealing with N-type methods for solving the nonsmooth variational inclusion (1.0.3) and analyzed the local and semi-local convergence results, see in [11, 42, 98].

To solve the nonsmooth variational inclusion (1.0.3), Geoffroy and Piétrus in [42] considered the method as follows:

$$0 \in A(s_k, s_{k+1}) + \xi(s_{k+1}) \quad \text{for each } k = 0, 1, 2, \ldots, \tag{5.1.2}$$

where $A : \mathcal{S} \times \mathcal{S} \to \mathcal{T}$ is an approximation of ζ. They presented a local convergence result under some assumptions and the assumptions are ζ and the set valued map, where ζ admits an (n, α)-PBA and the set-valued map $(A(s^*, \cdot) + \xi(\cdot))^{-1}$ is M-pseudo-Lipschitz around $(0, s^*)$. For the first time, Dontchev [26] introduced the iterative procedure (5.1.2) for solving (1.0.3). For this procedure (5.1.2) he presented the nonsmooth analogue of the Kantorovich-type theorem by assuming the Aubin continuity of the map $(A(s_0, \cdot) + \xi(\cdot))^{-1}$ at $(0, s_1)$, where s_1 is the first iterate of (5.1.2).

Let $s \in \Upsilon \subseteq \mathcal{S}$. Suppose that $\mathcal{M}(s)$ is a subset of Υ which is defined by

$$\mathcal{M}(s) := \{d \in \Upsilon : 0 \in A(s, s + d) + \xi(s + d)\}.$$

Usually, the method (5.1.2) guarantees the existence of a sequence and the sequence is a convergent. Therefore, for a starting point near to a solution, we know that, the sequences are not uniquely defined, which is constructed by the method (5.1.2). For example, Dontchev presented a convergence result which is established in [42, Theorem 3.3] and the result confirms the existence of a convergent sequence. Thus, in view of numerical computation, this kind of Newton-type methods are not convenient in practical application. This drawback allows us to propose the iterative procedure "so-called" extended Newton-type method (ENM) to solve the nonsmooth variational inclusion (1.0.3):

Algorithm 4 (The Extended Newton-type Method)(ENM)

Iter. 1. Pick $\eta \in [1, \infty)$, $s_0 \in \Upsilon$, and place $i := 0$.

Iter. 2. In case $0 \in \mathcal{M}(s_i)$, then stop; otherwise, go to the next Stair 3.

Iter. 3. In case $0 \notin \mathcal{M}(s_i)$, choose d_i such that $d_i \in \mathcal{M}(s_i)$

$$\text{and} \quad \|d_i\| \leq \eta \ \text{dist}\,(0, \mathcal{M}(s_i)).$$

Iter. 4. Set $s_{i+1} := s_i + d_i$.

Iter. 5. $i + 1$ is replaced by i and repeat this cycle Iter. 2.

Many effective works on semi-local analysis have been investigated for some special cases such as N-type method for nonlinear least square problems (cf. [25]), the ENM for solving variational inclusions (cf. [102]) and the Newton method for nonsmooth equations (cf. [11]). Rashid *et al.* [110] introduced the GN type method for approximating the solution of (1.0.3) in the case of smooth function and obtained the semi-local and local convergence results. Rashid introduced the GN method for nonsmooth generalized equations in his PhD thesis [98, Theorem 3.2.1], and obtained the semi-local and local convergence results. Moreover, Rashid [103] introduced an extended Newton-type method for solving the nonsmooth generalized equation (1.0.3) and achieved the semi-local and local convergence results. In recent time, Alom and Rashid [3] have been presented the general Gauss-type proximal point method for solving (1.0.3) in the case of smooth function and evaluate the semi-local and local convergence results. As our best knowledge, there is no other study on semi-local analysis for solving the nonsmooth variational inclusion (1.0.3) by using extended Newton-type method (ENM). Thus, the contribution, presented in this study, seems new.

In this chapter, we present semilocal and local convergence of Algorithm 4 under some mild conditions for the function ζ and the set-valued mapping $(\zeta+\xi)^{-1}$. In fact, the main motivation of this research is to analyze the semilocal and local convergence of the sequence generated by Algorithm 4 for solving the nonsmooth variational inclusion (1.0.3) using the notion of (n,α)-PBA introduced by Geoffroy and Piétrus [42] and Lipschitz-like property. Based on the information around the initial point, the main result is the convergence criterion, developed in the section 3, which provides some sufficient conditions, for a starting point near to the solution, ensuring the convergence to the solution of any sequence constructed by Algorithm 4. As a result, local convergence result for the ENM is obtained.

This work is arranged as follows: In section 5.2, we will show the existence and prove the convergence of the sequence generated by the Algorithm 4, which is introduced in section 5.1.1, by using (n,α)-PBA as well as the concept of Lipschitz-like property for set-valued mappings. The summary of the fundamental results in the present work are presented in section 5.4.

5.2 Convergence Analysis of ENM

Let $n \in \mathbb{N}^*$, $\alpha > 0$ and $\zeta : \Upsilon \subseteq \mathcal{S} \to \mathcal{T}$ is a nonsmooth function that admits (n,α)-PBA on Υ with a constant $L > 0$, where Υ is an open neighborhood of a point $\bar{s} \in \mathcal{S}$. Let $s \in \mathcal{S}$ and we define the mapping R_s as follows:

$$R_s(\cdot) := A(s,\cdot) + \xi(\cdot). \tag{5.2.1}$$

Then

$$\mathcal{M}(s) = \Big\{d \in \mathcal{S} \colon 0 \in R_s(s+d)\Big\} = \Big\{d \in \mathcal{S} \colon s+d \in R_s^{-1}(0)\Big\}. \tag{5.2.2}$$

Furthermore, the following equivalence is clear:

$$z \in R_s^{-1}(t) \Longleftrightarrow t \in A(s,z) + \xi(z) \quad \text{for any } z \in \mathcal{S} \text{ and } t \in \mathcal{T}. \tag{5.2.3}$$

In particular,

$$\bar{s} \in R_{\bar{s}}^{-1}(\bar{t}) \ \text{ for each } \ (\bar{s},\bar{t}) \in \text{gph}\ (\zeta + \xi).$$

Let $(\bar{s},\bar{t}) \in \text{gph}\ (\zeta+\xi)$ and let $r_{\bar{s}} > 0$, $r_{\bar{t}} > 0$. Furthermore, throughout in this section we assume that $\mathbb{B}_{r_{\bar{s}}}(\bar{s}) \subseteq \Upsilon \cap \text{dom}\,\xi$. Suppose that $\pi_{n,\alpha}$ is defined in Definition 2.0.3.

Define

$$\bar{r} := \min\left\{ r_{\bar{t}} - \frac{L\, {r_{\bar{s}}}^{n+\alpha}\,(3^{n+\alpha} + 2^{n+\alpha})}{\pi_{n,\alpha}\, 2^{n+\alpha}},\ \frac{r_{\bar{s}}(2^\alpha - MLr_{\bar{s}}^\alpha)}{4.2^\alpha M} \right\}. \tag{5.2.4}$$

Then

$$\bar{r} > 0 \Longleftrightarrow L < \min\left\{ \frac{r_{\bar{t}}\,\pi_{n,\alpha}\, 2^{n+\alpha}}{{r_{\bar{s}}}^{n+\alpha}\,(3^{n+\alpha}+2^{n+\alpha})},\ \frac{2^\alpha}{MLr_{\bar{s}}^\alpha} \right\}. \tag{5.2.5}$$

We know that the variational inclusion (1.0.3) is an abstract model for various problems. From now on, we make the following conditions.

(i) ζ admits a (n,α)-PBA with modulus L, on some open neighborhood Υ of $\bar{s}$, which is denoted by A;

(ii) ξ has closed graph;

(iii) The set valued map $(\zeta+\xi)^{-1}$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ with constant M.

The following lemma plays an important role to the convergence analysis of the ENM and the method defined by Algorithm 4. Dontchev proved that the following procedure is a refinement of [26, Lemma 1].

Lemma 5.2.1. *Suppose the assumptions* (i)-(iii) *hold and let* $\bar{r}$ *be defined in* (5.2.4)*, so that* (5.2.5) *is satisfied. Let* $s \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$*. Then* $R_s^{-1}(\cdot)$ *is Lipschitz-like on* $\mathbb{B}_{\bar{r}}(\bar{t})$ *relative to* $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ *with constant* $\frac{2^\alpha M}{2^\alpha - MLr_{\bar{s}}^\alpha}$*, that is,*

$$e(R_s^{-1}(t_1) \cap \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}), R_s^{-1}(y_2)) \leq \frac{2^\alpha M}{2^\alpha - MLr_{\bar{s}}^\alpha}\|t_1 - t_2\| \quad \textit{for any } t_1,\ t_2 \in \mathbb{B}_{\bar{r}}(\bar{t}).$$

Proof. Since ζ has a (n, α)-PBA on an open neighbourhood of $\bar{s} \in (\zeta + \xi)^{-1}(\bar{t})$ with a constant L and the map $(\zeta + \xi)^{-1}$ is Lipschitz-like around $(\bar{t}, \bar{s})$ with a constant M, then by Remark 2.0.4 we get that $R_s^{-1}(\cdot)$ is Lipschitz-like around $(\bar{t}, \bar{s})$ with a constant $M < L$, i.e, $\exists$ constants $r_{\bar{s}} > 0$, $r_{\bar{t}} > 0$ and M such that

$$e(R_{\bar{s}}^{-1}(t_1) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}(t_2)) \leq M\|t_1 - t_2\| \quad \text{for all } t_1,\, t_2 \in \mathbb{B}_{r_{\bar{t}}}(\bar{t}). \tag{5.2.6}$$

Note, by (5.2.4) and (5.2.5), that $\bar{r} > 0$. Now let

$$t_1,\ t_2 \in \mathbb{B}_{\bar{r}}(\bar{t}) \quad \text{and} \quad s' \in R_s^{-1}(t_1) \cap \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}). \tag{5.2.7}$$

It is sufficent to prove that there exist $s'' \in R_s^{-1}(t_2)$ such that

$$\|s' - s''\| \leq \frac{2^\alpha M}{2^\alpha - MLr_{\bar{s}}^\alpha}\|t_1 - t_2\|.$$

At the last stage, we shall verify, there exists a sequence $\{s_k\} \subset \mathbb{B}_{r_{\bar{s}}}(\bar{s})$ such that

$$t_2 \in A(s, s_{k-1}) - A(\bar{s}, s_{k-1}) + A(\bar{s}, s_k) + \xi(s_k), \tag{5.2.8}$$

and

$$\|s_k - s_{k-1}\| \leq M\|t_1 - t_2\|\Big(\frac{MLr_{\bar{s}}^\alpha}{2^\alpha}\Big)^{k-2} \tag{5.2.9}$$

for every $k = 2, 3, 4, \ldots$ the inequality hold. We proceed by mathematical induction.

Denote

$$z_i := t_i - A(s, s') + A(\bar{s}, s') \quad \text{for each } \ i = 1, 2.$$

Note by (5.2.7) that

$$\begin{aligned} \|s - s'\| &\leq \|s - \bar{s}\| + \|\bar{s} - s'\| \\ &\leq \frac{r_{\bar{s}}}{2} + \frac{r_{\bar{s}}}{2} \leq r_{\bar{s}}. \end{aligned} \tag{5.2.10}$$

It follows, from (5.2.7) and the relation $\bar{r} \leq r_{\bar{t}} - \dfrac{L{r_{\bar{s}}}^{n+\alpha}\left(3^{n+\alpha} + 2^{n+\alpha}\right)}{\pi_{n,\alpha}\, 2^{n+\alpha}}$ by (5.2.4) that

$$\begin{aligned} \|z_i - \bar{t}\| &\leq \|t_i - \bar{t}\| + \|A(s, s') - A(\bar{s}, s')\| \\ &\leq \bar{r} + \|\zeta(s') - A(s, s')\| + \|\zeta(s') - A(\bar{s}, s')\| \\ &\leq \bar{r} + \frac{L}{\pi_{n,\alpha}}\Big(\|s - s'\|^{n+\alpha} + \|\bar{s} - s'\|^{n+\alpha}\Big) \\ &\leq \bar{r} + \frac{L}{\pi_{n,\alpha}}\Big({r_{\bar{s}}}^{n+\alpha} + \big(\frac{r_{\bar{s}}}{2}\big)^{n+\alpha}\Big) \\ &= \bar{r} + \frac{L{r_{\bar{s}}}^{n+\alpha}\big(2^{n+\alpha} + 1\big)}{\pi_{n,\alpha}} \\ &\leq r_{\bar{t}}. \end{aligned}$$

This implies that $z_i \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$ for each $i = 1, 2$. Letting $s_1 := s'$. Then $s_1 \in R_s^{-1}(t_1)$ by (5.2.7) and it follows from (5.2.3) that

$$t_1 \in A(s, s_1) + \xi(s_1),$$

we can be written the inequality as like as follows

$$t_1 - A(s, s_1) + A(\bar{s}, s_1) \in A(\bar{s}, s_1) + \xi(s_1).$$

According to the definition of z_1, we get that $z_1 \in A(\bar{s}, s_1) + \xi(s_1)$. Hence $s_1 \in R_{\bar{s}}^{-1}(z_1)$ by (5.2.3). This together with (5.2.7) implies that

$$s_1 \in R_{\bar{s}}^{-1}(z_1) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}).$$

According to the concept of Lipschitz-like property of $R_{\bar{s}}^{-1}(\cdot)$ and noting that $z_1,\, z_2 \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$, it follows from (5.2.6) that there exists $s_2 \in R_{\bar{s}}^{-1}(z_2)$ such that

$$\|s_2 - s_1\| \leq M\|z_1 - z_2\| = M\|t_1 - t_2\|.$$

Furthermore, from the definition of z_2 and noting $s_1 = s'$, we get that

$$s_2 \in R_{\bar{s}}^{-1}(z_2) = R_{\bar{s}}^{-1}(t_2 - A(s, s_1) + A(\bar{s}, s_1)),$$

which together with (5.2.3) implies that

$$t_2 \in A(s, s_1) - A(\bar{s}, s_1) + A(\bar{s}, s_2) + \xi(s_2).$$

This shows that (5.2.8) and (5.2.9) are true with created points s_1 and s_2.

Suppose that the points $s_1, s_2, ..., s_m$ have created so that (5.2.8) and (5.2.9) are true for $k = 2, 3, \ldots, m$. We need to create s_{m+1} such that (5.2.8) and (5.2.9) are also true for $k = m + 1$. To do this, setting

$$z_i^m := t_2 - A(s, s_{m+i-1}) + A(\bar{s}, s_{m+i-1}) \quad \text{for each } i = 0, 1.$$

Then, by the inductional assumption together with the concept of (n, α)-PBA of A, we get that

$$\begin{aligned} \|z_0^m - z_1^m\| &= \|[A(s, s_{m-1}) - A(\bar{s}, s_{m-1})] - [A(s, s_m) - A(\bar{s}, s_m)]\| \\ &\leq L\|s - \bar{s}\|^\alpha \|s_m - s_{m-1}\| \leq \frac{Lr_{\bar{s}}^\alpha}{2^\alpha}\|s_m - s_{m-1}\| \\ &\leq \|t_1 - t_2\|\Big(\frac{MLr_{\bar{s}}^\alpha}{2^\alpha}\Big)^{m-1}. \end{aligned} \tag{5.2.11}$$

We have $\|s_1 - \bar{s}\| \leq \frac{r_{\bar{s}}}{2}$ and $\|t_1 - t_2\| \leq 2\bar{r}$ from (5.2.7) and using (5.2.9) we get

$$\begin{aligned} \|s_m - \bar{s}\| &\leq \sum_{k=2}^{m} \|s_k - s_{k-1}\| + \|s_1 - \bar{s}\| \\ &\leq 2M\bar{r}\sum_{k=2}^{m}\Big(\frac{MLr_{\bar{s}}^\alpha}{2^\alpha}\Big)^{k-2} + \frac{r_{\bar{s}}}{2} \\ &\leq \frac{2.2^\alpha M\bar{r}}{2^\alpha - MLr_{\bar{s}}^\alpha} + \frac{r_{\bar{s}}}{2}. \end{aligned}$$

By (5.2.4), we have $4.2^\alpha M\bar{r} \leq r_{\bar{s}}(2^\alpha - MLr_{\bar{s}}^\alpha)$ and then (5.2.12) becomes

$$\|s_m - \bar{s}\| \leq r_{\bar{s}}. \tag{5.2.12}$$

Consequently,

$$\|s_m - s\| \leq \|s_m - \bar{s}\| + \|\bar{s} - s\| \leq \frac{3}{2}r_{\bar{s}}. \tag{5.2.13}$$

Furthermore, using (5.2.7), (5.2.12) and (5.2.13), we get that, for each $i = 0, 1$,

$$\begin{aligned}
\|z_i^m - \bar{t}\| &\leq \|t_2 - \bar{t}\| + \|A(s, s_{m+i-1}) - A(\bar{s}, s_{m+i-1})\| \\
&\leq \bar{r} + \|\zeta(s_{m+i-1}) - A(s, s_{m+i-1})\| + \|\zeta(s_{m+i-1}) - A(\bar{s}, s_{m+i-1})\| \\
&\leq \bar{r} + \frac{L}{\pi_{n,\alpha}}\Big(\|s - s_{m+i-1}\|^{n+\alpha} + \|\bar{s} - s_{m+i-1}\|^{n+\alpha}\Big) \\
&\leq \bar{r} + \frac{L}{\pi_{n,\alpha}}\Big(\Big(\frac{3}{2}r_{\bar{s}}\Big)^{n+\alpha} + r_{\bar{s}}{}^{n+\alpha}\Big) \\
&= \bar{r} + \frac{L\Big(3^{n+\alpha} + 2^{n+\alpha}\Big)r_{\bar{s}}{}^{n+\alpha}}{\pi_{n,\alpha}\, 2^{n+\alpha}} \\
&\leq r_{\bar{t}}.
\end{aligned}$$

It follows that $z_i^m \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$ for each $i = 0, 1$. Whereas the assumption (5.2.8) holds for $k = m$, we get

$$t_2 \in A(s, s_{m-1}) - A(\bar{s}, s_{m-1}) + A(\bar{s}, s_m) + \xi(s_m).$$

we can write the inequality as follows

$$t_2 - A(s, s_{m-1}) + A(\bar{s}, s_{m-1}) \in A(\bar{s}, s_m) + \xi(s_m);$$

Then by definition of z_0^m, it follows that $z_0^m \in A(\bar{s}, s_m) + \xi(s_m)$. This, together with (5.2.3) and (5.2.12), yields that

$$s_m \in R_{\bar{s}}^{-1}(z_0^m) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}),$$

Using (5.2.6) again, inasmuch as $z_0^m, z_1^m \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$, there exists an element $s_{m+1} \in R_{\bar{s}}^{-1}(z_1^m)$ such that

$$\|s_{m+1} - s_m\| \leq M\|z_0^m - z_1^m\| \leq M\|t_1 - t_2\|\Big(\frac{MLr_{\bar{s}}^{\alpha}}{2^{\alpha}}\Big)^{m-1}, \tag{5.2.14}$$

where the last inequality holds by (5.2.11). By the definition of z_1^m, we have

$$s_{m+1} \in R_{\bar{s}}^{-1}(z_1^m) = R_{\bar{s}}^{-1}(t_2 - A(s, s_m) + A(\bar{s}, s_m)),$$

which together with (5.2.3) implies

$$t_2 \in A(s, s_m) - A(\bar{s}, s_m) + A(\bar{s}, s_{m+1}) + \xi(s_{m+1}). \tag{5.2.15}$$

This together with (5.2.14) completes the induction step and the existence of sequence $\{s_k\}$ satisfying (5.2.8) and (5.2.9).

Whereas $\dfrac{MLr_{\bar{s}}^{\alpha}}{2^{\alpha}} < 1$, we conclude from (5.2.9) that $\{s_k\}$ is a Cauchy sequence. Define $s'' := \lim_{k\to\infty} s_k$. Note that ξ has closed graph. Then, taking limit in (5.2.8), we get $t_2 \in A(s, s'') + \xi(s'')$ and so $s'' \in R_s^{-1}(t_2)$. Moreover,

$$\begin{aligned}\|s' - s''\| &\le \limsup_{m\to\infty} \sum_{k=2}^{m} \|s_k - s_{k-1}\| \\ &\le \limsup_{m\to\infty} \sum_{k=2}^{m} \Big(\frac{MLr_{\bar{s}}^{\alpha}}{2^{\alpha}}\Big)^{k-2} M\|t_1 - t_2\| \\ &\le \frac{2^{\alpha}M}{2^{\alpha} - MLr_{\bar{s}}^{\alpha}}\|t_1 - t_2\|.\end{aligned}$$

The Lemma 5.2.1 is proved. □

Before going to prove the main theorem in this chapter, we define the map $G_s : \mathcal{S} \to \mathcal{T}$, for each $s \in \mathcal{S}$, by

$$G_s(\cdot) := A(\bar{s}, \cdot) - A(s, \cdot). \tag{5.2.16}$$

and the set-valued map $\Psi_s : \mathcal{S} \rightrightarrows 2^{\mathcal{S}}$ by

$$\Psi_s(\cdot) = R_{\bar{s}}^{-1}[G_s(\cdot)]. \tag{5.2.17}$$

Then we have that

$$\begin{aligned}\|G_s(s') - G_s(s'')\| &= \|[A(\bar{s}, s') - A(s, s')] - [A(\bar{s}, s'') - A(s, s'')]\| \\ &\le L\|\bar{s} - s\|^{\alpha}\|s' - s''\| \quad \text{for any } s',\, s'' \in \mathcal{S}.\end{aligned} \tag{5.2.18}$$

The main result of this chapter read as follows, which provides some sufficient conditions ensuring the convergence of the ENM for nonsmooth variational inclusions (1.0.3) from starting point s_0.

Theorem 5.2.1. *Suppose that* $\eta > 1$. *Let* $\bar{s} \in \mathcal{S}$, Υ *be an open and convex subset of* $\mathcal{S}$ *containing* $\bar{s}$ *and let* ζ *be a function which has* (n, α)*-PBA on* Υ *with a constant* $L > 0$. *Assume that the map* ξ *has closed graph and the map* $R_{\bar{s}}^{-1}(\cdot)$ *is Lipschitz-like on* $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ *relative to* $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$ *with constant* $M > 0$. *Let* $\bar{r}$ *be defined by* (5.2.4) *so that* (5.2.5) *holds. Let* $\delta > 0$ *be such that*

(a) $\delta \leq \min\left\{\frac{r_{\bar{s}}}{4}, \frac{\bar{r}.\pi_{n,\alpha}}{4^{n+\alpha}}, \left(\frac{r_{\bar{t}}\,\pi_{n,\alpha}}{L\,(3^{n+\alpha}+2^{n+\alpha}+1)}\right)^{\frac{1}{n+\alpha}}, 1\right\}$,

(b) $(M+1)L(2^{\alpha+1}\eta\delta^{\alpha} + r_{\bar{s}}^{\alpha}) \leq 2^{\alpha}$,

(c) $\|\bar{t}\| < \frac{L}{\pi_{n,\,\alpha}}\delta^{n+\alpha}$.

Suppose that

$$\lim_{s\to\bar{s}} \operatorname{dist}(\bar{t}, A(s,s)+\xi(s)) = 0. \tag{5.2.19}$$

Then ∃'s some $\hat{\delta} > 0$ such that any sequence $\{s_m\}$ constructed by Algorithm 4 with a starting point $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s})$ converges to a solution s^* of nonsmooth variational inclusions (1.0.3), that is, s^* satisfies $0 \in \zeta(s^*) + \xi(s^*)$.

Proof. By assumption (b), it can be easily written that

$$ML(2^{\alpha+1}\eta\delta^{\alpha} + r_{\bar{s}}^{\alpha}) \leq (M+1)L(2^{\alpha+1}\eta\delta^{\alpha} + r_{\bar{s}}^{\alpha}) \leq 2^{\alpha}. \tag{5.2.20}$$

Set

$$b := \frac{2^{\alpha}\eta ML\delta^{\alpha}}{2^{\alpha} - MLr_{\bar{s}}^{\alpha}}. \tag{5.2.21}$$

It follows from (5.2.20) that

$$b \leq \frac{1}{2} \tag{5.2.22}$$

Since $\pi_{n,\alpha}\|\bar{t}\| < L\delta^{n+\alpha}$ by assumption (c) and (5.2.19) holds, there exists $0 < \hat{\delta} \leq \delta$ be such that

$$\operatorname{dist}(0, A(s_0,s_0)+\xi(s_0)) \leq \frac{L}{\pi_{n,\alpha}}\delta^{n+\alpha} \quad \text{for each } s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s}) \tag{5.2.23}$$

Let $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s})$. We will proceed by mathematical induction. For (1.0.3) we will show that Algorithm 4 generates at least one sequence and any sequence $\{s_m\}$ generated by Algorithm 4 for (1.0.3) satisfies the following assertions:

$$\|s_m - \bar{s}\| \leq 2\delta \tag{5.2.24}$$

and

$$\|s_{m+1} - s_m\| \leq b\left(\frac{1}{\pi_{n,\alpha}}\right)^{(n+\alpha)^m}\delta, \tag{5.2.25}$$

for every $m = 0, 1, 2, ...$ For this motive we define

$$r_s := \frac{3}{2}\left(\frac{ML}{\pi_{n,\alpha}}\|s-\bar{s}\|^{n+\alpha} + M\|\bar{t}\|\right), \quad \text{for each } s \in \mathcal{S}. \tag{5.2.26}$$

Owing to the fact $4\delta \leq r_{\bar{s}}$ in assumption (a) and $\eta > 1$, by assumption (b) we can write as follows

$$\begin{aligned}
(M+1)L\,2^\alpha.3\,\delta^\alpha &\leq (M+1)L.2^\alpha\big(2\delta^\alpha + \delta^\alpha\big)\\
&= (M+1)L\Big(2^{\alpha+1}\delta^\alpha + (2\delta)^\alpha\Big)\\
&\leq (M+1)L\Big(2^{\alpha+1}\eta\delta^\alpha + (4\delta)^\alpha\Big)\\
&\leq (M+1)L\big(2^{\alpha+1}\eta\delta^\alpha + {r_{\bar{s}}}^\alpha\big)\\
&\leq 2^\alpha.
\end{aligned}$$

The above inequality gives either

$$ML\delta^\alpha \leq \frac{2^\alpha}{2^\alpha.3} = \frac{1}{3} \quad \text{or } L\delta^\alpha \leq \frac{2^\alpha}{2^\alpha.3} = \frac{1}{3} \tag{5.2.27}$$

By the facts $\pi_{n,\alpha}\|\bar{t}\| < L\delta^{n+\alpha}$ from condition (c)and (5.2.27), the inequality (5.2.26) reduces to, for each $s \in \mathbb{B}_{2\delta}(\bar{s})$

$$\begin{aligned}
r_s = {} & \frac{3}{2}\left(\frac{ML}{\pi_{n,\alpha}}\|s-\bar{s}\|^{n+\alpha} + M\|\bar{t}\|\right)\\
\leq {} & \frac{3}{2}\left(\frac{ML}{\pi_{n,\alpha}}\|s-\bar{s}\|^{n+\alpha} + \frac{ML}{\pi_{n,\alpha}}\delta^{n+\alpha}\right)\\
\leq {} & \frac{3}{2}\left(\frac{ML}{\pi_{n,\alpha}}(2\delta)^{n+\alpha} + \frac{ML}{\pi_{n,\alpha}}\delta^{n+\alpha}\right)\\
= {} & \frac{3}{2}\frac{ML}{\pi_{n,\alpha}}\delta^\alpha\left(2^{n+\alpha}+1\right).\delta^n
\end{aligned} \tag{5.2.28}$$

Since $\delta^n \leq \delta$, we get that,

$$\begin{aligned}
\leq {} & \frac{3}{2}\frac{ML}{\pi_{n,\alpha}}\delta^\alpha\left(2^{n+\alpha}+1\right).\delta\\
\leq {} & \frac{3}{2}.\frac{1}{3\pi_{n,\alpha}}\left(2^{n+\alpha}+1\right).\delta\\
\leq {} & \frac{1}{2\pi_{n,\alpha}}\left(2^{n+\alpha}+1\right).\delta\\
\leq {} & 2\delta, \quad \text{for each } s \in \mathbb{B}_{2\delta}(\bar{s}).
\end{aligned} \tag{5.2.29}$$

It is trivial that (5.2.24) is true for $m = 0$. To show, (5.2.25) holds for $m = 0$, firstly we need to verify that s_1 exists, that is, we need to show that $\mathcal{M}(s_0) \neq \emptyset$. To do this, we consider the mapping Ψ_{s_0} defined by (5.2.17) and apply Lemma 2.0.4 to the map Ψ_{s_0} with $\eta_0 = \bar{s}$. Let us check that both assumptions (2.0.4) and (2.0.5) of Lemma 2.0.4, with $r := r_{s_0}$ and $\lambda := \dfrac{1}{3}$ hold. Noting that $\bar{s} \in R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{r_{s_0}}(\bar{s})$ by (5.2) and by the definition of the excess e and the map Ψ_{s_0}, we obtain

$$\begin{aligned}
\operatorname{dist}(\bar{s}, \Psi_{s_0}(\bar{s})) &\leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(\bar{s})) \\
&\leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{2\delta}(\bar{s}), R_{\bar{s}}^{-1}[G_{s_0}(\bar{s})]) \\
&\leq e(R_{\bar{s}}^{-1}(\bar{t}) \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}[G_{s_0}(\bar{s})]) \\
&\leq M\|\bar{t} - G_{s_0}(\bar{s})\|. \qquad (5.2.30)
\end{aligned}$$

by the notion of (n, α)-PBA of ζ withconstant L, we get that

$$\begin{aligned}
\|G_{s_0}(s) - \bar{t}\| &= \|A(\bar{s}, s) - A(s_0, s) - \bar{t}\| \\
&\leq \|A(\bar{s}, s) - A(s_0, s)\| + \|\bar{t}\| \\
&\leq \|\zeta(s) - A(\bar{s}, s)\| + \|\zeta(s) - A(s_0, s)\| + \|\bar{t}\| \\
&\leq \frac{L}{\pi_{n,\alpha}}\|\bar{s} - s\|^{n+\alpha} + \frac{L}{\pi_{n,\alpha}}\|s_0 - s\|^{n+\alpha} + \|\bar{t}\| \\
&\leq \frac{L}{\pi_{n,\alpha}}\Big(\|\bar{s} - s\|^{n+\alpha} + \|s_0 - s\|^{n+\alpha}\Big) + \|\bar{t}\|. \qquad (5.2.31)
\end{aligned}$$

Note that $L\,\delta^{n+\alpha}(2^{n+\alpha} + 3^{n+\alpha} + 1) \leq \pi_{n,\alpha}\, r_{\bar{t}}$ because of assumption (a), $\pi_{n,\alpha}\|\bar{t}\| < L\delta^{n+\alpha}$ by assumption (c) and $\|s_0 - \bar{s}\| \leq \hat{\delta} \leq \delta$. It follows from (5.2.31), for each $s \in \mathbb{B}_{r_{s_0}}(\bar{s}) \subseteq \mathbb{B}_{2\delta}(\bar{s})$, that

$$\begin{aligned}
\|G_{s_0}(s) - \bar{t}\| &\leq \frac{L}{\pi_{n,\alpha}}\Big(\|\bar{s} - s\|^{n+\alpha} + \big(\|s_0 - \bar{s}\| + \|\bar{s} - s\|\big)^{n+\alpha}\Big) + \|\bar{t}\| \\
&\leq \frac{L}{\pi_{n,\alpha}}\Big((2\delta)^{n+\alpha} + (\delta + 2\delta)^{n+\alpha}\Big) + \|\bar{t}\| \\
&= \frac{L}{\pi_{n,\alpha}}\Big((2\delta)^{n+\alpha} + (3\delta)^{n+\alpha}\Big) + \|\bar{t}\| \\
&\leq \frac{L}{\pi_{n,\alpha}}\delta^{n+\alpha}(2^{n+\alpha} + 3^{n+\alpha}) + \frac{L}{\pi_{n,\alpha}}\delta^{n+\alpha} \\
&= \frac{L}{\pi_{n,\alpha}}\delta^{n+\alpha}(2^{n+\alpha} + 3^{n+\alpha} + 1) \leq r_{\bar{t}}. \qquad (5.2.32)
\end{aligned}$$

This implies that

$$G_{s_0}(s) \in \mathbb{B}_{r_{\bar{t}}}(\bar{t}), \quad \text{foreach } s \in \mathbb{B}_{r_{s_0}}(\bar{s}). \tag{5.2.33}$$

Especially, let $s = \bar{s}$ in (5.2.31). Then we get that

$$\|G_{s_0}(\bar{s}) - \bar{t}\| \leq \frac{L}{\pi_{n,\alpha}}\|s_0 - s\|^{n+\alpha} + \|\bar{t}\| \tag{5.2.34}$$

$$\leq \frac{L}{\pi_{n,\alpha}}\delta^{n+\alpha} + \frac{L}{\pi_{n,\alpha}}\delta^{n+\alpha} \leq \frac{2L}{\pi_{n,\alpha}}\delta^{n+\alpha} \leq r_{\bar{t}}. \tag{5.2.35}$$

and hence

$$G_{s_0}(\bar{s}) \in \mathbb{B}_{r_{\bar{t}}}(\bar{t}).$$

Hence, by the assumed Lipschitz-like property of $R_{\bar{s}}^{-1}$ and (5.2.34), we have from (5.2.30) that

$$\begin{aligned} \text{dist}(\bar{s}, \Psi_{s_0}(\bar{s})) &\leq M\|\bar{t} - G_{s_0}(\bar{s})\| \\ &\leq \frac{ML}{\pi_{n,\alpha}}\|s_0 - x\|^{n+\alpha} + M\|\bar{t}\| \\ &= \left(1 - \frac{1}{3}\right) r_{s_0} = (1-\lambda)r; \end{aligned}$$

that is, the assumption (2.0.4) of Lemma 2.0.4 is satisfied.

Below, we will show that the assumption (2.0.5) of Lemma 2.0.4 holds. To do this, let $s', s'' \in \mathbb{B}_{r_{s_0}}(\bar{s})$. Then from assumption (a) and (5.2.29), we have that $s', s'' \in \mathbb{B}_{r_{s_0}}(\bar{s}) \subseteq \mathbb{B}_{2\delta}(\bar{s}) \subseteq \mathbb{B}_{r_{\bar{s}}}(\bar{s})$ and $G_{s_0}(s')$, $G_{s_0}(s'') \in \mathbb{B}_{r_{\bar{t}}}(\bar{t})$ by (5.2.33). This, together with the assumed Lipschitz-like property of $R_{\bar{s}}^{-1}$, implies that

$$\begin{aligned} e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(s'')) &\leq e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), \Psi_{s_0}(s'')) \\ &= e(R_{\bar{s}}^{-1}[G_{s_0}(s')] \cap \mathbb{B}_{r_{\bar{s}}}(\bar{s}), R_{\bar{s}}^{-1}[G_{s_0}(s'')]) \\ &\leq M\|G_{s_0}(s') - G_{s_0}(s'')\|. \end{aligned} \tag{5.2.36}$$

Applying (5.2.18), we get that

$$\|G_{s_0}(s') - G_{s_0}(s'')\| \leq L\|\bar{s} - s_0\|^{\alpha}\|s' - s''\|.$$

With the help of first relation in (5.2.27) and combining the above two inequalities we get,

$$\begin{aligned} e(\Psi_{s_0}(s') \cap \mathbb{B}_{r_{s_0}}(\bar{s}), \Psi_{s_0}(s'')) &\leq ML\|\bar{s} - s_0\|^{\alpha}\|s' - s''\| \\ &\leq ML\delta^{\alpha}\|s' - s''\| \\ &\leq \frac{1}{3}\|s' - s''\| = \lambda\|s' - s''\|. \end{aligned}$$

Under this process we get that the assumption (2.0.5) of Lemma 2.0.4 is also satisfied. Inasmuch both assumptions (2.0.4) and (2.0.5) of Lemma 2.0.4 are satisfied, we can say that Lemma 2.0.4 is applicable and therefore, we conclude that ∃'s $\hat{s}_1 \in \mathbb{B}_{r_{s_0}}(\bar{s})$ such that $\hat{s}_1 \in \Psi_{s_0}(\hat{s}_1)$, that is, $0 \in A(s_0, \hat{s}_1) + \xi(\hat{s}_1)$ and so $\hat{s}_1 - s_0 \in \mathcal{M}(s_0)$. This fact reflects that $\mathcal{M}(s_0) \neq \emptyset$.

Whereas $\eta > 1$ and $\mathcal{M}(s_0) \neq \emptyset$, we can select $d_0 \in \mathcal{M}(s_0)$ such that

$$\|d_0\| \leq \eta \operatorname{dist}(0, \mathcal{M}(s_0)).$$

For Algorithm 4, $s_1 := s_0 + d_0$ is defined. Hence s_1 is generated for (1.0.3).

Moreover, according the definition of $\mathcal{M}(s_0)$, we can obtain

$$\begin{aligned} \mathcal{M}(s_0) &:= \Big\{ d_0 \in \Upsilon : 0 \in A(s_0, s_0 + d_0) + \xi(s_0 + d_0) \Big\} \\ &= \Big\{ d_0 \in \Upsilon : s_0 + d_0 \in R_{s_0}^{-1}(0) \Big\}, \end{aligned}$$

so

$$\operatorname{dist}(0, \mathcal{M}(s_0)) = \operatorname{dist}(s_0, R_{s_0}^{-1}(0)). \tag{5.2.37}$$

Now we are ready to show that for $m = 0$ the inequality (5.2.25) is hold. Note that $\bar{r} > 0$ by assumption (a). Then (5.2.5) is satisfied by (5.2.4). Lemma 5.2.1 states us that the mapping $R_s^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\dfrac{2^\alpha M}{2^\alpha - MLr_{\bar{s}}^\alpha}$ for each $s \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ when $R_{\bar{s}}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{r_{\bar{t}}}(\bar{t})$ relative to $\mathbb{B}_{r_{\bar{s}}}(\bar{s})$. Particularly, $R_{s_0}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\dfrac{2^\alpha M}{2^\alpha - MLr_{\bar{s}}^\alpha}$ as $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s}) \subseteq \mathbb{B}_{\delta}(\bar{s}) \subseteq \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ by assumption (a) and the choice of $\hat{\delta}$.

Moreover, assumptions (a), (c) and the 2nd relation of the inequality (5.2.27) imply that

$$\begin{aligned} \|\bar{t}\| &\leq \frac{L}{\pi_{n,\alpha}}\,\delta^{n+\alpha} = \frac{L}{\pi_{n,\alpha}}\,\delta^{\alpha}.\delta^{n} \\ &\leq \frac{L}{\pi_{n,\alpha}}\,\delta^{\alpha}.\delta \leq \frac{1}{3\,\pi_{n,\alpha}}\,.\delta \\ &\leq \frac{1}{3\,\pi_{n,\alpha}}\,.\frac{\bar{r}\pi_{n,\alpha}}{4^{n+\alpha}} \leq \frac{\bar{r}}{3} \end{aligned} \tag{5.2.38}$$

Now (5.2.23) becomes

$$\operatorname{dist}(0, R_{s_0}(s_0)) = \operatorname{dist}(0, A(s_0, s_0) + \xi(s_0)) \leq \frac{L}{\pi_{n,\alpha}}\,\delta^{n+\alpha} \leq \frac{\bar{r}}{3}. \tag{5.2.39}$$

Noting that $s_0 \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ as mentioned earlier and by (5.2.38)) we have that $0 \in \mathbb{B}_{\frac{\bar{r}}{3}}(\bar{t})$.

Applying Lemma 2.0.1 then we have

$$\text{dist}(s_0, {R_{s_0}}^{-1}(0)) \leq \frac{2^\alpha M}{2^\alpha - MLr_{\bar{s}}^\alpha}\ \text{dist}(0, R_{s_0}(s_0)) \tag{5.2.40}$$

According to Algorithm 4 and using (5.2.37) and (5.2.40) we have

$$\begin{aligned}\|s_1 - s_0\| = \|d_0\| &\leq \eta\ \text{dist}(0, \mathcal{M}(s_0)) = \eta\ \text{dist}(s_0, R_{s_0}^{-1}(0)) \\ &\leq \frac{2^\alpha \eta M}{2^\alpha - MLr_{\bar{s}}^\alpha}\ \text{dist}(0, R_{s_0}(s_0)) \\ &\leq \frac{2^\alpha \eta ML}{\pi_{n,\alpha}(2^\alpha - MLr_{\bar{s}}^\alpha)}\ \delta^{n+\alpha} \\ &\leq \frac{2^\alpha \eta ML}{\pi_{n,\alpha}(2^\alpha - MLr_{\bar{s}}^\alpha)}\ \delta^n.\delta^\alpha \\ &\leq \frac{2^\alpha \eta M\, L\delta^\alpha}{\pi_{n,\alpha}(2^\alpha - MLr_{\bar{s}}^\alpha)}\ \delta,\ \ [\text{Since}\, \delta^n \leq \delta].\end{aligned} \tag{5.2.41}$$

From (5.2.22) and (5.2.41) we get,

$$\begin{aligned}\|s_1 - s_0\| = \|d_0\| &\leq \frac{b}{\pi_{n,\alpha}}\,\delta \\ &\leq b\Big(\frac{1}{\pi_{n,\alpha}}\Big)\delta.\end{aligned}$$

This shows that (5.2.25) is hold for $m = 0$.

Let the points $s_1, s_2, \ldots, s_k$ have obtained by Algorithm 4 satisfying (5.1.2) such that (5.2.24) and (5.2.25) are hold for $m = 0, 1, 2, \ldots, k-1$. We show that assertions (5.2.24) and (5.2.25) are also hold for $m = k$. Because (5.2.24) and (5.2.25) are true for every $m \leq k-1$, we get from the following inequality

$$\|s_k - \bar{s}\| \ \leq\ \sum_{i=0}^{k-1} \|d_i\| + \|s_0 - \bar{s}\| \leq b\delta \sum_{i=0}^{k-1} \Big(\frac{1}{\pi_{n,\alpha}}\Big)^{(n+\alpha)^i} + \delta \leq 2\delta, \tag{5.2.42}$$

and so $s_k \in \mathbb{B}_{2\delta}(\bar{s})$. This shows that (5.2.24) holds for $m = k$.

The next step is that, we show that for $m = k$ the assertion (5.2.25) is also hold . Let $s_k \in \mathbb{B}_{r_{s_k}}(\bar{s})$. If we apply Lemma 2.0.4 to the map Ψ_{s_k} with $\eta = \bar{s}$, $r := r_{s_k}$ and $\lambda := \dfrac{1}{3}$, then by the correlated argument for the case $k = 0$ one can find that $\mathcal{M}(s_k) \neq \emptyset$. Because of $s_k \in \mathbb{B}_{r_{s_k}}(\bar{s}) \subseteq \mathbb{B}_{2\delta}(\bar{s}) \subseteq \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$, Lemma 5.2.1 permit us to say that $R_{s_k}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{\bar{r}}(\bar{t})$ relative to $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ with constant $\dfrac{2^\alpha M}{2^\alpha - MLr_{\bar{s}}^\alpha}$.

Moreover, inasmuch as $-A(s_{k-1}, s_k) \in \xi(s_k)$, using the idea of (n, α)-PBA of ζ, the inequality $4^{n+\alpha}\delta \leq \bar{r}\pi_{n,\alpha}$ from assumption (a), we obtain that

$$\begin{aligned}
\operatorname{dist}\big(0, R_{s_k}(s_k)\big) &= \operatorname{dist}\big(0, A(s_k, s_k) + \xi(s_k)\big) \\
&\leq \|A(s_k, s_k) - A(s_{k-1}, s_k)\| \\
&= \frac{L}{\pi_{n,\alpha}}\|\zeta(s_k) - A(s_{k-1}, s_k)\|^{n+\alpha} \leq \frac{L}{\pi_{n,\alpha}}\|s_k - s_{k-1}\|^{n+\alpha} \\
&\leq \frac{L}{\pi_{n,\alpha}}\big(\|s_k - \bar{s}\| + \|\bar{s} - s_{k-1}\|\big)^{n+\alpha} \leq \frac{L}{\pi_{n,\alpha}}\big(2\delta + 2\delta\big)^{n+\alpha} \\
&= \frac{L}{\pi_{n,\alpha}}4^{n+\alpha}\delta^{n+\alpha} \leq \frac{L}{\pi_{n,\alpha}}\,\delta^{\alpha}4^{n+\alpha}\delta \\
&= \frac{1}{3\,\pi_{n,\alpha}}\,4^{n+\alpha}\,\frac{\bar{r}.\pi_{n,\alpha}}{4^{n+\alpha}} \leq \frac{\bar{r}}{3}.
\end{aligned} \tag{5.2.43}$$

It is noted earlier that $s_k \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Moreover, (5.2.38) implies that $0 \in \mathbb{B}_{\frac{\bar{r}}{3}}(\bar{t})$. This, together with (5.2.43), implies that Lemma 2.0.1 is applicable for the map $R_{s_k}^{-1}(\cdot)$ and hence we have that

$$\operatorname{dist}\big(s_k, R_{s_k}^{-1}(0)\big) \;\leq\; \frac{2^{\alpha}M}{2^{\alpha} - MLr_{\bar{s}}^{\alpha}}\operatorname{dist}\big(0, R_{s_k}(s_k)\big). \tag{5.2.44}$$

Because of $\mathcal{M}(s_k) \neq \emptyset$, Algorithm 4 ensures the existence of a point s_{k+1} which satisfies the inequality as follows

$$\begin{aligned}
\|s_{k+1} - s_k\| = \|d_k\| &\leq \eta\ \operatorname{dist}\big(0, \mathcal{M}(s_k)\big) = \eta\ \operatorname{dist}\big(s_k, R_{s_k}^{-1}(0)\big) \\
&\leq \frac{2^{\alpha}\eta M}{2^{\alpha} - MLr_{\bar{s}}^{\alpha}}\operatorname{dist}\big(0, R_{s_k}(s_k)\big) \\
&= \frac{2^{\alpha}\eta M}{2^{\alpha} - MLr_{\bar{s}}^{\alpha}}\operatorname{dist}\big(0, A(s_k, s_k) + \xi(s_k)\big) \\
&\leq \frac{2^{\alpha}\eta M}{2^{\alpha} - MLr_{\bar{s}}}\|A(s_k, s_k) - A(s_{k-1}, s_k)\| \\
&= \frac{2^{\alpha}\eta M}{2^{\alpha} - MLr_{\bar{s}}^{\alpha}}\|\zeta(s_k) - A(s_{k-1}, s_k)\| \\
&\leq \frac{2^{\alpha}\eta LM}{\pi_{n,\alpha}(2^{\alpha} - MLr_{\bar{s}}^{\alpha})}\|s_k - s_{k-1}\|^{n+\alpha} \\
&\leq \frac{b}{\delta^{\alpha}\pi_{n,\alpha}}\Big(b(\frac{1}{\pi_{n,\alpha}})^{(n+\alpha)^{k-1}}\,\delta\Big)^{n+\alpha} \\
&\leq \frac{b}{\delta^{\alpha}\pi_{n,\alpha}}\Big(b(\frac{1}{\pi_{n,\alpha}})^{(n+\alpha)^{k-1}}\Big)^{n+\alpha}\delta^{n+\alpha}
\end{aligned} \tag{5.2.45}$$

$$
\begin{aligned}
&\leq \frac{b}{\pi_{n,\alpha}}\Big(b(\frac{1}{\pi_{n,\alpha}})^{(n+\alpha)^{k-1}}\Big)^{n+\alpha}\delta^n \\
&\leq \frac{b}{\pi_{n,\alpha}}\Big(b(\frac{1}{\pi_{n,\alpha}})^{(n+\alpha)^{k-1}}\Big)^{n+\alpha}\delta \ [\text{Since}\, \delta^n \leq \delta] \\
&\leq b(\frac{1}{\pi_{n,\alpha}})^{(n+\alpha)^k}\delta.
\end{aligned}
$$

This shows that (5.2.25) holds for $m = k$. By this process, we can get from (5.2.25) that $\{s_m\}$ is a Cauchy sequence and hence convergent to some s^*. where as the graph of ξ is closed, we can pass to the limit in $s_{k+1} \in R_{s_k}^{-1}(0)$ obtaining that s^* is a solution of (1.0.3). So, the proof is completed. □

Especially, when $\bar{s}$ is a solution of (1.0.3), that is, $\bar{t} = 0$, Theorem 5.2.1 is reduced to the following corollary, which gives the local convergent result of the ENM for solving nonsmooth generalized equation (1.0.3).

Corollary 5.2.1. *Suppose that $\eta > 1$ and $\bar{s}$ be a solution of the variational inclusion* (1.0.3). *Let Υ be an open and convex subset of $\mathcal{S}$ containing $\bar{s}$ and $\tilde{r} > 0$ be such that $\mathbb{B}_{\tilde{r}}(\bar{s})$ is an open and convex set. Assume that the function ζ is continuous which has a (n, α)-PBA on $\mathbb{B}_{\tilde{r}}(\bar{s})$ with a constant $L > 0$, the map ξ has closed graph. Assume that the map $R_{\bar{s}}^{-1}(\cdot)$ is Lipschitz-like around $(0, \bar{s})$ with constant M. Suppose that*

$$\lim_{s \to \bar{s}} \text{dist}(0, A(s, s) + \xi(s)) = 0. \tag{5.2.46}$$

Then there exists some $\hat{\delta} > 0$ such that any sequence $\{s_m\}$ generated by Algorithm 4 starting from $s_0 \in \mathbb{B}_{\hat{\delta}}(\bar{s})$ converges to a solution s^* of nonsmooth generalized equation (1.0.3), that is, s^* satisfies that $0 \in \zeta(s^*) + \xi(s^*)$.

Proof. By hypothesis $R_{\bar{s}}^{-1}(\cdot)$ is pseudo-Lipschitz around $(0, \bar{s})$. Then there exists constants r_0, $\hat{r}_{\bar{s}}$ and M such that $R_{\bar{s}}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{r_0}(\bar{t})$ relative to $\mathbb{B}_{\hat{r}_{\bar{s}}}(\bar{s})$ with constant M. Then, for each $0 < r \leq \hat{r}_{\bar{s}}$, one has that

$$e(R_{\bar{s}}^{-1}(t_1) \cap \mathbb{B}_r(\bar{s}), R_{\bar{s}}^{-1}(t_2)) \leq M\|t_1 - t_2\| \quad \text{for any } t_1, t_2 \in \mathbb{B}_{r_0}(0), \tag{5.2.47}$$

that is, the map $R_{\bar{s}}^{-1}(\cdot)$ is Lipschitz-like on $\mathbb{B}_{r_0}(0)$ relative to $\mathbb{B}_r(\bar{s})$ with constant M.

Let $L \in (0, 1)$ and choose $r_{\bar{s}} \in (0, \hat{r}_{\bar{s}})$ such that

$$\frac{r_{\bar{s}}}{2} \leq \tilde{r},\ 2^{n+\alpha}\pi_{n,\alpha}\, r_0 - L(3^{n+\alpha} + 2^{n+\alpha}){r_{\bar{s}}}^{n+\alpha} > 0$$

and A is a (n, α)-PBA of ζ on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Then, define

$$\bar{r} = \min\Big\{r_0 - \frac{L\, {r_{\bar{s}}}^{n+\alpha}\,(3^{n+\alpha} + 2^{n+\alpha})}{\pi_{n,\alpha}\, 2^{n+\alpha}}, \frac{r_{\bar{s}}(2^{\alpha} - MLr_{\bar{s}}^{\alpha})}{4.2^{\alpha} M}\Big\} > 0. \tag{5.2.48}$$

and

$$\min\left\{\frac{r_{\bar{s}}}{4}, \frac{\bar{r}.\pi_{n,\alpha}}{4^{n+\alpha}}, \Big(\frac{r_0\,\pi_{n,\alpha}}{L\,(3^{n+\alpha} + 2^{n+\alpha} + 1)}\Big)^{\frac{1}{n+\alpha}}\right\}$$

Thus we can select $0 < \delta \leq 1$ such that

$$\delta \leq \min\left\{\frac{r_{\bar{s}}}{4}, \frac{\bar{r}.\pi_{n,\alpha}}{4^{n+\alpha}}, \Big(\frac{r_0\,\pi_{n,\alpha}}{L\,(3^{n+\alpha} + 2^{n+\alpha} + 1)}\Big)^{\frac{1}{n+\alpha}}\right\}. \tag{5.2.49}$$

and

$$(M+1)L(2^{\alpha+1}\eta\delta^{\alpha} + {r_{\bar{s}}}^{\alpha}) \leq 2^{\alpha}.$$

Now it is our routine work to check all the conditions of Theorem 5.2.1 are hold. Thus, Theorem 5.2.1 is applicable to complete the proof of the corollary 5.2.1. □

5.3 Application of (n, α)-point-based approximation (P-BA)

This section is dedecated to present applications of (n, α)-PBA. In particular, when the Fréchet derivative of ζ is (ℓ, α)-Hölder, the function A is a $(1, \alpha)$- PBA for ζ . Moreover, when ζ is twice Fréchet differentiable function such that $\nabla^2\zeta$ is (K, α)-Hölder, then the function A is $(2, \alpha)$-PBA for ζ. In addition, application of (n, α)-PBA is provided for normal maps.

5.3.1 Application of (n, α)-PBA for differentiable function

Let $0 < \alpha < 1$ and Υ be a convex subset of $\mathcal{S}$. Let $p, q \in \Upsilon$.

(1) Suppose that the Fréchet derivative of ζ is (ℓ, α)-Hölder continuous. We show that the function

$$A : (p, q) \longmapsto \zeta(p) + \nabla\zeta(p)(q - p)$$

is a $(1, \alpha)$-PBA for ζ. In this case, by using the Algorithm 4 we can infer that there exists a sequence $\{s_k\}$ which converges superlinearly and this result recovers the convergence result of Geoffroy and Piétrus in [42].

In this regards, define the function $\Lambda(p, q)$ by

$$\Lambda(p, q) = \|\zeta(q) - A(p, q)\|.$$

It follows that

$$\begin{aligned}
\Lambda(p, q) = \quad & \|\zeta(q) - \zeta(p) - \nabla\zeta(p)(q - p)\| \\
= \quad & \| \int_0^1 \Big(\nabla\zeta(p + f(q - p)) - \nabla\zeta(p)\Big)(q - p)df\| \\
\leq \quad & \|q - p\| \int_0^1 \|\nabla\zeta(p + f(q - p)) - \nabla\zeta(p)\|df \\
\leq \quad & \|q - p\| \int_0^1 \ell\|f(q - p)\|^{\alpha} df \\
\leq \quad & \|q - p\|^{1+\alpha}\ell \int_0^1 f^{\alpha} df \\
\leq \quad & \frac{\ell}{(\alpha + 1)}\|q - p\|^{1+\alpha}.
\end{aligned}$$

This yields that A satisfies the first property of $(1, \alpha)$-PBA on Υ. To proof the second property of $(1, \alpha)$-PBA, we assume that $t, z \in \Upsilon$. Then, we have that

$$\begin{aligned}
\Lambda'(p, q, t, z) = \quad & \|A(p, t) - A(q, t) - A(p, z) + A(q, z)\|, \\
= \quad & \|\zeta(p) + \nabla\zeta(p)(t - p) - \zeta(q) - \nabla\zeta(q)(t - q) - \zeta(p) - \nabla\zeta(p)(z - p) \\
& +\zeta(q) + \nabla\zeta(q)(z - q)\| \\
\leq \quad & \|(\nabla\zeta(p) - \nabla\zeta(q))(t - z)\| \leq \|\nabla\zeta(p) - \nabla\zeta(q)\|\|t - z\| \\
\leq \quad & \ell\|p - q\|^{\alpha}\|t - z\|
\end{aligned}$$

This shows that the second property of $(1, \alpha)$-PBA for ζ also holds. Therefore, we say that when the Fréchet derivative of ζ is (ℓ, α)-Hölder with exponent $\alpha \in (0, 1)$, the function $A : (p, q) \longmapsto \zeta(p) + \nabla\zeta(p)(q - p)$ is a $(1, \alpha)$-PBA.

(2) Let $r_{\bar{s}} > 0$ be such that $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}) \subseteq \mathcal{S}$. Suppose that ζ is twice Fréchet differentiable function on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ such that $\nabla^2\zeta$ is (K, α)-Hölder on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ and with exponent $\alpha \in$

$(0, 1)$. Choose $\ell > 0$ and $L > 0$ be such that

$$L > \ell + K(r_{\bar{s}} + 1).$$

Let $p, q \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ and define the function

$$A(p, q) = \zeta(p) + \nabla\zeta(p)(q - p) + \frac{1}{2}\nabla^2\zeta(p)(q - p)^2. \tag{5.3.1}$$

Then, Theorem 5.2.1 ensures the existence of a sequence $\{s_k\}$ which converges super-quadratically and the result of Theorem 5.2.1 coincides with the result of [41, 105].

To show the first property of $(2, \alpha)$-PBA, denote $\triangle(p, q) = \|\zeta(q) - A(p, q)\|$. Then we have that

$$\triangle(p, q) = \|\zeta(q) - \zeta(p) - \nabla\zeta(p)(q - p) - \frac{1}{2}\nabla^2\zeta(p)(q - p)^2\|. \tag{5.3.2}$$

Since, $\| \int_0^1 \Big((1-f)\nabla^2\zeta(p+f(q-p))(q-p)^2\Big)df\| = \|\zeta(q) - \zeta(p) - \nabla\zeta(p)(q-p)\|$, then (5.3.2) reduces to

$$\begin{aligned}
\triangle(p, q) = \quad & \| \int_0^1 \Big((1-f)\nabla^2\zeta(p + f(q-p))(q-p)^2\Big)df - \frac{1}{2}\nabla^2\zeta(p)(q-p)^2\| \\
= \quad & \| \int_0^1 \Big((1-f)\nabla^2\zeta(p + f(q-p)) - (1-f)\nabla^2\zeta(p)\Big)(q-p)^2 df\| \\
\leq \quad & \|q - p\|^2 \int_0^1 \|(1-f)\nabla^2\zeta(p + f(q-p)) - (1-f)\nabla^2\zeta(p)\|df \\
\leq \quad & \|q - p\|^2 \int_0^1 \|(1-f)\nabla^2\zeta(p + f(q-p)) - \nabla^2\zeta(p)\Big)df \\
\leq \quad & K\|q - p\|^2 \int_0^1 (1-f)\|f(q-p)\|^\alpha df \\
\leq \quad & K\|q - p\|^{2+\alpha} \int_0^1 (1-f)f^\alpha df \\
\leq \quad & \frac{K}{(\alpha+1)(\alpha+2)}\|q - p\|^{2+\alpha} \\
\leq \quad & \frac{L}{(\alpha+1)(\alpha+2)}\|q - p\|^{2+\alpha}.
\end{aligned}$$

Therefore, A satisfies the first property of a $(2, \alpha)$-PBA on Υ.

For the proof of second property, we assume that $a,\ b$ be any elements of $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$, Then, we get that

$$\begin{aligned}
\triangle'(p,q,a,b) = \quad & \|A(p,a)-A(q,a)-A(p,b)+A(q,b)\|, \\
= \quad & \|\zeta(p)+\nabla\zeta(p)(a-p)+\frac{1}{2}\nabla^2\zeta(p)(a-p)^2-\zeta(q)-\nabla\zeta(q)(a-q) \\
& -\frac{1}{2}\nabla^2\zeta(q)(a-q)^2-\zeta(p)-\nabla\zeta(p)(b-p)-\frac{1}{2}\nabla^2\zeta(p)(b-p)^2 \\
& +\zeta(q)+\nabla\zeta(q)(b-q)+\frac{1}{2}\nabla^2\zeta(q)(b-q)^2\| \\
= \quad & \|[\nabla\zeta(p)-\nabla\zeta(q)](a-b)+\frac{1}{2}[\nabla^2\zeta(p)(a-p)^2-\nabla^2\zeta(q)(a-q)^2 \\
& -\nabla^2\zeta(p)(b-p)^2+\nabla^2\zeta(q)(b-q)^2]\| \\
= \quad & \|[\nabla\zeta(p)-\nabla\zeta(q)](a-b)+\frac{1}{2}[\nabla^2\zeta(p)(a-q+q-p,a-p) \\
& -\nabla^2\zeta(p)(b-q+q-p,b-p)+\nabla^2\zeta(q)(b-q,b-p+p-q) \\
& -\nabla^2\zeta(q)(a-q,a-p+p-q)]\| \\
= \quad & \|[\nabla\zeta(p)-\nabla\zeta(q)](a-b)+\frac{1}{2}[\nabla^2\zeta(p)(a-q,a-p) \\
& +\nabla^2\zeta(p)(q-p,a-p)-\nabla^2\zeta(p)(b-q,b-p) \\
& -\nabla^2\zeta(p)(q-p,b-p)+\nabla^2\zeta(q)(b-q,b-p)+\nabla^2\zeta(q)(b-q,p-q) \\
& -\nabla^2\zeta(q)(a-q,a-p)-\nabla^2\zeta(q)(a-q,p-q)]\| \\
= \quad & \|[\nabla\zeta(p)-\nabla\zeta(q)](a-b)\|+\frac{1}{2}[\nabla^2\zeta(q)(b-q,b-p) \\
& -\nabla^2\zeta(p)(b-q,b-p)+\nabla^2\zeta(p)(a-q,a-p) \\
& -\nabla^2\zeta(q)(a-q,a-p)+\nabla^2\zeta(p)(q-p,a-p)-\nabla^2\zeta(p)(q-p,b-p) \\
& +\nabla^2\zeta(q)(b-q,p-q)-\nabla^2\zeta(q)(a-q,p-q)]\| \\
= \quad & \|[\nabla\zeta(p)-\nabla\zeta(q)](a-b)+\frac{1}{2}[\nabla^2\zeta(q)-\nabla^2\zeta(p)](b-q,b-p) \\
& +\frac{1}{2}[\nabla^2\zeta(p)-\nabla^2\zeta(q)](a-q,a-p)+\frac{1}{2}\nabla^2\zeta(p)(q-p,a-b) \\
& +\frac{1}{2}\nabla^2\zeta(q)(b-a,p-q)]\| \\
= \quad & \|[\nabla\zeta(p)-\nabla\zeta(q)](a-b)+\frac{1}{2}[\nabla^2\zeta(q)-\nabla^2\zeta(p)](b-q,b-p) \\
& +\frac{1}{2}[\nabla^2\zeta(p)-\nabla^2\zeta(q)](a-b+b-q,a-p)+\frac{1}{2}\nabla^2\zeta(p)(q-p,a-b) \\
& +\frac{1}{2}\nabla^2\zeta(q)(b-a,p-q)]\|
\end{aligned}$$

This also can be written as

$$\begin{aligned}\triangle'(p,q,a,b) = \quad & \|[\nabla\zeta(p)-\nabla\zeta(q)](a-b)+\frac{1}{2}[\nabla^2\zeta(q)-\nabla^2\zeta(p)](b-q,b-a)\\ &+\frac{1}{2}[\nabla^2\zeta(p)-\nabla^2\zeta(q)](a-b,a-p)+\frac{1}{2}\nabla^2\zeta(p)(q-p,a-b)\\ &+\frac{1}{2}\nabla^2\zeta(q)(b-a,p-q)]\|\end{aligned}$$

Since there exist an open subset $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}) \subseteq \mathcal{S}$ and a positive number K such that $\|\nabla^2\zeta\| \leq K$ on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Let $a,b \in \mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$. Then, $\|a-b\| \leq r_{\bar{s}}$. Then, by applying the notion of (ℓ,α)-Hölder continuity property of $\nabla\zeta$ and (K,α)-Hölder continuity property of $\nabla^2\zeta$, we get

$$\begin{aligned}\triangle'(p,q,a,b) \leq \quad & \|[\nabla\zeta(p)-\nabla\zeta(q)](a-b)\|+\frac{1}{2}\|\nabla^2\zeta(q)-\nabla^2\zeta(p)\|\|b-q\|\|b-a\|\\ &+\frac{1}{2}\|\nabla^2\zeta(p)-\nabla^2\zeta(q)\|\|a-b\|\|a-p\|+\frac{1}{2}\|\nabla^2\zeta(p)\|\|q-p\|\|a-b\|\\ &+\frac{1}{2}\|\nabla^2\zeta(q)\|\|b-a\|\|p-q\|\\ \leq \quad & \ell\|p-q\|^\alpha\|a-b\|+\frac{K}{2}\|p-q\|^\alpha\|b-q\|\|b-a\|\\ &+\frac{K}{2}\|p-q\|^\alpha\|b-a\|\|a-p\|+\frac{K}{2}\|q-p\|^\alpha\|a-b\|\\ &+\frac{K}{2}\|p-q\|^\alpha\|b-a\|\\ \leq \quad & \ell\|p-q\|^\alpha\|a-b\|+\frac{K}{2}r_{\bar{s}}\|p-q\|^\alpha\|a-b\|+K\|p-q\|^\alpha\|a-b\|\\ \leq \quad & \Big(\ell+K(r_{\bar{s}}+1)\Big)\|p-q\|^\alpha\|a-b\|\\ \leq \quad & L\|p-q\|^\alpha\|a-b\|, \quad \text{for all } a,b\in\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s}).\end{aligned}$$

This shows that the second property of $(2,\alpha)$-PBA is satisfied. Thus, both of properties for (n,α)-PBA hold on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$ when $n=2$ and $0<\alpha<1$. Hence, A is $(2,\alpha)$-PBA for ζ on $\mathbb{B}_{\frac{r_{\bar{s}}}{2}}(\bar{s})$.

5.3.2 Application of (n,α)-PBA for Normal Maps

In this subsection we deal with a class of nonsmooth functions, i.e. normal maps. Huge number of mathematician have studied by normal maps to obtain solutions of variational inequalities and comprehensive accounts on this topic can be found in [38, 57–60, 118, 122].

At first Robinson [122] discussed about the normal maps . Here we recall the notion of normal maps which was introduced by Robinson [118, 122].

Definition 5.3.1. *Suppose $\mathcal{C}$ be a nonempty closed convex subset of a Banach space $\mathcal{S}$ and let Π be the metric projector from $\mathcal{S}$ onto $\mathcal{C}$. Assume that Υ be an open subset of $\mathcal{S}$ meeting $\mathcal{C}$ and let ζ be a function from Υ to $\mathcal{S}$. $\zeta_{\mathcal{C}}$ is the normal map which is defined from the set $\Pi^{-1}(\Upsilon)$ to $\mathcal{S}$ by*

$$\zeta_{\mathcal{C}}(s) = \zeta(\Pi(s)) + (s - \Pi(s)). \tag{5.3.1}$$

Furthermore, variational problem is as follows

$$\text{find} \quad t_0 \in \mathcal{C} : \langle \zeta(t_0), c - t_0 \rangle \geq 0, \quad \text{for all } c \in \mathcal{C}$$

is completely equivalent to the normal-map equation $\zeta_{\mathcal{C}}(s_0) = 0$ through the transformation $s_0 = t_0 - \zeta(t_0)$. For nonlinear optimization involving normal maps, Robinson has shown that how the first-order necessary optimality conditions as well as linear and nonlinear complementarity problems and more general variational inequalities, can all be expressed compactly and conveniently in the form of equations $\zeta_{\mathcal{C}}(s) = 0$.

Nevertheless, sometimes the use of normal maps enables one to gain insight into special properties of problem classes that might have remained obscure in the formalism of variational inequalities. A particular illustration of this is the characterization of the local and global homeomorphism properties of linear normal maps, this concept given in [122] and improved in [95, 96].

In [103, Proposition 4.1], Rashid proved that for any function ζ admitting a PBA on a nonempty closed convex subset $\mathcal{C}$ of a Hilbert space H, the normal map associated with ζ admits a PBA on H. In our study we will show that the same result holds when we replace the normal maps $\zeta_{\mathcal{C}} + \xi$ in lieu of the normal maps $\zeta_{\mathcal{C}}$. Rashid [99, 103] reformulate the normal maps $\zeta_{\mathcal{C}} + \xi$ by simple conversion of the definition of normal maps given by Robinson [122]. In [99, 103] Rashid assumed the concept of point-based approximation and p-PBA. Here we extend that concept to (n, α)-PBA which is reformulated by Rashid [99, 103], then we show that if ζ have a (n, α)-PBA, then one can easily be designed a (n, α)-PBA for $\zeta_{\mathcal{C}} + \xi$.

The normal maps $\zeta_{\mathcal{C}} + \xi$ reformulated by Rashid [99] is as follows.

Definition 5.3.2. *Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space $\mathcal{S}$ and let Π be the metric projector from $\mathcal{S}$ onto $\mathcal{C}$. Let Υ be an open subset of $\mathcal{S}$ meeting $\mathcal{C}$ and let $\zeta : \Upsilon \to \mathcal{S}$ and $\xi : \Upsilon \rightrightarrows \mathcal{S}$. The normal map $\zeta_{\mathcal{C}} + \xi$ is defined from the set $\Pi^{-1}(\Upsilon)$ to $\mathcal{S}$ by*

$$(\zeta_{\mathcal{C}} + \xi)(s) = \zeta(\Pi(s)) + \xi(\Pi(s)) + (s - \Pi(s)). \tag{5.3.2}$$

We are now able to design a (n, α)-PBA for the normal map $\zeta_{\mathcal{C}} + \xi$ provided that a (n, α)-PBA exists for ζ. The following proposition are taken from [99, Proposition 4.3].

Proposition 5.3.1. *Suppose $\mathcal{S}$ be a Banach space and $\mathcal{C}$ be a nonempty closed convex subset of $\mathcal{S}$ and let Π be the metric projector on $\mathcal{C}$ which is nonexpansive. Assume that $A : \mathcal{C} \times \mathcal{C} \to \mathcal{S}$, $\zeta : \mathcal{C} \to \mathcal{S}$ be functions and let $\xi : \mathcal{C} \rightrightarrows \mathcal{S}$ be a set-valued map which has closed graph. If A is a (n, α)-PBA for ζ on $\mathcal{C}$ with a constant L, then the function $H : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ defined by $H(t, s) = \big(A(\Pi(t), \cdot)_{\mathcal{C}} + \xi(\cdot)\big)(s)$ is a (n, α)-PBA for $\zeta_{\mathcal{C}} + \xi$ on $\mathcal{S}$ with the same constant L.*

Proof. Let $t, s \in \mathcal{S}$. By the definition of normal map, $(\zeta_{\mathcal{C}} + \xi)(s)$ and $H(t, s)$ are respectively defined as follows

$$(\zeta_{\mathcal{C}} + \xi)(s) = \zeta(\Pi(s)) + \xi(\Pi(s)) + (s - \Pi(s)),$$

and

$$H(t, s) = A\big(\Pi(t), \Pi(s)\big) + \xi(\Pi(s)) + (s - \Pi(s)).$$

Hypothetically we know that A has the two properties for ζ which is given in Definition 2.0.15 with a constant L. Now we need to show that H also has these same two properties for $\zeta_{\mathcal{C}} + \xi$ with the constant L. Whereas A is the (n, α)-PBA for ζ on $\mathcal{C}$, then using the notion of the non-expansiveness of the metric projector and the first property of (n, α)-PBA we get that

$$\begin{aligned}
&\|(\zeta_{\mathcal{C}} + \xi)(s) - H(t, s)\| \\
= \;& \|\zeta(\Pi(s)) + \xi(\Pi(s)) + (s - \Pi(s)) - [A\big(\Pi(y), \Pi(s)\big) + \xi(\Pi(s)) + (s - \Pi(s))]\| \\
= \;& \|\zeta(\Pi(s)) - A\big(\Pi(y), \Pi(s)\big)\| \leq \frac{L}{\pi_{n,\alpha}} \|\Pi(y) - \Pi(s)\|^{n+\alpha} \\
\leq \;& \frac{L}{\pi_{n,\alpha}} \|t - s\|^{n+\alpha}.
\end{aligned}$$

We notice that H satisfies the first property of (n, α)-PBA. After that for proving the second property, we suppose that $s, s' \in \mathcal{S}$. To this end, let $t, z \in \mathcal{S}$. We will prove that $H(s, \cdot) -$

$H(s',\cdot)$ is Lipschitz continuous on $\mathcal{S}$ with lipschitz constant $L\|s-s'\|^{\alpha}$. Again using the concept of non expansiveness of metric projector and second property of (n,α)-PBA , we obtain that

$$\begin{aligned}
& \|[H(s,t)-H(s',t)]-[H(s,z)-H(s',z)] \\
= \ & \|[A\big(\Pi(s),\Pi(t)\big)+\xi(\Pi(t))+(y-\Pi(t))-A\big(\Pi(s'),\Pi(t)\big)-\xi(\Pi(t)) \\
& -(y-\Pi(t))]-[A\big(\Pi(s),\Pi(z)\big)+\xi(\Pi(z))+(z-\Pi(z))-A\big(\Pi(s'),\Pi(z)\big) \\
& -\xi(\Pi(z))-(z-\Pi(z))]\| \\
= \ & \|[A\big(\Pi(s),\Pi(t)\big)-A\big(\Pi(s'),\Pi(t)\big)]-[A\big(\Pi(s),\Pi(z)\big)-A\big(\Pi(s'),\Pi(z)\big)\| \\
\leq \ & L\|\Pi(s)-\Pi(s')\|^{\alpha}\|\Pi(t)-\Pi(z)\| \leq L\|s-s'\|^{\alpha}\|t-z\|.
\end{aligned}$$

This process shows that the second property of the (n,α)-PBA is satisfied. So the both properties in Definition 2.0.15 are fulfilled for H, In this conclusion now we can say that H is a (n,α)-PBA for $\zeta_{\mathcal{C}}+\xi$ on $\mathcal{S}$. The proof is completed. □

5.4 Numerical Experiment

In this section, to present the numerical experiment we recall some necessary notations and notions . Let a Fréchet differentiable function at $s\in\mathbb{R}^n$ be $\psi:\mathbb{R}^n\to\mathbb{R}^m$. Let the set of all points $s\in\mathbb{R}^n$ is denoted by P_ψ at which the derivative $\psi'(s)$ exists. The B-subdifferential of ψ at $s\in\mathbb{R}^n$, denoted by $\partial_B\psi(s)$, is the set

$$\partial_B\psi(s)=\left\{\ J\in\mathbb{R}^{m\times n}: J=\lim_{k\to+\infty}\psi'(s_k)\quad \text{for some } \{s_k\}\subset P_\psi \quad \text{such that}\{s_k\}\to s\right\}$$

Then, Clarke's generalized Jacobian of ψ at $s\in\mathbb{R}^n$ is the set $\partial\psi(s)$=conv $\partial_B\psi(s)$. If ψ is differentiable near s, and ψ' is continuous at s, then obviously $\partial\psi(s)=\partial_B\psi(s)=\{\psi'(s)\}$. Otherwise, $\partial_B\psi(s)$ is not necessarily a singleton, even if ψ is differentiable at s. In this case, $\psi'(s)\in\partial_B\psi(s)$ holds. Now, in order to illustrate the theoretical result of the extended Newton-type method, we consider the following example in one dimension.

Example 5.4.1. *Let* $S = T = \mathbb{R}, s_0 = -1.7, \eta = 5, L = 0.5, r_{\bar{s}} = 3, n = 1, \alpha = 0.9$ *and* $M = 1$. *Let* $\zeta : \mathbb{R} \to \mathbb{R}$ *and* $\xi : \mathbb{R} \rightrightarrows \mathbb{R}$ *be defined, respectively, by*

$$\zeta(s) = \begin{cases} \dfrac{s}{7} + s^2, & \text{if } s < 0, \\ \\ \dfrac{10s^2}{7} - 2s, & \text{if } s \geq 0 \end{cases} \quad \text{and } \xi(s) = \{\frac{s}{14} - \frac{1}{7}, s + \frac{1}{7}\}. \tag{5.4.3}$$

Then Algorithm 4 generates a sequence which converges superlinearly to $s^* = -0.5000$ *and* $s^* = -1.0000$ *, respectively, with initial points* $s_0 = -1.7$ *and* $s_0 = -1.5$ *in the case* $s < 0$. *On the other hand, Algorithm 4 generates a superlinear convergent sequence which converges to* $s^* = 01.4204$ *and* $s^* = 0.5000$ *, respectively, with initial points* $s_0 = 1.5$ *and* $s_0 = 1.7$ *in the case* $s \geq 0$.

Solution: It is manifest that ζ is not differentiable at $s = 0$ and hence ζ is nonsmooth function on $\mathbb{R}$. But this function is differentiable on $\mathbb{R} - \{0\}$ and hence $\partial_B\zeta(s) = \{\zeta'(s)\}$. So, we get

$$\partial_B\zeta(s) = \{\zeta'(s)\} = \begin{cases} \dfrac{1}{7} + 2s, & \text{if } s < 0, \\ \\ \dfrac{20s}{7} - 2, & \text{if } s \geq 0 \end{cases}$$

We mark that

$$\Gamma(s) := (\zeta + \xi)(s) \begin{cases} \{s^2 + \dfrac{3s}{14} - \dfrac{1}{7}, \quad s^2 + \dfrac{8s}{7} + \dfrac{1}{7}\}, & \text{if } s < 0, \\ \\ \{\dfrac{10s^2}{7} - \dfrac{27s}{14} - \dfrac{1}{7}, \dfrac{10s^2}{7} - s + \dfrac{1}{7}\}, & \text{if } s \geq 0 \end{cases}$$

Initially, we study the set-valued mapping $\Gamma(s) = s^2 + \dfrac{3s}{14} - \dfrac{1}{7}$ for the case $x < 0$ and note that Γ has a closed graph at $(\bar{s}, \bar{t})$ with $\bar{s} = -1$ and $\bar{t} = 0.64$. Thus, $(-1, 0.64) \in$ gph Γ and if $(\zeta + \xi)^{-1}$ is Lipschitz-like then Γ is Lipschitz-like at $(0.64, -1)$. By taking $A(s, \cdot) = \zeta(s) + \partial_B\zeta(s)(\cdot - s)$, it is easily shown that $R_{\bar{s}}(\cdot) = \zeta(\bar{s}) + \partial_B\zeta(\bar{s})(\cdot - \bar{s}) + \xi(\cdot)$ is Lipschitz-like at $(\bar{t}, \bar{s})$ for $\bar{t} = 0.64$ and $\bar{s} = -1$. Therefore, the assumptions of Theorem 5.2.1 hold. From the definition of $\mathcal{M}(s_k)$, we get

$$\begin{aligned} \mathcal{M}(s_k) &= \{d_k \in \mathbb{R} : 0 \in \zeta(s_k) + \partial_B\zeta(s_k)d_k + \xi(s_k + d_k)\} \\ &= \{d_k \in \mathbb{R} : d_k = \frac{2 - 3s_k - 14s_k^2}{3 + 28s_k}\} \end{aligned}$$

Alternatively, if $M(s_k) \neq \emptyset$ we take

$$0 \in \zeta(s_k) + \partial_B \zeta(s_k)(s_{k+1} - s_k) + \xi(s_{k+1})$$
$$\Rightarrow s_{k+1} = \frac{2 + 14s_k^2}{3 + 28s_k}$$

Also, from (5.2.45) with $0 \leq \alpha \leq 1$ we consume

$$\|d_k\| \leq \frac{2^\alpha \eta L M}{\pi_{n,\alpha}(2^\alpha - MLr_{\bar{s}}^{\alpha})} \|d_{k-1}\|^{n+\alpha}$$

Hereafter, for the given values of $L, M, \eta, r_{\bar{s}}, n$ and α, w get that Algorithm 4 generates a superlinearly convergent sequence with initial point $s_0 = -1.7$ in a neighborhood of $\bar{s} = -1.9$. Then the following Tables 5.1 and 5.2, obtained by using Matlab code, indicates that the solution of the variational inclusion $\Gamma(s) \in 0$ has the solutions $s^* = -1.0000$ and $s^* = -0.5000$ in the case $s < 0$ and $s^* = 0.5000$ and $s^* = 1.4202$ in the case $s \geq 0$. The graphs of Γ are plotted in Figure 1.

Remark 5.4.1. *If we set $\alpha = 1$ in Example 5.4.1, we get the quadratic convergence of Algorithm 4.*

Table 5.1 Numerical results for Example 5.4.1 for the case $s < 0$

iteration no.	s_k	$\Gamma = s^2 + \frac{3s}{14} - \frac{1}{7}$	s_k	$\Gamma = s^2 + \frac{8s}{7} + \frac{1}{7}$
1	-1.7000	2.3829	-1.5000	0.6786
2	-0.9520	0.5595	-1.1346	0.1335
3	-0.6209	0.1096	-1.0161	0.0140
4	-0.5142	0.0114	-1.0003	0.0002
5	-0.5002	0.0002	-1.0000	0.0000
6	-0.5000	0.0000	-1.0000	0.0000
7	-0.5000	0.0000	-1.0000	0.0000

Table 5.2 Numerical results for Example 5.4.1 for the case $s \geq 0$

iteration no.	s_k	$\Gamma = \frac{10s^2}{7} - \frac{27s}{14} - \frac{1}{7}$	s_k	$\Gamma = \frac{10s^2}{7} - s + \frac{1}{7}$
1	1.5000	0.1786	1.7000	2.5714
2	1.4242	0.0082	1.0333	0.6349
3	1.4204	0.0000	0.7081	0.1511
4	1.4204	0.0000	0.5605	0.0311
5	1.4204	0.0000	0.5087	0.0038
6	1.4204	-0.0000	0.5002	0.0001
7	1.4204	0.0000	0.5000	0.0000
8	1.4204	0.0000	0.5000	0.0000

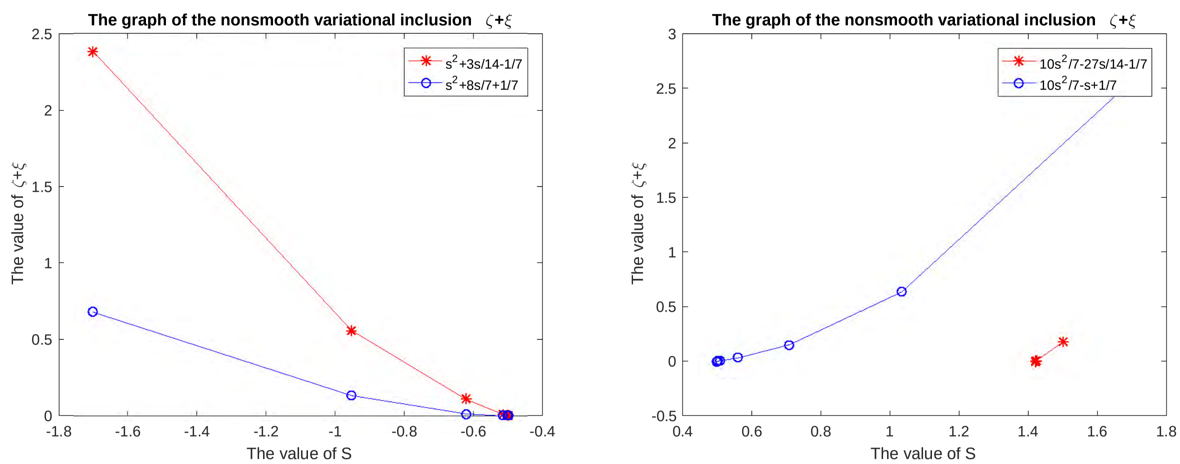

Figure 5.1: (Color online) Superlinear rate of convergence of Algorithm 4 at -1.0000 (-0.5000) and 0.5000 (1.4204)

5.5 Concluding Remarks

We have established semilocal and local convergence of the ENM for solving the nonsmooth variational inclusion (1.0.3) under the conditions $\eta > 1$, $(\zeta + \xi)^{-1}$ is Lipschitz-like and the nonsmooth function ζ has a (n, α)-PBA. Moreover, when $0 < \alpha < 1$ and $\nabla\zeta$ is (ℓ, α)-Hölder, we have presented an application of (n, α)-PBA for smooth function with $n = 1$, that is, we have shown A is a $(1, \alpha)$-PBA. In this case Theorem 5.2.1 provides the superlinear convergent result and this result extends the convergence theorem of Geoffroy and Piétrus [42]. On the other hand, for $n = 2$ and $0 < \alpha < 1$, if ζ is twice Fréchet differentiable function and $\nabla^2\zeta$ is (K, α)-Hölder, we have given an application of (n, α)-PBA, that is, we have shown A is a $(2, \alpha)$-PBA. In this case Theorem 5.2.1 yields the superquadratic convergent result and we have given a numerical experiment to illustrates the theoretical result. Therefore, this result extends the convergence result of [41, 105]. Finally, we have given another application of normal maps for $\zeta_{\mathcal{C}} + \xi$ which extends the concept of PBA reformulated by Rashid [103]. That is, we have shown that if ζ has a (n, α)-PBAs, it is easy to construct a (n, α)-PBA for the $\zeta_{\mathcal{C}} + \xi$.

Chapter 6

Conclusions

In this dissertation, we deal with two types of variational inclusions. We introduce and study several types of iterative procedure for solving these variational inclusions. Newton-type method (3.1.3) are applied for approximating the solution of the variational inclusion problem (1.0.1) and we have established local convergence results of the Newton-type method under the assumptions that $R_{s^*}^{-1}(\cdot)$ is pseudo-Lipschitz and $\nabla\zeta$ is continuous, Lipschitz continuous and Hölder continuous respectively and g is admissible for FODD. More clearly, we have shown that the Newton-type method defined by the method (3.1.3) converges linearly, quadratically and superlinearly to the solution of (1.0.1) if $\nabla\zeta$ is continuous, Lipschitz continuous and Hölder continuous respectively, together with a divided difference admissible function g. This study improves and extends the results corresponding to [43]; see more details in [63].

For solving the variational inclusion (1.0.1) we introduce an iterative method "so-called" EN-type method defined by Algorithm 2. The semilocal and local convergence results for the EN-type method are established under the conditions that $\eta > 1$, $\nabla\zeta$ is continuous and Lipschitz continuous, g admits first order divided difference as well as $R_{\bar{s}}^{-1}$ is Lipschitz-like. This work extends and improves the result corresponding to [13, 105]; see more details in [62]. On the other hand, for solving the variational inclusion (1.0.1) we introduce another iterative method defined by Algorithm 3 under the assumptions that $\nabla\zeta$ is (L, q)-Hölder continuous and g admits the first-order divided difference satisfying q-Hölderian condition. We present the semilocal and local convergence analysis of the method. To validate our theoretical result we have given numerical experiments and this results extends and improves the corresponding ones in [62, 103]. To know in detail, the reader could refer to our paper [64].

Moreover, to approximate the solution of the nonsmooth variational inclusions (1.0.3) we introduce the iterative procedure "so-called" extended Newton-type method (ENM) defined by the Algorithm 4 in Chapter 5. In this literature we have established semilocal and local convergence of the extended Newton-type method method for solving the nonsmooth variational inclusion (1.0.3) under the conditions $\eta > 1$, $(\zeta + \xi)^{-1}$ is Lipschitz-like and the nonsmooth function ζ has a (n, α)-PBA. Moreover, when $0 < \alpha < 1$ and $\nabla\zeta$ is (ℓ, α)-Hölder, we have presented an application of (n, α)-PBA for smooth function with $n = 1$, that is, we have shown A is a $(1, \alpha)$-PBA. In this case Theorem 5.2.1 provides the superlinear convergent result and this result extends the convergence theorem of Geoffroy and Piétrus [42]. On the other hand, for $n = 2$ and $0 < \alpha < 1$, if ζ is twice Fréchet differentiable function and $\nabla^2\zeta$ is (K, α)-Hölder, we have given an application of (n, α)-PBA, that is, we have shown A ia s $(2, \alpha)$-PBA. In this case Theorem 5.2.1 yields the superquadratic convergent result and this result extends and improves the convergence result of [41, 105]. Finally, we have given another application of (n, α)-PBA for normal maps $\zeta_{\mathcal{C}} + \xi$, which extends the concept of PBA reformulated by Rashid [103]. That is, we have shown that if ζ has a (n, α)-PBAs, it is easy to construct a (n, α)-PBA for the $\zeta_{\mathcal{C}} + \xi$. At the end we have given a numerical experiment to illustrates our theoretical result.

Our future research is to study EN-type method for solving variational inclusion using set-valued approximations. More clearly, if the single-valued function involved in (5.1.1) is an another set-valued mapping, introducing and studying an EN-type method, for solving such type of variational inclusion problems, is an important task for our future research.

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