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A Study on Fixed Point Iterative Procedures

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University of Rajshahi

M. PHIL.
THESIS

A Study on Fixed Point Iterative Procedures

M. PHIL. THESIS

This thesis is submitted in partial fulfillment of the requirements for the degree of
Master of Philosophy in Mathematics

Submitted By

Miss. Saleha Khatun

To

The Department of Mathematics
University of Rajshahi
Rajshahi-6025, Bangladesh

June, 2016

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CERTIFICATE

I have the pleasure to certify that the Master of Philosophy thesis entitled **"A Study on Fixed Point Iterative Procedures"** submitted by Miss. Saleha Khatun in fulfillment of the requirement for the degree of Master of Philosophy in Mathematics has been completed under my supervision. I believe that the research work is an original one and it has not been submitted elsewhere for any kind of degree.

I wish him a bright future and every success in life.

....

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STATEMENT OF ORIGINALITY

I declare that the works in my Master of Philosophy thesis entitled **“A Study on Fixed Point Iterative Procedures”** is original and accurate to the best of my knowledge. I also declare that the materials contained in my research work have not been written by any person for any kind of degree.

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June, 2016 The Author

CONTENTS

ABSTRACT

Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach's fixed point theorem. Fixed point iterative procedures are one of the early achievements of fixed point theory for their usefulness to construct the solving technique of different nonlinear problems. Most of the physical problems of applied sciences and engineering are usually formulated as functional equations. Such equations can be written in the form of fixed point equations in an easy manner. It is always desired to develop an iterative procedure which approximates the solution of these equations in fewer numbers of steps. From this point of view, the main objective of our research is to fit a best fixed point iterative procedure whose working ability (rate of convergence) is better than that of the analogous fixed point iterative procedures. There exist a numeral number of fixed point iterative procedures in literature. But there raised a natural question that, "Which is the best fixed point iterative procedure under the equivalent situation?". Tofind the answer of that question already many works have been completed by various renowned researchers; see for instance [12, 27, 37, 39, 47, 49] and their references. By the inspiration of these works here we have proposed a new three-step fixed point iterative procedure whose rate of convergence is better than that of analogous fixed point iterative procedures in case of contraction mapping. Using our new fixed point iterative procedure we have also established some weak and strong convergence theorems for non-expansive mapping andwe applythese results to find the solutions of constrained minimization problems and feasibilityproblems. In the last part of our research, we have studied the fixed point iterative procedures with errors and proveda convergence theorem of multi-step Noor fixed point iterative procedure with errors for Zamfirescu operator, which generates the convergence theorems of rest fixed point iterative procedures with errors for the same operator.

INTRODUCTION

Many problems of physical interest are converted into nonlinear problems that can be solved precisely by fixed point iterative procedure; this means a suitable fixed point iterative procedure play a vital role to solve these types of nonlinear problems. The study of variational inequality and complementarity problems of mappingssatisfying certain constraints has been at the center of rigorous research activity.Given the fact, complementarity and variational inequality problems which areextremely useful in optimization theory can be found by solving an equationwith some special form of nonlinear function.It is very important to developsome faster iterative procedure to find the approximate solution.From theseperceptive, here we introduce a newthree-step fixed point iterative procedure and prove that it is faster than Picard [48], Mann [53], Ishikawa [40], Noor [24], Agarwal*et al.* [37], Abbas *et al.* [27] and Thakur*et al.* [12] iterative procedures.

The celebrated Banach's fixed point theorem established by Polish mathematician Stefan Banach in 1922 [37] is one of the most useful results in fixed point theory. It has many advantages to solve nonlinear problems, but it has one weakness, that it is applicable for continuous mappingonly. To overcome this situation W. R. Mann [53] invented a new iterative procedure in 1953. Later, in 1974 S. Ishikawa [40] devised a new iterative procedure to establish the convergence of Lipschitzian pscudocontractive map when Mann iteration scheme failed to converge. After S. Ishikawa in 2000, M.A. Noor [24] introduced and analyzed three-step Noor iterative procedure to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces when Ishikawa Iterative procedure failed.Afterwards many iterative procedures have been formulated to solve many complex problems; see for instance the monograph of V. Berinde [49]. There are many research works in literature on the approximation of fixed points of contraction mapping and non-expansive mappingby using Mann's iterative procedure [53], Ishikawa's iterative procedure [40] , Noor's iterative procedure [24], Agarwal *et al.*'s iterative procedure [37], Abbas *et al.*'s iterative procedure [27] and recently Thakur *et al.*'s iterative procedure [12] have also been studied in the same purposes, see for instance [8, 12, 14, 15, 21, 24, 27, 31, 37, 40, 41, 44, 45, 47, 57, 58, 61] and their references. From this continuation, in this work, we have proposed a new three-step iterative procedure whose rate of convergence is better than that of above mentioned iterative procedures. Besides this work here we have also studied the iterative procedures with errors and established a convergence theorem of multi-step Noor fixed point iterative procedure with errors [42] for more general Zamfirescu operator [46] which generates the convergence theorem of other fixed point iterative procedure with errors. Throughout this thesis $\mathbb{N}$ will denote the set of natural numbers and $\mathbb{R}$ will denote the set of real numbers.

For convenience of discussion we divide our total work into seven individual chapters. In the Chapter-1, we have presented some known definitions and some fundamental results of fixed point theory, which have used as the tools of our main work.

In the Chapter-2 we introduced different fixed point iterative procedures. Here we have also proved the convergence theorem and stability theorem of our new iterative procedure by using Zamfirescu operator.

Chapter-3 is one of the main parts of our research. In this chapter, we have shown that the rate of convergence of our new iterative procedure is fastest than that of all comparable iterative procedures for contraction mapping by using the sense of V. Briende [44] in all ways of comparison thinking.

In the Chapter-4 we established some weak and strong convergence theorems by using our new iterative procedure under different well-defined conditions.

In the Chapter-5 wepresented different fixed point iterative procedure with errors.

Chapter-6 is another principal part of our research. Here we have established a principle of convergence of multi-step Noor fixed point iterative procedure for more general zamfirescu operator which is able to generate the principle of convergence of other fixed point iterative procedures with errors.

Chapter-7 is the final chapter of this thesis, which is furnished with application andconclusion.

LIST OF FIGURES AND THEIR SIGNIFICANCE

CHAPTER-1

MATHEMATICAL PRELIMINARIES AND SOME FUNDAMENTAL RESULTS

The main aim of this Chapter is to state some basic definitions and some fundamental results of fixed point theory to keep this thesis in sequence and for the convenience of references. For the conciseness, all of the theorems are stated without proof.

1.1 Some basic definitions

Definition 1.1.1.[55]A **vector space or linear space**is a set$X$ together with two operations, addition and scalar multiplication such that for all $x, y \in X$and all $\alpha \in \mathbb{R}$(set of real numbers) both $x + y$ and $\alpha x$are in$X$, and for all $x, y, z \in X$and all $\alpha, \beta \in \mathbb{R}$the following properties are satisfied:

(i) $x + y = y + x$;

(ii) $(x + y) + z = x + (y + z)$;

(iii) $(\alpha + \beta)x = \alpha x + \beta x$;

(iv) $\alpha(x + y) = \alpha x + \alpha y$;

(v) $\alpha(\beta x) = (\alpha\beta)x$;

(vi) there exists a $0 \in X$such that for all $x^/ \in X,\ 0 + x^/ = x^/$;

(vii) there exists a $-x \in X$such that for all $x \in X,\ x + (-x) = 0$;

(viii**)** there exists a $1 \in \mathbb{R}$such that for all $x \in X, 1 \cdot x = x$.

Definition 1.1.2.[54] Let X be a nonempty set. The mapping $d: X \times X \to \mathbb{R}$is said to be a **metric** on X if it satisfies the following conditions:

(i) $d(x, y) \geq 0, \forall\, x, y \in X$;

(ii) $d(x, y) = 0$, iff $x = y$, $\forall\, x, y \in X$;

(iii) $d(x, y) = d(y, x), \forall\, x, y \in X$;

(iv) $d(x, y) \leq d(x, z) + d(z, y), \forall\, x, y, z \in X$.

Here the set X together with the metric di.e., the order pair $(X, d)$is called a **metric space.**

Example 1.1.3. If $X = \mathbb{R}$(Set of real numbers) and the metric define by

$d(x, y) = |x - y|, \forall\, x, y \in X$.

Then (X, d)form a metric space.

Definition 1.1.4. [55]Let Xbe a vector space. A function $\|\cdot\|: X \to [0, \infty)$is called a **norm** if and only if for all $x, y \in X$and all $\alpha \in \mathbb{R}$, the following rules hold:

(i) $\|x + y\| \leq \|x\| + \|y\|, \forall\, x, y \in X$;

(ii) $\|\alpha x\| = |\alpha|\|x\|, \forall\, x \in X$ and $\alpha \in \mathbb{R}$;

(iii) $\|x\| = 0 \Rightarrow x = 0$.

The pair $(X, \|\cdot\|)$is then called a **normed linear space**. A normed linear space $(X, \|\cdot\|)$defines a metric space $(X, d)$with d defined by $d(x, y) = \|x - y\|$.

Definition 1.1.5. [55]A vectorspace $X$is called an **inner product space or unitary space** if to each ordered pair of vectors x and $y \in X$there is associated a number$\langle x, y\rangle$ the so-called **inner product or scalar product** of x and$y$, such that the following rules hold:

(i) $\langle y, x\rangle = \overline{\langle x, y\rangle}$ (the bar denotes complex conjugation);

(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \ \ \forall\, x, y, , z \in X$;

(iii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle\ \forall x, y \in X$and $\alpha \in \mathbb{R}$;

(iv) $\langle x, y\rangle \geq 0\ \forall\, x \in X$;

(v) $\langle x, x\rangle = 0$ iff $x = 0$.

Definition 1.1.6.[54] A sequence $\{x_n\}$in a metric space $(X,d)$is said to be **convergent** if there is a point $x \in X$with the following property:

For every $\varepsilon > 0$there is an integer Nsuch that $n \geq N$implies that$d(x_n, x) < \varepsilon$.In this case we also say that the sequence $\{x_n\}$is converges to $x \in X$, or that $x \in X$is the limit of $\{x_n\}$, and we write $x_n \to x$, or $\lim_{n\to\infty} x_n = x$.

If the sequence $\{x_n\}$does not convergent, then it is said to be **divergent**.

A sequence $\{x_n\}$ in a metric space (X,d) is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there is an integer Nsuch that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

A metric space$(X,d)$is said to be a **complete metric space** if every Cauchy sequence in X is convergent.

Definition 1.1.7.[55]Let (H,d)be a metric space. If this metric space is complete, i.e., if every Cauchy sequence converges in$H$, where the metric $d$is defined by the inner product of the space, then the space $(H,d)$is called a **Hilbert Space**.

Definition 1.1.8.[55]A normed linear space Xwhich is complete as a metric space and the metric is defined by its norm, is called a **Banach Space**.

Definition 1.1.9.[55]Let X be a norm linear space. Then the **closed unit ball** of X is denoted by $\bar{B}$ and defined as the set

$$\bar{B} = \{x \in X : \|x\| \leq 1\}.$$

Definition 1.1.10.[55] A subset C of a linear space X is called convex set if for all $x, y \in C$ and for all real numbers $0 \leq t \leq 1$, we have

$$tx + (1-t)y \in C.$$

Definition 1.1.11. [55]A Banach space$B$is said to be **uniformly convex** if,$\|x_n\| \leq 1, \|y_n\| \leq 1$ and $\|x_n + y_n\| \to 2$as $n \to \infty$imply$\|x_n - y_n\| \to 0, \forall\, x_n, y_n \in B$.

Definition 1.1.12.[57]Let X be a Banach space,$S_X = \{x \in X : \|x\| = 1\}$ be a unit sphere on X and X^* be the dual of X, that is the space of all continuous linear functional on X. Then the space X is said to be a smooth Banach space if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \quad (1.1)$$

exists for each x and y in S_X. In this case, the norm of X is called **Gâteaux differentiable norm**.

Definition 1.1.13.[57]The Banach space X is said to has**Fréchet differentiable norm** if for each $x \in X$, the limit defined by (1.1) exists and attained uniformly for $y \in S_X$, and in this case it is also well-known that

$$\langle y, J(x)\rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x+y\|^2 \leq \langle y, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|y\|) \qquad (1.2)$$

for all $x, y \in X$, where J is the Fréchet derivative of the functional$\frac{1}{2}\|\cdot\|^2$ $at\ x \in X$, $\langle\cdot,\cdot\rangle$ is the dual pairing between X and X^*, b is an increasing function defined on$[0,\infty)$such that $\lim_{t\to 0}\frac{b(t)}{t} = 0$.

Definition 1.1.14. [64] The Banach space B is said to satisfy the Opial's condition, if for each sequence $\{x_n\}_{n=0}^{\infty}$ in B, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\| \qquad (1.3)$$

for all$y \in B$ with $y \neq x$.

Definition 1.1.15. [64] A Banach space B is said to have the Kadec-Klee property if for every sequence $\{x_n\}$ in B, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$together imply $x_n \to x$ as $n \to \infty$.

Definition 1.1.16.[57]Let B be a nonempty bounded closed convex subset of a uniformly convex Banach space$X$. Then the mapping $T: B \to X$ is called demiclosed at $y \in X$ if for each sequence $\{x_n\}$in B and each $x \in X$, $x_n \rightharpoonup x$, and $Tx_n \to y$ implies that$x \in B$and $Tx = y$.

Definition 1.1.17.[12] Let B be a nonempty bounded closed convex subset of a uniformly convex Banach space$X$. Then the mapping $T: B \to B$ is called semicompact if any sequence $\{x_n\}$in B, such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, has a subsequence converging strongly to some $p \in B$.

Definition 1.1.18. [15] Let B be a subset of a Norm space$X$. Then the mapping $T: B \to B$ is said to satisfy Condition (I) if there exists a non-decreasing function $h: [0, \infty) \to [0, \infty)$ with $h(0) = 0, h(r) > 0$ for all $r \in (0, 1)$ such that $\|x - Tx\| \geq h(d(x,\ F(T)))$ for all x ∈ B, where $d\big(x,\ F(T)\big) = inf\{\|x - p\|: p \in F(T)\}$.

1.2 Fixed Points

Definition 1.2.1.[9]Consider a mapping $T$of a set Minto itself or into some set containing M. Then the solution of the equation $Tx = x$is called a **fixed point**or an **invariant point**(sometimes shortened to **fixpoint**) of$T$ in M for all$x \in M$. Briefly, the point $x \in M$is called a **fixed point** of the mapping $T : M \to M$ iff $Tx = x$. Geometrically, the intersecting point of the curve $y = Tx$and the straight line $y = x$is a fixed point of T.

The **set of fixed points of** T is denoted by $F(T)$,where $F(T) = \{x: Tx = x\}$.

Example 1.2.2. Let the mapping Tbe defined on the real numbers by $Tx = x^2 - 3x + 4$, then $x = 2$ is a fixed point of T, because $T2 = 2$.Consider $y = Tx$and we obtain thefollowing figure:

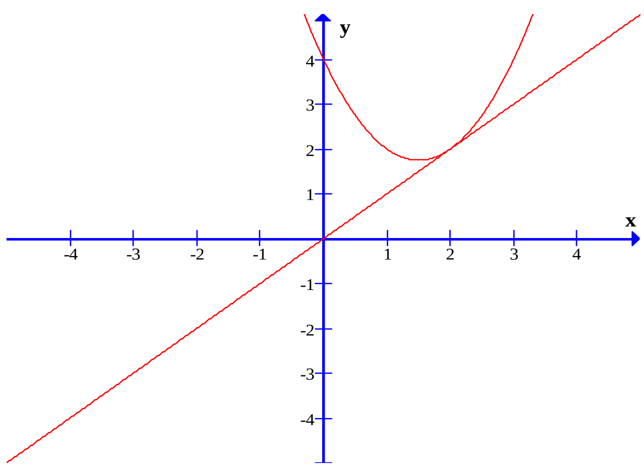

Figure-1.1: Fixed point of $Tx = x^2 - 3x + 4$.

Not all functions have fixed points; as for example, if T is a function defined on the real numbers as $Tx = x + 1$ then it has no fixed points, since x is never equal to $x + 1$ for any real number. In graphical terms, a fixed point means the point $(x, Tx)$is on the line$y = x$, or in other words the graph of T has a point in common with that line. The example $Tx = x + 1$ is a case where the graph and the line are a pair ofparallel lines. Points which come back to the same value after a finite number of iterations of the function are known as periodic points; a fixed point is a periodic point with period equal to one.

Theorem 1.2.3.[9] Let Mbe a metric space. Suppose that T is a continuous mapping of M into a compact subset of M and that, for each $\varepsilon > 0$, there exists $x(\varepsilon)$such that

$$d\big(Tx(\varepsilon), x(\varepsilon)\big) < \varepsilon. \tag{1.4}$$

Then Thas a fixed point.

Definition 1.2.4.[9] The points $x(\varepsilon)$satisfying (1.4) of Theorem 1.2.3 is calledε**-fixed points** for T.

1.3 Convergence of fixed point[9]

A formal definition of convergence can be stated as follows:

Suppose $\{p_n\}$ as $n$goes from 0 to$\infty$is a sequence that converges to p, with $p_n \neq 0, \forall\, n \in \mathbb{N}$. If positive constants λ and α exist with $\lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n-p|} = \lambda$ then $\{p_n\}$as $n$goes from 0 to∞ converges to $p$of order α, with asymptotic error constant λ. There is a nice checklist for checking the convergence of a fixed point p for a function$Tx = x$.

1) First check that, $Tp = p$

2) Check for linear convergence. Start by finding $|T^{/}(p)|$. If

$\lvert T^{/}(p)\rvert \in (0,1]$	then we have linear convergence
$\lvert T^{/}(p)\rvert > 1$	series diverges
$\lvert T^{/}(p)\rvert = 0$	then we have at least linear convergence and maybe something better, we should check for quadratic

3) If we find that we have something better than linear we should check for quadratic convergence. Start by finding $|T^{//}(p)|$. If

$\lvert T^{//}(p)\rvert \neq 0$	then we have quadratic convergence provided that $T^{//}(p)$ is continuous
$\lvert T^{//}(p)\rvert = 0$	then we have something even better than quadratic convergence
$\lvert T^{//}(p)\rvert$does not exist	then we have convergence that is better than linear but still not quadratic

1.4 Contraction Mappings

Definition 1.4.1.[9, 54]Let Mbe a metric space. A mapping $T: M \to M$is called a **contraction mapping** if ∃a positive real number $0 \leq \lambda < 1$such that

$$d(Tx, Ty) \leq \lambda d(x, y), \forall\, x, y \in M,$$

where$d(x, y)$denotes the metric between $x$and y.

If $M$is a normed space, then$T$is contraction if

$$\|Tx - Ty\| \leq \lambda \|x - y\|.$$

If $T$is linear, this reduces to $\|Tx\| \leq \lambda \|x\|, \forall\, x \in M$. Thus, a linear operator $T: M \to M$is contraction if its norm satisfies

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x=1} \|Tx\|.$$

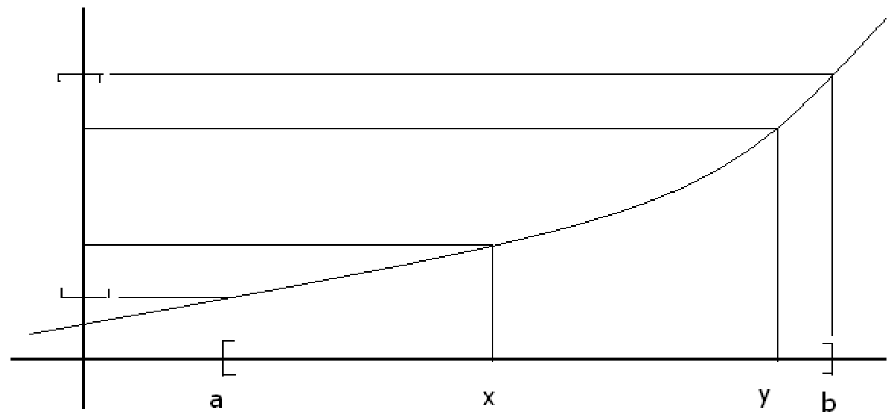

Figure-1.2: Contraction Mapping.

Example 1.4.2. Consider the cosine function on [0, 1]. Graphs of $y = cosx$ and $y = x$ show there is one intersecting point in [0, 1], which means cosine function has a fixed point in [0, 1]. We will show this point can be obtained through iteration.

Since cosine is a decreasing function. Therefore, for $0 \leq x < 1$we have $cos1 \leq cosx < 1$with$cosx \approx 0.54$that is $cos: [0,1] \to [0,1]$. For$x, y \in [0,1]$ the Mean-value theorem tells us

$$cosx - cosy = cos^{/}(t)(x - y) = (-sint)(x - y).$$

for some$t$between$x$ and y. Thus $|cosx - cosy| = |sint||x - y|$.

Since t is between$x$ and $y$and sine is increasing on this interval (it increases from 0 up to$\frac{\pi}{2} \approx 1.57 > 1$) we have$|sint| = sint \leq sin1 \approx 0.8414$.

Therefore, $|cosx - cosy| \leq 0.8414|x - y|$.

So, cosine is a contraction mapping on [0, 1], which is complete. Hence, there is a unique $a \in [0,1]$such that $cosa = a$.

A beautiful application of contraction mappings to the construction of fractals (interpreted as fixed points in a metric space whose points are compact subsets of the plane).

1.5 Non-expansive Mappings

Definition 1.5.1.[9] Let Mbe a metric space. A mapping $T: M \to M$is called a **non-expansive** if

$$d(Tx, Ty) \leq d(x, y), \forall\, x, y \in M,$$

where $d(x, y)$denotes the metric between x and y.

If M is a normed space, then T is **non-expansive** if

$$\|Tx - Ty\| \leq \|x - y\|.$$

If T is linear, this reduces to

$$\|Tx\| \leq \|x\|, \forall x \in M.$$

Thus, a linearoperator $T: M \to M$is **non-expansive** if its norm satisfies $\|T\| \leq 1$.

The non-expansive mapping $T: M \to M$is called **strictly non-expansive** if

$$d(Tx, Ty) \leq d(x, y) \Rightarrow x = y, \forall x, y \in M.$$

If$M$is a normed space, then the condition for strictly non-expansive reduces to

$$\|Tx - Ty\| \leq \|x - y\| \text{iff } x \neq y.$$

If $T$is linear, this reduces to$\|Tx\| \leq \|x\|, \forall$ non zero $x \in M$.

Example 1.5.2. Contraction mapping, isometrics and orthogonal projections all are non-expansive mappings. A fixed point of a non-expansive mapping need not be unique.

Definition 1.5.3.[42]Let C be a subset of real normed linear space X. A mapping $T: C \to C$ is said to be **asymptotically non-expansive** on Cif there exists a sequence $\{r_n\}$in $[0, \infty)$ with $lim_{n\to\infty} r_n = 0$such that for each$x, y \in C$,

$$\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|, \ \forall\, n \geq 1.$$

In this case,

(i) if $r_n \equiv 0$, then T is known as a **non-expansive mapping**.

(ii) T is called **asymptotically non-expansive mapping in the intermediate sense**if provided T is uniformly continuous and

$$\lim sup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

Example 1.5.4.[42] Let $X = \mathbb{R}, C = \left[\frac{-1}{\pi}, \frac{1}{\pi}\right]$and $|k| < 1$. For each $x \in C$, define

$$T(x) = \begin{cases} kx \sin\frac{1}{x} & if\ x \neq 0, \\ 0 & if\ x = 0. \end{cases}$$

Then $T$is asymptotically non-expansive in the intermediate sense.

1.6 Some well-known fixed point theorems

Theorem 1.6.1. (Brouwer fixed point theorem)[9]

The Brouwer fixed point theorem is one of the early achievements of algebraic topology and is the basic of the more general fixed point theorems that are important in Functional analysis as well as numerical analysis. This theorem is named after Dutch Mathematician L. E. J. Brouwer (1910).

Statement.*A continuous mapping of a convex, closed set into itself necessarily has a fixed point.*

Examples.

1. A continuous mapping that maps the set [0, 1] into itself has a fixed point.
2. A continuous mapping that maps a disk into itself has a fixed point.
3. A continuous mapping that maps a spherical ball into itself has a fixed point.

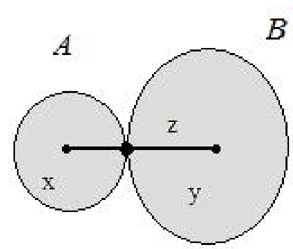

Figure-1.3: Geometrical interpretation of Brouwer fixed point theorem.

Theorem 1.6.2. (Banach fixed point theorem) [9, 38]

Banach fixed point theorem is one of the pivotal results of Mathematical analysis. It is widely considered as the source of metric fixed point theory. Also its significance lies in its vast applicability in a number of branches of Mathematics.This theorem was first stated by Polish mathematician Stefan Banach in 1922. He established this theorem as a part of his doctoral thesis. It is also known as Contraction mapping theorem. Here we state and prove this celebrated theorem.

Statement. *Let* (X, d) *be non-empty complete metric space and* $T: X \to X$ *be a contraction mapping on* X, *i.e. there is a non-negative real number* $0 \leq \lambda < 1$ *such that*

$$d(Tx, Ty) \leq \lambda d(x, y), \forall\, x, y \in X \qquad (1.5)$$

Then the mapping T *admits one and only one (unique) fixed point in* X. *For any* $x_1 \in X$ *the sequence of iterates* $x_1, T(x_1), T\big(T(x_1)\big), \cdots$ *converges to the fixed point of* T.

Theorem 1.6.2 has many advantage to solving nonlinear problems, but it has one disadvantage - the contractive condition (1.5) forces to continuous mapping on X.

Definition 1.6.3.[34]In 1968, R. Kannan [34] obtained a fixed point theorem which extends the Theorem 1.6.2 to mappings that need not be continuous, by considering instead of (1.5) the condition as follows:

There exists $\beta \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)], \text{ for all } x, y \in X. \qquad (1.6)$$

If a mapping T satisfies (1.6), then it is known as **Kannan mapping**.

By applying Kannan's theorem, a lot of papers were committed to obtaining fixed point theorems for various classes of contractive type conditions that do not require the continuity of T, see for instance, Rus [16], and references therein. One of them is Chatterjea's fixed point theorem.

Definition 1.6.4.[39]Due to Chatterjea's fixed point theorem [39] we have, there exists $\gamma \in \left(0, \frac{1}{2}\right)$ such that

$$d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)], \text{ for all } x, y \in X. \tag{1.7}$$

The mapping which satisfies (1.7) is known as **Chatterjea mapping**.

In 1972, T. Zamfirescu [46] combined the above three contractive definitions, defined by (1.5), (1.6) and (1.7), and obtained a very interesting result as follows:

Theorem 1.6.5.[46]*Let (X, d) be a complete metric space and let $T: X \to X$ be a mapping on X for which there exist the real numbers α, β and γ satisfying $a \in (0, 1)$, $b, c \in (0, \frac{1}{2})$such that for each pair $x, y \in X$, at least one of the following is true:*

(z_1) $d(Tx, Ty) \leq ad(x, y)$,

(z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then the mapping T hasa unique fixed point $p \in X$ and the Picard iteration scheme $\{x_n\}$defined by

$$x_{n+1} = Tx_n, \; n \in \mathbb{N}$$

converges to p, for any $x_1 \in X$.

Definition 1.6.7. [46] An operator T which satisfies the contractive conditions$(z_1), (z_2)$, and(z_3) of Theorem 1.6.3 is said to be a **Zamfirescu operator**.

1.7Rate of Convergence

In numerical analysis, the speed at which a convergentsequence approaches its limit is called the rate of convergence. Although strictly speaking, a limit does not give information about any finite first part of the sequence, this concept is of practical importance if we deal with a sequence of successive approximations for an iterative method, as then typically fewer iterations are needed to yield a useful approximation if the rate of convergence is higher. This may even make the difference between needing ten or a million iterations.

Definition 1.7.1.[49] Suppose that the sequence$\{x_n\}$ converges to the numberξ. We say that this sequence converges linearly to ξ, if there exists a number $\mu \in (0,1)$such that

$$\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|} = \mu. \qquad (1.8)$$

The number μ is called the **rate of convergence**. If the above holds with$\mu = 0$, then the sequence is said to converge super linearly. One says that the sequence converges sub linearly if it converges, but $\mu = 1$.

The next definition is used to distinguish super linear rates of convergence.

Definition 1.7.2.[49]We say that the sequence converges with order q for $q > 1$ to ξ if

$$\lim_{k\to\infty} \frac{|x_{k+1}-\xi|}{|x_k-\xi|} = \mu \text{with} \mu > 0. \qquad (1.9)$$

In particular, convergence with order 2 is called quadratic convergence, and convergence with order 3 is called cubic convergence.

This is sometimes called Q-linear convergence, Q-quadratic convergence, etc., to distinguish it from the definition below. The Q stands for "quotient," because the definition uses the quotient between two successive terms.

Definition 1.7.3.[49] The drawback of the above definitions is that these do not catch some sequences which still converge reasonably fast, but whose “speed” is variable, such as the sequence $\{b_k\}$below. Therefore, the definition of rate of convergence is sometimes extended as follows.

Under the definition 1.7.1, the sequence $\{x_k\}$converges with at least order $q$if there exists a sequence $\{\varepsilon_k\}$such that$|x_n - \xi| \le \varepsilon_k, \forall k,$ and the sequence

$\{\varepsilon_k\}$converges to zero with order$q$ according to the above "simple" definition. To distinguish it from that definition, this is sometimes called R-linear convergence, R-quadratic convergence, etc. (with the R standing for “root”).

Example 1.7.4. Consider the following sequences:

$$a_0 = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{4},\ a_3 = \frac{1}{8},\ a_4 = \frac{1}{16},\ a_5 = \frac{1}{32},\ldots,a_k = \frac{1}{2^k},\ldots$$
$$b_0 = 1,\ b_1 = 1,\ b_2 = \frac{1}{4},\ b_3 = \frac{1}{4},\ b_4 = \frac{1}{16},\ b_5 = \frac{1}{16},\ldots,b_k = \frac{1}{4^{floor(k/2)}},\ldots$$
$$c_0 = \frac{1}{2},\ c_1 = \frac{1}{4},\ c_2 = \frac{1}{16},\ c_3 = \frac{1}{256},\ c_4 = \frac{1}{65536},\ \ldots..,c_k = \frac{1}{2^{2^k}},\ldots$$
$$d_0 = 1,\ d_1 = \frac{1}{2},\ d_2 = \frac{1}{3},\ d_3 = \frac{1}{4},\ d_4 = \frac{1}{5},\ d_5 = \frac{1}{6},\ldots,d_k = \frac{1}{k+1},\ldots$$

The sequence $\{a_k\}$converges linearly to 0 with rate$\frac{1}{2}$. The sequence $\{b_k\}$also converges linearly to 0 with rate $\frac{1}{2}$ under the extended definition, but not under the simple definition. The sequence $\{c_k\}$converges super linearly. In fact, it is quadratically convergent. Finally, the sequence$\{d_k\}$converges sub-linearly.

1.8T-Stability

Definition 1.8.1. [1, 28] Let Bbe a Banach space, T be a self-map on B, and assume that $x_{n+1} = f(T, x_n)$defines some iteration schemes involving T. For example,$f(T, x_n) = T(x_n)$. Suppose that $F(T)$, the fixed point set of T, is nonempty and that the sequence $\{x_n\}$converges to a fixed point $p$of T Let $\{y_n\}$ be an arbitrary sequence in B and define

$$\varepsilon_n = \|y_{n+1} - f(T, y_n)\| \tag{1.10}$$

for $n = 0, 1, 2, \cdots$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that$\lim_{n\to\infty} y_n = p$, then the iteration process $x_{n+1} = f(T, x_n)$is said to be T**- stable**.

Examples 1.8.2. Picard iterativeprocedures, Mann iterative procedure, Ishikawa iterative procedure, Noor Iterative procedure all are$T$- stable.

CHAPTER-1

MATHEMATICAL PRELIMINARIES AND SOME FUNDAMENTAL RESULTS

The main aim of this Chapter is to state some basic definitions and some fundamental results of fixed point theory to keep this thesis in sequence and for the convenience of references. For the conciseness, all of the theorems are stated without proof.

1.1 Some basic definitions

Definition 1.1.1.[55]A **vector space or linear space**is a set$X$ together with two operations, addition and scalar multiplication such that for all $x, y \in X$and all $\alpha \in \mathbb{R}$(set of real numbers) both $x + y$ and αx are in X, and for all $x, y, z \in X$and all $\alpha, \beta \in \mathbb{R}$the following properties are satisfied:

(i) $x + y = y + x$;

(ii) $(x + y) + z = x + (y + z)$;

(iii) $(\alpha + \beta)x = \alpha x + \beta x$;

(iv) $\alpha(x + y) = \alpha x + \alpha y$;

(v) $\alpha(\beta x) = (\alpha\beta)x$;

(vi) there exists a $0 \in X$such that for all $x^{/} \in X,\ 0 + x^{/} = x^{/}$;

(vii) there exists a $-x \in X$such that for all $x \in X,\ x + (-x) = 0$;

(viii**)** there exists a $1 \in \mathbb{R}$such that for all $x \in X, 1 \cdot x = x$.

Definition 1.1.2. [54] Let X be a nonempty set. The mapping $d: X \times X \to \mathbb{R}$is said to be a **metric** on X if it satisfies the following conditions:

CHAPTER-2

SOME FIXED POINT ITERATIVE PROCEDURES

A fundamental principle in mathematical sciences is iterative procedure. As the name suggests, a process which is repeated until an answer is achieved is called iterative procedure. Iterative procedure is used to find roots of equation, solution of linear and nonlinear system of equations and solution of differential equations. All the numerical iterative procedure is formulated to compare with any one of the fixed point iterative procedure and to establish the fixed point iterative procedure, different types of fixed point theorems are used as very important tools. So, we can say that numerical iterative proceduresare the achievement of fixed point theory. In this chapter, we will recall some fixed point iterative procedures and their convergence theorems and finally introduce our new iterative procedure.

2.1 Picard iterative procedure

Definition 2.1.1.[48]Let$T: X \to X$ be a given operator and Xbe a metric space or normed linear space or Banach space. Then the sequence$\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \qquad (2.1)$$

for all$n \in \mathbb{N}$ and$x_1 \in X$ is called **Picard iterative Procedure**.

The following results are established by V. Berinde [49] and T. Zamfirescu [46], about the convergence of Picard iterative procedure.

Theorem 2.1.2 [49]*Let$p = \xi$be a root of the equation $f(p) = 0$and let I be an interval containing the point $p = \xi$. Let $T(p)$and $T^{/}(p)$ be continuous in I where $T(p)$is defined by the equation $T(p) = p$which is equivalent to $f(p) = 0$. Let $p = \xi$be a fixed point of$T$. Then if $|T^{/}(p)| < 1$ for all $p \in I$, the sequence of* *approximation* $\{x_n\}$ *defined by the Picard iterative procedure(2.1) converges to the root* $p = \xi$*(fixed point of* T*), provided that the initial approximation* p_0*is chosen in* I.

Theorem 2.1.3[46]*Let*X *be a Banach space and* $T: X \to X$ *be a Zamfirescu operators. Then* T*have a unique fixed point* p*and the Picard iterative procedure*$\{x_n\}$ *defined by (2.1) converge to* p *for any* $p_1 \in X$.

2.2Kranoselskii's iterative Procedure

Definition 2.2.1.[23]Let $T: X \to X$be a given operator and Xbe a metric space or normed linear space or Banach space. Then the sequence $\{x_n\}$ defined by

$$x_{n+1} = (1-\delta)x_n + \delta T x_n, \quad (2.2)$$

for all$n \in \mathbb{N}$ and$x_1 \in X$ and $\delta \in (0,1)$is called **Kranoselskii's iterative procedure**.

Here we state and prove a convergence theorem of Kranoselskii's iterative procedure (2.2)for Zamfirescu operator.

Theorem 2.2.2.*Let* E*be an arbitrary Banach space,* X*be a closed convex subset of* E*, and* $T: X \to X$*an Zamfirescu operator. Let* $\{x_n\}$ *be Kranoselskii's iterative procedure (2.2) and*$x_1 \in X$*with* $\delta \in (0,1)$. *Then*$\{x_n\}$*converges strongly to the fixed point of* T.

Proof. By theorem 2.1.3, we know that T has a unique fixed point in X, say u. Consider $x, y \in X$. Since T is a Zamfirescu operator, therefore at least one of the conditions(z_1),$(z_2)$and (z_3) is satisfied by T.

If (z_2) holds, then

$$\|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|]$$

$$\le b\big[\|x - Tx\| + [\|y - x\| + \|x - Tx\| + \|Tx - Ty\|]\big]$$

$$\Rightarrow \|Tx - Ty\| \le \frac{b}{(1-b)}\|x - y\| + 2\frac{b}{(1-b)}\|x - Tx\|. \quad (2.3)$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \leq \frac{c}{(1-c)}\|x - y\| + 2\frac{c}{(1-c)}\|x - Tx\| \tag{2.4}$$

Let us denote

$$\mu = max\left\{a, \frac{b}{(1-b)}, \frac{c}{(1-c)}\right\} \tag{2.5}$$

Then we have, $0 \leq \mu < 1$ and in view of(z_1), (2.20) and (2.21) we get the following inequality

$$\|Tx - Ty\| \leq \mu\|x - y\| + 2\mu\|x - Tx\| \text{holds } \forall\, x, y \in B. \text{(2.6)}$$

Now let $\{x_n\}_{n=0}^{\infty}$be the Kranoselskii's iterativeproceduredefined by (2.2) and$x_0 \in X$ arbitrary. Then

$$\begin{aligned}\|x_{n+1} - u\| &= \|(1-\delta)x_n + \delta T x_n - (1 - \delta + \delta)u\| \\ &\leq (1-\delta)\|x_n - u\| + \delta\|Tx_n - u\| \end{aligned} \tag{2.7}$$

Take, $x = u$and $y = x_n$ $y = x_n$ in (2.6), we obtain

$$\|Tx_n - u\| \leq \mu\|x_n - u\|, \text{where } \mu \text{is given by (2.5).} \tag{2.8}$$

Now, combining (2.7) and (2.8), we get

$$\begin{aligned}\|x_{n+1} - u\| &\leq (1-\delta)\|x_n - u\| + \delta\mu\|x_n - u\| \\ &= (1 - \delta + \delta\mu)\|x_n - u\| \\ &= (1 - (1-\mu)\delta)\|x_n - u\|, n \in \mathbb{N}. \end{aligned} \tag{2.9}$$

Inductively we obtain,

$$\|x_{n+1} - u\| \leq \prod_{k=0}^{n}(1 - (1-\mu)\delta)^k \|x_1 - u\|, n \in \mathbb{N}. \tag{2.10}$$

As $\mu < 1$ and $\delta \in (0, 1)$, hence we obtain

$$\lim_{n\to\infty} \prod_{k=0}^{n}(1 - (1-\mu)\delta)^k \|x_1 - u\| = 0.$$

This by (2.10) implies that,

$$\lim_{n\to\infty} \|x_{n+1} - u\| = 0,$$

This implies that $\{x_n\}$ is converges strongly to u.

This completes the theorem. ■

2.3 Mann iterative procedure

The Mann iterative procedure known as one-step iterative procedure invented in 1953, was used to prove the convergence of the sequence to a fixed point of many valued mapping for which the Banach fixed point theorem 1.6.2failed.

Definition 2.3.1 [53]Let$B$ be a nonempty closed convex subset of a norm space or Banach space$X$ and T be a mapping on B. The**Manniterativeprocedure** is defined as follows.

For any arbitrary $x_1 \in B$, the iterative sequence $\{x_n\}$ constructed by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n; \ \ \forall\, n \in \mathbb{N}, \tag{2.11}$$

where the sequence $\{\alpha_n\} \subset (0,1)$ is convergent, such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The following results are obtained by M. Zulfikar AliandMd. Asaduzzaman[32] about the convergence of Mann iterative procedure (2.11).

Theorem 2.3.2.[32]*Let B be a nonempty bounded closed convex subset of a Banach space $X$and T be a map on Bsatisfying the contractive definition*

$$\|Tx-Ty\| \le k \, max \left\{ \begin{array}{c} c\|x-y\|, \\ (\|x-Tx\|+\|y-Ty\|), \\ (\|x-Ty\|+\|y-Tx\|) \end{array} \right\} \tag{2.12}$$

for all $x, y \in S$, where $c \ge 0, \ 0 \le k < 1$. Let $\{x_n\}$ be a sequence in Bdefined by Mann iterative procedure (2.11). Now, if $\{x_n\}$converges, then it converges to a fixed point of$T$.

Theorem 2.3.3.[32]*Let B be a nonempty closed convex subset of a uniformly convex Banach space X. Let $T: B \to B$satisfying the contractive definition (2.12) and such that $T(B)$ is relatively compact. If $F(T)$ the fixed point set of T is* *nonempty, then Mann iterative procedure defined by(2.11)converges to a fixed point of*T.

The following results are obtained by B.E. Rhoades [5] and VasileBerinde [46] about the convergence of Mann iterative procedure (2.11).

Theorem 2.3.4.[5]*Let*X *be a uniformly convex Banach space,* K*a closed convex subset of* X*and* $T: K \to K$*be a Zamfirescu operator. Let*$\{x_n\}$*be Mann iterative proceduredefined by (2.11) and* $x_1 \in K$*with* $\{\alpha_n\}$*satisfying*

(a)$\alpha_0 = 1$;*(b)* $0 < \alpha_n < 1$ for $n \geq 1$; *(c)*$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$.

Then$\{x_n\}$ *converges strongly to the fixed point of* T.

Theorem 2.3.5.[50]*Let* X*be an arbitrary Banach space,* K *a closed convex subset of* X, *and* $T: K \to K$*be a Zamfirescu operator. Let* $\{x_n\}$ *be Mann iterative procedure defined by (2.11) and* $x_1 \in K$, *with* $\{\alpha_n\} \subset [0,1]$*satisfying*$\sum_{n=1}^{\infty} \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to the fixed point of* T.

2.4Ishikawaiterative procedure

In 1974 Ishikawa devised the two-step iterative procedure to establish the convergence of Lipschitzianpscudocontractive map where the Mann iterative procedure failed to converge.

Definition 2.4.1 [40] Let$B$ be a nonempty closed convex subset of a norm space X and T be a mapping on B.The**Ishikawaiterative procedure**is defined as follows. For any arbitrary $x_1 \in B$, the iterative sequence $\{x_n\}$ constructed by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n; \quad \forall\, n \in \mathbb{N}, \end{aligned}\right\} \tag{2.13}$$

where the sequences$\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ are convergent, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Here, we state some convergence theorems of Ishikawa iterative procedure, which are given by B.E. Rhoades [5] and VasileBerinde [51].

Theorem 2.4.2.[5]*Let Xbe a uniformly convex Banach space, Kbe a closed convex subset of X and $T: K \to K$be a Zamfirescu operator. Let $\{x_n\}$be* Ishikawa iterative procedure *defined by (2.13) and $x_1 \in K$ with $\{\alpha_n\}$and $\{\beta_n\}$are sequences of numbers in$(0,1)$satisfying$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then $\{x_n\}$ converges strongly to the fixed point of T.*

Theorem 2.4.3 [51]*Let X be an arbitrary Banach space, K a closed convex subset of X, and $T: K \to K$be aZamfirescu operator. Let $\{x_n\}$ be the Ishikawa iterative procedure defined by (2.13) and$x_1 \in K$ with $\{\alpha_n\}$and $\{\beta_n\}$are sequences of positive numbers satisfying$\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the fixed point of T.*

2.5Nooriterative procedure

In 2000, M. A. Noor [24] introduced and analyzed the three-step iterative procedure to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle.

Definition 2.5.1 [24]Let$B$ be a nonempty closed convex subset of a norm space or Banach space$X$ and T be a mapping on B.The **Nooriterative procedure** is defined as follows.

For any arbitrary $x_1 \in B$, the iterative sequence $\{x_n\}$ constructed by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\} \tag{2.14}$$

where the sequences $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$and $\{\gamma_n\} \subset (0,1)$ are convergent, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The following result is obtained by M. Asauzzaman and M. Zulfikar Ali[33], about the convergence of Noor iterative procedure (2.14).

Theorem 2.5.2. [33]*Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T: B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T, where $F(T)$ denotes the set of fixed points of T. Let $\{x_n\}$ be the Noor iterative procedure defined by (2.14) and $x_1 \in B$. Then the sequence $\{x_n\}$ converges strongly to the fixed point $p \in F(T)$.*

2.6 Agarwal *et al.* iterative procedure

In 2007, Agarwal *et al.* [37] introduced the following two-step iterative procedure. They proved that the rate of convergence of this iterative procedure is same as that of the Picard iterative procedure (2.1) and faster than the Mann iterative procedure (2.11) for contraction mapping.

Definition 2.6.1.[37]Let B be a nonempty closed convex subset of a norm space or Banach space X and T be a mapping on B. The **Agarwal *et al.* iterative procedure** is defined as follows.

For any arbitrary $x_1 \in B$, the iterative sequence $\{x_n\}$ constructed by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n; \quad n \in \mathbb{N}, \end{aligned}\right\} \qquad (2.15)$$

where the sequences $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ are convergent, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

2.7 Abbas *et al.* iterative procedure

In 2014, Abbas *et al.* [27]introduced a three-step iterative procedure and showed that the rate of convergence of their iterative procedure is faster than that of Agarwal *et al.* iterative procedure (2.15).

Definition 2.7.1.[27]Let B be a nonempty closed convex subset of a norm space or Banach space$X$ and T be a mapping on B.The **Abbas *et al.*iterative procedure**is defined as follows.

For any arbitrary$x_1 \in B$, the iterative sequence $\{x_n\}$ constructed by

$$\left.\begin{array}{c} x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n, \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n;\ \forall\, n \in \mathbb{N}, \end{array}\right\} \tag{2.16}$$

where the sequences$\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\gamma_n\} \subset (0,1)$ are convergent, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The following result is established by Abbas *et al.*[23], about the rate of convergence of Abbas *et al.* iterative procedure (2.16).

Theorem 2.7.2. [27]*Let C be a nonempty closed convex subset of a uniformly Banach space E. Let T be a contraction with a contractionfactor$k \in (0,1)$andfixed point q. Let$\{u_n\}$ be defined by the iterative procedure (2.15) and $\{x_n\}$ by (2.16), where$\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε* in (0,1). *Then $\{x_n\}$ converges faster than $\{u_n\}$.*

The following results are established by Abbas *et al.*[27], about the weak and strong convergence of Abbas *et al.* iterative procedure (2.16).

Theorem 2.7.3. [27]*Let C be a nonempty closed convex subset of a uniformly Banach space E. Let T be a non-expansive self-mapping of$C$. Let$\{x_n\}$defined by the iterative procedure (2.16), where$\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some ε in* $(0,1)$. *If $F(T) \neq \emptyset$,then* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Theorem 2.7.4. [27]*Let E be a uniformly Banach space andlet C,$T$and $\{x_n\}$be taken as in Theorem 2.7.3. Assume that (a) E satisfies Opial's condition or (b) E*

has a $Fre\acute{c}het$ *differentiable norm. If* $F(T) \neq \emptyset$*,then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Theorem 2.7.5. [27]*Let* E *be a uniformly Banach space andlet* $C, T, F(T)$ *and* $\{x_n\}$*be taken as in Theorem 2.7.3. Then the sequence* $\{x_n\}$ *converges to a point of* $F(T)$ *if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$*where* $d(x_n, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Theorem 2.7.6. [27]*Let* E *be a real uniformly convex Banach space andlet* $C, T, F(T)$,$\{x_n\}$*be taken as in Theorem 2.7.3. Let* T*satisfy Condition (I), then* $\{x_n\}$ *converges strongly to a fixed point of* T.

2.8Thakur *et al.* iterative procedure

Recently, Thakur *et al.* [10] introduced another three-step iterative procedure and showed that the rate of convergence of their iterative procedure is faster than all the above mentioned iterative procedure defined by (2.1),(2.11) and (2.13) to (2.16) for contraction mapping.

Definition 2.8.1. [12] Let B be a nonempty closed convex subset of a norm space or Banach space$X$ and T be a mapping on B.The **Thakur *et al.* iterative procedure** is defined as follows.

For any arbitrary$x_1 \in B$, the iterative sequence $\{x_n\}$ constructed by,

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)z_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\} \tag{2.17}$$

where the sequences$\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$ and $\{\gamma_n\} \subset (0,1)$ are convergent, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

The following result is given by Thakur *et al.* [12], about the rate of convergence of Thakur *et al.* iterative procedure.

Theorem 2.8.2. [12]*Let C be a nonempty closed convex subset of a norm space E. Let T be a contraction with a contraction constant $k \in (0, 1)$ and the fixed point p. Let$\{u_n\}$ be defined by the iterative procedure (2.16) and $\{x_n\}$ by (2.17), where$\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\}$are in $[\varepsilon, 1-\varepsilon]$ for all $n \in \mathbb{N}$ and for some $\varepsilon \in (0,1)$. Then $\{x_n\}$ converges faster than $\{u_n\}$.*

2.9 Our new three-step iterative procedure

In order to preserve the continuation of the above mentioned works, here we have proposed a new three-step iterative procedure. Our new iterative procedure is defined as follows.

Definition 2.9.1. Let B be a nonempty closed convex subset of a norm space or Banach space$X$ and T be a mapping on B.For any arbitrary$x_1 \in B$, the iterative sequence $\{x_n\}$ constructed by

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)Tx_n + \beta_n Tz_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n Tx_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\} \tag{2.18}$$

where the sequences$\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$ and $\{\gamma_n\} \subset (0, 1)$ are convergent, such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Here, we state and prove a convergence theorem of our new iterative procedure (2.18) for Zamfirescu operator.

Theorem 2.9.2.*Let$X$ be an arbitrary Banach space, Bbe a nonempty closed convex subset of $X$and$T: B \to B$be aZamfirescuoperator.Let $p \in F(T)$be a fixed point of T* and *$\{x_n\}$be our new iterative procedure defined by(2.18)and $x_1 \in B$. Then the sequence$\{x_n\}$converges strongly to the fixed point $p \in F(T)$.*

Proof. By theorem 2.1.3, we know that T has a unique fixed point in$B$, say p. Consider $x, y \in B$. Since T is aZamfirescu operator, therefore at least one of the conditions(z_1),$(z_2)$and (z_3) is satisfied by T.

If (z_2) holds, then

$$\|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|]$$

$$\le b\big[\|x - Tx\| + [\|y - x\| + \|x - Tx\| + \|Tx - Ty\|]\big]$$

$$\Rightarrow \|Tx - Ty\| \le \frac{b}{(1-b)}\|x - y\| + 2\frac{b}{(1-b)}\|x - Tx\| \tag{2.19}$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \le \frac{c}{(1-c)}\|x - y\| + 2\frac{c}{(1-c)}\|x - Tx\| \tag{2.20}$$

Let us denote

$$\mu = max\left\{a, \frac{b}{(1-b)}, \frac{c}{(1-c)}\right\} \tag{2.21}$$

Then we have, $0 \le \mu < 1$ and in view of(z_1), (2.20) and (2.21) we get the following inequality

$$\|Tx - Ty\| \le \mu\|x - y\| + 2\mu\|x - Tx\| \text{holds } \forall\, x, y \in B. \tag{2.22}$$

Now, let$\{x_n\}$be sequence defined by our new iterative procedure (2.18) and $x_1 \in B$arbitrary. Then we have

$$\|x_{n+1} - p\| = \|(1 - \alpha_n)Tx_n + \alpha_n Ty_n - (1 - \alpha_n + \alpha_n)p\|$$

$$\le (1 - \alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\| \tag{2.23}$$

Putting$x = x_n$and $y = p$in (2.22), we obtain

$$\|Tx_n - p\| \le \mu\|x_n - p\|, \text{ where } \mu \text{is given by (2.21).} \tag{2.24}$$

Again, putting with $x = y_n$and $y = p$in (2.22), we obtain

$$\|Ty_n - p\| \le \mu\|y_n - p\|, \text{ where } \mu \text{is given by (2.21).} \tag{2.25}$$

Further, we have

$$\|y_n - p\| = \|(1 - \beta_n)Tx_n + \beta_n Tz_n - (1 - \beta_n + \beta_n)p\|$$

$$\le (1 - \beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\|. \tag{2.26}$$

Again,putting $x = z_n$and $y = p$in (2.22), we get

$$\|Tz_n - p\| \le \mu\|z_n - p\|. \tag{2.27}$$

Combining (2.24), (2.25), (2.26) and (2.27) we obtain,

$$\|Ty_n - p\| \le \mu[(1-\beta_n)\mu\|x_n - p\| + \mu\beta_n\|z_n - p\|]. \tag{2.28}$$

But, we have

$$\|z_n - p\| = \|(1-\gamma_n)x_n + \gamma_n T z_n - (1-\gamma_n+\gamma_n)p\|$$
$$\le (1-\gamma_n)\|x_n - p\| + +\gamma_n\|Tx_n - p\|. \tag{2.29}$$

From, (2.28) and (2.29), we get

$$\|Ty_n - p\| \le \mu\left[(1-\beta_n)\mu\|x_n - p\| + \mu\beta_n\begin{bmatrix}(1-\gamma_n)\|x_n - p\| \\ +\gamma_n\|Tx_n - p\|\end{bmatrix}\right]. \tag{2.30}$$

From, (2.24) and (2.30),we get

$$\|Ty_n - p\| \le \mu\left[(1-\beta_n)\mu\|x_n - p\| + \mu\beta_n\begin{bmatrix}(1-\gamma_n)\|x_n - p\| \\ +\gamma_n\mu\|x_n - p\|\end{bmatrix}\right]. \tag{2.31}$$

Now, combining (2.23), (2.24) and (2.31), we obtain

$$\|x_{n+1} - p\| \le (1-\alpha_n)\mu\|x_n - p\| + \alpha_n\left[\mu^2\begin{bmatrix}(1-\beta_n)\|x_n - p\| \\ +\beta_n\begin{bmatrix}(1-\gamma_n)\|x_n - p\| \\ +\mu\gamma_n\|x_n - p\|\end{bmatrix}\end{bmatrix}\right].$$
$$= \mu[1-(1-\mu)(1+\mu\beta_n+\mu^2\beta_n\gamma_n)\alpha_n]\|x_n - p\|. \tag{2.32}$$

Since, $\mu[1-(1-\mu)(1+\mu\beta_n+\mu^2\beta_n\gamma_n)\alpha_n] \le [1-(1-\mu)\alpha_n]$.

Hence from (2.32), we get

$$\|x_{n+1} - p\| \le [1-(1-\mu)\alpha_n]\|x_n - p\|, \ \ n \in \mathbb{N}. \tag{2.33}$$

By (2.33) we inductively obtain

$$\|x_{n+1} - p\| \le \prod_{k=0}^{n}[1-(1-\mu)\alpha_k]\|x_1 - p\|, \ n \in \mathbb{N}. \tag{2.34}$$

Using the fact that $0 \le \mu < 1, \alpha_n, \beta_n, \gamma_n \in (0,1)and\sum_{n=0}^{\infty}\alpha_n = \infty$, we obtain that,

$$\lim_{n\to\infty}\prod_{k=0}^{n}[1-(1-\mu)\alpha_k] = 0. \tag{2.35}$$

Now from (2.34) and (3.35), we obtain

$$\lim_{n\to\infty}\|x_{n+1} - p\| = 0,$$

This implies that$\{x_n\}$converges strongly to the fixed point$p$.

This completes our proof. ■

Definition 2.9.3. Let Bbe a Banach space, T be a self-map on B and assume that $x_{n+1} = f(T, x_n)$representsour new iterativeprocedure(2.18) involving T. Suppose that the fixed point set$F(T)$of$T$is nonempty and that the sequence $\{x_n\}$converges to a fixed point $p \in F(T)$. Let $\{w_n\}$ be an arbitrary sequence in B and define $\varepsilon_n = \|w_{n+1} - f(T, w_n)\|$for $n \in \mathbb{N}$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that$\lim_{n\to\infty} w_n = 0$, then our new iterative procedure represented by$x_{n+1} = f(T, x_n)$is said to be T- **stable**.

Lemma 2.9.4. Let Xbe a Banach space, B be a nonempty, convex subset of $X$and $T: B \to B$be a Zamfirescu operator. If our new iterative procedure (2.18) converges, then $\lim_{n\to\infty} \varepsilon_n = 0$, where $\varepsilon_n$is given by the definition 2.9.3.

Proof. Let $\lim_{n\to\infty} w_n = w_n{}^*$. Then according to the definition 2.9.3, we have

$$
\begin{aligned}
0 \leq \varepsilon_n &= \|w_{n+1} - (1-\alpha_n)Tw_n - \alpha_n Tu_n\| \\
&= \|w_{n+1} - Tw_n + \alpha_n(Tw_n - Tu_n)\| \\
&\leq \|w_{n+1} - Tw_n\| + \alpha_n\mu\|w_n - u_n\| \\
&= \|w_{n+1} - Tw_n\| + \alpha_n\mu\|w_n - (1-\beta_n)Tw_n - \beta_n Tv_n\| \\
&\leq \|w_{n+1} - Tw_n\| + \alpha_n\mu\|w_n - Tw_n\| + \alpha_n\beta_n\mu\|Tw_n - Tv_n\| \\
&\leq \|w_{n+1} - Tw_n\| + \alpha_n\mu\|w_n - Tw_n\| + \alpha_n\beta_n\mu^2\|w_n - v_n\|
\end{aligned}
$$

$$
\begin{aligned}
&= \|w_{n+1} - Tw_n\| + \alpha_n\mu\|w_n - Tw_n\| + \alpha_n\beta_n\mu^2 \left\| \begin{matrix} w_n - (1-\gamma_n)w_n \\ -\gamma_n Tw_n \end{matrix} \right\| \\
&= \|w_{n+1} - Tw_n\| + \alpha_n\mu\|w_n - Tw_n\| + \alpha_n\beta_n\gamma_n\mu^2\|w_n - Tw_n\| \\
&= \|w_{n+1} - Tw_n\| + (\alpha_n\mu + \alpha_n\beta_n\gamma_n\mu^2)\|w_n - Tw_n\| \\
&\leq \|w_{n+1} - w_n\| + \|w_n - Tw_n\| + (\alpha_n\mu + \alpha_n\beta_n\gamma_n\mu^2)\|w_n - Tw_n\| \\
&= \|w_{n+1} - w_n\| + (1 + \alpha_n\mu + \alpha_n\beta_n\gamma_n\mu^2)\|w_n - Tw_n\| \\
&\leq \|w_{n+1} - w_n{}^*\| + \|w_n{}^* - w_n\| + \begin{pmatrix} 1 + \alpha_n\mu \\ +\alpha_n\beta_n\gamma_n\mu^2 \end{pmatrix} \begin{bmatrix} \|w_n - w_n{}^*\| \\ +\|w_n{}^* - Tw_n\| \end{bmatrix} \\
&\to 0 \text{ as } n \to \infty,
\end{aligned}
$$

i.e., $\lim_{n\to\infty} \varepsilon_n = 0$.

Now, we state and prove a Stabilitytheoremof our new iterative procedure (2.18).

Theorem 2.9.5.*Let X be an arbitrary Banach space, Bbe a nonempty closed convex subset of$X$and$T: B \to B$be a zamfirescuoperator. Let$p \in F(T)$be a fixed point of T. Then prove that the iterative sequence$\{x_n\}$defined by our new iterative procedure (2.18) involving the operator $T$is T-stable.*

Proof. From the definition 2.9.3, we can say that theour new iterative procedure (2.18) will be T-stable if$\lim_{n\to\infty} \varepsilon_n = 0$,where$\varepsilon_n = \|x_{n+1} - (1-\alpha_n)x_n - \alpha_n T y_n\|$implies that$\lim_{n\to\infty} x_n = p \in F(T)$.

Now, from the lemma 2.9.4, we observed that if our new iterative proceduredefined by (2.18)converges to a fixed point of T then$\lim_{n\to\infty} \varepsilon_n = 0$. But, in our theorem 2.9.2 we have already shown that our new iterative proceduredefined by (2.18) is strongly convergent to a fixed point of T.

So, by combining our theorem 2.9.2 and lemma 2.9.4, we obtain $\lim_{n\to\infty} \varepsilon_n = 0$and this proves that our new iterative procedure (2.18)is$T$-stable.■

CHAPTER-3

RATE OF CONVERGENCE OF FIXED POINT ITERATIVE PROCEDURES

In this chapter we demonstrate that our iterative procedure(2.18) converges to the fixed point faster than that of Picarditerative procedure (2.1), Manniterative procedure (2.11), Ishikawaiterative procedure (2.13), Nooriterative procedure (2.14), Agarwal*et al.*iterative procedure (2.15), Abbas *et al.*iterative procedure (2.16), and Thakur *et al.* iterative procedure (2.17) for contraction mappings in the sense of Berinde [48].

3.1 A discussion on error estimate of fixed point iterative procedure

The error estimate and stability of fixed points appear to have been given systematically, mainly for the Picard iterative procedure(2.1) (sequence of successive approximations), in conjunction with various contractions. For illustration here we give the following example.

Example 3.1.1. If $T: X \to X$is an α-contraction on a complete metric space (X, d),that is, there exists a constant $0 \leq \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \forall\, x, y \in X,$$

then by Banach fixed point theorem (Theorem 1.6.2) we know that

(a) $F(T) = \{x^*\}$, Where $F(T)$denotes the set of fixed point of T.

(b)$x_{n+1} = Tx_n$ (Picard iterative procedure (2.1)) converges to $x^*$for all$x_1 \in X$.

(c) Both the a priori and a posteriori error estimates

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha^n} d(x_1, x_2),\ n \in \mathbb{N}, \qquad (3.1)$$

$$d(x_n, x^*) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n),\ n \in \mathbb{N}, \qquad (3.2)$$

hold.

Remark 3.1.2. The errors $d(x_n, x^*)$are decreasing as rapidly as the term of geometric progression with ratio α, that is $\{x_n\}$converges to x^*at least as rapidly as the geometric series. The convergence is however linear,

$$d(x_n, x^*) \leq d(x_{n-1}, x^*), n \in \mathbb{N}.$$

If Tsatisfies a weaker contractive condition, e.g., $T$is non-expensive, then Picard iterative procedure (2.1) does not converges generally or even if it converges, its limit is not fixed point of T. More general iterative procedures are needed.

3.2 Rate of convergence of fixed point iterative procedures

The problem of studying the rate of convergence of fixed point iterative procedures arises in two different contexts:

(1) For large classes of operator (quasi-contractive type operators) not only Picard iterative procedure (2.1), but also the Manniterative procedure (2.11), Ishikawaiterative procedure (2.13), Nooriterative procedure (2.14), Agarwal*et al.*iterative procedure (2.15), Abbas *et al.*iterative procedure (2.16), and Thakur *et al.*iterative procedure (2.17) can be used to approximate the fixed points.In such situation, it is of theoretical importance to compare these methods in order to establish, if possible which one converges faster.

(2) For a certain fixed point iterative procedure (Picard, Kranoselskij's, Mann, Ishikawa, Noor etc.) we do not know an analytical error estimate of the form (3.1) and (3.2) of example 3.1.1.In this case we can try an empirical study of the rate of convergence.

Now, we give a theorem, which have been stated by the help of Banach fixed point theorem. By this theorem we are able to provide some useful information about the rate of convergence of fixed point iterative procedures towards the fixed point.

Theorem 3.2.1. *Let Tbe a contraction mapping on a complete metric space M with contraction constant $\lambda$and fixed point$a$. For any $x_1 \in M$ with T-iterates $\{x_n\}$, we have the error estimates*

$d(x_n, a) \leq \frac{\lambda^n}{1-\lambda} d(x_1, T(x_1)),$ (3.3)

$d(x_n, a) \leq \lambda d(x_{n-1}, a),$ (3.4)

and $d(x_n, a) \leq \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n).$ (3.5)

3.3 Sense of B.E. Rhoades about the rate of convergence of two fixed point iterative procedures

In 1976 B.E. Rhoades [4] introduce the following technique to check the rate of convergence of fixed point iterative procedures:

Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by two fixed point iterative procedures, which are converge to a certain fixed point$p$ of a given operator T. To compare the rate of convergence of these fixed point iterative procedures, B.E. Rhoades [4] considered that the iterative procedure represented by $\{x_n\}$is betterthan the iterative procedure represented by$\{y_n\}$if

$\|x_n - p\| \leq \|y_n - p\|, \ \forall n.$ (3.6)

3.4Sense of V. Berinde about the rate of convergence of two fixed point iterative procedures

After B.E. Rhoades [4] in 2004 V. Berinde [48] established a technique to compare the rate of convergence of two fixed point iterative procedures. To define V. Berinde technique we need the following definition about rate of convergence of two sequences of real numbers:

Definition 3.4.1.[48, 49] Let$\{r_n\}$and $\{s_n\}$be two sequences of real numbers that converge to $r$and s, respectively, and assume that there exists a limit

$l = \lim_{n\to\infty} \frac{|r_n - r|}{|s_n - s|}$ (3.7)

(i) If $l = 0$, then it can be said that$\{r_n\}$converges faster to r than $\{s_n\}$to s.

(ii) If $0 < l < \infty$, then it can be said that$\{r_n\}$ and$\{s_n\}$have the same rate of convergence.

In the case (i), the notation $r_n - r = o(s_n - s)$ will be used and if $l = \infty$, then the sequence $\{s_n\}$ converges faster than $\{r_n\}$, that is $s_n - s = o(r_n - r)$.

Suppose that, for two fixed point iterative procedures represented by the sequences $\{x_n\}$ and $\{y_n\}$, both converging to the same fixed point p of a given operator T, the error estimates

$$\|x_n - p\| \leq r_n,\ n \in \mathbb{N}, \tag{3.8}$$

$$\|y_n - p\| \leq s_n,\ n \in \mathbb{N}, \tag{3.9}$$

are available, where $\{r_n\}$ and $\{s_n\}$ are sequences of positive numbers (converging to zero).

Then, in view of definition 3.4.1 V. Berinde [48] adopted the following concept about rate of convergence of two fixed point iterative procedures.

Definition 3.4.2.[48, 49]Let $\{x_n\}$ and $\{y_n\}$ be two sequences generated by two fixed point iterative procedures that converge to the same fixed point p and satisfy (3.8) and (3.9), respectively. If $\{r_n\}$ converges faster than $\{s_n\}$, then it can be said that $\{x_n\}$ converges faster than $\{y_n\}$ to the fixed point p.

3.5 Recent development ofthe rate of convergence of fixed point iterative procedures

In recent years, definition 3.4.2 has been used as a standard tool to compare the rate of convergence of two fixed point iterative procedures. Using this technique Sahu [11] established that the Agarwal *et al.* iterative procedure (2.15) converges faster than the Mann iterative procedure (2.11) and the Picard iterative procedure (2.1). Using a similar technique Abbas and Nazir [27] established that the Abbas *et al.* iterative procedure (2.16) converges faster than the Agarwal *et al.* iterative procedure (2.15) and hence it converges faster than the Mann iterative procedure (2.11) and the Picard iterative procedure (2.1) also. Recently, in 2014 D. Thakur, B.S. Thakur, and M. Postolache [12] established that the Thakur *et al.* iterative procedure (2.17) converges faster than Abbas *et al.* iterative procedure (2.16) and supported that claim by the following example.

Example 3.5.3. Let $X = \mathbb{R}$ and $B = [1, 50]$. Let $T: B \to B$ be a mapping defined by $T(x) = \sqrt{x^2 - 8x + 40}$ for all $x \in B$. For the initial value $x_1 = 40$ and $\alpha_n =$

$0.85,\ \beta_n = 0.65,\ \gamma_n = 0.45$ for $n = 0, 1, 2, \ldots$, Thakur*et al.* iterative procedure (2.17) is faster than Abbas *et al.* iterative procedure (2.16)

So, by the above mentioned discussion we can comment that the Thakur*et al.* iterative procedure (2.17) is faster than that of all the iterative procedures(2.1), (2.11), and (2.13) to (2.16). Finally, by the rest two sections of this chapter we have shown that the rate of convergence of our new iterative procedure (2.18) is faster than that of Thakur*et al.* iterative procedure (2.17) for contraction mapping in the sense of V. Berinde [48], that is the rate of convergence of our new iterative procedure (2.18) is faster than that of all the iterative proceduresdefined by (2.1), (2.11), and (2.13) to (2.17).

3.6 Analytical Comparison of rate of convergence of our new iterative procedure withThakur *et al.* iterative procedure

In this section, we have analytically shown that our proposed new iterative procedure (2.18) converges faster than Thakur*et al.* iterative procedure (2.17) in the sense of V. Berinde [48].

Theorem 3.6.1.*Let B be a nonempty closed convex subset of a norm space X. Let T be a contraction mapping on B with the contractive constant $\lambda \in (0, 1)$ and p be a fixed point of T. Let $\{u_n\}$ be the sequence generated by Thakur et al. iterative procedure(2.17) and $\{x_n\}$ be the sequence generated byour new iterative procedure(2.18), where$\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Then the iterative procedurerepresented by $\{x_n\}$ converges faster than the iterative procedurerepresented by$\{u_n\}$. That is our new iterative procedure (2.18) converges faster than Thakur et al. iterative procedure (2.17).*

Proof. From the Thakur *et al.* iterative procedure (2.17), we have

$$\left.\begin{array}{r} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Tv_n, \\ v_n = (1-\beta_n)w_n + \beta_n Tw_n, \\ w_n = (1-\gamma_n)u_n + \gamma_n Tu_n;\ \forall\, n \in \mathbb{N}. \end{array}\right\} \qquad (3.10)$$

Now, using (3.10) and according to the Theorem 2.3 of Thakur *et al.* [9], we obtain

$$\|u_{n+1} - p\| \leq k^n[1-(1-k^2)\alpha\beta\gamma]^n \|u_1 - p\|, \forall\, n \in \mathbb{N}. \tag{3.11}$$

Let $s_n = k^n[1-(1-k^2)\alpha\beta\gamma]^n$. (3.12)

Now from the 3rd equation of our new iterative procedure (2.18), we have

$$\|z_n - p\| = \|(1-\gamma_n)x_n + \gamma_n T x_n - p\|$$
$$\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\|$$
$$\leq (1-\gamma_n)\|x_n - p\| + \gamma_n k\|x_n - p\|$$
$$\leq [1-(1-k)\gamma_n]\|x_n - p\| \tag{3.13}$$

From the 2nd equation of our new iterative procedure (2.18), we have

$$\|y_n - p\| = \|(1-\beta_n)Tx_n + \beta_n T z_n - p\|$$
$$\leq (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\|$$
$$\leq (1-\beta_n)k\|x_n - p\| + \beta_n k\|z_n - p\|$$
$$\leq (1-\beta_n)k\|x_n - p\| + \beta_n k[1-(1-k)\gamma_n]\|x_n - p\|$$
$$= k[1-(1-k)\beta_n\gamma_n]\|x_n - p\| \tag{3.14}$$

Finally, from the 1st equation of our new iterative procedure (2.18), we have

$$\|x_{n+1} - p\| = \|(1-\alpha_n)Tx_n + \alpha_n T y_n - p\|$$
$$\leq (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\|$$
$$\leq (1-\alpha_n)k\|x_n - p\| + \alpha_n k\|y_n - p\|$$
$$\leq (1-\alpha_n)k\|x_n - p\| + \alpha_n k^2[1-(1-k)\beta_n\gamma_n]\|x_n - p\|$$
$$= k\big[1-\alpha_n + \alpha_n k[1-(1-k)\beta_n\gamma_n]\big]\|x_n - p\|$$
$$\leq k\big[1-\alpha_n k + \alpha_n k[1-(1-k)\beta_n\gamma_n]\big]\|x_n - p\|$$
$$= k[1-\alpha_n k + \alpha_n k - k(1-k)\alpha_n\beta_n\gamma_n]\|x_n - p\|$$
$$= k[1-k(1-k)\alpha_n\beta_n\gamma_n]\|x_n - p\|$$
$$\leq k[1-k^2(1-k)\alpha_n\beta_n\gamma_n]\|x_n - p\|$$
$$\leq k^n[1-k^2(1-k)\alpha\beta\gamma]^n\|x_1 - p\|$$

i.e., $\|x_{n+1} - p\| \leq k^n[1-k^2(1-k)\alpha\beta\gamma]^n\|x_1 - p\|$. (3.15)

Again, let $r_n = k^n[1-k^2(1-k)\alpha\beta\gamma]^n$. (3.16)

Now from (3.12) and (3.16) we obtain,

$$\frac{r_n}{s_n} = \frac{k^n\left[1-k^2(1-k)\alpha\beta\gamma\right]^n}{k^n[1-(1-k^2)\alpha\beta\gamma]^n}$$

$$= \frac{\left[1-k^2(1-k)\alpha\beta\gamma\right]^n}{[1-(1-k^2)\alpha\beta\gamma]^n} \qquad (3.17)$$

Taking limit as n tends to ∞ on both sides of (3.17), we get

$$\lim_{n\to\infty}\frac{r_n}{s_n} = 0.$$

Therefore, according to the definitions (3.4.1) and (3.4.2) we can conclude that, our iterative procedurerepresented by $\{x_n\}$ converges faster than the Thakur *et al.* iterative procedurerepresented by$\{u_n\}$.■

3.7 Numerical and graphical Comparison of rate of convergence of our new iterative procedurewith Thakur *et al.* iterative procedure

For proper numerical comparison here we consider the Example 3.5.3 (Example 3 of Thakur*et al.* [12]) and explain numerically and graphically that the rate of convergence of our iterative procedure (2.18) is faster than that of the Picarditerative procedure (2.1), the Manniterative procedure (2.11), the Ishikawaiterative procedure (2.13), the Nooriterative procedure (2.14), the Agarwal*et al.*iterative procedure (2.15), the Abbas *et al.*iterative procedure (2.16), and the Thakur*et al.*iterative procedure (2.17). The numerical and graphical comparisons are shown in the Table 3.7.1 and Figure 3.1 respectively. From this comparison it has been shown that all of above mentioned iterative procedures converge to the fixed point $p = 5$ after different number of iterative steps, but our new iterative procedure (2.17) converges to this fixed point after minimum number of iterative steps.

Table 3.7.1: Comparative results given by different iterative procedures for example3.5.3.

Step No.	Picard	Mann	Ishikawa	Noor	Agarwal et al.	Abbas et al.	Thakur et al.	New iteration
1	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000
2	36.3318042492	36.8820336118	34.8751575132	33.9816211055	34.3249281505	34.2399531822	32.9458774280	32.3527454021
3	32.7008496221	33.7905308732	29.8335259837	28.0882816012	29.2099663794	28.5873914017	26.0696692526	24.9217277929
4	29.1159538575	30.7306375124	24.9067432334	22.3811620460	24.2839479878	23.0905148078	19.4826041425	17.8627609012
5	25.5892777970	27.7090706072	20.1467307646	16.9736024952	19.5407939247	17.8350979079	13.4242477938	11.5463790857
6	22.1381326176	24.7347891266	15.6449263114	12.0962209155	15.0671172216	12.9887334680	8.4745882697	6.9395376930
7	18.7880774656	21.8200359935	11.5741197024	8.2289280979	11.0428213030	8.9032413368	5.7279660470	5.2225332988
8	15.5784221001	18.9820007784	8.2638548016	6.0182077910	7.8146051254	6.2123720180	5.0765141830	5.0138389524
9	12.5721859009	16.2455313784	6.1736938982	5.2517005165	5.8654600160	5.2064678069	5.0064676549	5.0007808271
10	9.8733161157	13.6475866165	5.3185408455	5.0576355955	5.1788064630	5.0252795464	5.0005330507	5.0000437773
11	7.6482574613	11.2442765494	5.0768890301	5.0129587850	5.0349290827	5.0028941692	5.0000438381	5.0000024535
12	6.1081734180	9.1201110370	5.0179832209	5.0029016212	5.0075152828	5.0003285479	5.0000036046	5.0000001375
13	5.3333287129	7.3913650188	5.0041744485	5.0006491038	5.0017032369	5.0000372607	5.0000002964	5.0000000077
14	5.0771808572	6.1732610225	5.0009673150	5.0001451769	5.0003922866	5.0000042253	5.0000000244	5.0000000004
15	5.0160062399	5.4814708358	5.0002240577	5.0000324684	5.0000907312	5.0000004791	5.0000000020	5.0000000000
16	5.0032258274	5.1725897008	5.0000518932	5.0000072614	5.0000210066	5.0000000543	5.0000000002	5.0000000000
17	5.0006461643	5.0576419946	5.0000120186	5.0000016240	5.0000048648	5.0000000062	5.0000000000	5.0000000000
18	5.0001292729	5.0187159301	5.0000027835	5.0000003632	5.0000011267	5.0000000007	5.0000000000	5.0000000000
19	5.0000258562	5.0060176595	5.0000006447	5.0000000812	5.0000002609	5.0000000001	5.0000000000	5.0000000000
20	5.0000051713	5.0019286052	5.0000001493	5.0000000182	5.0000000604	5.0000000000	5.0000000000	5.0000000000
21	5.0000010343	5.0006174572	5.0000000346	5.0000000041	5.0000000140	5.0000000000	5.0000000000	5.0000000000
22	5.0000002069	5.0001976174	5.0000000080	5.0000000009	5.0000000032	5.0000000000	5.0000000000	5.0000000000
23	5.0000000414	5.0000632408	5.0000000019	5.0000000002	5.0000000008	5.0000000000	5.0000000000	5.0000000000
24	5.0000000083	5.0000202374	5.0000000004	5.0000000000	5.0000000002	5.0000000000	5.0000000000	5.0000000000
25	5.0000000017	5.0000064760	5.0000000001	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
26	5.0000000003	5.0000020723	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
27	5.0000000001	5.0000006631	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
28	5.0000000000	5.0000002122	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
29	5.0000000000	5.0000000679	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
30	5.0000000000	5.0000000217	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
31	5.0000000000	5.0000000070	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
32	5.0000000000	5.0000000022	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
33	5.0000000000	5.0000000007	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
34	5.0000000000	5.0000000002	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
35	5.0000000000	5.0000000001	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000
36	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000

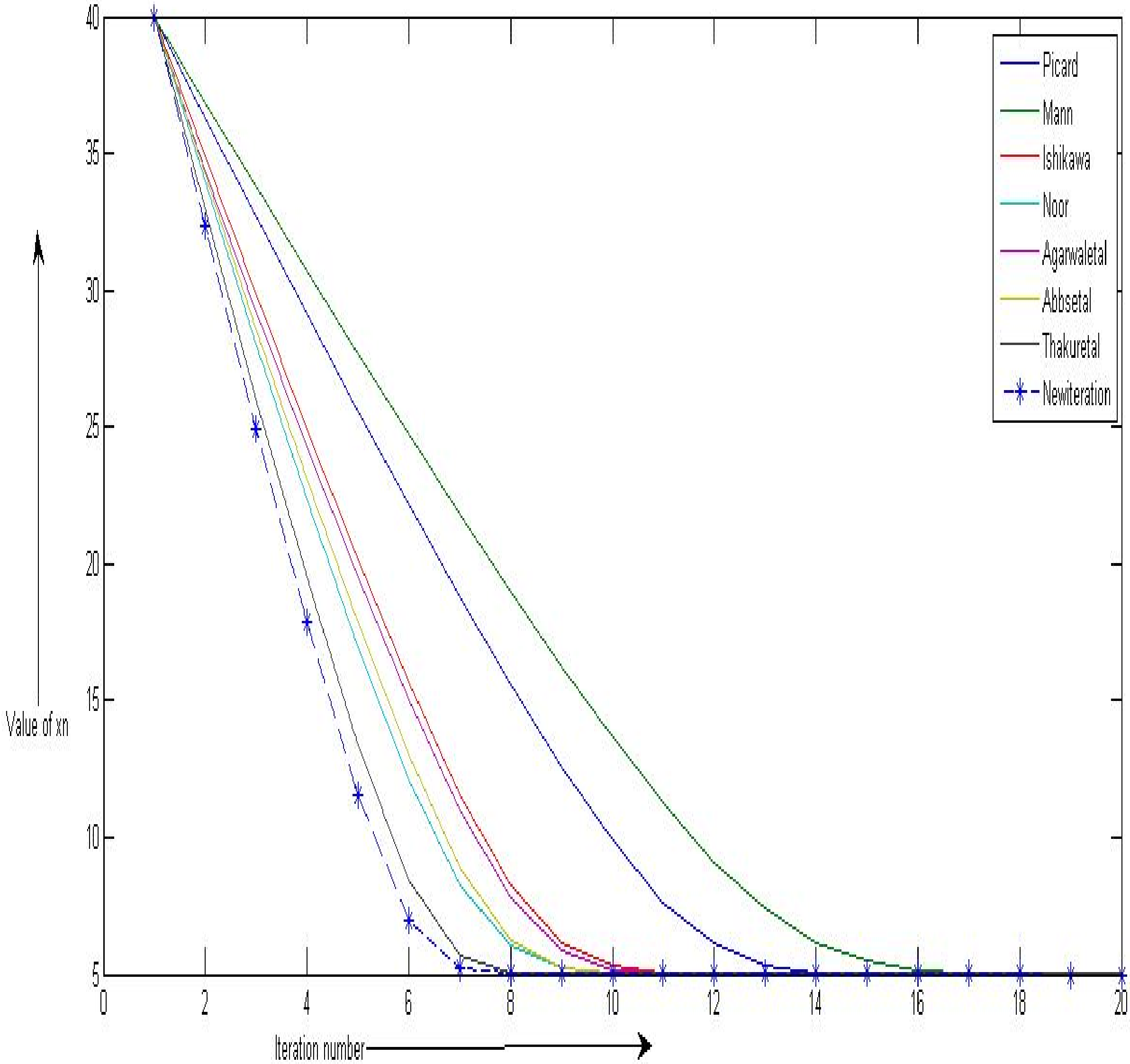

Figure-3.1: Graphical representation of convergence behavior of different iterativeprocedures along with our new iterative procedure (2.18) for the contraction mapping described in the Example 3.5.3.

CHAPTER-4

SOME CONVERGENCE THEOREMS OF NEW FIXED POINT ITERATIVE PROCEDURE VIA NON-EXPANSIVE MAPPING

In this chapter, we have established some weak and strong convergence theorems for non-expansive mapping using our new iterative procedure (2.18), which have extended the results of several authors cited in [6, 29, 30, 36, 41, 47, 56, 58, 59,64].

4.1 Some essential lemmas

In this section we state and prove some lemmas which are used as tools to prove the weak and strong convergence theorems of our new iterative procedure.

Lemma 4.1.1.*Let B be a nonempty closed convex subset of a norm space X. Let T be a non-expansive mapping on B, $\{x_n\}$ be a sequence defined by the iterative procedure (2.18), where$\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$ and $F(T)$ is non-empty. Then the limit $lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$.*

Proof. Let p be a fixed point of T, i.e. $p \in F(T)$. Then for all $n \in \mathbb{N}$, from our iterative procedure (2.18) we have,

$\|z_n - p\| = \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\|$

$\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\|$

$\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\|$

i.e.,$\|z_n - p\| \leq \|x_n - p\|$,(4.1)

and

$\|y_n - p\| = \|(1-\beta)Tx_n + \beta_n Tz_n - p\|$

$\leq (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\|$

$\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\|$

$\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\|$

i.e.,$\|y_n - p\| \leq \|x_n - p\|$(4.2)

Now combining (4.2) and the 3rd equation of (2.18), we have

$\|x_{n+1} - p\| = \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\|$

$\leq (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\|$

$\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\|$

$\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\|$

$=\|x_n - p\|$

i.e., $\|x_{n+1} - p\| \leq \|x_n - p\|$. (4.3)

Taking limit as n tends to ∞ on both sides of (4.3), we get

$\lim_{n\to\infty}\|x_{n+1} - p\| \leq \lim_{n\to\infty}\|x_n - p\|$.

This proves that the limit $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$. ■

Now we state a lemma of Schu [18, 19], which will be need to prove our next lemma.

Lemma 4.1.2.[18] *Suppose that* X *is uniformly convex Banach space and* $0 < p \leq t_n \leq q < 1$ *for all* $n \in \mathbb{N}$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of* X *such that* $\limsup_{n\to\infty}\|x_n\| \leq r$, $\limsup_{n\to\infty}\|y_n\| \leq r$ *and* $\limsup_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = r$ *hold for some* $r \geq 0$. *Then* $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 4.1.3.*Let* B *be a nonempty closed convex subset of a uniformly convex Banach space* X. *Let* T *be a non-expansive mapping on* B, $\{x_n\}$ *be a sequence defined by our new iterative procedure (2.18), where*$\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ *for all* $n \in \mathbb{N}$ *and for some* $\xi \in (0,1)$ *and* $F(T)$ *is non-empty . Then* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Proof. Let p be a fixed point of T, i.e. $p \in F(T)$. Then by our Lemma 4.1, we have $\lim_{n\to\infty}\|x_n - p\|$ exists. Assume that $\lim_{n\to\infty}\|x_n - p\| = l$.

Now, from (4.1) and (4.2), we get

$$\limsup_{n\to\infty}\|z_n - p\| \le l \tag{4.4}$$

and

$$\limsup_{n\to\infty}\|y_n - p\| \le l. \tag{4.5}$$

Since T is a non-expansive mapping and p is a fixed point of T, therefore we have

$$\|Tx_n - p\| \le \|x_n - p\| \tag{4.6}$$

and

$$\|Ty_n - p\| \le \|y_n - p\|. \tag{4.7}$$

Taking lim sup as n tends to ∞ on both sides of (4.6) and (4.7), and combining with (4.4) and (4.5), we get

$$\limsup_{n\to\infty}\|Tx_n - p\| \le l \tag{4.8}$$

and

$$\limsup_{n\to\infty}\|Ty_n - p\| \le l. \tag{4.9}$$

Since

$$\lim_{n\to\infty}\|x_{n+1} - p\| = \lim_{n\to\infty}\|x_n - p\| = l$$

$$\text{i.e., } \lim_{n\to\infty}\|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| = l. \tag{4.10}$$

So, from (4.8), (4.9), (4.10) and the Lemma 4.1.2, we have

$$\lim_{n\to\infty}\|Tx_n - Ty_n\| = 0. \tag{4.11}$$

We have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\| \\ &\le \|Tx_n - p\| + \alpha_n\|Tx_n - Ty_n\|,\end{aligned}$$

which yields

$$l \le \liminf_{n\to\infty}\|Tx_n - p\|. \tag{4.12}$$

Combining (4.8) and (4.12), we obtain

$$\lim_{n\to\infty}\|Tx_n - p\| = l. \tag{4.13}$$

Now, we have

$\|Tx_n - p\| \leq \|Tx_n - Ty_n\| + \|Ty_n - p\| \leq \|Tx_n - Ty_n\| + \|y_n - p\|,$

which yields

$l \leq \liminf_{n\to\infty} \|y_n - p\|.$

(4.14)

From (4.5) and (4.14), we obtain

$$\lim_{n\to\infty} \|y_n - p\| = l. \quad (4.15)$$

Again, since T is a non-expansive mapping and p is a fixed point of T, therefore we have

$\limsup_{n\to\infty} \|Tz_n - p\| \leq \limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\|,$

which yields

$$\limsup_{n\to\infty} \|Tz_n - p\| \leq l. \quad (4.16)$$

Since

$\lim_{n\to\infty} \|y_n - p\| = l.$

$$\Rightarrow \lim_{n\to\infty} \|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| = l$$

$$\Rightarrow \lim_{n\to\infty} \|(1-\beta_n)(Tx_n - p) + \beta_n(Tz_n - p)\| = l. \quad (4.17)$$

From (4.8), (4.16), (4.17) and the Lemma 4.1.2, we obtain

$$\lim_{n\to\infty} \|Tx_n - Tz_n\| = 0. \quad (4.18)$$

Again, since T is a non-expansive mapping and p is a fixed point of T, therefore we have

$\|Tx_n - p\| \leq \|Tx_n - Tz_n\| + \|Tz_n - p\|$

$\leq \|Tx_n - Tz_n\| + \|z_n - p\| \leq \|z_n - p\|,$

which yields

$l \leq \liminf_{n\to\infty} \|z_n - p\|.$

(4.19)

From (4.4) and (4.19), we obtain

$$\lim_{n\to\infty} \|z_n - p\| = l. \quad (4.20)$$

Now, from (4.20) we have,

$l = \lim_{n\to\infty} \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\|$

$= \lim_{n\to\infty} \|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\|.$ (4.21)

From (4.8), (4.21) and the Lemma 4.1.2, we obtain

$$\lim_{n\to\infty} \|x_n - T\mathrm{x}_n\| = 0.$$

This proofs our lemma.■

Now, we state a lemma of Bruck [35], which will be help to prove our next lemma.

Lemma 4.1.4.[35] *Let B be a nonempty bounded closed convex subset of a uniformly convex Banach space X and $T: B \to X$ be a non-expansive mapping. Then there is a strictly increasing and continuous convex function $g: [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$g(\|T(tx + (1-t)y) - (tTx + (1-t)Ty)\|) \le \|x - y\| - \|Tx - Ty\|,$$

for all $x, y \in B$ and $t \in [0,1]$.

Lemma 4.1.5. *Let B be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a non-expansive mapping on B, $\{x_n\}$ be a sequence defined by our new iterative procedure (2.18), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$ and $F(T)$ is non-empty. Then for any $p_1, p_2 \in F(T)$, $lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ exists, for all $t \in [0,1]$.*

Proof. By Lemma 4.1.1 we have, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$ and hence $\{x_n\}$ is bounded. So there exists a real number $m > 0$ such that $\{x_n\} \subseteq D$, where D is a closed convex nonempty subset of B. Let $a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$ for all $t \in [0,1]$. Then $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - p_2\|$, hence from Lemma 4.1.1 $\lim_{n\to\infty} a_n(1)$ exist.

Therefore, in order to complete this lemma, it is sufficient to show that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in [0,1]$.

Now for each $n \in \mathbb{N}$, we define the maps $Q_n, R_n,$ and S_n on D by

$$\left.\begin{aligned} S_n x &= (1-\alpha_n)Tx + \alpha_n T R_n x \\ R_n x &= (1-\beta_n)Tx + \beta_n T Q_n x \\ Q_n x &= (1-\gamma_n)x + \gamma_n Tx, \end{aligned}\right\} \tag{4.22}$$

for all $x \in D$.

Since T is non-expansive, so for all $x, y \in D$, we observe that,

$$\begin{aligned} \|Q_n x - Q_n y\| &= \|(1-\gamma_n)x + \gamma_n Tx - (1-\gamma_n)y + \beta_n Ty\| \\ &\le \|x-y\|, \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} \|R_n x - R_n y\| &= \|(1-\beta_n)Tx + \beta_n T Q_n x - (1-\beta_n)Ty + \beta_n T Q_n y\| \\ &\le (1-\beta_n)\|Tx - Ty\| + \beta_n \|T Q_n x - T Q_n y\| \\ &\le (1-\beta_n)\|x-y\| + \beta_n \|Q_n x - Q_n y\| \\ &\le (1-\beta_n)\|x-y\| + \beta_n \|x-y\| \\ &\le \|x-y\|. \end{aligned} \tag{4.24}$$

Hence by using (4.22) and (4.24), we obtain

$$\|S_n x - S_n y\| \le \|x-y\|, \forall\, x, y \in D. \tag{4.25}$$

Now, if we set

$$W_{n,m} = S_{n+m-1} S_{n+m-2} \dots S_n,$$

and

$$b_{n,m} = \left\|W_{n,m}(t x_n + (1-t)p_1) - \left(t W_{n,m} x_n + (1-t)p_1\right)\right\|,$$

for all $m, n \in \mathbb{N}$, then we obtain $W_{n,m} x_n = x_{n+m}$ and $W_{n,m} p = p, \forall\, p \in F(T)$. Hence every fixed point of T is also fixed point of $W_{n,m}$, and we have

$$\left\|W_{n,m} x - W_{n,m} y\right\| \le \|x-y\|,\ \forall\, x, y \in D. \tag{4.26}$$

By Lemma 4.1.4, there exists a strictly increasing continuous convex function $g: [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that

$$\begin{aligned} g\left(b_{n,m}\right) &\le \|x_n - p_1\| - \left\|W_{n,m} x_n - W_{n,m} p_1\right\| \\ &= \|x_n - p_1\| - \|x_{n+m} - p_1\|. \end{aligned} \tag{4.27}$$

Again, since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$, then by (4.27) we obtain

$$\lim_{n,m\to\infty} g\left(b_{n,m}\right) = 0. \tag{4.28}$$

Applying the property of g in (4.27), we have

$$\lim_{n,m\to\infty} b_{n,m} = 0. \tag{4.29}$$

Now, we have

$$\begin{aligned} a_{n+m}(t) &= \|tx_{n+m} + (1-t)p_1 - p_2\| \\ &= \left\|tW_{n,m}x_n + (1-t)p_1 - p_2\right\| \\ &\le b_{n,m} + \left\|W_{n,m}(tx_n + (1-t)p_1) - p_2\right\| \\ &\le b_{n,m} + \left\|W_{n,m}(tx_n + (1-t)p_1) - W_{n,m}p_2\right\| \\ &\le b_{n,m} + \|(tx_n + (1-t)p_1) - p_2\| \\ &= b_{n,m} + a_n(t), (4.30) \end{aligned}$$

for all $t \in (0,1)$.

But, we have

$$\begin{aligned} \limsup_{m\to\infty} a_m(t) &= \limsup_{m\to\infty} a_{n+m}(t) \\ &\le \limsup_{m\to\infty}(b_{n,m} + a_n(t)). \end{aligned} \tag{4.31}$$

Now, from (4.29) and (4.31), we have

$$\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t).$$

This implies that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in (0,1)$. Therefore, $\lim_{n\to\infty} a_n(t)$ exists for all $t \in [0,1]$. This completes our proof.■

Now, according to Lemma 2.3 of Khan and Kim [43], we establish the following lemma, which will be needed to prove our next results.

Lemma 4.1.6. *Let B be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a non-expansive mapping on B, $\{x_n\}$ be a sequence defined by our new iterative procedure (2.18), where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$ and $F(T)$ is non-empty and $lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Then for any $p_1, p_2 \in F(T)$, $lim_{n\to\infty}\langle x_n, J(p_1 - p_2)\rangle$ exists; in particular,*

$$\langle p - q, J(p_1 - p_2)\rangle = 0$$

for all $p, q \in \omega_w(x_n)$, the set of all weak limits of $\{x_n\}$.

Proof. Let $t \in [0,1]$. Put $x = p_1 - p_2$ with $p_1 \neq p_2$ and $h = t(x_n - p_1)$ in the inequality (1.2), we get

$$\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle$$

$$\leq \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2$$

$$\leq \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, J(p_1 - p_2)\rangle + b(t\|x_n - p_1\|). \quad (4.32)$$

But $\limsup_{n\geq 1}\|x_n - p_1\| \leq M'$ for some $M' > 0$, so from (4.32), we have

$$\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle$$

$$\leq \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2$$

$$\leq \frac{1}{2}\|p_1 - p_2\|^2 + t\liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle + b(tM').$$

That is,

$$\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle$$

$$\leq \liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle + \frac{b(tM')}{tM'}M'. \quad (4.33)$$

Now, if we take the limit as $t \to 0$ on both sides of (4.33), then we get,

$$\limsup_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle \leq \liminf_{n\to\infty}\langle x_n - p_1, J(p_1 - p_2)\rangle.$$

This implies that $\lim_{n\to\infty}\langle x_n, J(p_1 - p_2)\rangle$ exists for all $p_1, p_2 \in F(T)$; in particular, we have $\langle p - q, J(p_1 - p_2)\rangle = 0$ for all $p, q \in \omega_w(x_n)$. ■

Lemma 4.1.7. [20] *Let B be a nonempty closed convex subset of a uniformly convex Banach space X and T a non-expansive mapping on B. Then $I - T$ is demiclosed at zero.*

Lemma 4.1.8. [37] *Let X be a reflexive Banach space satisfying the Opial's condition, B a nonempty convex subset of X, and $T: B \to X$ an operator such that $I - T$ is demiclosed at zero and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in B such that $lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $lim_{n\to\infty}\|x_n - p\| = 0$ exists for all $p \in F(T)$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Lemma 4.1.9. [60] *Let X be a real reflexive Banach space such that its dual X^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in $X$and $x^*, y^* \in \omega_w(x_n)$, where $\omega_w(x_n)$ denotes the $w-limit$ set of $\{x_n\}$. Suppose $lim_{n\to\infty}\|tx_n + (1-t)x^* - y^*\| = 0$ exists for all $t \in [0,1]$. Then $x^* = y^*$.*

4.2Weak convergence theorem of our new iterative procedure

In this sectionwe establish a weak convergence theorem of our new iterative procedure (2.18) for non-expansive mapping under differentconditions.

Theorem 4.2.1.*Let B be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a non-expansive mapping on B, $\{x_n\}$ be a sequence defined by our new iterative procedure (2.18), where$\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$ and $F(T)$ is non-empty. If any one of the following conditions holds:*

(i) X satisfies the Opial's condition defined by Definition 1.1.14,

(ii) X has a $Fr\acute{e}chet$ differentiable norm defined by Definition 1.1.13,

(iii) the dual $X^*$of X satisfies the Kadec-Klee property defined by Definition 1.1.15.

Then $\{x_n\}$converges weakly to a point of $F(T)$.

Proof. Let $p \in F(T)$, then by Lemma 4.1.1, $\lim_{n\to\infty}\|x_n - p\|$ exists.

In order to complete our proof, we have to show that $\{x_n\}$has a unique weak subsequential limit in $F(T)$, i.e., $\{x_n\}$converges weakly to a unique fixed point of T.

Let u and v be two weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$of $\{x_n\}$, respectively. Now, by Lemma 4.1.3, we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, and by Lemma 4.1.7 we have $I - T$ is demiclosed with respect to zero. Hence, for the subsequences $\{x_{n_i}\}$and $\{x_{n_j}\}$, we obtain $Tu = u$ and $Tv = v$ respectively, i.e., $u, v \in F(T)$.

Now, we prove the uniqueness and for this first assume that the condition (i) holds.

If $u \neq v$, then by Opial's condition, we have

$$\lim_{n\to\infty}\|x_n - u\| = \lim_{i\to\infty}\left\|x_{n_i} - u\right\| < \lim_{i\to\infty}\left\|x_{n_i} - v\right\|$$
$$= \lim_{n\to\infty}\|x_n - v\|$$
$$= \lim_{j\to\infty}\left\|x_{n_j} - v\right\| < \lim_{j\to\infty}\left\|x_{n_j} - u\right\|$$
$$= \lim_{n\to\infty}\|x_n - u\|$$

i.e., $\lim_{n\to\infty}\|x_n - u\| < \lim_{n\to\infty}\|x_n - u\|$.

This is a contradiction. Hence $u = v$.

Next, we assume that the condition (ii) holds.

Now, by Lemma 4.1.8, we have$\langle p - q, J(p_1 - p_2)\rangle = 0$, for all $p, q \in \omega_w(x_n)$. Therefore, $\|u - v\|^2 = \langle u - v, J(u - v)\rangle = 0$ implies $u - v$.

Finally, we assume that the condition (iii) holds.

By Lemma 4.1.5, we have, $\lim_{n\to\infty}\|tx_n + (1-t)u - v\|$ exists for all $t \in [0,1]$and $u, v \in F(T)$. Hence by Lemma 4.1.9, we have $u = v$, and by Lemma 4.1.8, we have $\{x_n\}$converges weakly to a unique fixed point of T.

This completes the proof. ■

4.3 Strong convergence theorems for our new iterative procedure

In this sectionwe establish some strong convergence theorem of our new iterative procedure (2.18) for non-expansive mappingunder differentconditions.

Theorem 4.3.1 *LetB be a nonempty closed convex subset of a uniformly convex Banach space* X. *Let* T *be a non-expansive mapping on* B, $\{x_n\}$ *be a sequence defined by our new iterative procedure (2.18), where*$\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ *for all* $n \in \mathbb{N}$ *and for some* $\xi \in (0,1)$ *and* $F(T)$ *is non-empty. If* T *is semicompact, then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. According to our assumption and by Lemma 4.1.3, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Now, since T is semicompact, hence $\{x_n\}_{n=0}^{\infty}$ has a subsequence $\left\{x_{n_j}\right\}$ (say) converging to some $p \in B$ as B is closed. Then by the continuity of T, we have

$$\lim_{j\to\infty} \left\|Tx_{n_j} - Tp\right\| = 0. \quad (4.34)$$

By applying Lemma 4.1.3 in (4.34), we obtain

$$\|Tp - p\| = 0.$$

This confirm that p is a fixed point of$T$, i.e., $p \in F(T)$.

Now, by Lemma 4.1.1, we can say that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F(T)$. Therefore, $\{x_n\}$ must converges to $p \in F(T)$.

This completes the proof.■

Theorem 4.3.2*Let$B$ be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a non-expansive mapping on B, $\{x_n\}$ be a sequence defined by our new iterative procedure (2.18), where$\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$ and $F(T)$ is non-empty. Then $\{x_n\}_{n=0}^{\infty}$ converges to a point of $F(T)$ if and only if*$\liminf_{n\to\infty} d\big(x_n, F(T)\big) = 0$, *where* $d\big(x_n, F(T)\big) = inf\{\|x_n - p\|: p \in F(T)\}$.

Proof. First suppose that, $\liminf_{n\to\infty} d\big(x_n, F(T)\big) = 0$. Now, by our assumption and Lemma 4.1.3, we have $\lim_{n\to\infty}\|x_n - p\|$ exists for all$p \in F(T)$, thus $\lim_{n\to\infty} d\big(x_n, F(T)\big)$ exists. But by hypothesis,$\liminf_{n\to\infty} d\big(x_n, F(T)\big) = 0$, therefore $\lim_{n\to\infty} d\big(x_n, F(T)\big) = 0$.

Now, we have to show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in$B$. Since $\lim_{n\to\infty} d\big(x_n, F(T)\big) = 0$, hence for given $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have

$$d\big(x_n, F(T)\big) < \frac{\varepsilon}{2}.$$

Particularly,

$$\inf\{\|x_n - p\|: p \in F(T)\} < \frac{\varepsilon}{2}.$$

Hence, there exists a $p_1 \in F(T)$ such that

$$\left\|x_{n_0} - p_1\right\| < \frac{\varepsilon}{2}. \tag{4.35}$$

Now, for $m, n \geq n_0$, we have

$$\|x_m - x_n\| \leq \|x_m - p_1\| + \|x_n - p_1\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e., $\|x_m - x_n\| \leq \varepsilon$.

This proves that $\{x_n\}$ is a Cauchy sequence in$B$.

Now, since B is a closed subset of a Banach space, that is B is a closed subset of a complete space. Hence the sequence $\{x_n\}$ is a convergent sequence in$B$, and for $p_2 \in B$ we have

$$\lim_{n\to\infty} x_n = p_2. \tag{4.36}$$

Again, since $F(T)$ is closed, therefore $\lim_{n\to\infty} d\big(x_n, F(T)\big) = 0$ and (4.35) gives

$$d\big(p_2,\ F(T)\big) = 0, i.e., p_2 \in\ F(T).$$

Therefore, $\{x_n\}$ converges to a point of $F(T)$.

Conversely, suppose $\{x_n\}$ converges to a point of$F(T)$, hence for all $p_3 \in\ F(T)$, we have

$$\lim_{n\to\infty} x_n = p_3$$

$$\Longrightarrow \lim_{n\to\infty} \|x_n - p_3\| = 0. \tag{4.37}$$

But, we have$d\big(x_n, F(T)\big) = \inf\{\|x_n - p_3\| : p_3 \in F(T)\}$, therefore from (4.37), we obtain

$$\lim_{n\to\infty} d\big(x_n, F(T)\big) = 0$$

$$\Longrightarrow \liminf_{n\to\infty} d\big(x_n, F(T)\big) = 0.$$

This completes the proof.■

Theorem 4.3.3*Let$B$ be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a non-expansive mapping on B, $\{x_n\}$ be a sequence defined by our new iterative procedure (2.7), where$\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$ and $F(T)$ is non-empty. If T satisfy*

Condition (I)defined by Definition 1.1.18, then $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. By Lemma 4.1.3, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{4.38}$$

Since T satisfy Condition (I), hence from (4.38), we have

$$\lim_{n\to\infty} h\left(d\left(x_n, F(T)\right)\right) \le \lim_{n\to\infty}\|x_n - Tx_n\| = 0,$$

$$\Rightarrow \lim_{n\to\infty} h\left(d\left(x_n, F(T)\right)\right) = 0.$$

$$\Rightarrow h\left(\lim_{n\to\infty} d\left(x_n, F(T)\right)\right) = 0. \tag{4.39}$$

According to the Condition (I), we have $h: [0,\infty) \to [0,\infty)$ is a non-decreasing function with $h(0) = 0, h(r) > 0$ for all $r \in (0,1)$. Hence from (4.39), we obtain

$$\lim_{n\to\infty} d\left(x_n, F(T)\right) = 0,$$

$$\Rightarrow \liminf_{n\to\infty} d\left(x_n, F(T)\right) = 0.$$

Therefore, we observe that all the conditions of our theorem 4.3.1 already have satisfied.Hence by the theorem 4.3.1, we can say that $\{x_n\}$ converges strongly to a fixed point of T.

This completes the proof.■

CHAPTER-5

SOME FIXED POINT ITERATIVE PROCEDURES WITH ERRORS

In this chapter, we will recall some fixed point iterative procedures with errors and state the convergence theorem of S. Plubtieng and R. Wangkeeree [42] for multi-step Noor fixed point iterative procedure with errors.

5.1 Multi-step Noor fixed point iterative procedure with errors

In 2006, S. Plubtieng and R. Wangkeeree [42] introduced the following multi-step Noor fixed point iterative procedure with errors:

Definition 5.1.1. [42] Let B be a nonempty subset of an arbitrary normed space X and let T be a mapping from B into itself. Then the multi-step Noor fixed point iterative procedure with errors is defined as follows.

For a given, $u_0 \in B$, and a fixed $m \in \mathbb{N}$, compute the iterative sequences $\left\{u_n^{(1)}\right\}, \dots, \left\{u_n^{(m)}\right\}$defined by

$$\left.\begin{aligned} u_n^{(1)} &= a_n^{(1)} T u_n + b_n^{(1)} u_n + c_n^{(1)} v_n^{(1)} \\ u_n^{(2)} &= a_n^{(2)} T u_n^{(1)} + b_n^{(2)} u_n + c_n^{(2)} v_n^{(2)} \\ u_n^{(3)} &= a_n^{(3)} T u_n^{(2)} + b_n^{(3)} u_n + c_n^{(3)} v_n^{(3)} \\ &\vdots \\ &\vdots \\ u_n^{(m-1)} &= a_n^{(m-1)} T u_n^{(m-2)} + b_n^{(m-1)} u_n + c_n^{(m-1)} v_n^{(m-1)} \\ u_{n+1} = u_n^{(m)} &= a_n^{(m)} T u_n^{(m-1)} + b_n^{(m)} u_n + c_n^{(m)} v_n^{(m)}, n \in \mathbb{N} \end{aligned}\right\} \quad (5.1)$$

where,$\left\{v_n^{(1)}\right\}, \left\{v_n^{(2)}\right\}, \left\{v_n^{(3)}\right\}, \dots, \left\{v_n^{(m)}\right\}$ are bounded sequences in B and $\left\{a_n^{(i)}\right\}, \left\{b_n^{(i)}\right\}, \left\{c_n^{(i)}\right\}$ are appropriate real sequences in $(0,1)$such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ for each $i \in \{1, 2, \cdots, m\}$.

The following lemma and theorem established by S. Plubtieng and R. Wangkeeree[42] to prove the convergence of multi-step Noor fixed point iterative procedure with errors for asymptotically non-expansive mapping in the indeterminate sense.

Lemma 5.1.2.[42]*Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T: C \to C$ be continuous asymptotically non-expansive mapping in the intermediate sense. Put*

$$G_n = sup_{x,y \in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0, \ \forall\, n \geq 1,$$

so that $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\left\{u_n^{(k)}\right\}$be defined by (5.1) with the following restrictions:

(i) $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1$ foralli $\in \{1, 2, 3, \dots, m\}$ and for all $n \geq 1$;

(ii)$n \geq 1 \sum_{n=1}^{\infty} c_n^{(i)} < \infty$for all $i \in \{1, 2, 3, \dots, m\}$.

If $p \in F(T)$, then $lim_{n\to\infty} \|x_n - p\|$exists.

Theorem 5.1.3. [42]*Let X be a uniformly convex Banach space, C a nonempty closed bounded convex subset of X and $T: C \to C$ be continuous asymptotically non-expansive mapping in the intermediate sense. Put*

$$G_n = sup_{x,y \in C}(\|T^n x - T^n y\| - \|x - y\|) \vee 0, \ \forall\, n \geq 1,$$

sothat $\sum_{n=1}^{\infty} G_n < \infty$. Let the sequence $\left\{u_n^{(k)}\right\}$be defined by (5.1) whenever $\left\{a_n^{(i)}\right\}, \left\{b_n^{(i)}\right\}, \left\{c_n^{(i)}\right\}$ satisfy the same assumptions as in Lemma 5.1.2 for each $i \in \{1, 2, 3, \dots, m\}$ and the additional assumption that $0 < a \leq a_n^{(m-1)}, a_n^{(m)} \leq b < 1$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T for each $k = 1, 2,3, \dots, m$.

5.2 Noor iterative procedure with errors defined by Cho *et al.*

In 2004, Y.J. Cho, H. Zhou, G. Guo [63] introduced the following three-step Noor fixed point iterative procedure with errors:

Definition 5.2.1.[63] Let B be a nonempty subset of an arbitrary normed space X and let T be a mapping from B into itself. Then the Noor fixed point iterative procedure with errors is defined as follows.

For a given, $u_0 \in B$,compute the iterative sequences $\left\{u_n^{(1)}\right\}$, $\left\{u_n^{(2)}\right\}, \left\{u_n^{(3)}\right\}$ defined by

$$\left.\begin{aligned} u_n^{(1)} &= a_n^{(1)} T u_n + \left(1 - a_n^{(1)} - c_n^{(1)}\right) u_n + c_n^{(1)} v_n^{(1)} \\ u_n^{(2)} &= a_n^{(2)} T u_n^{(1)} + \left(1 - a_n^{(2)} - c_n^{(2)}\right) u_n + c_n^{(2)} v_n^{(2)} \\ u_{n+1} = u_n^{(3)} &= a_n^{(3)} T u_n^{(2)} + \left(1 - a_n^{(3)} - c_n^{(3)}\right) u_n + c_n^{(3)} v_n^{(2)}, n \in \mathbb{N} \end{aligned}\right\} \quad (5.2)$$

where$\left\{a_n^{(i)}\right\}$, $\left\{c_n^{(i)}\right\}$are appropriate real sequences in $(0, 1)$for all $i \in \{1,2,3\}$..

If $c_n^{(1)} = c_n^{(2)} = c_n^{(3)} \equiv 0$, then (5.2) reduces to following Noor iterative procedurewhich is given by Noor et al. [19-22]:

$$\left.\begin{aligned} u_n^{(1)} &= a_n^{(1)} T u_n + \left(1 - a_n^{(1)}\right) u_n \\ u_n^{(2)} &= a_n^{(2)} T u_n^{(1)} + \left(1 - a_n^{(2)}\right) u_n \\ u_{n+1} = u_n^{(3)} &= a_n^{(3)} T u_n^{(2)} + \left(1 - a_n^{(3)}\right) u_n,\ n \in \mathbb{N} \end{aligned}\right\} \quad (5.3)$$

where$\left\{a_n^{(i)}\right\}$ are appropriate real sequences in $(0, 1)$for all $i \in \{1,2,3\}$.

5.3 Ishikawa iterative procedure with errors defined by Y. Xu

In 1998, Y. Xu [62] introduced the following two-step Ishikawa fixed point iterative procedure with errors:

Definition 5.3.1. [62] Let B be a nonempty subset of an arbitrary normed space X and let T be a mapping from B into itself. Then the Ishikawa fixed point iterative procedure with errors defined by Y. Xu is defined as follows.

For a given, $u_0 \in B$ compute the iterative sequences $\left\{u_n^{(1)}\right\}$, $\left\{u_n^{(2)}\right\}$defined by

$$\left.\begin{aligned} u_n^{(1)} &= a_n^{(1)}Tu_n + b_n^{(1)}u_n + c_n^{(1)}v_n^{(1)} \\ u_{n+1} = u_n^{(2)} &= a_n^{(2)}Tu_n^{(1)} + b_n^{(2)}u_n + c_n^{(2)}v_n^{(2)},\ n \in \mathbb{N} \end{aligned}\right\} \tag{5.4}$$

where$\left\{v_n^{(1)}\right\}, \left\{v_n^{(2)}\right\}$ are bounded sequences in B and $\left\{a_n^{(i)}\right\}, \left\{b_n^{(i)}\right\}, \left\{c_n^{(i)}\right\}$are appropriate real sequences in $[0, 1]$for all $i \in \{1,2\}$.

If $b_n^{(i)} = 1 - a_n^{(i)}$ and $c_n^{(i)} \equiv 0$ for all $i = 1, 2$, then (5.4) reduces to the following two-step Ishikawa iterative procedure defined by S. Ishikawa [11]:

$$\left.\begin{aligned} u_n^{(1)} &= a_n^{(1)}Tu_n + (1 - a_n^{(1)})u_n \\ u_{n+1} = u_n^{(2)} &= a_n^{(2)}Tu_n^{(1)} + (1 - a_n^{(2)})u_n,\ n \in \mathbb{N} \end{aligned}\right\} \tag{5.5}$$

where$\left\{a_n^{(i)}\right\}$ are appropriate real sequences in $(0, 1)$for all $i \in \{1,2\}$.

5.4 Ishikawa iterative procedure with errors defined by L.S. Lu

In 1995, L.S. Lu [22] introduced the following two-step Ishikawa fixed point iterative procedure with errors:

Definition 5.4.1. [22] Let B be a nonempty subset of an arbitrary normed space X and let T be a mapping from B into itself. Then the Ishikawa fixed point iterative procedure with errors defined by L.S. Lu is defined as follows.

For a given, $u_0 \in B$ compute the iterative sequences $\left\{u_n^{(1)}\right\}$, $\left\{u_n^{(2)}\right\}$defined by

$$\left.\begin{aligned} u_n^{(1)} &= a_n^{(1)}Tu_n + b_n^{(1)}u_n + v_n^{(1)} \\ u_{n+1} = u_n^{(2)} &= a_n^{(2)}Tu_n^{(1)} + b_n^{(2)}u_n + v_n^{(2)},\ n \in \mathbb{N} \end{aligned}\right\} \tag{5.6}$$

where$\left\{v_n^{(1)}\right\}, \left\{v_n^{(2)}\right\}$ are bounded sequences in B and$\left\{a_n^{(i)}\right\}, \left\{b_n^{(i)}\right\}$are appropriate real sequences in $(0, 1)$for all $i \in \{1,2\}$.

5.5 Mann iterative procedure with errors defined by Y. Xu

In 1998, Y. Xu [62] introduced the following one-step Mann fixed point iterative procedure with errors:

Definition 5.5.1. [62] Let B be a nonempty subset of an arbitrary normed space X and let T be a mapping from B into itself. Then the Mann fixed point iterative procedure with errors given by Y. Xu[62] is defined as follows.

For a given, $u_0 \in B$ compute the iterative sequence $\left\{u_n^{(1)}\right\}$defined by

$$u_{n+1} = u_n^{(1)} = a_n^{(1)} T u_n + b_n^{(1)} u_n + c_n^{(1)} v_n^{(1)},\ n \in \mathbb{N}\Big\} \tag{5.7}$$

where$\left\{v_n^{(1)}\right\}$ is bounded sequence in B and$\left\{a_n^{(1)}\right\}, \left\{b_n^{(1)}\right\}, \left\{c_n^{(1)}\right\}$are appropriate real sequences in$(0, 1)$.

If $b_n^{(1)} = 1 - a_n^{(1)}$and $c_n^{(1)} = 0$, then (5.7) reduces to the following Mann iterative procedure defined by W. R. Mann [53]:

$$u_{n+1} = u_n^{(1)} = a_n^{(1)} T u_n + (1 - a_n^{(1)}) u_n,\ n \in \mathbb{N}\Big\} \tag{5.8}$$

where $\left\{a_n^{(1)}\right\}$ are appropriate real sequences in$(0, 1)$.

If$b_n^{(1)} = 1 - a_n^{(1)}, c_n^{(1)} = 0$and $a_n^{(1)} = \lambda \in (0, 1)$then (5.8) reduces to the following Krasnoselskij's iterative procedure defined by M. A. Krasnoselskij [23]:

$$u_{n+1} = u_n^{(1)} = \lambda T u_n + (1 - \lambda) u_n,\ n \in \mathbb{N}\Big\}. \tag{5.9}$$

5.6Mann iterative procedure with errors defined by L.S. Lu

In 1995, L.S. Lu [22] introduced the following one-step Mann fixed point iterative procedure with errors:

Definition 5.4.1. [22] Let B be a nonempty subset of an arbitrary normed space X and let T be a mapping from B into itself. Then the Mann fixed point iterative procedure with errors defined by L.S. Lu [18]is defined as follows.

For a given, $u_0 \in B$ compute the iterative sequence$\left\{u_n^{(1)}\right\}$defined by

$$u_{n+1} = u_n^{(1)} = a_n^{(1)} T u_n + b_n^{(1)} u_n + v_n^{(1)},\ n \in \mathbb{N}\Big\} \tag{5.10}$$

where $\left\{v_n^{(1)}\right\}$ is bounded sequence in B and $\left\{a_n^{(1)}\right\}, \left\{b_n^{(1)}\right\}$ are appropriate real sequences in $(0, 1)$.

5.7 Generalization of multi-step Noor fixed point iterative procedure with errors

The Mann iterative procedure (5.8), Ishikawa iterative procedure (5.5), Noor iterative procedure (5.3), Krasnoselskij's iterative procedure (5.9), Mann iterative procedure with errors defined by Y. Xu (5.7), Mann iterative procedure with errors defined by L.S. Lu (5.10), Ishikawaiterative procedure with errors defined by Y. Xu (5.4), Ishikawaiterative procedure with errors defined by L.S. Lu (5.6) and Noor iterative procedure with errors defined Cho *et al.* (5.2) all are special case of multi-step Noor fixed point iterative procedure with errors (5.1), which will be trustworthy by the following discussion:

If $m = 3$ and $b_n^{(i)} = 1 - a_n^{(i)} - c_n^{(i)}$ for all $i = 1, 2, 3$ then multi-step Noor fixed point iterative procedure with errors(5.1) reduces to Noor iterative procedure with errors defined by Cho *et al.*(5.2).

If $m = 3$, $b_n^{(i)} = 1 - a_n^{(i)} - c_n^{(i)}$for all $i = 1, 2, 3$ and $c_n^{(1)} = c_n^{(2)} = c_n^{(3)} \equiv 0$, then multi-step Noor fixed point iterative procedure with errors(5.1) reduces to Noor iterative procedure defined by Noor *et al.*(5.3).

If $m = 2$ thenthe multi-step Noor fixed point iterative procedure with errors (5.1) reduces to Ishikawa iterative procedure with errors defined by Y. Xu (5.4).

If $m = 2$, and $c_n^{(1)} = c_n^{(2)} \equiv 1$ then the multi-step Noor fixed point iterative procedure with errors (5.1) reduces to Ishikawa iterative procedure with errors defined by L.S. Lu (5.6).

If $m = 2$, $b_n^{(i)} = 1 - a_n^{(i)} - c_n^{(i)}$for all $i = 1, 2$and $c_n^{(1)} = c_n^{(2)} \equiv 0$, then the multi-step Noor fixed point iterative procedure with errors (5.1) reduces to Ishikawa iterative procedure (5.5).

If$m = 1$ then the multi-step Noor fixed point iterative procedure with errors (5.1) reduces to Mann iterative procedure with errors defined by Y. Xu(5.7).

If $m = 1$, $c_n^{(1)} = 1$then the multi-step Noor fixed point iterative procedure with errors (5.1)reduces to Mann iterative procedure with errors defined by L.S. Lu (5.10).

If $m = 1$, $b_n^{(1)} = 1 - a_n^{(1)} - c_n^{(1)}$and $c_n^{(1)} = 0$, then the multi-step Noor fixed point iterative procedure with errors (5.1) reduces to Mann iterative procedure(5.8).

If $m = 1$, $b_n^{(1)} = 1 - a_n^{(1)} - c_n^{(1)}$, $c_n^{(1)} = 0$ and $a_n^{(1)} = \lambda \in (0, 1)$then the multi-step Noor fixed point iterative procedure with errors (5.1)reduces to Krasnoselskij's iterative procedure (5.9).

Therefore, it is clear from above discussion that multi-step Noor fixed point iterative procedurewith errors (5.1) is a general iterative procedure among the analogous iterative procedures. From this point of view, in the next chapter we have established convergence theorem of multi-step Noor fixed point iterative procedurewith errors for more general Zamfirescu operator, which generates the convergence theorem of other relevant iterative procedures for Zamfirescu operator.

CHAPTER-6

CONVERGENCE THEOREM OF MULTI-STEP NOOR FIXED POINT ITERATIVE PROCEDURE WITH ERRORS VIA ZAMFIRESCU OPERATORS

In this chapter,we establish a general theorem to approximate fixed point of Zamfirescu operators on an arbitrary normed space through the multi-step Noor fixed point iterative procedure with errors in the sense of S. Plubtieng and R. Wangkeeree [42]. Our result generalizes and improves the corresponding results of A. Rafiq [2], Y. Xu [58], L. S. Liu [22], M. O. Osilike [29] and various authors in literature.

6.1 Background of our convergence theorem of multi-step Noor fixed point iterative procedure with errors

One of the most studied classes of quasi contractive type operators is that of Zamfirescu operator, for which all important fixed point iteration procedures, i.e., the Picard [46], Mann [53], Ishikawa [40], Noor [24-26] iterative procedures are known to converge to the unique fixed point of T. T. Zamfirescu showed in [46] that an operator satisfying conditions in Theorem 1.6.5 has a unique fixed point that can be approximated using the Picard iteration scheme. Later, Rhoades [4, 5] proved that the Mann and Ishikawa iterative procedures can also be used to approximate fixed points of Zamfirescu operator.The class of operators satisfying Zamfirescuconditions is independent; see for instance Rhoades [5]. The class of strictly pseudocontractive operators has been extensively studied by several authors in the last years. For a recent survey and a comprehensive bibliography, we refer to the V. Berinde’s monograph [49]. In 2003, 2004 and 2007, V. Berinde [49-52] proved the convergence theorems in arbitrary Banach spaces of the Mann and Ishikawa iterativeprocedures associated to Zamfirescu operator for extending the results of B.E. Rhoades [5]. In 2006, A. Rafiq [2] extends the result of V. Berinde [49-52]. Recently in 2013, Asaduzzaman*et al.* [33] extend the result of V. Berinde [49-52] and A. Rafiq [2] for Noor iterative procedure using Zamfirescu operator as follows.

Theorem 6.1.1.[33] *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T : B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T, where $F(T)$ denotes the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iterative procedure defined by (1.3)and $x_0 \in B$. Then the Noor iterative procedure $\{x_n\}_{n=0}^{\infty}$ strongly converges to the fixed point $p \in F(T)$.*

There is a certain gap in the above described results. Actually in the above described results, different types of fixed point iterative procedures associated with Zamfirescu operator have been considered without errors. To fill up this gap here we have established a general convergence theorem to approximate fixed point of Zamfirescu operator on an arbitrary normed space through the multi-step Noor fixed point iterative procedure with errors in the sense of S. Plubtieng and R. Wangkeeree [42], which generates the rest. So, the main purpose of our present chapter is to recognized a convergence theorem for muti-step Noor fixed point iterative procedure with errors defined by (5.1) in the class of Zamfirescu operator on arbitrary normed spaces. Our result generalizes and improves upon, among others, the corresponding results of A. Rafiq [3].

Now we state a lemma of M. O. Osilike [28], which is needed to prove our theorem.

Lemma 6.1.2. [28] *Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n) r_n + s_n t_n + k_n \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty, lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $lim_{n\to\infty} r_n = 0$.

6.2 Convergence theorem for multi-step Noor fixed point iterative procedure with errors

In this section we sate and prove a convergence theorem for multi-step iterative procedure by using Zamfirescu operator, which have generate the analogous results of different fixed point iterative procedures.

Theorem 6.2.1. *Let B be a nonempty closed convex subset of an arbitrary normed space X. Let $T: B \to B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$ be a sequence defined by multi-step Noor fixed point iterative procedurewith errors (5.1), for each $k = 1, 2, 3, \dots, m$ and $n \in \mathbb{N}$. If $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and*

$$\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right),$$

for each $k = 1, 2, 3, \dots, m$ and $n \in \mathbb{N}$. Then $\left\{u_n^{(k)}\right\}$ converges strongly to a fixed point of T.

Proof. According to our assumption T is a Zamfirescu operator, so by Theorem 1.6.5, we know that T has a unique fixed point in B, say p

$$\text{i.e., } Tp = p. \tag{6.1}$$

Now, we combine the Zamfirescu conditions according to the approach of V. Berinde [49-52]. Since T is a Zamfirescu operator, hence T is satisfied at least one of the Zamfirescu conditions $(z_1), (z_2)$ and (z_3) defined by the Theorem 1.6.5.

If T satisfies (z_2), then for all $x, y \in B$ we have

$$\begin{aligned} \|Tx - Ty\| &\leq b[\|x - Tx\| + \|y - Ty\|] \\ &\leq b[\|x - Tx\| + \|y - x\| + \|x - Tx\| + \|Tx - Ty\|], \end{aligned}$$

which implies

$$\|Tx - Ty\| \leq \frac{b}{1-b}\|x - y\| + \frac{2b}{1-b}\|x - Tx\| . \tag{6.2}$$

If T satisfies (z_3), then for all $x, y \in B$ similarly we obtain

$$\|Tx - Ty\| \leq \frac{c}{1-c}\|x - y\| + \frac{2c}{1-c}\|x - Tx\| . \tag{6.3}$$

Now, if we take

$$\delta = max\left\{a,\ \frac{b}{1-b}, \frac{c}{1-c}\right\}. \tag{6.4}$$

Then we have $0 \leq \delta < 1$ and in view of (z_1) and (6.2) to (6.4), we obtained the following inequality.

$$\|Tx - Ty\| \leq \delta\|x - y\| + 2\delta\|x - Tx\| . \tag{6.5}$$

If we suppose$\left\{u_n^{(k)}\right\}$be a multi-step Noor fixed point iterative procedurewith errors defined by (5.1) and $u_0 \in B$ arbitrary, then we have

$$\|u_{n+1} - p\| = \left\|a_n^{(m)} T u_n^{(m-1)} + b_n^{(m)} u_n + c_n^{(m)} v_n^{(m)} - p\right\|. \tag{6.6}$$

Since $a_n^{(m)} + b_n^{(m)} + c_n^{(m)} = 1$, hence from (2.6) we have

$$\|u_{n+1} - p\| = \left\|(1 - a_n^{(m)})(u_n - p) + a_n^{(m)}\left(Tu_n^{(m-1)} - p\right) + c_n^{(m)}(v_n^{(m)} - u_n)\right\|$$

$$\leq (1 - a_n^{(m)})\|u_n - p\| + a_n^{(m)}\left\|Tu_n^{(m-1)} - p\right\| + c_n^{(m)}\left\|v_n^{(m)} - u_n\right\|. \tag{6.7}$$

But according to our assumption, we have

$$\left\|v_n^{(m)} - u_n\right\| = 0(a_n^{(m)})$$

Hence from (6.7), we have

$$\|u_{n+1} - p\| \leq (1 - a_n^{(m)})\|u_n - p\| + a_n^{(m)}\left\|Tu_n^{(m-1)} - p\right\| + c_n^{(m)} 0(a_n^{(m)}). \tag{6.8}$$

Now, if we put $x = u_n^{(m-1)}$and $y = p$ in (6.5), we obtain

$$\left\|Tu_n^{(m-1)} - Tp\right\| \leq \delta\left\|u_n^{(m-1)} - p\right\|, \tag{6.9}$$

whereδ is given by (6.4).

Combining (6.8) and (6.9), we obtain

$$\|u_{n+1} - p\| \leq (1 - a_n^{(m)})\|u_n - p\| + a_n^{(m)}\delta\left\|u_n^{(m-1)} - p\right\| + c_n^{(m)} 0(a_n^{(m)}) \tag{6.10}$$

Further by the definition of multi-step Noor fixed point iterative procedurewith errors (5.1), we have

$$\left\|u_n^{(m-1)} - p\right\| = \left\|a_n^{(m-1)} T u_n^{(m-2)} + b_n^{(m-1)} u_n + c_n^{(m-1)} v_n^{(m-1)} - p\right\| \tag{6.11}$$

Since $a_n^{(m-1)} + b_n^{(m-1)} + c_n^{(m-1)} = 1$, hence from (6.11) we have

$$\left\|u_n^{(m-1)} - p\right\| \leq (1 - a_n^{(m-1)})\|u_n - p\| + a_n^{(m-1)}\left\|Tu_n^{(m-2)} - p\right\|$$

$$+c_n^{(m-1)} \left\| v_n^{(m-1)} - u_n \right\|. \tag{6.12}$$

But according to our assumption $\left\| v_n^{(m-1)} - u_n \right\| = 0(a_n^{(m-1)})$, hence from (6.12) we have

$$\left\| u_n^{(m-1)} - p \right\| \leq (1 - a_n^{(m-1)}) \| u_n - p \|$$

$$+a_n^{(m-1)} \left\| Tu_n^{(m-2)} - p \right\| + c_n^{(m-1)} 0(a_n^{(m-1)}) \tag{6.13}$$

Now, if we put $x = u_n^{(m-2)}$ and $y = p$ in (6.5), then we have

$$\left\| Tu_n^{(m-2)} - Tp \right\| \leq \delta \left\| u_n^{(m-2)} - p \right\|, \tag{6.14}$$

where δ is given by (6.4).

Combining (6.13) and (6.14), we obtain

$$\left\| u_n^{(m-1)} - p \right\| \leq (1 - a_n^{(m-1)}) \| u_n - p \| + a_n^{(m-1)} \delta \left\| u_n^{(m-2)} - p \right\|$$

$$+c_n^{(m-1)} 0(a_n^{(m-1)}). \tag{6.15}$$

From (6.10) and (6.15), we have

$$\| u_{n+1} - p \| \leq (1 - a_n^{(m)}) \| u_n - p \| + a_n^{(m)} \delta \begin{bmatrix} (1 - a_n^{(m-1)}) \| u_n - p \| \\ +a_n^{(m-1)} \delta \left\| u_n^{(m-2)} - p \right\| \end{bmatrix}$$

$$+ c_n^{(m)} 0(a_n^{(m)}) + a_n^{(m)} c_n^{(m-1)} 0(a_n^{(m-1)})$$

$$= (1 - a_n^{(m)} + a_n^{(m)} \delta (1 - a_n^{(m-1)})) \| u_n - p \|$$

$$+\delta^2 a_n^{(m)} a_n^{(m-1)} \left\| u_n^{(m-2)} - p \right\|$$

$$+ c_n^{(m)} 0(a_n^{(m)}) + a_n^{(m)} c_n^{(m-1)} 0(a_n^{(m-1)}) \tag{6.16}$$

Further by the definition of multi-step Noor fixed point iterative procedurewith errors (5.1), we have

$$\left\| u_n^{(m-2)} - p \right\| = \left\| a_n^{(m-2)} Tu_n^{(m-3)} + b_n^{(m-2)} u_n + c_n^{(m-2)} v_n^{(m-2)} - p \right\| \tag{6.17}$$

Since $a_n^{(m-2)} + b_n^{(m-2)} + c_n^{(m-2)} = 1$, hence from (6.17) we have

$$\left\| u_n^{(m-2)} - p \right\| \leq (1 - a_n^{(m-2)}) \| u_n - p \| + a_n^{(m-2)} \left\| Tu_n^{(m-3)} - p \right\|$$

$$+c_n^{(m-2)} \left\| v_n^{(m-2)} - u_n \right\|. \tag{6.18}$$

But according to our assumption $\left\|v_n^{(m-2)} - u_n\right\| = 0(a_n^{(m-2)})$, hence from (6.18) we have

$$\left\|u_n^{(m-2)} - p\right\| \leq (1 - a_n^{(m-2)})\|u_n - p\| + a_n^{(m-2)} \left\|Tu_n^{(m-3)} - p\right\|$$
$$+c_n^{(m-2)}0(a_n^{(m-2)}) \qquad (6.19)$$

Now, if we put $x = u_n^{(m-3)}$and $y = p$ in (6.5), then we have

$$\left\|Tu_n^{(m-3)} - Tp\right\| \leq \delta \left\|u_n^{(m-3)} - p\right\|, \qquad (6.20)$$

whereδ is given by (2.4).

Combining (6.19) and (6.20), we obtain

$$\left\|u_n^{(m-2)} - p\right\| \leq (1 - a_n^{(m-2)})\|u_n - p\| + a_n^{(m-2)}\delta \left\|u_n^{(m-3)} - p\right\|$$
$$+c_n^{(m-2)}0(a_n^{(m-2)}). \qquad (6.21)$$

From (6.16) and (6.21), we have

$$\|u_{n+1} - p\|$$
$$\leq (1 - a_n^{(m)})\|u_n - p\|$$
$$+a_n^{(m)}\delta\left[(1 - a_n^{(m-1)})\|u_n - p\| + a_n^{(m-1)}\delta\left[\begin{array}{c}(1 - a_n^{(m-2)})\|u_n - p\| \\ +a_n^{(m-2)}\delta\left\|u_n^{(m-3)} - p\right\|\end{array}\right]\right]$$
$$+c_n^{(m)}0(a_n^{(m)}) + a_n^{(m)}c_n^{(m-1)}0(a_n^{(m-1)}) + \delta^2 a_n^{(m)}a_n^{(m-1)}c_n^{(m-2)}0(a_n^{(m-2)})$$
$$= (1 - a_n^{(m)} + \delta a_n^{(m)}(1 - a_n^{(m-1)}) + \delta^2 a_n^{(m)}a_n^{(m-1)}(1 - a_n^{(m-2)}))\|u_n - p\|$$
$$+\delta^3 a_n^{(m)}a_n^{(m-1)}a_n^{(m-2)}\left\|u_n^{(m-3)} - p\right\| + c_n^{(m)}0\left(a_n^{(m)}\right)$$
$$+a_n^{(m)}c_n^{(m-1)}0(a_n^{(m-1)}) + \delta^2 a_n^{(m)}a_n^{(m-1)}c_n^{(m-2)}0(a_n^{(m-2)}) \qquad (6.22)$$

Now if we continue the above process until the initial equation of multi-step Noor fixed point iterative procedurewith errors (5.1) have been used, then the inequality (6.22) can written as follows.

$$\|u_{n+1} - p\| \leq [1 - a_n^{(m)} + \delta a_n^{(m)}(1 - a_n^{(m-1)}) + \delta^2 a_n^{(m)}a_n^{(m-1)}\left(1 - a_n^{(m-2)}\right)$$
$$+ \cdots + \delta^{m-1}a_n^{(m)}a_n^{(m-1)} \ldots a_n^{(3)}a_n^{(2)}\left(1 - a_n^{(1)}\right)]\|u_n - p\|$$

$$+\delta^m a_n^{(m)} a_n^{(m-1)} \dots a_n^{(3)} a_n^{(2)} a_n^{(1)} \|u_n - p\|$$

$$+c_n^{(m)} 0\left(a_n^{(m)}\right) a_n^{(m)} c_n^{(m-1)} 0\left(a_n^{(m-1)}\right)$$

$$+\delta^2 a_n^{(m)} a_n^{(m-1)} c_n^{(m-2)} 0\left(a_n^{(m-2)}\right)$$

$$+\cdots+\delta^{m-1} a_n^{(m)} a_n^{(m-1)} \dots a_n^{(3)} a_n^{(2)} c_n^{(1)} 0(a_n^{(1)})$$

$$= (1 - a_n^{(m)} + \delta a_n^{(m)}(1 - a_n^{(m-1)}) + \delta^2 a_n^{(m)} a_n^{(m-1)}\left(1 - a_n^{(m-2)}\right)$$

$$+\cdots+\delta^{m-1} a_n^{(m)} a_n^{(m-1)} \dots a_n^{(3)} a_n^{(2)}\left(1 - a_n^{(1)}\right)$$

$$+\delta^m a_n^{(m)} a_n^{(m-1)} \dots a_n^{(3)} a_n^{(2)} a_n^{(1)}) \|u_n - p\| + c_n^{(m)} 0\left(a_n^{(m)}\right)$$

$$+a_n^{(m)} c_n^{(m-1)} 0\left(a_n^{(m-1)}\right) + \delta^2 a_n^{(m)} a_n^{(m-1)} c_n^{(m-2)} 0\left(a_n^{(m-2)}\right)$$

$$+\cdots+\delta^{m-1} a_n^{(m)} a_n^{(m-1)} \dots a_n^{(3)} a_n^{(2)} c_n^{(1)} 0(a_n^{(1)}),$$

which implies,

$$\|u_{n+1} - p\| \le [1 - (1-\delta) a_n^{(m)}\left(1 - \delta a_n^{(m-1)}\right)\left(1 - \delta a_n^{(m-2)}\right)\left(1 - \delta a_n^{(m-3)}\right) \dots$$

$$\dots (1 - \delta a_n^{(1)})] \|u_n - p\| + c_n^{(m)} 0\left(a_n^{(m)}\right)$$

$$+a_n^{(m)} c_n^{(m-1)} 0\left(a_n^{(m-1)}\right) + \delta^2 a_n^{(m)} a_n^{(m-1)} c_n^{(m-2)} 0\left(a_n^{(m-2)}\right)$$

$$+\cdots+\delta^{m-1} a_n^{(m)} a_n^{(m-1)} \dots a_n^{(3)} a_n^{(2)} c_n^{(1)} 0(a_n^{(1)}). \quad (6.23)$$

But it is clear that,

$$[1 - (1-\delta) a_n^{(m)}\left(1 - \delta a_n^{(m-1)}\right)\left(1 - \delta a_n^{(m-2)}\right)\left(1 - \delta a_n^{(m-3)}\right) \dots (1 - \delta a_n^{(1)})]$$

$$\le [1 - (1-\delta)^m a_n^{(m)}].$$

Hence form (6.23), we obtain

$$\|u_{n+1} - p\| \le [1 - (1-\delta)^m a_n^{(m)}] \|u_n - p\| + c_n^{(m)} 0\left(a_n^{(m)}\right)$$

$$+a_n^{(m)} c_n^{(m-1)} 0\left(a_n^{(m-1)}\right) + \delta^2 a_n^{(m)} a_n^{(m-1)} c_n^{(m-2)} 0\left(a_n^{(m-2)}\right)$$

$$+\cdots+\delta^{m-1} a_n^{(m)} a_n^{(m-1)} \dots a_n^{(3)} a_n^{(2)} c_n^{(1)} 0(a_n^{(1)}), n \in \mathbb{N}. \; (6.24)$$

By (6.24) inductively, we obtain

$$\|u_{n+1} - p\| \le \prod_{r=0}^{n}\left[1 - (1-\delta)^m a_r^{(m)}\right] \|u_0 - p\|$$

$$+\delta^2 a_n^{(m)} a_n^{(m-1)} c_n^{(m-2)} 0\left(a_n^{(m-2)}\right)$$

$+\cdots+\delta^{m-1} a_n^{(m)} a_n^{(m-1)} \ldots a_n^{(3)} a_n^{(2)} c_n^{(1)} 0(a_n^{(1)}),\ n \in \mathbb{N}$. (6.25)

Now since $0 \leq \delta < 1,\ a_n^{(m)} \in (0,1)$and $\sum_{n=1}^{\infty} a_n^{(m)} = \infty$, hence by Lemma 6.1.2 we can write

$$\lim_{n\to\infty} \prod_{r=0}^{n} [1-(1-\delta)^m a_r^{(m)}] = 0. \quad (6.26)$$

Taking limit as $n \to \infty$ on both sides of (6.25) and using (6.26), we get

$$\lim_{n\to\infty} \|u_{n+1} - p\| = 0.$$

This implies that $\left\{u_n^{(k)}\right\},\ k = 1,2,3,\ldots,m$converges strongly to $p \in F(T)$.

This completes our proof.■

Corollary 6.2.2.*Let B be a nonempty closed convex subset of an arbitrary normed space$X$. Let $T: B \to B$ satisfies the Kannan's contractive conditions defined by (1.6). Let $\left\{u_n^{(k)}\right\}$be a sequence defined by multi-step Noor fixed point iterative procedurewith errors (5.1), for each $k = 1,2,3,\ldots,m$and $n \in \mathbb{N}$. If $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\left\|v_n^{(k)} - u_n\right\| = 0\left(a_n^{(k)}\right)$, for each $k = 1,2,3,\ldots,m$ and $n \in \mathbb{N}$. Then $\left\{u_n^{(k)}\right\}$converges strongly to a fixed point of T.*

Corollary 6.2.3.*Let B be a nonempty closed convex subset of an arbitrary normed space$X$. Let $T: B \to B$ satisfies the Chatterjea's contractive conditions defined by (1.7) respectively. Let $\left\{u_n^{(k)}\right\}$be a sequence defined by multi-step Noor fixed point iterative procedurewith errors (5.1), for each $k = 1,2,3,\ldots,m$and $n \in \mathbb{N}$. If $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, and $\left\|v_n^{(k)} - u_n\right\| = 0\left(a_n^{(k)}\right)$, for each $k = 1,2,3,\ldots,m$ and $n \in \mathbb{N}$. Then $\left\{u_n^{(k)}\right\}$converges strongly to a fixed point of T.*

Corollary 6.2.4.*Let B be a nonempty closed convex subset of an arbitrary normed space$X$. Let $T: B \to B$ be a Zamfirescu operator. Let $\left\{u_n^{(k)}\right\}$be a sequence defined* *by Noor iterative procedurewith errors (5.2), for each* $k = 1,2,3$*and* $n \in \mathbb{N}$. *If* $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and* $\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right)$, *for each* $k = 1,2,3$ *and* $n \in \mathbb{N}$. *Then* $\left\{u_n^{(k)}\right\}$*converges strongly to a fixed point of* T.

Corollary 6.2.5.*Let* B *be a nonempty closed convex subset of an arbitrary normed space*X. *Let* $T: B \to B$ *be a Zamfirescu operator. Let* $\left\{u_n^{(k)}\right\}$*be a sequence defined by Noor iterative procedure(5.3), for each* $k = 1,2,3$*and* $n \in \mathbb{N}$. *If* $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and* $\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right)$, *for each* $k = 1,2,3$ *and* $n \in \mathbb{N}$. *Then* $\left\{u_n^{(k)}\right\}$*converges strongly to a fixed point of* T.

Corollary 6.2.6.*Let* B *be a nonempty closed convex subset of an arbitrary normed space*X. *Let* $T: B \to B$ *be a Zamfirescu operator. Let* $\left\{u_n^{(k)}\right\}$*be a sequence defined by Ishikawa iterative procedurewith errorsdefined by Y. Xu* (5.4), *for each* $k = 1,2$*and* $n \in \mathbb{N}$. *If* $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and* $\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right)$, *for each* $k = 1,2$ *and* $n \in \mathbb{N}$. *Then* $\left\{u_n^{(k)}\right\}$*converges strongly to a fixed point of* T.

Corollary 6.2.7.*Let* B *be a nonempty closed convex subset of an arbitrary normed space*X. *Let* $T: B \to B$ *be a Zamfirescu operator. Let* $\left\{u_n^{(k)}\right\}$*be a sequence defined by Ishikawa iterative procedure with errorsdefined by L.S.Lu(5.6), for each* $k = 1,2$*and* $n \in \mathbb{N}$. *If* $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and* $\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right)$, *for each* $k = 1,2$ *and* $n \in \mathbb{N}$. *Then* $\left\{u_n^{(k)}\right\}$*converges strongly to a fixed point of* T.

Corollary 6.2.8.*Let* B *be a nonempty closed convex subset of an arbitrary normed space*X. *Let* $T: B \to B$ *be a Zamfirescu operator. Let* $\left\{u_n^{(k)}\right\}$*be a sequence defined* *by Ishikawa iterative procedure (5.5), for each* $k = 1, 2$*and* $n \in \mathbb{N}$. *If* $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and*

$$\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right),$$

for each $k = 1, 2$ *and* $n \in \mathbb{N}$. *Then* $\left\{u_n^{(k)}\right\}$*converges strongly to a fixed point of* T.

Corollary 6.2.9.*Let* B *be a nonempty closed convex subset of an arbitrary normed space*X. *Let* $T: B \to B$ *be a Zamfirescu operator. Let* $\left\{u_n^{(k)}\right\}$*be a sequence defined by Mann iterative procedure with errorsdefined by Y. Xu (5.7), for each* $k = 1$*and* $n \in \mathbb{N}$. *If* $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and*

$$\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right),$$

for each $k = 1$ *and* $n \in \mathbb{N}$. *Then* $\left\{u_n^{(k)}\right\}$*converges strongly to a fixed point of* T.

Corollary 6.2.10.*Let* B *be a nonempty closed convex subset of an arbitrary normed space*X. *Let* $T: B \to B$ *be a Zamfirescu operator. Let* $\left\{u_n^{(k)}\right\}$*be a sequence defined by Mann iterative procedure with errorsdefined by L.S. Lu (5.10), for each* $k = 1$*and* $n \in \mathbb{N}$. *If* $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and*

$$\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right),$$

for each $k = 1$ *and* $n \in \mathbb{N}$. *Then* $\left\{u_n^{(k)}\right\}$*converges strongly to a fixed point of* T.

Corollary 6.2.11.*Let* B *be a nonempty closed convex subset of an arbitrary normed space*X. *Let* $T: B \to B$ *be a Zamfirescu operator. Let* $\left\{u_n^{(k)}\right\}$*be a sequence defined by Mann iterative procedure(5.8), for each* $k = 1$*and* $n \in \mathbb{N}$. *If* $F(T) \neq \emptyset$, $\sum_{n=1}^{\infty} a_n^{(k)} = \infty$, *and*

$$\left\| v_n^{(k)} - u_n \right\| = 0\left(a_n^{(k)}\right),$$

for each $k = 1$ *and* $n \in \mathbb{N}$. *Then* $\left\{u_n^{(k)}\right\}$*converges strongly to a fixed point of* T.

6.3 Some Remarks

In this section we give some remarks on our Theorem 6.2.1.

Remark 6.3.1. The contractive condition (1.5) makes the mapping T a continuous function on Xwhile this is not the case with the contractive conditions (1.6), (1.7) and (6.5).

Remark 6.3.2. The Kannan's and the Chatterjea's contractive conditions defined by (1.6) and (1.7) respectively are both included in the class of Zamfirescu operator and so their convergence theorems for the multi-step Noor fixed point iterative procedurewith errors are obtained in Corollary 6.2.2 and Corollary 6.2.3 respectively.

Remark 6.3.3 Theorem 3 of A. Rafiq [3] in the context of Mann iterative procedure with errors on a closed convex normed space has been obtained in Corollary 6.2.10.

Remark 6.3.4. Theorem 3 of A. Rafiq [2] in the context of Noor iterative procedure on a closed convex normed space has been obtained in Corollary 6.2.5.

5. In Corollary 2.7, Theorem 2 of V. Berinde [51] is generalized to the setting of normed spaces.

Remark 6.3.6. In Corollary 6.2.11, Theorem 2 of V. Berinde [50] and Theorem 2.1 of V. Berinde [52] are generalized to the setting of normed spaces.

Remark 6.3.7. In Corollary 6.2.4, the result of Y.J. Cho, H. Zhou, G. Guo [63] is generalized to the class of Zamfirescu operator.

Remark 6.3.8. Corollary 6.2.6 and Corollary 6.2.9 are used to generalize the result of Y. Xu [62] to the setting for the class of Zamfirescu operator.

Remark 6.3.9. Corollary 6.2.7 and Corollary 6.2.10 are used to generalize the result of L. S. Liu [22] to the setting for the class of Zamfirescu operator.

Remark 6.3.10. Our Theorem 6. 2.1 also generalized the result of M. O. Osilike [28-30].

CHAPTER-7

APPLICATION AND CONCLUSION

In this chapter, we give some applications of our new fixed point iterative procedure (2.18) and finally conclude our whole work.

7.1 Application of our new fixed point iterative procedure to constrained optimization problemsand split feasibility problems

This section is allocated to some applications of our new fixed point iterative procedure (2.18). Let Hbe a real Hilbert spacewith inner product $\langle\cdot,\cdot\rangle$and norm$\|\cdot\|$, respectively. Let Cbe a nonempty closedconvex subset of $H$and $T: C \to H$a nonlinear operator. $T$is said to be:

(1) L-Lipschitzian if there exists a constant$L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \forall\, x, y \in C.$$

An L-Lipschitzian will be contraction if $L \in (0, 1)$, and non-expansive if $L = 1$.

(2) Monotone if $\langle Tx - Ty, x - y\rangle \geq 0, \forall\, x, y \in C$.

(3) λ-strongly monotone if there exists a constant $\lambda > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \lambda\|x - y\|^2, \forall\, x, y \in C.$$

(4) ν-inverse strongly monotone (ν-ism) if there exists a constant $\nu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \nu\|Tx - Ty\|^2, \forall\, x, y \in C.$$

The variational inequality problem defined by $C$and Twill be denoted by$VI(C, T)$. These were initially studied by Kinderlehrer and Stampachhia [10]. Thevariational inequality problem $VI(C, T)$ is the problem of finding a vector $\boldsymbol{z}$in Csuch that $\langle T\boldsymbol{z}, \boldsymbol{z} - k\rangle \geq 0$ for all $k \in C$. The set of all such vectors which solvevariational inequality $VI(C, T)$problem is denoted by$\Omega(C, T)$. The variationalinequality problem is connected with various kinds of problems such as the convexminimization problem, the complementarity problem, the problem of finding a point $u \in H$ satisfying $0 = Tu$ and so on. The existence and approximation of solutions are important aspects in the study of variational inequalities. The variational inequality problem $VI(C,T)$ is equivalent to the fixed point problem, that is to find $x^* \in C$ such that

$$x^* = F_\mu x^* = P_C(I - \mu T)x^*,$$

where $\mu > 0$ is a constant and P_C is the metric projection from H onto C and $F_\mu \coloneqq P_C(I - \mu T)$. If T is L-Lipschitzian and λ-strongly monotone, then the operator F_μ is a contraction on C provided that $0 < \mu < 2\lambda/L^2$. In this case, an application of Banach contraction principle (Theorem 1.6.2) implies that $\Omega(C,T) = \{x^*\}$ and the sequence of the Picard iteration process, given by

$$x_{n+1} = F_\mu x_n, \quad n \in \mathbb{N}$$

converges strongly to x^*.

Construction of fixed points of non-expansive operators is an important subject in the theory of non-expansive operators and has applications in a number of applied areas such as image recovery and signal processing (see [7, 8, 13]). For instance, split feasibility problem of C and T (denoted by $SFP(C,T)$) is

to find a point x in C such that $Tx \in Q$ (7.1)

where C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of another Hilbert space H_2 and $T: H_1 \to H_2$ is a bounded linear operator. The $SFP(C,T)$ is said to be consistent if (7.1) has a solution. It is easy to see that $SFP(C,T)$ is consistent if and only if the following fixed point problem has a solution:

find $x \in C$ such that $x = P_C\big(I - \gamma T^*\big(I - P_Q\big)T\big)x,$ (7.2)

where P_C and P_Q are the orthogonal projections onto C and Q, respectively; $\gamma > 0$, and T^* is the adjoint of T. Note that for sufficient small $\gamma > 0$, the operator $P_C\big(I - \gamma T^*\big(I - P_Q\big)T\big)$ in (7.2) is non-expansive.

In the view of Lemma 4.1.3, we have the following sharper results which contain our iterative procedure (2.18) faster than the one of the iterative procedures defined by (2.17), (2.16), (2.15), (2.14), (2.13), (2.11) and (2.1). These results deal withvariational inequality problems.

Theorem 7.1.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T: C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Thenfor $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and definedby*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu T)x_n + \alpha_n P_C(I-\mu T)y_n, \\ y_n &= (1-\beta_n)P_C(I-\mu T)x_n + \beta_n P_C(I-\mu T)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu T)x_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\}$$

converges weakly to $x^* \in \Omega(C,\ T)$.

Proof. Since $T \coloneqq P_C(I-\mu T)$ is non-expansive, hence the result follows fromTheorem 4.2.1. ■

The Theorem 7.1.1 leads the following corollaries for iterative procedures defined by (2.16), (2.17), (2.14), (2.15), (2.13), (2.11) and (2.1) respectively.

Corollary 7.1.2. *Let C be a nonempty closed convex subset of a Hilbert space H and $T: C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Thenfor $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and definedby*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu T)y_n + \alpha_n P_C(I-\mu T)z_n, \\ y_n &= (1-\beta_n)P_C(I-\mu T)x_n + \beta_n P_C(I-\mu T)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu T)x_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\}$$

converges weakly to $x^* \in \Omega(C,\ T)$.

Corollary 7.1.3. *Let C be a nonempty closed convex subset of a Hilbert space H and $T: C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$, $\{\beta_n\}$,$\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Thenfor $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and definedby*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu T)x_n + \alpha_n P_C(I-\mu T)y_n, \\ y_n &= (1-\beta_n)z_n + \beta_n P_C(I-\mu T)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu T)x_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\}$$

converges weakly to $x^* \in \Omega(C,\ T)$.

Corollary 7.1.4. *Let C be a nonempty closed convex subset of a Hilbert space H and $T: C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and defined by*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C(I-\mu T)y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n P_C(I-\mu T)z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu T)x_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\}$$

converges weakly to $x^* \in \Omega(C,\ T)$.

Corollary 7.1.5. *Let C be a nonempty closed convex subset of a Hilbert space H and $T: C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$, $\{\beta_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and defined by*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu T)x_n + \alpha_n P_C(I-\mu T)y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n P_C(I-\mu T)x_n;\quad n \in \mathbb{N}, \end{aligned}\right\}$$

converges weakly to $x^* \in \Omega(C,\ T)$.

Corollary 7.1.6. *Let C be a nonempty closed convex subset of a Hilbert space H and $T: C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\}$, $\{\beta_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and defined by*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C(I-\mu T)y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n P_C(I-\mu T)x_n;\quad \forall\, n \in \mathbb{N}, \end{aligned}\right\}$$

converges weakly to $x^* \in \Omega(C,\ T)$.

Corollary 7.1.7. *Let C be a nonempty closed convex subset of a Hilbert space H and $T: C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Suppose $\{\alpha_n\} \in [\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and defined by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(I-\mu T)x_n;\ \forall\, n \in \mathbb{N},$$

converges weakly to $x^* \in \Omega(C,\ T)$.

Corollary 7.1.8. *Let C be a nonempty closed convex subset of a Hilbert space H and $T: C \to H$ a L-Lipschitzian and λ-strongly monotone operator. Then for $\mu \in (0, 2\lambda/L^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$ and defined by*

$$x_{n+1} = P_C(I - \mu T)x_n; \quad \forall\, n \in \mathbb{N},$$

converges weakly to $x^* \in \Omega(C,\ T)$.

7.1.9 Application to constrained optimization problems

Let C be a closed convex subset of a Hilbert space H, P_C the metric projection of H onto C and $T: C \to H$ a ν-ism where $\nu > 0$ is a constant. It is well known that $P_C(I - \mu T)$ is non-expansive operator provided that $\mu \in (0, 2\nu)$.

The algorithms for signal and image processing are often iterative constrained optimization processes designed to minimize a convex differentiable function T over a closed convex set C in H. It is well known that every L-Lipschitzian operator is $2/L$-ism. Therefore, we have the following result which generates the sequence of vectors in the constrained or feasible set C which converges weakly to the optimal solution which minimizes T.

Theorem 7.1.10. *Let C be a closed convex subset of a Hilbert space H and T a convex and differentiable function on an open set D containing the set C. Assume that ∇T is an L-Lipschitz operator on D, $\mu \in (0,\ 2/L)$ and minimizers of T relative to the set C exist. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu\nabla T)x_n + \alpha_n P_C(I-\mu\nabla T)y_n,\\ y_n &= (1-\beta_n)P_C(I-\mu\nabla T)x_n + \beta_n P_C(I-\mu\nabla T)z_n,\\ z_n &= (1-\gamma_n)x_n + \gamma_n P_C(I-\mu\nabla T)x_n;\ \forall\, n \in \mathbb{N}, \end{aligned}\right\}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\xi, 1-\xi]$ for all $n \in \mathbb{N}$ and for some $\xi \in (0,1)$. Then the sequence $\{x_n\}$ converges weakly to a minimizer of T.

Proof. Since $T \coloneqq P_C(I - \mu\nabla T)$ is non-expansive, hence the result follows from Theorem 4.2.1. ■

7.1.11Application to split feasibility problems

Recall that a mapping Tin aHilbert space $H$is said to be averaged if Tcan be written as $(1-\alpha)I+\alpha S$, where$\alpha \in (0,1)$ and $S$is a non-expansive map on H. Set$q(x):=\frac{1}{2}\left\|\left(T-P_Q T\right)x\right\|,\ x\in C$.

Consider the minimization problem

$$\text{find}\min_{x\in C} q(x).$$

By [17], the gradient of $q$is $\nabla q=T^*\left(I-P_Q\right)T$, where $T^*$is the adjoint of T.Since $I-P_Q$is non-expansive, it follows that $\nabla q$is L-Lipschitzian with$L=\|T\|^2$.Therefore, $\nabla q$is $1/L$-ism and for any $0<\mu<2/L$, $I-\mu\nabla q$is averaged. Therefore,the composition $P_C(I-\mu\nabla q)$ is also averaged. Set$T\coloneqq P_C(I-\mu\nabla q)$. Note thatthe solution set of $SFP(C,T)$is $F(T)$.

We now present an iterative procedure that can be used to find solutions of $SFP(C,T)$.

Theorem 7.1.12.*Assume that $SFP(C,T)$is consistent. Suppose$\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$are sequences in $[\xi,1-\xi]$ for all $n\in\mathbb{N}$ and for some $\xi\in(0,1)$. Let$\{x_n\}$ be a sequence in C generated by*

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)P_C(I-\mu\nabla q)x_n+\alpha_n P_C(I-\mu\nabla q)y_n,\\ y_n &= (1-\beta_n)P_C(I-\mu\nabla q)x_n+\beta_n P_C(I-\mu\nabla q)z_n,\\ z_n &= (1-\gamma_n)x_n+\gamma_n P_C(I-\mu\nabla q)x_n;\ \forall\, n\in\mathbb{N}, \end{aligned}\right\}$$

where $0<\mu<2/\|T\|^2$. Then the sequence$\{x_n\}$converges weakly to a solution of $SFP(C,T)$.

Proof. Since $T\coloneqq P_C(I-\mu\nabla q)$ is non-expansive, hence the result follows fromTheorem 4.2.1.■

7.2 Conclusion

7.2.1 Conclusion on our new three-step iterative procedure

By our Theorem 2.9.2, we have shown that the sequence of our new three-step iterative procedure (2.18) converges strongly for Zamfirescu operator.

From the comparison Table 3.7.1, we observe that the Picard iterative procedure (2.1), the Mann iterative procedure (2.11), the Ishikawa iterative procedure (2.13), the Noor iterative procedure (2.14), the Agarwal*et al*. iterative procedure (2.15), the Abbas *et al.* iterative procedure (2.16) and the Thakur*et al.*iterative procedure (2.17) converge to the fixed point after 28^{th} iteration, 36^{th} iteration, 26^{th} iteration, 24^{th} iteration, 20^{th} iteration, 20^{th} iteration and 17^{th} iteration respectively, where as our proposed new three-step iterative procedure (2.18) converge to the fixed point after 15^{th} iteration under the same situation. So, we can conclude that the rate of convergence of our new three-step iterative procedure (2.18) is fastest among all the above mentioned iterativeprocedures for contraction mapping. Our Theorem 3.6.1 proves this argument analytically. By Theorem 4.2.1 we establish the weak convergence of our new three-step iterative procedure (2.18) for non-expansive mapping under different conditions and by Theorem 4.3.1, Theorem 4.3.2, and Theorem 4.3.3 we establish the strong convergence of our new three-step iterative procedure (2.18) for non-expansive mapping under different conditions.

7.2.2 Conclusion on the convergence theorem of multi-step fixed point iterative procedure with errors

Our Theorem 6.2.1 improves the Theorem 3 of A. Rafiq [3] by extending it from Mann iterativeprocedure with errors to multi-step Noor fixed point iterative procedurewith errors. Since the iterative procedureswith errors (5.2) to (5.10) are special cases of the multi-step Noor iterative procedure with errors (5.1), therefore our Theorem 6.2.1 made by the multi-step Noor iterative procedure with errors (5.1) associated with Zamfirescu operator generalizes all Theorems made by the iterative procedures with errors (5.2) to (5.10) associated with Zamfirescu operator. Furthermore, by our Theorem 6.2.1, various results in the literature are also extended and generalized in the following way:

1. The fixed point theorems of Kannan's operatordefined in Definition 1.6.3 and Chatterjea's operator defined in Definition 1.6.4 are extended to the larger class of Zamfirescu operator associated with multi-step Noor fixed point iterative procedurewith errors.
2. The fixed point theorems of V. Berinde [48, 50-52] are extended from the Mann and Ishikawa iterativeprocedure to multi-step Noor fixed point iterative procedurewith errors.
3. The fixed point theorem of A. Rafiq [2] is extended from the Mann, Ishikawa and Noor iterative procedure to multi-step Noor fixed point iterative procedurewith errors.
4. The fixed point theorem of Y.J. Cho, H. Zhou, G. Guo [63] is generalized and extended from three-step iterative procedure with errors in asymptotically non-expansive mapping to multi-step Noor fixed point iterative procedurewith errors in Zamfirescu operator.
5. The fixed point theorems of Y. Xu [62] and L. S. Liu [22] are generalized from the Mann and Ishikawa iterative procedures with errors to multi-step Noor fixed point iterative procedurewith errors.
6. The fixed point theorems of M. O. Osilike [28-30]are generalized from the Mann and Ishikawa iterative procedure with errors to multi-step Noor fixed point iterative procedurewith errors.

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