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On Generalizations of Separation Properties, Compactness and Connectedness

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ON GENERALIZATIONS OF SEPARATION PROPERTIES, COMPACTNESS AND CONNECTEDNESS

Ph.D. THESIS

This Thesis is Submitted to the Department of Mathematics, University of Rajshahi for the Degree of Doctor of Philosophy in Mathematics.

Submitted
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DEDICATED

TO

MY BELOVED PARENTS

CONTENTS

DECLARATION

I do hereby declare that the whole work submitted as a thesis entitled "**ON GENERALIZATIONS OF SEPARATION PROPERTIES, COMPACTNESS AND CONNECTEDNESS**" to the Department of Mathematics, University of Rajshahi, Bangladesh for the Degree of Doctor of Philosophy (Ph.D.) in Mathematics is an original research work of mine and have not been previously submitted elsewhere for the award of any other degree. From this thesis three (3) papers have published in a reputed peer reviewed international journal and four (4) papers are prepared to be submitted for publication.

Sanjoy Kumar Biswas

CERTIFICATE

This to certify that the thesis entitled **"ON GENERALIZATIONS OF SEPARATION PROPERTIES, COMPACTNESS AND CONNECTEDNESS"** has been prepared by Sanjoy Kumar Biswas under our supervision for submission to the Department of Mathematics, University of Rajshahi, Bangladesh for the Degree of Doctor of Philosophy (Ph.D.) in Mathematics. It is also certified that the materials include in this thesis are original works of the researcher and have not been previously submitted for the award of any other degree. From this thesis three (3) papers have published in a reputed peer reviewed international journal and four (4) papers are prepared to be submitted for publication.

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ABSTRACT

The thesis is concerned with generalizations of some important and interesting properties of separation, compactness and connectedness in a span of seven chapters.

In the first chapter pseudo regular and pseudo normal topological spaces have been defined. Their properties have been studied and a number of important theorems regarding these spaces have been established.

In the second chapter strongly pseudo-regular and strongly pseudo-normal topological spaces have been introduced and their properties have been studied. A number of important theorems have been proved in this regard.

This is the third chapter. In this chapter strictly pseudo-regular and strictly pseudo-normal topological spaces have been defined and their properties have been studied. In the former class a compact set can be separated from an external point by a continuous function, while in the latter, two disjoint compact sets can be separated by a continuous function. Many important properties have been proved.

The fourth chapter introduces the notions of nearly regular topological spaces of the first kind and the second kind and studies their properties. A number of important theorems regarding these spaces have been established.

In this fifth chapter two new generalizations of normal spaces have been defined and studied. The spaces in these classes have been termed nearly normal topological spaces of the first kind and the second kind respectively.

This is the sixth chapter. In this chapter further new generalizations of normal spaces have been made. These have been called slightly normal spaces of the first kind, the second kind and the third kind respectively. A number of important properties of these spaces have been proved.

In this chapter seven compactness has been generalized to pseudo-compactness and c-compactness, and a continuum i.e., a connected compact space has been generalized to pseudo-continuum. Several properties of these three classes of spaces have been studied.

CHAPTER ONE

Pseudo Regular and Pseudo Normal Topological Spaces

1.1. Introduction

Regular and normal topological spaces have been generalized in various ways. p-regular, p-normal, β-normal and γ-normal spaces ([5], [8], [9], [15], [36]) are several examples of some of these.

In this chapter we have introduced pseudo regular and pseudo normal spaces and studied their important properties. Many results have been proved about these spaces. We have also established characterizations of such spaces. Parallel study of further generalizations using preopen and semi open sets etc. are intended to be done in near future. We have used the terminology and definitions of text book of S. Majumdar and N. Akhter [22], Munkres [10], Dugundji [11], Simmons [7], Kelley [12] and Hocking-Young [13].

Unless otherwise stated, every compact set considered in this chapter will have at least two elements.

1.2. Preliminaries

We start with the definitions of almost γ-normal, almost p-normal, almost β-normal, γ-normal, p-normal, β-normal spaces.

A subset A of a topological space X is said to be **regular open** (resp. **regular closed**) if A=int(cl(A)) (resp. cl(int(A)),**preopen** (briefly **p-open**) if A$\subseteq$int(cl(A)), β**-open** if A$\subseteq$cl(int(cl(A))), γ**-open** if A$\subseteq$cl(int(A))$\cup$ int(cl(A))[5].

A topological spaces X is said to be **almost** γ**-normal** (resp. **almost p-normal**, **almost** β**-normal** [5]) if for any two disjoint closed subsets A and B of X, one of which is regularly closed, there exist disjoint γ-open (resp. p-open, β-open) sets U and V of X such that A$\subseteq$U and B$\subseteq$V.

A topological spaces X is said to be γ**-normal** (resp. **p-normal**, β**-normal** [5]) if for every pair of disjoint closed subsets A and B of X, there exist disjoint γ-open (resp. p-open, β-open) sets U and V of X such that A$\subseteq$ U and B$\subseteq$V.

Their inter-relationships are also mentioned below:

normal$\Rightarrow$almost normal
$\Downarrow$ $\Downarrow$
p-normality$\Rightarrow$almost p-normality
$\Downarrow$ $\Downarrow$
γ-normality$\Rightarrow$ almost γ-normality
$\Downarrow$ $\Downarrow$
β-normality$\Rightarrow$ almost β-normality

We now define pseudo regular spaces and proceed to study them.

1.3. Pseudo Regular Spaces

Definition 1.3.1: A topological space X will be called **pseudo regular** if every compact subset K of X and every $x \in X$ with $x \notin K$ can be separated by disjoint open sets.

Example 1.3.1: Let K be a compact subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$ such that $x \notin K$. Since $\mathbb{R}^n$ is T_1, $\{x\}$ is closed and since $\mathbb{R}^n$ is normal and K is closed (by Heine-Borel Theorem), $\{x\}$ and K can be separated by disjoint open sets. Thus $\mathbb{R}^n$ is **pseudo regular**.

Example 1.3.2: Let X = {a, b, c, d} and $\mathfrak{I} = \{X, \phi, \{a,b\}, \{c,d\}\}$. Then (X, $\mathfrak{I}$) is **regular** but not **pseudo regular.**

For, {a, c} is compact, b $\notin \{a,c\}$, but {a, c} and b cannot be separated by disjoint open sets.

Example 1.3.3: Let X =$\mathbb{R}$, $\mathfrak{I}$ be the topology generated by $\mathfrak{I}_0 \cup \zeta$ where $\mathfrak{I}_0$ is the usual topology on $\mathbb{R}$ and $\zeta = \{\{x\} \mid x \in \mathbb{R} - \mathbb{Q}\}$. Then $\mathbb{Q}$ is closed, since $\mathbb{R} - \mathbb{Q}$ is open. $\mathbb{Q}$ cannot be separated from an irrational point since the only open set which contains $\mathbb{Q}$ is $\mathbb{R}$. Therefore X is not regular. The compact sets in X are the closed and bounded subsets of $\mathbb{R}$, i.e., finite unions of closed intervals, e.g., $[a_1,b_1] \cup \cdots \cup [a_n, b_n]$. Let K be a compact subset of X and let $x \notin K$. So let K be given by K=$[a_1,b_1] \cup \cdots \cup [a_n, b_n]$. Let $\delta_i = d(x,[a_i,b_i])$, the distance of x from $[a_i,b_i]$ and let $\delta = \frac{1}{2}\min\{\delta_1, \ldots, \delta_n\}$. Then G=$(a_1 - \delta, b_1 + \delta) \cup \ldots \cup (a_n - \delta, b_n + \delta)$ and H=$(x-\delta, x+\delta)$, separate K and x. Thus X is **pseudo regular** but **not regular**.

Theorem 1.3.1: Every pseudo regular compact space is regular.

Proof: Let X be compact and pseudo regular. Let K be a closed subset of X and let $x \in X$ with $x \notin K$ .Since X is compact, K is compact. Again, since X is pseudo regular, there exist disjoint open sets G and H such that $x \in G$ and $K \subseteq H$. Therefore X is regular.

Theorem 1.3.2: Every regular Hausdorff space is pseudo regular.

Proof: Let X be a regular Hausdorff space. Let K be a compact subset of X and $x \in X$, $x \notin K$. Since X is Hausdorff, K is closed. Now, since X is regular, there exist disjoint open sets G and H such that $x \in G$ and $K \subseteq H$. Therefore X is pseudo regular.

Theorem 1.3.3: A topological space X is pseudo regular if and only if for every $x \in X$ and any compact set K not containing x, there exists an open set H of X such that $x \in H \subseteq \overline{H} \subseteq K^c$.

Proof: Let X be pseudo regular and let K be compact in X. Let $x \notin K$ i.e., $x \in K^c$. Since X is pseudo regular, there exist open sets U, V such that $x \in U, K \subseteq V$ and $U \cap V = \phi$. Then $U \subseteq V^c \subseteq K^c$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq K^c$. Writing U=H we have $x \in H \subseteq \overline{H} \subseteq K^c$.

Now, let for every $x \in X$ and any compact set K not containing x, there exists open set H such that $x \in H \subseteq \overline{H} \subseteq K^c$. Since K is a compact set and $x \notin K$. Then $x \in K^c$. According to the condition, there exists open set H such that $x \in H \subseteq \overline{H} \subseteq K^c$. Let $\overline{H}^c$ = G. Then G is open, $K \subseteq G$ and $G \cap H = \phi$. Thus X is pseudo regular.

Theorem 1.3.4: The product space X of any non-empty collection $\{X_i\}$ of topological spaces is pseudo regular if and only if each X_i is pseudo regular.

Proof: Let $\{X_i\}$ be a non-empty collection of pseudo regular spaces and X = ΠX_i. We show that X is a pseudo regular space. Let K be a compact set not containing a point $x \in X$. Let $K_i = \Pi_i(K)$, $x_i \notin K_i$. Since the projection maps are continuous $\Pi_i(K) = K_i$ is a compact subset of X_i. Since $x \notin K$, there exists i_0 such that $x_{i_0} \notin K_{i_0}$. Since X_{i_0} is pseudo regular, there exist disjoint open sets G_{i_0}, H_{i_0} in X_i such that $x_{i_0} \in H_{i_0}$, $K_{i_0} \subseteq G_{i_0}$. For each $i \neq i_0$, let G_i, H_i be open sets such that $x_i \in H_i, K_i \subseteq G_i$. Let G=$\Pi_i G_i$ and H=$\Pi_i H_i$. Then $G \cap H = \phi$, since $G_{i_0} \cap H_{i_0} = \phi$. Now, $K \subseteq G$, $x \in H$. Hence X is pseudo regular.

Conversely, if X is pseudo regular, then we show that for each i, X_i is pseudo regular. For each i, let K_i be a compact subset of X_i and $x_i \in X_i$ but $x_i \notin K_i$. Let $K = \Pi_i K_i$ and x=$\{x_i\}$, $x \in X$ but $x \notin K$. Then K is compact by Tychonoff Theorem. Since X is pseudo regular, there exist disjoint open sets G and H such that $x \in G$ and $K \subseteq H$ and G=$\Pi_i G_i$, H=$\Pi_i H_i$, G_i, H_i are open sets in X_i such that $x_i \in H_i, K_i \subseteq G_i$ and $G_i \cap H_i = \phi$. Therefore X_i is pseudo regular.

Theorem 1.3.5: Any subspace of a pseudo regular space is pseudo regular.
Proof: Let X be a pseudo regular space and $Y \subseteq X$. Let $y \in Y$ and B is a compact subset of Y such that $y \notin B$. Since B is compact in Y, so B is compact in X. Since X is pseudo regular, there exist disjoint open sets G and H of X such that $y \in G$ and $B \subseteq H$. Let U=$G \cap Y$ and V=$H \cap Y$. Then U and V are disjoint open sets of Y where $y \in U$ and $B \subseteq V$. Hence Y is pseudo regular.

Corollary 1.3.1: Let X be a topological space and A, B are two pseudo regular subspaces of X. Then $A \cap B$ is pseudo regular.
Proof: $A \cap B$ being a subspace of both A and B, $A \cap B$ is pseudo regular by the above Theorem 1.3.5.

Theorem 1.3.6: Let X be a pseudo regular space and R is an equivalence relation of X. Then R is a closed subset of X×X.

Proof: We shall prove that R^c is open. So, let $(x,y)\in R^c$. It is sufficient to show that there exist two open sets G and H of X such that $x\in G$ and $y\in H$ and G×H$\subseteq R^c$. Let p:X→$\frac{X}{R}$ be the projection map. Since $(x,y)\in R^c$, $p(x)\neq p(y)$ i.e., $x\notin p^{-1}(p(y))$. Again, since $\{y\}$ is compact and p is a continuous mapping, p(y) is compact. Also, let $\{G_i\}$ be an open cover of $p^{-1}(p(y))$ in X, and let $\overline{G_i}=p(G_i)$. Then $\{\overline{G_i}\}$ is an open cover of p(y) in $\frac{X}{R}$. Since p(y) is a singleton element in $\frac{X}{R}$, there exists $\overline{G_{i_0}}$ such that $p(y)\in\overline{G_{i_0}}$ in $\frac{X}{R}$. Then by the definition of the topology in $\frac{X}{R}$ and the nature of the map p, (i) G_{i_0} is open in X, (ii) $G_{i_0}=p^{-1}(\overline{G_{i_0}})$ and (iii) $p^{-1}(p(y))\subseteq G_{i_0}$ in X. Hence $p^{-1}(p(y))$ is compact in X. So by the pseudo regularity of X, there exist disjoint open sets G and H in X such that $x\in G$ and $p^{-1}(p(y))\subseteq H$. Hence $y\in p^{-1}(p(y))\subseteq H$ i.e., $y\in H$. Since $G\cap H=\phi$, $p(G)\cap p(H)=\phi$. Therefore G×H$\subseteq R^c$ and so $(x,y)\in G\times H\subseteq R^c$.

Corollary 1.3.2: Let X be a pseudo regular space and R is an equivalence relation of X. Then $\frac{X}{R}$ is Hausdorff.

Proof: Let $\bar{x}$ and $\bar{y}$ be two distinct points of $\frac{X}{R}$. Then $\bar{x}=p(x)$ and $\bar{y}=p(y)$ for some $x,y\in X$ such that $x\neq y$ and $(x,y)\in R^c$. By the proof of the above Theorem 1.3.6, there exist disjoint open sets $\overline{G}$ and $\overline{H}$ in $\frac{X}{R}$ such that $\bar{x}\in\overline{G}$ and $\bar{y}\in\overline{H}$. Thus $\frac{X}{R}$ is Hausdorff. [$\overline{G}$=p(G) and $\overline{H}$=p(H) of the above theorem].

We now define pseudo normal spaces and proceed to study them.

1.4. Pseudo Normal Spaces

Definition 1.4.1: A topological space X will be called **pseudo normal** if each pair of disjoint compact subsets of X can be separated by disjoint open sets.

Example 1.4.1: Since $\mathbb{R}^n$ is normal and every compact subset of $\mathbb{R}^n$ is closed, $\mathbb{R}^n$ is pseudo normal.

Example 1.4.2: Let X = {a, b, c, d} and $\mathfrak{I} = \{X, \phi, \{a,b\}, \{c,d\}\}$. Then (X, $\mathfrak{I}$) is a **normal space**. Here {a, c}and {b, d}are two disjoint compact sets in X, but there do not exist disjoint open sets containing these compact sets. Therefore (X, $\mathfrak{I}$) is **not pseudo normal**.

Theorem 1.4.1: Every pseudo normal compact space is normal.
Proof: Let X be compact and pseudo normal. Let A, B be two disjoint closed subsets of X. Since X is compact, A and B are compact. Again, since X is pseudo normal, A and B can be separated by disjoint open sets. Therefore X is normal.

Theorem 1.4.2: Every normal Hausdorff space is pseudo normal.
Proof: Let X be Hausdorff and normal. Let A, B be two disjoint compact subsets of X. Since X is Hausdorff, A, B are closed. Again, since X is normal, there exist disjoint open sets G and H in X such that $A \subseteq G$ and $B \subseteq H$. Therefore X is pseudo normal.

Theorem 1.4.3: A topological space X is pseudo normal if and only if each pair of disjoint compact sets K_1 and K_2, there exists open set U such that $K_1 \subseteq U \subseteq \overline{U} \subseteq K_2{}^c$.

Proof: Let X be a pseudo normal space and K_1, K_2 be two compact subsets of X and $K_1 \cap K_2 = \phi$. Since X is pseudo normal, there exist open sets U, V such that $K_1 \subseteq U,\ K_2 \subseteq V$ and $U \cap V = \phi$. Then $U \subseteq V^c \subseteq K_2{}^c$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq K_2{}^c$. Hence we have $K_1 \subseteq U \subseteq \overline{U} \subseteq K_2{}^c$.

Conversely, suppose that for each pair K_1 and K_2 of disjoint compact subsets of X, there exists an open set H of X such that $K_1 \subseteq U \subseteq \overline{U} \subseteq K_2{}^c$. We shall show that X is pseudo normal. Here $K_1 \subseteq H$ and $K_2 \subseteq \overline{H}^c$. Let $\overline{H}^c = G$. Then G is open, $K_2 \subseteq G$ and $G \cap H = \phi$. [For $x \in H \cap \overline{H}^c \Rightarrow x \in H$ and $x \in \overline{H}^c$. But $x \in H \Rightarrow x \in \overline{H}$. So $x \notin \overline{H}^c$ which is a contradiction, so $G \cap H = \phi$]

Theorem 1.4.4: Every open image of a pseudo normal space is pseudo normal.

Proof: Let X be a pseudo normal space and Y a topological space and let $f : X \to Y$ be an open and onto mapping. Let K_1 and K_2 be two disjoint compact subsets in Y. Then $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are compact in X. Since X is pseudo normal, there exist open subsets U and V of X such that $f^{-1}(K_1) \subseteq U$ and $f^{-1}(K_2) \subseteq V$ and $U \cap V = \phi$. Again, since f is open, f(U) and f(V) are open in Y and $K_1 \subseteq ff^{-1}(K_1) \subseteq f(U),\ K_2 \subseteq ff^{-1}(K_2) \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Hence Y is pseudo normal.

Corollary 1.4.1: Every quotient space of a pseudo normal space is pseudo normal.

Proof: Let X be a pseudo normal space and R is an equivalence relation on X. Since the projection map p:X$\rightarrow \frac{X}{R}$ is open and onto, the corollary then follows from the above Theorem 1.4.4.

Although a subspace of a normal space need not be normal (see Majumdar and Akhter [22], p.109), we have the following theorem:

Theorem 1.4.5: Every subspace of a pseudo normal space is pseudo normal.

Proof: Let X be a pseudo normal space and $Y \subseteq X$. Let K_1 and K_2 be two disjoint compact subsets in Y. Since K_1 and K_2 are compact in Y, these are compact in X too. Since X is pseudo normal, there exist disjoint open sets U and W such that $K_1 \subseteq U$ and $K_2 \subseteq W$. Then $U \cap Y$ and $W \cap Y$ are disjoint open sets in Y with property that $K_1 \subseteq U \cap Y$ and $K_2 \subseteq W \cap Y$. Hence Y is pseudo normal.

Comment 1.4.1: A continuous image of a pseudo regular (pseudo normal) space need not be pseudo regular (pseudo normal).

For if (X, T_1) is a pseudo regular (pseudo normal) space and (X, T_2) a space with the indiscrete topology, then the identity map $1_x : X \rightarrow X$ is continuous and onto. But (X, T_2) is not pseudo regular (pseudo normal).

Theorem 1.4.6: Each compact Hausdorff space is pseudo normal.

Proof: Let X be a compact Hausdorff space and A, B be two disjoint compact subsets of X. Let $x \in A$ and $y \in B$. Then $x \neq y$. Since X is Hausdorff, there exist disjoint open sets G_y and H_y such that $x \in G_y$ and $y \in G_y$. Obviously $\{H_y : y \in B\}$ is an open cover of B. Since B is compact, so there exists a finite sub-cover $\{H_{y_1}, H_{y_2}, \ldots, H_{y_m}\}$ of B. Let $H_x = H_{y_1} \cup H_{y_2} \cup \ldots \cup H_{y_m}$ and $G_x = G_{y_1} \cap G_{y_2} \cap \ldots \cap G_{y_m}$. Then $B \subseteq H_x$, $x \in G_x$ and $H_x \cap G_x = \phi$ i.e., X is pseudo regular. So for each $x \in A$, there exist two disjoint open sets G_x and H_x of X such that $x \in G_x$ and $B \subseteq H_x$. Hence $\{G_x : x \in A\}$ is an open cover of A. Since A is compact, so there exists a finite sub-cover $\{G_{x_1}, G_{x_2}, \ldots, G_{x_n}\}$ of this cover A. Let $G = G_{x_1} \cup G_{x_2} \cup \ldots \cup G_{x_n}$ and $H = H_{x_1} \cap H_{x_2} \cap \ldots \cap H_{x_n}$. Then G, H are open sets of X and $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$.

Remark 1.4.1: It follows from the above proof that every compact Hausdorff space is pseudo regular.

Theorem 1.4.7: Every locally compact Hausdorff space is pseudo regular.

Proof: Let X be a locally compact Hausdorff space. Then there exists one point compactification X_∞ of X and X_∞ is Hausdorff and compact. According to above Remark 1.4.1, X_∞ is pseudo regular. Again, according to Theorem 1.3.5, as a subspace of X_∞, X is pseudo regular.

Theorem 1.4.7: Let X be a topological space such that for every compact subset K of X, X-K contains at least two elements, if each X is pseudo normal then X is pseudo regular.

Proof: Let X be a pseudo normal space. Let K be a compact subset of X and let $x \in X$ such that $x \notin K$. Then there exists y such that y$\notin K$ and y$\neq$x. Then {x, y} being finite with two elements, is a compact subset of X such that {x, y}$\cap K = \phi$. Since X is pseudo normal, there exist open sets G and H with{x, y}$\subseteq G$, $K \subseteq H$, $G \cap H = \phi$. Since x$\in G$, $K \subseteq H$, $G \cap H = \phi$, hence X is pseudo regular.

Example 1.4.3: Let X = {a, b, c} and $\Im = \{X, \phi, \{a\}, \{b, c\}\}$. Then (X, $\Im$) is a **pseudo normal** space. But X is **not pseudo regular**. For, {a, b} compact $c \notin \{a, b\}$, but {a, b} and c cannot be separated by disjoint open sets in X.

Theorem 1.4.8: Every metric space is both pseudo regular and pseudo normal.

Proof: Since every metric space is Hausdorff, every compact set is closed. Again, since every metric space is regular, normal, therefore it is pseudo regular and pseudo normal by Theorem 1.3.2 and Theorem 1.4.2 respectively.

1.5. Almost Pseudo Regular Spaces and Almost Pseudo Normal Spaces

Here we consider two classes of topological spaces one of which lies between the class of Hausdorff spaces and the class of pseudo regular spaces, while the other lies between the class of Hausdorff spaces and the class of pseudo normal spaces.

Definition 1.5.1: A topological space X will be called **almost pseudo regular** if for every finite set A with at least two elements and for every x $\notin A$, there exist disjoint open sets G and H such that A$\subseteq G$ and x$\in H$.

Example 1.5.1: Let X= $\mathbb{Q}$, r, s$\in$ $\mathbb{Q}$, r<s , and let $V_{r,s}= \{q \in \mathbb{Q} | r < q < s\}$. Let $\mathfrak{I}$, the topology generated by $\{X, \emptyset, \{V_{r,s}\} | r, s \in \mathbb{Q}, | r < s\}$ .Let A={$q_1, \cdots, q_n$}, n≥ 2, and let $q \in \mathbb{Q}$, $q \notin A$ .Suppose $q_1<q_2<\cdots<q_n$. If $q<q_1$, let $\delta = q_1 - q$. Then $q \in V_{q-\delta/2, q+\delta/2}$ and A$\subseteq V_{q_1-\delta/2, q_n+\delta/2}$. If $q>q_n$, we construct the required open sets similarly.

If $q_i<q<q_{i+1}$, for some i,$1 \leq i < n$, let $\delta = \min(q - q_i, q_{i+1} - q)$. Then $q \in (q - \delta/2, q + \delta/2)$=V, say $q_1, \cdots, q_i \in (q_1 - \delta/2, q_2 + \delta/2) = V_1$ and $q_{i+1}, \cdots, q_n$ $\in (q_{i+1} - \delta/2, q_n + \delta/2) = V_2$, say. Then $q \in V$, $A \subseteq V_1 \cup V_2$ and $V \cap (V_1 \cup V_2) = \phi$.

Definition 1.5.2: A topological space X will be called **almost pseudo normal** if for every two finite disjoint sets A and B each with at least two elements , there exist disjoint open sets G and H such that A$\subseteq G$ and B$\subseteq H$.

Example 1.5.2: It can be shown that the above Example 1.5.1 is almost pseudo normal too.

Many of the properties of the pseudo regular and pseudo normal spaces are expected to hold but we are not proving these here. We will follow up these in near future.

CHAPTER TWO

Strongly Pseudo-Regular and Strongly Pseudo-Normal Topological Spaces

2.1. Introduction

This is the second chapter of our thesis on generalizations and specializations of regular and normal topological spaces. Earlier, regular and normal topological spaces have been generalized in various other ways. p-regular, p-normal, β-normal and γ-normal spaces ([5], [8], [9], [15], [36]) are several examples of some of these.

In this chapter we have introduced strongly pseudo-regular and strongly pseudo-normal spaces and studied their important properties. Many important results about these spaces have been established.

Unless otherwise stated, every compact set considered in this chapter will have at least two elements.

We shall now define and study strongly pseudo-regular spaces as specializations of pseudo regular spaces (see [23]).

2.2. Strongly Pseudo-Regular Spaces

Definition 2.2.1: A topological space X will be called **strongly pseudo-regular** if, for each compact set K and for every $x \in X$ with $x \notin K$, there exist open sets G and H such that $x \in G$ and $K \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$.

Example 2.2.1: X= $\mathbb{R}$ with usual topology is strongly pseudo-regular. To see this, let K be a non-empty compact subset of X and let $x \in X,\ x \notin K$. Then, by Heine-Borel Theorem, K is closed and bounded. Hence, K may be written as K= $\bigcup_{i=1}^{\infty}[a_i, b_i]$, where $[a_i, b_i] \cap [a_j, b_j] = \phi$ if $i \neq j$.

Let $a_{i_0} = \min_i\{a_{i_0}\}$, $b_{j_0} = \max_j\{b_{j_0}\}$. Since $x \notin K$, one of the following three conditions must hold:

(i) $x < a_{i_0}$

(ii) $x > b_{j_0}$

(iii) there exist a_{j_1}, a_{k_1} and b_{j_1}, b_{k_1} such that $a_{j_1} \leq b_{j_1} < a_{k_1} \leq b_{k_1}$, and [a_{j_1}, b_{j_1}] and [a_{k_1} , b_{k_1}] are consecutive intervals in K, and $b_{j_1} < x < a_{k_1}$.

If (i) holds, let $\partial_1 = \frac{1}{3}\left|x - a_{i_0}\right|$, and let U1=($x - \partial_1, x + \partial_1$), V1=($a_{i_0} - \partial_1, b_{j_0} + \partial_1$). Then U1, V1 are open and $\overline{U_1} \cap \overline{V_1} = \phi$. Also $x \in U_1,\ K \subseteq V_1$.

If (ii) holds, let $\partial_2 = \frac{1}{3}\left|x - b_{j_0}\right|$, and let U2=($x - \partial_2, x + \partial_2$), V2=($a_{i_0} - \partial_2, b_{j_0} + \partial_2$). Then U2, V2 are open, $x \in U_2,\ K \subseteq V_2$ and $\overline{U_2} \cap \overline{V_2} = \phi$.

If (iii) holds, let $\partial_3 = \frac{1}{3}\min\{x - b_{j_1}, a_{k_1} - x\}$, and let U3=($x - \partial_3, x + \partial_3$) and V3= ($a_{i_0} - \partial_3, b_{j_1} + \partial_3$) $\cup$ ($a_{k_1} - \partial_3, b_{j_0} + \partial_3$). Then U3, V3 are open, $x \in U_3,\ K \subseteq V_3$ and $\overline{U_3} \cap \overline{V_3} = \phi$.

Thus, X is strongly pseudo-regular.

Theorem 2.2.1: Every strongly pseudo-regular space is pseudo regular but the converse is not true in general.

Proof: The first part is obvious. To prove the converse, let X={a, b, c, d} and $\Im = \{X, \phi, \{a\}, \{b,c\}, \{a,b,c\}\}$. Then (X, $\Im$) is a topological space. The closed subsets of X are $X, \phi, \{b,c,d\}, \{a,d\}, \{d\}$. Let K=$\{a\}$. Then K is compact and b $\notin K$ .Then we have open sets G=$\{a\}$, H=$\{b,c\}$ such that K$\subseteq G$, b$\in$H and G$\cap$H$=\phi$. Hence X is pseudo regular. G and H are the only disjoint open sets which contain K and b respectively.

Now, we have $\overline{H} = \{b,c,d\}$, $\overline{G} = \{a,d\}$ and $\overline{G} \cap \overline{H} = \{d\} \neq \phi$. Hence X is not strongly pseudo-regular.

Theorem 2.2.2: Any subspace of a strongly pseudo-regular space is strongly pseudo-regular.

Proof: Let X be a strongly pseudo-regular space and $Y \subseteq X$. Let $y \in Y$ and K be a compact subset of Y such that $y \notin K$. Since K is compact in Y, so K is compact in X. Since X is strongly pseudo-regular, there exist open sets G and H of X such that $y \in G$ and $K \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Let U=$G \cap Y$ and V= $H \cap Y$. Then U and V are open sets of Y where $y \in U$ and $K \subseteq V$ and $\overline{U} \cap \overline{V} = \phi$. Hence Y is strongly pseudo-regular.

Corollary 2.2.1: Let X be a topological space and A, B are two strongly pseudo-regular subspaces of X. Then $A \cap B$ is strongly pseudo-regular.

Proof: Since $A \cap B$ being a subspace of both A and B, $A \cap B$ is strongly pseudo-regular by the above Theorem 2.2.2.

Theorem 2.2.3: A topological space X is strongly pseudo-regular if, for each $x \in X$ and for any compact set K not containing x, there exists an open set H of X such that $x \in H \subseteq \overline{H} \subseteq K^c$.

Proof: Let X be strongly pseudo-regular and let K be compact in X. Let $x \notin K$ i.e., $x \in K^c$. Since X is strongly pseudo regular, there exist open sets U, V such that $x \in U, K \subseteq V$ and $\overline{U} \cap \overline{V} = \phi$ and so $U \cap V = \phi$.Then $U \subseteq V^c \subseteq K^c$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq K^c$. Writing U=H we have $x \in H \subseteq \overline{H} \subseteq K^c$.

Theorem 2.2.4: A topological space X is strongly pseudo-regular if X is completely Hausdorff .

Proof: Let X be a completely Hausdorff space and let K be a compact subset of X. Let x, y be two distinct points of X with $y \in K$ and $x \notin K$ .Since X is completely Hausdorff, there exist open sets G_y and H_y such that $x \in G_y$ and $y \in H_y$ and $\overline{G_y} \cap \overline{H_y} = \phi$. Let $\{H_y : y \in K\}$ is an open cover of K.
Since K is compact, so there exist a finite subcover $\{H_{y_1}, H_{y_2}, \ldots, H_{y_n}\}$ of K.
Let $H = H_{y_1} \cup H_{y_2} \cup \ldots \cup H_{y_n}$ an $G = G_{y_1} \cap G_{y_2} \cap \ldots \cap G_{y_n}$. Then $K \subseteq H$, $x \in G$ and we claim that $\overline{G} \cap \overline{H} = \phi$. If $\overline{G} \cap \overline{H} \neq \phi$, let $z \in \overline{G} \Rightarrow z \in \overline{G_{y_1}} \cap \ldots \cap \overline{G_{y_n}}$ and $z \in \overline{H} \Rightarrow z \in \overline{H_{y_i}}$, for some y_i. This implies $z \in \overline{G_{y_i}} \cap \overline{H_{y_i}}$, which is a contradiction. Therefore $\overline{G} \cap \overline{H} = \phi$. Hence X is strongly pseudo-regular.

Theorem 2.2.5: The product space X of any non-empty collection $\{X_i\}$ of topological spaces is strongly pseudo-regular if and only if each X_i is strongly pseudo-regular.

Proof: Let $\{X_i\}$ be a non-empty collection of strongly pseudo-regular spaces and X $= \Pi X_i$. We show that X is a strongly pseudo-regular space. Let K be a compact set not containing a point $x \in X$. Let $K_i = \Pi_i(K)$, $x_i \notin K_i$. Since the projection maps are continuous, $\Pi_i(K) = K_i$ is a compact subset of X_i. Since $x \notin K$, there exists i_0 such that $x_{i_0} \notin K_{i_0}$. Since X_{i_0} is strongly pseudo-regular, there exist open sets G_{i_0}, H_{i_0} in X_i such that $x_{i_0} \in H_{i_0}$, $K_{i_0} \subseteq G_{i_0}$ and $\overline{G_{i_0}} \cap \overline{H_{i_0}} = \phi$. For each $i \neq i_0$, let G_i, H_i be open sets such that $x_i \in H_i, K_i \subseteq G_i$. Let G=$\Pi_i G_i$ and H=$\Pi_i H_i$. Then $\overline{G} \cap \overline{H} = \phi$, since $\overline{G_{i_0}} \cap \overline{H_{i_0}} = \phi$ and $K \subseteq G$, $x \in H$. Hence X is strongly pseudo-regular.

Conversely, if X is strongly pseudo-regular, then we show that for each i , X_i is strongly pseudo-regular. For each i, let K_i be a compact subset of X_i and $x_i \in X_i$ but $x_i \notin K_i$. Let $K = \Pi_i K$ and x=$\{x_i\}$ then $x \in X$ but $x \notin K$. Then K compact by Tychonoff Theorem. Since X is strongly pseudo-regular, there exist open sets G and H such that $x \in G$ and $K \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$ and G=$\Pi_i G_i$, H=$\Pi_i H_i$, G_i, H_i are open sets in X_i such that $x_i \in H_i, K_i \subseteq G_i$ and $\overline{G_i} \cap \overline{H_i} = \phi$. Therefore X_i is strongly pseudo-regular.

Theorem 2.2.6: Let X be a strongly pseudo-regular space and R is an equivalence relation of X. Then R is a closed subset of X×X.

Proof: We shall prove that R^c is open. So, let $(x, y) \in R^c$. It is sufficient to show that there exist two open sets G and H of X such that $x \in G$ and $y \in H$ and G×H$\subseteq R^c$. Let p:X$\rightarrow \dfrac{X}{R}$ be the projection map. Since $(x, y) \in R^c$, $p(x) \neq p(y)$ i.e., $x \notin p^{-1}(p(y))$. Again, since $\{y\}$ is compact and p is a continuous mapping, p(y) is compact. Also, let $\{G_i\}$be an open cover of $p^{-1}(p(y))$ in X, and let $\overline{G_i} = p(G_i)$. Then $\{\overline{G_i}\}$ is an open cover of p(y) in $\dfrac{X}{R}$.

Since p(y) is a singleton element in $\dfrac{X}{R}$, there exists $\overline{G_{i_0}}$ such that $p(y) \in \overline{G_{i_0}}$ in

$\frac{X}{R}$. Then by the definition of the topology in $\frac{X}{R}$ and the nature of the map p, (i) G_{i_0} is open in X, (ii) $G_{i_0} = p^{-1}(\overline{G_{i_0}})$ and (iii) $p^{-1}(p(y)) \subseteq G_{i_0}$ in X. Hence $p^{-1}(p(y))$ is compact in X. So by the strongly pseudo-regularity of X, there exist open sets G and H in X such that $x \in G$ and $p^{-1}(p(y)) \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Hence $y \in p^{-1}(p(y)) \subseteq H$ i.e., $y \in H$. Since $\overline{G} \cap \overline{H} = \phi$, $p(\overline{G}) \cap p(\overline{H}) = \phi$. Therefore G×H$\subseteq R^c$ and so $(x, y) \in G \times H \subseteq R^c$.

Corollary 2.2.2: Let X be a strongly pseudo-regular space and R is an equivalence relation of X. Then $\frac{X}{R}$ is completely Hausdorff.

Proof: Let cls$x$ and cls$y$ be two distinct points of $\frac{X}{R}$. Then clsx=$p(x)$ and clsy=$p(y)$ for some $x, y \in X$ such that $x \neq y$ and $(x, y) \in R^c$. By the proof of the above Theorem 2.2.6, there exist open sets G_x and G_y in $\frac{X}{R}$ such that cls$x$ $\in G_x$ and cls$y \in G_y$ and $\overline{G_x} \cap \overline{G_y} = \phi$. Thus $\frac{X}{R}$ is completely Hausdorff.

We shall now define a new class of specialized pseudo normal spaces (see [23]), viz., strongly pseudo-normal spaces and proceed to study them.

2.3. Strongly Pseudo-Normal Spaces

Definition 2.3.1: A topological space X will be called **strongly pseudo-normal** if, for each pair of disjoint compact subsets K_1, K_2 of X, there exist open sets G and H such that $K_1 \subseteq G$, $K_2 \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$.

Example 2.3.1: X= ℝ with usual topology is strongly pseudo-normal. To see this, let K_1 and K_2 be two non-empty disjoint compact sets in X. Then, K_1 and K_2 may be written as $K_1 = \bigcup_{i=1}^{n} [a_i, b_i]$, $K_2 = \bigcup_{j=1}^{n} [c_j, d_j]$ where $[a_i, b_i] \cap [a_{i'}, b_{i'}] = \phi$ if $i \neq i'$, $[c_j, d_j] \cap [c_{j'}, d_{j'}] = \phi$ if $j \neq j'$ and $[a_i, b_i] \cap [c_j, d_j] = \phi$ for each i and j.

For each consecutive pair $[a_i, b_i]$ and $[c_j, d_j]$ in the natural ordering in ℝ, let

$$\partial_{ij} = \frac{1}{3} \inf \{ |x - y| : x \in [a_i, b_i], y \in [c_j, d_j] \}$$

and let

$$V_{ij} = (a_i - \partial_{ij,} \, b_i + \partial_{ij,}),$$

$$W_{ij} = (c_j - \partial_{ij,} \, d_j + \partial_{ij,}).$$

Then, each V_{ij} and each W_{ij} are open, and

$V_{ij} \cap W_{ij} = \phi$. Let $V = \bigcup_{i,j} V_{ij}$ and $W = \bigcup_{i,j} W_{ij}$. Then, V and W are open,

$$\overline{V} \cap \overline{W} = (\bigcup_{i,j} \overline{V_{ij}}) \cap (\bigcup_{i,j} \overline{W_{ij}}) = \bigcup_{i,j} (\overline{V_{ij}} \cap \overline{W_{ij}}) = \phi \text{ and } K_1 \subseteq V, \; K_2 \subseteq W.$$

Thus, X is strongly pseudo-normal.

In the above $\overline{V} = \bigcup_{i,j} \overline{V_{ij}}$, $\overline{W} = \bigcup_{i,j} \overline{W_{ij}}$, because of the nature of V_{ij}'s and W_{ij}'s.

Theorem 2.3.1: Every strongly pseudo-normal space is pseudo normal but the converse is not true in general.

Proof: Let X be strongly pseudo-normal. Let K_1, K_2 be two disjoint compact subsets of X. Since X is strongly pseudo-normal, there exist open sets G and H such that $K_1 \subseteq G$, $K_2 \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$ .Since $\overline{G} \cap \overline{H} = \phi$, so $G \cap H = \phi$ Thus X is pseudo normal.

Conversely, let X={a, b, c, d} and $\Im = \{X, \phi, \{a\}, \{b,c\}, \{a,b,c\}\}$. Then $\Im$ is a topology on X. The closed subsets of X are $X, \phi, \{b,c,d\}, \{a,d\}, \{d\}$. Let $K_1 = \{a\}$, $K_2 = \{b\}$.Then K_1 and K_2 are two disjoint compact subsets of X .We have open sets G=$\{a\}$, H=$\{b,c\}$such that $K_1 \subseteq G$, $K_2 \subseteq H$ and $G \cap H = \phi$. Hence X is pseudo normal. Clearly, G and H are the only disjoint open sets which separate K_1 and K_2 respectively.

We have $\overline{H} = \{b,c,d\}$, $\overline{G} = \{a,d\}$and so $\overline{G} \cap \overline{H} = \{d\} \neq \phi$. Hence X is not strongly pseudo-normal.

Theorem 2.3.2: Every open image of a strongly pseudo-normal space is strongly pseudo-normal.

Proof: Let X be a strongly pseudo-normal space and Y a topological space and let $f : X \to Y$ be an open and onto mapping. Let K_1 and K_2 be two disjoint compact subsets in Y. Since f is open, f^{-1} is continuous, $f^{-1}(K_1)$ and $f^{-1}(K_2)$ are compact in X. Since X is strongly pseudo-normal, there exist open subsets U and V of X such that $f^{-1}(K_1) \subseteq U$ and $f^{-1}(K_2) \subseteq V$ and $\overline{U} \cap \overline{V} = \phi$. Again, since f is open, f(U) and f(V) are open in Y and $K_1 \subseteq ff^{-1}(K_1) \subseteq f(U)$, $K_2 \subseteq ff^{-1}(K_2) \subseteq f(V)$. Now $f(\overline{U}) \cap f(\overline{V}) = \phi$. Since f is open , f is also closed. Therefore $f(\overline{U})$ is closed, hence $f(\overline{U}) = \overline{f(\overline{U})}$. Since

$f(U) \subseteq f(\overline{U})$, $\overline{f(U)} \subseteq \overline{f(\overline{U})} = f(\overline{U})$. Similarly $\overline{f(V)} \subseteq \overline{f(\overline{V})} = f(\overline{V})$. Therefore $\overline{f(U)} \cap \overline{f(V)} = \phi$. Hence Y is strongly pseudo-normal.

Corollary 2.3.1: Every quotient space of a strongly pseudo-normal space is strongly pseudo-normal.

Proof: Let X be a strongly pseudo-normal space and R is an equivalence relation on X. Since the projection map p:X$\rightarrow \frac{X}{R}$ is open and onto, the corollary then follows from the above Theorem 2.3.2.

Although a subspace of a normal space need not be normal (see [22], p. 109), we have the following theorem:

Theorem 2.3.3: Every subspace of a strongly pseudo-normal space is strongly pseudo-normal.

Proof: Let X be a strongly pseudo-normal space and $Y \subseteq X$. Let K_1 and K_2 be two disjoint compact subsets in Y. Since K_1 and K_2 are compact in Y, these are compact in X too. Since X is strongly pseudo-normal, there exist open sets U and V such that $K_1 \subseteq U$ and $K_2 \subseteq V$ and $\overline{U} \cap \overline{V} = \phi$. Let G=$U \cap Y$ and H=$V \cap Y$. Then G and H are open sets in Y with property that $K_1 \subseteq G$ and $K_2 \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Hence Y is strongly pseudo-normal.

Comment 2.3.1: A continuous image of a strongly pseudo-regular (strongly pseudo-normal) space need not be strongly pseudo-regular (strongly pseudo-normal).

For, if(X,T_1) is a strongly pseudo-regular (strongly pseudo-normal) space and (X,I) a space with the indiscrete topology, then the identity map

$1_x : (X,T_1) \to (X,I)$ is continuous and onto. But (X,I) is not strongly pseudo-regular (strongly pseudo-normal).

Theorem 2.3.4: A topological space X is strongly pseudo-normal if for each pair of disjoint compact sets K_1 and K_2, there exists an open set U such that $K_1 \subseteq U \subseteq \overline{U} \subseteq {K_2}^c$.

Proof: Let X be a strongly pseudo-normal space and K_1, K_2 be two compact subsets of X and $K_1 \cap K_2 = \phi$. Since X is strongly pseudo-normal, there exist open sets U, V such that $K_1 \subseteq U,\ K_2 \subseteq V$ and $\overline{U} \cap \overline{V} = \phi$ and so $U \cap V = \phi$.

Then $U \subseteq V^c \subseteq {K_2}^c$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq {K_2}^c$. Hence we have $K_1 \subseteq U \subseteq \overline{U} \subseteq {K_2}^c$.

Theorem 2.3.5: A topological space X is strongly pseudo-normal if X is completely Hausdorff .

Proof: Let X be a completely Hausdorff space and let A, B be two disjoint compact subsets of X. Let $x \in A$ and $y \in B$. Then $x \neq y$. Since X is completely Hausdorff, there exist open sets G_y and H_y such that $x \in G_y$ and $y \in G_y$ and $\overline{G_y} \cap \overline{H_y} = \phi$. Obviously $\{H_y : y \in B\}$ is an open cover of B. Since B is compact, so there exists a finite sub-cover $\{H_{y_1}, H_{y_2}, \dots, H_{y_m}\}$ of B. Let $H_x = H_{y_1} \cup H_{y_2} \cup \dots \cup H_{y_m}$ and $G_x = G_{y_1} \cap G_{y_2} \cap \dots \cap G_{y_m}$. Then $B \subseteq H_x,\ x \in G_x$ and $\overline{G_x} \cap \overline{H_x} = \phi$ i.e., X is strongly pseudo-regular. So for each $x \in A$, there exist two open sets G_x and H_x of X such that $x \in G_x$ and $B \subseteq H_x$ and $\overline{G_x} \cap \overline{H_x} = \phi$. Hence $\{G_x : x \in A\}$ is an open cover of A. Since A is compact, so there exists a finite sub-cover $\{G_{x_1}, G_{x_2}, \dots, G_{x_n}\}$ of this cover A. Let $G = G_{x_1} \cup G_{x_2} \cup \dots \cup G_{x_n}$ and $H = H_{x_1} \cap H_{x_2} \cap \dots \cap H_{x_n}$. Then G, H are open sets of X and $A \subseteq G,\ B \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Hence X is strongly pseudo-normal.

Theorem 2.3.6: Every strongly pseudo-normal space is strongly pseudo-regular.

Proof: Let X be a strongly pseudo-normal space. Let K be a compact subset of X and let $x \in X$ such that $x \notin K$. Therefore $\{x\}$and K are disjoint compact subsets of X. Since X is strongly pseudo-normal, there exist open sets G and H in X such that $\{x\} \subseteq G$ and $K \subseteq H$ i.e., $x \in G$ and $K \subseteq H$ and $\overline{G} \cap \overline{H} = \phi$. Hence X is strongly pseudo-regular.

Theorem 2.3.7: Every normal T_1- space is strongly pseudo-regular.

Proof: Let X be normal and T_1. Then by Theorem 3.10 of ([22], p. 108), for each $x \in X$ and for each open set G with $x \in G_x$, there exists an open set H_x in X such that $x \in H_x \subseteq \overline{H_x} \subseteq G_x$ ….(1)

Let K be a compact subset of X and let $y \in X$ such that $y \notin K$. We note that X is Hausdorff, hence for each $x \in K$, there exist open sets G_x and V_x such that $x \in G_x$, $y \in V_x$ and $G_x \cap V_x = \phi$. By (1), there exists an open set H_x in X such that $x \in H_x \subseteq \overline{H_x} \subseteq G_x$. Clearly ϰ=$\{H_x \mid x \in K\}$ and so G=$\{G_{x_1}, \ldots, G_{x_n}\}$ is an open cover of K. K being compact, ϰ has a finite sub-cover, say, $\{H_{x_1}, \ldots\ldots\ldots, H_{x_n}\}$. Let G=$G_{x_1} \cup \ldots \cup G_{x_n}$ and V=$V_{x_1} \cap \ldots \cap V_{x_n}$. Then G and V are open sets in K and $G \cap V = \phi$. Also if H=$H_{x_1} \cup \ldots \cup H_{x_n}$, then H is open, H$\supseteq$ K and $x \in V$ and $H \cap V = \phi$. Since $\{\overline{H_{x_1}}, \ldots, \overline{H_{x_n}}\} \subseteq G$, $\overline{H_{x_1}} \cup \ldots \cup \overline{H_{x_n}}$ is contained in G and is disjoint from V. $\overline{H_{x_1}} \cup \ldots \cup \overline{H_{x_n}}$ is a closed set containing $H_{x_1} \cup \ldots \cup H_{x_n}$ and so $\overline{H} \subseteq \overline{H_{x_1}} \cup \ldots \cup \overline{H_{x_n}}$. Hence $\overline{H} \cap V = \phi$. Now, there exists an open set W in X such that $x \in W \subseteq \overline{W} \subseteq V$. Then $\overline{H_{x_1} \cup \ldots \cup H_{x_n}} \cap \overline{W} = \phi$ i.e., $\overline{H} \cap \overline{W} = \phi$. Therefore X is strongly pseudo-regular.

CHAPTER THREE

Strictly Pseudo-Regular and Strictly Pseudo-Normal Topological Spaces

3.1. Introduction

This is the third chapter of our thesis. Here we have defined and studied two new classes of topological spaces. A topological space X will be called strictly pseudo-regular if for each compact set K and for every $x \in X$ with $x \notin K$, there exists a continuous function $f : X \to [0,1]$ such that f(x) = 0 and f(K)=1. X will be called strictly pseudo-normal if for each pair of disjoint compact subsets K_1, K_2 of X, there exists a continuous function $f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. We have established various properties of these spaces. The strictly pseudo-regular spaces resemble the completely regular spaces. Many such properties hold for these two classes. For Hausdorff spaces, 'completely regular' and 'completely normal' are synonymous with 'strictly pseudo-regular' and 'strictly pseudo- normal' spaces respectively. For compact spaces 'strictly pseudo-regular' implies 'completely regular' and 'strictly pseudo- normal' implies 'completely normal' . Every metric space is both strictly pseudo-regular and strictly pseudo-normal.

We shall now define and study strictly pseudo-regular spaces as a generalization of completely regular spaces.

3.2. Strictly Pseudo-Regular Spaces

Definition 3.2.1: A topological space X will be called **strictly pseudo-regular** if for each compact set K and for every $x \in X$ with $x \notin K$, there exists a continuous function $f : X \to [0,1]$ such that f(x) = 0 and f(K)=1.

Example 3.2.1: Let K be a compact subset of ℝ and let $x \in \mathbb{R}$ such that $x \notin K$. Since ℝ is Hausdorff, K is closed and since ℝ is completely regular, there exists a continuous function $f : X \to [0,1]$ such that f(x) = 0 and f(K)=1. Thus ℝ is strictly pseudo-regular.

Theorem 3.2.1: Every strictly pseudo-regular compact space is completely regular.

Proof: Let X be compact and strictly pseudo-regular. Let K be a closed subset of X and let $x \in X$ with $x \notin K$ .Since X is compact, K is compact. Again, since X is strictly pseudo-regular, there exists a continuous function $f : X \to [0,1]$ such that f(x)=0 and f(K)=1. Therefore X is completely regular.

Theorem 3.2.2: Every completely regular Hausdorff space is strictly pseudo-regular.

Proof: Let X be a completely regular Hausdorff space. Let K be a compact subset of X and $x \in X$ with $x \notin K$. Since X is Hausdorff, K is closed. Now, since X is completely regular, there exists a continuous function $f : X \to [0,1]$ such that f(x)=0 and f(K)=1. Therefore X is strictly pseudo-regular.

Theorem 3.2.3: A topological space X is strictly pseudo-regular if for each $x \in X$ and any compact set K not containing x, there exists an open set H of X such that $x \in H \subseteq \overline{H} \subseteq K^c$.

Proof: Let X be a strictly pseudo-regular space and let K be compact in X. Let $x \notin K$ i.e., $x \in K^c$. Since X is strictly pseudo-regular, there exists a continuous function $f : X \to [0,1]$ such that f(x)=0 and f(K) = 1. Let $a, b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $x \in$ f^{-1} ([0,a)) and K $\subseteq f^{-1}$ ((b,1]). Let U = f^{-1} ([0,a)) and V = f^{-1} ((b,1]) . Then $x \in U, K \subseteq V$ and $U \cap V = \phi$. Then $U \subseteq V^c \subseteq K^c$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq K^c$. Writing U=H, we have $x \in H \subseteq \overline{H} \subseteq K^c$.

Theorem 3.2.4: Any subspace of a strictly pseudo-regular space is strictly pseudo-regular.

Proof: Let X be a strictly pseudo-regular space and $Y \subseteq X$. Let $y \in Y$ and K be a compact subset of Y such that $y \notin K$. Since $y \in Y$, so $y \in X$ and since K is compact in Y, so K is compact in X. Since X is strictly pseudo-regular, there exists a continuous function $f : X \to [0,1]$ such that f(y)=0 and f(K)=1. Therefore the restriction function $\overline{f}$ of f is a continuous function $\overline{f} : Y \to [0,1]$ such that f(y)=0 and f(K)=1. Hence Y is strictly pseudo-regular.

Corollary 3.2.1: Let X be a topological space and A, B are two strictly pseudo-regular subspaces of X. Then $A \cap B$ is strictly pseudo-regular.

Proof: Since $A \cap B$ being a subspace of both A and B, $A \cap B$ is strictly pseudo-regular by the above Theorem 3.2.4.

Theorem 3.2.5: Every strictly pseudo-regular space is Hausdorff.

Proof: Let X be a strictly pseudo-regular space. Let $x, y \in X$ with $x \neq y$. Then $\{x\}$ is a compact set and $y \notin \{x\}$. Since X is strictly pseudo-regular, there exists a continuous function $f: X \rightarrow [0,1]$ such that f(y)=0 and f({x})=1. Let $a,b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $y \in$ f^{-1} ([0,a)) and $\{x\} \subseteq$ f^{-1} ((b,1]) i.e., $x \in$ f^{-1} ((b,1]) . Therefore X is Hausdorff.

Theorem 3.2.6: Every strictly pseudo-regular space is pseudo regular.

Proof: Let X be a strictly pseudo-regular space. Let K be a compact subset of X and $x \in X$ with $x \notin K$. Since X is strictly pseudo-regular, there exists a continuous function $f: X \rightarrow [0,1]$ such that f(x)=0 and f(K)=1. Let $a,b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $x \in$ f^{-1} ([0,a)) and K$\subseteq$ f^{-1} ((b,1]) .Therefore X is pseudo regular.

We shall now define strictly pseudo-normal spaces as a class of specialized pseudo normal spaces (see [23]) and proceed to study them.

3.3. Strictly Pseudo-Normal Spaces

Definition 3.3.1: A topological space X will be called **strictly pseudo-normal** if for each pair of disjoint compact subsets K_1, K_2 of X, there exists a continuous function $f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1.

Example 3.3.1: Let K_1, K_2 be two disjoint compact subsets of ℝ. Since ℝ is Hausdorff, K_1, K_2 are also closed and since ℝ is completely normal, there exists a continuous function $f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Thus ℝ is strictly pseudo-normal.

Theorem 3.3.1: Every strictly pseudo-normal compact space is completely normal.

Proof: Let X be compact and strictly pseudo-normal. Let K_1, K_2 be two disjoint closed subsets of X. Since X is compact, K_1, K_2 are also compact. Again, since X is strictly pseudo-normal, there exists a continuous function $f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1.Therefore X is completely normal.

Theorem 3.3.2: Every completely normal Hausdorff space is strictly pseudo-normal.

Proof: Let X be Hausdorff and completely normal. Let K_1, K_2 be two disjoint compact subsets of X. Since X is Hausdorff, K_1, K_2 are closed. Again, since X is completely normal, there exists a continuous function

$f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Therefore X is strictly pseudo-normal.

Theorem 3.3.3: A topological space X is strictly pseudo-normal if each pair of disjoint compact sets K_1 and K_2, there exists an open set U such that $K_1 \subseteq U \subset \overline{U} \subseteq K_2^c$.

Proof: Let X be a strictly pseudo-normal space and K_1, K_2 be two compact subsets of X and $K_1 \cap K_2 = \phi$. Then $K_1 \subseteq K_2^c$. Since X is strictly pseudo-normal, there exists a continuous function $f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Let $a, b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $K_1 \subseteq$ f^{-1} ([0,a)) and $K_2 \subseteq$ f^{-1} ((b,1]). Let U = f^{-1} ([0,a)) and V = f^{-1} ((b,1]) . Then $K_1 \subseteq U,\ K_2 \subseteq V$ and $U \cap V = \phi$. Then $U \subseteq V^c \subseteq K_2^c$. So $\overline{U} \subseteq \overline{V^c} = V^c \subseteq K_2^c$. Hence we have $K_1 \subseteq U \subset \overline{U} \subseteq K_2^c$.

Although a subspace of a normal space need not be normal (see [22], p.109), we have the following theorem:

Theorem 3.3.4: Every subspace of a strictly pseudo-normal space is strictly pseudo-normal.

Proof: Let X be a strictly pseudo-normal space and $Y \subseteq X$. Let K_1 and K_2 be two disjoint compact subsets of Y. Since K_1 and K_2 are compact in Y, these are compact in X too. Since X is strictly pseudo-normal, there exists a continuous function $f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Therefore the restriction function $\overline{f}$ of f is a continuous function $\overline{f} : Y \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Hence Y is strictly pseudo-normal.

Theorem 3.3.5: Every strictly pseudo-normal space is Hausdorff.

Proof: Let X be a strictly pseudo-normal space. Let $x, y \in X$ with $x \neq y$. Then {x} and {y} are two disjoint compact subsets of X . Since X is strictly pseudo-normal, there exists a continuous function $f : X \to [0,1]$ such that f({x})=0 and f({y})=1. Let $a,b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and obviously $\{x\} \subseteq$ f^{-1} ([0,a)) i.e., $x \in$ f^{-1} ([0,a)) and $\{y\} \subseteq$ f^{-1} ((b,1]) i.e., $y \in$ f^{-1} ((b,1]) . Therefore X is Hausdorff.

Theorem 3.3.6: Every strictly pseudo-normal space is pseudo normal.

Proof: Let X be a strictly pseudo-normal space. Let K_1 and K_2 be two disjoint compact subsets of X. Since X is strictly pseudo-normal, there exists a continuous function $f : X \to [0,1]$ such that f(K_1)=0 and f(K_2)=1. Let $a,b \in [0,1]$ and $a < b$. Then [0,a) and (b,1] are two disjoint open sets of [0,1]. Since f is continuous, f^{-1} ([0,a)) and f^{-1} ((b,1]) are two disjoint open sets of X and $K_1 \subseteq$ f^{-1} ([0,a)), $K_2 \subseteq$ f^{-1} ((b,1]) .Therefore X is pseudo normal.

Theorem 3.3.7: Every strictly pseudo-normal space is strictly pseudo-regular.

Proof: Let X be a strictly pseudo-normal space. Let K be a compact subset of X and let $x \in X$ such that $x \notin K$. Therefore $\{x\}$ and K are disjoint compact subsets of X. Since X is strictly pseudo-normal, there exists a continuous function $f : X \to [0,1]$ such that f({x})=0 and f(K)=1 i.e., f(x)=0 and f(K)=1. Hence X is strictly pseudo-regular.

Theorem 3.3.8: Every metric space is both strictly pseudo-regular and strictly pseudo-normal.

Proof: Since every strictly pseudo-regular, strictly pseudo-normal spaces is pseudo-regular, pseudo-normal (Theorem 3.2.6 and Theorem 3.3.6 respectively), and since every metric space is both pseudo regular and pseudo normal (Theorem 1.4.8), therefore, it is strictly pseudo-regular and strictly pseudo-normal.

CHAPTER FOUR

Nearly Regular Topological Spaces of the First Kind and the Second Kind

4.1. Introduction

Regular topological spaces form a very important and interesting class of spaces in topology. The class of p-regular spaces is an example of generalization of this class ([5]).

In this chapter we shall introduce a number of new important generalizations of regular spaces. We shall provide examples of such spaces and establish some of their important properties. The generalizations to be introduced by us in this chapter is nearly regular topological spaces of the first kind and the second kind.

We now define nearly regular spaces of the first kind and proceed to study them.

4.2. Nearly Regular Spaces of the First Kind

Definition 4.2.1: A topological space X will be called **nearly regular of the first kind (n. r .f. k.)** if there exists a nontrivial closed set F_0 such that for each $x \in X$, $x \notin F_0$, there exist disjoint open sets G and H such that $x \in G$ and $F_0 \subseteq H$.

Example 4.2.1: Let

$X = \mathbb{R}, \Im = \langle \{\mathbb{R}, \emptyset, (1,2), (1,2)^c\} \cup A = \{\{x\} | x \in \mathbb{R} - (1,2)\} \cup B = \{\mathbb{R} - \{y_1, y_2\} | y_1 \in (1,2), y_2 \in (1,2)^c\} \rangle$

Here (1,2) is a closed set F_0. The points of A are the only points disjoint from F_0. Each of these points can be separated from F_0 by disjoint open sets. Let $x \in A$ then $\{x\}$ and F_0 are desired open sets. Then **X is n. r. f. k.** but **not regular.**

Let F= $(1,2)^c - [\{x_1\} \cup \dots \dots \cup \{x_n\}], x_i \in (1,2)^c$

$= (1,2)^c \cap [\{x_1\} \cup \dots \dots \cup \{x_n\}]^c$

Then F is closed $x_1 \notin F$. x_1 and F cannot be separated by disjoint open sets.

Theorem 4.2.1: Every regular space is nearly regular space of the first kind but the converse is not true in general.

Proof: Let X be a regular space. Let F_0 be a closed subset of X and let $x \in X$ such that $x \notin F_0$. Now, since X is regular, there exist disjoint open sets G and H such that $x \in G$ and $F_0 \subseteq H$. Therefore X is nearly regular space of the first kind.

To see that the converse is always not true, let $X = \{a,b,c,d,e\}$, $\Im = \{X, \Phi, \{a,b\}, \{a,b,e\}, \{e\}, \{a,b,c,d\}\}$. Then $(X, \Im)$ is a topological space in which the closed sets of X are X, Φ, $\{c,d,e\}, \{e\}, \{a,b,c,d\}, \{c,d\}$.

The closed set $\{a,b,c,d\}$ and e can be separated by $\{a,b,c,d\}$ and $\{e\}$, but the closed set $\{c,d,e\}$ and a cannot be separated by disjoint open sets. Thus $(X, \Im)$ is nearly regular space of the first kind but not regular.

Theorem 4.2.2: A topological space X is nearly regular space of the first kind if and only if there exists a closed set F_0 such that for each $x \in X$ with $x \notin F_0$, there exists an open set G such that $x \in G \subseteq \overline{G} \subseteq F_0^{\,c}$.

Proof: First, suppose that X is nearly regular space of the first kind. Then there exists a closed set F_0 in X such that for each $x \in X$ with $x \notin F_0$, there exist open sets G and H such that $x \in G$, $F_0 \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq F_0^{\,c}$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq F_0^{\,c}$. Thus, $x \in G \subseteq \overline{G} \subseteq F_0^{\,c}$.

Conversely, suppose that there exists a closed subset F_0 of X such that for each $x \in X$ with $x \notin F_0$, there exist an open set G in X such that $x \in G \subseteq \overline{G} \subseteq F_0^{\,c}$. Let $\overline{G}^c$ = H. Then H is open, $G \cap H = \phi$ and $x \in G$ and $F_0 \subseteq H$. Hence X is nearly regular space of the first kind.

Theorem 4.2.3: Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces, and let X=$\prod_{i \in I} X_i$ be the product space. If X_i nearly regular of the first kind, for each i, then X is nearly regular of the first kind.

Proof: Since each X_i is nearly regular of the first kind, there exists, for each i, a closed set F_i of X_i such that for each $x_i \in X_i$ with $x_i \notin F_i$ there are open sets U_i, V_i in X_i such that $x_i \in U_i$, $F_i \subseteq V_i$, $U_i \cap V_i = \phi$ ....(1)

Let F=$\prod_{i \in I} F_i$. Then F is closed in X. Let $\overline{x} \in X$ such that $\overline{x} \notin F$. Let $\overline{x} = \{\overline{x}_i\}$.

Then there exists i_0 such that $\overline{x}_{i_0} \notin F_{i_0}$. By (1), there are open sets G_{i_0}, H_{i_0} in X_{i_0} such that $x_{i_0} \in G_{i_0}$, $F_{i_0} \subseteq H_{i_0}$, $G_{i_0} \cap H_{i_0} = \phi$. For each $j \in I$, $j \neq i_0$, let G_j and H_j be open sets in X_j such that $x_j \in G_j$, $F_j \subseteq H_j$. Then $G = \prod_{i \in I} G_i$, $H = \prod_{i \in I} H_i$ are open sets in X such that $x \in G$, $F \subseteq H$ and $G \cap H = \phi$. Therefore, X is nearly regular space of the first kind.

Theorem 4.2.4: Let $\{X_i\}_{i\in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i\in I} X_i$ be the product space. If X is nearly regular of the first kind, then at least one of the X_i's is nearly regular of the first kind.

Proof: Let X be a nearly regular space of the first kind. Then there exists a closed set F in X such that for every $x \in X, x \notin F$, there are open sets U and V in X such that $x \in U$ and $F \subseteq V$ and $U \cap V = \phi$. Let, for each $i \in I, \pi_i(F) = F_i$ where $\pi_i : X \to X_i$ is the projection map. Then each F_i is closed in X_i. For each $i \in I$, let $x_i \in X_i$ be such that $x_i \notin F_i$. Let $x = \{x_i\}$. Then $x \notin F$. Since X is nearly regular space of the first kind, there are open sets G and H in X such that $x \in G, F \subseteq H$ and $G \cap H = \phi$. Let $\pi_i(G) = G_i$, $\pi_i(H) = H_i$. Then G_i and H_i are open in X_i, for each $i \in I$. Since $G \cap H = \phi$ there exists $i_0 \in I$ such that $G_{i_0} \cap H_{i_0} = \phi$. Clearly, $x_{i_0} \in G_{i_0}$, $F_{i_0} \subseteq H_{i_0}$. Hence X_{i_0} is nearly regular of the first kind.

Theorem 4.2.5: Every subspace of a nearly regular space of the first kind is nearly regular space of the first kind.

Proof: Let X be a nearly regular first kind space and Y a subspace of X. Since X is nearly regular first kind space, there exists a closed set F in X which can be separated from each point of X which is not contained in F. Then for each $y \in Y \subseteq X$, $y \notin F$, there exist open sets U_1, U_2 in X such that $y \in U_1$, $F \subseteq U_2$ with $U_1 \cap U_2 = \phi$. Let $F_0 = Y \cap F$. Then F_0 is closed in Y and clearly $y \notin F_0$ .Also let $V_1 = Y \cap U_1$, $V_2 = Y \cap U_2$. Then V_1 and V_2 are disjoint open sets in Y where $y \in V_1$, $F_0 \subseteq V_2$. Hence Y is nearly regular space of the first kind.

Corollary 4.2.1: Let X be a topological space and A, B are two nearly regular subspaces of X of the first kind. Then $A \cap B$ is nearly regular space of the first kind.

Proof: $A \cap B$ being a subspace of both A and B, $A \cap B$ is nearly regular space of the first kind by the above Theorem 4.2.5.

Theorem 4.2.6: Let X be a nearly regular T_1-space of the first kind and R is an equivalence relation of X. If the projection mapping p:X$\rightarrow \frac{X}{R}$ is closed. Then R is a closed subset of X×X.

Proof: We shall prove that R^c is open. So, let $(x, y) \in R^c$. It is sufficient to show that there exist two open sets G and H of X such that $x \in G$ and $y \in H$ and G×H$\subseteq R^c$. For that $p(G) \cap p(H) = \phi$. Since $(x, y) \in R^c$, $p(x) \neq p(y)$ i.e; $x \notin p^{-1}(p(y))$.Again, since $\{y\}$ is closed and since p closed mapping, p(y) is closed and since p is a continuous mapping, $p^{-1}(p(y))$ is closed. So by the nearly regularity of X of the first kind, there exist disjoint open sets G and U in X such that $x \in G$ and $p^{-1}(p(y)) \subseteq U$.Since p is a closed mapping, there exists an open set V containing p(y) such that $p^{-1}(p(y)) \subseteq p^{-1}(V) \subseteq U$. Writing $p^{-1}(V) = H$, we have G×H$\subseteq R^c$.

Corollary 4.2.2: Let X be a nearly regular T_1-space of the first kind. R is an equivalence relation of X and p:X$\rightarrow \frac{X}{R}$ is closed and open mapping. Then $\frac{X}{R}$ is Hausdorff.

Proof: Since p:X$\rightarrow \frac{X}{R}$ is closed, by the proof of the above Theorem 4.2.6, R is a closed subset of X×X. Let $p(x)$ and $p(y)$ be two distinct points of $\frac{X}{R}$.

Therefore $(x, y) \notin R$. Since R is a closed subset of X×X, there exist open sets G, H in X such that $x \in G$ and $y \in H$ and G×H$\subseteq R^c$. So $p(x) \in p(G)$, $p(y) \in p(H)$. Since p is open, p(G) and p(H) are open sets of $\frac{X}{R}$ and since

G×H$\subseteq R^c$, $p(G) \cap p(H) = \phi$.Thus $\frac{X}{R}$ is Hausdorff.

We now define nearly regular spaces of the second kind and proceed to study them.

4.3. Nearly Regular Spaces of the Second Kind

Definition 4.3.1: A topological space X will be called **nearly regular of the second kind (n. r. s. k.)** if there exists a point $x_0 \in X$ such that for each nontrivial closed set F in X with $x_0 \notin F$, there exist disjoint open sets G and H such that $x_0 \in G$ and $F \subseteq H$.

Example 4.3.1: Let x_0 be a point in $\mathbb{R}^n$ such that for every closed set F in $\mathbb{R}^n$, $x_0 \notin F$. Since $\mathbb{R}^n$ is T_1, $\{x_0\}$ is closed and since $\mathbb{R}^n$ is normal and F is closed, $\{x_0\}$ and F can be separated by disjoint open sets. Thus $\mathbb{R}^n$ is **nearly regular of the second kind**.

Example 4.3.2: The Example 4.2.1 of **n. r. f. k.** is **not n. r. s. k**.

The closed sets of the form:

(1,2), $(1,2)^c$, $\mathbb{R} - \{x_{n_1}, \dots \dots, x_{n_k}\}$, $\{y_1, y_2\}$.

Let $z \in \mathbb{R}$. Let F=$\{y_1, y_2\}$. If $z \in (1,2)$, then z and F can be separated by disjoint open sets.

If $z \notin (1,2)$, then z = y_2 for some $\mathbb{R} - \{y_1, y_2\} \in B$. So, z cannot be separated from F=$\{y_1, y_2\}$.

Theorem 4.3.1: Every regular space is nearly regular space of the second kind but the converse is not true in general.

Proof: Let X be a regular space. Let x_0 be a point in X and let F be a closed subset in X such that $x_0 \notin F$. Now, since X is regular, there exist disjoint open sets G and H such that $x_0 \in G$ and $F \subseteq H$. Therefore X is nearly regular of the second kind.

To see that the converse is always not true,

let $X = \mathbb{R}, \Im = \langle \{\mathbb{R}, \emptyset, \{x_0\}, \{x_0\}^c\} \cup \{(n, n+1)^c | n \in \mathbb{N}\}, x_0 \notin \mathbb{N} \rangle$

Let $x_0 \notin (n_0, n_0+1)$. The closed sets are finite unions of $\{x_0\}$ and (n, n+1), $n \in \mathbb{N}$. x_0 and (n_0,n_0+1) are separated by $\{x_0\}$ and $\{x_0\}^c$. Thus X is n.r.s.k. But is not regular. Because if x = 5 and F = (5, 6), then F is closed and x $\notin$ F. But X and F can be separated by disjoint open sets.

Theorem 4.3.2: A topological space X is nearly regular space of the second kind if and only if there exists a point x_0 in X such that for each nontrivial closed set F in X with $x_0 \notin F$, there exists an open set G such that $x_0 \in G \subseteq \overline{G} \subseteq F^c$.

Proof: First, suppose that X be a nearly regular space of the second kind. Then there exists a point $x_0 \in X$ such that for each nontrivial closed set F in X with $x_0 \notin F$, there exist open sets G and H such that $x_0 \in G, F \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq F^c$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq F^c$. Thus, $x_0 \in G \subseteq \overline{G} \subseteq F^c$.

Conversely, suppose that there exists a point x_0 in X such that for each nontrivial closed set F in X with $x_0 \notin F$, there exists an open set G such that $x_0 \in G \subseteq \overline{G} \subseteq F^c$. Let $\overline{G}^c$ = H. Then H is open, $G \cap H = \phi$ and $x_0 \in G$ and $F \subseteq H$. Hence X is nearly regular space of the second kind.

Theorem 4.3.3: Let $\{X_i\}_{i\in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i\in I} X_i$ be the product space. If X_i nearly regular of the second kind, for each i, then X is nearly regular of the second kind.

Proof: Since each X_i is nearly regular of the second kind, there exists, for each i, a point x_i in X_i such that for each closed subset F_i in X_i with $x_i \notin F_i$ there are open sets U_i, V_i in X_i such that $x_i \in U_i$, $F_i \subseteq V_i$, $U_i \cap V_i = \phi$ …………………………(1)

Let F= $\prod_{i\in I} F_i$. Then F is closed in X. Let $x \in X$ such that $x \notin F$. Let $x = \{x_i\}$.

Then there exists i_0 such that $x_{i_0} \notin F_{i_0}$. By (1), there are open sets G_{i_0}, H_{i_0} in X_{i_0} such that $x_{i_0} \in G_{i_0}$, $F_{i_0} \subseteq H_{i_0}$, $G_{i_0} \cap H_{i_0} = \phi$. For each $j \in I,\ j \neq i_0$, let G_j and H_j be open sets in X_j such that $x_j \in G_j$, $F_j \subseteq H_j$.

Then $G = \prod_{i\in I} G_i$, $H = \prod_{i\in I} H_i$ are open sets in X such that $x \in G$, $F \subseteq H$ and $G \cap H = \phi$. Therefore, X is nearly regular space of the second kind.

Theorem 4.3.4: Let $\{X_i\}_{i\in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i\in I} X_i$ be the product space. If X is nearly regular of the second kind, then at least one of the X_i's is nearly regular of the second kind.

Proof: The proof of the Theorem 4.3.4 of the above is almost similar to the proof of the Theorem 4.2.4.

Theorem 4.3.5: Any subspace of a nearly regular space of the second kind is nearly regular space of the second kind.

Proof: The proof of the Theorem 4.3.5 follows from the proof of the Theorem 4.2.5.

Corollary 4.3.1: Let X be a topological space and A, B are two nearly regular subspaces of X of the second kind. Then $A \cap B$ is nearly regular space of the second kind.

Proof: The proof of the Corollary 4.3.1 of the above is almost similar to the proof of the Corollary 4.2.1.

Theorem 4.3.6: Let X be a nearly regular T_1-space of the second kind and R is an equivalence relation of X. If the projection mapping p:X$\rightarrow \frac{X}{R}$ is closed. Then R is a closed subset of X$\times$X.

Proof: The proof of the Theorem 4.3.6 is most similar to the proof of the Theorem 4.2.6.

Corollary 4.3.2: Let X be a nearly regular T_1-space of the second kind. R is an equivalence relation of X and p:X$\rightarrow \frac{X}{R}$ is closed and open mapping. Then $\frac{X}{R}$ is Hausdorff.

Proof: The proof of the Corollary 4.3.2 follows from the proof of the Corollary 4.2.2.

Theorem 4.3.7: Every metric space is both n. r. f. k. and n. r. s. k.

Proof: Since every metric space is regular, therefore, it is n. r. f. k. and n. r. s. k.

CHAPTER FIVE

Nearly Normal Topological Spaces of the First Kind and the Second Kind

5.1. Introduction

A number of generalizations of normal topological spaces have been defined and studied earlier. p-normal, β-normal, γ-normal and mildly normal spaces ([5], [8], [9], [15], [36]) are several examples of some of these.

In this fifth chapter we have defined two new generalizations of normal spaces. These have been called nearly normal topological spaces of the first kind and the second kind. We have provided examples and established many properties of such spaces.

We now define nearly normal spaces of the first kind and proceed to study them.

5.2. Nearly Normal Spaces of the First Kind

Definition 5.2.1: A topological space X will be called **nearly normal of the first kind (n. n. f. k)** if there exists a nontrivial closed set F_0 in X such that, for each nontrivial closed set F in X which is disjoint from F_0, F_0 and F can be separated by disjoint open sets in X. This space will be denoted by **(X, $\boldsymbol{F_0}$).**

Theorem 5.2.1: Every normal space is nearly normal space of the first kind but the converse is not true in general.

Proof: Let X be a normal space. Let F_0 be a closed set in X such that, for every closed set H in X such that $F_0 \cap H = \phi$. Now, since X is normal, there exist disjoint open sets G_1, G_2 in X such that $F_0 \subseteq G_1$ and $H \subseteq G_2$. Therefore X is nearly normal space of the first kind.

To see that the converse is always not true,

let $X = \mathbb{R}, \Im = \langle \mathbb{R}, \emptyset, (1,2), (1,2)^c, (2,3), (2,3)^c, (2,4), (2,4)^c, (2,7), (2,7)^c, (4,5)^c \rangle$

Let F_0=(1,2) . Clearly F_0 is closed. $(1,2)^c, (2,3), (2,4), (4,5), (2,7)$ are nontrivial closed sets in X. F_0 can be separated from each of them by open sets, but $(2,4) \cap (4,5)$ and $(2,3) \cap (4,5)$ are disjoint closed sets which can't be separated by disjoint open sets. **Hence (X, F_0) is n. n. f. k. but not normal.**

[Many such examples can be easily constructed.]

Theorem 5.2.2: A topological space X is nearly normal space of the first kind if and only if there is a nontrivial closed set F_0 in X such that, for every nontrivial closed set F in X which are disjoint from F_0 and an open set G such that $F_0 \subseteq G \subseteq \overline{G} \subseteq (F)^c$.

Proof: First, suppose that X is nearly normal space of the first kind. Then there is a nontrivial closed set F_0 in X such that, for every nontrivial closed set F in X such that $F_0 \cap F = \phi$ and there are open sets G, H in X such that $F_0 \subseteq G$ and $F \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq (F)^c$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq (F)^c$. Thus, $F_0 \subseteq G \subseteq \overline{G} \subseteq (F)^c$.

Conversely, suppose that there is a nontrivial closed set F_0 in X such that, for every nontrivial closed set F in X which are disjoint from F_0 and an open set G such that $F_0 \subseteq G \subseteq \overline{G} \subseteq (F)^c$. Here $F_0 \subseteq G$ and $F \subseteq \overline{G}^c$. Let $\overline{G}^c = H$. Then H is open, $F \subseteq H$ and $G \cap H = \phi$. Hence X is nearly normal space of the first kind.

Theorem 5.2.3: Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i \in I} X_i$ be the product space. If X_i is nearly normal of the first kind, for each i, then X is nearly normal of the first kind.

Proof: Since each X_i is nearly normal of the first kind, there exists, for each $i \in I$, a nontrivial closed set F_i of X_i such that for each nontrivial closed set H_i in X_i with $F_i \cap H_i = \phi$, there are open sets U_i, V_i in X_i such that $F_i \subseteq U_i$, $H_i \subseteq V_i$ and $U_i \cap V_i = \phi$ ………………………….(1)

Let F= $\prod_{i \in I} F_i$. Then F is a nontrivial closed in X. Let K be a nontrivial closed subset of X such that $F \cap K = \phi$. Let, for each $i \in I, \pi_i(K) = K_i$ where $\pi_i : X \to X_i$ is the projection map. Then K_i is nontrivial closed in X_i. By (1), there are open sets W_i, $W_i^{/}$ in X_i such that $F_i \subseteq W_i$, $K_i \subseteq W_i^{/}$. Let $W = \prod_{i \in I} W_i$, $W^{/} = \prod_{i \in I} W_i^{/}$. Then $F \subseteq W$, $K \subseteq W^{/}$ and $W \cap W^{/} = \phi$. Therefore, X is nearly normal space of the first kind.

Theorem 5.2.4: Every open and one-one image of a nearly normal space of the first kind is nearly normal space of the first kind.

Proof: Let X be a nearly normal space of the first kind and Y a topological space and let $f: X \to Y$ be an open and onto mapping. Since X is nearly normal space of the first kind, there is a nontrivial closed set F in X such that, for every nontrivial closed set H in X such that $F \cap H = \phi$, there are open sets U, V in X such that $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$ .Since f is open, $f(F^c)$ and $f(H^c)$ are open in Y. So $(f(F^c))^c$ and $(f(H^c))^c$ are closed in Y.

Now, $F^c \cup H^c$=X and so $f(F^c \cup H^c)$ =Y, i.e., $f(F^c) \cup f(H^c)$=Y. Hence $(f(F^c))^c \cap (f(H^c))^c = \phi$. Let $y_0 \in (f(F^c))^c$. Then $y_0 \notin f(F^c)$ i.e., there exists $x_0 \in F^c$, $f(x_0) \neq y_0$. Hence $x_1 \in F$ such that $f(x_1) = y_0$, since f is onto. Thus $y_0 \in f(F)$. Hence $(f(F^c))^c \subseteq f(F)$. Similarly, $(f(H^c))^c \subseteq f(H)$.

Now, $f(F) \subseteq f(U)$, $f(H) \subseteq f(V)$, f being open and one-one, $f(U)$, $f(V)$ are open and disjoint in Y. Thus for a nontrivial closed set $(f(F^c))^c$ in Y such that, for every nontrivial closed sets $(f(H^c))^c$ in Y such that $(f(F^c))^c \cap (f(H^c))^c = \phi$, there are open sets $f(U)$, $f(V)$ in Y such that $(f(F^c))^c \subseteq f(U)$, $(f(H^c))^c \subseteq f(V)$ and $f(U) \cap f(V) = \phi$. Hence Y is nearly normal space of the first kind.

Corollary 5.2.1: Every quotient space of a nearly normal space of the first kind is nearly normal space of the first kind.

Proof: Let X be a nearly normal space of the first kind and R is an equivalence relation on X. Since the projection map p:X$\to \frac{X}{R}$ is open and onto, the corollary then follows from the above Theorem 5.2.4.

Theorem 5.2.5: Let X be a nearly normal space of the first kind and Y is a subspace of X. Then Y is a nearly normal space of the first kind.

Proof: Since X is nearly normal space of the first kind, there is a nontrivial closed set F in X such that, for every nontrivial closed set H in X such that $F \cap H = \phi$, there are open sets U, V in X such that $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$. Let $F' = Y \cap F$ and $H' = Y \cap H$. Then for a nontrivial closed set F' in Y such that, for every nontrivial closed set H$'$ in Y such that $F' \cap H' = \phi$. Also let $U' = Y \cap U$, $V' = Y \cap V$. Then U$'$, V$'$ are open sets in Y and $U' \cap V' = \phi$ and $F' \subseteq U'$, $H' \subseteq V'$. Hence Y is nearly normal space of the first kind.

Remark 5.2.1: The corresponding theorem does not hold for normal spaces. The validity of the proof in Theorem 5.2.5 above depends on the separablity of a particular pair of disjoint closed spaces by disjoint open spaces (See Ex. of Munkres [10]).

Comment 5.2.1: A continuous image of a nearly regular space of the first kind (nearly normal space of the first kind) need not be nearly regular space of the first kind (nearly normal space of the first kind).

For if (X, T_1) is a nearly regular space of the first kind (nearly normal space of the first kind) and (X, T_2) a space with the indiscrete topology, then the identity map $1_x : X \to X$ is continuous and onto. But (X, T_2) is not nearly regular space of the first kind (nearly normal space of the first kind).

Theorem 5.2.6: Each compact Hausdorff space is nearly normal space of the first kind.

Proof: Let X be a compact Hausdorff space and let for a nontrivial closed subset A, there is a nontrivial closed subset B in X which is disjoint from A. Let $x \in A$ and $y \in B$. Then $x \neq y$. Since X is Hausdorff, there exist disjoint open sets G_y and H_y such that $x \in G_y$ and $y \in G_y$. Obviously $\{H_y : y \in B\}$ is an open cover of B.

Since B is a closed subset of X, B is compact. So there exist a finite subcover $\{H_{y_1}, H_{y_2}, \dots, H_{y_m}\}$ of B. Let $H_x = H_{y_1} \cup H_{y_2} \cup \dots \cup H_{y_m}$ and $G_x = G_{y_1} \cap G_{y_2} \cap \dots \cap G_{y_m}$. Then $B \subseteq H_x$, $x \in G_x$ and $H_x \cap G_x = \phi$ i.e., X is nearly regular space of the first kind. So for each $x \in A$, there exist two disjoint open sets G_x and H_x of X such that $x \in G_x$ and $B \subseteq H_x$. Hence $\{G_x : x \in A\}$ is an open cover of A. Since A is a closed subset of X, A is compact. So there exist a finite subcover $\{G_{x_1}, G_{x_2}, \dots, G_{x_n}\}$ of this cover A. Let $G = G_{x_1} \cup G_{x_2} \cup \dots \cup G_{x_n}$ and $H = H_{x_1} \cap H_{x_2} \cap \dots \cap H_{x_n}$. Then G, H are open sets of X and $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$. Hence the proof.

Remark 5.2.2: It follows from the above proof that every compact Hausdorff space is nearly regular space of the first kind.

Theorem 5.2.7: Every locally compact Hausdorff space is nearly regular space of the first kind.

Proof: Let X be a locally compact Hausdorff space. Then there exists one point compactification X_∞ of X. Then, X_∞ is Hausdorff and compact. According to the above Remark 5.2.2, X_∞ is nearly regular space of first kind. Again, according to Theorem 5.2.5, as a subspace of X_∞ , X is nearly regular space of the first kind.

Theorem 5.2.8: Let X be a T_1- space. Then X is nearly normal space of the first kind if and only if X is nearly regular space of the first kind.

Proof: First, suppose that X be a nearly normal space of the first kind. Let x be a point in X and let F_0 be a nontrivial closed subset of X such that $x \notin F_0$. Since X is T_1- space, {x} is closed subset of X. We have $\{x\} \cap F = \phi$. Since X is nearly normal space of the first kind, there are open sets G and H such that $\{x\} \subseteq G,\ F_0 \subseteq H,\ G \cap H = \phi$ i.e., $x \in G,\ F_0 \subseteq H,\ G \cap H = \phi$. Hence X is nearly regular space of the first kind.

Conversely, suppose that X be a nearly regular space of the first kind. Let x be a point in X and let F_0 be a nontrivial closed subset of X such that $x \notin F_0$. Since X is T_1- space, {x} is closed subset of X. We have $\{x\} \cap F_0 = \phi$. Since X is nearly regular space of the first kind, there exist open sets G and H such that x $\in G,\ F_0 \subseteq H,\ G \cap H = \phi$ i.e., $\{x\} \subseteq G,\ F_0 \subseteq H,\ G \cap H = \phi$. Hence X is nearly normal space of the first kind.

Theorem 5.2.9: Every metric space is nearly normal space of the first kind.

Proof: Since every metric space is normal, therefore it is nearly normal space of the first kind.

We now define nearly normal spaces of the second kind and proceed to study them.

5.3. Nearly Normal Spaces of the Second Kind

Definition 5.3.1: A topological space X will be called **nearly normal of the second kind (n. n. s. k)** if for each nontrivial closed set F_1, there exists a nontrivial closed set F_2 in X which is disjoint from F_1 such that F_1 and F_2 can be separated by disjoint open sets in X.

Example 5.3.1: Every n. n. f. k. is n. n. s. k.

[We are to construct an example of an n. n. s. k. space which is not n. n. f. k.]

Theorem 5.3.1: Every normal space is nearly normal space of the second kind but the converse is not true in general.

Proof: Let X be a normal space. Let F be a closed set in X such that, there exists a closed set H in X such that $F \cap H = \phi$. Now, since X is normal, there exist disjoint open sets G_1, G_2 in X such that $F \subseteq G_1$ and $H \subseteq G_2$. Therefore X is nearly normal space of the second kind.

To see that the converse is always not true,

the proof is most similar to the proof of the last part of Theorem 5.2.1 of **n. n. f. k.**

Theorem 5.3.2: A topological space X is nearly normal space of the second kind if and only if for each nontrivial closed set F in X such that, there is a nontrivial closed set F_0 in X which is disjoint from F and an open set G such that $F \subseteq G \subseteq \overline{G} \subseteq F_0^{\ c}$.

Proof: First, suppose that X is nearly normal space of the second kind. Then for each nontrivial closed set F in X such that, there is a nontrivial closed set F_0 in X such that $F_0 \cap F = \phi$ and there are open sets G, H in X such that $F \subseteq G$ and $F_0 \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq F_0^{\ c}$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq F_0^{\ c}$. Thus, $F \subseteq G \subseteq \overline{G} \subseteq F_0^{\ c}$.

Conversely, suppose that for each nontrivial closed set F in X such that, there is a nontrivial closed set F_0 in X which is disjoint from F and an open set G such that $F \subseteq G \subseteq \overline{G} \subseteq F_0^{\ c}$. Here $F \subseteq G$ and $F_0 \subseteq \overline{G}^c$. Let $\overline{G}^c = H$. Then H is open, $F_0 \subseteq H$ and $G \cap H = \phi$. Hence X is nearly normal space of the second kind.

Theorem 5.3.3: Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i \in I} X_i$ be the product space. If X_i is nearly normal of the second kind, for each i, then X is nearly normal of the second kind.

Proof: Since each X_i is nearly normal of the second kind, for each $i \in I$, for each nontrivial closed set F_i of X_i such that there exists a nontrivial closed set H_i in X_i with $F_i \cap H_i = \phi$, there are open sets U_i, V_i in X_i such that $F_i \subseteq U_i$, $H_i \subseteq V_i$ and $U_i \cap V_i = \phi$..............................(1)

Let F=$\prod_{i\in I} F_i$. Then F is closed in X. Let K be a nontrivial closed subset of X such that $F \cap K = \phi$. Let, for each $i \in I, \pi_i(K) = K_i$ where $\pi_i : X \to X_i$ is the projection map. Then K_i is closed in X_i. By (1), there are open sets W_i, $W_i^/$ in X_i such that $F_i \subseteq W_i$, $K_i \subseteq W_i^/$. Let $W = \prod_{i\in I} W_i$, $W^/ = \prod_{i\in I} W_i^/$. Then $F \subseteq W$, $K \subseteq W^/$ and $W \cap W^/ = \phi$. Therefore, X is nearly normal space of the second kind.

Theorem 5.3.4: Every open and one-one image of a nearly normal space of the second kind is nearly normal space of the second kind.

Proof: The proof of the Theorem 5.3.4 of the above is almost similar to the proof of the Theorem 5.2.4.

Corollary 5.3.1: Every quotient space of a nearly normal space of the second kind is nearly normal space of the second kind.

Proof: The proof of the Corollary 5.3.1 is most similar to the proof of the Corollary 5.2.1.

Theorem 5.3.5: Let X be a nearly normal space of the second kind and Y is a subspace of X. Then Y is a nearly normal space of the second kind.

Proof: The proof of the Theorem 5.3.5 follows from the proof of the Theorem 5.2.5.

Remark 5.3.1: The corresponding theorem does not hold for normal spaces. The validity of the proof in Theorem 5.3.5 above depends on the separablity of a particular pair of disjoint closed spaces by disjoint open spaces (See Ex. of Munkres [10]).

Comment 5.3.1: A continuous image of a nearly regular space of the second kind (nearly normal space of the second kind) need not be nearly regular space of the second kind (nearly normal space of the second kind).

For if (X,T_1) is a nearly regular space of the second kind (nearly normal space of the second kind) and (X,T_2) a space with the indiscrete topology, then the identity map $1_x : X \to X$ is continuous and onto. But (X,T_2) is not nearly regular space of the second kind (nearly normal space of the second kind).

Theorem 5.3.6: Each compact Hausdorff space is nearly normal space of the second kind.

Proof: The proof of the Theorem 5.3.6 is most similar to the proof of the Theorem 5.2.6.

Theorem 5.3.7: Every locally compact Hausdorff space is nearly regular space of the second kind.

Proof: The proof of the Theorem 5.3.7 of the above is almost similar to the proof of the Theorem 5.2.7.

Theorem 5.3.8: Let X be a T_1- space and x_0 be a point in X. X is nearly normal space of the second kind if and only if X is nearly regular space of the second kind.

Proof: The proof of the Theorem 5.3.8 is almost similar to the proof of the Theorem 5.2.8.

Theorem 5.3.9: Every metric space is nearly normal space of the second kind.

Proof: Since every metric space is normal, therefore, it is nearly normal space of the second kind.

CHAPTER SIX

Slightly Normal Topological Spaces of the First Kind and the Second Kind and the Third Kind

6.1. Introduction

Two types of generalizations of normal spaces different from those considered in the last chapter have been defined in this chapter. A topological space X in which a particular pair of disjoint closed subsets can be separated by disjoint open sets will be called a slightly normal space of the first kind. Generalizing this concept, we shall call a topological space X a slightly normal space of the second kind (third kind) if there is a finite (countable) collection of mutually disjoint closed subsets in X, for which each pair can be separated by disjoint open sets. We have studied these classes closely, and established a number of important properties of these spaces which resemble those of normal spaces.

We now define slightly normal spaces of the first kind and proceed to study them.

6.2. Slightly Normal Spaces of the First Kind

Definition 6.2.1: A topological space X will be called **slightly normal of the first kind (s. n. f. k.)** if there exist two disjoint nontrivial closed sets F_1, F_2 in X such that F_1 and F_2 can be separated by disjoint open sets. This space will be denoted by **(X; F_1, F_2).**

Example 6.2.1: Every normal space is **slightly normal space of the first kind**.

Example 6.2.2: Let $X = \mathbb{R}, \mathfrak{I} = \langle \mathbb{R}, \emptyset, (1,4), (1,4)^c, (5,7)^c \rangle$

Then the disjoint closed sets (1,4) and $(1,4)^c$ can be separated by disjoint open sets, but (1,4) and (5,7) are disjoint closed sets and these can't be separated by disjoint open sets. **Thus $(X, \mathfrak{I})$ is s. n. f. k. but not normal. Here X=$(X; (1,4), (1,4)^c)$.**

Comment 6.2.1: It is easy to see that there are infinitely many s. n. f. k. which are not normal.

Theorem 6.2.1: Every normal space is slightly normal space of the first kind but the converse is not true in general.

Proof: Let X be a normal space. Let F_1, F_2 be two disjoint nontrivial closed sets in X. Now, since X is normal, there exist disjoint open sets G_1, G_2 in X such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. Therefore X is slightly normal space of the first kind.

To see that the converse is always not true,

let $X = \{a,b,c,d,e\}$, $\mathfrak{I} = \{X, \Phi, \{a,b\}, \{a,b,e\}, \{e\}, \{a,b,c,d\}, \{b,c,d,e\}, \{b\}, \{b,e\}, \{b,c,d\}\}$.

Then $(X, \mathfrak{I})$ is a topological space in which the closed sets of X are X, Φ, $\{c,d,e\}, \{e\}, \{a,b,c,d\}, \{c,d\}$,{a},{a, c, d, e},{a, c, d},{a, b} .

The closed sets $\{a,b,c,d\}$ and {e} can be separated by $\{a,b,c,d\}$ and {e}, but the closed sets $\{c,d,e\}$ and {a} cannot be separated by disjoint open sets. Thus $(X, \mathfrak{I})$ is slightly normal space of the first kind but not normal.

Theorem 6.2.2: A topological space X is slightly normal space of the first kind if and only if there exist two disjoint nontrivial closed sets F_1, F_2 and an open set G such that $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2{}^c$.

Proof: First, suppose that X is slightly normal space of the first kind. Then there exist disjoint nontrivial closed sets F_1, F_2 and open sets G, H in X such that $F_1 \subseteq G$ and $F_2 \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq F_2{}^c$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq F_2{}^c$. Thus, $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2{}^c$.

Conversely, suppose that there exist disjoint nontrivial closed sets F_1, F_2 and an open set G in X such that $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2{}^c$. Here $F_1 \subseteq G$ and $F_2 \subseteq \overline{G}^c$. Let $\overline{G}^c = H$. Then H is open, $F_2 \subseteq H$ and $G \cap H = \phi$. Hence X is slightly normal space of the first kind.

Theorem 6.2.3: Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i \in I} X_i$ be the product space. If each X_i is slightly normal of the first kind, then X is slightly normal of the first kind.

Proof: Since each X_i is slightly normal of the first kind, there exist for each i, two nontrivial closed sets F_i, H_i and two open sets U_i, V_i in X_i such that $F_i \subseteq U_i$, $H_i \subseteq V_i$, $F_i \cap H_i = \phi$, $U_i \cap V_i = \phi$.

Let F= $\prod_{i \in I} F_i$, H= $\prod_{i \in I} H_i$. Then F and H are closed sets in X. Clearly, $F \cap H = \phi$. Let $U = \prod_{i \in I} U_i$, $V = \prod_{i \in I} V_i$. Then, U and V are open sets in X, and $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$. Therefore, X is slightly normal space of the first kind.

Theorem 6.2.4: Every open and one-one image of a slightly normal space of the first kind is slightly normal space of the first kind.

Proof: Let X be a slightly normal space of the first kind and Y a topological space and let $f: X \to Y$ be an open and onto mapping. Since X is slightly normal space of the first kind, there exist disjoint nontrivial closed sets F_1, F_2 and disjoint open sets G_1, G_2 in X such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. Since *f* is open, $f(F_1^c)$ and $f(F_2^c)$ are open in Y. So $(f(F_1^c))^c$ and $(f(F_2^c))^c$ are closed in Y.

Now, $F_1^c \cup F_2^c$=X and so $f(F_1^c \cup F_2^c)$ =Y, i.e., $f(F_1^c) \cup f(F_2^c)$=Y. Hence $(f(F_1^c))^c \cap (f(F_2^c))^c = \Phi$. Let $y \in (f(F_1^c))^c$. Then $y \notin f(F_1^c)$ i.e., for every $x \in F_1^c$, $f(x) \neq y$. Hence there exists $x_1 \in F_1$ such that $f(x_1) = y$, since f is onto. Thus $y \in f(F_1)$. Hence $(f(F_1^c))^c \subseteq f(F_1)$. Similarly, $(f(F_2^c))^c \subseteq f(F_2)$.

Now, $f(F_1) \subseteq f(G_1)$, $f(F_2) \subseteq f(G_2)$, f being open and one-one, $f(G_1)$, $f(G_2)$ are open and disjoint in Y. Thus for the disjoint nontrivial closed sets $(f(F_1^c))^c$, $(f(F_2^c))^c$ in Y and there exist disjoint open sets $f(G_1)$, $f(G_2)$ in Y such that $(f(F_1^c))^c \subseteq f(G_1)$ and $(f(F_2^c))^c \subseteq f(G_2)$. Hence Y is slightly normal space of the first kind.

Corollary 6.2.1: Every quotient space of a slightly normal space of the first kind is slightly normal space of the first kind.

Proof: Let X be a slightly normal space of the first kind and R is an equivalence relation on X. Since the projection map p:X$\to \dfrac{X}{R}$ is open and onto, the corollary then follows from the above Theorem 6.2.4.

Theorem 6.2.5: Let X be a slightly normal space of the first kind and Y is a subspace of X. Then Y is a slightly normal space of the first kind.

Proof: Since X is slightly normal space of the first kind, there exist disjoint nontrivial closed sets F_1, F_2 and disjoint open sets G_1, G_2 in X such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. Let $H_1 = Y \cap F_1$ and $H_2 = Y \cap F_2$. Then H_1, H_2 are closed in Y and $H_1 \cap H_2 = \Phi$. Also let $V_1 = Y \cap G_1$, $V_2 = Y \cap G_2$. Then $V_1 \cap V_2 = \Phi$ and $H_1 \subseteq V_1$, $H_2 \subseteq V_2$. Hence Y is slightly normal space of the first kind.

Remark 6.2.1: The corresponding theorem does not hold for normal spaces. The validity of the proof in Theorem 6.2.5 above depends on the separablity of a particular pair of disjoint closed spaces by disjoint open spaces (See Ex. of Munkres [10]).

Comment 6.2.1: A continuous image of a slightly normal space of the first kind need not be slightly normal space of the first kind.

For if (X, T_1) is a slightly normal space of the first kind) and (X, T_2) a space with the indiscrete topology, then the identity map $1_x : X \to X$ is continuous and onto. But (X, T_2) is not slightly normal space of the first kind.

Theorem 6.2.6: Each compact Hausdorff space is slightly normal space of the first kind.

Proof: Let X be a compact Hausdorff space and let A, B be two disjoint closed subsets of X. Let $x \in A$ and $y \in B$. Then $x \neq y$. Since X is Hausdorff, there exist disjoint open sets G_y and H_y such that $x \in G_y$ and $y \in H_y$. Obviously $\{H_y : y \in B\}$ is an open cover of B.

Since B is a closed subset of X, B is compact. So there exists a finite sub-cover $\{H_{y_1}, H_{y_2}, \ldots, H_{y_m}\}$ of B. Let $H_x = H_{y_1} \cup H_{y_2} \cup \ldots \cup H_{y_m}$ and $G_x = G_{y_1} \cap G_{y_2} \cap \ldots \cap G_{y_m}$. Then $B \subseteq H_x$, $x \in G_x$ and $H_x \cap G_x = \phi$. So for each $x \in A$ there exist two disjoint open sets G_x and H_x of X such that $x \in G_x$ and $B \subseteq H_x$. Hence $\{G_x : x \in A\}$ is an open cover of A. Since A is a closed subset of X, A is compact. So there exists a finite sub-cover $\{G_{x_1}, G_{x_2}, \ldots, G_{x_n}\}$ of this cover A. Let $G = G_{x_1} \cup G_{x_2} \cup \ldots \cup G_{x_n}$ and $H = H_{x_1} \cap H_{x_2} \cap \ldots \cap H_{x_n}$. Then G, H are open sets of X and $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$.

Theorem 6.2.7: Every metric space is slightly normal space of the first kind.

Proof: Since every metric space is normal, therefore it is slightly normal space of the first kind.

We now define slightly normal spaces of the second kind and proceed to study them.

6.3. Slightly Normal Spaces of the Second Kind

Definition 6.3.1: A topological space X will be called **slightly normal of the second kind (s. n. s. k)** if there exists a finite collection $\mathcal{F}$ of pairwise disjoint nontrivial closed sets in X such that, for each pair F_1, F_2 in $\mathcal{F}$, F_1 and F_2 can be separated by disjoint open sets in X. This space will be denoted by **(X,$\mathcal{F}$).**

Example 6.3.1:

Let $X = \mathbb{R}, \mathfrak{J} = \langle \mathbb{R}, \emptyset, \mathbb{Q}, \mathbb{Q}^c, (1,2), (1,2)^c, (3,4), (3,4)^c, \cdots, (15,16), (15,16)^c \rangle$

Let $\mathcal{F} = \{(1,2), (3,4), (5,6), \cdots, (15,16)\}$. Then $\mathcal{F}$ is a finite collection of pairwise disjoint nontrivial closed sets in X such that, for each distinct pair F_1, F_2 in $\mathcal{F}$, F_1 and F_2 can be separated by disjoint open sets, since each of these is open as well. **Thus $(X, \mathfrak{J})$ is s. n. s. k. However X is not normal,** for let A = $[(1,2) \cap \mathbb{Q}] \cup [(3,4) \cap \mathbb{Q}^c]$ and B = $[(1,2) \cap \mathbb{Q}^c] \cup [(3,4) \cap \mathbb{Q}]$. Then A and B are disjoint closed sets but they can't be separated by disjoint open sets.

Here X=$(\boldsymbol{X}; (\mathbf{1}, \mathbf{2}), (\mathbf{3}, \mathbf{4}), (\mathbf{5}, \mathbf{6}), \cdots, (\mathbf{15}, \mathbf{16}))$.

Comment 6.3.1: Obviously an infinite number of such examples can be constructed.

Theorem 6.3.1: A topological space X is slightly normal space of the second kind if and only if there exists a finite collection $\mathcal{F}$ of pairwise disjoint nontrivial closed sets F_1, F_2 and an open set G such that $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2^{\,c}$.

Proof: First, suppose that X is slightly normal space of the second kind. Then there exists a finite collection $\mathcal{F}$ of pairwise disjoint nontrivial closed sets such that for each pair F_1, F_2 in $\mathcal{F}$, there exist open sets G, H in X such that $F_1 \subseteq G$ and $F_2 \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq F_2{}^c$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq F_2{}^c$. Thus, $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2{}^c$.

Conversely, suppose that there exists a finite collection $\mathcal{F}$ of pairwise disjoint nontrivial closed sets such that for each pair F_1, F_2 in $\mathcal{F}$, there exist an open set G in X such that $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2{}^c$. Here $F_1 \subseteq G$ and $F_2 \subseteq \overline{G}^c$. Let $\overline{G}^c = H$. Then H is open, $F_2 \subseteq H$ and $G \cap H = \phi$. Hence X is slightly normal space of the second kind.

Theorem 6.3.2: Let $\{X_i\}_{i\in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i\in I} X_i$ be the product space. If each X_i is slightly normal of the second kind, then X is slightly normal of the second kind.

Proof: Since each X_i is slightly normal of the second kind, there exists for each i, a finite collection $\mathcal{F}$ of pairwise disjoint nontrivial closed sets such that for each pair F_i, H_i in $\mathcal{F}$, there exist open sets U_i, V_i in X_i such that $F_i \subseteq U_i$, $H_i \subseteq V_i$, $F_i \cap H_i = \phi$, $U_i \cap V_i = \phi$.

Let F= $\prod_{i\in I} F_i$, H= $\prod_{i\in I} H_i$. Then F and H are closed sets in X. Clearly, $F \cap H = \phi$. Let $U = \prod_{i\in I} U_i$, $V = \prod_{i\in I} V_i$. Then, U and V are open sets in X, and $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$. Therefore, X is slightly normal space of the second kind.

Theorem 6.3.3: Every open and one-one image of a slightly normal space of the second kind is slightly normal space of the second kind.

Proof: The proof of the Theorem 6.3.3 of the above is almost similar to the proof of the Theorem 6.2.4.

Corollary 6.3.1: Every quotient space of a slightly normal space of the second kind is slightly normal space of the second kind.

Proof: The proof of the Corollary 6.3.1 follows from the proof of the Corollary 6.2.1.

Theorem 6.3.4: Let X be a slightly normal space of the second kind and Y is a subspace of X. Then Y is a slightly normal space of the second kind.

Proof: The proof of the Theorem 6.3.4 is most similar to the proof of the Theorem 6.2.5.

Remark 6.3.1: The corresponding theorem does not hold for normal spaces. The validity of the proof in Theorem 6.3.4 above depends on the separablity of a particular pair of disjoint closed spaces by disjoint open spaces (See Ex. of Munkres [10]).

Comment 6.3.1: A continuous image of a slightly normal space of the second kind need not be slightly normal space of the second kind.

For if (X,T_1) is a slightly normal space of the second kind and (X,T_2) a space with the indiscrete topology, then the identity map $1_x : X \to X$ is continuous and onto. But (X,T_2) is not slightly normal space of the second kind.

Theorem 6.3.5: Each compact Hausdorff space is slightly normal space of the second kind.

Proof: The proof of the Theorem 6.3.5 of the above is almost similar to the proof of the Theorem 6.2.6.

We now define slightly normal spaces of the third kind and proceed to study them.

6.4. Slightly Normal Spaces of the Third Kind

Definition 6.4.1: A topological space X will be called **slightly normal of the third kind (s. n. t. k)** if there exists a countable collection $\mathcal{C}$ of pairwise disjoint nontrivial closed sets in X such that, for each pair F_1, F_2 in $\mathcal{C}$**,** F_1 and F_2 can be separated by disjoint open sets in X. This space will be denoted by **(X,$\mathcal{C}$).**

Example 6.4.1: Let $X = \mathbb{R}, \mathfrak{I} = \langle\{\mathbb{R}, \emptyset, \mathbb{Q}, \mathbb{Q}^c\} \cup \{(n, n+1), (n, n+1)^c | n \in \mathbb{N}\}\rangle$ Let $\mathcal{C} = \{(n, n+1) | n \in \mathbb{N}\}$. **Then (X,$\mathcal{C}$) is clearly s. n. t. k. But X is not normal,** for let A = [(1,2) ∩ $\mathbb{Q}$]∪[(2,3) ∩ $\mathbb{Q}^c$] and B = [(1,2) ∩ $\mathbb{Q}^c$] ∪[(2,3) ∩ $\mathbb{Q}$]. Then A and B are disjoint closed sets but they can't be separated by disjoint open sets.

Example 6.4.2:

Let $X = \mathbb{C}, \mathfrak{I} = \left\langle\{\mathbb{C}, \emptyset, \mathbb{Q}, \mathbb{Q}^c\} \cup \left\{\{D_n = \left\{z \in \mathbb{C} \middle| |z - n| < \frac{1}{3}\right\}, D_n^c\} | n \in \mathbb{N}\right\}\right\rangle$

Let $\mathcal{C} = \{D_n | n \in \mathbb{N}\}$. Then $\mathcal{C}$ is a countable collection of pairwise disjoint closed sets such that, for each pair D_{n_1} and D_{n_2} $(n_1 \neq n_2)$ can be separated by disjoint open sets since each D_n is both open and closed. **Hence X is s. n. t. k. However X is not normal** since $\mathbb{Q} \cap D_1$ and $\mathbb{Q}^c \cap D_1$ are disjoint closed sets which can't be separated by disjoint open sets.

Theorem 6.4.1: A topological space X is slightly normal space of the third kind if and only if there exists a countable collection $\mathcal{C}$ of pairwise disjoint nontrivial closed sets F_1, F_2 and an open set G such that $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2^{\,c}$.

Proof: First, suppose that X is slightly normal space of the third kind. Then there exists a countable collection $\mathcal{C}$ of pairwise disjoint nontrivial closed sets such that for each pair F_1, F_2 in $\mathcal{C}$, there exist open sets G, H in X such that $F_1 \subseteq G$ and $F_2 \subseteq H$ and $G \cap H = \phi$. It follows that $G \subseteq H^c \subseteq F_2^{\,c}$. Hence $G \subseteq \overline{G} \subseteq H^c \subseteq F_2^{\,c}$. Thus, $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2^{\,c}$.

Conversely, suppose that there exists a countable collection $\mathcal{C}$ of pairwise disjoint nontrivial closed sets such that for each pair F_1, F_2 in $\mathcal{C}$, there exist an open set G in X such that $F_1 \subseteq G \subseteq \overline{G} \subseteq F_2^{\,c}$. Here $F_1 \subseteq G$ and $F_2 \subseteq \overline{G}^c$. Let $\overline{G}^c = H$. Then H is open, $F_2 \subseteq H$ and $G \cap H = \phi$. Hence X is slightly normal space of the third kind.

Theorem 6.4.2: Let $\{X_i\}_{i \in I}$ be a non-empty family of topological spaces, and let X= $\prod_{i \in I} X_i$ be the product space. If each X_i is slightly normal of the third kind, then X is slightly normal of the third kind.

Proof: Since each X_i is slightly normal of the third kind, there exists for each i, a countable collection $\mathcal{C}$ of pairwise disjoint nontrivial closed sets such that for each pair F_i, H_i in $\mathcal{C}$, there exist open sets U_i, V_i in X_i such that $F_i \subseteq U_i$, $H_i \subseteq V_i$, $F_i \cap H_i = \phi$, $U_i \cap V_i = \phi$.

Let F= $\prod_{i \in I} F_i$, H= $\prod_{i \in I} H_i$. Then F and H are closed sets in X. Clearly, $F \cap H = \phi$. Let $U = \prod_{i \in I} U_i$, $V = \prod_{i \in I} V_i$. Then, U and V are open sets in X, and $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$. Therefore, X is slightly normal space of the third kind.

Theorem 6.4.3: Every open and one-one image of a slightly normal space of the third kind is slightly normal space of the third kind.

Proof: The proof of the Theorem 6.4.3 is most similar to the proof of the Theorem 6.2.4.

Corollary 6.4.1: Every quotient space of a slightly normal space of the third kind is slightly normal space of the third kind.

Proof: The proof of the Corollary 6.4.1 of the above is almost similar to the proof of the Corollary 6.2.1.

Theorem 6.4.4: Let X be a slightly normal space of the third kind and Y is a subspace of X. Then Y is a slightly normal space of the third kind.

Proof: The proof of the Theorem 6.4.4 follows from the proof of the Theorem 6.2.5.

Remark 6.4.1: The corresponding theorem does not hold for normal spaces. The validity of the proof in Theorem 6.4.4 above depends on the separablity of a particular pair of disjoint closed spaces by disjoint open spaces (See Ex. of Munkres [10]).

Comment 6.4.1: A continuous image of a slightly normal space of the third kind need not be slightly normal space of the third kind.

For if (X, T_1) is a slightly normal space of the third kind and (X, T_2) a space with the indiscrete topology, then the identity map $1_x : X \to X$ is continuous and onto. But (X, T_2) is not slightly normal space of the third kind.

Theorem 6.4.5: Each compact Hausdorff space is slightly normal space of the third kind.

Proof: The proof of the Theorem 6.4.5 is most similar to the proof of the Theorem 6.2.6.

CHAPTER SEVEN

Pseudo-Compact Spaces, C-Compact Spaces and Pseudo-Continua

7.1. Introduction

In this chapter two generalizations of compact spaces have been considered. Such spaces have been called pseudo-compact and c-compact. Another generalization of compact spaces viz., H-closed spaces has been introduced and studied ([18], [35]) a number of years ago.

We have shown that both pseudo-compact spaces and c-compact spaces are distinct from each of compact spaces and H-closed spaces. Properties of pseudo-compact spaces and c-compact spaces have been studied here.

Definitions of a number of generalized connectedness viz., locally connectedness, connectedness imkleinen, path-connectedness, locally path-connectedness, Cantor's connectedness have been given. Generalizing a continuum i.e., a connected compact space, we have defined a pseudo-continuum. Some properties of pseudo-continua have been proved.

7.2. Pseudo-Compact Spaces

Definition 7.2.1: A collection $\{G_\alpha\}$ will be called **pseudo-open-cover** of a topological space X if $\{\overline{G_\alpha}\}$ covers X.

X will be called **pseudo-compact** if every pseudo-open-cover of X has a finite sub-pseudo-cover.

A pseudo-open-cover need not be an open cover, as the following example shows:

Example 7.2.1: Let X=ℝ, and $\mathfrak{I}$ = the topology generated by $\{(x, x+1) | x \in \mathbb{Z}\}$. Then $\{(x, x+1) | x \in \mathbb{Z}\}$ is **not an open cover** of X, but it is a **pseudo-open-cover** of X, since $\overline{(x, x+1)} = [x, x+1]$ for all $x \in \mathbb{Z}$ and $\bigcup_{x \in Z} [x, x+1] = X$.

Example 7.2.2: Let X= [0,1] and $\mathfrak{I} = \{X, \emptyset, (0,1)\}$ be a topology on X. Then $\{(0,1)\}$ is the only pseudo-open cover of X. Since this is finite, X is **pseudo-compact.**

Definition 7.2.2: A topological space X is **H-closed** (Gangully and Jana [35]) if every open cover $\{G_\alpha\}$ of X has a finite sub-collection $\{G_{\alpha_1}, \ldots, G_{\alpha_n}\}$ such that $\overline{G_{\alpha_1}} \cup \ldots \cup \overline{G_{\alpha_n}} = X$.

An H-closed space need neither be pseudo-compact nor be compact. The following example proves the truth of this statement:

Example 7.2.3: Let X=ℝ, and $\mathfrak{I}$ = the topology generated by $\{\{\mathbb{Q}\} \cup \{(x, x+1) | x \in \mathbb{Z}\}\}$. Then , for each open cover $\mathcal{C}$ of X, either C contains {ℝ} or $\mathcal{C} = \{\mathbb{Q}\} \cup \{(x, x+1) | x \in \mathbb{Z}\}$, or $\mathcal{C} = \{\mathbb{Q} \cup \{\cup\, (x, x+1) | x \in \mathbb{Z}\}\}$.

Since $\overline{\mathbb{R}} = \mathbb{R} = X$ and $\overline{\mathbb{Q}} = \mathbb{R} = X$, (X, $\mathfrak{I}$) is **H-closed**.

Now, $\bigcup_{x \in Z}\{\overline{(x, x+1)} = [x, x+1]\} = \mathbb{R} = X$. Hence $\{(x, x+1) | x \in \mathbb{Z}\}$ is a pseudo-open-cover of X. But it does not have a finite sub-pseudo-cover. Hence X is **not pseudo-compact**. Also, since $\mathcal{C}$ does not have a finite subcover, X is **not compact**.

A pseudo-compact space may not be compact. This is illustrated by the following example:

Example 7.2.4: Let $X = \mathbb{R} \cup \{i\}, i = \sqrt{-1}$, and let $\mathfrak{I}$ be the topology generated by $\{\{(x, x+1) | x \in \mathbb{Z}\} \cup \{\mathbb{Q} \cup \{i\}\}\}$. Then $\mathcal{C} = \{(x, x+1) | x \in \mathbb{Z}\} \cup \{\mathbb{Q} \cup \{i\}\}$ is a pseudo-open-cover of X. For, $\bigcup_{x \in Z}\{\overline{(x, x+1)} \mid x \in Z\} \cup \{\overline{Q \cup \{i\}}\} = \bigcup_{x \in Z}\{[x, x+1] \mid x \in Z\} \cup \{\mathbb{R} \cup \{i\}\} = \mathbb{R} \cup \{\mathbb{R} \cup \{i\}\} = \mathbb{R} \cup \{i\} = X$.

Any other pseudo-open-cover of X must contain $\mathbb{Q} \cup \{i\}$ or X as its member. Hence every pseudo-open-cover of X has a finite sub-pseudo-cover, viz., $\{\mathbb{Q} \cup \{i\}\}$, or $\{X\}$, or $\{\{\mathbb{Q} \cup \{i\}\}, X\}$. So, X is **pseudo-compact**.

However, $\mathcal{C}$ is an open cover of X but $\mathcal{C}$ does not have a finite sub-cover. Thus, X is **not compact**.

A compact space need not be pseudo-compact. This is shown by the following example:

Example 7.2.5: Let X =[0,1] and let $\mathfrak{I}$ be the topology on X which is induced by the usual topology on $\mathbb{R}$. Then X is **compact** by Heine-Borel Theorem. We shall show that X is not pseudo-compact.

Let $\mathcal{C} = \left\{\left(0, \frac{1}{2}\right)\right\} \cup \{(\frac{1}{2} + \frac{1}{2^{n+1}}, 1) | n \in \mathbb{N}\}$ and

$\mathcal{D} = \left\{[0, \frac{1}{2}]\right\} \cup \{[\frac{1}{2} + \frac{1}{2^{n+1}}, 1] | n \in \mathbb{N}\}$

$= \left\{\overline{V} \mid V \in C\right\}$.

Then $\bigcup_{V \in C} \overline{V} = [0,1] = X$.

Thus $\mathcal{C}$ is pseudo-open-cover of X. But it does not have a finite sub-pseudo-cover. Hence X is **not pseudo-compact**.

Theorem 7.2.1: Every pseudo-compact space is H-closed.

Proof: Let X be a pseudo-compact space. Let $\{G_\alpha\}$ be a pseudo-open-cover of X. So $\{\overline{G_\alpha}\}$ covers X. Since X is pseudo-compact, there exists a finite sub-collection $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ of X such that $\overline{G_{\alpha_1}} \cup \dots \cup \overline{G_{\alpha_n}} = X$. Hence X is H-closed.

Theorem 7.2.2: Every pseudo-compact subspace of a completely Hausdorff space is closed.

Proof: Let X be a completely Hausdorff space and K be a pseudo-compact subspace of X. We show that K^c is open. Let $x \in K^c, y \in K$. Then $x \neq y$. Since X is completely Hausdorff, so there exist open sets G_y, H_y such that $x \in G_y, y \in H_y$ and $\overline{G_y} \cap \overline{H_y} = \phi$. Clearly $\{H_y : y \in K\}$ is a pseudo-open cover of K. Since K is pseudo-compact, so there exists $y_1, \dots, y_n \in K$ such that $K \subseteq \overline{H_{y_1}} \cup \dots \cup \overline{H_{y_n}}$. Since $\left(\overline{G_{y_1}} \cap \dots \cap \overline{G_{y_n}}\right) \cap (\overline{H_{y_1}} \cup \dots \cup \overline{H_{y_n}}) = \phi$, $\overline{G_{y_1}} \cap \dots \cap \overline{G_{y_n}} \subseteq K^c$. So $G_{y_1} \cap \dots \cap G_{y_n} \subseteq K^c$. Since $G_{y_1} \cap \dots \cap G_{y_n}$ is open and $x \in G_{y_1} \cap \dots \cap G_{y_n}$. So K^c is open.

Theorem 7.2.3: A continuous image of a pseudo-compact space is pseudo-compact.

Proof: Let X be a pseudo-compact space, Y a topological space and let $f: X \to Y$ be a continuous mapping. We show that as a subspace of Y, f(X) is pseudo-compact. Let $\{G_\alpha\}$ be a pseudo-open-cover of f(X). Since f is continuous, $\{f^{-1}(G_\alpha)\}$ is a pseudo-open cover of X. Therefore $\overline{\{f^{-1}(G_\alpha)\}}$ covers X.

Now, we shall show that $\cup\overline{f^{-1}(G_\alpha)} = \cup f^{-1}\left(\overline{G_\alpha}\right)$. The continuity of f implies $\overline{f^{-1}(G_\alpha)} \subseteq f^{-1}\left(\overline{G_\alpha}\right)$. Let $x \in f^{-1}\left(\overline{G_\alpha}\right)$. Then $f(x) \in \overline{G_\alpha}$. Hence for every neighborhood V of f(x), $V \cap G_\alpha \neq \phi$. Let U be any neighborhood of x. Then $f(x) \in f(U)$. Now, $f(U) \cap ff^{-1}(G_\alpha) = f\left(U \cap f^{-1}(G_\alpha)\right)$ and $f(U) \cap ff^{-1}(G_\alpha) \neq \phi$, since f is open. Therefore, $f\left(U \cap f^{-1}(G_\alpha)\right) \neq \phi \Rightarrow U \cap f^{-1}(G_\alpha) \neq \phi$. Hence $x \in \overline{f^{-1}(G_\alpha)}$.

Since X is pseudo-compact, so there exists a finite sub-pseudo-cover $\{f^{-1}(G_{\alpha_1}), \ldots, f^{-1}(G_{\alpha_n})\}$ of $\{f^{-1}(G_\alpha)\}$ such that $\overline{f^{-1}(G_{\alpha_1})} \cup \ldots \cup \overline{f^{-1}(G_{\alpha_n})} = f^{-1}\left(\overline{G_{\alpha_1}}\right) \cup \ldots \cup f^{-1}\left(\overline{G_{\alpha_n}}\right) \subseteq X$. Hence $f(X) \subseteq \overline{G_{\alpha_1}} \cup \ldots \cup \overline{G_{\alpha_n}}$. Therefore f(X) is pseudo-compact.

Theorem 7.2.4: A closed subspace of a pseudo-compact space is pseudo-compact.
Proof: The proof can be constructed exactly as in the case of compact spaces.

Theorem 7.2.5: (Generalizations of Theorem 4.1.16, [3]) Let X and Y be pseudo-compact topological spaces. Then the product space $X \times Y$ is pseudo-compact.
Proof: The proof is similar to that of Theorem 4.1.16 [3]. Still, we are writing the proof for completeness. For each $x_0 \in X, y \to (x_0, y)$ is a surjective continuous function and Y is a pseudo-compact space implies $x_0 \times Y$ is a pseudo-compact subset of $X \times Y$. Let $\mathcal{C}$ be a collection of basic open sets in $X \times Y$ such that $X \times Y = \bigcup_{U \times V \in C} \overline{U \times V}$

.

This implies that $x_0 \times Y \subseteq \bigcup_{U\times V \in C} \overline{U \times V}$ implies there exist $U_1 \times V_1, \dots, U_n \times V_n$ such that $x_0 \times Y \subseteq \overline{(U_1 \times V_1)} \cup \dots \cup \overline{(U_n \times V_n)}$ ....(1)

Also, if for some i, $(U_i \times V_i) \cap (x_0 \times Y) = \phi$, then we do not require to include such an $U_i \times V_i$ in our finite sub-cover $\{U_i \times V_i\}, i = 1,2, \dots, n$. So we may assume that each $(U_i \times V_i) \cap (x_0 \times Y) \neq \phi$. This implies that $x_0 \in U_i$ for all i=1,2, ⋯ ,n and hence $x_0 \in W_{x_0} = \bigcap_{i=1}^{n} U_i$. Now it is clear that $W_{x_0} \times Y \subseteq \overline{(U_1 \times V_1)} \cup \dots \cup \overline{(U_n \times V_n)}$. Consider $(x, y) \in W_{x_0} \times Y$. Then $x \in U_i$ for all i and $y \in Y$. Hence from equation (1) $(x_0, y) \in \overline{U_j \times V_j}$ for some j. This implies $(x, y) \in \overline{U_j \times V_j}$ for the same j. That is for each $x_0 \in X$, the tube $W_{x_0} \times Y$ is covered by the closures of finitely many members of $\mathcal{C}$. So, $\overline{W_{x_0} \times Y}$ too is covered by the closures of finitely many members of $\mathcal{C}$. Since $\overline{W_{x_0}} \times \mathrm{Y} \subseteq \overline{W_{x_0} \times Y}$, the same is true for about $\overline{W_{x_0}} \times \mathrm{Y}$.

Now, we shall prove that $X \times Y$ is covered by the closure of finitely many such tubes $W_x \times Y$. Now $\{W_x : x \in X\}$ is a pseudo-open-cover for X. Hence X is a pseudo-compact space implies there exist $x_1, x_2, \dots, x_k \in X$ such that $X = \bigcup_{i=1}^{k} \overline{W_{x_i}}$. Now $(x, y) \in X \times Y \Rightarrow x \in \overline{W_{x_i}}$ for some i, $1 \le i \le k$ and hence $(x, y) \in \overline{W_{x_i}} \times Y$. This implies that $X \times Y = \bigcup_{i=1}^{k} \overline{\mathrm{W}_{\mathrm{x}_i}} \times Y$ and hence $X \times Y$ is covered by the closure of finitely many members of $\mathcal{C}$. This proves that $X \times Y$ is pseudo-compact.

Corollary 7.2.1: If X_1, ⋯ ,X_n are pseudo-compact topological spaces. Then the product space $X_1 \times \dots \times X_n$ is pseudo-compact.

Proof: It follows from the above Theorem 7.2.5 by induction.

Theorem 7.2.6: Let X be a topological space. Let A, B be pseudo-compact subspaces of X. Then $A \cup B$ is pseudo-compact.

Proof: Let $\mathcal{C} = \{C_\alpha\}$ be a pseudo-open-cover of $A \cup B$. Then $\mathcal{C}$ is a pseudo-open-cover of both A and B. Since A and B are pseudo-compact then $\mathcal{C}$ contains finite sub-pseudo-covers $\mathcal{C}_1$ and $\mathcal{C}_2$ of A and B respectively. Then $\mathcal{C}_1 \cup \mathcal{C}_2$ is a finite sub-pseudo-cover of $\mathcal{C}$ and, $\mathcal{C}_1 \cup \mathcal{C}_2$ covers $A \cup B$. Hence $A \cup B$ is pseudo-compact.

Corollary 7.2.2: If X is a topological space and $A_1, \cdots, A_n$ are pseudo-compact subspaces of X. Then $A_1 \cup \cdots \cup A_n$ is pseudo-compact.

Proof: It follows from the above Theorem 7.2.6 by induction.

7.3. C-Compact Spaces

Definition 7.3.1: A topological space X is called **c-compact** if every closed cover of X has a finite sub-cover.

Example 7.3.1: Let X=ℝ, and $\mathfrak{I}$ = the topology generated by the collection $\{(x, x+1) | x \in \mathbb{Z}\} \cup \{\mathbb{Z}\}$ consider $(x, x+1)$ as closed sets.

Let A be a subset of ℝ containing finite numbers of integers, say $x_1, x_2, \ldots\ x_n$ where $x_1 < x_2 < \cdots < x_n$ and bounded by x_1 and x_n.

Let $\mathcal{C}$ be a closed cover of A. Then $\mathcal{C} = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)\} \cup \{\mathbb{Z}\}$. Then every closed cover of A is finite. Thus A is **c-compact.**

Clearly, many such examples can be constructed.

We shall now give below an example of a topological space which is **compact but not c-compact:**

Example 7.3.2: Let X=[0,1] , and $\mathfrak{I}$ = the topology on X which is induced by the usual topology on ℝ . Then X is **compact** by Heine-Borel Theorem.

However, X is **not c-compact**. For, if

$\mathcal{C} = [0, \frac{1}{2}] \cup \{[\frac{1}{2} + \frac{1}{2^{n+1}}, 1] | n \in \mathbb{N}\}$. Then $\mathcal{C}$ is a closed cover of X, but it does not have a finite sub-cover. To see that $\mathcal{C}$ is a closed cover of X, we note that

$$[0, \frac{1}{2}] \cup \{ \bigcup_{n=1}^{\infty} \{[\frac{1}{2} + \frac{1}{2^{n+1}}, 1] | n \in \mathbb{N}\}\}$$

$=[0, \frac{1}{2}] \cup (\frac{1}{2}, 1] = [0,1]$.

Theorem 7.3.1: Every open subspace of a c-compact space is c-compact.

Proof: Let X be a c-compact space and K be an open subset of X. If $\{F_\alpha\}$ is a closed cover of K, we can take the collection $\{X-K\}\cup\{F_\alpha\}$ as a closed cover of X. Since X is c-compact, so there exist a finite sub-collection $\{X-K, F_{\alpha_1}, \dots, F_{\alpha_n}\}$ of X such that $X = X - K \cup F_{\alpha_1} \cup \dots \cup F_{\alpha_n}$. So we have a finite sub-collection $\{F_{\alpha_1}, \dots, F_{\alpha_n}\}$ such that $K \subseteq F_{\alpha_1} \cup \dots \cup F_{\alpha_n}$. Hence K is c-compact.

Theorem 7.3.2: A continuous image of a c-compact space is c-compact.

Proof: Let X be a c-compact space, Y a topological space and let $f : X \to Y$ be a continuous mapping. We show that as a subspace of Y, f(X) is c-compact. Let $\{F_\alpha\}$ be a closed cover of f(X). Since f is continuous, $\{f^{-1}(F_\alpha)\}$ is a closed cover of X. Since f is continuous, $\{f^{-1}(F_\alpha)\}$ is a closed cover of X. Since X is c-compact, so there exists a finite sub-cover $\{f^{-1}(F_{\alpha_1}), \dots, f^{-1}(F_{\alpha_n})\}$ of $\{f^{-1}(F_\alpha)\}$. Hence $\{F_{\alpha_1}, \dots, F_{\alpha_n}\}$ is a finite sub-cover of $\{F_\alpha\}$. Therefore f(X) is c-compact.

Theorem 7.3.3: Every c-compact space is pseudo-compact.

Proof: Let X be a c-compact space and let $\mathcal{C}$ be a collection of open sets in X such that $\mathcal{F} = \{\overline{G_\alpha} : G_\alpha \in \mathcal{C}\}$ is a cover of X. Thus $\mathcal{C}$ is a pseudo-open-cover of X. Then $\mathcal{F}$ is a closed cover of X. X being c-compact, $\mathcal{F}$ has a finite sub-cover viz., $\{\overline{G_{\alpha_1}}, \cdots, \overline{G_{\alpha_n}}\}$. Hence $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ is a pseudo-sub-cover of X. So X is pseudo-compact.

Theorem 7.3.4: Let X and Y be c-compact topological spaces. Then the product space $X \times Y$ is c-compact.

Proof: Since every c-compact space is pseudo-compact by above Theorem 7.3.3, the proof can be constructed exactly as that of the Theorem 7.2.5.

Corollary 7.3.1: If $X_1, \cdots, X_n$ are c-compact topological spaces. Then the product space $X_1 \times ... \times X_n$ is c-compact.

Proof: It follows from the above Theorem 7.3.4 by induction.

7.4. Pseudo-Continua

Definition 7.4.1: ((Majumdar and Akhter [22], Munkres [10]) Let X be a topological space. X is said to be **connected** if it can not be expressed as the union of a pair of disjoint nonempty open subsets of X.

Definition 7.4.2: ((Majumdar and Akhter [22], Munkres [10]) A topological space X is said to be **locally connected at x** if every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its point, it is said simply to be **locally connected.**

Definition 7.4.3: ((Majumdar and Akhter [22], Munkres [10]) Given points x and y of the space X, a **path** in X from x to y is a continuous map $f:[a,b]\to X$ of some closed interval in the real line into X, such that f(a)=x and f(b)=y. A space X is said to be **path connected** if every pair of points of X can be joined by a path in X.

Definition 7.4.4: ((Majumdar and Akhter [22], Munkres [10]) A topological space X is said to be **locally path connected at x** if every neighborhood U of x, there is a path connected neighborhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be **locally path connected.**

Definition 7.4.5: (Majumdar and Akhter [22]) **Cantor** gave a special definition of **connectedness** for metric spaces. According to him, a metric space X is connected if for any two distinct points a, b of X and for any $\in > 0$, there exists a finite sequence $a = x_1, x_2, \cdots, x_n = b$ of X such that $d(x_i, x_{i+1}) < \in$, $i = 1, \cdots, n-1$.

Definition 7.4.6: (Majumdar and Akhter [22]) A topological space X is said to be **connected imkleinen** at a point $x \in X$ if for every open set U containing x, there is an open set V containing x such that for every $y \in V$, there exists a connected subset C such that $x, y \in C$.

Definition 7.4.7: (Hocking and Young [13], p.43) A connected compact space is called a **continuum**.

Definition 7.4.8: A connected pseudo-compact space will be called a **pseudo-continuum**.

Example 7.4.1: Let X= [0,1] and $\mathfrak{I} = \{X, \emptyset, (0,1)\}$ be the topology on X. Then {(0,1)} is the only pseudo-open cover of X. Since this is finite, X is **pseudo-compact.** Clearly, X is connected. Hence X is a **pseudo-continuum**.

Theorem 7.4.1: A continuous image of a pseudo-continuum is a pseudo-continuum.

Proof: Since every continuous image of a pseudo-compact space is pseudo-compact [Theorem 7.2.3] and every continuous image of a connected space is connected (Theorem 23.5, Munkres [10], Theorem 1.6, p.72, Majumdar and Akhter [22]), the theorem follows.

We recollect the following theorem on connectedness:

Theorem 7.4.2: (Theorem 1.4, p.71, Majumdar and Akhter [22], Theorem 23.3, Munkres [10]) Let $\{A_i\}$ be a collection of connected subsets of a topological space X. If $\bigcap_i A_i$ is non-empty, then $\bigcup_i A_i$ is connected.

So we have

Theorem 7.4.3: Let X be a topological space. Let A and B be two subspaces of X such that

(i) A, B are pseudo-continua

(ii) $A \cap B \neq \emptyset$.

Then $A \cup B$ is pseudo-continuum.

Proof: Since A, B are pseudo-continua, Theorem 7.2.6 implies that $A \cup B$ is pseudo-compact and since $A \cap B \neq \emptyset$, Theorem 7.4.2 implies that $A \cup B$ is connected. Thus $A \cup B$ is pseudo-continuum.

Corollary 7.4.1: If X is a topological space and $A_1, \cdots, A_n$ are subspaces of X such that

(i) $A_1, A_2, \cdots, A_n$ are pseudo-continua

(ii) $A_1 \cap \cdots \cap A_n \neq \emptyset$.

Then $A_1 \cup \cdots \cup A_n$ is pseudo-continuum.

Proof: It follows from the above Theorem 7.4.3 by induction.

Theorem 7.4.4: Let X and Y be pseudo-continua. Then the product space $X \times Y$ is pseudo-continuum.

Proof: Since product of two pseudo-compact spaces is pseudo-compact [Theorem 7.2.5] and product of two connected space is connected (Theorem 23.6, Munkres [10], Theorem 1.9, p.73, Majumdar and Akhter [22]), the theorem follows.

Corollary 7.4.2: If $X_1, \cdots, X_n$ are pseudo-continua. Then the product space $X_1 \times \dots \times X_n$ is pseudo-continuum.

Proof: It follows from the above Theorem 7.4.4 by induction.

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