Department of Applied Mathematics MPhil Thesis

2014

# A Study on Turbulent and Magneto-hydrodynamic turbulent Flow in Incompressible Fluid

Mumtahinah, Mst.
University of Rajshahi

# A STUDY ON TURBULENT AND MAGNETO-HYDRODYNAMIC TURBULENT FLOW IN INCOMPRESSIBLE FLUID

**M. Phil. Thesis**

*Submitted by*

**MST. MUMTAHINAH**

**DEPARTMENT OF APPLIED MATHEMATICS**
**UNIVERSITY OF RAJSHAHI, RAJSHAHI-6205**
**BANGLADESH**

**June, 2014**

# A STUDY ON TURBULENT AND MAGNETO-HYDRODYNAMIC TURBULENT FLOW IN INCOMPRESSIBLE FLUID

**M. Phil. Thesis**

*Submitted for Partial fulfilment of the requirements for the degree of Master of Philosophy*
in
**APPLIED MATHEMATICS**

by

**MST. MUMTAHINAH**

*Under the supervision of*

**DR. M. ABUL KALAM AZAD**
Associate Professor
Department of Applied Mathematics
University of Rajshahi
Rajshahi-6205, Bangladesh

*Dedicated to*

**MY PARENTS**

# *DECLARATION*

*I do hereby declare that the thesis entitled "**A STUDY ON TURBULENT AND MAGNETO-HYDRODYNAMIC TURBULENT FLOW IN INCOMPRESSIBLE FLUID**" submitted in the Department of Applied Mathematics, Rajshahi University, Rajshahi, Bangladesh for the degree of **MASTER OF PHILOSOPHY** in Applied Mathematics under the supervision of **DR. MD. ABUL KALAM AZAD**, Associate Professor, Department of Applied Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh. This research work is an original one and it has not been submitted in any other university or institution for any degree.*

*Date:* *Mst. Mumtahinah*

*M. Phil Research Fellow*

*Dr. Md. Abul Kalam Azad*
Associate Professor
Department of Applied Mathematics
University of Rajshahi
Rajshahi-6205, Bangladesh

Phone : +88-0721-711155 (Off)
+88-0721-751375 (Res)
Cell : +88-01556-333862
Fax : +88-0721-750064
E-mail: : azad267@gmail.com

Date......................

# *CERTIFICATE*

Certified that the Thesis entitled "A Study on Turbulent and Magneto-hydrodynamic turbulent Flow in Incompressible Fluid" submitted by Mst. Mumtahinah in fulfilment of the requirements for the degree of Master of Philosophy in the Department of Applied Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh has been completed under my supervision. I strongly believe that this research work is an original one and it has not been submitted elsewhere for any degree.

(Dr. Md. Abul Kalam Azad)
Supervisor

# ACKNOWLEDGEMENTS

At first I would like to pay my gratitude to all mighty Allah for giving me energy, strength, endurance and ability to complete this thesis.

I would like to express my sincere admiration, deepest sense of gratitude, indebtness and gratefulness to my supervisor Dr. Abul Kalam Azad, Associate Professor, Department of Applied Mathematics, University of Rajshahi, Rajshahi, Bangladesh, for his helpful and valuable suggestions, encouraging guidance and generous advice throughout the progress of research work.

I am deeply indebted to Professor Dr. M. Hossain Ali, Chairman, Department of Applied Mathematics, Rajshai University for their sympathetic co-operation and support.

I express my deepest sense of heartiest gratitude to Professor Shamsul Alam Sarker, Professor Zillur Rahman and all other teachers of the department for their valuable suggestions and encouragements.

I am thankful to all of my family member specially to my parents.

I am thankful to the university of Rajshahi and in particular to the Department of Applied Mathematics for providing all facilities and co-operation during the course of my M.Phil. program.



Department of Applied Mathematics
University of Rajshahi, Rajshahi-6205 (Mst. Mumtahinah)
Bangladesh

# *PREFACE*

The thesis entitled **"A Study on Turbulent and Magneto-hydrodynamic turbulent Flow in Incompressible Fluid"** is being presented for the award of the degree of Master of Philosophy in Applied Mathematics. It is the outcome of my research works conducted in the Department of Applied Mathematics, Rajshahi University, Rajshahi, Bangladesh under the supervision of Dr. Abul Kalam Azad, Associate Professor, Department of Applied Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh.

**The thesis has been divided into five chapters.**

The **first chapter** is a general introductory chapter and gives the general idea of turbulence, distribution functions and their principal concepts. Some results and theories which are needed in the subsequent chapters have been included in this chapter. Types and examples of turbulence, different stages of Reynolds number, Reynolds equation, averaging rules, Coriolis effect etc have been briefly discussed. Distribution functions, Joint distribution functions, equation of motion of dust particles, spectral representation of turbulence and Fourier Transformation of the Navier-Stockes equation have also been discussed. Lastly, a brief review of the past researchers related to this thesis have also been studied in this chapter. Throughout the work we have considered the flow of fluids to be isotropic and homogeneous. The notions generally adopted are those used by Taylor, Vonkarman, Hinze, Reynolds, Deissler, Sarker, Kisore, Batchelor, Coriolis and Lundgren.

The **Second chapter** consist of two parts. In **part A**, we have studied the decay of temperature fluctuations in dusty fluid homogeneous turbulence prior to the final period considering correlations between fluctuating quantities at two- and three-point. In this part we have tried to solve the correlation equations by converting it to spectral form by taking their Fourier transform. Lastly, by integrating the energy spectrum over all wave numbers, the energy decay law of temperature fluctuations in homogeneous turbulence before the final period in presence of dust particle is obtained.

In **part B**, we have studied the decay of temperature fluctuations in dusty fluid homogeneous turbulence before the final period in presence of Coriolis force and have considered correlations between fluctuating quantities at two- and three- points by neglecting the fourth order correlation in comparison to the second and third order correlations. The correlation equations for two- and three- point in a rotating system in presence of dust particles are obtained and these equations are converted to spectral form by taking their Fourier transforms. Finally by integrating the energy spectrum over all wave numbers, the energy decay law of temperature fluctuations in homogeneous dusty fluid turbulence before the final period in presence of Coriolis force is obtained.

The **Third chapter** consists of two parts. In **part A**, we have studied the joint distribution functions for simultaneous velocity, temperature, concentration fields in turbulent flow undergoing a first order reaction in presence of Coriolis force. The various properties of the constructed joint distribution functions have been discussed. In this chapter we have tried to derive the transport equations for one and two point joint distribution functions of velocity, temperature, concentration in convective turbulent flow due to first order reaction in presence of coriolis force.

In **part B**, we have an attempt to derive the transport equation for the joint distribution function of certain variables in convective turbulent flow undergoing a first order reaction in a rotating system in presence of dust particles. Equations for the evolution of one- point and two- point joint distribution function for velocity, temperature and concentration in convective turbulent flow field undergoing first-order reaction in a rotating system in presence of dust particles have been derived. Finally we have made a result with comparison of the equation for one- point distribution function in the case of zero coriolis force in the absence of the dust particles and negligible diffusivity.

In **Chapter four**, we have studied the statistical theory of certain variables for three-point distribution functions in MHD turbulent flow in a rotating system in presence of dust particles. In this chapter we have made an attempt to derive the transport equations for evolution of distribution functions for simultaneous velocity, magnetic, temperature and concentration fields in MHD turbulent flow due to Coriolis force in presence of dust particles and various properties of the distribution function have been discussed.

In **Chapter five**, we have made an attempt to discuss the summary about the whole thesis.

The following research papers are extracted from this research work. No. 1 and 2 have been published, and other three communicated for publication in different national and international journals:

1. Azad M.A.K. and **Mumtahinah Mst.** Decay of Temperature Fluctuations in Dusty Fluid Homogeneous Turbulence Prior to the Final Period. **Res. J. Appl. Sci. Engng. Tech.**, 6(8), 1490-6, 2013
2. Azad M.A.K. and **Mumtahinah Mst.** Decay Of Temperature Fluctuations In Dusty Fluid Homogeneous Turbulence Prior To The Ultimate Period In Presence Of Coriolis Force. . **Res. J. Appl. Sci. Engng. Tech.**, 7(10), 1932-39, 2013
3. Azad M.A.K. and **Mumtahinah Mst**. Transport Equation for the Joint Distribution Functions of Certain Variables in Convective Turbulent Flow in Presence of Coriolis Force Under Going a First Order Reaction. (Communicated for publication)
4. **Mumtahinah Mst** and Azad M.A.K.**.** Transport Equation for the Joint Distribution Functions of Certain Variables in Convective Dusty Fluid Turbulent Flow in a Rotating System Under Going a First Order Reaction. (Communicated for publication)
5. **Mumtahinah Mst** and Azad M.A.K.**.** Statistical Theory of Certain Variables for Three- Point Distribution Functions in MHD Turbulent Flow in a Rotating System in Presence of Dust Particles. (Communicated for publication)

Department of Applied Mathematics
University of Rajshahi, Rajshahi-6205 (Mst. Mumtahinah)
Bangladesh

# *CONTENT*

**Page**

## CHAPTER-I

# CHAPTER-I

# GENERAL INTRODUCTION

## 1.1 Basic Concept of Turbulence

Fluids are anything that flow conventionally classified as either liquids or gases, treated as continuous media, and their motion and state can be specified in terms of the velocity u, pressure p, density ρ, etc evaluated at every point in space x and time t. Fluid dynamics is the natural science of fluids that deals with fluid flow. It is a subdiscipline of fluid mechanics. Fluid dynamics is one of the most important area of Physics.

Our life would not exist without fluids, and without the behaviour that fluids exhibit, the air we breathe and the water we drink are fluids and it makes most of our body mass. Fluid phenomena often studied by physicists, astronomers, biologists and others who do not necessarily deal in the design and analysis of devices. Atmospheric scientists study global circulation for long-range weather prediction and analysis of climate change; mesoscale weather patterns for short-range weather prediction, tornado and hurricane warnings and pollutant transport. Ocean circulation patterns are studied in Oceanography to find out causes of El Ni˜no, effects of ocean currents on weather and climate, and effects of pollution on living organisms. Convection in the Earth's mantle is studied in Geophysics to understand plate tectonics, earthquakes, volcanoes and Production of the magnetic field. In biological sciences circulatory and respiratory systems in animals, and cellular processes are under the study area of Fluid Dynamics.

It is easily recognized that a complete listing of fluid applications would be nearly impossible simply because the presence of fluids in technological devices is ubiquitous. Internal combustion engines in all types of transportation systems (Turbojet, scramjet, rocket engines), Waste disposal (chemical treatment, incineration, sewage transport and treatment), Steam, gas and wind turbines, and hydroelectric facilities for electric power generation, Pipelines (crude oil and natural gas transferral, irrigation facilities and office building and household plumbing), Fluid/structure interaction (design of tall buildings, continental shelf oil-drilling rigs, aircraft and launch vehicle airframes and control systems, dams, bridges, etc.), Heating, ventilating and air-conditioning (HVAC) systems, Cooling systems for high-density electronic devices, Solar heat and geothermal heat utilization, Artificial hearts, kidney dialysis machines, insulin pumps - these all are just few examples of application of fluid dynamics in technologies.

Fluid dynamics offers a systematic structure that embraces empirical and semi-empirical laws derived from flow measurement and used to solve practical problems. The solution to a fluid dynamics problem typically involves calculating various properties of the fluid, such as velocity, pressure, density, and temperature, as functions of space and time.

The flow of fluids can be qualitatively characterized as laminar or turbulent. Laminar flow is typically either a very slow motion or involves a level of viscosity. Fluid particles move evenly and slide across each other in layers (lamina is Latin for layer, plate), and are therefore laminar. However, turbulent flows (turbulentus is Latin for uneven) are characterized by quick motion or a low effect of viscosity, when even minor perturbations in stream grow uncontrollably and cause unpredictable local behaviour of fluid and intensive eddy mixing in the whole area.

Nearly all macroscopic flows encountered in the natural world and in engineering practice are turbulent. Winds and currents in the atmosphere and ocean; flows through residential, commercial, and municipal water (and air) delivery systems; flows past transportation devices (cars, trains, aircraft, ships, etc.); and flows through turbines, engines, and reactors used for power generation and conversion are all turbulent.

Turbulence is an enigmatic state of fluid flow that may be simultaneously beneficial and problematic. For example, in airbreathing combustion systems, it is exploited for mixing reactants but, within the same device, it also leads to noise and efficiency losses. Within the earth's ocean and atmosphere, turbulence sets the mass, momentum, and heat transfer rates involved in pollutant dispersion and climate regulation.

Defining turbulence is a very critical job. As stated in Oxford dictionary Turbulence is violent or unsteady movement of air or water, or of some other fluid. Turbulent flow is flow that is "irregular" in time and space. However this is not exact mathematical definition. In Turbulence or turbulent flow is a flow regime characterized by chaotic property changes that includes low momentum diffusion, high momentum convection, and rapid variation of pressure and velocity in space and time. According to Webster's "New International Dictionary", turbulence means agitation, commotion and disturbance. This definition however, is too general and not sufficient to characterize turbulent fluid motion in the modern sense. Lesieur [62] with some humour stated, "Turbulence is a dangerous topic which is at the origin of serious fights in scientific meetings since it represents extremely different points of view, all of which have in common their complexity, as well as an inability to solve the problem. It is even difficult to agree on what exactly is the problem to be solved."

In 1937 Taylor and Vonkarman [106] gave the definition: "Turbulence is an irregular motion which in general makes its appearances in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighbouring streams of the same fluid flow past or over one another". As per the definition the flow has to satisfy the condition of irregularity. Irregularity is a very important feature of turbulence and because of it, describing the motion in all details as a function of time and space coordinates is impossible. However, using laws of probability the irregularity of turbulent can be described. It appears possible to indicate distinct average values of various important quantities, such as velocity, pressure, temperature etc. If turbulent motion were entirely irregular, it would be inaccessible to any mathematical treatment. Therefore, it is not sufficient just to say that turbulence is an irregular motion.

Hinze [42] suggested that, "Turbulent fluid motion is an irregular condition of flow in which various quantities show a random variation with time and space co-ordinates, so that, statistically distinct average values can be discerned". This definition incorporated both time and space co-ordinates and justified that turbulent motion is not only irregular in time.

Oertel et al [74] said that Turbulence is the swirling motion of fluids that occurs irregularly in space and time.

Reynolds [83] made the first systematic experimental investigation of turbulent flow. Taylor and Vonkarman [106], Stanisic [103], Deissler [32] developed the idea of turbulent flow. According to Stanisic the study of interaction between a magnetic field and turbulent motion of an electrically conducting fluid is called magneto-hydrodynamics. Deissler [32, 33] developed a theory for homogeneous turbulence, which was valid for times before the final period. Sarker and Kisore [96] studied the decay of MHD turbulence before the final period.

But a universally accepted definition of a turbulent flow is given as the flow in which variables like velocity, density, pressure etc. are random variables having some mean values. The ratio of the random part of the motion to the mean motion in a turbulent flow is called the intensity of the turbulence.

Actually Turbulence is better to express as a list of properties and attributes that can help to identify turbulent flows:

**Randomness:** Turbulent flow is unpredictable in the sense that small random perturbations during a particular period of time are amplified to that level, and after a certain period of time deterministic prediction of further development becomes impossible.

**Diffusitivity:** Mixing of transported scalar quantities occurs relatively more quickly than during molecular diffusion. The intensity of this mixing can be several orders of magnitude greater than mixing occurring as a result of molecular diffusion.

**Vorticity:** Turbulent flows are characterized by high local values of vorticity related to the presence of vortex structures. The field of vorticity is generally non-homogeneous and changes dynamically in time. Vortex structures tend to be referred to as coherent vortices or more generally coherent structures.

**Scale spectrum:** Vortex structures, which occur spontaneously in a turbulent flow field, are characterized by a wide scale of length measuring units. The structures size is characterized by dense spectrum typical for fractals. Turbulent flow field can be characterized as a dynamical system with a "very high" number of degrees of freedom.

**3D structure:** Vortex structures occur in the space of a turbulent flow field in random locations and with random orientation. The 3D structure of the vector field of velocity fluctuations originates from this situation. During certain boundary conditions, the structures greater than the certain limit size can be spatially arranged; for example, they can have a planar character.

**Dissipation:** Turbulence is a dissipative process, which means that the kinetic energy of the motion of a fluid is dissipated at the level of small vortices and changes to heat. Therefore, in order for turbulent flows to be conserved over the long term, it is necessary to supply energy to the system from outside. This is done in the area of large scales; energy is collected from the main stream. The energy is then transferred towards smaller scales with the help of cascade transfer.

**Non-linearity:** Turbulent flows are basically non-linear, and their occurrence is conditioned on the application of non-linearities, when a growth of small perturbations occurs. The development as well as the interaction of individual structures in the turbulent flow field can be described only with a non-linear mathematical model.

The flow of water in the river, clouds in the sky, burning flames, the starry universe – these are some examples of phenomena that we can label as turbulent. Ever since ancient times Turbulence has always been a fascinating phenomenon for people.

Though thinkers are continuously trying, the process of recognizing the laws of turbulence has not been finished due to its variability and complexity.

The fascinating complexity of turbulence has attracted the attention of scientists for centuries. One of the first known findings about the structure of turbulence in modern times was the observation of fluid flows by Leonardo da Vinci. Leonardo illustrated the flow of water as a moment when the turbulent flow field is comprised of various structures of various sizes. Figure 1.a.

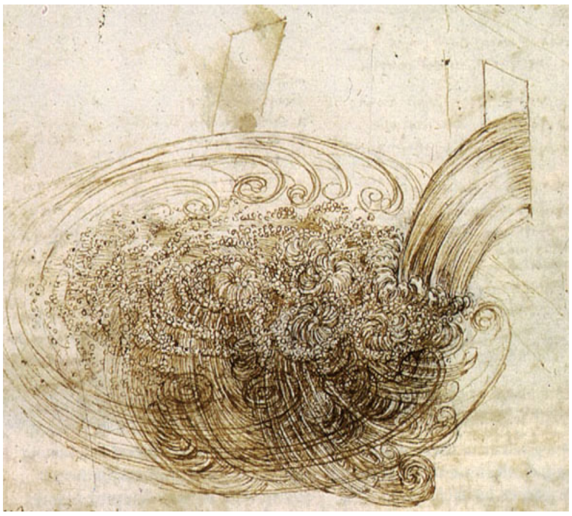


Figure 1.a Painting by Leonardo da Vinci showing turbulence in flow of water.

Another historical example of a regular structure in turbulent flow is the known red spot on Jupiter. It is basically an enormous storm – turbulent vortex (anti-cyclone) and has lasted at least 350 years (in 1655 it was first observed by French astronomer Cassini). Figure 1.b

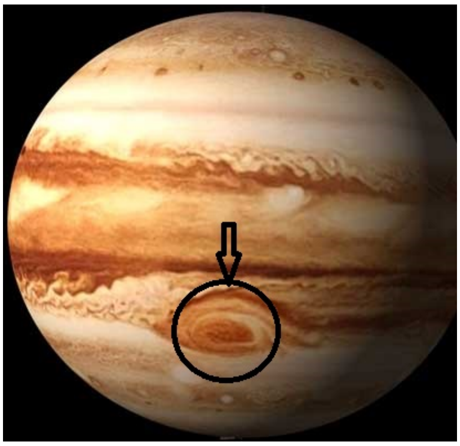


Figure 1.b: Turbulent vortex in Jupiter

The word turbulence is employed to label many different physical phenomena, which exhibit the common characteristics of disorder and complexity. It is the ubiquitous presence of spontaneous (intrinsic) fluctuations, distributed over a wide range of length and time scales makes the very nature of the turbulent fluctuations extremely peculiar. Turbulence has to do with non-linearity; there is no hint of the non-linear solutions in the linearized approximations, and strong departure from absolute statistical equilibrium.

Everyday experiences enable us to recognize turbulence. The smoke that rises from a cigarette or fire shows the irregular behavior of the moving air that carries it. Wind is subject to sharp local changes in direction and speed, which can have dramatic results for sailors and pilots. During transport by passenger aircraft, the term "turbulence" is often associated with buckling seatbelts. The term is also used when describing free streams and streaks. When water flows in a river, its presence has an important effect for the settling of sediment on the bottom. Quick flow of fluid around an obstacle or around an aviation profile creates turbulence in the between layer and creates a turbulent jolt causing increased resistance strength, which causes the flow fluid to affect the obstacle. The behavior of most oceanic and atmospheric flows cannot be exactly predicted, because they fall into the category of turbulent flows, and the same applies to flows of planetary scales. Small-scale turbulence in the Earth atmosphere represents a serious problem during astronomical observations conducted from the Earth surface, and it is a decisive factor to take into consideration when selecting an observatory. The atmospheres of planets such as Jupiter and Saturn, the solar atmosphere and the Earth outer atmosphere are turbulent. Galaxies typically have the shape of vortices similar to those that occur in turbulent streams, such as flows in a mixing layer of two streams of different velocity. These are formed as a result of turbulent phenomena. We can name a lot of other similar examples from aerodynamics, hydraulics, nuclear and chemical engineering, oceanology, meteorology, astrophysics, cosmology or geophysics. On the opposite field of the spectrum there are quantum vortices occurring in a superfluid fluid, which have dimensions expressible in multiples of the average size of an atom. The realm of turbulence therefore includes our observed universe, and turbulence is a typical kind of behavior of that universe in all of its degrees.

## 1.2 Example of Turbulence:

Turbulent motion is the most common motion in nature. Laminar flow is rather an exception and is limited to flow that can be characterized by very low velocity and thus Reynolds number (Re) values. In view of the definition of Re, this means that either the flow speeds are very low (such as melting of glaciers) or the typical dimension of the area is very small (such as motion of microorganisms in fluid) or the fluid shows extremely high viscosity (such as the motion of lubricant in bearings). Of course, a combination of these situations can also be considered.

**Grid turbulence:** A classic example of turbulent flow is flow behind a grid made from rods, which have regular square eyes. Behind individual rods, wakes are formed, which interact with each other and very quickly cause flow of a homogeneous structure (at a distance of about 20 spaces from the grid). The resulting flow, which is usually referred to as "grid turbulence", has certain beneficial properties. Mainly it is to a great extent homogeneous in a statistical sense in a level parallel to the grid generator of turbulence. The fluctuations also show a high level of isotropy, and deviations are in order of percentages.

For its beneficial properties as well as for its relatively easy achievability in laboratory conditions, grid turbulence has been considered an etalon of turbulent flow.

**Free shear layers:** The occurrence of free shear layers is unusually common, such as during surrounding of bodies or during flows through curved or non-prismatic (expansion) channels or at the boundary of an area of flow fluid in an unlimited space (jet). A free shear layer is nearly always unstable and results in the creation of vortex structures. In practice, we encounter free shear layers everywhere where a jet of fluid blown into a calm environment occurs or in connection with separation of a boundary layer.

**Boundary layers:** During flow in the boundary layer on the surrounded surface, the decisive parameter is the Reynolds number, where the length parameter can be thickness of the layer or the distance of the particular location from the beginning of the boundary layer, meaning for example from the leading edge. At a certain value of this parameter, a transition occurs of the boundary layer to turbulence. The boundary layer also has a turbulent structure.

**Wakes:** Wakes behind bluff bodies have a turbulent character with a dominant quasi-periodic low-frequency component. In relation to bluff bodies, the Reynolds number is decisive, where the length parameter is the transverse dimension of the bluff body. A typical situation is transverse surrounding of a cylinder, when a quasi-periodic von Kármán-Bénard vortex street occurs in the wake.

**Heat transfer:** Also during flow combined with heat transfer, we can often observe behaviour of fluid that can be described as turbulent. If fluid flow occurs as a result of heat transfer, it is referred to as natural convection. Thermal energy then causes fluid flow, which under certain conditions can be turbulent. An example is the surface of the sun, in which turbulent convective flow in the solar atmosphere is very apparent. This is caused by differences in temperatures between the surface of the sun and higher layers of its atmosphere and the lower temperatures on the surface in areas of sun spots. The photograph shows obvious turbulent sections and a cell structure in the background, which is related to Rayleigh-Bénard convection.

**Chemical turbulence:** Chemical reactions are processes with various non-linear dynamic characteristics. Non-linearities have their origin in the interaction of various particles between each other and in the behaviour of individual particles. An example is a Belousov-Zhabotinsky reaction, during which an oscillating reaction occurs without any variable external influences. It has been shown that for achievement of a homogeneous structure of a mixture (reactants are citric acid, potassium bromide, sulfuric acid and cerium ions), very intensive mixing is necessary, or otherwise the result is non-homogeneousness of both a stationary character (Turing structures) and a non-stationary character. Through intensive mixing, a structure can be maintained more or less homogeneous. Of course if we do not apply mixing, certain unstable frequencies occur in flows, which can culminate into quasi-stationary structures, which differ from each other based on their chemical contents. During a Belousov-Zhabotinsky reaction, when regular spiral structures are created as a result of periodic oscillations, which are related to global Hopf bifurcation. The chemical particles participating in the reaction differ by color. This is a very stable process also known as a “chemical clock”. If we breach the equilibrium of the particles entering into the reaction, then the reaction will either be stopped or will transit to a stormy turbulent regime.

**Burning:** Burning is another area with the occurrence of a whole range of turbulent conditions. It basically involves a combination of the two previously mentioned cases; it is a chemical reaction which is strongly exothermal and under normal circumstances irreversible.

## 1.3 Different Types of Turbulence

Taylor and Vonkarman [106] have stated that turbulence can be generated by the friction forces at fixed walls (fluid flow through conduits, fluid flow past solid surfaces) or by the flow of layers of fluids with different velocities past or over one another.

This definition indicates that there are two distinct types of turbulence.

(i) Wall turbulence: Turbulence is generated by the viscous effect due to presence of a solid wall is designated as wall turbulence.

(ii) Free turbulence: Turbulence in the absence of wall generated by the flow of layers of fluids at different velocities is called free turbulence.

## 1.4 Isotropic Turbulence

Batchelor [22] and Hinze [42] discussed homogenous isotropic turbulence in greater detail in their study. Isotropic turbulence is the simplest type of turbulence, because statistical features of it have no preferred direction or orientation. No average shear stress can occur and consequently, no velocity gradient is found in the mean velocity.

This mean velocity, if it occurs, remains constant throughout the field. A minimum number of quantities and relations are required to describe structure and behavior of Isotropic turbulence because of its simplicity and not having preference of any specific direction. However, actual turbulent flow showing true isotropy cannot be found - that indicates this type of turbulence has only hypothetical existence - though conditions may be made such that isotropy is more or less closely approached.

In isotropic turbulence the mean value of any function of the velocity components and their derivatives is unaltered by any rotation or reflection of the axes of references. Thus in particular, $\overline{u}_1^2 = \overline{u}_2^2 = \overline{u}_3^2$ and $\overline{u_1 u_2} = \overline{u_2 u_3} = \overline{u_3 u_1} = 0$.

So, if the turbulent fluctuations are completely isotropic, that is, if they do not have any directional preference, then the off-diagonal components of $\overline{u_i u_j}$ vanish, and the normal stresses are equal. This is illustrated in Figure 1.C

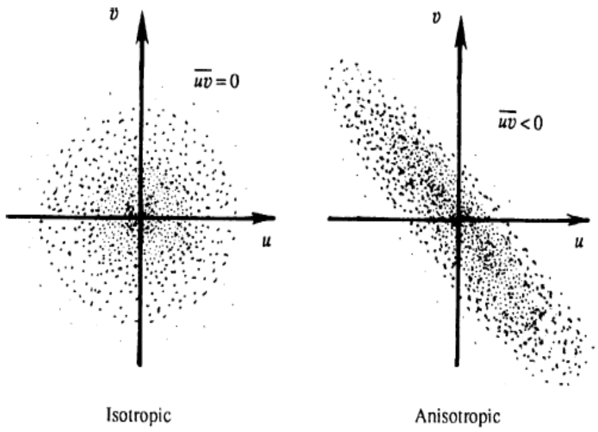



Figure 1.c: Isotropic Turbulence

Isotropy introduces a great simplicity into the calculations. The study of isotropic turbulence may also be of practical importance, since far from solid boundaries it has been observed that $\overline{u}_1^2, \overline{u}_2^2, \overline{u}_3^2$ tend to become equal to one another, e.g. in the natural winds at a sufficient height above the ground and in a pipe flow near the axis.

## 1.5 Homogeneous Turbulence

Turbulence which has quantitatively the same structure in all parts of the flow field is called homogeneous turbulence. In a homogeneous turbulent flow field, the statistical characteristic are invariant for any translation in the space occupied by the fluid. The conception of homogeneous turbulence is also idealized, in that there is no known method of realizing such a motion exactly.

However, the idealization of turbulence as being homogeneous or spatially stationary and isotropic allows some significant simplifications. Turbulence behind a grid towed through a nominally quiescent fluid bath is approximately homogeneous and isotropic, and turbulence in the interior of a real inhomogeneous turbulent flow is commonly assumed to be homogeneous and isotropic.

## 1.6 Convective Turbulent Flow

Convection is an important turbulent process. Turbulent convection or Rayleigh-Bénard convection in a fluid heated from below and cooled from above is found to play a major role in a great deal of natural and industrial processes, e.g., in the sun, planetary atmospheres, industrial manufacturing, and many other places. When the temperature difference exceeds a particular level, the heated fluid rises and the cooled fluid falls, thereby forms one or more convection cells. Increasing the difference causes the well-defined cells to become turbulent. Turbulent convection occurs in earth's outer core, atmosphere, and oceans, and is found in the outer layer of the sun and in giant planets. A very common example is found in the photosphere of the sun, where an irregular and continuously changing polygonal pattern of bright areas surrounded by darker boundaries is a dominant feature.

## 1.7 Laminar Flow and Turbulent Flow:

Viscous flows generically fall into two categories though the boundary between them is imperfectly defined - laminar and turbulent. The basic difference between the two categories is phenomenological. Reynolds [83] demonstrated it in a dramatic way by injecting a thin stream of dye into the flow of water through a tube (Figure 1.D). At low flow rates, the dye stream was observed to follow a well-defined straight path, indicating that the fluid moved in parallel layers (laminae) with no unsteady macroscopic mixing or overturning motion of the layers. Such smooth orderly flow is called laminar. However, if the flow rate was increased beyond a certain critical value, the dye streak broke up into irregular filaments and spread throughout the cross section of the tube, indicating the presence of unsteady, apparently chaotic three-dimensional macroscopic mixing motions. Such irregular disorderly flow is called a turbulent.

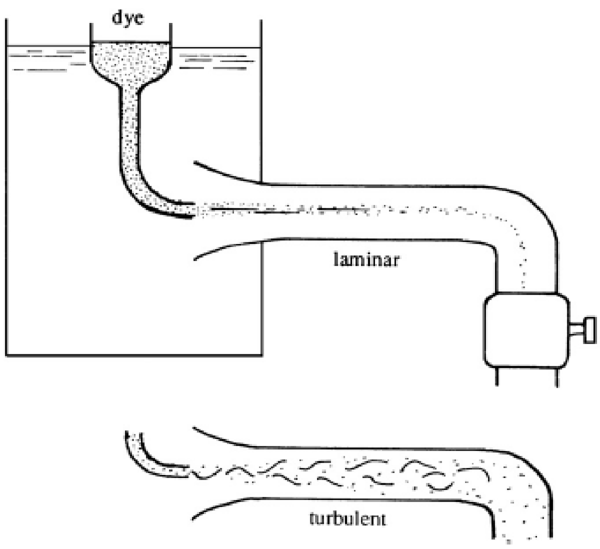



Figure 1.d: Reynold's experiment to distinguish between lamina and turbulent flows. At low flow rates (the upper drawing), the pipe flow was laminar and the dye filament moved smoothly through the pipe. At high flow rates (the lower drawing), the flow became turbulent and the dye filament was mixed throughout the cross section of the pipe.

Laminar flow or streamline flow occurs when a fluid flows in parallel layers, with no disruption between the layers [85]. At low velocities the fluid tends to flow without lateral mixing, and adjacent layers slide past one another like playing cards. There are no cross currents perpendicular to the direction of flow, nor eddies or swirls of fluids [38]. In laminar flow the motion of the particles of fluid is very orderly with all particles moving in straight lines parallel to the pipe walls [72]. Laminar flow tends to occur at lower velocities, below the onset of turbulent flow. Figure 1.e is showing Laminar Flow.

Turbulent flow is a less orderly flow regime that is characterized by eddies or small packets of fluid particles which result in lateral mixing. Turbulent flows can often be observed to arise from laminar flows as the Reynolds number is increased. The transition to turbulence happens because small disturbances to the flow are no longer damped by the flow, but begin to grow by taking energy from the original laminar flow. Figure 1.f shows Turbulent flow.

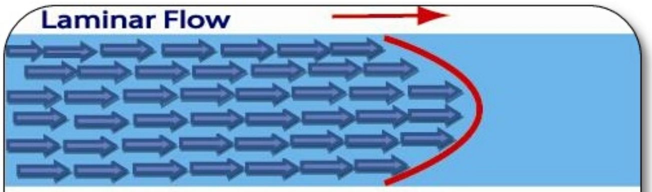



Figure 1.e

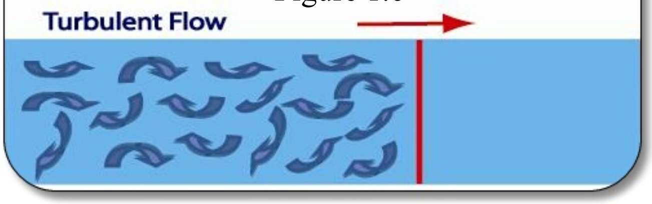



Figure 1.f

Reynolds demonstrated that the transition from laminar to turbulent flow always occurred at a fixed value of the ratio that bears his name, the Reynolds number, $Re = \frac{Vd}{v} \sim$ 2000 to 3000 where V is the velocity averaged over the tube's cross section, d is the tube diameter, and $v$ is the kinematic viscosity.

## 1.8 Reynolds Number and its Effect on Turbulent Flow

In the year 1883 a British physicist Osborne Reynolds [84] demonstrated that the transition from laminar to turbulent flow in a pipe depends upon the value of a mathematical quantity equal to the average velocity of flow times the diameter of the tube times the mass density of the fluid divided by its absolute viscosity.

Mathematically,

$$\text{Reynolds no.} = \frac{intertia\ force}{viscous\ force} = \frac{\rho v^2}{\frac{\mu v}{d}} = \frac{\rho v d}{\pi} = \frac{Vd}{\nu}$$

Where V → mean velocity of liquid

d → diameter of pipe

ν → kinematics viscosity of liquid

This mathematical quantity is a pure number without dimensions that represents the type of flow, i.e., either the flow is laminar or turbulent. The number is called Reynolds number after the name of its inventor. Reynolds [84] found that to be laminar flow the number should remain less than 2000. For the Reynolds number between 2000 to 2300, the flow is neither laminar nor turbulent. However, when the Reynolds number for a flow exceeds 2300, it becomes turbulent.

## 1.9 Critical Reynolds Number:

Reynolds conducted in a series of experiments in which water at rest in a tank was allowed to flow through a glass pipe. Reynolds argued that it was likely to exist a critical value of a certain non-dimensional quantity beyond which a laminar flow gives rise to a "sinuous" motion. It was found from Reynolds observations of the flow for tubes with different diameter, different velocities, with altered kinetic viscosity through changes in temperature that as the velocity of the fluid exceeds some critical value, the stationary and the regularity of the flow break off. Small (velocity) disturbances are no longer damped by the laminar flow, but grow by extracting kinetic energy from the mean flow. Disordered swirling motions, in which fluid particles follow complicated (non-brownian) trajectories, take place. The flow is then called turbulent. In this situation, velocity gradients are much larger and the Reynolds number at which there is a transition from laminar to turbulent flow is called Critical Reynolds Number.

The approximate value of the critical Reynolds number $Re_{cr}$ at which the laminar regime breaks down was established to be order of $2\times10^3$. Later with Reynolds apparatus, Ekman [37] was able to maintain laminar flow up to a critical Reynolds number of $4\times10^4$ when the testing conditions were made extremely free from disturbances. Therefore, critical Reynolds number are classified into two

(i) Upper critical Reynolds number

(ii) Lower critical Reynolds number

## 1.10 Upper Critical Reynolds Number

The upper critical Reynolds number is a number at which the flow enters from transition to turbulent flow. However, several more recent investigators [40, 81] have repeatedly demonstrated that there is no definite upper critical Reynolds number rather the numerical value depends largely on the test conditions affecting the initial turbulence of flow.

Obviously, the upper critical Reynolds number is a function of initial disturbances; its numerical values always increase with a decrease in disturbances. For engineering purposes, high numerical values of the upper critical Reynolds number are of limited practical significance; the transition from laminar to turbulent flow in a tube may be assumed to take place at 2100-4000.

## 1.11 Lower Critical Reynolds Number

The lower critical Reynolds number is a number which defines the below limit of laminar flow. In other words the critical Reynolds number at which the flow enters from laminar to transition period is known as a lower critical Reynolds number. At lower critical Reynolds number is taken to be approximately 2000.

In brief status of flow can be can be changed at various phases of Reynolds number. When it is smaller than the critical Reynolds number i.e. $R < Recr$, the flow is laminar. If the Reynolds number is greater than the critical Reynolds number i.e. $R > Recr$, the flow is turbulent. Transition normally takes place at Reynolds number 2000-4000.

## 1.12 Averaging Procedure:

Averaging method is unavoidable for the statistical formulation of the theory of turbulence. In turbulent flow the instantaneous velocity u is the sum of the time average part $\bar{u}$ and fluctuating velocity $u'$ i.e.

$$u = \bar{u} + u'. \qquad (1.12.1)$$

Where, $\bar{u} \rightarrow$ mean velocity

$u' \rightarrow$ fluctuating velocity and
$u \rightarrow$ velocity of motion

In taking the average of a turbulent quantity, the result depends not only on the scale used but also on the demand of averaging. Pai [75] introduced four different kinds of averaging procedure to study turbulent flows. These are time average, space average, space-time average and ensemble average.

Time average can be used for quasi-steady turbulent flow field. For a homogeneous turbulence flow field, space average can be considered. If the flow field is steady and homogeneous, space-time average is used. Lastly, if the flow field is neither steady nor homogeneous, we assume that averaging is taken over a large number of experiments that have initial and boundary conditions. This type of average is called ensemble average or statistical average. Ensemble average is more general than the time and space averages and very useful for the study in homogeneous, non-stationary turbulent flow. This type of averaging can be applied to any flow. However, like the time and space averages, the physical interpretation of the ensemble average is not so simple. In general the hierarchy of correlations completely determine any turbulent field. According to Leslie's [61] the assemble average is defined as-

$\langle u_i,(r,t)\rangle, \langle u_i(r,t)u_j(r',t)\rangle, \langle u_i(r,t)u_j(r',t)u_m,(r'',t)\rangle$

where $\langle \quad \rangle$ denote the ensemble average.

In homogeneous isotropic turbulence the first correlation represents the mean velocity, and is simply zero, the pair correlation $\langle u_i(r)u_j(r')\rangle$ is often considered to be a sufficient description of turbulent flow. The higher order correlations are assumed to give less and less information so that only a finite number of correlations are required to determine the statistical properties of turbulence.

The double correlation tensor $R_{ij}(\hat{r},\hat{x},t)$ for two-points separated by the space vector $\hat{r}$ is defined by

$$R_{ij}(\hat{r},\hat{x},t) = \langle u_i\left(\hat{x}-\frac{1}{2}\hat{r},t\right)u_j\left(\hat{x}+\frac{1}{2}\hat{r},t\right)\rangle$$

Similarly, the triple correction tensor $T_{ijk}$ or higher correlation tensors can be introduced.

The Fourier transform of $R_{ij}$ with respect to $\hat{r}$ defined by

$$\emptyset_{ij}(\hat{k},\hat{x},t) = \frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty} \bar{e}^{(\hat{k},\hat{r})} R_{ij}(\hat{r},\hat{x},t)d\hat{r}$$

represents the energy spectrum function $E(\hat{k},t)$ in the sense that it describes the distribution of kinetic energy over the various wave number component of turbulent flows. The Fourier transform defined above can be treated as generalized functions or distributions in the sense o Lighthill [63]. It follows from the inverse Fourier transform that

$$\frac{1}{2}\langle u^2\rangle = \frac{1}{2}\langle u_i(\hat{x})u_j(\hat{x})\rangle = \frac{1}{2}R_{ij}(o,\hat{x},t) = \int_0^{\infty} E(\hat{k},t)d\hat{k}.$$

So that $E(\hat{k},t)$ represents the density of contributions to the kinetic energy in the wave number of space k, thus the investigation of the energy spectrum function $E(\hat{k},t)$ is the central problem of the dynamics of turbulence.

The mathematical form of the four methods of averaging procedure are given below

(i) Time average for stationary turbulence

$$\frac{t}{u}(x,t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} u(x,t)ds.$$

The scale used in the averaging process determines the value of the period 2T.

(ii) Space average in which we take the average over all the spaces at a given time, i.e.

$$\frac{s}{u}(x,t) = \lim_{v_b\to\infty}\frac{1}{V_b}\int_{v_b} u(s,t)ds.$$

The scale used in the averaging process determines the volume of space $V_b$.

(iii) Space time average in which we take the average over a long period of time and over the space. i.e.,

$$\frac{s,t}{u}(x,t) = \lim_{T\to\infty, v_b\to\infty} \frac{1}{2T \cdot V_b} \int_{-Tv_b}^{+T} \int u(s,y)dsdy$$

The scale used determines both the values of T and $V_b$.

(iv) Statistical average in which we take the average over the whole collection of sample turbulent functions for a fixed time, i.e.

$$\frac{\omega}{u}(x,t,w) = \int_{\Omega} u(x,t,\omega)du(\omega)$$

over the whole Ω space of ω, the random parameter, where $\int_{\Omega} d\mu(\omega) = 1$.

The essential characteristic of the turbulent motion is that the turbulent fluctuations are random in nature. A turbulent velocity field can be regarded as a random vector field of a set of vectors in space and time. Any random vector field can be regarded as a field consisting of three random scalar fields as its components. A random scalar function $u(x,t,\omega)$ is a function of the spatial co-ordinates x and time t, which depends on a parameter ω. The parameter ω is chosen at random according to some probability law in a space.

In the experimental investigation we use time averages almost exclusively, space averages seldom and never statistical averages. In theory, we use almost exclusively the statistical averages.

For stationary homogeneous turbulence we may expect and assumed that the three averaging lead to the same result

$$\frac{t}{u} = \frac{s}{u} = \frac{\omega}{u}$$

which is Ergodic hypothesis.

## 1.13 Reynolds Rules of Averages

Osborne Reynolds [83] introduces elementary statistical motion into the consideration of turbulent flow. In the theoretical investigation of turbulence, he assumed that a turbulent flow instantaneously satisfies the Navier-Stokes equations and that the instantaneous velocity may be separated into a mean velocity and a turbulent fluctuating velocity. However, it is virtually impossible to predict the flow in detail at high Reynolds numbers, as there is an enormous range of length and time scales to be resolved.

If u, P, T and ρ be respectively the instantaneous velocity, pressure time and density, then the process of averaging are written as

$u = \bar{u} + u', P = \bar{P} + P', \rho = \bar{\rho} + \rho', T = \bar{T} + T'$ etc.

Here the quantities with bar denote mean values and the quantities with prime denote fluctuating values.

Furthermore, $\bar{u}' = \bar{P}' = \bar{T}' = 0$.

In the study of turbulence we often have to carry out an averaging procedure not only on single quantities but also on products of quantities.

In order to develop the rule of averaging, three arbitrary statistically dependent physical quantities e.g., A, B, C can be considered, each consisting of a mean and fluctuating part, i.e.

$$A = \bar{A} + a, B = \bar{B} + b \text{ and } C = \bar{C} + c \quad (1.13.1)$$

$$\text{then } \bar{A} = \overline{\bar{A} + a} = \bar{\bar{A}} + \bar{a} = \bar{A}, \text{ when } \bar{a} = 0 \quad (1.13.2)$$

In the above relations we used the properties that the average of the sum is equal to the sum of the averages and the average of a constant times B is equal to the constant times the average of B.

Then,

$$\overline{\bar{A}\bar{B}} = \bar{\bar{A}}\bar{\bar{B}} = \bar{A}\bar{B} \quad (1.13.3)$$

$$\overline{\bar{A}b} = \bar{\bar{A}}\bar{b} = \bar{A}\bar{b} = 0 \quad \therefore \bar{b} = 0 \quad (1.13.4)$$

$$\overline{\bar{B}a} = \bar{\bar{B}}\bar{a} = \bar{B}\bar{a} = 0 \quad \therefore \bar{a} = 0 \quad (1.13.5)$$

Similarly,

$$\overline{AB} = \overline{(\bar{A} + a)(\bar{B} + b)} = \overline{\bar{A}\bar{B}} + \overline{\bar{A}b} + \overline{a\bar{B}} + \overline{ab} = \bar{A}\bar{B} + \overline{ab} \quad (1.13.6)$$

Note that the average of a product is not equal to the product of the averages terms such as $\overline{\mathrm{ab}}$ are called correlations.

## 1.14 Reynolds Equations and Reynolds Stresses

We usually assume that in turbulent flow, instantaneous velocity components satisfy the Navier-Stockes equation

$$\frac{\partial U}{\partial t} + (U.\nabla)U = F - \frac{1}{\rho}\nabla p + v\nabla^2 U. \quad (1.14.1)$$

The tensor form the equation (1.14.1) can be written as

$$\frac{\partial \bar{u}_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = F - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial u_j\, \partial u_j} \quad (1.14.2)$$

Substituting the expression for the instantaneous velocity components $u_i = \bar{u}_i + u_i'$, into the Navier-Stockes equation (1.14.2) for an incompressible fluid after neglecting the body force and taking the mean values of these equations according to Reynolds rule of averaging (1.13.1) - (1.13.6), we have the following Reynolds equation of motion for the turbulent flow of an incompressible fluid.

$$\rho\left(\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j}\right) = -\frac{\partial \bar{P}}{\partial x_i} + \mu \frac{\partial^2 \bar{u}_i}{\partial x_j\, \partial x_j} + \frac{\partial}{\partial x_j}\left(\rho\overline{u_i' u_j'}\right), \quad (1.14.3)$$

here i and j run from 1 to 3 and Einstein's summation convection is used. The bar represents the mean value and the prime represents the turbulent fluctuation. Additional terms over the Navier-Stockes equations are due to Reynolds stress are $-\overline{\rho\mu_i'^2}$ and the eddy stresses are $-\overline{\rho u_i' u_j'}\,(i \neq j)$, where ρ is the density of the fluid. These stresses represent the rate of transfer of momentum across the corresponding surfaces because of turbulent velocity fluctuations.

The solutions of Reynolds equation represent the turbulent flow, but as in the case of Navier-Stockes equation it is not possible to solve Reynolds equations for many practical purposes. In general the Reynolds equations are not sufficient to determine the unknown variable $u_i, u_j$ $(i, j = 1,2,3)$, p and Reynolds stresses. This is one of the main difficulties in theoretical investigation of turbulent flow. In similar way, Reynolds equation of motion for the turbulent flow of a compressible fluid may be obtained. But the expressions for the eddy stresses (Reynolds stresses) of compressible fluid are much more complicated because the fluctuations of density should be considered.

## 1.15 Coriolis Force

In a rotating coordinate system there is an apparent force which deflects an object in internal motion from a straight line path, the resulting path is curve in the direction opposite to the direction of coordinate rotation, then the deflection force is called Coriolis (1792-1843), has traditionally been derived as a matter of coordinate transformation by essentially kinematical technique. This has the consequence that it's physical significance for processes in the atmosphere, as well for simple mechanical systems. It also helps to clarify the relation between angular momentum and rotational kinetic energy and how an inertial force can have a significant effect on the movement of a body and still without doing any work.

The mathematical expression of the Coriolis acceleration is $a_c = -2\Omega \times v$, where $a_c$ is the acceleration of the particle in the rotating system, v is the velocity of the particle in the rotating system, and $\Omega$ is the angular velocity vector which has magnitude equal to the rotation rate $\omega$ and is directed along the axis of rotation of the rotating reference frame, and $\times$ symbol represents the cross product operator.

Hence mathematically the Coriois force is $F_c = -2m\Omega \times v$, where m is the mass of the relevant object.

## 1.16 Coriolis Effect

The Coriolis effect is a deflection of moving objects when they are viewed in a rotating reference frame. In a reference frame with clockwise rotation, the deflection is to the left of the motion of the object; in one with counter-clockwise rotation, the deflection is to the right. The Rotation of the earth and the inertia of the mass experiencing the effect create Coriolis effect. When Newton's laws of motion govern the motion of an object in a (non accelerating) inertial frame of reference are transformed to a rotating frame of reference, the Coriolis and centrifugal forces appear. Both the forces are proportional to the mass of the object. The Coriolis force is proportional to the rotation rate and centrifugal force is proportional to its square. The Coriolis force acts in a direction perpendicular to the rotation axis and to the velocity of the body in the rotating frame and is proportional to object's speed in rotating frame. The centrifugal force acts outwards in the radial direction and is proportional to the distance of the body from the axis of the rotating frame. This effect is responsible for the rotation of large cyclones. The practical impact of the Coriolis effect is mostly caused by the horizontal acceleration component produced by horizontal motion.

There are other components of the Coriolis effect. Eastward-travelling objects will be deflected upwards (feel lighter), while westward-travelling objects will be deflected downwards (feel heavier). This is known as the Coriolis effect. This aspect of the Coriolis effect is greatest near the equator. The force produced by this effect is similar to the horizontal component but the much larger vertical forces due to gravity and pressure mean that it is generally unimportant dynamically. Coriolis effect is an inertial force described by the 19th century French engineer and mathematician Gustave-Gaspard Coriolis in 1835. Coriolis showed that if the ordinary Newtonian laws of motion of bodies are to be used in a rotating frame of reference, an inertial force acting to the right of the direction of body motion for counter clockwise rotation of the reference frame or to the left for clockwise rotation must be included in the equations of motion.

The effect of the Coriolis force is an apparent deflection of the path of an object that moves within a rotating coordinate system. The object does not actually deviate from its path but it appears to do so because of the motion of the coordinate system. The Coriolis deflection is therefore related to the motion of the object, the motion of the earth and the latitude (Figure 1.g). The coriolis effect has great significance in astrophysics and stellar dynamics in which it is a controlling factor in the directions of rotation of sunspots. It is also significant in the earth sciences especially meteorology, physical geology, and oceanography, in that the earth is a rotating frame of reference, and motions over the surface of the earth are subject to acceleration from the force indicated. Thus the Coriolis force figures prominently in studies of the dynamics of the atmosphere in which it affects prevailing winds, the rotation of storms and in the hydrosphere in which it affects the rotation of the oceanic currents.

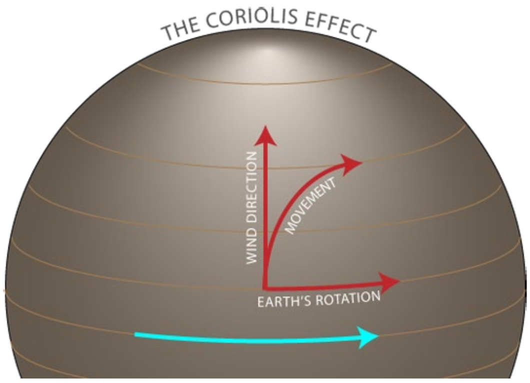



Figure 1.g: Coriolis Effect

## 1.17 Correlation Function

In 1935, Taylor [105] introduced new notions into the study of the statistical theory of turbulence. He successfully developed a statistical theory of turbulence which is applicable to continuous movements and which satisfies the equation of motion.

The first important new notion was that of studying the correlation or coefficient of correlation between two fluctuating quantities in turbulent flow. In his theory, Taylor makes much use of the correlation between the components of the fluctuating at neighbouring points. Denoting the components of the fluctuating velocity at one point p by u1, u2, u3 and another point $p^/$by $u_1^/, u_2^/, u_3^/$.

The correction function between any of the $u_i$ and $u_j^/$ where i. j = 1, 2 or 3 are defined as $\rho_{ij} = u_i u_j'$, (1.17.1)

where the bar denotes the average by certain process.

Sometimes it is convenient to use the correlation coefficient such as

$$R_{ij} = \frac{\overline{u_i u_j'}}{\sqrt{\overline{u_i^2}}\sqrt{\overline{u_j'^2}}} \tag{1.17.2}$$

By Cauchy inequality, we have

$$\overline{u_i u_j'} - \sqrt{\overline{u_i^2}}\sqrt{\overline{u_j'^2}} \leq 0 \tag{1.17.3}$$

hence $-1 \leq R_{ij} \leq 1$.

If we consider $u_i$ and $u_j'$ as the velocity components is a flow field, the correlation of equation (1.17.1) as a tensor of second rank. By a different process of averaging we obtain different kinds of correlation functions. If we consider $u_i$ and $u_j'$ are the velocity components at a given point in space, $u_i$ and $u_j'$ are functions of time; hence; we should take the time average in equation (1.17.1) to get the correlation function $\rho_{ij}$.

If we consider $u_i$ and $u_j'$ as the velocity components at a given time, $u_i$ and $u_j'$ are functions of space co-ordinates $x(x_1, x_2, x_3)$; hence we should take the space average in equation to get the correlation function. More generally if we consider $u_i$ and $u_j'$ as functions of both time t and spatial co-ordinates $x(x_1, x_2, x_3)$; we should take a space-time average in equation (1.17.1) to get the correlation function. The correlation function between the components of the fluctuating velocity at the same time two different points of the fluid, first introduce by Taylor [105] has been investigated extensively in the isotropic turbulence.

The correlation function between two fluctuating velocity components at the same point and at the same time gives the Reynolds stress. The correlation function between two fluctuating quantities may also be defined in a manner similar to above.

## 1.18 Distribution Function

In molecular kinetic theory in physics, a particle's distribution function is a function of seven variables, $f(x, y, z, t, v_x, v_y, v_z)$, which gives the number of particles per unit volume in phase space. It is the number of particles per unit volume having approximately the velocity $(v_x, v_y, v_z)$ near the place (x, y, z) ant time (t). The usual normalization of the distribution function is

$$n(x, y, z, t) = \int f dv_x dv_y dv_z,$$

$$N(t) = \int n dx\, dy\, dz$$

Here, N is the total number of particles and n is the number density of particles - the number of particles per unit volume or the density divided by the mass of individual particles.

A distribution function may be specialized with respect to a particular set of dimensions, e.g., take the quantum mechanical six dimensional phase spaces $f(x, y, z, t, p_x, p_y, p_z)$ and multiply by the total space volume to give the momentum distribution i.e. the number of particles in the momentum phase space having approximately the momentum $(p_x, p_y, p_z)$.

Particle distribution function are often used in plasma physics to describe wave particle interactions and velocity-space instabilities. Distribution function are also used in fluid mechanics, statistical mechanics and nuclear physics.

The basic distribution function uses the Boltzmann constant k and temperature T with the number density to modify the normal distribution,

$$f = \frac{n}{\sqrt{(2\pi kT)^3}} exp\left(-\frac{m({v_x}^2+{v_y}^2+{v_z}^2)}{2kT}\right).$$

Related distribution function may allow bulk fluid flow, in which case the velocity origin is shifted, so that the exponent's numerator is $m\left((v_x - u_x)^2 + (v_y - u_y)^2 + (v_z - u_z)^2\right)$; $(u_x, u_y, u_z)$ is the bulk velocity of the fluid. Distribution function may also feature non isotropic temperatures, in which each term in the exponent is divided by a different temperature.

The mathematical analogy of a distribution is a measure; the time evolution of a measure on a phase space is the topic of study in dynamical systems.

## 1.19 Joint Distribution Function

A joint distribution function is a function $D_{XY}(x, y)$in two random variables X and Y defined by

$D_{XY}(x, y) = P(X \leq x, Y \leq y),$

where x and y are arbitrary real numbers.

$D_X(x) = P(X \leq x) = P(X \leq x, Y \leq \infty) = \lim_{n \to \infty} D_{XY}(x, y),$

$D_Y(y) = P(Y \leq y) = P(X \leq \infty, Y \leq y), \lim_{n \to \infty} D_{XY}(x, y),$

here $D_X(x)$ is termed as the marginal distribution function of X corresponding to the joint distribution function $D_{XY}(x, y)$ and $D_Y(y)$ is termed as the marginal distribution function of Y corresponding to the joint distribution function $D_{XY}(x, y)$.

So that the joint probability function satisfies

$$D_{XY}[(x, y) \in C] = \iint_{(X,Y) \in C} P(X, Y)\, dXdY$$

$$D_{XY}[x \in A, y \in B] = \int_{Y \in B} \int_{X \in A} P(X, Y)\, dXdY$$

$D_{XY}(x, y) = P\{X \in (-\infty, x), Y \in (-\infty, y)\}$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} P(X, Y)\, dXdY$$

and

$D_{XY}(a \leq x \leq a + da, b \leq y \leq b + db) = \int_{b}^{b+db} \int_{a}^{a+da} P(X, Y)\, dXdY \approx P(a, b) dadb$

Two random variables X and Y are independent if

$D_{XY}(x, y) = D_X(x) D_Y(y)$ for all x and y

and joint probability density function by differentiation as follows

$$P(x,y) = \frac{\partial^2 D(x,y)}{\partial x\,\partial y} = \lim_{\delta x\to 0,\delta y\to 0} \frac{P(x \le X \le x+\delta x, y \le Y \le y+\delta y)}{\delta x \delta y}$$

A multiple distribution function is of the form

$D_{X_1.....X_n}(x_1, ......, x_n) = P(X_1 \le x_1, ......, X_n \le x_n)$.

## 1.20 Distribution Function in Turbulence and its Properties

The dynamical equations describing the time evolution of the finite dimensional probability distributions in turbulence were first proposed by Lundgren [66] and Monin [70, 71], Lundgren [66] considered a large ensemble of identical fluid system in turbulent state. In his consideration each number of the ensemble is an incompressible fluid in an infinite space with velocity $\hat{u}(\hat{r}, t)$ satisfying the continuity and Navier-Stockes equations. The only difference in the members of ensemble is the initial conditions that vary from member to member. He considered a function $F(\hat{u}(\hat{r}_1, t), \hat{u}(\hat{r}_2, t) - - -)$ whose ensemble is given as $\langle F(\hat{u}(r_1, t), (r_2, t) - - -)\rangle$ and defined one-point distribution function $f_1(\hat{r}_1 \cdot v_1, t)$ such that $f_1(\hat{r}_1 \cdot \hat{v}_1, t)d\hat{v}_1$ is the probability that the velocity at a point $\hat{r}_1$ at time t is in element $d\hat{v}_1$about $\hat{v}_1$and is given by $f_1(\hat{r}_1, \hat{v}_1, t) = \langle\delta\big(\hat{u}(\hat{r}_1, t)\big) - \hat{v}_1\rangle$

and two-points distribution function is given by

$$f_2(\hat{r}_1, \hat{v}_1, \hat{r}_2, \hat{v}_2, t) = \langle\delta(\hat{u}(r_1, t) - \hat{v}_1)\delta(\hat{u}(r_2, t) - \hat{v}_2)\rangle$$

In short one and two-point distribution function are denoted as $f_1^{(1)}$ and $f_2^{(1,2)}$. Here δ is the dirac-delta function, which is defined as

$$\int \delta\,(\bar{u} - \bar{\upsilon})d\bar{v} = \begin{cases} 1 & \text{at the point } \bar{u}=\bar{\upsilon} \\ 0 & \text{elsewhere} \end{cases}$$

and ⟨ ⟩ denote the ensemble average.

## 1.21 Dust Particles

Dust means dry fine powdery material. As stated in Oxford Dictionary, dust is Fine, dry powder consisting of tiny particles of earth or waste matter lying on the ground or on surfaces or carried in the air. Dust consists of particles in the atmosphere that comes from various sources such as soil dust lifted by wind, volcanic eruptions and pollution. Dust in homes, offices, and other human environments contains small amounts of plant pollen, human and animal hairs, textile fibres, paper fibres, minerals from outdoor soil, human skin cells, burnt meteorite, particles and many other materials which may be found in the local environment. Homogenous and passive dust particles in the boundary layers are entrained and adverted under the influence of a turbulent flow.

## 1.22 Equation of Motion of Dust Particles

Knowledge of the behaviour of discrete particles in a turbulent flow is of great interest to many branches of technology, particularly if there is a substantial difference between particles and the fluid. Saffman [86] derived an equation that described the motion of a fluid containing small dust particles, which is applicable to laminar flows as well as turbulent flow.

A more plausible explanation seems to be that the dust damps the turbulence. A dust particle in air or in any other gas has a much larger inertia than the equivalent volume of air will not therefore participate readily in turbulent fluctuations. The relative motion of dust particles and the air will dissipate energy because of the drag between dust and air and extract energy from turbulent fluctuations. If as certainly seems possible, the turbulent intensity is reduced than the Reynolds stresses will be decreased and the force required to maintain a given flow rate will likewise be reduced.

In order to formulate the problem in a reasonably simple manner and to bring out the essential features, we shall make simplifying assumption about the motion of dust particles. It will be supposed that their velocity and number density can be described by fields $u(\vec{x}, t)$ and $N(\vec{x}, t)$. We also assume that the bulk concentration (i.e. concentration of volume) of dust is very small so that the effect of dust particles on the gas is equivalent to an extra force $KN(\vec{\upsilon} - \vec{u})$ per unit volume, where $\vec{u}(\vec{x}, t)$ the velocity of the gas and K is constant. It is also supposed that the Reynolds number of the relative motion of dust and gas is small compared with unity, so that the force between the dust and gas is proportional to the velocity. Then with small bulk concentration and the neglect of the compressibility of the gas, the equations of motion and continuity of the gas are:

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + (u \cdot \nabla)\vec{u}\right] = -\nabla p + \nu\nabla^2\vec{u} + KN(\vec{v} - \vec{u}), \quad (1.22.1)$$

$$\text{div}\, \vec{u} = 0, \quad (1.22.2)$$

where p, ρ and μ are the pressure, density and viscosity of the clean gas respectively. If dust particles are spheres of radius ε, then by Stocke's drag formula, $K = 6\pi\mu\varepsilon$.

As will be seen below, the effect of the dust is measured by the mass concentration, say f. The bulk concentration if $f\frac{\rho}{\rho_1}$ where $\rho_1$ is the density of the material in the dust particles. For common materials $\frac{\rho}{\rho_1}$ will be of the order of several thousand or more, so that the mass concentration may be significant fraction of unity, while the bulk concentration is small. It is to be noted that for suspension in liquids, the bulk and mass concentration will roughly be the same. So that the qualitative differences in the motion of dusty gases and the suspensions in the liquids may be expected. For spherical, the Einstein increase in the viscosity is $\frac{5}{2}\mu f\frac{\rho}{\rho_1}$, which is negligible for a dusty gas but may be significant for a liquid suspension. The force exerted on the dust by the gas is equal and opposite to the force exerted on the gas by dust, so that the equation of motion of the dust is,

$$mN\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v}\right] = mN\vec{g} + KN(\vec{v} - \vec{u}), \quad (1.22.3)$$

where mN the mass of the dust per unit volume and $\vec{g}$ is the acceleration due to gravity. The buoyancy force is neglected since $\frac{\rho}{\rho_1}$ is small.

The equation of continuity of the dust is,

$$\frac{\partial N}{\partial t} + \text{div}(N\vec{v}) = 0 \quad (1.22.4)$$

Here, $\nu = \frac{\mu}{\rho}$ is kinetic viscosity of the clean gas and $\tau = \frac{M}{K}$ is called the relaxation time of the dust particles. It is measure of the time for the dust to adjust to changes in the gas velocity. For spherical particles of radius ε,

$$\tau = \frac{\frac{4}{3}\mu\varepsilon^3\rho_1}{6\pi\varepsilon\mu} \text{ or } \tau = \frac{2}{9}\frac{\varepsilon^2}{\nu}\frac{\rho_1}{\rho} \quad (1.22.5)$$

where $\frac{4}{3}\mu\varepsilon^3\rho$, mass of single spherical dust particle of radius $\varepsilon$; $\rho_1$, density of the material in the dust particles.

The effect of dust is described in two parameters f and $\tau$. The former describes how much dust is present and the latter is determined by the size of individual particles. Making the dust fine, will decrease $\tau$, and making coarse, will increase $\tau$ in a manner proportional to the surface area of the particles.

## 1.23 Order of Reaction and Rate of Reaction

The rate of a chemical reaction is the amount of substance reacted or produced per unit time. The rate law is an expression indicating how the rate depends on the concentrations of the reactants. The power of the concentration in the rate law expression is called the order with respect to the reactant. The rate of change of concentration as a function of time and may be expressed either in the form of disappearance of reactants or the appearance of new products. According to Bansal [21] the general reaction equation in which A and B are transformed to P give

$$aA + bB \rightarrow cP, \tag{1.23.1}$$

The reaction rate can be written as $-\frac{1}{a}\frac{d[A]}{dt}, -\frac{1}{b}\frac{d[B]}{dt}, +\frac{1}{c}\frac{d[P]}{dt}$

and the rate law may be written in the form of equation

$$-\frac{1}{a}\frac{d[A]}{dt} = k[A]^n[B]^m, \tag{1.23.2}$$

where $[A], [B]$ and $[P]$ denote the active concentrations in moles/litre species. A, B and P, t represent the time, n and m are integers, k is the proportionality constant referred to as the reaction rate constant or specific rate constant, and a, b, c are the stoichiometric coefficients.

Since the concentrations of A and B are diminishing, $-\frac{1}{a}\frac{d[A]}{dt}$, $-\frac{1}{b}\frac{d[B]}{dt}$, are negative number while $\frac{1}{c}\frac{d[P]}{dt}$ is positive, any of these derivatives may be used to express the rate of the reaction.

The order of a reaction is the algebraic sum of the exponents of all the concentration terms, which appear in the rate law (1.23.2). For the reaction given in equation (1.23.1)

$$-\frac{1}{a}\frac{d[A]}{dt} = k[A]^n[B]^m$$

where n is the order of the reaction with respect to A and m is the order of the reaction with respect to B. The overall order of the reaction is given by the sum (n + m).

A reaction is said to be of the first order if the rate of the reaction is proportional to the concentration of only one of the reacting substances. Let us consider a reaction in which A is being transformed to product P, $(A \rightarrow P)$. If C is the concentration of A, then the differential rate law can be written as

$$-\frac{dC}{dt} = k_1[C]$$

where $k_1$ is the first order rate constant and t the time.

This can be rearranged to

$$-\frac{dC}{C} = k_1 dt$$

Integrate both sides of the above equation to obtain

$-\ln C = k_1 + \theta$, where $\theta$ is a constant of integration.

## 1.24 Spectral Representation of the Turbulence

The solution of the Navier-Stockes equation is merely related to theoretical treatment of the turbulence. An alternative approach is based on the spectral form of the dynamical Navier-Stockes equation. The spectral form of the turbulence is still under-determined but it has a simple physical interpretation and is more convenient. The spectral approach is almost exclusively used for the description of homogeneous turbulence [56, 57]. The principal concepts of spectral representation in the study of turbulence are described below:

If we neglect the body forces from the Navier-Stockes equation (1.14.2) and multiply the $x_i$-component of Navier-Stockes equation written for the point P by $u_j'$ and multiply the $x_j'$ component of the equation written for the point $P'$ by $u_i'$ adding and taking ensemble averages

$$\frac{\partial}{\partial t}\overline{u_l u_j'} + \overline{u_j' u_l \frac{\partial u_l}{\partial x_l}} + \overline{u_l u_j' \frac{\partial u_j'}{\partial x_j'}} = -\frac{1}{\rho}\left[\overline{u_j' \frac{\partial p}{\partial x_l}} + \overline{u_l \frac{\partial p'}{\partial x_j'}}\right] + \nu\left[\overline{u_j' \frac{\partial^2 u_l}{\partial x_l^2}} + \overline{u_l \frac{\partial^2 u_j'}{\partial x_l^2}}\right] \quad (1.24.1)$$

Since in homogeneous turbulence the statistical quantities are independent of position in space and considering the point P and $P'$. Separated by a distance vector $\bar{r}$ and applying the laws of spatial covariance, a simplified form of equation (1.24.1) is obtained as:

$$\frac{\partial}{\partial t}\overline{u_l u_j'} = -\frac{\partial}{\partial r_1}\left(\overline{u_l u_j' u_l} - \overline{u_l u_j' u_l'}\right) + \frac{1}{\rho}\left[\frac{\partial \overline{p u_j'}}{\partial r_l} + \frac{\partial \overline{p' u_l}}{\partial r_j}\right] + 2\nu \frac{\partial^2 \overline{u_l u_j'}}{\partial r_l^2} \quad (1.24.2)$$

The covariance $\overline{u_l u_j'}$ is not suitable for direct analysis of quantitative estimate of the turbulent flows and it is better to use the three-dimensional Fourier transforms of $\overline{u_l u_j'}$ with respect to $\bar{r}$. The variable that corresponds to $\bar{r}$ in the three dimensional wave-number space is a vector $K = (K_1, K_2, K_3)$. We define the wave number spectral density as:

$$\phi_{ij}(\vec{K}) = \frac{1}{(2\pi)^3}\int \overline{u_l u_j'}\exp(-i\vec{K}.\bar{r})d\bar{r} = \frac{1}{(2\pi)^3}\iiint u_i u_j \exp\{-i(K_1 r_1 + K_2 r_2 + K_3 r_3)\}\, dr_1 dr_2 dr_3 \quad (1.24.3)$$

It can be shown that $\overline{u_l u_j'}$ has a continuous range of wavelength, $\phi_{ij}(\bar{K})$ has a continuous distribution in wave number space. We can rigorously regard $\phi_{ij}(\vec{K})dK_1 dK_2 dK_3$ as the contribution of elementary volume $dK_1 dK_2 dK_3$, centred at wave number $\hat{K}$ and therefore representing a wave number of length $\frac{2\pi}{|\bar{K}|}$, in the direction of vector $\vec{K}$ to the value of $\overline{u_l u_j'}$ hence the name "Spectral density". This is consistent with the behaviour of the inverse transform

$$\overline{u_l u_j'(r)} = \int_{-\infty}^{\infty} \phi_{ij}(\vec{K})\exp(i\vec{K}.r)d\vec{K} \quad (1.24.4)$$

The one dimensional wave number spectrum of $\overline{u_l u_j'}$ for a wave number component in the $x_l$ direction is

$$\phi_{ij}(K_l) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{u_l u_j'}(r_l)\exp(-i\vec{K}_l.r_l)\, dr_l \quad (1.24.5)$$

whose inverse is

$$u_i u_j'(r) = \int_{-\infty}^{\infty} \emptyset_{ij}(K_l)\exp(ik_l, r_1).\,dK_l \quad (1.24.6)$$

The equation (1.24.2) for unstrained homogeneous turbulence becomes on Fourier transforming as

$$\frac{\partial \phi_{ij}(\vec{K})}{\partial t} = \Gamma_{ij}(\vec{K}) + \Pi_{ij}(\vec{K}) - 2\nu K_l^2.\,\phi_{ij}(\vec{K}) \quad (1.24.7)$$

where $\Gamma$ and $\Pi$ are the transforms of the triple product and pressure terms respectively.

## 1.25 Fourier Transformation of the Navier-Stockes Equation:

The main reason for using Fourier transformation is that they convert differential operators into multipliers. The equations are so complicated in configuration (or coordinate) space that very little can be done with them and the transformation to wave number (or Fourier) space simplifies them very considerably.

Another and more mathematical argument shows that these transforms are right method of treating a homogeneous problem, associated with any correlation function, $\phi(\vec{x}, \vec{x}')$ is a sequence of Eigen functions $\phi(\vec{x}, \vec{x}')$ and their associated Eigen-values $\lambda(\vec{n})$. These quantities satisfy the equation.

$$\int \phi(\vec{x}, \vec{x}')\,\psi(\vec{n}, \vec{x})d^3\vec{x}' = \lambda(\vec{n})\psi(\vec{n}, \vec{x}) \quad (1.25.1)$$

and the orthonormalization relation

$$\int \psi(\vec{n}, \vec{x})\psi^*(\vec{m}, \vec{n})d^3\vec{x} = 1, \quad \text{if } m = n \quad (1.25.2)$$

= 0, otherwise.

These equations imply that $\phi$ is a scalar. Actually it is a tensor of order two, but this complicates the argument without introducing anything essentially new. The index $\vec{n}$ is in general a complex variable and $\psi^*$ denotes the complex conjugate of $\psi$ (strictly, $\psi^*$ is the adjoint of $\psi$, but since $\phi$ is real and symmetric the adjoint is simply the complex conjugate). The integrations in equations (1.25.1) and (1.25.2) are overall space, which may be finite or infinite. If the space is finite $\vec{n}$ is usually an infinite but countable sequence, while if space is infinite $\vec{n}$ will be a continuous variable, Here all the Eigen functions have real Eigen-values. If follows from (1.25.1) and (1.25.2) that,

$$\phi(\vec{x}\cdot\vec{x}) = \sum_{\text{all n}} \lambda(\vec{n}).\psi(\vec{n}.\vec{x})\,\psi^*(\vec{n},\vec{x}') \quad (1.25.3)$$

and this is the diagonal representation of the correlation function in terms of its Eigen functions. Evidently these functions are only defined 'within a phase' that is, a factor $\exp(i\gamma)$ can be added to $\psi(\vec{n},\vec{x})$ without altering $\phi(\vec{x},\vec{x}')$ provided $\gamma$ is real and independent of x. For a homogeneous field, $\phi$ is a function of $(\vec{x},\vec{x}')$ only and the problem is to find the Eigen functions which are also homogeneous within a phase in the sense that,

$$\psi(\vec{n},\vec{x}') = \exp(i\gamma)\,\psi(\vec{n},\vec{x}+a),$$

This equation is satisfied by the Fourier equation

$$\psi(\vec{n},\vec{x}) = \exp(i\vec{n}.\vec{x}) = \exp(i\vec{n}_j\vec{x}_j)$$

with $\gamma = -\vec{n}.\vec{a}$. In this situation (instance), therefore, "the index", $\vec{n}$ is a wave number. Equation (1.25.3) becomes

$$\phi(\vec{x},\vec{x}') = \sum \lambda(\vec{n})\exp\{in(\vec{x}-\vec{x}')\}$$

so that $\lambda(\vec{n})$ may be identified with $\phi(\vec{n})$, the Fourier transform of the correlation function.

Since we are considering homogeneous isotropic turbulence, the turbulent field must be infinite in extent. This produces, mathematical difficulties, which can only be resolved by using functional calculus. This difficulty is avoided by supposing that the turbulence is confined to the inside of a large box with sides $(a_1, a_2, a_3)$ and that it obeys cyclic boundary conditions on the sides of this box. The $a_i$ is allowed to tend to infinity at an appropriate point in the analysis. Thus the Fourier transform is defined by

$$U_i(\vec{x}) = (2\pi)^3(a_1, a_2, a_3)^{-1}\sum_{k} u_i(\vec{K})\exp(i\vec{K}.\vec{x}) \quad (1.25.4)$$

Here $\vec{K}$ is limited to wave vectors of the form

$$\frac{2n_1\pi}{a_1}, \frac{2n_2\pi}{a_2}, \frac{2n_3\pi}{a_3}$$

where $n_i$ are integers while the $a_i$ are the sides of the elementary box. As these sides become infinitely large equation (1.25.4) goes over into standard form,

$$U_i(\vec{x}) = \int u_i(\vec{K}).\exp(i\vec{K}.\vec{x})d^3\vec{K}. \quad (1.25.5)$$

The inverse (1.25.5) is

$$u_i(\vec{K}) = (2\pi)^{-3} \int_{box} u_i(\vec{x}) \exp(-i\vec{K}.\vec{x})\, d^3x \qquad (1.25.6)$$

The Fourier transform of Navier-Stockes equation may be written as

$$\left[\frac{d}{dt} + \nu K^2\right] u_i(\vec{K}) = M_{ijm}(\vec{K}). \sum^{\Delta} u_j(\vec{P}) . U_m(\vec{r}) \qquad (1.25.7)$$

where, $\overset{\Delta}{\Sigma}$ is a short notation for the integral operator in

$$\iint U_j(\vec{K})\, U_m(\vec{r}). \delta(\vec{K} - \vec{P} - \vec{r}). (d^3\vec{p})(d^3\vec{p}) \qquad (1.25.8)$$

where, $\delta K, p + r$ is the Kronecker delta symbol which is zero unless

$\vec{K} = \vec{p} + \vec{r}$

Here, $M_{ijm}(\vec{K})$ is a simple algebraic multiplier and not a differential operator. We have

$$M_{ijm}(\vec{K}) = -\frac{1}{2} i. P_{ijm}(\vec{K}) \qquad (1.25.9)$$

where, $P_{ijm}(\vec{K}) = K_m P_{ij}(\vec{K}) + K_j P_{im}(\vec{K})$

and $P_{ij} = \delta_{ij} - \frac{K_i K_j}{K^2}$

$P_{ij}(\vec{K})$ is the Fourier transform of $P_{ij}(\nabla)$ but $P_{ijm}(\vec{K})$ is not the transform of $P_{ijm}(\nabla)$.

As it is stands, equation (1.25.7) cannot describe stationary turbulence since it contains no input of energy to balance the dissipative effect of viscosity. In real life this input is provided by effects, such as the interaction of mean velocity gradient with the Reynolds stress, which are incompatible with the ideas of homogeneity and isotropy.

To avoid this difficulty we introduce in to the right hand side of equation (1.25.7) a hypothetical homogeneous isotropic stirring force $f_i$. Then the equation becomes

$$\left[\frac{d}{dt} + \nu K^2\right] u_i(\vec{K}) = M_{ijm}(\vec{K}) \Sigma^{\Delta} u_j(\vec{P})\, u_m(\vec{r}) + \partial_i(\vec{K}) \qquad (1.25.10)$$

## 1.26 A Brief Description of Past Researches Relevant to this Thesis Work

Turbulence is a leading topic in modern fluid dynamics research, and some of the best known physicists have worked in this area during the last century. Among them are G. I. Taylor, Kolmogorov, Reynolds, Prandtl, Vonkarman, Heisenberg, Landau, Millikan, and Onsagar.

The first systematic work on turbulence was carried out by British physicist Osborne Reynolds [83] in 1883. His experiments in pipe flows showed that the flow becomes turbulent or irregular when the dimensionless ratio, later named the Reynolds number by Sommerfeld, exceeds a certain critical value. This dimensionless number subsequently proved to be the parameter that determines the dynamic similarity of viscous flows. Reynolds also separated turbulent flow-dependent variables into mean and fluctuating components, and arrived at the concept of turbulent stress.

In 1921 Taylor [105], in a simple and elegant study of turbulent diffusion, introduced the idea of a correlation function. He showed that the root-mean-square distance of a particle from its source point initially increases with time as t, and subsequently as $t^{1/2}$, as in a random walk. Taylor continued his outstanding work in a series of papers during 1935-1936 in which he laid down the foundation of the statistical theory of turbulence.

Among the concepts he introduced were those of homogeneous and isotropic turbulence
and of a turbulence spectrum. Although real turbulent flows are not isotropic (turbulent shear stresses, in fact, vanish for isotropic flows), the mathematical techniques involved have proved valuable for describing the small scales of turbulence, which are isotropic or nearly so. In 1915 Taylor also introduced the mixing length concept, although credit goes to Prandtl for making full use of the idea.

During the 1920s Prandtl [80] and his student Vonkarman, working in Gottingen, Germany, developed semi-empirical theories of turbulence. The most successful of these was the mixing length theory, which is based on an analogy with the concept of mean free path in the kinetic theory of gases. By guessing at the correct form for the mixing length, Prandtl was able to deduce that the average turbulent velocity profile near a solid wall is logarithmic, one of the most reliable results for turbulent flows. It is for this reason that subsequent textbooks on fluid mechanics have for a long time glorified the mixing length theory. Recently, however, it has become clear that the mixing length theory is not helpful since there is really no rational way of predicting the form of the mixing length. In fact, the logarithmic law can be justified from dimensional considerations alone.

Some very important work was done by the British meteorologist Lewis Richardson [85]. In 1922 he wrote the very first book on numerical weather prediction named "*Weather Prediction by Numerical Process*". In this book he proposed that the turbulent kinetic energy is transferred from large to small eddies, until it is destroyed by viscous dissipation. This idea of a spectral energy cascade is at the heart of our present understanding of turbulence. However, Richardson's work was largely ignored at the time, and it was not until some 20 years later that the idea of a spectral cascade took a quantitative shape in the hands of Kolmogorov [57] and Obukhov [73] in Russia. Richardson also did another important piece of work that displayed his amazing physical intuition. On the basis of experimental data for the movement of balloons in the atmosphere, he proposed that the effective diffusion coefficient of a patch of turbulence is proportional to $l^{4/3}$, where $l$ is the scale of the patch. This is called Richardson's four-third law, which has been subsequently found to be in agreement with Kolmogorov's famous five-third law for the energy spectrum.

The Russian mathematician Kolmogorov, generally regarded as the greatest probabilist of the twentieth century, followed up on Richardson's idea of a spectral energy cascade. He hypothesized that the statistics of small scales are isotropic and depend on only two parameters - the kinematic viscosity and the average rate of kinetic energy dissipation per unit mass of fluid. Using this idea, in 1941 Kolmogorov [57] and Obukhov [73] independently derived that the spectrum in the inertial subrange must be proportional to $\varepsilon^{2/3}k^{-5/3}$, where $k$ is the wave number. This law name as five-thirds law is one of the most important results of turbulence theory and is in agreement with high Reynolds number observations.

Recent decades have seen much progress in theory, calculations, and measurements. Among these may be mentioned the work on the modeling, coherent structures, direct numerical simulations, and multidimensional diagnostics. Observations in the ocean and the atmosphere (which Vonkarman called “a giant laboratory for turbulence research”), in which the Reynolds numbers are very large, are shedding new light on the structure of stratified turbulence.

Recently, Azad and Sarker [2] derived the statistical theory of certain distribution function in MHD turbulence in a rotating system in presence of dust particles. Sarker and Azad [94] studied the decay of temperature fluctuations in homogeneous turbulence before the final period for the case of multi-point and multi-time considering rotating system and dust particle. Azad et al. [11], Azad et al. [12] and Azad and Sarker [5] also studied the decay of temperature fluctuations in dusty fluid MHD turbulence before the final period with taking rotating system. Kishore and Dixit [51], Kishore and Singh [49], Dixit and Upadhaya [34], Kishore and Golsefield [52] discussed the effect of coriolis force on acceleration covariance in ordinary and MHD turbulent flow. Kishore and Sarker [48] studied the rate of change of vorticity covariance in MHD turbulence in a rotating system. Sarker [90] studied the Thermal decay process of MHD turbulence in a rotating system. Sarker [89], Sarker and Rahman [95] considered dust particles on their own works.

The essential characteristic of turbulent flows is that turbulent fluctuations are random in nature and therefore by the application of statistical laws, it has been possible to give the idea of turbulent fluctuations. The turbulent flows in the absence of external agencies always decay. Batchelor and Townsend [23], Deissler [32, 33], Ghosh [39, 40] had given various analytical theories for the decay process of turbulence so far. Further Monin and Yaglom [70] gave the spectral approach for the decay process of turbulence. Also Sarker and Kishore [97] discussed the decay of MHD turbulence before the final period. The approach is phenomenological in the sense that they

considered the region where the variation of the mean temperature and mean velocity may be neglected because of the transportation of the thermal energy from place to place is very rapid.
Deissler [32, 33] developed a theory for homogeneous turbulence which was valid for times before the final period. Using Deissler's theory Loeffler and Deissler [64] studied the temperature fluctuations in homogeneous turbulence before the final period. Sarker and Rahman [88] studied the decay of temperature fluctuations in MHD turbulence before the final period. Sarker and Islam [92] considered the decay of dusty fluid turbulence before the final period in a rotating system.

Sarker and Rahman [95] discussed the decay of turbulence before the final period in presence of dust particles. Sarker and Islam [93] studied the effect of very strong magnetic field on acceleration covariance in MHD turbulence of dusty fluid turbulence in a rotating system. Further using Deissler's theory Kumar and Patel [59] studied the first order reactants in homogeneous turbulence before the final period for the case of multi-point and single time. The problem [59] also extended to the case of multi-point and multi-time concentration correlation in homogeneous turbulence by Kumar and Patel [60]. The numerical result of Kumar and Patel [60] carried out by Patel [76].

Following Deissler's approach Sarker and Islam [91] studied the decay of MHD turbulence before the final period for the case of multi-point and multi-time. Islam and Sarker [44] discussed the first order reactant in MHD turbulence before the final period of decay for the case of multi-point and multi-time. Sarker and Islam [92] also studied the decal of dusty fluid turbulence before the final period in a rotating system.

But at first Lundgren [65] derived the dynamical equations, which are describing the time evolution of the finite dimensional probability distribution of turbulent quantities. Lundgren [65] derived a hierarchy of coupled equations for multi-point turbulence velocity distribution function. Further Lundgren [66] considered a similar problem for non-homogeneous turbulence. Bigler [25] gave the hypothesis that in turbulent flow the thermo-chemical quantities can be related locally a few scalars.

Further Janicka, Kolbe and Kollmann [46] and Pope [78] have a more suitable model for the probability density function of scalars in turbulent reacting flows. Also Kishore [47] studied the distribution function in the statistical theory of MHD turbulence of an incompressible fluid. Pope [79] derived the transport equation for the joint probability density function of velocity and scalars in turbulent flow. Kishore and Singh [49] derived the transport equation for the bivariate joint distribution function of velocity and temperature in turbulent flow. Kishore and Singh [50] have been derived the transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow. Dixit and Upadhyay [35] considered the distribution function in the statistical theory of MHD turbulence of an incompressible fluid in the presence of the coriolis force. Kollmann and Janicka [56] derived the transport equation for the probability density function of a scalar in turbulent shear flow and considered a closure model based on gradient flux model.

But at this stage, one is met with the difficulty that the N-point distribution function depends upon the N+1-point distribution function and thus result is an unclosed system. This so-called "closer problem" is encountered in turbulence, kinetic theory and other non-linear system. Sarker and Kishore [96] discussed the distribution function in the statistical theory of convective MHD turbulence of an incompressible fluid. Further Sarker and Kishore [98] discussed the distribution function in the statistical theory of convective MHD turbulence of mixture of miscible incompressible fluid. Azad et al. [7,8,9] studied the first order reactant in MHD turbulence before the final period of decay considering rotating system and dust particles. Sarker et al. [99] studied the first order reactant in MHD turbulence before the final period of decay for the case of multi-point and multi-time in presence of dust particles. Aziz et al. [17, 18] extended their problem for the case of multi-point and multi-time for a rotating system. Aziz et al. [19, 20] studied the statistical theory of distribution function in magneto-hydrodynamic turbulence in a rotating system with dust particles undergoing a first order reaction. Azad et al. [10] premeditated the statistical theory of certain distribution function in MHD turbulent flow for velocity and concentration undergoing a first order reaction in a rotating system. Recently Azad et al. [15] studied the transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow in presence of dust particle.

By analyzing the above theories we have extracted the following chapters.

In part A of Chapter-II , we have studied the decay of temperature fluctuations in dusty fluid homogeneous turbulence prior to the final period considering correlations between fluctuating quantities at two- and three- point. We have obtained the energy decay law of temperature fluctuations in homogeneous turbulence before the final period in presence of dust particle.

In part B of Chapter-II, we have studied the decay of temperature fluctuations in dusty fluid homogeneous turbulence before the final period in presence of Coriolis force and have considered correlations between fluctuating quantities at two- and three-points by neglecting the fourth order correlation in comparison to the second and third order correlations. For solving the correlation equations are converted to spectral form by taking their Fourier transform. Finally we have put an effort to integrate the energy spectrum over all wave numbers, the energy decay law of temperature fluctuations in homogeneous dusty fluid turbulence before the final period in presence of Coriolis force is obtained.

In part A of Chapter-III, the joint distribution functions for simultaneous velocity, temperature, concentration fields in turbulent flow undergoing a first order reaction in presence of Coriolis force have been studied. The various properties of the constructed joint distribution functions have been discussed. In this chapter we have to derive transport equations for one and two point joint distribution functions of velocity, temperature, concentration in convective turbulent flow due to first order reaction in presence of Coriolis force.

In part B of chapter-III, we have studied the joint distribution functions for simultaneous velocity, temperature, concentration fields in turbulent flow undergoing a first order reaction in a rotating system in presence of dust particles. In this chapter, we have made an attempt to derive the transport equations for the joint distribution function of certain variables in convective turbulent flow undergoing a first order reaction in a rotating system in presence of dust particles.

In chapter-IV, we have studies the statistical theory of certain variables for three-point distribution functions in MHD turbulent flow in a rotating system in presence of dust particles. In this chapter we have derived the transport equations for evolution of three- point distribution function for simultaneous velocity magnetic, temperature and concentration field.

In Chapter-V, we have made an attempt to discuss the summary about the whole thesis.

# CHAPTER-II

# PART-A

## DECAY OF TEMPERATURE FLUCTUATIONS IN DUSTY FLUID HOMOGENEOUS TURBULENCE PRIOR TO THE FINAL PERIOD

### 2.1 Introduction:

Interest in motion of dusty viscous fluid has developed rapidly in recent years. Such situations occur in movement of dust-laden air, in problems of fluidization, in the use of dust in gas cooling system and in sedimentation problem in tidal rivers.

Taylor [105] has been pointed out that the equation of motion of turbulence relates the pressure gradient and the acceleration of the fluid particles and the mean–square acceleration can be determined from the observation of the diffusion of marked fluid particles. The behavior of dust particles in a turbulent flow depends on the concentration of the particles and the size of the particles with respect to the scale of turbulent fluid.

Corrsin [31] had made an analytical attack on the problem of turbulent temperature fluctuations using the approaches employed in the statistical theory of turbulence. His results pertain to the final period of decay and for the case of appreciable convective effects, to the “energy” spectral from in specific wave- number ranges.

Deissler [32, 33] developed a theory for homogeneous turbulence, which was valid for times before the final period. Following Deissler’s theory Loeffler and Deissler [64] studied the decay of temperature fluctuations in homogeneous turbulence before the final period. Sarker and Azad [94]; Azad and Sarker [3]; Azad and Sarker [4]; Azad et al [11]; Azad and Sarker [5]; Azad et al. [12] also studied the decay of temperature fluctuations in homogeneous and MHD dusty fluid turbulence. Azad et al [15] studied transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow in presence of dust particles. Bkar Pk et al [30] considered first-order reactant in homogeneous dusty fluid turbulence prior to the ultimate phase of decay for four-point correlation in a rotating system. Molla et al [68] studied the decay of temperature fluctuation in homogeneous turbulence before the final period in a Rotating System. Sarker et al [101] measured Homogeneous dusty fluid turbulence in a first order reactant for the case of multi Point and multi time prior to the final period of decay.

Saffman [86] derived an equation that describes the motion of a fluid containing small dust particle, which is applicable to laminar flows as well as turbulent flow. Kishore and Sarker [53] studied the rate of change of vorticity covariance in MHD turbulent flow of dusty incompressible fluid. Rahman [82] also studied the Rate of change of vorticity covariance in MHD turbulent flow of dusty fluid in a rotating system. Kishore and Sinha [54] also studied the rate of change of vorticity covariance of dusty fluid turbulence.

They had considered dust particles and Coriolis force in their won works. In their study, they considered two and three point correlations and neglecting fourth- and higher-order correlation terms compared to the second- and third-order correlation terms. Sinha [103] had considered the effect of dust particles on the acceleration of ordinary turbulence. Kishore and Singh [55] had studied the statistical theory of decay process of homogeneous hydro- magnetic turbulence. Dixit and Upadhyay [34] also had deliberated the effect of Coriolis force on acceleration covariance in MHD turbulent dusty flow with rotational symmetry. Kishore and Golsefied [52] considered the effect of Coriolis force on acceleration covariance in MHD turbulent flow of a dusty incompressible fluid. They had also considered dust particle in their own work.

In this chapter, by analyzing the above theories we have studied the decay of temperature fluctuations in homogeneous turbulence prior to the final period in presence of dust particle considering the correlations between fluctuating quantities at two- and three- point and single time. In solving the problem, it seems logical to use the approach which has already been employed with success for studying turbulence. In this work, Deissler's method is used to solving the problem. Through the study we have obtained the energy decay law of temperature fluctuations in homogeneous dusty fluid turbulence prior to the final period. In the result, it is shown that the energy decays more rapidly than clean fluid.

## METHODOLOGY

## 2.2 Correlation and Spectral equations:

For an incompressible fluid with constant properties and for negligible frictional heating, the energy equation may be written at the point P

$$\left[\frac{\partial \tilde{T}}{\partial t}+\tilde{u}_i\frac{\partial \tilde{T}}{\partial x_i}\right]=\frac{k}{\rho c_p}\frac{\partial^2 \tilde{T}}{\partial x_i \partial x_i} \qquad (2.2.1)$$

Where,

$\tilde{T}$ = Instantaneous values of temperature.

$\tilde{u}_i$ = Instantaneous velocity,

$\rho$ = Fluid density,

$c_p$ = Heat capacity at constant pressure,

$k$ = Thermal conductivity,

$x_i$ = Space co-ordinate,

$t$ = Time,

Separate these instantaneous values into time average and fluctuating components as $\tilde{T}=\overline{T}+T$ and $\tilde{u}_i=\overline{u}_i+u_i$ equation (2.2.1) may be written

$$\left[\frac{\partial \overline{T}}{\partial t}+\frac{\partial T}{\partial t}+\overline{u}_i\frac{\partial \overline{T}}{\partial x_i}+\overline{u}_i\frac{\partial T}{\partial x_i}+u_i\frac{\partial \overline{T}}{\partial x_i}+u_i\frac{\partial T}{\partial x_i}\right]=\gamma\left[\frac{\partial^2 \overline{T}}{\partial x_i \partial x_i}+\frac{\partial^2 T}{\partial x_i \partial x_i}\right] \qquad (2.2.2)$$

where, $\gamma=\dfrac{k}{\rho c_p}$,

From the case of homogeneity it follows that $\dfrac{\partial \overline{T}}{\partial x_i}=0$ and in addition the usual assumption is made that $\overline{T}$ is independent of time and that $\overline{u}_i=0$; Thus equation (2.2.2) simplifies to

$$\left[\frac{\partial T}{\partial t}+u_i\frac{\partial T}{\partial x_i}\right]=\frac{\nu}{P_r}\left[\frac{\partial^2 T}{\partial x_i \partial x_i}\right] \qquad (2.2.3)$$

where $P_r = \frac{\nu}{\gamma}$, Prandtl number, $\nu$ =Kinematic Viscosity.

Equation (2.2.3) holds at the arbitrary point P. For the point $P'$ the corresponding equation can be written as

$$\left[\frac{\partial T'}{\partial t}+u_i'\frac{\partial T'}{\partial x_i'}\right]=\frac{\nu}{P_r}\left[\frac{\partial^2 T'}{\partial x_i'\partial x_i'}\right] \qquad (2.2.4)$$

Multiplying equation (2.2.3) by $T'$, equation (2.2.4) by $T$ and taking time average and adding the two equations gives

$$\left[\frac{\partial \overline{TT'}}{\partial t}+u_i\frac{\partial \overline{TT'}}{\partial x_i}+u_i'\frac{\partial \overline{TT'}}{\partial x_i'}\right]=\frac{\nu}{P_r}\left[\frac{\partial^2 \overline{TT'}}{\partial x_i\partial x_i}+\frac{\partial^2 \overline{TT'}}{\partial x_i'\partial x_i'}\right] \qquad (2.2.5)$$

The continuity equation is

$$\frac{\partial u_i}{\partial x_i}=\frac{\partial u_i'}{\partial x_i'}=0 \qquad (2.2.6)$$

Substitution of equation (2.2.6) into (2.2.5) yields

$$\frac{\partial \overline{TT'}}{\partial t}+u_i\frac{\partial \overline{u_iTT'}}{\partial x_i}+\frac{\partial \overline{u_i'TT'}}{\partial x_i'}=\frac{\nu}{P_r}\left[\frac{\partial^2 \overline{TT'}}{\partial x_i\partial x_i}+\frac{\partial^2 \overline{TT'}}{\partial x_i'\partial x_i'}\right] \qquad (2.2.7)$$

By use of a new independent variable $r_i = x_i' - x_i$ i,e $\frac{\partial}{\partial x_i}=-\frac{\partial}{\partial r_i}$, $\frac{\partial}{\partial x_i'}=\frac{\partial}{\partial r_i}$

$$\frac{\partial \overline{TT'}}{\partial t}-\frac{\partial \overline{u_iTT'}}{\partial r_i}+\frac{\partial \overline{u_i'TT'}}{\partial r_i}=\frac{2\nu}{P_r}\frac{\partial^2 \overline{TT'}}{\partial r_i\partial r_i} \qquad (2.2.8)$$

This equation is converted into spectral form by use of the following three dimensional Fourier transforms

$$\overline{TT'}(\hat{r})=\int_{-\infty}^{\infty}\overline{\tau\tau'}(\hat{K})\ \exp\left[i\left(\hat{K}.\hat{r}\right)\right]d\hat{K} \qquad (2.2.9)$$

$$\overline{u_i'TT'}(\hat{r})=\int_{-\infty}^{\infty}\overline{\phi_i\tau\tau'}(\hat{K})\ \exp\left[i\left(\hat{K}.\hat{r}\right)\right]d\hat{K} \qquad (2.2.10)$$

And by interchanging $P$ and $P'$,

$$\overline{u_i'TT'}(\hat{r})=\overline{u_iTT'}(-\hat{r})$$

$$\overline{u_iTT'}(\hat{r})=\int_{-\infty}^{\infty}\overline{\phi_i\tau\tau'}(-\hat{K})\exp\left[i\left(\hat{K}.\hat{r}\right)\right]d\hat{K} \qquad (2.2.11)$$

$$\overline{u_iT'}(\hat{r})=\int_{-\infty}^{\infty}\overline{\phi_i\tau'}(-\hat{K})\exp\left[i\left(\hat{K}.\hat{r}\right)\right]d\hat{K} \qquad (2.2.11a)$$

$$\overline{u_i'T}(\hat{r}) = \int_{-\infty}^{\infty} \overline{\phi_i \tau}\left(\hat{K}\right) \exp\left[i\left(\hat{K}.\hat{r}\right)\right] d\hat{K} \qquad (2.2.11b)$$

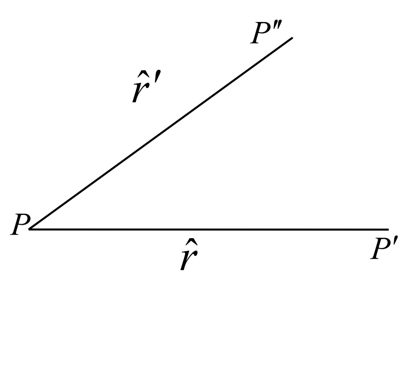
Substitution of equations (2.2.9) - (2.2.11b) into equation (2.2.8) leads to the spectral equation

$$\frac{\partial \overline{\tau\tau'}\left(\hat{K}\right)}{\partial t} + ik_i \left[\overline{\phi_i \tau\tau'}\left(-\hat{K}\right) - \overline{\phi_i \tau\tau}\left(\hat{K}\right)\right] = -\frac{2\nu}{\mathrm{P}_r} k^2\, \overline{\tau\tau'}\left(\hat{K}\right) \qquad (2.2.12)$$

Equation (2.2.12) is analogous to the two point spectral equation governing the decay of velocity fluctuations and therefore the quantity $\tau\tau'(k)$ may be interpreted as a temperature fluctuation "energy" contribution of thermal eddies of size $1/k$. Equation (2.2.12) expresses the time derivative of this "energy" as a function of the convective transfer to other wave numbers and the "dissipation" due to the action of thermal conductivity. The second term on the left hand side of equation (2.2.12) is the so called transfer to term while the term on the right hand side is "dissipation" term.

## 2.3 Three points correlation and spectral equations:

In order to obtain single time and three point correlation and spectral equation we consider three points $P$, $P'$ and $P''$ with position vectors $\hat{r}$ and $\hat{r}'$ are considered.




For the two points $P'$ and $P''$ we can write a relation according to equation (2.2.7),

$$\frac{\partial(T'T'')}{\partial t}+u_i\frac{\partial(u_i'T'T'')}{\partial x_i'}+\frac{\partial(u_i''T'T'')}{\partial x_i''}=\frac{\nu}{P_r}\left[\frac{\partial^2(T'T'')}{\partial x_i'\partial x_i'}+\frac{\partial^2(T'T'')}{\partial x_i''\partial x_i''}\right] \quad (2.3.1)$$

Equation (2.3.1) multiplied through by $u_j$, the j-th velocity fluctuation component at point. Then the equation at the point P can be written as

$$\frac{\partial(u_jT'T'')}{\partial t}+u_i\frac{\partial(u_ju_i'T'T'')}{\partial x_i'}+\frac{\partial(u_ju_i''T'T'')}{\partial x_i''}=\frac{\nu}{P_r}\left[\frac{\partial^2(u_jT'T'')}{\partial x_i'\partial x_i'}+\frac{\partial^2(u_jT'T'')}{\partial x_i''\partial x_i''}\right]+T'T''\frac{\partial u_j}{\partial t} \quad (2.3.2)$$

The momentum equation at point P, in presence of dust particles

$$\frac{\partial u_j}{\partial t}+\frac{\partial(u_ju_i)}{\partial x_i}=-\frac{1}{\rho}\frac{\partial P}{\partial x_j}+\nu\frac{\partial^2 u_j}{\partial x_i\partial x_i}+f(u_j-v_j)$$

$$\Rightarrow\frac{\partial u_j}{\partial t}=-\frac{\partial(u_ju_i)}{\partial x_i}-\frac{1}{\rho}\frac{\partial P}{\partial x_j}+\nu\frac{\partial^2 u_j}{\partial x_i\partial x_i}+f(u_j-v_j) \quad (2.3.3)$$

Here,

$u_j$

$v_j$ =turbulent velocity component

=dust velocity component

$f=\frac{kN}{\rho}$ (Dimension of frequency)

N, constant number density of dust particle

Substituted equation (2.3.3) into equation (2.3.2) the result on taking time averages is

$$\frac{\partial(\overline{u_jT'T''})}{\partial t}+u_i\frac{\partial(\overline{u_ju_i'T'T''})}{\partial x_i'}+\frac{\partial(\overline{u_ju_i''T''T'})}{\partial x_i''}=\frac{\nu}{P_r}\left[\frac{\partial^2(\overline{u_jT'T''})}{\partial x_i'\partial x_i'}+\frac{\partial^2(\overline{u_jT'T''})}{\partial x_i''\partial x_i''}\right]$$
$$-\frac{\partial(\overline{u_ju_iT'T''})}{\partial x_i}-\frac{1}{\rho}\frac{\partial(\overline{PT'T''})}{\partial x_j}+\nu\frac{\partial^2(\overline{u_jT'T''})}{\partial x_i\partial x_i}+f(\overline{u_jT'T''}-\overline{v_jT'T''}) \quad (2.3.4)$$

Making use of the relations $r_i=x_i'-x_i$ *and* $r_i'=x_i''-x_i'$ allows equation (2.3.4) can be written as

$$\frac{\partial(\overline{u_jT'T''})}{\partial t}-\frac{\nu}{P_r}\left\{(1+P_r)\frac{\partial^2(\overline{u_jT'T''})}{\partial r_i\partial r_i}+2P_r\frac{\partial^2(\overline{u_jT'T''})}{\partial r_i\partial r_i'}+(1+P_r)\frac{\partial^2(\overline{u_jT'T''})}{\partial r_i'\partial r_i'}\right\}=$$
$$-\frac{\partial(\overline{u_ju_i'T'T''})}{\partial r_i}-\frac{\partial(\overline{u_ju_i''T''T'})}{\partial r_i'}+\frac{\partial(\overline{u_ju_iT'T''})}{\partial r_i'}+\frac{\partial(\overline{u_ju_iT'T''})}{\partial r_i}$$
$$+\frac{1}{\rho}\frac{\partial(\overline{PT'T''})}{\partial r_j'}+\frac{1}{\rho}\frac{\partial(\overline{PT'T''})}{\partial r_j}+f(\overline{u_jT'T''}-\overline{v_jT'T''}) \quad (2.3.5)$$

Six-dimensional Fourier transforms for quantities this equation may be defined as

$$\overline{u_j T'T''} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\beta_j \theta'\theta''} \exp\left[i\left(\hat{K}\cdot\hat{r} + \hat{K}'\cdot\hat{r}'\right)\right] d\hat{K}\, d\hat{K} \qquad (2.3.6)$$

$$\overline{u_j u_i' T'T''} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\beta_j \beta_i' \theta'\theta''} \exp\left[i\left(\hat{K}\cdot\hat{r} + \hat{K}'\cdot\hat{r}'\right)\right] d\hat{K}\, d\hat{K} \qquad (2.3.7)$$

$$\overline{\mathrm{P}T'T''} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\alpha\,\theta'\theta''} \exp\left[i\left(\hat{K}\cdot\hat{r} + \hat{K}'\cdot\hat{r}'\right)\right] d\hat{K}\, d\hat{K} \qquad (2.3.8)$$

$$\overline{v_j u_i'' T'' T'} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\gamma_j \theta'\theta''} \exp\left[i\left(\hat{K}\cdot\hat{r} + \hat{K}'\cdot\hat{r}'\right)\right] d\hat{K}\, d\hat{K} \qquad (2.3.8a)$$

Interchanging the points $\mathrm{P}'$ *and* $\mathrm{P}''$ shows that

$$\overline{u_j u_i'' T'' T'} = \overline{u_j u_i' T'T''} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{\beta_j \beta_i' \theta'\theta''} \exp\left[i\left(\hat{K}\cdot\hat{r} + \hat{K}'\cdot\hat{r}'\right)\right] d\hat{K}\, d\hat{K} \qquad (2.3.8b)$$

Using equation (2.3.6) – (2.3.8.b) into equation (2.3.5) then the transformed equation can be written as

$$\frac{\partial\left(\overline{\beta_j \theta'\theta''}\right)}{\partial t} + \frac{\nu}{\mathrm{P}_r}\left\{\left(1+\mathrm{P}_r\right)k^2 + 2\mathrm{P}_r k_i k_i' + \left(1+\mathrm{P}_r\right)k'^2 - \frac{p_r}{\upsilon} f\right\}\overline{\beta_j \theta'\theta''} =$$

$$-\, i\left(k_i + k_i'\right)\overline{\beta_j \beta_i' \theta'\theta''} + i\left(k_i' + k_i\right)\overline{\beta_j \beta_i \theta'\theta''} + \frac{1}{\rho} i\left(k_j + k_j'\right)\overline{\alpha\theta'\theta''} - f\,\overline{\gamma_j \theta'\theta''} \qquad (2.3.9)$$

If the derivative with respect to $x_j$ is taken of the momentum equation (2.3.4) for point$\mathrm{P}$, and taking time average the resulting equation is

$$\frac{\partial^2\left(\overline{u_j u_i T'T''}\right)}{\partial x_j\,\partial x_i} = -\frac{1}{\rho}\frac{\partial^2\left(\overline{\mathrm{P}T'T''}\right)}{\partial x_j\,\partial x_j} \qquad (2.3.10)$$

In terms of the displacement vectors $\hat{r}$ *and* $\hat{r}'$ this becomes

$$\left[\frac{\partial^2}{\partial r_j'\,\partial r_i'} + 2\frac{\partial^2}{\partial r_j'\,\partial r_i} + \frac{\partial^2}{\partial r_j\,\partial r_i}\right]\overline{u_j u_i T' T''}$$

$$= -\frac{1}{\rho}\left[\frac{\partial^2}{\partial r_j'\,\partial r_j'} + 2\frac{\partial^2}{\partial r_j'\,\partial r_j} + \frac{\partial^2}{\partial r_j\,\partial r_j}\right]\overline{\mathrm{P}T'T''} \qquad (2.3.11)$$

Taking the Fourier transform of equation (2.3.11) and then solving for $\overline{\alpha\theta'\theta''}$ we get

$$\overline{\alpha\theta'\theta''} = \frac{-\rho\left[k'_j k'_i + 2k'_j k_i + k_j k_i\right]}{\left[k'_j k'_j + 2k'_j k_j + k_j k_j\right]}\overline{\beta_j\beta_i\theta'\theta''} \tag{2.3.12}$$

Equation (2.3.12) can be used to eliminate $\overline{\alpha\theta'\theta''}$ from equation (2.3.9).

## 2.4 Solution for times before the final period:

To obtain the equation for final period of decay the third-order fluctuation terms are neglected compared to the second-order terms. Analogously, it would be anticipated that for times before but sufficiently near to the final period the fourth-order fluctuation terms should be negligible in comparison with the third-order terms. If this assumption is made then equation (2.3.12) shows that the term $\overline{\alpha\theta'\theta''}$ associated with the pressure fluctuations, should also be neglected. Thus equation (2.3.9) simplifies to

$$\frac{\partial\left(\overline{\beta_j\theta'\theta''}\right)}{\partial t} + \frac{\nu}{\mathrm{P}_r}\left\{(1+\mathrm{P}_r)k^2 + 2\mathrm{P}_r k_i k'_i + (1+\mathrm{P}_r)k'^2 - \frac{p_r}{\upsilon}f\right\}\overline{\beta_j\theta'\theta''} = 0 \tag{2.4.1}$$

Where, $R\overline{\beta_j\theta'\theta''} = \overline{\gamma_j\theta'\theta''}$

and 1-R=S, R and S are arbitrary constant.

Inner multiplication of equation (2.4.1) by $k_j$ and integrating between $t_0$ *and* $t$ gives

$$k_j\overline{\beta_j\theta'\theta''}$$
$$= \left[k_j\overline{\beta_j\theta'\theta''}\right]_0 \exp\left\{-\frac{\nu}{\mathrm{P}_r}\left[(1+\mathrm{P}_r)k^2 + 2\mathrm{P}_r k_i k'_i\cos\xi + (1+\mathrm{P}_r)k'^2 - \frac{\mathrm{P}_r}{\upsilon}fs\right](t-t_0)\right\} \tag{2.4.2}$$

Now, letting $r' = 0$ in equation (2.2.6) and comparing the result with the equation (2.1.10) shows that

$$k_i\,\overline{\phi_i\tau\tau'}\left(\hat{K}\right) = \int_{-\infty}^{\infty} k_i\,\overline{\beta_i\theta'\theta''}\hat{K}\hat{K}'\,d\hat{K}' \tag{2.4.3}$$

Substituting of equation (2.4.2) and (2.4.3) into equation (2.1.12), we obtain

$$\frac{\partial\overline{\tau\tau'}\left(\hat{K}\right)}{\partial t} + \frac{2\nu}{\mathrm{P}_r}k^2\,\overline{\tau\tau'}\left(\hat{K}\right) = \int_{-\infty}^{\infty} ik_i\left[\overline{\beta_i\theta'\theta''} - \overline{\beta_i\theta'\left(-\hat{K}\right)\theta''\left(-\hat{K}'\right)}\right]_0$$
$$\times\exp\left\{-\frac{\nu(t-t_0)}{\mathrm{P}_r}\left[(1+\mathrm{P}_r)\left(k^2+k'^2\right) + 2\mathrm{P}_r k\,k'\cos\xi - \frac{\mathrm{P}_r}{\upsilon}fs\right]\right\}d\hat{K}' \tag{2.4.4}$$

Now, $d\hat{K}'\left(\equiv dk_1' dk_2' \, dk_3'\right)$ can be expressed in terms of $k' \; and \; \xi$ as

$$d\hat{K}' = -2\pi \, k'^2 \, d\left(\cos\xi\right) dk' \qquad (2.4.5)$$

Substituting equation (2.4.4) into (2.4.3) yields

$$\frac{\partial \overline{\tau\tau'}\left(\hat{K}\right)}{\partial t} + \frac{2\nu}{\mathrm{P}_r} k^2 \, \overline{\tau\tau'}\left(\hat{K}\right) = 2\int_{-\infty}^{\infty} 2i\,\pi\, k_i \left[\overline{\beta_i \, \theta'\theta''}\left(\hat{K}, \hat{K}'\right) - \overline{\beta_i \, \theta'\theta''}\left(-\hat{K}, -\hat{K}'\right)\right]_0$$
$$\times\left[\int_{-1}^{1} \exp\left\{-\frac{\nu\left(t-t_0\right)}{\mathrm{P}_r}\left[\left(1+\mathrm{P}_r\right)\left(k^2+k'^2\right) + 2\mathrm{P}_r k\, k' \cos\xi - \frac{\mathrm{P}_r}{\upsilon} fs\right]\right\} d\left(\cos\xi\right)\right] dk' \qquad (2.4.6)$$

In order to find the solutions completely and following Loeffler and Deissler [64], we assume that

$$i\,k_i \left[\overline{\beta_i \, \theta'\theta''}\left(\hat{K}, \hat{K}'\right) - \overline{\beta_i \, \theta'\theta''}\left(-\hat{K}, -\hat{K}'\right)\right]_0 = -\frac{\delta_0}{\left(2\pi\right)^2}\left(k^2 k'^4 - k^4 k'^2\right) \qquad (2.4.7)$$

where $\delta_0$ constant depending on the initial condition. The negative sign is placed in front of $\delta_0$ in order to make the transfer of energy from small to large wave no. for positive value of $\delta_0$. Substituting equation (2.4.7) into equation (2.4.6)

$$\frac{\partial \overline{\tau\tau'}\left(\hat{K}\right)}{\partial t} + \frac{2\nu}{\mathrm{P}_r} \cdot 2\pi k^2 \, \overline{\tau\tau'}\left(\hat{K}\right) = -2\delta_0 \int_0^{\infty} \left(k^2 k'^4 - k^4 k'^2\right)$$
$$\times\left[\int_{-1}^{1} \exp\left\{-\frac{\nu\left(t-t_0\right)}{\mathrm{P}_r}\left[\left(1+\mathrm{P}_r\right)\left(k^2+k'^2\right) + 2\mathrm{P}_r k\, k' \cos\xi - \frac{\mathrm{P}_r}{\upsilon} fs\right]\right\} d\left(\cos\xi\right)\right] dk' \qquad (2.4.8)$$

Multiplying both sides of equation (2.4.8) by $k^2$ and defining the spectral energy function

$$E = 2\pi k^2 \, \overline{\tau\tau'}\left(\hat{K}\right) \qquad (2.4.8.a)$$

and the resulting equation is

$$\frac{\partial E}{\partial t} + \frac{2\nu}{\mathrm{P}_r} k^2 \, E = w \qquad (2.4.9)$$

where

$$w = -2\delta_0 \int_0^{\infty} \left(k^2 k'^4 - k^4 k'^2\right) k^2 k'^2$$
$$\times\left[\int_{-1}^{1} \exp\left\{-\frac{\nu\left(t-t_0\right)}{\mathrm{P}_r}\left[\left(1+\mathrm{P}_r\right)\left(k^2+k'^2\right) + 2\mathrm{P}_r k\, k' \cos\xi - \frac{\mathrm{P}_r}{\upsilon} fs\right]\right\} d\left(\cos\xi\right)\right] dk' \qquad (2.4.9.a)$$

Integrating equation (2.4.9.a) w.r.to $\xi$, we have

$$w=-\frac{\delta_0}{2\upsilon(t-t_0)}\int_0^{\infty}\left(k^3k'^5-k^5k'^3\right)$$
$$\times\left[\exp\left\{-\frac{\nu(t-t_0)}{\mathrm{P}_r}\left[(1+\mathrm{P}_r)(k^2+k'^2)-2\mathrm{P}_r k\,k'-\frac{\mathrm{P}_r}{\upsilon}fs\right]\right\}\right]dk'+\frac{\delta_0}{\upsilon(t-t_0)}\int_0^{\infty}\left(k^3k'^5-k^5k'^3\right)$$
$$\times\left[\exp\left\{-\frac{\nu(t-t_0)}{\mathrm{P}_r}\left[(1+\mathrm{P}_r)(k^2+k'^2)+2\mathrm{P}_r k\,k'-\frac{\mathrm{P}_r}{\upsilon}fs\right]\right\}\right]dk' \qquad (2.4.9.b)$$

Again integrating equation (2.4.9.b) w.r.to $k'$ we have

$$w=-\frac{\delta_0\sqrt{\pi}\ \mathrm{P}_r^{\frac{5}{2}}}{2\upsilon^{\frac{3}{2}}(t-t_0)^{\frac{3}{2}}(1+\mathrm{P}_r)^{\frac{5}{2}}}\exp\left\{\frac{\mathrm{P}_r}{\upsilon}fs(t-t_0)\right\}$$
$$\times\exp\left\{\frac{-k\upsilon(1+2\mathrm{P}_r)(t-t_0)}{\mathrm{P}_r(1+\mathrm{P}_r)}\right\}$$
$$\times\left\{\frac{15\mathrm{P}_r k^4}{4\upsilon^2(t-t_0)^2(1+\mathrm{P}_r)}+\left[\frac{5\mathrm{P}_r^2}{(1+\mathrm{P}_r)^2}-\frac{3}{2}\right]\frac{k^6}{\upsilon(t-t_0)}+\left[\frac{\mathrm{P}_r^3}{(1+\mathrm{P}_r)^3}-\frac{\mathrm{P}_r}{(1+\mathrm{P}_r)}\right]k^8\right\} \qquad (2.4.10)$$

The equation (2.4.10) indicates that w must begin as $k^4$ for small $k$. The condition of w is fulfilled by the equation (2.4.10). It can be shown, using equation (2.4.10) that

$$\int_0^{\infty} w\,dk=0 \qquad (2.4.11)$$

It is be expected physically since w is a measure of the transfer of "energy" and the total energy transferred to all wave numbers must be zero.

The necessity for equation (2.4.11) to hold can be shown as follows if equation (2.2.10) is written for both $k\ and\ -k$, and resulting equations differentiated with respect to $r_i$ and added, the result is, for $\hat{r}=0$ $\left(\frac{\partial}{\partial r_i}=-\frac{\partial}{\partial x_i}\right)$

$$-2\frac{\partial}{\partial x_i}\overline{u_iTT}=\int_{-\infty}^{\infty}ik_i\left[\overline{\phi_i\tau\tau'(\hat{K})}-\overline{\phi_i\tau\tau'(-\hat{K})}\right]\ d\hat{K} \qquad (2.4.11.a)$$

Since according to the equations (2.4.8), (2.4.9) and (2.2.12),

$w \equiv 2\pi\, ik^2 k_i \left[ \overline{\phi_i \tau\tau'\left(-\hat{K}\right)} - \overline{\phi_i \tau\tau'\left(\hat{K}\right)} \right]$

So the equation (2.4.11.a) can be written as $-2\frac{\partial}{\partial x_i}\overline{u_i \tau\tau'} = \int_{-\infty}^{\infty} \frac{w}{2\pi k^2}\, dk$ as $d\hat{K} = 4\pi k^2 dk$ for $w = w(k,t)$ then the equation (2.4.11) becomes $\int_0^{\infty} w\, dk = -\frac{\partial}{\partial x_i}\overline{u_i \tau\tau'} = 0$

The linear equation (2.4.9) can be solved for w as

$$E = \exp\left[-\frac{2\upsilon k^2 (t-t_0)}{\mathrm{P}_r}\right] \int w \exp\left[\frac{2\upsilon k^2 (t-t_0)}{\mathrm{P}_r}\right] dt + J(k) \exp\left[-\frac{2\upsilon k^2 (t-t_0)}{\mathrm{P}_r}\right] \qquad (2.4.12)$$

Where, $J(k)$ is an arbitrary function of k.

For large times, Corrsin [31] has shown the correct form of the expression for E to be

$$E = \frac{N_0}{\pi} k^2 \exp\left[-\frac{2\upsilon k^2 (t-t_0)}{\mathrm{P}_r}\right] \qquad (2.4.13)$$

where $N_0$ is an constant which depends on the initial conditions. Using equation (2.4.13) to evaluate $J(k)$ in equation (2.4.12) yields

$$J(k) = \frac{N_0 K^2}{\pi} \qquad (2.4.14)$$

Now, substituting the values of w and $J(k)$ as given by the equation (2.4.10) and (2.4.14) into equation (2.4.12) gives the equation.

$$E(k,t) = \frac{N_0\, k^2}{\pi} \exp\left[-\frac{2\upsilon k^2 (t-t_0)}{\mathrm{P}_r}\right] + \frac{\delta_0 \sqrt{\pi}\ \mathrm{P}_r^{\frac{5}{2}}}{2\upsilon^{\frac{3}{2}} (1+\mathrm{P}_r)^{\frac{7}{2}}} \exp\{fs(t-t_0)\}$$

$$\times \exp\left\{\frac{-k^2 \upsilon (1+2\mathrm{P}_r)(t-t_0)}{\mathrm{P}_r (1+\mathrm{P}_r)}\right\}$$

$$\times \left[\frac{3\mathrm{P}_r k^4}{2\upsilon^2 (t-t_0)^{\frac{5}{2}}} + \frac{\mathrm{P}_r (7\mathrm{P}_r - 6) k^6}{3\upsilon (1+\mathrm{P}_r)(t-t_0)^{\frac{3}{2}}} - \frac{4(3\mathrm{P}_r^2 - 2\mathrm{P}_r + 3) k^8}{3(1+\mathrm{P}_r)^2 (t-t_0)^{\frac{1}{2}}} + \frac{8\sqrt{\upsilon}\,(3\mathrm{P}_r^2 - 2\mathrm{P}_r + 3) k^9 F(\eta)}{3(1+\mathrm{P}_r)^{\frac{5}{2}}\ \mathrm{P}_r^{\frac{1}{2}}}\right]$$

$$+\frac{\delta_0\sqrt{\pi}\ P_r^{\frac{5}{2}}}{4\upsilon^{\frac{3}{2}}(1+P_r)^{\frac{7}{2}}}\times\exp\{fs(t-t_0)\}\times\exp\left\{\frac{-k^2\upsilon(1+2P_r)(t-t_0)}{P_r(1+P_r)}\right\}$$

$$\times\left[\frac{3P_r k^4}{2\upsilon^2(t-t_0)^{\frac{5}{2}}}+\frac{P_r(7P_r-6)k^6}{3\upsilon(1+P_r)(t-t_0)^{\frac{3}{2}}}-\frac{4(3P_r^2-2P_r+3)k^8}{3(1+P_r)^2(t-t_0)^{\frac{1}{2}}}+\frac{8\sqrt{\upsilon}(3P_r^2-2P_r+3)k^9F(\eta)}{3(1+P_r)^{\frac{5}{2}}P_r^{\frac{1}{2}}}\right]$$

(2.4.15)

where,

$$F(\eta)=e^{-\eta^2\int_0^{\eta}e^{x^2}dx} \qquad (2.4.16)$$

$$\eta=k\sqrt{\frac{\upsilon(t-t_0)}{P_r(1+P_r)}} \qquad (2.4.17)$$

Putting $\hat{r}=0$ in equation (2.2.9) and we use the definition of E given by the equation (2.4.15), the result is

$$\frac{\overline{TT'}}{2}=\frac{\overline{T^2}}{2}=\int_0^{\infty}E(k)dk \qquad (2.4.18)$$

Substituting equation (2.4.15) into (2.4.18) gives

$$\frac{\overline{T}^2}{2}=\frac{N_0(P_r)^{\frac{3}{2}}}{8\sqrt{(2\pi)}\upsilon^{\frac{3}{2}}(t-t_0)^{\frac{3}{2}}}+\frac{\delta_0 R}{\upsilon^6(t-t_0)^5}\exp[fs]$$

$$\Rightarrow\overline{T}^2=A(t-t_0)^{-\frac{3}{2}}+B\exp[fs]\times(t-t_0)^{-5} \qquad (2.4.19)$$

where $A=\frac{N_0(P_r)^{\frac{3}{2}}}{4\sqrt{(2\pi)}\upsilon^{\frac{3}{2}}}$, $\quad B=\frac{2\delta_0 R}{\upsilon^6}$ and

$$R=\frac{\pi(P_r)^6}{2(1+P_r)(1+2P_r)^{\frac{5}{2}}}\left\{\frac{9}{16}+\frac{5P_r(7P_r-6)}{16(1+2P_r)}-\frac{35P_r(3P_r^2-2P_r+3)}{8(1+2P_r)^2}\right.$$
$$\left.+\frac{1.5422P_r(3P_r^2-2P_r+3)(1+2P_r)^{\frac{5}{2}}}{\sqrt{(\pi)}(1+P_r)^{\frac{11}{2}}}\left[1+\sum_{n=1}^{\infty}\frac{(11)\cdots\cdots[11+2(n-1)]}{(2n+1)n!(2)^{2n}(1+P_r)^n}\right]\right\}$$

(2.4.20)

R is a function of Prandtl no.

Equation (2.4.19) is the decay law of temperature fluctuation in homogeneous turbulence in presence of dust particle prior to the ultimate period. The first term of the right side of equation (2.4.19) corresponds to the temperature energy for two point correlation and the second terms represents the energy for the three point correlation. This second term becomes negligible at large times leaving the final period decay law previously found by Corrsin [31]. $\overline{T}^{2}$ is the total "energy" (the mean square of the temperature fluctuations).

## 2.5 Results and Discussion

Equation (2.4.19) is the decay law of temperature fluctuation in homogeneous turbulence before the final period in presence of the dust particle. In the absence of the dust particle, i.e. $f = 0$ , then the equation (2.4.19) becomes

$$\frac{\overline{T^2}}{2} = \frac{N_0 \left(\mathrm{P}_r\right)^{\frac{3}{2}} \upsilon^{-\frac{3}{2}}}{8\sqrt{(2\pi)}\left(t-t_0\right)^{\frac{3}{2}}} + \frac{\delta_0 R}{\upsilon^6 \left(t-t_0\right)^5}$$

$= A\left(t-t_0\right)^{-\frac{3}{2}} + B\left(t-t_0\right)^{-5}$ ,which was obtained earlier by Loeffler and Deissler [64].

Here, $A = \dfrac{N_0 \left(\mathrm{P}_r\right)^{\frac{3}{2}} \upsilon^{-\frac{3}{2}}}{8\sqrt{(2\pi)}}$ $and\ B = \dfrac{\delta_0 R}{\upsilon^6}$ .

Due to the effect of dust particle in homogeneous turbulence, the temperature energy decays more rapidly than the energy for clean fluid prior to the ultimate period. For large times, the second term in the equation (2.4.19) becomes negligible leaving the $-\frac{3}{2}$ power decay law for the ultimate period.

# CHAPTER-II

# PART-B

## DECAY OF TEMPERATURE FLUCTUATIONS IN DUSTY FLUID HOMOGENEOUS TURBULENCE PRIOR TO THE ULTIMATE PERIOD IN PRESENCE OF CORIOLIS FORCE

### 2.6 Introduction

In geophysical flows, the system is usually rotating with a constant angular velocity. Such large-scale flows are generally turbulent. When the motion is referred to axes, which rotate steadily with the bulk of the fluid, the Coriolis and centrifugal force must be supposed to act on the fluid. On a rotating earth the Coriolis force acts to change the direction of a moving body to the right in the Northern Hemisphere and to the left in the Southern Hemisphere. This force plays an important role in a rotating system of turbulent flow, while centrifugal force with the potential is incorporated in to the pressure.

In a turbulent flow the behaviour of the dust particles depends on the concentration of the particles and the size of the particles with respect to the scale of turbulent fluid. Saffman [86] derived an equation that describe the motion of a fluid containing small dust particle, which is applicable to laminar flows as well as turbulent flow.

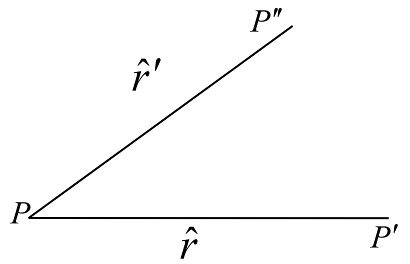
Kishore, and Sarker [53] studied the rate of change of vorticity covariance in MHD turbulent flow of dusty incompressible fluid. Also Rahman [82] studied the Rate of change of vorticity covariance in MHD turbulent flow of dusty fluid in a rotating system. Kishore and Sinha [54] also studied the rate of change of vorticity covariance of dusty fluid turbulence. Corrsin [31] had made an analytical attack on the problem of turbulent temperature fluctuations using the approaches employed in the statistical theory of turbulence. His results pertain to the final period of decay and for the case of appreciable convective effects, to the “energy” spectral from in specific wave-number ranges. Deissler [32, 33] developed a theory for homogeneous turbulence, which was valid for times before the final period. Following Deissler's theory Loeffler and Deissler [64] studied the decay of temperature fluctuations in homogeneous turbulence before the final period. Sarker and Azad [94]; Azad and Sarker [3]; Azad and Sarker [4]; Azad et al [11]; Azad and Sarker [5]; Azad et al. [12] also studied the decay of temperature fluctuations in homogeneous and MHD dusty fluid turbulence. Azad et al [15] studied the transport equatoin for the joint distribution function of velocity, temperature and concentration in convective tubulent flow in presence of dust particles. Molla et al [69] also studied decay of temperature fluctuations in homogeneous turbulenc before the final period in a rotating system. Bkar Pk. et al [30] studied first-order reactant in homogeneou dusty fluid turbulence prior to the ultimate phase of decay for four-point correlation in a rotating system.

They considered dust particles and Coriolis force on their won works. In their study, they considered two- and three- point correlations and neglecting fourth- and higher-order correlation terms compared to the second- and third-order correlation terms. Sinha [103] had considered the effect of dust particles on the acceleration of ordinary turbulence. Kishore and Singh [55] had studied the statistical theory of decay process of homogeneous hydro- magnetic turbulence. Dixit and Upadhyay [34] also had deliberated the effect of coriolis force on acceleration covariance in MHD turbulent dusty flow with rotational symmetry. Kishore and Golsefied [52] considered the effect of Coriolis force on acceleration covariance in MHD turbulent flow of a dusty incompressible fluid. Shimomura and Yoshizawa [102], discussed the statistical analysis of an isotropic turbulent viscosity in a rotating system.

In the present work, following the above theories we have studied the decay of temperature fluctuations in dusty fluid homogeneous turbulence prior to the final period in presence of Coriolis force considering the correlations between fluctuating quantities at two- and three- point and single time. In this work, Deissler's method is used to solving the problem. Through out the study we have obtained the energy decay law of temperature fluctuations in homogeneous dusty fluid turbulence prior to the final period due to Corilis force. In result, it has been shown that the energy decays more rapidly than non rotating clean fluid. It is the extension work of chapter two.

## 2.7 Three points correlation and spectral equations:

In order to obtain single time and three points correlation and spectral equation we consider three points $\mathrm{P}, \mathrm{P}'$ and $\mathrm{P}''$ with position vectors $\hat{r}$ and $\hat{r}'$ are considered.




For the two points $\mathrm{P}'$ and $\mathrm{P}''$ we can write a relation according to equation (2.2.7),

$$\frac{\partial(T'T'')}{\partial t}+u_i\frac{\partial(u_i'T'T'')}{\partial x_i'}+\frac{\partial(u_i''T'T'')}{\partial x_i''}=\frac{\nu}{\mathrm{P}_r}\left[\frac{\partial^2(T'T'')}{\partial x_i'\partial x_i'}+\frac{\partial^2(T'T'')}{\partial x_i''\partial x_i''}\right] \tag{2.7.1}$$

Equation (2.7.1) multiplied through by $u_j$, the j-th velocity fluctuation component at point $\mathrm{P}$. Then the equation can be written in a rotating system at the point $\mathrm{P}$.

$$\frac{\partial(u_jT'T'')}{\partial t}+u_i\frac{\partial(u_ju_i'T'T'')}{\partial x_i'}+\frac{\partial(u_ju_i''T'T'')}{\partial x_i''}=\frac{\nu}{\mathrm{P}_r}\left[\frac{\partial^2(u_jT'T'')}{\partial x_i'\partial x_i'}+\frac{\partial^2(u_jT'T'')}{\partial x_i''\partial x_i''}\right]+T'T''\frac{\partial u_j}{\partial t} \tag{2.7.2}$$

The momentum equation at point $\mathrm{P}$ in presence of dust particles and Corolis force both together

$$\frac{\partial u_j}{\partial t}+\frac{\partial(u_ju_i)}{\partial x_i}=-\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial x_j}+\nu\frac{\partial^2 u_j}{\partial x_i\partial x_i}-2\varepsilon_{mij}\Omega_m+f(u_j-v_j)$$

$$\Rightarrow\frac{\partial u_j}{\partial t}=-\frac{\partial(u_ju_i)}{\partial x_i}-\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial x_j}+\nu\frac{\partial^2 u_j}{\partial x_i\partial x_i}-2\varepsilon_{mij}\Omega_m+f(u_j-v_j) \tag{2.7.3}$$

Here,

$u_j$

$v_j$ =turbulent velocity component

=dust velocity component

$f=\dfrac{kN}{\rho}$ (Dimension of frequency)

$\varepsilon_{mij}$, alternating tensor, $\Omega_m$, angular velocity of a uniform rotation.

N, constant number density of dust particle

Substituted equation (2.7.2) into equation (2.7.3) the result on taking time averages is

$$\frac{\partial\left(\overline{u_jT'T''}\right)}{\partial t}+u_i\frac{\partial\left(\overline{u_ju_i'T'T''}\right)}{\partial x_i'}+\frac{\partial\left(\overline{u_ju_i''T''T'}\right)}{\partial x_i''}=\frac{\nu}{\mathrm{P}_r}\left[\frac{\partial^2\left(\overline{u_jT'T''}\right)}{\partial x_i'\partial x_i'}+\frac{\partial^2\left(\overline{u_jT'T''}\right)}{\partial x_i''\partial x_i''}\right]$$
$$-\frac{\partial\left(\overline{u_ju_iT'T''}\right)}{\partial x_i}-\frac{1}{\rho}\frac{\partial\left(\overline{\mathrm{P}T'T''}\right)}{\partial x_j}+\nu\frac{\partial^2\left(\overline{u_jT'T''}\right)}{\partial x_i\partial x_i}+f(\overline{u_jT'T''}-\overline{v_jT'T''})\quad-2\varepsilon_{mij}\Omega_m\left(\overline{u_jT'T''}\right) \tag{2.7.4}$$

Making use of the relations $r_i=x_i'-x_i$ *and* $r_i'=x_i''-x_i'$ allows equation (2.7.4) can be written as

$$\frac{\partial\left(\overline{u_jT'T''}\right)}{\partial t}-\frac{\nu}{\mathrm{P}_r}\left\{(1+\mathrm{P}_r)\frac{\partial^2\left(\overline{u_jT'T''}\right)}{\partial r_i\partial r_i}+2\mathrm{P}_r\frac{\partial^2\left(\overline{u_jT'T''}\right)}{\partial r_i\partial r_i'}+(1+\mathrm{P}_r)\frac{\partial^2\left(\overline{u_jT'T''}\right)}{\partial r_i'\partial r_i'}\right\}=$$
$$-\frac{\partial\left(\overline{u_ju_i'T'T''}\right)}{\partial r_i}-\frac{\partial\left(\overline{u_ju_i''T''T'}\right)}{\partial r_i'}+\frac{\partial\left(\overline{u_ju_iT'T''}\right)}{\partial r_i'}+\frac{\partial\left(\overline{u_ju_iT'T''}\right)}{\partial r_i}+\frac{1}{\rho}\frac{\partial\left(\overline{\mathrm{P}T'T''}\right)}{\partial r_j'}+\frac{1}{\rho}\frac{\partial\left(\overline{\mathrm{P}T'T''}\right)}{\partial r_j} \tag{2.7.5}$$
$$+f(\overline{u_jT'T''}-\overline{v_jT'T''})-2\varepsilon_{mij}\Omega_m\left(\overline{u_jT'T''}\right)$$

Six-dimensional Fourier transforms for quantities this equation may be defined as

$$\overline{u_jT'T''}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\beta_j\theta'\theta''}\exp\left[i\left(\hat{K}\cdot\hat{r}+\hat{K}'\cdot\hat{r}'\right)\right]d\hat{K}\,d\hat{K} \tag{2.7.6}$$

$$\overline{u_ju_i'T'T''}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\beta_j\beta_i'\theta'\theta''}\exp\left[i\left(\hat{K}\cdot\hat{r}+\hat{K}'\cdot\hat{r}'\right)\right]d\hat{K}\,d\hat{K} \tag{2.7.7}$$

$$\overline{pT'T''}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\alpha\theta'\theta''}\exp\left[i\left(\hat{K}\cdot\hat{r}+\hat{K}'\cdot\hat{r}'\right)\right]d\hat{K}\,d\hat{K} \tag{2.7.8}$$

Interchanging the points $\mathrm{P}'$ *and* $\mathrm{P}''$ shows that

$$\overline{v_ju_i''T''T'}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\gamma_j\theta'\theta''}\exp\left[i\left(\hat{K}\cdot\hat{r}+\hat{K}'\cdot\hat{r}'\right)\right]d\hat{K}\,d\hat{K} \tag{2.7.8a}$$

$$\overline{u_ju_i''T''T'}=\overline{u_ju_i'T'T''}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\beta_j\beta_i'\theta'\theta''}\exp\left[i\left(\hat{K}\cdot\hat{r}+\hat{K}'\cdot\hat{r}'\right)\right]d\hat{K}\,d\hat{K} \tag{2.7.8b}$$

Using equation (2.7.6) – (2.7.8b) into equation (2.7.5) then the transformed equation can be written as

$$\frac{\partial\left(\overline{\beta_j\theta'\theta''}\right)}{\partial t}+\frac{\nu}{\mathrm{P}_r}\left\{(1+\mathrm{P}_r)k^2+2\mathrm{P}_rk_ik_i'+(1+\mathrm{P}_r)k'^2+\frac{p_r}{\upsilon}(2\varepsilon_{mij}\Omega_m-f)\right\}\overline{\beta_j\theta'\theta''}=$$
$$-i(k_i+k_i')\overline{\beta_j\beta_i'\theta'\theta''}+i(k_i'+k_i)\overline{\beta_j\beta_i\theta'\theta''}+\frac{1}{\rho}i(k_j+k_j')\overline{\alpha\theta'\theta''}-f\overline{\gamma_j\theta'\theta''} \tag{2.7.9}$$

If the derivative with respect to $x_j$ is taken of the momentum equation (2.7.4) for point $\mathrm{P}$, and taking time average the resulting equation is

$$\frac{\partial^2\left(\overline{u_j u_i T'T''}\right)}{\partial x_j\,\partial x_i} = -\frac{1}{\rho}\frac{\partial^2\left(\overline{\mathrm{P}T'T''}\right)}{\partial x_j\,\partial x_j} \tag{2.7.10}$$

In terms of the displacement vectors $\hat{r}$ *and* $\hat{r}'$ this becomes

$$\left[\frac{\partial^2}{\partial r_j'\,\partial r_i'} + 2\frac{\partial^2}{\partial r_j'\,\partial r_i} + \frac{\partial^2}{\partial r_j\,\partial r_i}\right]\overline{u_j u_i T'T''} = -\frac{1}{\rho}\left[\frac{\partial^2}{\partial r_j'\,\partial r_j'} + 2\frac{\partial^2}{\partial r_j'\,\partial r_j} + \frac{\partial^2}{\partial r_j\,\partial r_j}\right]\overline{\mathrm{P}T'T''} \tag{2.7.11}$$

Taking the Fourier transform of equation (2.7.11) and then solving for $\overline{\alpha\theta'\theta''}$ we get

$$\overline{\alpha\theta'\theta''} = \frac{-\rho\left[k_j'k_i' + 2k_j'k_i + k_jk_i\right]}{\left[k_j'k_j' + 2k_j'k_j + k_jk_j\right]}\overline{\beta_j\beta_i\theta'\theta''} \tag{2.7.12}$$

Equation (2.7.12) can be used to eliminate $\overline{\alpha\theta'\theta''}$ from equation (2.7.9).

## 2.8 Solution for times prior to the ultimate period

To obtain the equation for final period of decay the third-order fluctuation terms are neglected compared to the second-order terms. Analogously, it would be anticipated that for times before but sufficiently near to the final period the fourth-order fluctuation terms should be negligible in comparison with the third-order terms. If this assumption is made then equation (2.7.12) shows that the term $\overline{\alpha\theta'\theta''}$ associated with the pressure fluctuations, should also be neglected. Thus equation (2.7.9) simplifies to

$$\frac{\partial\left(\overline{\beta_j\theta'\theta''}\right)}{\partial t} + \frac{\nu}{\mathrm{P}_r}\left\{(1+\mathrm{P}_r)k^2 + 2\mathrm{P}_r k_i k_i' + (1+\mathrm{P}_r)k'^2 + \frac{p_r}{\upsilon}(2\varepsilon_{mij}\Omega_m - fs)\right\}\overline{\beta_j\theta'\theta''} = 0 \tag{2.8.1}$$

$R\,\overline{\beta_j\theta'\theta''} = \overline{\gamma_j\theta'\theta''}$ where,

and 1-R=S, R and S are arbitrary constant.

Inner multiplication of equation (2.8.1) by $k_j$ and integrating between $t_0$ *and* $t$ gives

$$k_j\overline{\beta_j\theta'\theta''}$$
$$= \left[k_j\overline{\beta_j\theta'\theta''}\right]_0 \exp\left\{-\frac{\nu}{\mathrm{P}_r}\left[(1+\mathrm{P}_r)k^2 + 2\mathrm{P}_r k_i k_i'\cos\xi + (1+\mathrm{P}_r)k'^2 + \frac{\mathrm{P}_r}{\upsilon}(2\varepsilon_{mij}\Omega_m - fs)\right](t-t_0)\right\}$$

(2.8.2)

Now, letting $r' = 0$ in equation (2.7.6) and comparing the result with the equation (2.2.10) shows that

$$k_i\overline{\phi_i\tau\tau'}\left(\hat{K}\right) = \int_{-\infty}^{\infty} k_i\overline{\beta_i\theta'\theta''}\hat{K}\hat{K}'\,d\hat{K}' \tag{2.8.3}$$

Substituting of equation (2.8.2) and (2.8.3) into equation (2.2.12), we obtain

$$\frac{\partial \overline{\tau\tau'}\left(\hat{K}\right)}{\partial t}+\frac{2\nu}{\mathrm{P}_r}k^2\,\overline{\tau\tau'}\left(\hat{K}\right)=\int_{-\infty}^{\infty} ik_i\left[\overline{\beta_i\,\theta'\theta''}-\beta_i\theta'\left(-\hat{K}\right)\theta''\left(-\hat{K}'\right)\right]_0$$
$$\times\exp\left\{-\frac{\nu\left(t-t_0\right)}{\mathrm{P}_r}\left[\left(1+\mathrm{P}_r\right)\left(k^2+k'^2\right)+2\mathrm{P}_r k\,k'\cos\xi+\frac{\mathrm{P}_r}{\upsilon}(2\varepsilon_{mij}\Omega_m-fs)\right]\right\}d\hat{K}' \tag{2.8.4}$$

Now, $d\hat{K}'\left(\equiv dk_1'dk_2'\,dk_3'\right)$ can be expressed in terms of $k'$ *and* $\xi$ as

$$d\hat{K}'=-2\pi\,k'^2\,d\left(\cos\xi\right)dk' \tag{2.8.5}$$

Substituting equation (2.8.5) into (2.8.4) yields

$$\frac{\partial \overline{\tau\tau'}(\hat{K})}{\partial t}+\frac{2\nu}{\mathrm{P}_r}k^2\,\overline{\tau\tau'}\left(\hat{K}\right)=2\int_{-\infty}^{\infty}2i\,\pi\,k_i\left[\overline{\beta_i\,\theta'\theta''}\left(\hat{K},\hat{K}'\right)-\overline{\beta_i\,\theta'\theta''}\left(-\hat{K},-\hat{K}'\right)\right]_0$$
$$\times\left[\int_{-1}^{1}\exp\left\{-\frac{\nu\left(t-t_0\right)}{\mathrm{P}_r}\left[\left(1+\mathrm{P}_r\right)\left(k^2+k'^2\right)+2\mathrm{P}_r k\,k'\cos\xi+\frac{\mathrm{P}_r}{\upsilon}(\varepsilon_{mij}\Omega_m-fs)\right]\right\}d\left(\cos\xi\right)\right]dk' \tag{2.8.6}$$

In order to find the solutions completely and following Loeffler and Deissler [64], we assume that

$$i\,k_i\left[\overline{\beta_i\,\theta'\theta''}\left(\hat{K},\hat{K}'\right)-\overline{\beta_i\,\theta'\theta''}\left(-\hat{K},-\hat{K}'\right)\right]_0=-\frac{\delta_0}{(2\pi)^2}\left(k^2k'^4-k^4k'^2\right) \tag{2.8.7}$$

where $\delta_0$ constant depending on the initial condition. The negative sign is placed in front of $\delta_0$ in order to make the transfer of energy from small to large wave no. for positive value of $\delta_0$. Substituting equation (2.8.7) into equation (2.8.6)

$$\frac{\partial \overline{\tau\tau'}\left(\hat{K}\right)}{\partial t}+\frac{2\nu}{\mathrm{P}_r}\cdot 2\pi k^2\,\overline{\tau\tau'}\left(\hat{K}\right)=-2\delta_0\int_0^{\infty}\left(k^2k'^4-k^4k'^2\right)$$
$$\times\left[\int_{-1}^{1}\exp\left\{-\frac{\nu\left(t-t_0\right)}{\mathrm{P}_r}\left[\left(1+\mathrm{P}_r\right)\left(k^2+k'^2\right)+2\mathrm{P}_r k\,k'\cos\xi+\frac{\mathrm{P}_r}{\upsilon}(2\varepsilon_{mij}\Omega_m-fs)\right]\right\}d\left(\cos\xi\right)\right]dk' \tag{2.8.8}$$

Multiplying both sides of equation (2.8.8) by $k^2$ and defining the spectral energy function

$$E=2\pi k^2\,\overline{\tau\tau'}\left(\hat{K}\right) \tag{2.8.8a}$$

and the resulting equation is

$$\frac{\partial E}{\partial t}+\frac{2\nu}{\mathrm{P}_r}k^2\,E=w \tag{2.8.9}$$

where

$$w = -2\delta_0 \int_0^\infty \left(k^2 k'^4 - k^4 k'^2\right) k^2 k'^2$$

$$\times \left[ \int_{-1}^{1} \exp\left\{ -\frac{\nu(t-t_0)}{\mathrm{P}_r} \left[ (1+\mathrm{P}_r)\left(k^2 + k'^2\right) + 2\mathrm{P}_r k\, k' \cos\xi + \frac{\mathrm{P}_r}{\upsilon}(2\varepsilon_{mij}\Omega_m - fs \right] \right\} d(\cos\xi) \right] dk'$$

(2.8.9a)

Integrating equation (2.8.9a) w.r.to $\xi$, we have

$$w = -\frac{\delta_0}{2\upsilon(t-t_0)} \int_0^\infty \left(k^3 k'^5 - k^5 k'^3\right)$$

$$\times \left[ \exp\left\{ -\frac{\nu(t-t_0)}{\mathrm{P}_r} \left[ (1+\mathrm{P}_r)\left(k^2 + k'^2\right) - 2\mathrm{P}_r k\, k' + \frac{\mathrm{P}_r}{\upsilon}(2\varepsilon_{mij}\Omega_m - fs) \right] \right\} \right] dk' + \frac{\delta_0}{\upsilon(t-t_0)} \int_0^\infty \left(k^3 k'^5 - k^5 k'^3\right)$$

$$\times \left[ \exp\left\{ -\frac{\nu(t-t_0)}{\mathrm{P}_r} \left[ (1+\mathrm{P}_r)\left(k^2 + k'^2\right) + 2\mathrm{P}_r k\, k' + \frac{\mathrm{P}_r}{\upsilon}(2\varepsilon_{mij}\Omega_m - fs) \right] \right\} \right] dk' \qquad (2.8.9b)$$

Again integrating equation (2.8.9b) w.r.to $k'$ we have

$$w = -\frac{\delta_0 \sqrt{\pi}\ \mathrm{P}_r^{\frac{5}{2}}}{2\upsilon^{\frac{3}{2}} (t-t_0)^{\frac{3}{2}} (1+\mathrm{P}_r)^{\frac{5}{2}}} \exp\left\{ -\frac{\mathrm{P}_r}{\upsilon}(2\varepsilon_{mij}\Omega_m - fs)(t-t_0) \right\}$$

$$\times \exp\left\{ \frac{-k\upsilon(1+2\mathrm{P}_r)(t-t_0)}{\mathrm{P}_r(1+\mathrm{P}_r)} \right\} \qquad (2.8.10)$$

$$\times \left\{ \frac{15\mathrm{P}_r k^4}{4\upsilon^2 (t-t_0)^2 (1+\mathrm{P}_r)} + \left[ \frac{5\mathrm{P}_r^2}{(1+\mathrm{P}_r)^2} - \frac{3}{2} \right] \frac{k^6}{\upsilon(t-t_0)} + \left[ \frac{\mathrm{P}_r^3}{(1+\mathrm{P}_r)^3} - \frac{\mathrm{P}_r}{(1+\mathrm{P}_r)} \right] k^8 \right\}$$

The equation (2.8.9) indicates that w must begin as $k^4$ for small $k$. The condition of w is fulfilled by the equation (2.8.10). It can be shown, using equation (2.8.10) that

$$\int_0^\infty w\, dk = 0 \qquad (2.8.11)$$

It was to be expected physically since w is a measure of the transfer of "energy" and the total energy transferred to all wave numbers must be zero.

The necessity for equation (2.8.11) to hold can be shown as follows if equation (2.2.10) is written for both $\hat{K}\ and - \hat{K}$, and resulting equations differentiated with respect to $r_i$ and added, the result is, for $\hat{r} = 0$ $\left( \frac{\partial}{\partial r_i} = -\frac{\partial}{\partial x_i} \right)$

$$-2\frac{\partial}{\partial x_i}\overline{u_i TT} = \int_{-\infty}^{\infty} ik_i\left[\overline{\phi_i\tau\tau'(\hat{K})} - \overline{\phi_i\tau\tau'(-\hat{K})}\right] d\hat{K} \tag{2.8.11a}$$

Since according to the equations (2.8.8), (2.8.9) and (2.2.12),

$$w \equiv 2\pi\, ik^2 k_i\left[\overline{\phi_i\tau\tau'(-\hat{K})} - \overline{\phi_i\tau\tau'(\hat{K})}\right]$$

So the equation (2.8.11a) can be written as $-2\frac{\partial}{\partial x_i}\overline{u_i\tau\tau'} = \int_{-\infty}^{\infty}\frac{w}{2\pi k^2}\,dk$ as

$d\hat{K} = 4\pi k^2 dk$ for $w = w(k,t)$ then the equation (2.8.11a)

becomes $\int_0^{\infty} w\, dk = -\frac{\partial}{\partial x_i}\overline{u_i\tau\tau'} = 0$

The linear equation (2.8.9) can be solved for w as

$$E = \exp\left[-\frac{2\upsilon k^2(t-t_0)}{\mathrm{P}_r}\right]\int w\,\exp\left[\frac{2\upsilon k^2(t-t_0)}{\mathrm{P}_r}\right]dt + J(k)\,\exp\left[-\frac{2\upsilon k^2(t-t_0)}{\mathrm{P}_r}\right] \tag{2.8.12}$$

where, $J(k)$ is an arbitrary function of k.

For large times, Corrsin [31] has shown the correct form of the expression for E to be

$$E = \frac{N_0}{\pi}k^2\exp\left[-\frac{2\upsilon k^2(t-t_0)}{\mathrm{P}_r}\right] \tag{2.8.13}$$

where, $N_0$ is an constant which depends on the initial conditions. Using equation (2.8.13) to evaluate $J(k)$ in equation (2.812) yields

$$J(k) = \frac{N_0 K^2}{\pi} \tag{2.8.14}$$

Now, substituting the values of w and $J(k)$ as given by the equation (2.8.10) and (2.8.14) into equation (2.8.12) gives the equation.

$$E(k,t) = \frac{N_0 k^2}{\pi}\exp\left[-\frac{2\upsilon k^2(t-t_0)}{\mathrm{P}_r}\right] + \frac{\delta_0\sqrt{\pi}\,\mathrm{P}_r^{\frac{5}{2}}}{2\upsilon^{\frac{3}{2}}(1+\mathrm{P}_r)^{\frac{7}{2}}}\exp\{-(2\varepsilon_{mij}\Omega_m - fs)(t-t_0)\}$$
$$\times\exp\left\{\frac{-k^2\upsilon(1+2\mathrm{P}_r)(t-t_0)}{\mathrm{P}_r(1+\mathrm{P}_r)}\right\}$$
$$\times\left[\frac{3\mathrm{P}_r k^4}{2\upsilon^2(t-t_0)^{\frac{5}{2}}} + \frac{\mathrm{P}_r(7\mathrm{P}_r-6)k^6}{3\upsilon(1+\mathrm{P}_r)(t-t_0)^{\frac{3}{2}}} - \frac{4(3\mathrm{P}_r^2-2\mathrm{P}_r+3)k^8}{3(1+\mathrm{P}_r)^2(t-t_0)^{\frac{1}{2}}} + \frac{8\sqrt{\upsilon}\,(3\mathrm{P}_r^2-2\mathrm{P}_r+3)k^9F(\eta)}{3(1+\mathrm{P}_r)^{\frac{5}{2}}\mathrm{P}_r^{\frac{1}{2}}}\right]$$

$$+\frac{\delta_0\sqrt{\pi}\ \mathrm{P}_r^{\frac{5}{2}}}{4\upsilon^{\frac{3}{2}}(1+\mathrm{P}_r)^{\frac{7}{2}}}\times\exp\{-(2\varepsilon_{mij}\Omega_m-fs)\}\times\exp\left\{\frac{-k^2\upsilon(1+2\mathrm{P}_r)(t-t_0)}{\mathrm{P}_r(1+\mathrm{P}_r)}\right\}$$

$$\times\left[\frac{3\mathrm{P}_r k^4}{2\upsilon^2(t-t_0)^{\frac{5}{2}}}+\frac{\mathrm{P}_r(7\mathrm{P}_r-6)k^6}{3\upsilon(1+\mathrm{P}_r)(t-t_0)^{\frac{3}{2}}}-\frac{4(3\mathrm{P}_r^2-2\mathrm{P}_r+3)k^8}{3(1+\mathrm{P}_r)^2(t-t_0)^{\frac{1}{2}}}+\frac{8\sqrt{\upsilon}(3\mathrm{P}_r^2-2\mathrm{P}_r+3)k^9F(\eta)}{3(1+\mathrm{P}_r)^{\frac{5}{2}}\mathrm{P}_r^{\frac{1}{2}}}\right] \quad (2.8.15)$$

where,

$$F(\eta)=e^{-\eta^2\int_0^\eta e^{x^2}dx} \quad (2.8.16)$$

$$\eta=k\sqrt{\frac{\upsilon(t-t_0)}{\mathrm{P}_r(1+\mathrm{P}_r)}} \quad (2.8.17)$$

Putting $\hat{r}=0$ in equation (2.2.9) and we use the definition of E given by the equation (2.8.15), the result is

$$\frac{\overline{T\,T'}}{2}=\frac{\overline{T^2}}{2}=\int_0^\infty E(k)\,dk \quad (2.8.18)$$

Substituting equation (2.8.15) into (2.8.18) gives

$$\frac{\overline{T}^{\,2}}{2}=\frac{N_0\,(\mathrm{P}_r)^{\frac{3}{2}}}{8\sqrt{(2\pi)}\,\upsilon^{\frac{3}{2}}(t-t_0)^{\frac{3}{2}}}+\frac{\delta_0 R}{\upsilon^6(t-t_0)^5}\exp[-(2\varepsilon_{mij}\Omega_m-fs)]$$

$$\Rightarrow\overline{T}^{\,2}=A(t-t_0)^{-\frac{3}{2}}+B\exp[-(2\varepsilon_{mij}\Omega_m-fs]\times(t-t_0)^{-5} \quad (2.8.19)$$

where, $A=\dfrac{N_0\,(\mathrm{P}_r)^{\frac{3}{2}}}{4\sqrt{(2\pi)}\,\upsilon^{\frac{3}{2}}}$, $\quad B=\dfrac{2\delta_0 R}{\upsilon^6}$ and

$$R=\frac{\pi(\mathrm{P}_r)^6}{2(1+\mathrm{P}_r)(1+2\mathrm{P}_r)^{\frac{5}{2}}}\left\{\frac{9}{16}+\frac{5\mathrm{P}_r(7\mathrm{P}_r-6)}{16(1+2\mathrm{P}_r)}-\frac{35\,\mathrm{P}_r(3\mathrm{P}_r^2-2\mathrm{P}_r+3)}{8(1+2\mathrm{P}_r)^2}\right.$$

$$\left.+\frac{1.5422\,\mathrm{P}_r(3\mathrm{P}_r^2-2\mathrm{P}_r+3)(1+2\mathrm{P}_r)^{\frac{5}{2}}}{\sqrt{(\pi)}\,(1+\mathrm{P}_r)^{\frac{11}{2}}}\left[1+\sum_{n=1}^{\infty}\frac{(11)\cdots\cdots[11+2(n-1)]}{(2n+1)n!(2)^{2n}(1+\mathrm{P}_r)^n}\right]\right\} \quad (2.8.20)$$

R is a function of Prandtl no.

Equation (2.8.19) is the decay law of temperature fluctuation in homogeneous dusty fluid turbulence prior to the ultimate period in presence of Coriolis force. The first term of the right side of equation (2.8.19) corresponds to the temperature energy for two- point correlation and the second terms represents the energy for the three point correlation. This second term becomes negligible at large times leaving the final period decay law previously found by Corrsin [31]. $\overline{T}^{2}$ is the total "energy" (the mean square of the temperature fluctuations).

## 2.9 Result and Discussion

Equation (2.8.19) is the decay law of temperature fluctuation in homogeneous dusty fluid turbulence before the final period in presence of Coriolis force. In the absence of the dust particle and Coriolis force, i.e. $f=0$ then the equation (2.8.19) becomes

$$\frac{\overline{T}^{2}}{2}=\frac{N_0\left(\mathrm{P}_r\right)^{\frac{3}{2}}}{8\sqrt{(2\pi)}\,\upsilon^{\frac{3}{2}}\left(t-t_0\right)^{\frac{3}{2}}}+\frac{\delta_0 R}{\upsilon^{6}\left(t-t_0\right)^{5}}\exp\left[-2\varepsilon_{mij}\Omega_m\right] \tag{2.9.1}$$
$$\Rightarrow \overline{T}^{2}=A\left(t-t_0\right)^{-\frac{3}{2}}+B\exp\left[-2\varepsilon_{mij}\Omega_m\right]\times\left(t-t_0\right)^{-5}$$

which was obtained earlier by Molla et al [68]. In this work, they had studied the decay of temperature fluctuation in homogeneous turbulenc before the final period in a Rotating System. They considered two - and three - point correlations and neglecting fourth- and higher-order correlation terms compared to the second- and third-order correlation terms.and derivrd the above equation.

If $\Omega_m=0$, then the equation (2.8.19) becomes

$$\frac{\overline{T}^{2}}{2}=\frac{N_0\left(\mathrm{P}_r\right)^{\frac{3}{2}}}{8\sqrt{(2\pi)}\,\upsilon^{\frac{3}{2}}\left(t-t_0\right)^{\frac{3}{2}}}+\frac{\delta_0 R}{\upsilon^{6}\left(t-t_0\right)^{5}}\exp[fs]=A\left(t-t_0\right)^{-\frac{3}{2}}+B\exp[fs]\times\left(t-t_0\right)^{-5} \tag{2.9.2}$$

which was obtained earlier by Azad and Mumtahinah [13].

In the absence of the dust particle and the Coriolis force i.e. $f=0$ and $\Omega_m=0$, the equation (2.8.19) becomes

$$\frac{\overline{T^{2}}}{2}=\frac{N_0\left(\mathrm{P}_r\right)^{\frac{3}{2}}\upsilon^{-\frac{3}{2}}}{8\sqrt{(2\pi)}\left(t-t_0\right)^{\frac{3}{2}}}+\frac{\delta_0 R}{\upsilon^{6}\left(t-t_0\right)^{5}} \tag{2.9.3}$$

$=A\left(t-t_0\right)^{-\frac{3}{2}}+B\left(t-t_0\right)^{-5}$ which was obtained earlier by Loeffler and Deissler [64].

Here, $A=\dfrac{N_0\left(\mathrm{P}_r\right)^{\frac{3}{2}}\upsilon^{-\frac{3}{2}}}{8\sqrt{(2\pi)}}$ *and* $B=\dfrac{\delta_0 R}{\upsilon^{6}}$ .

Due to the effect of Coriolis force in homogeneous dusty fluid turbulence, the temperature energy fluctuations decays more rapidly then the energy for non rotating clean fluid prior to the ultimate period. For large times, the second term in the equation (2.9.3) becomes negligible leaving the $-\frac{3}{2}$ power decay law for the ultimate period.

In their study, they considered two and three point correlations and neglecting fourth- and higher-order correlation terms compared to the second- and third-order correlation terms.

Through the study we have obtained the equation (2.8.20) for energy decay law of temperature fluctuations in homogeneous dusty fluid turbulence prior to the final period in a rotating system. In this result, it has been shown that the energy decays more rapidly than clean fluid and non rotating system.

# CHAPTER-III

# PART-A

## TRANSPORT EQUATION FOR THE JOINT DISTRIBUTION FUNCTIONS OF CERTAIN VARIABLES IN CONVECTIVE TURBULENT FLOW IN PRESENCE OF CORIOLIS FORCE UNDER GOING A FIRST ORDER REACTION

### 3. 1 Introduction

In molecular kinetic theory in physics a particle's distribution function is a function of seven variables, $f(x, y, z, v_x, v_y, v_z)$ which gives the number of particles per unit volume in phase space. It is the number of particles per unit volume having approximately the velocity $(v_x, v_y, v_z)$ near the place $(x, y, z)$ and time $t$. Particle distribution functions are often used in plasma physics to describe wave-particle interactions and velocity-space instabilities. Distribution functions are also used in fluid mechanics, statistical mechanics, fluid and nuclear physics. In the past, several researchers discussed the distribution functions in the statistical theory of turbulence. G. K. Batchelor [24] studied the theory of homogeneous turbulence. Lundgren [66] derived the transport equation for the distribution of velocity in turbulent flow. Bigler [25] gave the hypothesis that in turbulent flames, the thermo chemical quantities can be related locally to few scalars and considered the probability density function of these scalars. Kishore [47] studied the distributions functions in the statistical theory of MHD turbulence of an incompressible fluid. S. B. Pope [78] studied the statistical theory of turbulence flames. Also, Pope [79] derived the transport equation for the joint probability density function of velocity and scalars in turbulent flow. Kollman and Janica [56] derived the transport equation for the probability density function of a scalar in turbulent shear flow and considered a closure model based on gradient flux model. Kishore and Singh [49] derived the transport equation for the bivariate joint distribution function of velocity and temperature in turbulent flow. Also Kishore and Singh [50] have been derived the transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow. The Coriolis force helps to clarify the relation between angular momentum and rotational kinetic energy and how an inertial force can have a significant affect on the movement of a body and still without doing any work. On a rotating earth the Coriolis force acts to change the direction of a moving body to the right in the Northern Hemisphere and to the left in the Southern Hemisphere. This deflection is not only instrumental in the large-scale atmospheric circulation, the development of storms, and the sea-breeze circulation - Atkinson [1], it can even affect the outcome of baseball tournaments. Also a first-order reaction is defined a reaction that proceeds at a rate that depends linearly only on one reactant concentration. Later, some researchers extended their works including coriolis force. In the continuation, Azad and Sarker [2] studied the Statistical theory of certain distribution functions in MHD turbulence in a rotating system in presence of dust particles. Sarker and Azad [87] studied the decay of MHD turbulence before the final period for the case of multi-point and multi-time in a rotating system. Sarker and Azad[87], Azad and Sarker[5] deliberated the decay of temperature fluctuations in homogeneous turbulence before the final period for the case of multi- point and multi- time in a rotating system and dust particles. Azad and Sarker [8] discussed the decay of temperature fluctuations in MHD turbulence before the final period in a rotating system. Also, Azad et al[8], Sarker et al [99], Azad et al [6], Aziz et al[17], Azad et al[9] discussed the First Order Reactant in MHD turbulence before the final period of decay for the case of multi-point multi-time and multi -point single time considering rotating system and dust particles. Following the above researchers, Aziz et al [18,20**]**, Azad et al [10] had further studied the statistical theory of certain distribution functions in MHD turbulent flow for velocity and concentration considering first order reaction with a rotating system and dust particles. Aziz et al [19] extended their study for the first order reactant in MHD turbulence before the final period of decay for the case of multi-point and multi-time in a rotating system in presence of dust particle. Sarker, Bkar Pk and Azad [101] studied the homogeneous dusty fluid turbulence in a first order reactant for the case of multi -point and multi -time prior to the final period of decay. Azad, Molla and Z. Rahman [15] studied the transport equatoin for the joint distribution function of velocity, temperature and concentration in convective tubulent flow in presence of dust particles. Molla, Azad and Z. Rahman [68] discussed the decay of temperature fluctuations in homogeneous turbulenc before the final period in a rotating system.

Bkar et al [29], Bkar et al [26, 29] premeditated the first-order reactant in homogeneou dusty fluid turbulence prior to the ultimate phase of decay for four-point correlation considering rotating system. Bkar PK, et al [28, 27] had studied the decay of MHD turbulence before the final period for four- point correlation among dust particle and rotating system. M. H. U. Molla et al [68] studied the transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow in presence of Coriolis force.

But at this stage, one is met with the difficulty that the N-point distribution function depends upon the N+1-point distribution function and thus result is an unclosed system. This so-called closer problem is encountered in turbulence, Kinetic theory and other non-linear system.

In this chapter, we have studied the joint distribution function for simultaneous velocity, temperature, concentration fields in turbulent flow in presence of Coriolis force undergoing a first order reaction. Finally, the transport equations for evolution of distribution functions have been derived and various properties of the distribution function have been discussed.

## METHODOLOGY

## 3.2 Basic equations

The equation of motion and field equations of temperature and concentration in presence of Coriolis force are shown by

$$\frac{\partial u_\alpha}{\partial t}+u_\alpha\frac{\partial u_\alpha}{\partial x_\beta}=-\frac{\partial}{\partial x_\beta}\int_0^\infty\frac{1}{4\pi}\frac{\partial}{\partial x'_\beta}\left\{u_\alpha(x',t)\frac{\partial}{\partial x'_\beta}\cdot u_\alpha(x',t)\right\}\frac{dx'_\beta}{|x_\beta-x'_\beta|}+\nu\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}u_\alpha-2\in_{m\alpha\beta}\Omega_m u_\alpha \tag{3.2.1}$$

$$\frac{\partial\theta}{\partial t}+u_\alpha\frac{\partial\theta}{\partial x_\beta}=f\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}\theta \tag{3.2.2}$$

$$\frac{\partial c}{\partial t}+u_\alpha\frac{\partial c}{\partial x_\beta}=\mathrm{D}\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}c-Rc \tag{3.2.3}$$

here u and x are vector quantities in the whole process. uα (x, t) = Fluctuating velocity component,θ (x, t) =Temperature fluctuation,c = Concentration of contaminants, $\nu$ = Kinematics viscosity, *f*= Coefficient of thermal conductivity, D= Diffusive coefficient for contaminants, $\in_{m\alpha\beta}$ = Alternating tensor, $\Omega_m$ = Angular velocity of a uniform rotation, R=constant reaction rate.

## 3.3 Formulation of the problem

We consider the turbulence and the concentration fields are homogeneous, also consider a large ensemble of mixture of miscible fluids in which each member is an infinite incompressible heat conducting fluid in turbulent state. The fluid velocity u, temperature θ and concentration c are randomly distributed functions of position and time and satisfy their field equations. Different members of ensemble are subjected to different initial conditions and the aim is to find out a way by which we can determine the ensemble averages at the initial time. The present aim is to construct a joint distribution functions, study its properties and derive an equation for its evolution of this joint distribution functions in presence of Coriolis force undergoing a first order reaction.

## 3.4 Joint distribution function in convective turbulence and their properties

It may be considered that the fluid velocity u, temperature θ, concentration c at each point of the flow field in turbulence. Lundgren [65] and Sarker and Kishore [97, 98] has studied the flow field on the basis of one variable character only (namely the fluid u) but we can study it for two or more variable characters as well. For the corresponding each point of the flow field, we have three measurable characteristics. We represent the three variables by v, $\phi$ and ψ and denote the pairs of these variables at the points $x^{(1)}, x^{(2)} --------, x^{(n)}$ as $\left(v^{(1)}, \phi^{(1)}, \psi^{(1)}\right)$, $\left(v^{(2)}, \phi^{(2)}, \psi^{(2)}\right)$ - - - - - - - -- -, $\left(v^{(n)}, \phi^{(n)}, \psi^{(n)}\right)$ at a fixed instant of time. It is possible that the same pair may be occurring more than once; therefore, we simplify the problem by an assumption that the distribution is discrete (in the sense that no pairs occur more than once). Instead of considering discrete points in the flow field if we consider the continuous distribution of the variables and ψ over the entire flow field, statistically behavior of the fluid may be described by the distribution function $F(v,\phi,\psi)$ which is normalized so that

$\int F(v,\phi,\psi)dv\,d\phi\,d\psi = 1$,

where the integration ranges over all the possible values of v, $\phi$ and ψ. We shall make use of the same normalization condition for the discrete distributions also. The joint distribution functions of the above quantities can be defined in terms of Dirac Delta-functions.

The one-point joint distribution function $F_1^{(1)}\left(v^{(1)},\phi^{(1)},\psi^{(1)}\right)$ is defined in such a way that $F_1^{(1)}\left(v^{(1)},\phi^{(1)},\psi^{(1)}\right)dv^{(1)}d\phi^{(1)}d\psi^{(1)}$ is the probability that the fluid velocity, temperature and concentration field at a time t are in the element $dv^{(1)}$ about $v^{(1)}$, $d\phi^{(1)}$ about $\phi^{(1)}$ and $d\psi^{(1)}$ about $\psi^{(1)}$ respectively and is given as

$$F_1^{(1)}\left(v^{(1)},\phi^{(1)},\psi^{(1)}\right) = \langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\rangle \tag{3.4.1}$$

where, δ is the Dirac delta-function defined as:

$$\int \delta(u-v)dv = \begin{cases} 1 & at\ the\ point\ u=v \\ 0 & otherwise \end{cases}$$

Two-point joint distribution function is given by

$$F_2^{(1,2)} = \langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\,\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\rangle \tag{3.4.2}$$

And three point distribution functions is shown by

$$\begin{aligned} F_3^{(1,2,3)} = \langle & \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\,\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right) \\ & \times\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\,\delta\left(c^{(3)}-\psi^{(3)}\right)\rangle \end{aligned} \tag{3.4.3}$$

Similarly, we can define an infinite numbers of multi-point joint distribution functions $F_4^{(1,2,3,4)}$, $F_5^{(1,2,3,4,5)}$ and so on. The joint distribution functions so constructed have the following properties:

**(A) Reduction properties**

Integration with respect to pair of variables at one-point, lowers the order of distribution function by one.

For example:

$$\int F_1^{(1)} dv^{(1)}\, d\phi^{(1)}\, d\psi^{(1)} = 1$$
$$\int F_2^{(1,2)} dv^{(2)}\, d\phi^{(2)}\, d\psi^{(2)} = F_1^{(1)}$$
$$\int F_3^{(1,2,3)} dv^{(3)}\, d\phi^{(3)}\, d\psi^{(3)} = F_2^{(1,2)}$$

and so on.

Also the integration with respect to any one of the variables reduces the number of Delta-functions from the distribution function by one as:

$$\int F_1^{(1)} dv^{(1)} = \langle \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right)\rangle$$
$$\int F_1^{(1)} d\phi^{(1)} = \langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right)\rangle$$

and

$$\int F_2^{(1,2)} dv^{(2)} = \langle \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right)\rangle \text{ and so on.}$$

**(B) Separation properties**

The pairs of variables at the two points are statistically independent of each other if these points are far apart from each other in the flow field i.e.,

$$\lim_{\left|x^{(2)} - x^{(1)}\right| \to \infty} F_2^{(1,2)} = F_1^{(1)} F_1^{(2)}$$

And similarly,

$$\lim_{\left|x^{(3)} - x^{(2)}\right| \to \infty} F_3^{(1,2,3)} = F_2^{(1,2)} F_1^{(3)} \quad etc.$$

**(C) Coincidence property**

When two points coincide in the flow field, the components at these points should be obviously the same that is $F_2^{(1,2)}$ must be zero. Thus:

$$v^{(2)} = v^{(1)}, \phi^{(2)} = \phi^{(1)} \quad and \quad \psi^{(2)} = \psi^{(1)}$$

But also $F_2^{(1,2)}$ must have the property

$$\int F_2^{(1,2)} dv^{(2)}\, d\phi^{(2)}\, d\psi^{(2)} = F_1^{(1)}$$

And hence it follows that:

$$\lim_{\left|x^{(2)} - x^{(1)}\right| \to \infty} F_2^{(1,2)} = F_1^{(1)}\, \delta\left(v^{(2)} - v^{(1)}\right) \delta\left(\phi^{(2)} - \phi^{(1)}\right) \delta\left(\psi^{(2)} - \psi^{(1)}\right)$$

Similarly

$$\underset{\left|x^{(3)}-x^{(2)}\right|\to\infty}{Lim} F_3^{(1,2,3)} = F_2^{(1,2)}\,\delta\left(v^{(3)}-v^{(1)}\right)\delta\left(\phi^{(3)}-\phi^{(1)}\right)\delta\left(\psi^{(3)}-\psi^{(1)}\right) \quad etc.$$

## 3.5 Continuity equation in terms of distribution functions

An infinite number of continuity equations can be derived for the convective turbulent flow and the continuity equations can be easily expressed in terms of distribution functions and are obtained directly by div u = 0.

$$\left\langle \frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}} \right\rangle = \left\langle \frac{\partial}{\partial x_\alpha^{(1)}} u_\alpha^{(1)} \int F_1^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \right\rangle = \frac{\partial}{\partial x_\alpha^{(1)}} \left\langle u_\alpha^{(1)} \int F_1^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \int \left\langle u_\alpha^{(1)} \right\rangle \left\langle F_1^{(1)} \right\rangle dv^{(1)} d\phi^{(1)} d\psi^{(1)} = \frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_1^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)}$$

$$= \int \frac{\partial F_1^{(1)}}{\partial x_\alpha^{(1)}} v_\alpha^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} = 0 \tag{3.5.1}$$

And similarly

$$\int \frac{\partial F_1^{(1)}}{\partial x_\alpha^{(1)}} \phi_\alpha^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} = 0 \tag{3.5.2}$$

Which are the first order continuity equations in which only one point distribution function is involved. For second-order continuity equations, if we multiply the continuity equation by

$$\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

And if we take the ensemble average, we obtain:

$$0 = \left\langle \delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}} \right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \left\langle \delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right) u_\alpha^{(1)} \right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \int \left\langle u_\alpha^{(1)}\, \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right) \right\rangle \left\langle \delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right) \right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_2^{(1,2)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{3.5.3}$$

$$0 = \frac{\partial}{\partial x_\alpha^{(1)}} \int \phi_\alpha^{(1)} F_2^{(1,2)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{3.5.4}$$

And similarly, the Nth-order continuity equations are

$$0 = \frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_N^{(1,2,---N)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{3.5.5}$$

And

$$0 = \frac{\partial}{\partial x_\alpha^{(1)}} \int \phi_\alpha^{(1)} F_N^{(1,2,---N)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{3.5.6}$$

The continuity equations are symmetric in their arguments i.e.

$$\frac{\partial}{\partial x_\alpha^{(r)}} \int \left( v_\alpha^{(r)} F_N^{(1,2,---s---r---N)} dv^{(r)} d\phi^{(r)} d\psi^{(r)} \right)$$

$$= \frac{\partial}{\partial x_\alpha^{(s)}} \int \left( v_\alpha^{(s)} F_N^{(1,2,---r---s---N)} dv^{(s)} d\phi^{(s)} d\psi^{(s)} \right) \tag{3.5.7}$$

Since, the divergence property is an important property and it is easily verified by the use of the property of distribution function as:

$$\frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_1^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} = \frac{\partial}{\partial x_\alpha^{(1)}} \left\langle u_\alpha^{(1)} \right\rangle = \left\langle \frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}} \right\rangle = 0 \tag{3.5.8}$$

And all the properties of the distribution function obtained in section (3.4.1) can also be easily verified.

## 3.6 Equations for the evolution of joint distribution functions

This, in fact is done by making use of the definitions of the constructed distribution functions, the transport equation for $F(v, \phi, \psi, x, t)$ is obtained from the definition of $F$ and from the transport equations (3.2.1), (3.2.2), (3.2.3). Differentiating equation (3.4.1) we get,

$$\frac{\partial}{\partial t}F_1^{(1)}=\frac{\partial}{\partial t}\left\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle=\left\langle\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle+\left\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$
$$=\left\langle-\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial u_\alpha^{(1)}}{\partial t}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle \tag{3.6.1}$$
$$+\left\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial \theta^{(1)}}{\partial t}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$+\left\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\frac{\partial c^{(1)}}{\partial t}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

Using equation (3.2.1), (3.2.2) and (3.2.3) in the equation (3.6.1) we get

$$\frac{\partial}{\partial t}F_1^{(1)}=\left\langle-\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\begin{bmatrix}-u_\alpha^{(1)}\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}-\frac{\partial}{\partial x_\beta^{(1)}}\int\frac{1}{4\pi}\frac{\partial}{\partial x_\beta^{(2)}}\left\{u_\alpha^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\cdot u_\alpha^{(2)}\right\}\\ \times\frac{dx_\beta^{(2)}}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}+\nu\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}u_\alpha^{(1)}-2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}\end{bmatrix}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\left\{-u_\alpha^{(1)}\frac{\partial\theta^{(1)}}{\partial x_\beta^{(1)}}+f\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\theta^{(1)}\right\}\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$+\left\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\left\{-u_\alpha^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}+\mathrm{D}\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}c^{(1)}\right\}\frac{\partial}{\partial\psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)-Rc^{(1)}\right\rangle$$

$$\therefore\frac{\partial F_1^{(1)}}{\partial t}+\left\langle-\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial\theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$+\left\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)u_\alpha^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial\psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$
$$+\left\langle\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\left[-\frac{\partial}{\partial x_\beta^{(1)}}\left\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}u_\alpha^{(2)}u_\alpha^{(2)}\frac{dx_\beta^{(2)}}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}\right\}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right]\right\rangle$$
$$+\left\langle\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\nu\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}u_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$

$$+\left\langle -\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$

$$+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)f\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\right)\theta^{(1)}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$

$$+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\mathrm{D}\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\right)c^{(1)}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle \tag{3.6.2}$$

$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)Rc^{(1)}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle=0$$

Various terms in the above equation can be simplified as that they may be expressed in terms of one point and two point distribution functions. The 2nd, 3rd and 4th terms on the left hand side of the above equation are simplified in a similar fashion and take the forms as follows

$$\left\langle -\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$

$$=\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle,\quad \left[\because \frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}}=-1\right] \tag{3.6.3}$$

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial \theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$

$$=\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle, \tag{3.6.4}$$

and

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)u_\alpha^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle \tag{3.6.5}$$

Adding equation (3.6.3), (3.6.4) and (3.6.5) we get,

$$\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$

$$+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$= \frac{\partial}{\partial x_\beta^{(1)}} \left\langle u_\alpha^{(1)} \left( \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right) \right\rangle$$

$$= \frac{\partial}{\partial x_\beta^{(1)}} v_\alpha^{(1)} F_1^{(1)} \qquad \left[ \textit{Applying the properties of distribution function} \right]$$

$$= v_\alpha^{(1)} \frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}} \tag{3.6.6}$$

We reduce the 5th and 6th terms on left hand side of equation (3.6.2),

$$\left\langle \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \left[ -\frac{\partial}{\partial x_\beta^{(1)}} \left\{ \frac{1}{4\pi} \int \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} u_\alpha^{(2)} u_\alpha^{(2)} \frac{dx_\beta^{(2)}}{\left| x_\beta^{(1)} - x_\beta^{(2)} \right|} \right\} \frac{\partial}{\partial v_\alpha^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \right] \right\rangle$$

$$= \frac{\partial}{\partial v_\alpha^{(1)}} \left[ -\frac{1}{4\pi} \int \left( \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\left| x_\beta^{(1)} - x_\beta^{(2)} \right|} \right) \right] \left( v_\alpha^{(2)} \frac{\partial}{\partial x_\beta^{(2)}} \right)^2 F_2^{(1,2)} dx^{(2)} dv^{(2)} \, d\phi^{(2)} \, d\psi^{(2)} \tag{3.6.7}$$

And

$$\left\langle \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \nu \left( \frac{\partial}{\partial x_\beta^{(1)}} \frac{\partial}{\partial x_\beta^{(1)}} u_\alpha^{(1)} \right) \frac{\partial}{\partial v_\alpha^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \right\rangle$$

$$= \left\langle \nu \frac{\partial}{\partial v_\alpha^{(1)}} \frac{\partial}{\partial x_\beta^{(1)}} \frac{\partial}{\partial x_\beta^{(1)}} u_\alpha^{(1)} \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(1)} - v^{(1)}\right) \right\rangle$$

$$= \nu \frac{\partial}{\partial v_\alpha^{(1)}} \left\langle \frac{\partial}{\partial x_\beta^{(1)}} \frac{\partial}{\partial x_\beta^{(1)}} u_\alpha^{(1)} \left[ \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(1)} - v^{(1)}\right) \right] \right\rangle$$

$$= \nu \frac{\partial}{\partial v_\alpha^{(1)}} \frac{\partial}{\partial x_\beta^{(1)}} \frac{\partial}{\partial x_\beta^{(1)}} \left\langle u_\alpha^{(1)} \left[ \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(1)} - v^{(1)}\right) \right] \right\rangle$$

$$= \nu \frac{\partial}{\partial v_\alpha^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \left\langle \begin{array}{l} \int u_\alpha^{(2)} \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \delta\left(u^{(2)} - v^{(2)}\right) \\ \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(1)} - v^{(1)}\right) dv^{(2)} d\phi^{(2)} d\psi^{(2)} \end{array} \right\rangle$$

$$= \frac{\partial}{\partial v_\alpha^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} \nu \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int v^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} \tag{3.6.8}$$

We reduce the 7th term on left hand side of equation (3.6.2),

$$\left\langle -\delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \right\rangle$$

$$= \left\langle 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}} \left[ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right] \right\rangle \tag{3.6.9}$$

$$= 2 \in_{m\alpha\beta} \Omega_m \frac{\partial}{\partial v_\alpha^{(1)}} \left\langle u_\alpha^{(1)} \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle$$

$$= 2 \in_{m\alpha\beta} \Omega_m \frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}} \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle = 2 \in_{m\alpha\beta} \Omega_m F_1^{(1)}$$

Similarly, 8th, 9th and 10th terms of left hand side of (3.6.2) can be simplified as follows:

$$\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right) f\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\right)\theta^{(1)}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$

$$=\frac{\partial}{\partial \phi^{(1)}}\underset{x^{(2)}\to x^{(1)}}{Lim} f\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int \phi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)} \tag{3.6.10}$$

$$\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right) \mathrm{D}\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\right)c^{(1)}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=\frac{\partial}{\partial \psi^{(1)}}\underset{x^{(2)}\to x^{(1)}}{Lim} \mathrm{D}\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int \psi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)} \tag{3.6.11}$$

and $\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)Rc^{(1)}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle=-R\psi^{(1)}\frac{\partial}{\partial \psi^{1}}F_1^{(1)}$ (3.6.12)

Substituting the results (3.6.6)-(3.6.12) in equation (3.6.2), we get the transport equation for one point distribution function $F_1^{(1)}(v,\phi,\psi)$ in turbulent flow in a rotating system undergoing a first order reaction

$$\begin{aligned}
&\frac{\partial F_1^{(1)}}{\partial t}+v_\alpha^{(1)}\frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}}+\frac{\partial}{\partial v_\alpha^{(1)}}\left[-\frac{1}{4\pi}\int\left(\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}\right)\right]\left(v_\alpha^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\right)^2 F_2^{(1,2)}dx^{(2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}\\
&+\frac{\partial}{\partial v_\alpha^{(1)}}\underset{x^{(2)}\to x^{(1)}}{Lim} \nu\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int v^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}\\
&+\frac{\partial}{\partial \phi^{(1)}}\underset{x^{(2)}\to x^{(1)}}{Lim} f\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int \phi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}\\
&+\frac{\partial}{\partial \psi^{(1)}}\underset{x^{(2)}\to x^{(1)}}{Lim} \mathrm{D}\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int \psi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}+2\in_{m\alpha\beta}\Omega_m F_1^{(1)}-R\psi^{(1)}\frac{\partial}{\partial \psi^{1}}F_1^{(1)}=0
\end{aligned} \tag{3.6.13}$$

Similarly, a transport equation for two-point distribution function $F_2^{(1,2)}$ in turbulent flow in rotating system undergoing a first order reaction can be derived by differentiating equation (3.4.2) and using equation (3.2.1),(3.2.2),(3.2.3) and simplifying in the same manner which is

$$\frac{\partial F_2^{(1,2)}}{\partial t}+\left(v_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}+v_\alpha^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\right)F_2^{(1,2)}+\frac{\partial}{\partial v_\alpha^{(1)}}\left[\begin{array}{l}-\frac{1}{4\pi}\int\left(\frac{\partial}{\partial x_\beta^{(3)}}\frac{\partial}{\left|x_\beta^{(1)}-x_\beta^{(3)}\right|}\right)\left(v_\alpha^{(3)}\frac{\partial}{\partial x_\beta^{(3)}}\right)^2\\ \times F_3^{(1,2,3)}dx^{(3)}dv^{(3)}\,d\phi^{(3)}\,d\psi^{(3)}\end{array}\right]$$

$$+\frac{\partial}{\partial v_\alpha^{(2)}}\left[-\frac{1}{4\pi}\int\left(\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\left|x_\beta^{(2)}-x_\beta^{(3)}\right|}\right)\left(v_\alpha^{(3)}\frac{\partial}{\partial x_\beta^{(3)}}\right)^2F_3^{(1,2,3)}dx^{(3)}dv^{(3)}\,d\phi^{(3)}\,d\psi^{(3)}\right] \quad (3.6.14)$$

$$+\nu\left(\frac{\partial}{\partial v_\alpha^{(1)}}\underset{x^{(3)}\to x^{(1)}}{Lim}+\frac{\partial}{\partial v_\alpha^{(2)}}\underset{x^{(3)}\to x^{(2)}}{Lim}\right)\frac{\partial}{\partial x_\beta^{(3)}}\frac{\partial}{\partial x_\beta^{(3)}}\int v^{(3)}F_3^{(1,2,3)}dv^{(3)}d\phi^{(3)}d\psi^{(3)}$$

$$+f\left(\frac{\partial}{\partial \phi^{(1)}}\underset{x^{(3)}\to x^{(1)}}{Lim}+\frac{\partial}{\partial \phi^{(2)}}\underset{x^{(3)}\to x^{(2)}}{Lim}\right)\frac{\partial}{\partial x_\beta^{(3)}}\frac{\partial}{\partial x_\beta^{(3)}}\int \phi^{(3)}F_3^{(1,2,3)}dv^{(3)}d\phi^{(3)}d\psi^{(3)}$$

$$+\mathrm{D}\left(\frac{\partial}{\partial \psi^{(1)}}\underset{x^{(3)}\to x^{(1)}}{Lim}+\frac{\partial}{\partial \psi^{(2)}}\underset{x^{(3)}\to x^{(2)}}{Lim}\right)\frac{\partial}{\partial x_\beta^{(3)}}\frac{\partial}{\partial x_\beta^{(3)}}\int \psi^{(3)}F_3^{(1,2,3)}dv^{(3)}d\phi^{(3)}d\psi^{(3)}$$

$$+2\in_{m\alpha\beta}\Omega_m F_2^{(1,2)}-R\psi^{(2)}\frac{\partial}{\partial\psi^2}F_2^{(1,2)}=0$$

Continuing this way, we can derive the equations for evolution of $F_3^{(1,2,3)}, F_4^{(1,2,3,4)}$ and so on. Logically, it is possible to have an equation for every $F_n$ $(n\ is\ an\ \text{integer})$ but the system of equations so obtained is not closed. It seems that certain approximations will be required thus obtained.

## 3.7 Results and Discussion

If the reaction rate R=0, the transport equation (3.6.13) for one point joint distribution function $F_1^{(1)}(v,\phi,\psi)$ in turbulent flow undergoing a first order reaction becomes

$$\frac{\partial F_1^{(1)}}{\partial t}+v_\alpha^{(1)}\frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}}+\frac{\partial}{\partial v_\alpha^{(1)}}\left[-\frac{1}{4\pi}\int\left(\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}\right)\right]\left(v_\alpha^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\right)^2F_2^{(1,2)}dx^{(2)}dv^{(2)}\,d\phi^{(2)}\,d\psi^{(2)}$$

$$+\frac{\partial}{\partial v_\alpha^{(1)}}\underset{x^{(2)}\to x^{(1)}}{Lim}\,\nu\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int v^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)} \quad (3.7.1)$$

$$+\frac{\partial}{\partial \phi^{(1)}}\underset{x^{(2)}\to x^{(1)}}{Lim}\,f\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int \phi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}$$

$$+\frac{\partial}{\partial \psi^{(1)}}\underset{x^{(2)}\to x^{(1)}}{Lim}\,\mathrm{D}\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int \psi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}+2\in_{m\alpha\beta}\Omega_m F_1^{(1)}=0$$

which was obtained earlier by M.H.U.Molla [67].

In the absence of the Coriolis force, $\Omega_m = 0$, then the transport equation for one point joint distribution function $F_1^{(1)}(v,\phi,\psi)$ in turbulent flow equation (3.6.12) becomes

$$\frac{\partial F_1^{(1)}}{\partial t} + v_\alpha^{(1)} \frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}} + \frac{\partial}{\partial v_\alpha^{(1)}} \left[ -\frac{1}{4\pi} \int \left( \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\left| x_\beta^{(1)} - x_\beta^{(2)} \right|} \right) \right] \left( v_\alpha^{(2)} \frac{\partial}{\partial x_\beta^{(2)}} \right)^2 F_2^{(1,2)} dx^{(2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)}$$

$$+ \frac{\partial}{\partial v_\alpha^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} \nu \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int v^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} \qquad (3.7.2)$$

$$+ \frac{\partial}{\partial \phi^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} f \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int \phi^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)}$$

$$+ \frac{\partial}{\partial \psi^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} \mathrm{D} \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int \psi^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} = 0$$

which was obtained earlier by N. Kishore and S.R. Singh [55].

To close the system of equations for the joint distribution functions some approximations are required. If we consider the collection of ionized particles i.e., in plasma turbulence case, it can be provided closure form easily by decomposing $F_2^{(1,2)}$ as $F_1^{(1)} F_1^{(2)}$. But such type of approximations can be possible if there is no interaction or correlation between two particles. If we decompose $F_2^{(1,2)}$ as

$$F_2^{(1,2)} = (1+\varepsilon) F_1^{(1)} F_1^{(2)} \qquad (3.7.3)$$

$$F_3^{(1,2,3)} = (1+\varepsilon)^2 F_1^{(1)} F_1^{(2)} F_1^{(3)} \qquad (3.7.4)$$

where ε is the correlation coefficient between the particles. If there is no correlation between the particles, ε will be zero and joint distribution function can be decomposed in usual way. Here, we are considering such type of approximation only to provide closed form of the equation i.e., to approximate two-point equation as one point equation. The transport equation for the joint distribution function of velocity, temperature, and concentration has been shown here to provide an advantageous basis for modeling the turbulent flows in presence of Coriolis force undergoing a first order reaction.

In this chapter, we have made an attempt for the modeling of various terms such as fluctuating pressure, viscosity and diffusivity in order to close the equation for joint distribution function of velocity, temperature and concentration. Since $F(v,\phi,\psi)$ contains all the statistical information about the velocity at each point, a turbulence model to determine the Reynolds stresses is not needed. However, since $F(v,\phi,\psi)$ is one point statistics, the length scale information is also not needed.

# CHAPTER-III

# PART-B

## TRANSPORT EQUATION FOR THE JOINT DISTRIBUTION FUNCTIONS OF CERTAIN VARIABLES IN CONVECTIVE DUSTY FLUID TURBULENT FLOW IN A ROTATING SYSTEM UNDER GOING A FIRST ORDER REACTION

### 3.8 Introduction

Now a day the two major and distinct areas of investigations in statistical mechanics are the kinetic theory of gases and the statistical theory of fluid mechanics. In the past several researchers discussed the distribution functions in the statistical theory of turbulence. A distribution function may be specialized with respect to a particular set of dimensions. Distribution functions may also feature non-isotropic temperatures, in which each term in the exponent is divided by a different temperature. Particle distribution functions are often used in plasma physics to describe wave-particle interactions and velocity-space instabilities. Distribution functions are also used in fluid mechanics, statistical mechanics and nuclear physics. The mathematical analog of a distribution is a measure; the time evolution of a measure on a phase space is the topic of study in dynamical systems. G. K. Batchelor [24] studied the theory of homogeneous turbulence. Lundgren [65] derived the transport equation for the distribution of velocity in turbulent flow. Bigler [25] gave the hypothesis that in turbulent flames, the thermo chemical quantities can be related locally to few scalars and considered the probability density function of these scalars. Kishore [47] studied the distributions functions in the statistical theory of MHD turbulence of an incompressible fluid. S. B. Pope [78] studied the statistical theory of turbulence flames. Also, Pope [79] derived the transport equation for the joint probability density function of velocity and scalars in turbulent flow. Kollman and Janica [56] derived the transport equation for the probability density function of a scalar in turbulent shear flow and considered a closure model based on gradient flux model. Kishore and Singh [55] derived the transport equation for the bivariate joint distribution function of velocity and temperature in turbulent flow. Also Kishore and Singh [50] have been derived the transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow. The Coriolis force helps to clarify the relation between angular momentum and rotational kinetic energy and how an inertial force can have a significant affect on the movement of a body and still without doing any work. On a rotating earth the Coriolis force acts to change the direction of a moving body to the right in the Northern Hemisphere and to the left in the Southern Hemisphere. Later, some researchers extended their works including Coriolis force. In the continuation, Azad and Sarker [2] studied the Statistical theory of certain distribution functions in MHD turbulence in a rotating system in presence of dust particles. Sarker and Azad [87] studied the decay of MHD turbulence before the final period for the case of multi-point and multi-time in a rotating system. Sarker and Azad[94], Azad and Sarker[4] deliberated the decay of temperature fluctuations in homogeneous turbulence before the final period for the case of multi- point and multi- time in a rotating system and dust particles. Azad and Sarker [5] discussed the decay of temperature fluctuations in MHD turbulence before the final period in a rotating system. Also, Azad et al [8], Sarker et al [99], Azad et al [6], Aziz et al [17], Azad et al [9] discussed the First Order Reactant in MHD turbulence before the final period of decay for the case of multi-point multi-time and multi- point single time considering rotating system and dust particles. Following the above researchers, Aziz et al [18, 20], Azad et al [10] had further studied the statistical theory of certain distribution functions in MHD turbulent flow for velocity and concentration considering first order reaction with a rotating system and dust particles. Aziz et al [19] extended their study for the first order reactant in MHD turbulence before the final period of decay for the case of multi-point and multi-time in a rotating system in presence of dust particle. Sarker, Bkar Pk. and Azad [101] studied the homogeneous dusty fluid turbulence in a first order reactant for the case of multi- point and multi-time prior to the final period of decay. Azad, Molla and Z. Rahman [15] studied the transport equatoin for the joint distribution function of velocity, temperature and concentration in convective tubulent flow in presence of dust particles. Molla, Azad and Z. Rahman [15] discussed the decay of temperature fluctuations in homogeneous turbulence before the final period in a rotating system. Bkar et al [30], Bkar et al [26, 29] premeditated the first-order reactant in homogeneous dusty fluid turbulence prior to the ultimate phase of decay for four-point correlation considering rotating system.

Bkar PK, et al [28, 27] had studied the decay of MHD turbulence before the final period for four- point correlation among dust particle and rotating system. M. H. U. Molla et al [67] studied the transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow in presence of Coriolis force.

In this chapter, we have been the derived transport equation for the joint distribution function of velocity temperature and concentration in convective turbulent flow in presence of dust particles undergoing a first order reaction in a rotating system. Various properties of the distribution function for velocity, temperature, concentration in convective turbulent flow in presence of dust particles have been discussed.

## METHODOLOGY

## 3.9 Basic equations

The equation of motion and field equations of temperature and concentration in a rotating system in presence of dust particles under going a first order reaction are shown by

$$\frac{\partial u_\alpha}{\partial t}+u_\alpha\frac{\partial u_\alpha}{\partial x_\beta}$$

$$=-\frac{\partial}{\partial x_\beta}\int_0^\infty\frac{1}{4\pi}\frac{\partial}{\partial x'_\beta}\left\{u_\alpha(x',t)\frac{\partial}{\partial x'_\beta}\cdot u_\alpha(x',t)\right\}\frac{dx'_\beta}{|x_\beta-x'_\beta|} \tag{3.9.1}$$

$$+\nu\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}u_\alpha-2\in_{m\alpha\beta}\Omega_m u_\alpha+f(u_\alpha-v_\alpha)$$

$$\frac{\partial\theta}{\partial t}+u_\alpha\frac{\partial\theta}{\partial x_\beta}=\gamma\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}\theta \tag{3.9.2}$$

$$\frac{\partial c}{\partial t}+u_\alpha\frac{\partial c}{\partial x_\beta}=D\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial x_\beta}c-Rc \tag{3.9.3}$$

with $\frac{\partial u_\alpha}{\partial x_\alpha}=\frac{\partial v_\alpha}{\partial x_\alpha}=0$

where, $u_\alpha(x,t)$= Component of turbulent velocity, $\theta(x,t)$ = Temperature fluctuation, c = Concentration of contaminants, $\nu$ = Kinematics viscosity, $f=\frac{KN}{\rho}$ = Dimension of frequency, N = Constant number of density of the dust particle, $\rho$ = Fluid density, D= Diffusive coefficient for contaminants, $\gamma = \frac{k_T}{\rho c_p}$ = Thermal diffusivity, $c_p$ =Specific heat at constant pressure, $v_\alpha$= Dust particle velocity, $k_T$ = Thermal conductivity, $\in_{m\alpha\beta}$ =Alternating tensor, $\Omega_m$ =Angular velocity of a uniform rotation, R=Constant reaction rate. Here u and x are vector quantities in the whole process.

## 3.10 Formulation of the problem

We consider the turbulence and the concentration fields are homogeneous. The fluid velocity u, temperature θ and concentration c are randomly distributed functions of position and time and satisfy their field equations. Different members of ensemble are subjected to different initial conditions and the aim is to find out a way by which we can determine the ensemble averages at the initial time. The present aim is to construct a joint distribution functions, study its properties and derive an equation for the transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow in a rotating system in presence of dust particles due to a first order reaction.

## 3.11 Continuity equation in terms of distribution functions

An infinite number of continuity equations can be derived for the convective turbulent flow and the continuity equations can be easily expressed in terms of distribution functions and are obtained directly by div u = 0.

$$\left\langle \frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}} \right\rangle = \left\langle \frac{\partial}{\partial x_\alpha^{(1)}} u_\alpha^{(1)} \int F_1^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \left\langle u_\alpha^{(1)} \int F_1^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \int \left\langle u_\alpha^{(1)} \right\rangle \left\langle F_1^{(1)} \right\rangle dv^{(1)} d\phi^{(1)} d\psi^{(1)}$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_1^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)}$$

$$= \int \frac{\partial F_1^{(1)}}{\partial x_\alpha^{(1)}} v_\alpha^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} = 0 \qquad (3.11.1)$$

and similarly $\int \frac{\partial F_1^{(1)}}{\partial x_\alpha^{(1)}} \phi_\alpha^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} = 0$ (3.11.2)

which are the first order continuity equations in which only one point distribution function is involved.

For second-order continuity equations, if we multiply the continuity equation by $\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\,\delta\left(c^{(2)} - \psi^{(2)}\right)$

and if we take the ensemble average, we obtain

$$0 = \left\langle\ \delta\left(u^{(2)} - v^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\,\delta\left(c^{(2)} - \psi^{(2)}\right)\frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}}\right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}}\left\langle\ \delta\left(u^{(2)} - v^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\,\delta\left(c^{(2)} - \psi^{(2)}\right)u_\alpha^{(1)}\right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \int \left\langle u_\alpha^{(1)}\ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\,\delta\left(c^{(1)} - \psi^{(1)}\right)\right\rangle\left\langle\ \delta\left(u^{(2)} - v^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\,\delta\left(c^{(2)} - \psi^{(2)}\right)\right\rangle$$

$$= \frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_2^{(1,2)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \qquad (3.11.3)$$

and similarly

$$0 = \frac{\partial}{\partial x_\alpha^{(1)}} \int \phi_\alpha^{(1)} F_2^{(1,2)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \qquad (3.11.4)$$

The Nth-order continuity equations are

$$0 = \frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_N^{(1,2,---N)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \qquad (3.11.5)$$

and

$$0 = \frac{\partial}{\partial x_\alpha^{(1)}} \int \phi_\alpha^{(1)} F_N^{(1,2,---N)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} \qquad (3.11.6)$$

The continuity equations are symmetric in their arguments i.e.

$$\frac{\partial}{\partial x_\alpha^{(r)}} \int \left(v_\alpha^{(r)} F_N^{(1,2,---s---r---N)} dv^{(r)} d\phi^{(r)} d\psi^{(r)}\right) = \frac{\partial}{\partial x_\alpha^{(s)}} \int \left(v_\alpha^{(s)} F_N^{(1,2,---r---s---N)} dv^{(s)} d\phi^{(s)} d\psi^{(s)}\right) \qquad (3.11.7)$$

Since, the divergence property is an important property and it is easily verified by the use of the property of distribution function as

$$\frac{\partial}{\partial x_\alpha^{(1)}}\int v_\alpha^{(1)} F_1^{(1)} dv^{(1)} d\phi^{(1)} d\psi^{(1)} = \frac{\partial}{\partial x_\alpha^{(1)}}\left\langle u_\alpha^{(1)}\right\rangle = \left\langle \frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}}\right\rangle = 0 \qquad (3.11.8)$$

and all the properties of the distribution function obtained in section (3.4.1) can also be easily verified.

## 3.12 Equations for the evolution of joint distribution functions

This, in fact is done by making use of the definitions of the constructed distribution functions, the transport equation for $F(v,\phi,\psi,x,t)$ is obtained from the definition of $F$ and from the transport equations (3.9.1), (3.9.2), (3.9.3). Differentiating equation (3.4.1) we get,

$$\frac{\partial}{\partial t}F_1^{(1)} = \frac{\partial}{\partial t}\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle = \left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle + \left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$
$$=\left\langle -\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial u_\alpha^{(1)}}{\partial t}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle + \left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial \theta^{(1)}}{\partial t}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\frac{\partial c^{(1)}}{\partial t}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle \qquad (3.12.1)$$

Using equation (3.9.1), (3.9.2) and (3.9.3) in the equation (3.12.1) we get,

$$\frac{\partial}{\partial t}F_1^{(1)} = \left\langle -\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\begin{bmatrix} -u_\alpha^{(1)}\dfrac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}} - \dfrac{\partial}{\partial x_\beta^{(1)}}\displaystyle\int \dfrac{1}{4\pi}\dfrac{\partial}{\partial x_\beta^{(2)}}\left\{u_\alpha^{(2)}\dfrac{\partial}{\partial x_\beta^{(2)}}\cdot u_\alpha^{(2)}\right\} \\ \times\dfrac{dx_\beta^{(2)}}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|} + v\dfrac{\partial}{\partial x_\beta^{(1)}}\dfrac{\partial}{\partial x_\beta^{(1)}}u_\alpha^{(1)} - 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)} \\ + f\left(\mathrm{u}_\alpha^{(1)}-v_\alpha^{(1)}\right) \end{bmatrix}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\left\{-u_\alpha^{(1)}\frac{\partial \theta^{(1)}}{\partial x_\beta^{(1)}} + \gamma\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\theta^{(1)}\right\}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\left\{-u_\alpha^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}} + \mathrm{D}\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}c^{(1)} - Rc^{(1)}\right\}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$\therefore \frac{\partial F_1^{(1)}}{\partial t} + \left\langle -\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial \theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle +$$
$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)u_\alpha^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$
$$+\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\left[-\frac{\partial}{\partial x_\beta^{(1)}}\left\{\frac{1}{4\pi}\int \frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}u_\alpha^{(2)}u_\alpha^{(2)}\frac{dx_\beta^{(2)}}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}\right\}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right]\right\rangle$$
$$+\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\nu\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}u_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)f\left(\mathrm{u}_\alpha^{(1)}-v_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\gamma\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\right)\theta^{(1)}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\mathrm{D}\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\right)c^{(1)}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle \tag{3.12.2}$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\mathrm{R}c^{(1)}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle = 0$$

Various terms in the above equation can be simplified as that they may be expressed in terms of one point and two point distribution functions. The 2nd, 3rd and 4th terms on the left hand side of the above equation are simplified in a similar fashion and take the forms as follows

$$\left\langle -\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$=\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle \quad \left[\because \frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}}=-1\right] \tag{3.12.3}$$

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial \theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$=\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle \tag{3.12.4}$$

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)u_\alpha^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$
$$=\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle \tag{3.12.5}$$

Adding equation (3.12.3), (3.12.4) and (3.12.5) we get,

$$\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)u_\alpha^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=\frac{\partial}{\partial x_\beta^{(1)}}\left\langle u_\alpha^{(1)}\left(\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right)\right\rangle$$
$$=\frac{\partial}{\partial x_\beta^{(1)}}v_\alpha^{(1)}F_1^{(1)} \qquad \left[\textit{Applying the properties of distribution function}\right] \tag{3.12.6}$$
$$=v_\alpha^{(1)}\frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}}$$

We reduce the 5th term on left hand side of equation (3.12.2),

$$\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\left[-\frac{\partial}{\partial x_\beta^{(1)}}\left\{\frac{1}{4\pi}\int \frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}u_\alpha^{(2)}u_\alpha^{(2)}\frac{dx_\beta^{(2)}}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}\right\}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right]\right\rangle$$
$$=\frac{\partial}{\partial v_\alpha^{(1)}}\left[-\frac{1}{4\pi}\int\left(\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}\right)\right]\left(v_\alpha^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\right)^2 F_2^{(1,2)}dx^{(2)}dv^{(2)}\,d\phi^{(2)}\,d\psi^{(2)} \tag{3.12.7}$$

We reduce the 6th term on left hand side of equation (3.12.2),

$$\left\langle \delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\nu\left(\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}u_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$=\left\langle \nu\frac{\partial}{\partial v_\alpha^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}u_\alpha^{(1)}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$=\nu\frac{\partial}{\partial v_\alpha^{(1)}}\left\langle \frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}u_\alpha^{(1)}\left[\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(1)}-v^{(1)}\right)\right]\right\rangle$$
$$=\nu\frac{\partial}{\partial v_\alpha^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\left\langle u_\alpha^{(1)}\left[\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(1)}-v^{(1)}\right)\right]\right\rangle$$

$$= \nu \frac{\partial}{\partial v_\alpha^{(1)}} \underset{x^{(2)}\to x^{(1)}}{Lim} \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \left\langle \begin{array}{l} \int u_\alpha^{(2)} \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \delta\left(u^{(2)} - v^{(2)}\right) \\ \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(1)} - v^{(1)}\right) dv^{(2)} d\phi^{(2)} d\psi^{(2)} \end{array} \right\rangle$$
$$= \frac{\partial}{\partial v_\alpha^{(1)}} \underset{x^{(2)}\to x^{(1)}}{Lim} \nu \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int v^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} \tag{3.12.8}$$

We reduce the 7th term on left hand side of equation (3.12.2),

$$\left\langle -\delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \right\rangle$$
$$= \left\langle 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}} \left[ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right] \right\rangle$$
$$= 2 \in_{m\alpha\beta} \Omega_m \frac{\partial}{\partial v_\alpha^{(1)}} \left\langle u_\alpha^{(1)} \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle \tag{3.12.9}$$
$$= 2 \in_{m\alpha\beta} \Omega_m \frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}} \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle$$
$$= 2 \in_{m\alpha\beta} \Omega_m F_1^{(1)}$$

We reduce the 8th term on left hand side of equation (3.12.2),

$$\left\langle \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) f\left(\mathrm{u}_\alpha^{(1)} - v_\alpha^{(1)}\right) \frac{\partial}{\partial v_\alpha^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \right\rangle$$
$$= \left\langle f\left(\mathrm{u}_\alpha^{(1)} - v_\alpha^{(1)}\right) \frac{\partial}{\partial v_\alpha^{(1)}} \left[ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right] \right\rangle \tag{3.12.10}$$
$$= f\left(\mathrm{u}_\alpha^{(1)} - v_\alpha^{(1)}\right) \frac{\partial}{\partial v_\alpha^{(1)}} \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle = f\left(\mathrm{u}_\alpha^{(1)} - v_\alpha^{(1)}\right) \frac{\partial}{\partial v_\alpha^{(1)}} F_1^{(1)}$$

Similarly, 9th ,10th and 11th terms of left hand side of (3.12.2) can be simplified as follows

$$\left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \gamma \left( \frac{\partial}{\partial x_\beta^{(1)}} \frac{\partial}{\partial x_\beta^{(1)}} \right) \theta^{(1)} \frac{\partial}{\partial \phi^{(1)}} \delta\left(\theta^{(1)} - \phi^{(1)}\right) \right\rangle$$
$$= \frac{\partial}{\partial \phi^{(1)}} \underset{x^{(2)}\to x^{(1)}}{Lim} \gamma \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int \phi^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} \tag{3.12.11}$$

$$\left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \mathrm{D} \left( \frac{\partial}{\partial x_\beta^{(1)}} \frac{\partial}{\partial x_\beta^{(1)}} \right) c^{(1)} \frac{\partial}{\partial \psi^{(1)}} \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle$$

$$= \frac{\partial}{\partial \psi^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} \mathrm{D} \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int \psi^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} \tag{3.12.12}$$

and

$$\left\langle - \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) Rc^{(1)} \frac{\partial}{\partial \psi^{(1)}} \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle = R\psi^{(1)} \frac{\partial}{\partial \psi^{(1)}} F_1^{(1)} \tag{3.12.13}$$

Substituting the results (3.12.6)-(3.12.13) in equation (3.12.2), we get the transport equation for one point distribution function $F_1^{(1)}(v,\phi,\psi)$ in turbulent flow in a rotating system

$$\begin{aligned}
&\frac{\partial F_1^{(1)}}{\partial t} + v_\alpha^{(1)} \frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}} + \frac{\partial}{\partial v_\alpha^{(1)}} \left[ -\frac{1}{4\pi} \int \left( \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\left| x_\beta^{(1)} - x_\beta^{(2)} \right|} \right) \right] \left( v_\alpha^{(2)} \frac{\partial}{\partial x_\beta^{(2)}} \right)^2 F_2^{(1,2)} dx^{(2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} \\
&+ \frac{\partial}{\partial v_\alpha^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} \nu \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int v^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} + \frac{\partial}{\partial \phi^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} \gamma \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int \phi^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} \\
&+ \frac{\partial}{\partial \psi^{(1)}} \underset{x^{(2)} \to x^{(1)}}{Lim} \mathrm{D} \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\partial x_\beta^{(2)}} \int \psi^{(2)} F_2^{(1,2)} dv^{(2)} d\phi^{(2)} d\psi^{(2)} + 2 \in_{m\alpha\beta} \Omega_m F_1^{(1)} \qquad (3.12.14) \\
&+ f\left(\mathrm{u}_\alpha^{(1)} - v_\alpha^{(1)}\right) \frac{\partial}{\partial v_\alpha^{(1)}} F_1^{(1)} - R\psi^{(1)} \frac{\partial}{\partial \psi^{(1)}} F_1^{(1)} = 0
\end{aligned}$$

Similarly, a transport equation for two-point distribution function $F_2^{(1,2)}$ in turbulent flow in presence of dust particles can be derived by differentiating equation (3.4.2) and using equation (3.9.1), (3.9.2), (3.9.3) and simplifying in the same manner which is

$$\begin{aligned}
&\frac{\partial F_2^{(1,2)}}{\partial t} + \left( v_\alpha^{(1)} \frac{\partial}{\partial x_\beta^{(1)}} + v_\alpha^{(2)} \frac{\partial}{\partial x_\beta^{(2)}} \right) F_2^{(1,2)} + \frac{\partial}{\partial v_\alpha^{(1)}} \left[ \begin{array}{l} -\frac{1}{4\pi} \int \left( \frac{\partial}{\partial x_\beta^{(1)}} \frac{\partial}{\left| x_\beta^{(1)} - x_\beta^{(3)} \right|} \right) \left( v_\alpha^{(3)} \frac{\partial}{\partial x_\beta^{(3)}} \right)^2 \\ \times F_3^{(1,2,3)} dx^{(3)} dv^{(3)} d\phi^{(3)} d\psi^{(3)} \end{array} \right] \\
&+ \frac{\partial}{\partial v_\alpha^{(2)}} \left[ -\frac{1}{4\pi} \int \left( \frac{\partial}{\partial x_\beta^{(2)}} \frac{\partial}{\left| x_\beta^{(2)} - x_\beta^{(3)} \right|} \right) \left( v_\alpha^{(3)} \frac{\partial}{\partial x_\beta^{(3)}} \right)^2 F_3^{(1,2,3)} dx^{(3)} dv^{(3)} d\phi^{(3)} d\psi^{(3)} \right] \\
&+ \nu \left( \frac{\partial}{\partial v_\alpha^{(1)}} \underset{x^{(3)} \to x^{(1)}}{Lim} + \frac{\partial}{\partial v_\alpha^{(2)}} \underset{x^{(3)} \to x^{(2)}}{Lim} \right) \frac{\partial}{\partial x_\beta^{(3)}} \frac{\partial}{\partial x_\beta^{(3)}} \int v^{(3)} F_3^{(1,2,3)} dv^{(3)} d\phi^{(3)} d\psi^{(3)} \\
&+ \gamma \left( \frac{\partial}{\partial \phi^{(1)}} \underset{x^{(3)} \to x^{(1)}}{Lim} + \frac{\partial}{\partial \phi^{(2)}} \underset{x^{(3)} \to x^{(2)}}{Lim} \right) \frac{\partial}{\partial x_\beta^{(3)}} \frac{\partial}{\partial x_\beta^{(3)}} \int \phi^{(3)} F_3^{(1,2,3)} dv^{(3)} d\phi^{(3)} d\psi^{(3)} \\
&+ \mathrm{D} \left( \frac{\partial}{\partial \psi^{(1)}} \underset{x^{(3)} \to x^{(1)}}{Lim} + \frac{\partial}{\partial \psi^{(2)}} \underset{x^{(3)} \to x^{(2)}}{Lim} \right) \frac{\partial}{\partial x_\beta^{(3)}} \frac{\partial}{\partial x_\beta^{(3)}} \int \psi^{(3)} F_3^{(1,2,3)} dv^{(3)} d\phi^{(3)} d\psi^{(3)}
\end{aligned}$$

$$+2\in_{m\alpha\beta}\Omega_m F_2^{(1,2)}+f\left(u_\alpha^{(1)}-v_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(2)}}F_2^{(1,2)}-R\psi^{(2)}\frac{\partial}{\partial\psi^{(2)}}F_2^{(1,2)}=0 \qquad (3.12.15)$$

Continuing this way, we can derive the equations for evolution of $F_3^{(1,2,3)}, F_4^{(1,2,3,4)}$ and so on. Logically, it is possible to have an equation for every $F_n$ $(n \text{ is an integer})$ but the system of equations so obtained is not closed. It seems that certain approximations will be required for closing the system.

## 3.13 Results and Discussion

If the reaction rate R=0, the transport equation for one point joint distribution function $F_1^{(1)}(v,\phi,\psi)$ in turbulent flow undergoing a first order reaction, equation (3.12.14) becomes

$$\frac{\partial F_1^{(1)}}{\partial t}+v_\alpha^{(1)}\frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}}+\frac{\partial}{\partial v_\alpha^{(1)}}\left[-\frac{1}{4\pi}\int\left(\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}\right)\right]\left(v_\alpha^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\right)^2F_2^{(1,2)}dx^{(2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}$$
$$+\frac{\partial}{\partial v_\alpha^{(1)}}\lim_{x^{(2)}\to x^{(1)}}\nu\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int v^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}+\frac{\partial}{\partial\phi^{(1)}}\lim_{x^{(2)}\to x^{(1)}}\gamma\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int\phi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}$$
$$+\frac{\partial}{\partial\psi^{(1)}}\lim_{x^{(2)}\to x^{(1)}}D\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int\psi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}+2\in_{m\alpha\beta}\Omega_m F_1^{(1)}+f\left(u_\alpha^{(1)}-v_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}F_1^{(1)}=0,$$
$$(3.13.1)$$

which was obtained earlier by M.H.U.Molla [67]

If the system is non rotating and the fluid is clean then $\Omega_m=0$ & $f=0$ and the transport equation (3.12.14) for one point join distribution function $F_1^{(1)}(v,\phi,\psi)$ in turbulent flow becomes

$$\frac{\partial F_1^{(1)}}{\partial t}+v_\alpha^{(1)}\frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}}+\frac{\partial}{\partial v_\alpha^{(1)}}\left[-\frac{1}{4\pi}\int\left(\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\left|x_\beta^{(1)}-x_\beta^{(2)}\right|}\right)\right]\left(v_\alpha^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\right)^2F_2^{(1,2)}dx^{(2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}$$
$$+\frac{\partial}{\partial v_\alpha^{(1)}}\lim_{x^{(2)}\to x^{(1)}}\nu\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int v^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}$$
$$+\frac{\partial}{\partial\phi^{(1)}}\lim_{x^{(2)}\to x^{(1)}}\gamma\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int\phi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}$$
$$+\frac{\partial}{\partial\psi^{(1)}}\lim_{x^{(2)}\to x^{(1)}}D\frac{\partial}{\partial x_\beta^{(2)}}\frac{\partial}{\partial x_\beta^{(2)}}\int\psi^{(2)}F_2^{(1,2)}dv^{(2)}d\phi^{(2)}d\psi^{(2)}=0, \qquad (3.13.2)$$

which was obtained earlier by Kishore and Singh [49]. For closing the transport equations for the joint distribution functions, some approximations are required. If the particles are ionized i.e., in plasma turbulence case, it can be provided closure form easily by decomposing $F_2^{(1,2)}$ *as* $F_1^{(1)}F_1^{(2)}$ But such type of approximations can be possible if there is no interaction or correlation between two particles. If we decompose $F_2^{(1,2)}$ as

$$F_2^{(1,2)} = (1+\varepsilon)F_1^{(1)}F_1^{(2)} \tag{3.13.3}$$

$$F_3^{(1,2,3)} = (1+\varepsilon)^2 F_1^{(1)}F_1^{(2)}F_1^{(3)} \tag{3.13.4}$$

where ε is the correlation coefficient between the particles. If there is no correlation between the particles, ε will be zero and joint distribution function can be decomposed in usual way. Here, we are considering such type of approximation only to provide closed the form of the equation i.e., to approximate two-point equation as one point equation. The transport equation for the joint distribution functions of velocity, temperature, and concentration have been shown here to provide an advantageous basis for modeling the turbulent flows in presence of dust particles and a rotating system due to a first order reaction.

# CHAPTER-IV

## STATISTICAL THEORY OF CERTAIN VARIABLES FOR THREE- POINT DISTRIBUTION FUNCTIONS IN MHD TURBULENT FLOW IN A ROTATING SYSTEM IN PRESENCE OF DUST PARTICLES

### 4.1 Introduction

At present, two major and distinct areas of investigations in non-equilibrium statistical mechanics are the kinetic theory of gases and statistical theory of fluid mechanics. In molecular kinetic theory in physics, a particle's distribution function is a function of several variables. Particle distribution functions are often used in plasma physics to describe wave-particle interactions and velocity-space instabilities. Distribution functions are also used in fluid mechanics, statistical mechanics and nuclear physics. A distribution function may be specialized with respect to a particular set of dimensions. Distribution functions may also feature non-isotropic temperatures, in which each term in the exponent is divided by a different temperature. The mathematical analogy of a distribution is a measure, the time evolution of a measure on a phase space is the topic of study in dynamical systems. Various analytical theories in the statistical theory of turbulence have been discussed in the past by Hopf [43], Kraichanan [58], Edward [36] and Herring [41]. Further Lundgren [65] derived a hierarchy of coupled equations for multi-point turbulence velocity distribution functions, which resemble with BBGKY hierarchy of equations of Ta-You [107] in the kinetic theory of gasses. Bigler [25] gave the hypothesis that in turbulent flames, the thermo chemical quantities can be related locally to few scalars and considered the probability density function of these scalars. Kishore [47] studied the Distributions functions in the statistical theory of MHD turbulence of an incompressible fluid. Pope [79] derived the transport equation for the joint probability density function of velocity and scalars in turbulent flow. Kollman and Janicka [56] derived the transport equation for the probability density function of a scalar in turbulent shear flow and considered a closure model based on gradient – flux model. Kishore and Singh [49] derived the transport equation for the bivariate joint distribution function of velocity and temperature in turbulent flow. Also Kishore and Singh [50] have been derived the transport equation for the joint distribution function of velocity, temperature and concentration in convective turbulent flow. Dixit and Upadhyay [35] considered the distribution functions in the statistical theory of MHD turbulence of an incompressible fluid in the presence of the coriolis force. Sarker and Kishore [96] discussed the distribution functions in the statistical theory of convective MHD turbulence of an incompressible fluid. Also Sarker and Kishore [98] studied the distribution functions in the statistical theory of convective MHD turbulence of mixture of a miscible incompressible fluid. Sarker and Islam [100] studied the Distribution functions in the statistical theory of convective MHD turbulence of an incompressible fluid in a rotating system. Azad and Sarker [2] discussed Statistical theory of certain distribution functions in MHD turbulence in a rotating system in presence of dust particles. Islam and Sarker [45] studied distribution functions in the statistical theory of MHD turbulence for velocity and concentration undergoing a first order reaction.

The above researchers have done their research for two- point distribution functions. But in this chapter, we have studied the statistical theory for three- point distribution function of certain variables in MHD turbulence in a rotating system in presence of dust particles. At this stage, one is met with the difficulty that the N-point distribution function depends upon the (N+1)-point distribution function and thus result is an unclosed system. This so-called closer problem is encountered in turbulence, Kinetic theory and other non-linear system.

In present research, the main purpose is to study the statistical theory of three- point distribution function for simultaneous velocity, magnetic, temperature and concentration fields in MHD turbulence in a rotating system in presence of dust particles. Finally, the transport equations for evolution of distribution functions have been derived and various properties of the distribution function have been discussed.

## METHODOLOGY

## 4.2 Basic Equations:

The equations of motion and continuity for viscous incompressible dusty fluid MHD turbulent flow, the diffusion equations for the temperature and concentration in a rotating system are given by

$$\frac{\partial u_\alpha}{\partial t}+\frac{\partial}{\partial x_\beta}\left(u_\alpha u_\beta - h_\alpha h_\beta\right)=-\frac{\partial w}{\partial x_\alpha}+\nu\nabla^2 u_\alpha - 2\in_{m\alpha\beta}\Omega_m u_\alpha + f\left(u_\alpha - v_\alpha\right) \tag{4.2.1}$$

$$\frac{\partial h_\alpha}{\partial t}+\frac{\partial}{\partial x_\beta}\left(h_\alpha u_\beta - u_\alpha h_\beta\right)=\lambda\nabla^2 h_\alpha\ , \tag{4.2.2}$$

$$\frac{\partial \theta}{\partial t}+u_\beta\frac{\partial \theta}{\partial x_\beta}=\gamma\nabla^2\theta\ , \tag{4.2.3}$$

$$\frac{\partial c}{\partial t}+u_\beta\frac{\partial c}{\partial x_\beta}=D\nabla^2 c \tag{4.2.4}$$

with $\frac{\partial u_\alpha}{\partial x_\alpha}=\frac{\partial v_\alpha}{\partial x_\alpha}=\frac{\partial h_\alpha}{\partial x_\alpha}=0$ (4.2.5)

where

$u_\alpha(x,t)$, α – component of turbulent velocity

$h_\alpha(x,t)$, α – component of magnetic field

$\theta(x,t)$, temperature fluctuation

c, concentration of contaminants

$v_\alpha$, dust particle velocity

$\in_{m\alpha\beta}$, alternating tensor

$f=\frac{KN}{\rho}$, dimension of frequency

N, constant number of density of the dust particle

$w(\hat{x},t)=P/\rho+\frac{1}{2}\left|\vec{h}\right|^2+\frac{1}{2}\left|\hat{\Omega}\times\hat{x}\right|^2$, total pressure

$P(\hat{x},t)$, hydrodynamic pressure

ρ, fluid density

Ω, angular velocity of a uniform rotation

ν, Kinetic viscosity

$\lambda=(4\pi\mu\sigma)^{-1}$, magnetic diffusivity

$\gamma=\frac{k_T}{\rho c_p}$, thermal diffusivity,

$c_p$, specific heat at constant pressure,

$k_T$, thermal conductivity

σ, electrical conductivity

μ, magnetic permeability

D, diffusive co-efficient for contaminants.

The repeated suffices are assumed over the values 1, 2 and 3 and unrepeated suffices may take any of these values. Here u, h and x are vector quantities in the whole process.

The total pressure w which, occurs in equation (4.2.1) may be eliminated with the help of the equation obtained by taking the divergence of equation (4.2.1)

$$\nabla^2 w = -\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(u_\alpha u_\beta - h_\alpha h_\beta\right) = -\left[\ \frac{\partial u_\alpha}{\partial x_\beta}\frac{\partial u_\beta}{\partial x_\alpha} - \frac{\partial h_\alpha}{\partial x_\beta}\frac{\partial h_\beta}{\partial x_\alpha}\ \right] \tag{4.2.6}$$

In a conducting infinite fluid only the particular solution of the Equation (4.2.6) is related, so that

$$w = \frac{1}{4\pi}\int\left[\ \frac{\partial u'_\alpha}{\partial x'_\beta}\frac{\partial u'_\beta}{\partial x'_\alpha} - \frac{\partial h'_\alpha}{\partial x'_\beta}\frac{\partial h'_\beta}{\partial x'_\alpha}\ \right]\frac{\partial \bar{x}'}{\left|\bar{x}' - \bar{x}\right|} \tag{4.2.7}$$

Hence equation (4.2.1) – (4.2.4) becomes

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}\left(u_\alpha u_\beta - h_\alpha h_\beta\right) = -\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha}\int\left[\ \frac{\partial u'_\alpha}{\partial x'_\beta}\frac{\partial u'_\beta}{\partial x'_\alpha} - \frac{\partial h'_\alpha}{\partial x'_\beta}\frac{\partial h'_\beta}{\partial x'_\alpha}\ \right]\frac{d\bar{x}'}{\left|\bar{x}' - \bar{x}\right|} + \nu\nabla^2 u_\alpha$$
$$-2 \in_{m\alpha\beta} \Omega_m u_\alpha + f\left(u_\alpha - v_\alpha\right) \tag{4.2.8}$$

$$\frac{\partial h_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}\left(h_\alpha u_\beta - u_\alpha h_\beta\right) = \lambda\nabla^2 h_\alpha \tag{4.2.9}$$

$$\frac{\partial \theta}{\partial t} + u_\beta\frac{\partial \theta}{\partial x_\beta} = \gamma\nabla^2\theta \tag{4.2.10}$$

$$\frac{\partial c}{\partial t} + u_\beta\frac{\partial c}{\partial x_\beta} = D\nabla^2 c \tag{4.2.11}$$

## 4.3 Formulation of the Problem

We consider the turbulence and the concentration fields are homogeneous, the chemical reaction and the local mass transfer have no effect on the velocity field and the reaction rate and the diffusivity are constant. We also consider a large ensemble of identical fluids in which each member is an infinite incompressible reacting and heat conducting fluid in turbulent state. The fluid velocity u, Alfven velocity h, temperature θ and concentration C are randomly distributed functions of position and time and satisfy their field. Different members of ensemble are subjected to different initial conditions and the aim is to find out a way by which we can determine the ensemble averages at the initial time.

Certain microscopic properties of conducting fluids such as total energy, total pressure, stress tensor which are nothing but ensemble averages at a particular time can be determined with the help of the bivariate distribution functions (defined as the averaged distribution functions with the help of Dirac delta-functions). The present aim is to construct the distribution functions, study its properties and derive an equation for its evolution of this distribution functions.

## 4.4 Distribution Function in MHD Turbulence and Their Properties

In MHD turbulence, we may consider the fluid velocity u, Alfven velocity h, temperature θ and concentration c at each point of the flow field. Then corresponding to each point of the flow field, we have four measurable characteristics. We represent the four variables by v, g, $\phi$ and ψ and denote the pairs of these variables at the points

$\overline{x}^{(1)},\overline{x}^{(2)},------,\overline{x}^{(n)}$ as $\left(\overline{v}^{(1)},\overline{g}^{(1)},\phi^{(1)},\psi^{(1)}\right)$,

$\left(\overline{v}^{(2)},\overline{g}^{(2)},\phi^{(2)},\psi^{(2)}\right)$,$------\left(\overline{v}^{(n)},\overline{g}^{(n)},\phi^{(n)},\psi^{(n)}\right)$ at a fixed instant of time.

It is possible that the same pair may be occur more than once; therefore, we simplify the problem by an assumption that the distribution is discrete (in the sense that no pairs occur more than once). Symbolically we can express the bivariate distribution as

$$\left\{ \left(\overline{v}^{(1)},\overline{g}^{(1)},\phi^{(1)},\psi^{(1)}\right), \left(\overline{v}^{(2)},\overline{g}^{(2)},\phi^{(2)},\psi^{(2)}\right), ------\left(\overline{v}^{(n)},\overline{g}^{(n)},\phi^{(n)},\psi^{(n)}\right) \right\}$$

Instead of considering discrete points in the flow field, if we consider the continuous distribution of the variables $\bar{v}, \bar{g}, \phi$ and ψ over the entire flow field, statistically behavior of the fluid may be described by the distribution function $F(\bar{v}, \bar{g}, \phi, \psi)$ which is normalized so that

$$\int F(\bar{v}, \bar{g}, \phi, \psi) d\bar{v} d\bar{g} d\phi d\psi = 1$$

where the integration ranges over all the possible values of v, g, $\phi$ and ψ. We shall make use of the same normalization condition for the discrete distributions also.

The distribution functions of the above quantities can be defined in terms of Dirac delta function.

The one-point distribution function $F_1^{(1)}\left(v^{(1)}, g^{(1)}, \phi^{(1)}, \psi^{(1)}\right)$, defined so that $F_1^{(1)}\left(v^{(1)}, g^{(1)}, \phi^{(1)}, \psi^{(1)}\right) dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)}$ is the probability that the fluid velocity, Alfven velocity, temperature and concentration at a time t are in the element $dv^{(1)}$ about $v^{(1)}$, $dg^{(1)}$ about $g^{(1)}$, $d\phi^{(1)}$ about $\phi^{(1)}$ and $d\psi^{(1)}$ about $\psi^{(1)}$ respectively and is given by

$$F_1^{(1)}\left(v^{(1)}, g^{(1)}, \phi^{(1)}, \psi^{(1)}\right) = \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle \tag{4.4.1}$$

where δ is the Dirac delta-function defined as

$$\int \delta(\bar{u} - \bar{v}) d\bar{v} = \begin{cases} 1 & \text{at the point } \bar{u} = \bar{v} \\ 0 & \text{elsewhere} \end{cases}$$

Two-point distribution function is given by

$$F_2^{(1,2)} = \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(2)} - v^{(2)}\right) \delta\left(h^{(2)} - g^{(2)}\right) \right.$$
$$\left. \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \right\rangle \tag{4.4.2}$$

and three point distribution function is given by

$$F_3^{(1,2,3)} = \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(2)} - v^{(2)}\right) \delta\left(h^{(2)} - g^{(2)}\right) \right.$$
$$\left. \times \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \delta\left(u^{(3)} - v^{(3)}\right) \delta\left(h^{(3)} - g^{(3)}\right) \delta\left(\theta^{(3)} - \phi^{(3)}\right) \delta\left(c^{(3)} - \psi^{(3)}\right) \right\rangle \tag{4.4.3}$$

Similarly, we can define an infinite numbers of multi-point distribution functions

$F_4^{(1,2,3,4)}$, $F_5^{(1,2,3,4,5)}$ and so on. The following properties of the constructed distribution functions can be deduced from the above definitions:

## (A) Reduction Properties:

Integration with respect to pair of variables at one-point, lowers the order of distribution function by one. For example,

$$\int F_1^{(1)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} = 1 \ ,$$

$$\int F_2^{(1,2)} dv^{(2)} dg^{(2)} d\phi^{(2)} d\psi^{(2)} = F_1^{(1)} \ ,$$

$$\int F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} = F_2^{(1,2)}$$

and so on. Also the integration with respect to any one of the variables, reduces the number of Delta-functions from the distribution function by one as

$$\int F_1^{(1)} dv^{(1)} = \left\langle \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle \ ,$$

$$\int F_1^{(1)} dg^{(1)} = \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle \ ,$$

$$\int F_1^{(1)} d\phi^{(1)} = \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \right\rangle ,$$

and

$$\int F_2^{(1,2)} dv^{(2)} = \left\langle \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(h^{(2)} - g^{(2)}\right) \right.$$
$$\left. \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \right\rangle$$

## (B) Separation Properties:

If two points are far apart from each other in the flow field, the pairs of variables at these points are statistically independent of each other i.e.,

$$\lim_{\left|\vec{x}^{(2)} \to \bar{x}^{(1)}\right| \to \infty} F_2^{(1,2)} = F_1^{(1)} F_1^{(2)}$$

and similarly,

$$\lim_{\left|\bar{x}^{(3)} \to \bar{x}^{(2)}\right| \to \infty} F_3^{(1,2,3)} = F_2^{(1,2)} F_1^{(3)} \quad \text{etc.}$$

## (C) Co-incidence Properties:

When two points coincide in the flow field, the components at these points should be obviously the same that is $F_2^{(1,2)}$ must be zero.

Thus $\bar{v}^{(2)} = \bar{v}^{(1)}$, $g^{(2)} = g^{(1)}$, $\phi^{(2)} = \phi^{(1)}$ and $\psi^{(2)} = \psi^{(1)}$, but $F_2^{(1,2)}$ must also have the property.

$$\int F_2^{(1,2)} dv^{(2)} dg^{(2)} d\phi^{(2)} d\psi^{(2)} = F_1^{(1)}$$

and hence it follows that

$$\lim_{\left|\bar{x}^{(2)} \to \bar{x}^{(1)}\right| \to \infty} \int F_2^{(1,2)} = F_1^{(1)} \delta\left(v^{(2)} - v^{(1)}\right) \delta\left(g^{(2)} - g^{(1)}\right) \delta\left(\phi^{(2)} - \phi^{(1)}\right) \delta\left(\psi^{(2)} - \psi^{(1)}\right)$$

Similarly,

$$\lim_{\left|\bar{x}^{(3)} \to \bar{x}^{(2)}\right| \to \infty} \int F_3^{(1,2,3)} = F_2^{(1,2)} \delta\left(v^{(3)} - v^{(1)}\right) \delta\left(g^{(3)} - g^{(1)}\right) \delta\left(\phi^{(3)} - \phi^{(1)}\right) \delta\left(\psi^{(3)} - \psi^{(1)}\right) \text{ etc.}$$

## (D) Symmetric Conditions:

$$F_n^{(1,2,r,------s,-----n)} = F_n^{(1,2,-----s,---r,---n)}.$$

## (E) Incompressibility Conditions:

(i) $$\int \frac{\partial F_n^{(1,2,---n)}}{\partial x_\alpha^{(r)}} v_\alpha^{(r)} d\bar{v}^{(r)} d\bar{h}^{(r)} = 0$$

(ii) $$\int \frac{\partial F_n^{(1,2,---n)}}{\partial x_\alpha^{(r)}} h_\alpha^{(r)} d\bar{v}^{(r)} d\bar{h}^{(r)} = 0$$

## 4.5 Continuity Equation in Terms of Distribution Functions

The continuity equations can be easily expressed in terms of distribution functions. An infinite number of continuity equations can be derived for the convective MHD turbulent flow and are obtained directly by using $\operatorname{div} u = 0$

Taking ensemble average of equation (4.2.5), we get

$$0=\left\langle \frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}}\right\rangle=\left\langle \frac{\partial}{\partial x_\alpha^{(1)}} u_\alpha^{(1)} \int F_1^{(1)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \right\rangle$$

$$=\frac{\partial}{\partial x_\alpha^{(1)}}\left\langle u_\alpha^{(1)} \int F_1^{(1)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \right\rangle$$

$$=\frac{\partial}{\partial x_\alpha^{(1)}}\int \left\langle u_\alpha^{(1)} \right\rangle\left\langle F_1^{(1)} \right\rangle dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)}$$

$$=\frac{\partial}{\partial x_\alpha^{(1)}}\int v_\alpha^{(1)} F_1^{(1)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)}$$

$$=\int \frac{\partial F_1^{(1)}}{\partial x_\alpha^{(1)}} v_\alpha^{(1)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{4.5.1}$$

and similarly,

$$0=\int \frac{\partial F_1^{(1)}}{\partial x_\alpha^{(1)}} g_\alpha^{(1)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{4.5.2}$$

which are the first order continuity equations in which only one point distribution function is involved.

For second-order continuity equations, if we multiply the continuity equation by $\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$

and if we take the ensemble average, we obtain

$$o=\left\langle \delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}}\right\rangle$$

$$=\frac{\partial}{\partial x_\alpha^{(1)}}\left\langle \delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)u_\alpha^{(1)}\right\rangle$$

$$=\frac{\partial}{\partial x_\alpha^{(1)}}\Big[\int\Big\langle u_\alpha^{(1)}\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)$$
$$\times\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\Big\rangle dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)}\Big]$$

$$=\frac{\partial}{\partial x_\alpha^{(1)}}\int v_\alpha^{(1)} F_2^{(1,2)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{4.5.3}$$

and similarly,

$$o=\frac{\partial}{\partial x_\alpha^{(1)}}\int g_\alpha^{(1)} F_2^{(1,2)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{4.5.4}$$

The Nth – order continuity equations are

$$o = \frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_N^{(1,2,---,N)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{4.5.5}$$

and

$$o = \frac{\partial}{\partial x_\alpha^{(1)}} \int g_\alpha^{(1)} F_N^{(1,2,.........,N)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \tag{4.5.6}$$

The continuity equations are symmetric in their arguments i.e.

$$\frac{\partial}{\partial x_\alpha^{(r)}} \left( v_\alpha^{(r)} F_N^{(1,2,..........;r.N)} dv^{(r)} dg^{(r)} d\phi^{(r)} d\psi^{(r)} \right)$$

$$= \frac{\partial}{\partial x_\alpha^{(s)}} \int v_\alpha^{(s)} F_N^{(1,2,.....r,s,.....N)} dv^{(s)} dg^{(s)} d\phi^{(s)} d\psi^{(s)} \tag{4.5.7}$$

Since the divergence property is an important property and it is easily verified by the use of the property of distribution function as

$$\frac{\partial}{\partial x_\alpha^{(1)}} \int v_\alpha^{(1)} F_1^{(1)} dv^{(1)} dg^{(1)} d\phi^{(1)} d\psi^{(1)} \frac{\partial}{\partial x_\alpha^{(1)}} \langle \; u_\alpha^{(1)} \; \rangle = \langle \; \frac{\partial u_\alpha^{(1)}}{\partial x_\alpha^{(1)}} \; \rangle = o \tag{4.5.8}$$

and all the properties of the distribution function obtained in section (4.4) can also be verified.

## 4.6 Equations for evolution of one – point distribution functions $F_1^{(1)}$:

We shall make use of equation (4.2.8) - (4.2.11) to convert these into a set of equations for the variation of the distribution function with time. This, in fact, is done by making use of the definitions of the constructed distribution functions, differentiating them partially with respect to time, making some suitable operations on the right-hand side of the equation so obtained and lastly replacing the time derivative of $u, h, \theta$ and c from the equations (4.2.8) - (4.2.11).

Differentiating equation (4.4.1) with respect to time, we get,

$$\frac{\partial F_1^{(1)}}{\partial t} = \frac{\partial}{\partial t} \langle \; \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \; \rangle$$

$$= \langle\ \delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(u^{(1)} - v^{(1)}\right)\ \rangle$$
$$+ \langle\ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(h^{(1)} - g^{(1)}\right)\ \rangle$$
$$+ \langle\ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(\theta^{(1)} - \phi^{(1)}\right)\ \rangle$$
$$+ \langle\ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\frac{\partial}{\partial t}\delta\left(c^{(1)} - \psi^{(1)}\right)\ \rangle$$

$$= \langle -\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial u^{(1)}}{\partial t}\frac{\partial}{\partial v^{(1)}}\delta\left(u^{(1)} - v^{(1)}\right)\ \rangle$$
$$+ \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial h^{(1)}}{\partial t}\frac{\partial}{\partial g^{(1)}}\delta\left(h^{(1)} - g^{(1)}\right)\ \rangle$$
$$+ \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial \theta^{(1)}}{\partial t}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)} - \phi^{(1)}\right)\ \rangle$$
$$+ \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\frac{\partial c^{(1)}}{\partial t}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)} - \psi^{(1)}\right)\ \rangle \qquad (4.6.1)$$

Using equations (4.2.8) to (4.2.11) in the equation (4.6.1), we get

$$\frac{\partial F_1^{(1)}}{\partial t} = \langle -\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\{\ -\frac{\partial}{\partial x_\beta^{(1)}}\left(u_\alpha^{(1)}u_\beta^{(1)} - h_\alpha^{(1)}h_\beta^{(1)}\right)$$
$$-\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(1)}}\int\ [\ \frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial u_\beta^{(1)}}{\partial x_\alpha^{(1)}} - \frac{\partial h_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial h_\beta^{(1)}}{\partial x_\alpha^{(1)}}\ ]\frac{d\bar{x}'}{\left|\bar{x}' - \bar{x}\right|} + \nu\nabla^2 u_\alpha^{(1)} - 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}$$
$$+ f\left(u_\alpha^{(1)} - v_\alpha^{(1)}\right)\ \}\times\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)} - v^{(1)}\right)\ \rangle$$
$$+ \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\{\ -\frac{\partial}{\partial x_\beta^{(1)}}\left(h_\alpha^{(1)}u_\beta^{(1)} - u_\alpha^{(1)}h_\beta^{(1)}\right) + \lambda\nabla^2 h_\alpha^{(1)}\ \}$$
$$\times\frac{\partial}{\partial g_\alpha^{(1)}}\delta\left(h^{(1)} - g^{(1)}\right)\ \rangle + \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\{-u_\beta^{(1)}\frac{\partial \theta^{(1)}}{\partial x_\beta^{(1)}} + \gamma\nabla^2\theta^{(1)}\ \}$$
$$\times\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)} - \phi^{(1)}\right)\ \rangle + \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\{-u_\beta^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}} + D\nabla^2 c\ \}$$
$$\times\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)} - \psi^{(1)}\right)\ \rangle$$

$$= \langle \ \delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})\frac{\partial u_\alpha^{(1)} u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$+\langle -\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})\frac{\partial h_\alpha^{(1)} h_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$+\langle \ \delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(1)}}\int \left[ \ \frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial u_\beta^{(1)}}{\partial x_\alpha^{(1)}} - \frac{\partial h_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial h_\beta^{(1)}}{\partial x_\alpha^{(1)}} \ \right]$$

$$\times \frac{d\bar{x}'}{|\bar{x}' - \bar{x}|}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$+\langle -\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)}) \times \nu\nabla^2 u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$+\langle \ \delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)}) \times 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$+\langle -\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)}) \times f(u_\alpha{}^{(1)} - v_\alpha{}^{(1)})\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$+\langle \ \delta(u^{(1)} - v^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)}) \times \frac{\partial h_\alpha^{(1)} u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial g_\alpha^{(1)}}\delta(h^{(1)} - g^{(1)}) \ \rangle$$

$$+\langle -\delta(u^{(1)} - v^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)}) \times \frac{\partial u_\alpha^{(1)} h_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial g_\alpha^{(1)}}\delta(h^{(1)} - g^{(1)}) \ \rangle$$

$$+\langle -\delta(u^{(1)} - v^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)}) \times \lambda\nabla^2 h_\alpha^{(1)} \frac{\partial}{\partial g_\alpha^{(1)}}\delta(h^{(1)} - g^{(1)}) \ \rangle$$

$$+\langle \ \delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(c^{(1)} - \psi^{(1)}) \times u_\beta^{(1)} \frac{\partial \theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \phi^{(1)}}\delta(\theta^{(1)} - \phi^{(1)}) \ \rangle$$

$$+\langle -\delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(c^{(1)} - \psi^{(1)}) \times \gamma\nabla^2\theta^{(1)} \frac{\partial}{\partial \phi^{(1)}}\delta(\theta^{(1)} - \phi^{(1)}) \ \rangle$$

$$+\langle \ \delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)}) \times u_\beta^{(1)} \frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \psi^{(1)}}\delta(c^{(1)} - \psi^{(1)}) \ \rangle$$

$$+\langle -\delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)}) \times D\nabla^2 c^{(1)} \frac{\partial}{\partial \psi^{(1)}}\delta(c^{(1)} - \psi^{(1)}) \ \rangle \qquad (4.6.2)$$

Various terms in the above equation can be simplified as that they may be expressed in terms of one point and two point distribution functions.

The 1st term in the above equation is simplified as follows:

$$\langle \ \delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})\frac{\partial u_\alpha^{(1)} u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$= \langle u_\beta^{(1)}\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$= \langle -u_\beta^{(1)}\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\delta(u^{(1)} - v^{(1)}) \ \rangle$$

$$= \langle -u_\beta^{(1)} \delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)} - v^{(1)}\right) \rangle ;(\text{since} \frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}} = 1)$$

$$= \langle -\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)} - v^{(1)}\right) \rangle \tag{4.6.3}$$

Similarly, seventh, tenth and twelfth terms of right hand-side of equation (4.6.2) can be simplified as follows;

$$\langle \ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial h_\alpha^{(1)} u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial g_\alpha^{(1)}}\delta\left(h^{(1)} - g^{(1)}\right) \rangle$$

$$= \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(h^{(1)} - g^{(1)}\right) \rangle \tag{4.6.4}$$

Tenth term,

$$\langle \ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)u_\beta^{(1)}\frac{\partial \theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)} - \phi^{(1)}\right) \rangle$$

$$= \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)} - \phi^{(1)}\right) \rangle \tag{4.6.5}$$

and twelfth term

$$\langle \ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)u_\beta^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)} - \psi^{(1)}\right) \rangle$$

$$= \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)} - \psi^{(1)}\right) \rangle \tag{4.6.6}$$

Adding these equations from (4.6.3) to (4.6.6), we get

$$\langle -\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)} - v^{(1)}\right) \rangle$$

$$+ \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(h^{(1)} - g^{(1)}\right) \rangle$$

$$+ \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)} - \phi^{(1)}\right) \rangle$$

$$+ \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)} - \psi^{(1)}\right) \rangle$$

$$= -\frac{\partial}{\partial x_\beta^{(1)}}\langle u_\beta^{(1)} \langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right) \rangle$$

$$= -\frac{\partial}{\partial x_\beta^{(1)}} v_\beta^{(1)} F_1^{(1)}$$

[ Applying the properties of distribution functions ]

$$= -v_{\beta}^{(1)} \frac{\partial F_1^{(1)}}{\partial x_{\beta}^{(1)}} \tag{4.6.7}$$

Similarly second and eighth terms on the right hand-side of the equation (4.6.2) can be simplified as

$$\left\langle -\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial h_{\alpha}^{(1)} h_{\beta}^{(1)}}{\partial x_{\beta}^{(1)}} \frac{\partial}{\partial v_{\alpha}^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \right\rangle$$

$$= -g_{\beta}^{(1)} \frac{\partial g_{\alpha}^{(1)}}{\partial v_{\alpha}^{(1)}} \frac{\partial}{\partial x_{\beta}^{(1)}} F_1^{(1)} \tag{4.6.8}$$

and

$$\left\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial u_{\alpha}^{(1)} h_{\beta}^{(1)}}{\partial x_{\beta}^{(1)}} \frac{\partial}{\partial g_{\alpha}} \delta\left(h^{(1)} - g^{(1)}\right) \right\rangle$$

$$= -g_{\beta}^{(1)} \frac{\partial v_{\alpha}^{(1)}}{\partial g_{\alpha}^{(1)}} \frac{\partial}{\partial x_{\beta}^{(1)}} F_1^{(1)} \tag{4.6.9}$$

Fourth term can be reduced as

$$\left\langle -\nu\nabla^2 u_{\alpha}^{(1)} \delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\frac{\partial}{\partial v_{\alpha}^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \right\rangle$$

$$= -\nu \frac{\partial}{\partial v_{\alpha}^{(1)}} \left\langle \nabla^2 u_{\alpha}^{(1)} \left[\ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\ \right] \right\rangle$$

$$= -\nu \frac{\partial}{\partial v_{\alpha}^{(1)}} \frac{\partial^2}{\partial x_{\beta}^{(1)} \partial x_{\beta}^{(1)}} \left\langle u_{\alpha}^{(1)} \left[\ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\ \right] \right\rangle$$

$$= -\nu \frac{\partial}{\partial v_{\alpha}^{(1)}} \lim_{\underline{x}^{(2)} \to \underline{x}^{(1)}} \frac{\partial^2}{\partial x_{\beta}^{(2)} \partial x_{\beta}^{(2)}} \left\langle u_{\alpha}^{(2)} \left[\ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\ \right] \right\rangle$$

$$= -\nu \frac{\partial}{\partial v_{\alpha}^{(1)}} \lim_{\underline{x}^{(2)} \to \underline{x}^{(1)}} \frac{\partial^2}{\partial x_{\beta}^{(2)} \partial x_{\beta}^{(2)}} \left\langle \int u_{\alpha}^{(2)} \delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right) \right.$$

$$\left. \times \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right) dv^{(2)} dg^{(2)} d\phi^{(2)} d\psi^{(2)} \right\rangle$$

$$= -\nu \frac{\partial}{\partial v_{\alpha}^{(1)}} \lim_{\underline{x}^{(2)} \to \underline{x}^{(1)}} \frac{\partial^2}{\partial x_{\beta}^{(2)} \partial x_{\beta}^{(2)}} \int v_{\alpha}^{(2)} F_2^{(1,2)} dv^{(2)} dg^{(2)} d\phi^{(2)} d\psi^{(2)} \tag{4.6.10}$$

Ninth, eleventh and thirteen terms of the right hand side of equation (4.6.2)

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\lambda\nabla^2 h_\alpha^{(1)}\frac{\partial}{\partial g_\alpha^{(1)}}\delta\left(h^{(1)}-g^{(1)}\right)\right\rangle$$

$$=\left\langle -\lambda\nabla^2 h_\alpha^{(1)}\frac{\partial}{\partial g_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=-\lambda\frac{\partial}{\partial g_\alpha^{(1)}}\lim_{\bar{x}^{(2)}\to\bar{x}^{(1)}}\frac{\partial^2}{\partial x_\beta^{(2)}\partial x_\beta^{(2)}}\int g_\alpha^{(2)}F_2^{(1,2)}dv^{(2)}dg^{(2)}d\phi^{(2)}d\psi^{(2)} \qquad (4.6.11)$$

Eleventh term,

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\gamma\nabla^2\theta^{(1)}\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$

$$=\left\langle -\gamma\nabla^2\theta^{(1)}\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$

$$=-\gamma\frac{\partial}{\partial\phi^{(1)}}\lim_{\bar{x}^{(2)}\to\bar{x}^{(1)}}\frac{\partial^2}{\partial x_\beta^{(2)}\partial x_\beta^{(2)}}\int\phi^{(2)}F_2^{(1,2)}dv^{(2)}dg^{(2)}d\phi^{(2)}d\psi^{(2)} \qquad (4.6.12)$$

Thirteenth term,

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)D\nabla^2 c^{(1)}\frac{\partial}{\partial\psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=\left\langle -D\nabla^2 c^{(1)}\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$

$$=-D\frac{\partial}{\partial\psi^{(1)}}\lim_{\bar{x}^{(2)}\to\bar{x}^{(1)}}\frac{\partial^2}{\partial x_\beta^{(2)}\partial x_\beta^{(2)}}\int\psi^{(2)}F_2^{(1,2)}dv^{(2)}dg^{(2)}d\phi^{(2)}d\psi^{(2)} \qquad (4.6.13)$$

We reduce the third term of right hand side of equation (4.6.2), we get

$$\left\langle \delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(1)}}\int\left[\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial u_\beta^{(1)}}{\partial x_\alpha^{(1)}}-\frac{\partial h_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial h_\beta^{(1)}}{\partial x_\alpha^{(1)}}\right]\frac{d\bar{x}'}{\left|\bar{x}'-\bar{x}\right|}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$

$$=\frac{\partial}{\partial v_\alpha^{(1)}}\left[\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(1)}}\left(\frac{1}{\left|\bar{x}^{(2)}-\bar{x}^{(1)}\right|}\right)\left(\frac{\partial v_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial v_\beta^{(2)}}{\partial x_\alpha^{(2)}}-\frac{\partial g_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial g_\beta^{(2)}}{\partial x_\alpha^{(2)}}\right)F_2^{(1,2)}dx^{(2)}dv^{(2)}dg^{(2)}d\phi^{(2)}d\psi^{(2)}\right.$$

(4.6.14)

Fifth and sixth terms of right hand side of equation (4.6.2)

$$\left\langle \delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\times 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$

$$=\left\langle 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}\frac{\partial}{\partial v_\alpha^{(1)}}\left[\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right]\right\rangle$$

$$=2\in_{m\alpha\beta}\Omega_m\frac{\partial}{\partial v_\alpha^{(1)}}\left\langle u_\alpha^{(1)}\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=2\in_{m\alpha\beta}\Omega_m\frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}}\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=2\in_{m\alpha\beta}\Omega_m F_1^{(1)} \tag{4.6.15}$$

and

$$\left\langle -\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)f\left(u_\alpha^{(1)}-v_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$

$$=-\left\langle f\left(u_\alpha^{(1)}-v_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}\left[\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right]\right\rangle$$

$$=-f\left(u_\alpha^{(1)}-v_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=-f\left(u_\alpha^{(1)}-v_\alpha^{(1)}\right)\frac{\partial}{\partial v_\alpha^{(1)}}F_1^{(1)} \tag{4.6.16}$$

Substituting the results (4.6.3) to (4.6.16) in equation (4.6.2) we get the transport equation for one point distribution function $F_1^{(1)}(v,g,\phi,\psi)$ in MHD turbulent flow in a rotating system in presence of dust particles as

$$\frac{\partial F_1^{(1)}}{\partial t}+v_\beta^{(1)}\frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}}+g_\beta^{(1)}\left(\frac{\partial g_\alpha^{(1)}}{\partial v_\alpha^{(1)}}+\frac{\partial v_\alpha^{(1)}}{\partial g_\alpha^{(1)}}\right)\frac{\partial F_1^{(1)}}{\partial x_\beta^{(1)}}-\frac{\partial}{\partial v_\alpha^{(1)}}\left[\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(1)}}\left(\frac{1}{\left|\overline{x}^{(2)}-\overline{x}^{(1)}\right|}\right)\right.$$

$$\times\left(\frac{\partial v_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial v_\beta^{(2)}}{\partial x_\alpha^{(2)}}-\frac{\partial g_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial g_\beta^{(2)}}{\partial x_\alpha^{(2)}}\right)F_2^{(1,2)}dx^{(2)}dv^{(2)}dg^{(2)}d\phi^{(2)}d\psi^{(2)}$$

$$+v\frac{\partial}{\partial v_\alpha^{(1)}}\lim_{\overline{x}^{(2)}\to\overline{x}^{(1)}}\frac{\partial^2}{\partial x_\beta^{(2)}\partial x_\beta^{(2)}}\int v_\alpha^{(2)}F_2^{(1,2)}dv^{(2)}dg^{(2)}d\phi^{(2)}d\psi^{(2)}$$

$$+\lambda\frac{\partial}{\partial g_\alpha^{(1)}}\lim_{\overline{x}^{(2)}\to\overline{x}^{(1)}}\frac{\partial^2}{\partial x_\beta^{(2)}\partial x_\beta^{(2)}}\int g_\alpha^{(2)}F_2^{(1,2)}dv^{(2)}dg^{(2)}d\phi^{(2)}d\psi^{(2)}$$

$$+\gamma \frac{\partial}{\partial \phi^{(1)}} \lim_{\bar{x}^{(2)} \to \bar{x}^{(1)}} \frac{\partial^2}{\partial x_\beta^{(2)} \partial x_\beta^{(2)}} \int \phi^{(2)} F_2^{(1,2)} dv^{(2)} dg^{(2)} d\phi^{(2)} d\psi^{(2)}$$

$$+D \frac{\partial}{\partial \psi^{(1)}} \lim_{\bar{x}^{(2)} \to \bar{x}^{(1)}} \frac{\partial^2}{\partial x_\beta^{(2)} \partial x_\beta^{(2)}} \int \psi^{(2)} F_2^{(1,2)} dv^{(2)} dg^{(2)} d\phi^{(2)} d\psi^{(2)}$$

$$+2 \in_{m\alpha\beta} \Omega_m F_1^{(1)} + f\left( u_\alpha^{(1)} - v_\alpha^{(1)} \right) \frac{\partial}{\partial v_\alpha^{(1)}} F_1^{(1)} = 0 \qquad (4.6.17)$$

## 4.7 Equations for two-point distribution function $F_2^{(1,2)}$:

Differentiating equation (4.4.2) with respect to time, we get,

$$\frac{\partial F_2^{(1,2)}}{\partial t} = \frac{\partial}{\partial t} \Big\langle\ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(2)} - v^{(2)}\right)$$
$$\delta\left(h^{(2)} - g^{(2)}\right) \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \Big\rangle$$
$$= \Big\langle\ \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(2)} - v^{(2)}\right) \delta\left(h^{(2)} - g^{(2)}\right) \delta\left(\theta^{(2)} - \phi^{(2)}\right)$$
$$\delta\left(c^{(2)} - \psi^{(2)}\right) \frac{\partial}{\partial t} \delta\left(u^{(1)} - v^{(1)}\right) \Big\rangle + \Big\langle\ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right)$$
$$\delta\left(u^{(2)} - v^{(2)}\right) \delta\left(h^{(2)} - g^{(2)}\right) \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \frac{\partial}{\partial t} \delta\left(h^{(1)} - g^{(1)}\right) \Big\rangle$$
$$+ \Big\langle\ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(2)} - v^{(2)}\right) \delta\left(h^{(2)} - g^{(2)}\right) \delta\left(\theta^{(2)} - \phi^{(2)}\right)$$
$$\delta\left(c^{(2)} - \psi^{(2)}\right) \frac{\partial}{\partial t} \delta\left(\theta^{(1)} - \phi^{(1)}\right) \Big\rangle + \Big\langle\ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right)$$

$$\delta\left(u^{(2)} - v^{(2)}\right) \delta\left(h^{(2)} - g^{(2)}\right) \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \frac{\partial}{\partial t} \delta\left(c^{(1)} - \psi^{(1)}\right) \Big\rangle$$
$$+ \Big\langle\ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right) \delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(h^{(2)} - g^{(2)}\right) \delta\left(\theta^{(2)} - \phi^{(2)}\right)$$
$$\delta\left(c^{(2)} - \psi^{(2)}\right) \frac{\partial}{\partial t} \delta\left(u^{(2)} - v^{(2)}\right) \Big\rangle + \Big\langle\ \delta\left(u^{(1)} - v^{(1)}\right) \delta\left(h^{(1)} - g^{(1)}\right) \delta\left(\theta^{(1)} - \phi^{(1)}\right)$$
$$\delta\left(c^{(1)} - \psi^{(1)}\right) \delta\left(u^{(2)} - v^{(2)}\right) \delta\left(\theta^{(2)} - \phi^{(2)}\right) \delta\left(c^{(2)} - \psi^{(2)}\right) \frac{\partial}{\partial t} \delta\left(h^{(2)} - g^{(2)}\right) \Big\rangle$$

$$+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\right.$$
$$\left.\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial}{\partial t}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right\rangle+\left\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right.$$
$$\left.\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\frac{\partial}{\partial t}\delta\left(c^{(2)}-\psi^{(2)}\right)\right\rangle$$

$$=\left\langle -\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right.$$
$$\left.\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial u^{(1)}}{\partial t}\frac{\partial}{\partial v^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right.$$
$$\left.\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial h^{(1)}}{\partial t}\frac{\partial}{\partial g^{(1)}}\delta\left(h^{(1)}-g^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right.$$
$$\left.\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial \theta^{(1)}}{\partial t}\frac{\partial}{\partial \phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right.$$
$$\left.\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial c^{(1)}}{\partial t}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right.$$
$$\left.\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial u^{(2)}}{\partial t}\frac{\partial}{\partial v^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right.$$
$$\left.\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial h^{(2)}}{\partial t}\frac{\partial}{\partial g^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\right.$$
$$\left.\delta\left(c^{(2)}-\psi^{(2)}\right)\frac{\partial \theta^{(2)}}{\partial t}\frac{\partial}{\partial \phi^{(2)}}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right\rangle$$
$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\right.$$
$$\left.\delta\left(\theta^{(2)}-\phi^{(2)}\right)\frac{\partial c^{(2)}}{\partial t}\frac{\partial}{\partial \psi^{(2)}}\delta\left(c^{(2)}-\psi^{(2)}\right)\right\rangle$$

Using equations (4.2.8) to (4.2.11) we get,

$$=\left\langle -\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\right.$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\left\{-\frac{\partial}{\partial x_\beta^{(1)}}\left(u_\alpha^{(1)}u_\beta^{(1)}-h_\alpha^{(1)}h_\beta^{(1)}\right)-\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(1)}}\int\left[\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial u_\beta^{(1)}}{\partial x_\alpha^{(1)}}-\frac{\partial h_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial h_\beta^{(1)}}{\partial x_\alpha^{(1)}}\right]\right.$$
$$\left.\left.\times\frac{d\bar{x}''}{\left|\bar{x}''-\bar{x}\right|}+\nu\nabla^2u_\alpha^{(1)}-2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}+f\left(u_\alpha^{(1)}-v_\alpha^{(1)}\right)\right\}\times\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\right\rangle$$

$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\{\ -\frac{\partial}{\partial x_\beta^{(1)}}\left(h_\alpha^{(1)}u_\beta^{(1)}-u_\alpha^{(1)}h_\beta^{(1)}\right)+\lambda\nabla^2 h_\alpha^{(1)}\ \}\times\frac{\partial}{\partial g_\alpha^{(1)}}\delta(h^{(1)}-g^{(1)})\ \rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\{-u_\beta^{(1)}\frac{\partial\theta^{(1)}}{\partial x_\beta^{(1)}}+\gamma\nabla^2\theta^{(1)}\ \}\times\frac{\partial}{\partial\phi^{(1)}}\delta(\theta^{(1)}-\phi^{(1)})\ \rangle+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(u^{(2)}-v^{(2)})$$
$$\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})\{-u_\beta^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}+D\nabla^2 c\ \}\frac{\partial}{\partial\psi^{(1)}}\delta(c^{(1)}-\psi^{(1)})\ \rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\{\ -\frac{\partial}{\partial x_\beta^{(2)}}\left(u_\alpha^{(2)}u_\beta^{(2)}-h_\alpha^{(2)}h_\beta^{(2)}\right)-\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(2)}}\int\ [\ \frac{\partial u_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial u_\beta^{(2)}}{\partial x_\alpha^{(2)}}-\frac{\partial h_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial h_\beta^{(2)}}{\partial x_\alpha^{(2)}}\ ]\frac{d\bar{x}''}{|\bar{x}''-\bar{x}'|}$$
$$+\nu\nabla^2 u_\alpha^{(2)}-2\in_{m\alpha\beta}\Omega_m u_\alpha^{(2)}+f\left(u_\alpha^{(2)}-v_\alpha^{(2)}\right)\ \}\times\frac{\partial}{\partial v_\alpha^{(2)}}\delta(u^{(2)}-v^{(2)})\ \rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\{\ -\frac{\partial}{\partial x_\beta^{(2)}}\left(h_\alpha^{(2)}u_\beta^{(2)}-u_\alpha^{(2)}h_\beta^{(2)}\right)+\lambda\nabla^2 h_\alpha^{(2)}\ \}\times\frac{\partial}{\partial g_\alpha^{(2)}}\delta(h^{(2)}-g^{(2)})\ \rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\{-u_\beta^{(2)}\frac{\partial\theta^{(2)}}{\partial x_\beta^{(2)}}+\gamma\nabla^2\theta^{(2)}\ \}\times\frac{\partial}{\partial\phi^{(2)}}\delta(\theta^{(2)}-\phi^{(2)})\ \rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})$$
$$\delta(\theta^{(2)}-\phi^{(2)})\{-u_\beta^{(2)}\frac{\partial c^{(2)}}{\partial x_\beta^{(2)}}+D\nabla^2 c^{(2)}\ \}\times\frac{\partial}{\partial\psi^{(2)}}\delta(c^{(2)}-\psi^{(2)})\ \rangle$$

$$=\langle\ \delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times\frac{\partial u_\alpha^{(1)}u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)}-v^{(1)})\ \rangle$$
$$+\langle -\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times\frac{\partial h_\alpha^{(1)}h_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)}-v^{(1)})\ \rangle$$
$$+\langle\ \delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(1)}}\int\ [\ \frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial u_\beta^{(1)}}{\partial x_\alpha^{(1)}}-\frac{\partial h_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial h_\beta^{(1)}}{\partial x_\alpha^{(1)}}\ ]\frac{d\bar{x}''}{|\bar{x}''-\bar{x}|}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)}-v^{(1)})\ \rangle$$
$$+\langle -\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})$$
$$\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})\times\nu\nabla^2 u_\alpha^{(1)}\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)}-v^{(1)})\ \rangle$$

$$+\Big\langle\ \delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\big(u^{(1)}-v^{(1)}\big)\Big\rangle$$

$$+\Big\langle-\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times f\big(u_\alpha{}^{(1)}-v_\alpha{}^{(1)}\big)\frac{\partial}{\partial v_\alpha^{(1)}}\delta\big(u^{(1)}-v^{(1)}\big)\Big\rangle$$

$$+\Big\langle\ \delta\big(u^{(1)}-v^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\frac{\partial h_\alpha^{(1)}u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial g_\alpha^{(1)}}\delta\big(h^{(1)}-g^{(1)}\big)\Big\rangle$$

$$+\Big\langle-\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\frac{\partial u_\alpha^{(1)}h_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial g_\alpha^{(1)}}\delta\big(h^{(1)}-g^{(1)}\big)\Big\rangle$$

$$+\Big\langle-\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\lambda\nabla^2 h_\alpha^{(1)}\frac{\partial}{\partial g_\alpha^{(1)}}\delta\big(h^{(1)}-g^{(1)}\big)\Big\rangle$$

$$+\Big\langle\ \delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times u_\beta^{(1)}\frac{\partial\theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial\phi^{(1)}}\delta\big(\theta^{(1)}-\phi^{(1)}\big)\Big\rangle$$

$$+\Big\langle-\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\gamma\nabla^2\theta^{(1)}\frac{\partial}{\partial\phi^{(1)}}\delta\big(\theta^{(1)}-\phi^{(1)}\big)\Big\rangle$$

$$+\Big\langle\ \delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times u_\beta^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial\psi^{(1)}}\delta\big(c^{(1)}-\psi^{(1)}\big)\Big\rangle$$

$$+\Big\langle-\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times D\nabla^2 c^{(1)}\frac{\partial}{\partial\psi^{(1)}}\delta\big(c^{(1)}-\psi^{(1)}\big)\Big\rangle$$

$$+\Big\langle\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\frac{\partial u_\alpha^{(2)}u_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\big(u^{(2)}-v^{(2)}\big)\Big\rangle$$

$$+\Big\langle-\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\frac{\partial h_\alpha^{(2)}h_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\big(u^{(2)}-v^{(2)}\big)\Big\rangle$$

$$+\Big\langle\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$

$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(2)}}\int\Big[\ \frac{\partial u_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial u_\beta^{(2)}}{\partial x_\alpha^{(2)}}-\frac{\partial h_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial h_\beta^{(2)}}{\partial x_\alpha^{(2)}}\ \Big]\frac{d\bar{x}''}{\left|\bar{x}''-\bar{x}'\right|}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\big(u^{(2)}-v^{(2)}\big)\Big\rangle$$

$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times \nu\nabla^2 u_\alpha^{(2)}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\rangle$$
$$+\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(2)}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times f\left(u_\alpha{}^{(2)}-v_\alpha{}^{(2)}\right)\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\rangle$$
$$+\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\frac{\partial h_\alpha^{(2)}u_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial g_\alpha^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\frac{\partial u_\alpha^{(2)}h_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial g_\alpha^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\lambda\nabla^2 h_\alpha^{(2)}\frac{\partial}{\partial g_\alpha^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\rangle$$
$$+\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_\beta^{(2)}\frac{\partial\theta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial\phi^{(2)}}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\gamma\nabla^2\theta^{(2)}\frac{\partial}{\partial\phi^{(2)}}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\rangle$$
$$+\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\times u_\beta^{(2)}\frac{\partial c^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial\psi^{(2)}}\delta\left(c^{(2)}-\psi^{(2)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\times D\nabla^2 c^{(2)}\frac{\partial}{\partial\psi^{(2)}}\delta\left(c^{(2)}-\psi^{(2)}\right)\rangle \qquad (4.7.1)$$

Various terms in the above equation can be simplified as that they may be expressed in terms of one point , two point and three point distribution functions.

The 1st term in the above equation is simplified as follows:

$$\langle\ \delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\frac{\partial u_\alpha^{(1)}u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle$$
$$=\langle u_\beta^{(1)}\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle$$
$$=\langle -u_\beta^{(1)}\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle;(\text{since}\ \frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}}=1)$$
$$=\langle -\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle \qquad (4.7.2).$$

Similarly, seventh, tenth and twelfth terms of right hand-side of equation (4.7.1) can be simplified as follows;

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\frac{\partial h_\alpha^{(1)}u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial g_\alpha^{(1)}}\delta\left(h^{(1)}-g^{(1)}\right)\ \rangle$$
$$=\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(h^{(1)}-g^{(1)}\right)\ \rangle \qquad (4.7.3).$$

Tenth term,

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_\beta^{(1)}\frac{\partial\theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\ \rangle$$
$$=\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\ \rangle \qquad (4.7.4)$$

and twelfth term

$$=\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\ \rangle \qquad (4.7.5).$$

Adding these equations from (4.7.2) to (4.7.5), we get

$$\langle -\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta(u^{(1)}-v^{(1)})\rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta(h^{(1)}-g^{(1)})\rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta(\theta^{(1)}-\phi^{(1)})\rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta(c^{(1)}-\psi^{(1)})\rangle$$

$$=-\frac{\partial}{\partial x_\beta^{(1)}}\langle u_\beta^{(1)}\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})$$
$$\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})\rangle$$

$$=-\frac{\partial}{\partial x_\beta^{(1)}}v_\beta^{(1)}F_2^{(1,2)}$$

[ Applying the properties of distribution functions ]

$$=-v_\beta^{(1)}\frac{\partial F_2^{(1,2)}}{\partial x_\beta^{(1)}} \tag{4.7.6}$$

Similarly, 14th, 20th, 23th and 25th terms of right hand-side of equation (4.7.1) can be simplified as follows;

$$\langle \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times\frac{\partial u_\alpha^{(2)}u_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial v_\alpha^{(2)}}\delta(u^{(2)}-v^{(2)})\rangle$$
$$=\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times u_\beta^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\delta(u^{(2)}-v^{(2)})\rangle \tag{4.7.7}$$

20th term,

$$\langle\ \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\times\frac{\partial h_\alpha^{(2)}u_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial g_\alpha^{(2)}}\delta(h^{(2)}-g^{(2)})\rangle$$

$$= \langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_{\beta}^{(2)}\frac{\partial}{\partial x_{\beta}^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\rangle \qquad (4.7.8)$$

23th term,

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_{\beta}^{(2)}\frac{\partial\theta^{(2)}}{\partial x_{\beta}^{(2)}}\frac{\partial}{\partial\phi^{(2)}}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\rangle$$

$$= \langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times u_{\beta}^{(2)}\frac{\partial}{\partial x_{\beta}^{(2)}}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\rangle \qquad (4.7.9)$$

and 25th term,

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\times u_{\beta}^{(2)}\frac{\partial c^{(2)}}{\partial x_{\beta}^{(2)}}\frac{\partial}{\partial\psi^{(2)}}\delta\left(c^{(2)}-\psi^{(2)}\right)\rangle$$

$$= \langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\times u_{\beta}^{(2)}\frac{\partial}{\partial x_{\beta}^{(2)}}\delta\left(c^{(2)}-\psi^{(2)}\right)\rangle \qquad (4.7.10)$$

Adding these equations from (4.7.7) to (4.7.10), we get

$$-\frac{\partial}{\partial x_{\beta}^{(2)}}\langle u_{\beta}^{(2)}\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)$$
$$\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\rangle$$
$$= -v_{\beta}^{(2)}\frac{\partial F_2^{(1,2)}}{\partial x_{\beta}^{(2)}} \qquad (4.7.11)$$

Similarly, 2nd ,8th ,15th and 21st terms of right hand-side of equation (4.7.1) can be simplified as follows;

$$\langle -\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times\frac{\partial h_{\alpha}^{(1)}h_{\beta}^{(1)}}{\partial x_{\beta}^{(1)}}\frac{\partial}{\partial v_{\alpha}^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\rangle$$

$$=-g_{\beta}^{1}\frac{\partial g_{\alpha}^{(1)}}{\partial v_{\alpha}^{(1)}}\frac{\partial F_{2}^{(1,2)}}{\partial x_{\beta}^{(1)}} \tag{4.7.12}$$

8th term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\frac{\partial u_{\alpha}^{(1)}h_{\beta}^{(1)}}{\partial x_{\beta}^{(1)}}\frac{\partial}{\partial g_{\alpha}^{(1)}}\delta\big(h^{(1)}-g^{(1)}\big)\Big\rangle$$

$$=-g_{\beta}^{1}\frac{\partial v_{\alpha}^{(1)}}{\partial g_{\alpha}^{(1)}}\frac{\partial F_{2}^{(1,2)}}{\partial x_{\beta}^{(1)}} \tag{4.7.13}$$

15th term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\frac{\partial h_{\alpha}^{(2)}h_{\beta}^{(2)}}{\partial x_{\beta}^{(2)}}\frac{\partial}{\partial v_{\alpha}^{(2)}}\delta\big(u^{(2)}-v^{(2)}\big)\Big\rangle$$

$$=-g_{\beta}^{2}\frac{\partial g_{\alpha}^{(2)}}{\partial v_{\alpha}^{(2)}}\frac{\partial F_{2}^{(1,2)}}{\partial x_{\beta}^{(2)}} \tag{4.7.14}$$

and 21st term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\frac{\partial u_{\alpha}^{(2)}h_{\beta}^{(2)}}{\partial x_{\beta}^{(2)}}\frac{\partial}{\partial g_{\alpha}^{(2)}}\delta\big(h^{(2)}-g^{(2)}\big)\Big\rangle$$

$$=-g_{\beta}^{2}\frac{\partial v_{\alpha}^{(2)}}{\partial g_{\alpha}^{(2)}}\frac{\partial F_{2}^{(1,2)}}{\partial x_{\beta}^{(2)}} \tag{4.7.15}$$

Fourth term can be reduced as

$$\Big\langle -\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(c^{(2)}-\psi^{(2)}\big)\times\nu\nabla^{2}u_{\alpha}^{(1)}\frac{\partial}{\partial v_{\alpha}^{(1)}}\delta\big(u^{(1)}-v^{(1)}\big)\Big\rangle$$

$$=-\nu\frac{\partial}{\partial v_{\alpha}^{(1)}}\Big\langle\nabla^{2}u_{\alpha}^{(1)}\big[\ \delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)$$
$$\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)\delta\big(c^{(2)}-\psi^{(2)}\big)\ \big]\ \Big\rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \frac{\partial^2}{\partial x_\beta^{(1)} \partial x_\beta^{(1)}} \langle u_\alpha^{(1)} [\ \delta(u^{(1)} - v^{(1)}) \delta(h^{(1)} - g^{(1)}) \delta(\theta^{(1)} - \phi^{(1)}) \delta(c^{(1)} - \psi^{(1)})$$

$$\delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)}) \delta(c^{(2)} - \psi^{(2)})\ ]\ \rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \lim_{\overline{x}(3) \to \overline{x}(1)} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \langle u_\alpha^{(3)} [\ \delta(u^{(1)} - v^{(1)}) \delta(h^{(1)} - g^{(1)}) \delta(\theta^{(1)} - \phi^{(1)})$$

$$\delta(c^{(1)} - \psi^{(1)}) \delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)}) \delta(c^{(2)} - \psi^{(2)})\ ]\ \rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \lim_{\overline{x}(3) \to \overline{x}(1)} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \langle \int\ u_\alpha^{(3)} \delta(u^{(3)} - v^{(3)}) \delta(h^{(3)} - g^{(3)}) \delta(\theta^{(3)} - \phi^{(3)})$$

$$\delta(c^{(3)} - \psi^{(3)}) \delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)}) \delta(c^{(2)} - \psi^{(2)}) \delta(u^{(1)} - v^{(1)})$$

$$\delta(h^{(1)} - g^{(1)}) \delta(\theta^{(1)} - \phi^{(1)}) \delta(c^{(1)} - \psi^{(1)}) dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)}\ \rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \lim_{\overline{x}(3) \to \overline{x}(1)} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \int v_\alpha^{(3)} F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \qquad (4.7.16)$$

Similarly, 9th ,11th ,13th ,17th ,22nd ,24th and 26th terms of right hand-side of equation (4.7.1) can be simplified as follows;

$$\langle -\delta(u^{(1)} - v^{(1)}) \delta(\theta^{(1)} - \phi^{(1)}) \delta(c^{(1)} - \psi^{(1)}) \delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)})$$

$$\delta(c^{(2)} - \psi^{(2)}) \times \lambda \nabla^2 h_\alpha^{(1)} \frac{\partial}{\partial g_\alpha^{(1)}} \delta(h^{(1)} - g^{(1)})\ \rangle$$

$$= -\lambda \frac{\partial}{\partial g_\alpha^{(1)}} \lim_{\overline{x}(3) \to \overline{x}(1)} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \int g_\alpha^{(3)} F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \qquad (4.7.17)$$

11th term,

$$\langle -\delta(u^{(1)} - v^{(1)}) \delta(h^{(1)} - g^{(1)}) \delta(c^{(1)} - \psi^{(1)}) \delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)})$$

$$\delta(c^{(2)} - \psi^{(2)}) \times \gamma \nabla^2 \theta^{(1)} \frac{\partial}{\partial \phi^{(1)}} \delta(\theta^{(1)} - \phi^{(1)})\ \rangle$$

$$= -\gamma \frac{\partial}{\partial \phi^{(1)}} \lim_{\overline{x}^{(3)} \to \overline{x}^{(1)}} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \int \phi^{(3)} F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \qquad (4.7.18)$$

13th term,

$$\Big\langle -\delta\big(u^{(1)} - v^{(1)}\big)\delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(u^{(2)} - v^{(2)}\big)\delta\big(h^{(2)} - g^{(2)}\big)\delta\big(\theta^{(2)} - \phi^{(2)}\big)$$
$$\delta\big(c^{(2)} - \psi^{(2)}\big) \times D\nabla^2 c^{(1)} \frac{\partial}{\partial \psi^{(1)}} \delta\big(c^{(1)} - \psi^{(1)}\big) \Big\rangle$$

$$= -D \frac{\partial}{\partial \psi^{(1)}} \lim_{\overline{x}^{(3)} \to \overline{x}^{(1)}} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \int \psi^{(3)} F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \qquad (4.7.19)$$

17th term,

$$\Big\langle -\delta\big(u^{(1)} - v^{(1)}\big)\delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(c^{(1)} - \psi^{(1)}\big)\delta\big(h^{(2)} - g^{(2)}\big)\delta\big(\theta^{(2)} - \phi^{(2)}\big)$$
$$\delta\big(c^{(2)} - \psi^{(2)}\big) \times \nu\nabla^2 u_\alpha^{(2)} \frac{\partial}{\partial v_\alpha^{(2)}} \delta\big(u^{(2)} - v^{(2)}\big) \Big\rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(2)}} \lim_{\overline{x}^{(3)} \to \overline{x}^{(2)}} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \int v_\alpha^{(3)} F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \qquad (4.7.20)$$

22nd term,

$$\Big\langle -\delta\big(u^{(1)} - v^{(1)}\big)\delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(c^{(1)} - \psi^{(1)}\big)\delta\big(u^{(2)} - v^{(2)}\big)\delta\big(\theta^{(2)} - \phi^{(2)}\big)$$
$$\delta\big(c^{(2)} - \psi^{(2)}\big) \times \lambda\nabla^2 h_\alpha^{(2)} \frac{\partial}{\partial g_\alpha^{(2)}} \delta\big(h^{(2)} - g^{(2)}\big) \Big\rangle$$

$$= -\lambda \frac{\partial}{\partial g_\alpha^{(2)}} \lim_{\overline{x}^{(3)} \to \overline{x}^{(2)}} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \int g_\alpha^{(3)} F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \qquad (4.7.21)$$

24th term,

$$\Big\langle -\delta\big(u^{(1)} - v^{(1)}\big)\delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(c^{(1)} - \psi^{(1)}\big)\delta\big(u^{(2)} - v^{(2)}\big)\delta\big(h^{(2)} - g^{(2)}\big)$$
$$\delta\big(c^{(2)} - \psi^{(2)}\big) \times \gamma\nabla^2 \theta^{(2)} \frac{\partial}{\partial \phi^{(2)}} \delta\big(\theta^{(2)} - \phi^{(2)}\big) \Big\rangle$$

$$= -\gamma \frac{\partial}{\partial \phi^{(2)}} \lim_{\bar{x}^{(3)} \to \bar{x}^{(2)}} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \int \phi^{(3)} F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \tag{4.7.22}$$

26th term,

$$\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)$$
$$\delta\left(\theta^{(2)} - \phi^{(2)}\right) \times D\nabla^2 c^{(2)} \frac{\partial}{\partial \psi^{(2)}} \delta\left(c^{(2)} - \psi^{(2)}\right) \rangle$$

$$= -D \frac{\partial}{\partial \psi^{(2)}} \lim_{\bar{x}^{(3)} \to \bar{x}^{(2)}} \frac{\partial^2}{\partial x_\beta^{(3)} \partial x_\beta^{(3)}} \int \psi^{(3)} F_3^{(1,2,3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \tag{4.7.23}$$

We reduce the third term of right - hand side of equation (4.7.1),

$$\langle \ \delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$
$$\delta\left(c^{(2)} - \psi^{(2)}\right) \times \frac{1}{4\pi} \frac{\partial}{\partial x_\alpha^{(1)}} \int \left[ \frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}} \frac{\partial u_\beta^{(1)}}{\partial x_\alpha^{(1)}} - \frac{\partial h_\alpha^{(1)}}{\partial x_\beta^{(1)}} \frac{\partial h_\beta^{(1)}}{\partial x_\alpha^{(1)}} \right] \frac{d\bar{x}''}{\left|\bar{x}'' - \bar{x}\right|} \frac{\partial}{\partial v_\alpha^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \rangle$$

$$= \frac{\partial}{\partial v_\alpha^{(1)}} \left[ \frac{1}{4\pi} \int \frac{\partial}{\partial x_\alpha^{(1)}} \left( \frac{1}{\left|\bar{x}^{(3)} - \bar{x}^{(1)}\right|} \right) \left( \frac{\partial v_\alpha^{(3)}}{\partial x_\beta^{(3)}} \frac{\partial v_\beta^{(3)}}{\partial x_\alpha^{(3)}} - \frac{\partial g_\alpha^{(3)}}{\partial x_\beta^{(3)}} \frac{\partial g_\beta^{(3)}}{\partial x_\alpha^{(3)}} \right) F_3^{(1,2,3)} \right.$$
$$\left. \times dx^{(3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \ \right] \tag{4.7.24}$$

Similarly,16th term,

$$\langle \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$
$$\delta\left(c^{(2)} - \psi^{(2)}\right) \times \frac{1}{4\pi} \frac{\partial}{\partial x_\alpha^{(2)}} \int \left[ \frac{\partial u_\alpha^{(2)}}{\partial x_\beta^{(2)}} \frac{\partial u_\beta^{(2)}}{\partial x_\alpha^{(2)}} - \frac{\partial h_\alpha^{(2)}}{\partial x_\beta^{(2)}} \frac{\partial h_\beta^{(2)}}{\partial x_\alpha^{(2)}} \right] \frac{d\bar{x}''}{\left|\bar{x}'' - \bar{x}'\right|} \frac{\partial}{\partial v_\alpha^{(2)}} \delta\left(u^{(2)} - v^{(2)}\right) \rangle$$

$$= \frac{\partial}{\partial v_\alpha^{(2)}} \left[ \frac{1}{4\pi} \int \frac{\partial}{\partial x_\alpha^{(2)}} \left( \frac{1}{\left|\bar{x}^{(3)} - \bar{x}^{(2)}\right|} \right) \left( \frac{\partial v_\alpha^{(3)}}{\partial x_\beta^{(3)}} \frac{\partial v_\beta^{(3)}}{\partial x_\alpha^{(3)}} - \frac{\partial g_\alpha^{(3)}}{\partial x_\beta^{(3)}} \frac{\partial g_\beta^{(3)}}{\partial x_\alpha^{(3)}} \right) F_3^{(1,2,3)} \right.$$
$$\left. \times dx^{(3)} dv^{(3)} dg^{(3)} d\phi^{(3)} d\psi^{(3)} \ \right] \tag{4.7.25}$$

Fifth and sixth terms of right hand side of equation (4.7.1), we get

$$\langle \ \delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$
$$\delta\left(c^{(2)} - \psi^{(2)}\right) \times 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \rangle$$

$$= \langle\ 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}} [\ \delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})$$
$$\delta(u^{(2)} - v^{(2)})\delta(h^{(2)} - g^{(2)})\delta(\theta^{(2)} - \phi^{(2)})\delta(c^{(2)} - \psi^{(2)})\ ]\ \rangle$$
$$= 2 \in_{m\alpha\beta} \Omega_m \frac{\partial}{\partial v_\alpha^{(1)}} \langle\ u_\alpha^{(1)} \delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})$$
$$\delta(u^{(2)} - v^{(2)})\delta(h^{(2)} - g^{(2)})\delta(\theta^{(2)} - \phi^{(2)})\delta(c^{(2)} - \psi^{(2)})\rangle$$
$$= 2 \in_{m\alpha\beta} \Omega_m \frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}} \langle\ \delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})$$
$$\delta(u^{(2)} - v^{(2)})\delta(h^{(2)} - g^{(2)})\delta(\theta^{(2)} - \phi^{(2)})\delta(c^{(2)} - \psi^{(2)})\ \rangle$$
$$= 2 \in_{m\alpha\beta} \Omega_m F_2^{(1,2)} \tag{4.7.26}$$

and sixth term,

$$\langle -\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})\delta(u^{(2)} - v^{(2)})\delta(h^{(2)} - g^{(2)})\delta(\theta^{(2)} - \phi^{(2)})$$
$$\delta(c^{(2)} - \psi^{(2)}) \times f(\ u_\alpha^{(1)} - v_\alpha^{(1)}\ )\frac{\partial}{\partial v_\alpha^{(1)}} \delta(u^{(1)} - v^{(1)})\ \rangle$$
$$= -\langle\ f(\ u_\alpha^{(1)} - v_\alpha^{(1)}\ )\frac{\partial}{\partial v_\alpha^{(1)}} [\ \delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})$$
$$\delta(u^{(2)} - v^{(2)})\delta(h^{(2)} - g^{(2)})\delta(\theta^{(2)} - \phi^{(2)})\delta(c^{(2)} - \psi^{(2)})]\ \rangle$$

$$= -f(\ u_\alpha^{(1)} - v_\alpha^{(1)}\ )\frac{\partial}{\partial v_\alpha^{(1)}} \langle\ \delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})$$
$$\delta(u^{(2)} - v^{(2)})\delta(h^{(2)} - g^{(2)})\delta(\theta^{(2)} - \phi^{(2)})\delta(c^{(2)} - \psi^{(2)})\rangle$$

$$= -f(\ u_\alpha^{(1)} - v_\alpha^{(1)}\ )\frac{\partial}{\partial v_\alpha^{(1)}} F_2^{(1,2)} \tag{4.7.27}$$

Similarly, 18th and 19th terms of right hand side of equation (4.7.1),

$$\langle \delta(u^{(1)} - v^{(1)})\delta(h^{(1)} - g^{(1)})\delta(\theta^{(1)} - \phi^{(1)})\delta(c^{(1)} - \psi^{(1)})\delta(h^{(2)} - g^{(2)})\delta(\theta^{(2)} - \phi^{(2)})$$
$$\delta(c^{(2)} - \psi^{(2)}) \times 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(2)} \frac{\partial}{\partial v_\alpha^{(2)}} \delta(u^{(2)} - v^{(2)})\ \rangle$$

$$= 2 \in_{m\alpha\beta} \Omega_m F_2^{(1,2)} \tag{4.7.28}$$

and 19th term,

$$\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$

$$\delta\left(c^{(2)}-\psi^{(2)}\right)\times f\left(\ u_\alpha^{(2)}-v_\alpha^{(2)}\ \right)\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\ \rangle$$

$$=-f\left(\ u_\alpha^{(2)}-v_\alpha^{(2)}\ \right)\frac{\partial}{\partial v_\alpha^{(2)}}F_2^{(1,2)} \tag{4.7.29}$$

Substituting the results (4.7.2) – (4.7.29) in equation (4.7.1) we get the transport equation for two point distribution function $F_2^{(1,2)}(v,g,\phi,\psi)$ in MHD turbulent flow in a rotating system in presence of dust particles as

$$\frac{\partial F_2^{(1,2)}}{\partial t}+\left(\ v_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}+v_\beta^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\ \right)F_2^{(1,2)}+g_\beta^{(1)}\left(\ \frac{\partial g_\alpha^{(1)}}{\partial v_\alpha^{(1)}}+\frac{\partial v_\alpha^{(1)}}{\partial g_\alpha^{(1)}}\ \right)\frac{\partial}{\partial x_\beta^{(1)}}F_2^{(1,2)}$$

$$+g_\beta^{(2)}\left(\ \frac{\partial g_\alpha^{(2)}}{\partial v_\alpha^{(2)}}+\frac{\partial v_\alpha^{(2)}}{\partial g_\alpha^{(2)}}\ \right)\frac{\partial}{\partial x_\beta^{(2)}}F_2^{(1,2)}-\frac{\partial}{\partial v_\alpha^{(1)}}\Big[\ \frac{1}{4\pi}\int\ \frac{\partial}{\partial x_\alpha^{(1)}}\left(\ \frac{1}{\left|\overline{x}^{(3)}-\overline{x}^{(1)}\right|}\ \right)$$

$$\times\left(\ \frac{\partial v_\alpha^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial v_\beta^{(3)}}{\partial x_\alpha^{(3)}}-\frac{\partial g_\alpha^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial g_\beta^{(3)}}{\partial x_\alpha^{(3)}}\ \right)F_3^{(1,2,3)}dx^{(3)}dv^{(3)}dg^{(3)}d\phi^{(3)}d\psi^{3)}\ \Big]$$

$$-\frac{\partial}{\partial v_\alpha^{(2)}}\Big[\ \frac{1}{4\pi}\int\ \frac{\partial}{\partial x_\alpha^{(2)}}\left(\ \frac{1}{\left|\overline{x}^{(3)}-\overline{x}^{(2)}\right|}\ \right)\left(\ \frac{\partial v_\alpha^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial v_\beta^{(3)}}{\partial x_\alpha^{(3)}}-\frac{\partial g_\alpha^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial g_\beta^{(3)}}{\partial x_\alpha^{(3)}}\ \right)$$

$$\times F_3^{(1,2,3)}dx^{(3)}dv^{(3)}dg^{(3)}d\phi^{(3)}d\psi^{3)}\ \Big]$$

$$+\nu\left(\ \frac{\partial}{\partial v_\alpha^{(1)}}\lim_{\overline{x}^{(3)}\to\overline{x}^{(1)}}+\frac{\partial}{\partial v_\alpha^{(2)}}\lim_{\overline{x}^{(3)}\to\overline{x}^{(2)}}\ \right)\frac{\partial^2}{\partial x_\beta^{(3)}\partial x_\beta^{(3)}}\int v_\alpha^{(3)}F_3^{(1,2,3)}dv^{(3)}dg^{(3)}d\phi^{(3)}d\psi^{(3)}$$

$$+\lambda\left(\ \frac{\partial}{\partial g_\alpha^{(1)}}\lim_{\overline{x}^{(3)}\to\overline{x}^{(1)}}+\frac{\partial}{\partial g_\alpha^{(2)}}\lim_{\overline{x}^{(3)}\to\overline{x}^{(2)}}\ \right)\frac{\partial^2}{\partial x_\beta^{(3)}\partial x_\beta^{(3)}}\int g_\alpha^{(3)}F_3^{(1,2,3)}dv^{(3)}dg^{(3)}d\phi^{(3)}d\psi^{(3)}$$

$$+\gamma\left(\ \frac{\partial}{\partial \phi^{(1)}}\lim_{\overline{x}^{(3)}\to\overline{x}^{(1)}}+\frac{\partial}{\partial \phi^{(2)}}\lim_{\overline{x}^{(3)}\to\overline{x}^{(2)}}\ \right)\frac{\partial^2}{\partial x_\beta^{(3)}\partial x_\beta^{(3)}}\int \phi^{(3)}F_3^{(1,2,3)}dv^{(3)}dg^{(3)}d\phi^{(3)}d\psi^{(3)}$$

$$+D\left(\ \frac{\partial}{\partial \psi^{(1)}}\lim_{\overline{x}^{(3)}\to\overline{x}^{(1)}}+\frac{\partial}{\partial \psi^{(2)}}\lim_{\overline{x}^{(3)}\to\overline{x}^{(2)}}\ \right)\frac{\partial^2}{\partial x_\beta^{(3)}\partial x_\beta^{(3)}}\int \psi^{(3)}F_3^{(1,2,3)}dv^{(3)}dg^{(3)}d\phi^{(3)}d\psi^{(3)}$$

$$+4\in_{m\alpha\beta}\Omega_m F_2^{(1,2)}$$

$$+\left[\; f\left(\; u_{\alpha}^{(1)}-v_{\alpha}^{(1)}\;\right)\frac{\partial}{\partial v_{\alpha}^{(1)}}+f\left(\; u_{\alpha}^{(2)}-v_{\alpha}^{(2)}\;\right)\frac{\partial}{\partial v_{\alpha}^{(2)}}\;\right]F_{2}^{(1,2)}=0 \qquad (4.7.30)$$

## 4.8 Equations for three-point distribution function $F_3^{(1,2,3)}$:

Differentiating equation (4.4.3) with respect to time, we get

$$\frac{\partial F_3^{(1,2,3)}}{\partial t}=\frac{\partial}{\partial t}\langle\; \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\rangle$$
$$=\langle\; \delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\frac{\partial}{\partial t}\delta\left(u^{(1)}-v^{(1)}\right)\;\rangle$$
$$+\langle\; \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\frac{\partial}{\partial t}\delta\left(h^{(1)}-g^{(1)}\right)\;\rangle$$
$$+\langle\; \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\frac{\partial}{\partial t}\delta\left(c^{(1)}-\psi^{(1)}\right)\;\rangle$$
$$+\langle\; \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\frac{\partial}{\partial t}\delta\left(u^{(2)}-v^{(2)}\right)\;\rangle$$
$$+\langle\; \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\frac{\partial}{\partial t}\delta\left(h^{(2)}-g^{(2)}\right)\;\rangle$$
$$+\langle\; \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\frac{\partial}{\partial t}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\;\rangle$$
$$+\langle\; \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\frac{\partial}{\partial t}\delta\left(c^{(2)}-\psi^{(2)}\right)\rangle$$
$$+\langle\; \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\frac{\partial}{\partial t}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\;\rangle$$

$$+\langle\ \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})$$
$$\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial}{\partial t}\delta(u^{(3)}-v^{(3)})\ \rangle$$
$$+\langle\ \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})$$
$$\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial}{\partial t}\delta(h^{(3)}-g^{(3)})\ \rangle$$
$$+\langle\ \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})$$
$$\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial}{\partial t}\delta(\theta^{(3)}-\phi^{(3)})\ \rangle$$
$$+\langle\ \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})$$
$$\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\frac{\partial}{\partial t}\delta(c^{(3)}-\psi^{(3)})\rangle$$

$$=\langle-\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial u^{(1)}}{\partial t}\frac{\partial}{\partial v^{(1)}}\delta(u^{(1)}-v^{(1)})\ \rangle$$
$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial h^{(1)}}{\partial t}\frac{\partial}{\partial g^{(1)}}\delta(h^{(1)}-g^{(1)})\ \rangle$$
$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial \theta^{(1)}}{\partial t}\frac{\partial}{\partial \phi^{(1)}}\delta(\theta^{(1)}-\phi^{(1)})\ \rangle$$
$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial c^{(1)}}{\partial t}\frac{\partial}{\partial \psi^{(1)}}\delta(c^{(1)}-\psi^{(1)})\ \rangle$$
$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial u^{(2)}}{\partial t}\frac{\partial}{\partial v^{(2)}}\delta(u^{(2)}-v^{(2)})\ \rangle$$
$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial h^{(2)}}{\partial t}\frac{\partial}{\partial g^{(2)}}\delta(h^{(2)}-g^{(2)})\ \rangle$$
$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial \theta^{(2)}}{\partial t}\frac{\partial}{\partial \phi^{(2)}}\delta(\theta^{(2)}-\phi^{(2)})\ \rangle$$

$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial c^{(2)}}{\partial t}\frac{\partial}{\partial \psi^{(2)}}\delta(c^{(2)}-\psi^{(2)})\ \rangle$$

$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial u^{(3)}}{\partial t}\frac{\partial}{\partial v^{(3)}}\delta(u^{(3)}-v^{(3)})\ \rangle$$

$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial h^{(3)}}{\partial t}\frac{\partial}{\partial g^{(3)}}\delta(h^{(3)}-g^{(3)})\ \rangle$$

$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(c^{(3)}-\psi^{(3)})\frac{\partial \theta^{(3)}}{\partial t}\frac{\partial}{\partial \phi^{(3)}}\delta(\theta^{(3)}-\phi^{(3)})\ \rangle$$

$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\frac{\partial c^{(3)}}{\partial t}\frac{\partial}{\partial \psi^{(3)}}\delta(c^{(3)}-\psi^{(3)})\ \rangle$$

Using equations (4.2.8) to (4.2.11), we get

$$=\langle -\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})$$
$$\{\ -\frac{\partial}{\partial x_\beta^{(1)}}\left(u_\alpha^{(1)}u_\beta^{(1)}-h_\alpha^{(1)}h_\beta^{(1)}\right)-\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(1)}}\int\ \left[\ \frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial u_\beta^{(1)}}{\partial x_\alpha^{(1)}}-\frac{\partial h_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial h_\beta^{(1)}}{\partial x_\alpha^{(1)}}\ \right]\frac{d\bar{x}'''}{|\bar{x}'''-\bar{x}|}$$
$$+\nu\nabla^2u_\alpha^{(1)}-2\in_{m\alpha\beta}\Omega_m u_\alpha^{(1)}+f\left(u_\alpha^{(1)}-v_\alpha^{(1)}\right)\ \}\times\frac{\partial}{\partial v_\alpha^{(1)}}\delta(u^{(1)}-v^{(1)})\ \rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})$$
$$\{\ -\frac{\partial}{\partial x_\beta^{(1)}}\left(h_\alpha^{(1)}u_\beta^{(1)}-u_\alpha^{(1)}h_\beta^{(1)}\right)+\lambda\nabla^2h_\alpha^{(1)}\ \}\times\frac{\partial}{\partial g_\alpha^{(1)}}\delta(h^{(1)}-g^{(1)})\ \rangle$$
$$+\langle -\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})$$
$$\times\left\{-u_\beta^{(1)}\frac{\partial \theta^{(1)}}{\partial x_\beta^{(1)}}+\gamma\nabla^2\theta^{(1)}\ \right\}\times\frac{\partial}{\partial \phi^{(1)}}\delta(\theta^{(1)}-\phi^{(1)})\ \rangle$$

$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)$$
$$\left\{-u_\beta^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}+D\nabla^2 c^{(1)}\right\}\times\frac{\partial}{\partial\psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)$$
$$\{\ -\frac{\partial}{\partial x_\beta^{(2)}}\left(u_\alpha^{(2)}u_\beta^{(2)}-h_\alpha^{(2)}h_\beta^{(2)}\right)-\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(2)}}\int\left[\ \frac{\partial u_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial u_\beta^{(2)}}{\partial x_\alpha^{(2)}}-\frac{\partial h_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial h_\beta^{(2)}}{\partial x_\alpha^{(2)}}\ \right]\frac{d\overline{x}'''}{\left|\overline{x}'''-\overline{x}'\right|}$$
$$+\nu\nabla^2 u_\alpha^{(2)}-2\in_{m\alpha\beta}\Omega_m u_\alpha^{(2)}+f\left(u_\alpha^{(2)}-v_\alpha^{(2)}\right)\ \}\times\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)$$
$$\left\{-u_\beta^{(2)}\frac{\partial c^{(2)}}{\partial x_\beta^{(2)}}+D\nabla^2 c^{(2)}\ \right\}\frac{\partial}{\partial\psi^{(2)}}\delta\left(c^{(2)}-\psi^{(2)}\right)\rangle$$
$$\{\ -\frac{\partial}{\partial x_\beta^{(2)}}\left(h_\alpha^{(2)}u_\beta^{(2)}-u_\alpha^{(2)}h_\beta^{(2)}\right)+\lambda\nabla^2 h_\alpha^{(2)}\ \}\frac{\partial}{\partial g_\alpha^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)$$
$$\times\left\{-u_\beta^{(2)}\frac{\partial\theta^{(2)}}{\partial x_\beta^{(2)}}+\gamma\nabla^2\theta^{(2)}\ \right\}\times\frac{\partial}{\partial\phi^{(2)}}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)$$
$$\{\ -\frac{\partial}{\partial x_\beta^{(3)}}\left(u_\alpha^{(3)}u_\beta^{(3)}-h_\alpha^{(3)}h_\beta^{(3)}\right)-\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(3)}}\int\left[\ \frac{\partial u_\alpha^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial u_\beta^{(3)}}{\partial x_\alpha^{(3)}}-\frac{\partial h_\alpha^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial h_\beta^{(3)}}{\partial x_\alpha^{(3)}}\ \right]\frac{d\overline{x}'''}{\left|\overline{x}'''-\overline{x}''\right|}$$
$$+\nu\nabla^2 u_\alpha^{(3)}-2\in_{m\alpha\beta}\Omega_m u_\alpha^{(3)}+f\left(u_\alpha^{(3)}-v_\alpha^{(3)}\right)\ \}\times\frac{\partial}{\partial v_\alpha^{(3)}}\delta\left(u^{(3)}-v^{(3)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)$$
$$\{\ -\frac{\partial}{\partial x_\beta^{(3)}}\left(h_\alpha^{(3)}u_\beta^{(3)}-u_\alpha^{(3)}h_\beta^{(3)}\right)+\lambda\nabla^2 h_\alpha^{(3)}\ \}\frac{\partial}{\partial g_\alpha^{(3)}}\delta\left(h^{(3)}-g^{(3)}\right)\rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)$$

$$\times \left\{ -u_{\beta}^{(3)} \frac{\partial \theta^{(3)}}{\partial x_{\beta}^{(3)}} + \gamma \nabla^2 \theta^{(3)} \right\} \times \frac{\partial}{\partial \phi^{(3)}} \delta\left(\theta^{(3)} - \phi^{(3)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)$$
$$\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)$$
$$\left\{ -u_{\beta}^{(3)} \frac{\partial c^{(3)}}{\partial x_{\beta}^{(3)}} + D \nabla^2 c^{(3)} \right\} \frac{\partial}{\partial \psi^{(3)}} \delta\left(c^{(3)} - \psi^{(3)}\right) \Big\rangle$$

$$= \Big\langle \ \delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)$$
$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \frac{\partial u_{\alpha}^{(1)} u_{\beta}^{(1)}}{\partial x_{\beta}^{(1)}} \frac{\partial}{\partial v_{\alpha}^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)$$
$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \frac{\partial h_{\alpha}^{(1)} h_{\beta}^{(1)}}{\partial x_{\beta}^{(1)}} \frac{\partial}{\partial v_{\alpha}^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \Big\rangle$$

$$+ \Big\langle \ \delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)$$
$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \frac{1}{4\pi} \frac{\partial}{\partial x_{\alpha}^{(1)}} \int \left[ \frac{\partial u_{\alpha}^{(1)}}{\partial x_{\beta}^{(1)}} \frac{\partial u_{\beta}^{(1)}}{\partial x_{\alpha}^{(1)}} - \frac{\partial h_{\alpha}^{(1)}}{\partial x_{\beta}^{(1)}} \frac{\partial h_{\beta}^{(1)}}{\partial x_{\alpha}^{(1)}} \right]$$
$$\times \frac{d\bar{x}'''}{\left|\bar{x}''' - \bar{x}\right|} \frac{\partial}{\partial v_{\alpha}^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \Big\rangle + \Big\langle -\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)$$
$$\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \nu \nabla^2 u_{\alpha}^{(1)} \frac{\partial}{\partial v_{\alpha}^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \Big\rangle$$

$$+ \Big\langle \ \delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)$$
$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times 2 \in_{m\alpha\beta} \Omega_m u_{\alpha}^{(1)} \frac{\partial}{\partial v_{\alpha}^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)$$
$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times f\left(u_{\alpha}{}^{(1)} - v_{\alpha}{}^{(1)}\right) \frac{\partial}{\partial v_{\alpha}^{(1)}} \delta\left(u^{(1)} - v^{(1)}\right) \Big\rangle$$

$$+ \Big\langle \ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)$$
$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \frac{\partial h_{\alpha}^{(1)} u_{\beta}^{(1)}}{\partial x_{\beta}^{(1)}} \frac{\partial}{\partial g_{\alpha}^{(1)}} \delta\left(h^{(1)} - g^{(1)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)$$
$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \frac{\partial u_{\alpha}^{(1)} h_{\beta}^{(1)}}{\partial x_{\beta}^{(1)}} \frac{\partial}{\partial g_{\alpha}^{(1)}} \delta\left(h^{(1)} - g^{(1)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)\delta\left(c^{(2)} - \psi^{(2)}\right)$$
$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \lambda \nabla^2 h_{\alpha}^{(1)} \frac{\partial}{\partial g_{\alpha}^{(1)}} \delta\left(h^{(1)} - g^{(1)}\right) \Big\rangle$$

$$+\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial\theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\ \rangle$$

$$+\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\gamma\nabla^2\theta^{(1)}\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\ \rangle$$

$$+\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial\psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\ \rangle$$

$$+\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times D\nabla^2c^{(1)}\frac{\partial}{\partial\psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\ \rangle$$

$$+\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial u_\alpha^{(2)}u_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\ \rangle$$

$$+\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial h_\alpha^{(2)}h_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\ \rangle$$

$$+\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(2)}}\int\left[\ \frac{\partial u_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial u_\beta^{(2)}}{\partial x_\alpha^{(2)}}-\frac{\partial h_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial h_\beta^{(2)}}{\partial x_\alpha^{(2)}}\ \right]$$

$$\times\frac{d\bar{x}'''}{\left|\bar{x}'''-\bar{x}'\right|}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\ \rangle$$

$$+\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\nu\nabla^2u_\alpha^{(2)}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\ \rangle$$

$$+\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(2)}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\ \rangle$$

$$+\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times f\left(u_\alpha{}^{(2)}-v_\alpha{}^{(2)}\right)\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\ \rangle$$

$$+\langle\ \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times\frac{\partial h_\alpha^{(2)}u_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial g_\alpha^{(2)}}\delta(h^{(2)}-g^{(2)})\ \rangle$$

$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times\frac{\partial u_\alpha^{(2)}h_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial g_\alpha^{(2)}}\delta(h^{(2)}-g^{(2)})\ \rangle$$

$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(\theta^{(2)}-\phi^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times\lambda\nabla^2 h_\alpha^{(2)}\frac{\partial}{\partial g_\alpha^{(2)}}\delta(h^{(2)}-g^{(2)})\ \rangle$$

$$+\langle\ \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times u_\beta^{(2)}\frac{\partial\theta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial\phi^{(2)}}\delta(\theta^{(2)}-\phi^{(2)})\ \rangle$$

$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(c^{(2)}-\psi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times\gamma\nabla^2\theta^{(2)}\frac{\partial}{\partial\phi^{(2)}}\delta(\theta^{(2)}-\phi^{(2)})\ \rangle$$

$$+\langle\ \delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times u_\beta^{(2)}\frac{\partial c^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial\psi^{(2)}}\delta(c^{(2)}-\psi^{(2)})\ \rangle$$

$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(u^{(3)}-v^{(3)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times D\nabla^2 c^{(2)}\frac{\partial}{\partial\psi^{(2)}}\delta(c^{(2)}-\psi^{(2)})\ \rangle$$

$$+\langle\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times\frac{\partial u_\alpha^{(3)}u_\beta^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial}{\partial v_\alpha^{(3)}}\delta(u^{(3)}-v^{(3)})\ \rangle$$

$$+\langle-\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times\frac{\partial h_\alpha^{(3)}h_\beta^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial}{\partial v_\alpha^{(3)}}\delta(u^{(3)}-v^{(3)})\ \rangle$$

$$+\langle\delta(u^{(1)}-v^{(1)})\delta(h^{(1)}-g^{(1)})\delta(\theta^{(1)}-\phi^{(1)})\delta(c^{(1)}-\psi^{(1)})\delta(u^{(2)}-v^{(2)})\delta(h^{(2)}-g^{(2)})\delta(\theta^{(2)}-\phi^{(2)})$$
$$\delta(c^{(2)}-\psi^{(2)})\delta(h^{(3)}-g^{(3)})\delta(\theta^{(3)}-\phi^{(3)})\delta(c^{(3)}-\psi^{(3)})\times\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(3)}}\int\ \left[\ \frac{\partial u_\alpha^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial u_\beta^{(3)}}{\partial x_\alpha^{(3)}}-\frac{\partial h_\alpha^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial h_\beta^{(3)}}{\partial x_\alpha^{(3)}}\ \right]$$

$$\times \frac{d\overline{x}'''}{\left|\overline{x}''' - \overline{x}''\right|} \frac{\partial}{\partial v_{\alpha}^{(3)}} \delta\left(u^{(3)} - v^{(3)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \nu\nabla^2 u_{\alpha}^{(3)} \frac{\partial}{\partial v_{\alpha}^{(3)}} \delta\left(u^{(3)} - v^{(3)}\right) \Big\rangle$$

$$+ \Big\langle \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times 2 \in_{m\alpha\beta} \Omega_m u_{\alpha}^{(3)} \frac{\partial}{\partial v_{\alpha}^{(3)}} \delta\left(u^{(3)} - v^{(3)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times f\left(u_{\alpha}{}^{(3)} - v_{\alpha}{}^{(3)}\right) \frac{\partial}{\partial v_{\alpha}^{(3)}} \delta\left(u^{(3)} - v^{(3)}\right) \Big\rangle$$

$$+ \Big\langle \ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \frac{\partial h_{\alpha}^{(3)} u_{\beta}^{(3)}}{\partial x_{\beta}^{(3)}} \frac{\partial}{\partial g_{\alpha}^{(3)}} \delta\left(h^{(3)} - g^{(3)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \frac{\partial u_{\alpha}^{(3)} h_{\beta}^{(3)}}{\partial x_{\beta}^{(3)}} \frac{\partial}{\partial g_{\alpha}^{(3)}} \delta\left(h^{(3)} - g^{(3)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \lambda\nabla^2 h_{\alpha}^{(3)} \frac{\partial}{\partial g_{\alpha}^{(3)}} \delta\left(h^{(3)} - g^{(3)}\right) \Big\rangle$$

$$+ \Big\langle \ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times u_{\beta}^{(3)} \frac{\partial \theta^{(3)}}{\partial x_{\beta}^{(3)}} \frac{\partial}{\partial \phi^{(3)}} \delta\left(\theta^{(3)} - \phi^{(3)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times \gamma\nabla^2 \theta^{(3)} \frac{\partial}{\partial \phi^{(3)}} \delta\left(\theta^{(3)} - \phi^{(3)}\right) \Big\rangle$$

$$+ \Big\langle \ \delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right) \times u_{\beta}^{(3)} \frac{\partial c^{(3)}}{\partial x_{\beta}^{(3)}} \frac{\partial}{\partial \psi^{(3)}} \delta\left(c^{(3)} - \psi^{(3)}\right) \Big\rangle$$

$$+ \Big\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(u^{(3)} - v^{(3)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right) \times D\nabla^2 c^{(3)} \frac{\partial}{\partial \psi^{(3)}} \delta\left(c^{(3)} - \psi^{(3)}\right) \Big\rangle \qquad (4.8.1)$$

Various terms in the above equation can be simplified as that they may be expressed in terms of one, two, three and four point distribution functions.

The 1st term in the above equation is simplified as follows

$$\langle\ \delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial u_\alpha^{(1)}u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle$$

$$=\langle u_\beta^{(1)}\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle$$

$$=\langle -u_\beta^{(1)}\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle;(\text{since}\,\frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}}=1)$$

$$=\langle -\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle \qquad (4.8.2)$$

Similarly, 7th, 10th, 12th terms of right hand-side of equation (4.8.1) can be simplified as follows;

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial h_\alpha^{(1)}u_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial g_\alpha^{(1)}}\delta\left(h^{(1)}-g^{(1)}\right)\ \rangle$$

$$=\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(h^{(1)}-g^{(1)}\right)\ \rangle \qquad (4.8.3)$$

10th term,

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial\theta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\ \rangle$$

$$=\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$

$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right) \qquad (4.8.4)$$

and 12th term

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial c^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial \psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\ \rangle$$

$$=\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\ \rangle \quad (4.8.5)$$

Adding these equations from (4.8.2) to (4.8.5), we get

$$\langle -\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(h^{(1)}-g^{(1)}\right)\ \rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\ \rangle$$
$$+\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\ \rangle$$

$$=-\frac{\partial}{\partial x_\beta^{(1)}}\langle u_\beta^{(1)}\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\ \rangle$$

$$=-\frac{\partial}{\partial x_\beta^{(1)}}v_\beta^{(1)}F_3^{(1,2,3)}$$

[ Applying the properties of distribution functions ]

$$=-v_\beta^{(1)}\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(1)}} \quad (4.8.6)$$

Similarly, 14th, 20th , 23rd and 25th terms of right hand-side of equation (4.8.1) can be simplified as follows:

$$\Big\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial u_\alpha^{(2)}u_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\Big\rangle$$

$$=\Big\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\Big\rangle \tag{4.8.7}$$

20th term,

$$\Big\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial h_\alpha^{(2)}u_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial g_\alpha^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\Big\rangle$$
$$=\Big\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\Big\rangle \tag{4.8.8}$$

23th term,

$$\Big\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(2)}\frac{\partial\theta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial\phi^{(2)}}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\Big\rangle$$
$$=\Big\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\delta\left(\theta^{(2)}-\phi^{(2)}\right)\Big\rangle \tag{4.8.9}$$

and 25th term,

$$\Big\langle \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(2)}\frac{\partial c^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial\psi^{(2)}}\delta\left(c^{(2)}-\psi^{(2)}\right)\Big\rangle$$
$$=\Big\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}\delta\left(c^{(2)}-\psi^{(2)}\right)\Big\rangle \tag{4.8.10}$$

Adding equations (4.8.7) to (4.8.10), we get

$$-\frac{\partial}{\partial x_\beta^{(2)}}\langle u_\beta^{(2)}\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\ \rangle$$
$$=-v_\beta^{(2)}\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(2)}} \tag{4.8.11}$$

Similarly, 27th, 33rd , 36th and 38th terms of right hand-side of equation (4.8.1) can be simplified as follows;

$$\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial u_\alpha^{(3)}u_\beta^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(3)}-v^{(3)}\right)\ \rangle$$

$$=\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(3)}\frac{\partial}{\partial x_\beta^{(3)}}\delta\left(u^{(3)}-v^{(3)}\right)\ \rangle \tag{4.8.12}$$

33rd term,

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial h_\alpha^{(3)}u_\beta^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial}{\partial g_\alpha^{(3)}}\delta\left(h^{(3)}-g^{(3)}\right)\ \rangle$$
$$=\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(3)}\frac{\partial}{\partial x_\beta^{(3)}}\delta\left(h^{(3)}-g^{(3)}\right)\ \rangle \tag{4.8.13}$$

36th term,

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(3)}\frac{\partial\theta^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial}{\partial\phi^{(3)}}\delta\left(\theta^{(3)}-\phi^{(3)}\right)\ \rangle$$
$$=\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times u_\beta^{(3)}\frac{\partial}{\partial x_\beta^{(3)}}\delta\left(\theta^{(3)}-\phi^{(3)}\right)\ \rangle \tag{4.8.14}$$

and 38th term,

$$\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\times u_\beta^{(3)}\frac{\partial c^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial}{\partial \psi^{(3)}}\delta\left(c^{(3)}-\psi^{(3)}\right)\ \rangle$$

$$=\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\times u_\beta^{(3)}\frac{\partial}{\partial x_\beta^{(3)}}\delta\left(c^{(3)}-\psi^{(3)}\right)\ \rangle \qquad (4.8.15)$$

Adding equations (4.8.12) to (4.8.15), we get

$$-\frac{\partial}{\partial x_\beta^{(3)}}\langle u_\beta^{(3)}\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)$$
$$\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\ \rangle$$
$$=-v_\beta^3\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(3)}} \qquad (4.8.16)$$

Similarly, 2nd ,8th ,15th ,21st ,28th and 34th terms of right hand-side of equation (4.8.1) can be simplified as follows;

$$\langle-\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial h_\alpha^{(1)}h_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\ \rangle$$
$$=-g_\beta^{(1)}\frac{\partial g_\alpha^{(1)}}{\partial v_\alpha^{(1)}}\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(1)}} \qquad (4.8.17)$$

8th term,

$$\langle-\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{\partial u_\alpha^{(1)}h_\beta^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial}{\partial g_\alpha}\delta\left(h^{(1)}-g^{(1)}\right)\rangle$$

$$=--g_\beta^{(1)}\frac{\partial v_\alpha^{(1)}}{\partial g_\alpha^{(1)}}\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(1)}} \qquad (4.8.18)$$

15th term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)\delta\big(c^{(2)}-\psi^{(2)}\big)$$
$$\delta\big(u^{(3)}-v^{(3)}\big)\delta\big(h^{(3)}-g^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times\frac{\partial h_\alpha^{(2)}h_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\big(u^{(2)}-v^{(2)}\big)\Big\rangle$$

$$= - g_\beta^{(2)}\frac{\partial g_\alpha^{(2)}}{\partial v_\alpha^{(2)}}\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(2)}} \tag{4.8.19}$$

21st term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)\delta\big(c^{(2)}-\psi^{(2)}\big)$$
$$\delta\big(u^{(3)}-v^{(3)}\big)\delta\big(h^{(3)}-g^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times\frac{\partial u_\alpha^{(2)}h_\beta^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial}{\partial g_\alpha^{(2)}}\delta\big(h^{(2)}-g^{(2)}\big)\Big\rangle$$

$$= - g_\beta^{(2)}\frac{\partial v_\alpha^{(2)}}{\partial g_\alpha^{(2)}}\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(2)}} \tag{4.8.20}$$

28th term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(c^{(2)}-\psi^{(2)}\big)\delta\big(h^{(3)}-g^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times\frac{\partial h_\alpha^{(3)}h_\beta^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial}{\partial v_\alpha^{(3)}}\delta\big(u^{(3)}-v^{(3)}\big)\Big\rangle$$

$$= - g_\beta^{(3)}\frac{\partial g_\alpha^{(3)}}{\partial v_\alpha^{(3)}}\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(3)}} \tag{4.8.21}$$

and 34th term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(c^{(2)}-\psi^{(2)}\big)\delta\big(u^{(3)}-v^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times\frac{\partial u_\alpha^{(3)}h_\beta^{(3)}}{\partial x_\beta^{(3)}}\frac{\partial}{\partial g_\alpha^{(3)}}\delta\big(h^{(3)}-g^{(3)}\big)\Big\rangle$$

$$= - g_\beta^{(3)}\frac{\partial v_\alpha^{(3)}}{\partial g_\alpha^{(3)}}\frac{\partial F_3^{(1,2,3)}}{\partial x_\beta^{(3)}} \tag{4.8.22}$$

Fourth term can be reduced as

$$\Big\langle -\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)\delta\big(c^{(2)}-\psi^{(2)}\big)$$
$$\delta\big(u^{(3)}-v^{(3)}\big)\delta\big(h^{(3)}-g^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times\nu\nabla^2u_\alpha^{(1)}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\big(u^{(1)}-v^{(1)}\big)$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \langle \nabla^2 u_\alpha^{(1)} [\ \delta(u^{(1)} - v^{(1)}) \delta(h^{(1)} - g^{(1)}) \delta(\theta^{(1)} - \phi^{(1)}) \delta(c^{(1)} - \psi^{(1)}) \delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)})$$
$$\delta(\theta^{(2)} - \phi^{(2)}) \delta(c^{(2)} - \psi^{(2)}) \delta(u^{(3)} - v^{(3)}) \delta(h^{(3)} - g^{(3)}) \delta(\theta^{(3)} - \phi^{(3)}) \delta(c^{(3)} - \psi^{(3)})\ ]\ \rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \frac{\partial^2}{\partial x_\beta^{(1)} \partial x_\beta^{(1)}} \langle u_\alpha^{(1)} [\ \delta(u^{(1)} - v^{(1)}) \delta(h^{(1)} - g^{(1)}) \delta(\theta^{(1)} - \phi^{(1)}) \delta(c^{(1)} - \psi^{(1)}) \delta(u^{(2)} - v^{(2)})$$
$$\delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)}) \delta(c^{(2)} - \psi^{(2)}) \delta(u^{(3)} - v^{(3)}) \delta(h^{(3)} - g^{(3)}) \delta(\theta^{(3)} - \phi^{(3)}) \delta(c^{(3)} - \psi^{(3)})\ ]\ \rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \lim_{\underline{x}^{(4)} \to \underline{x}^{(1)}} \frac{\partial^2}{\partial x_\beta^{(4)} \partial x_\beta^{(4)}} \langle u_\alpha^{(4)} [\ \delta(u^{(1)} - v^{(1)}) \delta(h^{(1)} - g^{(1)}) \delta(\theta^{(1)} - \phi^{(1)}) \delta(c^{(1)} - \psi^{(1)})$$
$$\delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)}) \delta(c^{(2)} - \psi^{(2)}) \delta(u^{(3)} - v^{(3)}) \delta(h^{(3)} - g^{(3)})$$
$$\delta(\theta^{(3)} - \phi^{(3)}) \delta(c^{(3)} - \psi^{(3)})\ ]\ \rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \lim_{\underline{x}^{(4)} \to \underline{x}^{(1)}} \frac{\partial^2}{\partial x_\beta^{(4)} \partial x_\beta^{(4)}} \langle \int u_\alpha^{(4)} \delta(u^{(4)} - v^{(4)}) \delta(h^{(4)} - g^{(4)}) \delta(\theta^{(4)} - \phi^{(4)}) \delta(c^{(4)} - \psi^{(4)})$$

$$\delta(u^{(3)} - v^{(3)}) \delta(h^{(3)} - g^{(3)}) \delta(\theta^{(3)} - \phi^{(3)}) \delta(c^{(3)} - \psi^{(3)}) \delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)})$$
$$\delta(c^{(2)} - \psi^{(2)}) \delta(u^{(1)} - v^{(1)}) \delta(h^{(1)} - g^{(1)}) \delta(\theta^{(1)} - \phi^{(1)}) \delta(c^{(1)} - \psi^{(1)}) dv^{(4)} dg^{(4)} d\phi^{(4)} d\psi^{(4)}\ \rangle$$

$$= -\nu \frac{\partial}{\partial v_\alpha^{(1)}} \lim_{\underline{x}^{(4)} \to \underline{x}^{(1)}} \frac{\partial^2}{\partial x_\beta^{(4)} \partial x_\beta^{(4)}} \int v_\alpha^{(4)} F_4^{(1,2,3,4)} dv^{(4)} dg^{(4)} d\phi^{(4)} d\psi^{(4)} \qquad (4.8.23)$$

Similarly, 9th ,11th ,13th ,17th ,22nd ,24th ,26th ,30th ,35th ,37th and 39th terms of right hand-side of equation (4.8.1) can be simplified as follows;

$$\langle -\delta(u^{(1)} - v^{(1)}) \delta(\theta^{(1)} - \phi^{(1)}) \delta(c^{(1)} - \psi^{(1)}) \delta(u^{(2)} - v^{(2)}) \delta(h^{(2)} - g^{(2)}) \delta(\theta^{(2)} - \phi^{(2)}) \delta(c^{(2)} - \psi^{(2)})$$
$$\delta(u^{(3)} - v^{(3)}) \delta(h^{(3)} - g^{(3)}) \delta(\theta^{(3)} - \phi^{(3)}) \delta(c^{(3)} - \psi^{(3)}) \times \lambda \nabla^2 h_\alpha^{(1)} \frac{\partial}{\partial g_\alpha^{(1)}} \delta(h^{(1)} - g^{(1)})\ \rangle$$

$$= -\lambda \frac{\partial}{\partial g_\alpha^{(1)}} \lim_{\underline{x}^{(4)} \to \underline{x}^{(1)}} \frac{\partial^2}{\partial x_\beta^{(4)} \partial x_\beta^{(4)}} \int g_\alpha^{(4)} F_4^{(1,2,3,4)} dv^{(4)} dg^{(4)} d\phi^{(4)} d\psi^{(4)} \qquad (4.8.24)$$

11th term,

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\right.$$

$$\left.\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\gamma\nabla^2\theta^{(1)}\frac{\partial}{\partial\phi^{(1)}}\delta\left(\theta^{(1)}-\phi^{(1)}\right)\right\rangle$$

$$=-\gamma\frac{\partial}{\partial\phi^{(1)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(1)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int\phi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.25}$$

13th term,

$$+\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\right.$$

$$\left.\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times D\nabla^2c^{(1)}\frac{\partial}{\partial\psi^{(1)}}\delta\left(c^{(1)}-\psi^{(1)}\right)\right\rangle$$

$$=-D\frac{\partial}{\partial\psi^{(1)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(1)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(3)}}\int\psi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.26}$$

17th term,

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\right.$$

$$\left.\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\nu\nabla^2u_\alpha^{(2)}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\right\rangle$$

$$=-\nu\frac{\partial}{\partial v_\alpha^{(2)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(2)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int v_\alpha^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.27}$$

22nd term,

$$\left\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\right.$$

$$\left.\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\lambda\nabla^2h_\alpha^{(2)}\frac{\partial}{\partial g_\alpha^{(2)}}\delta\left(h^{(2)}-g^{(2)}\right)\right\rangle$$

$$=-\lambda\frac{\partial}{\partial g_\alpha^{(2)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(2)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int g_\alpha^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.28}$$

24nd term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(c^{(2)}-\psi^{(2)}\big)$$
$$\delta\big(u^{(3)}-v^{(3)}\big)\delta\big(h^{(3)}-g^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times\gamma\nabla^2\theta^{(2)}\frac{\partial}{\partial\phi^{(2)}}\delta\big(\theta^{(2)}-\phi^{(2)}\big)\Big\rangle$$

$$=-\gamma\frac{\partial}{\partial\phi^{(2)}}\lim_{\overline{x}^{(4)}\to\overline{x}^{(2)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int\phi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.29}$$

26th term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(u^{(3)}-v^{(3)}\big)\delta\big(h^{(3)}-g^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times D\nabla^2c^{(2)}\frac{\partial}{\partial\psi^{(2)}}\delta\big(c^{(2)}-\psi^{(2)}\big)\Big\rangle$$

$$=-D\frac{\partial}{\partial\psi^{(2)}}\lim_{\overline{x}^{(4)}\to\overline{x}^{(2)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int\psi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.30}$$

30th term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(c^{(2)}-\psi^{(2)}\big)\delta\big(h^{(3)}-g^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times\nu\nabla^2u_\alpha^{(3)}\frac{\partial}{\partial v_\alpha^{(3)}}\delta\big(u^{(3)}-v^{(3)}\big)\Big\rangle$$

$$=-\nu\frac{\partial}{\partial v_\alpha^{(3)}}\lim_{\overline{x}^{(4)}\to\overline{x}^{(3)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int v_\alpha^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.31}$$

35th term,

$$\Big\langle -\delta\big(u^{(1)}-v^{(1)}\big)\delta\big(h^{(1)}-g^{(1)}\big)\delta\big(\theta^{(1)}-\phi^{(1)}\big)\delta\big(c^{(1)}-\psi^{(1)}\big)\delta\big(u^{(2)}-v^{(2)}\big)\delta\big(h^{(2)}-g^{(2)}\big)\delta\big(\theta^{(2)}-\phi^{(2)}\big)$$
$$\delta\big(c^{(2)}-\psi^{(2)}\big)\delta\big(u^{(3)}-v^{(3)}\big)\delta\big(\theta^{(3)}-\phi^{(3)}\big)\delta\big(c^{(3)}-\psi^{(3)}\big)\times\lambda\nabla^2h_\alpha^{(3)}\frac{\partial}{\partial g_\alpha^{(3)}}\delta\big(h^{(3)}-g^{(3)}\big)\Big\rangle$$

$$=-\lambda\frac{\partial}{\partial g_\alpha^{(3)}}\lim_{\overline{x}^{(4)}\to\overline{x}^{(3)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int g_\alpha^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.32}$$

37th term,

$$\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\gamma\nabla^2\theta^{(3)}\frac{\partial}{\partial\phi^{(3)}}\delta\left(\theta^{(3)}-\phi^{(3)}\right)\rangle$$

$$=-\gamma\frac{\partial}{\partial\phi^{(3)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(3)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int\phi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.33}$$

39th term,

$$\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\times D\nabla^2c^{(3)}\frac{\partial}{\partial\psi^{(3)}}\delta\left(c^{(3)}-\psi^{(3)}\right)\rangle$$

$$=-D\frac{\partial}{\partial\psi^{(3)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(3)}}\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int\psi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)} \tag{4.8.34}$$

We reduce the third term of right hand side of equation (4.8.1),

$$\langle\ \delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)$$
$$\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(1)}}\int\left[\frac{\partial u_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial u_\beta^{(1)}}{\partial x_\alpha^{(1)}}-\frac{\partial h_\alpha^{(1)}}{\partial x_\beta^{(1)}}\frac{\partial h_\beta^{(1)}}{\partial x_\alpha^{(1)}}\right]\frac{d\bar{x}'''}{\left|\bar{x}'''-\bar{x}\right|}\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\rangle$$

$$=\frac{\partial}{\partial v_\alpha^{(1)}}\left[\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(1)}}\left(\frac{1}{\left|\bar{x}^{(4)}-\bar{x}^{(1)}\right|}\right)\left(\frac{\partial v_\alpha^{(4)}}{\partial x_\beta^{(4)}}\frac{\partial v_\beta^{(4)}}{\partial x_\alpha^{(4)}}-\frac{\partial g_\alpha^{(4)}}{\partial x_\beta^{(4)}}\frac{\partial g_\beta^{(4)}}{\partial x_\alpha^{(4)}}\right)F_4^{(1,2,3,4)}dx^{(4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}\right] \tag{4.8.35}$$

16th term,

$$\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times\frac{1}{4\pi}\frac{\partial}{\partial x_\alpha^{(2)}}\int\left[\frac{\partial u_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial u_\beta^{(2)}}{\partial x_\alpha^{(2)}}-\frac{\partial h_\alpha^{(2)}}{\partial x_\beta^{(2)}}\frac{\partial h_\beta^{(2)}}{\partial x_\alpha^{(2)}}\right]$$
$$\times\frac{d\bar{x}'''}{\left|\bar{x}'''-\bar{x}'\right|}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\rangle$$

$$= \frac{\partial}{\partial v_\alpha^{(2)}} \Big[ \frac{1}{4\pi} \int \frac{\partial}{\partial x_\alpha^{(2)}} \Big( \frac{1}{\left| \bar{x}^{(4)} - \bar{x}^{(2)} \right|} \Big) \Big( \frac{\partial v_\alpha^{(4)}}{\partial x_\beta^{(4)}} \frac{\partial v_\beta^{(4)}}{\partial x_\alpha^{(4)}} - \frac{\partial g_\alpha^{(4)}}{\partial x_\beta^{(4)}} \frac{\partial g_\beta^{(4)}}{\partial x_\alpha^{(4)}} \Big)$$
$$F_4^{(1,2,3,4)} dx^{(4)} dv^{(4)} dg^{(4)} d\phi^{(4)} d\psi^{(4)} \Big] \quad (4.8.36)$$

Similarly, 29th term,

$$\Big\langle \delta\big(u^{(1)} - v^{(1)}\big)\delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(c^{(1)} - \psi^{(1)}\big)\delta\big(u^{(2)} - v^{(2)}\big)\delta\big(h^{(2)} - g^{(2)}\big)$$
$$\delta\big(\theta^{(2)} - \phi^{(2)}\big)\delta\big(c^{(2)} - \psi^{(2)}\big)\delta\big(h^{(3)} - g^{(3)}\big)\delta\big(\theta^{(3)} - \phi^{(3)}\big)\delta\big(c^{(3)} - \psi^{(3)}\big)$$
$$\times \frac{1}{4\pi} \frac{\partial}{\partial x_\alpha^{(3)}} \int \Big[ \frac{\partial u_\alpha^{(3)}}{\partial x_\beta^{(3)}} \frac{\partial u_\beta^{(3)}}{\partial x_\alpha^{(3)}} - \frac{\partial h_\alpha^{(3)}}{\partial x_\beta^{(3)}} \frac{\partial h_\beta^{(3)}}{\partial x_\alpha^{(3)}} \Big] \frac{d\bar{x}'''}{\left| \bar{x}''' - \bar{x}'' \right|} \frac{\partial}{\partial v_\alpha^{(3)}} \delta\big(u^{(3)} - v^{(3)}\big) \Big\rangle$$

$$= \frac{\partial}{\partial v_\alpha^{(3)}} \Big[ \frac{1}{4\pi} \int \frac{\partial}{\partial x_\alpha^{(3)}} \Big( \frac{1}{\left| \bar{x}^{(4)} - \bar{x}^{(3)} \right|} \Big) \Big( \frac{\partial v_\alpha^{(4)}}{\partial x_\beta^{(4)}} \frac{\partial v_\beta^{(4)}}{\partial x_\alpha^{(4)}} - \frac{\partial g_\alpha^{(4)}}{\partial x_\beta^{(4)}} \frac{\partial g_\beta^{(4)}}{\partial x_\alpha^{(4)}} \Big)$$
$$F_4^{(1,2,3,4)} dx^{(4)} dv^{(4)} dg^{(4)} d\phi^{(4)} d\psi^{(4)} \Big] \quad (4.8.37)$$

Fifth and sixth terms of right hand side of equation (4.8.1),

$$\Big\langle \delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(c^{(1)} - \psi^{(1)}\big)\delta\big(u^{(2)} - v^{(2)}\big)\delta\big(h^{(2)} - g^{(2)}\big)\delta\big(\theta^{(2)} - \phi^{(2)}\big)\delta\big(c^{(2)} - \psi^{(2)}\big)$$
$$\delta\big(u^{(3)} - v^{(3)}\big)\delta\big(h^{(3)} - g^{(3)}\big)\delta\big(\theta^{(3)} - \phi^{(3)}\big)\delta\big(c^{(3)} - \psi^{(3)}\big) \times 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}} \delta\big(u^{(1)} - v^{(1)}\big) \Big\rangle$$

$$= \Big\langle 2 \in_{m\alpha\beta} \Omega_m u_\alpha^{(1)} \frac{\partial}{\partial v_\alpha^{(1)}} \Big[ \delta\big(u^{(1)} - v^{(1)}\big)\delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(c^{(1)} - \psi^{(1)}\big)\delta\big(u^{(2)} - v^{(2)}\big)$$
$$\delta\big(h^{(2)} - g^{(2)}\big)\delta\big(\theta^{(2)} - \phi^{(2)}\big)\delta\big(c^{(2)} - \psi^{(2)}\big)\delta\big(u^{(3)} - v^{(3)}\big)\delta\big(h^{(3)} - g^{(3)}\big)\delta\big(\theta^{(3)} - \phi^{(3)}\big)\delta\big(c^{(3)} - \psi^{(3)}\big) \Big] \Big\rangle$$

$$= 2 \in_{m\alpha\beta} \Omega_m \frac{\partial}{\partial v_\alpha^{(1)}} \Big\langle u_\alpha^{(1)} \delta\big(u^{(1)} - v^{(1)}\big)\delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(c^{(1)} - \psi^{(1)}\big)\delta\big(u^{(2)} - v^{(2)}\big)$$
$$\delta\big(h^{(2)} - g^{(2)}\big)\delta\big(\theta^{(2)} - \phi^{(2)}\big)\delta\big(c^{(2)} - \psi^{(2)}\big)\delta\big(u^{(3)} - v^{(3)}\big)\delta\big(h^{(3)} - g^{(3)}\big)\delta\big(\theta^{(3)} - \phi^{(3)}\big)\delta\big(c^{(3)} - \psi^{(3)}\big) \Big\rangle$$

$$= 2 \in_{m\alpha\beta} \Omega_m \frac{\partial u_\alpha^{(1)}}{\partial v_\alpha^{(1)}} \Big\langle \delta\big(u^{(1)} - v^{(1)}\big)\delta\big(h^{(1)} - g^{(1)}\big)\delta\big(\theta^{(1)} - \phi^{(1)}\big)\delta\big(c^{(1)} - \psi^{(1)}\big)\delta\big(u^{(2)} - v^{(2)}\big)\delta\big(h^{(2)} - g^{(2)}\big)$$
$$\delta\big(\theta^{(2)} - \phi^{(2)}\big)\delta\big(c^{(2)} - \psi^{(2)}\big)\delta\big(u^{(3)} - v^{(3)}\big)\delta\big(h^{(3)} - g^{(3)}\big)\delta\big(\theta^{(3)} - \phi^{(3)}\big)\delta\big(c^{(3)} - \psi^{(3)}\big) \Big\rangle$$

$$= 2 \in_{m\alpha\beta} \Omega_m F_3^{(1,2,3)} \quad (4.8.38)$$

and sixth term,

$$\langle -\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times f(\ u_\alpha^{(1)}-v_\alpha^{(1)}\ )\frac{\partial}{\partial v_\alpha^{(1)}}\delta\left(u^{(1)}-v^{(1)}\right)\rangle$$

$$=-\langle\ f(\ u_\alpha^{(1)}-v_\alpha^{(1)}\ )\frac{\partial}{\partial v_\alpha^{(1)}}[\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)$$
$$\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)]\ \rangle$$

$$=-f(\ u_\alpha^{(1)}-v_\alpha^{(1)}\ )\frac{\partial}{\partial v_\alpha^{(1)}}\langle\ \delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)$$
$$\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\rangle$$

$$=-f(\ u_\alpha^{(1)}-v_\alpha^{(1)}\ )\frac{\partial}{\partial v_\alpha^{(1)}}F_3^{(1,2,3)} \tag{4.8.39}$$

Similarly, 18th ,19th ,31st and 32nd terms of right hand side of equation (4.8.1),

$$\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(2)}\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\rangle$$

$$=2\in_{m\alpha\beta}\Omega_m F_3^{(1,2,3)} \tag{4.8.40}$$

19th term,

$$\langle -\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)\delta\left(c^{(2)}-\psi^{(2)}\right)$$
$$\delta\left(u^{(3)}-v^{(3)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times f(\ u_\alpha^{(2)}-v_\alpha^{(2)}\ )\frac{\partial}{\partial v_\alpha^{(2)}}\delta\left(u^{(2)}-v^{(2)}\right)\rangle$$

$$=-f(\ u_\alpha^{(2)}-v_\alpha^{(2)}\ )\frac{\partial}{\partial v_\alpha^{(2)}}F_3^{(1,2,3)} \tag{4.8.41}$$

31st term,

$$\langle\delta\left(u^{(1)}-v^{(1)}\right)\delta\left(h^{(1)}-g^{(1)}\right)\delta\left(\theta^{(1)}-\phi^{(1)}\right)\delta\left(c^{(1)}-\psi^{(1)}\right)\delta\left(u^{(2)}-v^{(2)}\right)\delta\left(h^{(2)}-g^{(2)}\right)\delta\left(\theta^{(2)}-\phi^{(2)}\right)$$
$$\delta\left(c^{(2)}-\psi^{(2)}\right)\delta\left(h^{(3)}-g^{(3)}\right)\delta\left(\theta^{(3)}-\phi^{(3)}\right)\delta\left(c^{(3)}-\psi^{(3)}\right)\times 2\in_{m\alpha\beta}\Omega_m u_\alpha^{(3)}\frac{\partial}{\partial v_\alpha^{(3)}}\delta\left(u^{(3)}-v^{(3)}\right)\rangle$$

$$= 2 \in_{m\alpha\beta} \Omega_m F_3^{(1,2,3)} \tag{4.8.42}$$

32nd term,

$$\langle -\delta\left(u^{(1)} - v^{(1)}\right)\delta\left(h^{(1)} - g^{(1)}\right)\delta\left(\theta^{(1)} - \phi^{(1)}\right)\delta\left(c^{(1)} - \psi^{(1)}\right)\delta\left(u^{(2)} - v^{(2)}\right)\delta\left(h^{(2)} - g^{(2)}\right)\delta\left(\theta^{(2)} - \phi^{(2)}\right)$$

$$\delta\left(c^{(2)} - \psi^{(2)}\right)\delta\left(h^{(3)} - g^{(3)}\right)\delta\left(\theta^{(3)} - \phi^{(3)}\right)\delta\left(c^{(3)} - \psi^{(3)}\right) \times f\left( u_\alpha^{(3)} - v_\alpha^{(3)} \right)\frac{\partial}{\partial v_\alpha^{(3)}}\delta\left(u^{(3)} - v^{(3)}\right) \rangle$$

$$= -f\left( u_\alpha^{(3)} - v_\alpha^{(3)} \right)\frac{\partial}{\partial v_\alpha^{(3)}} F_3^{(1,2,3)} \tag{4.8.43}$$

Substituting the results (4.8.1) – (4.8.43) in equation (4.8.1) we get the transport equation for three point distribution function $F_3^{(1,2,3)}(v, g, \phi, \psi)$ in MHD turbulent flow in a rotating system in presence of dust particles as

$$\frac{\partial F_3^{(1,2,3)}}{\partial t} + \left( v_\beta^{(1)} \frac{\partial}{\partial x_\beta^{(1)}} + v_\beta^{(2)} \frac{\partial}{\partial x_\beta^{(2)}} v_\beta^{(3)} \frac{\partial}{\partial x_\beta^{(3)}} \right) F_3^{(1,2,3)} + \left[ g_\beta^{(1)} \left( \frac{\partial g_\alpha^{(1)}}{\partial v_\alpha^{(1)}} + \frac{\partial v_\alpha^{(1)}}{\partial g_\alpha^{(1)}} \right) \frac{\partial}{\partial x_\beta^{(1)}} \right.$$

$$\left. + g_\beta^{(2)} \left( \frac{\partial g_\alpha^{(2)}}{\partial v_\alpha^{(2)}} + \frac{\partial v_\alpha^{(2)}}{\partial g_\alpha^{(2)}} \right) \frac{\partial}{\partial x_\beta^{(2)}} + g_\beta^{(3)} \left( \frac{\partial g_\alpha^{(3)}}{\partial v_\alpha^{(3)}} + \frac{\partial v_\alpha^{(3)}}{\partial g_\alpha^{(3)}} \right) \frac{\partial}{\partial x_\beta^{(3)}} \right] F_3^{(1,2,3)}$$

$$+ \nu \left( \frac{\partial}{\partial v_\alpha^{(1)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(1)}} + \frac{\partial}{\partial v_\alpha^{(2)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(2)}} + \frac{\partial}{\partial v_\alpha^{(3)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(3)}} \right)$$

$$\times \frac{\partial^2}{\partial x_\beta^{(4)} \partial x_\beta^{(4)}} \int v_\alpha^{(4)} F_4^{(1,2,3,4)} dv^{(4)} dg^{(4)} d\phi^{(4)} d\psi^{(4)}$$

$$+ \lambda \left( \frac{\partial}{\partial g_\alpha^{(1)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(1)}} + \frac{\partial}{\partial g_\alpha^{(2)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(2)}} + \frac{\partial}{\partial g_\alpha^{(3)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(3)}} \right)$$

$$\times \frac{\partial^2}{\partial x_\beta^{(4)} \partial x_\beta^{(4)}} \int g_\alpha^{(4)} F_4^{(1,2,3,4)} dv^{(4)} dg^{(4)} d\phi^{(4)} d\psi^{(4)}$$

$$+ \gamma \left( \frac{\partial}{\partial \phi^{(1)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(1)}} + \frac{\partial}{\partial \phi^{(2)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(2)}} + \frac{\partial}{\partial \phi^{(3)}} \lim_{\bar{x}^{(4)} \to \bar{x}^{(3)}} \right)$$

$$\times \frac{\partial^2}{\partial x_\beta^{(4)} \partial x_\beta^{(4)}} \int \phi^{(4)} F_4^{(1,2,3,4)} dv^{(4)} dg^{(4)} d\phi^{(4)} d\psi^{(4)}$$

$$+D\left(\frac{\partial}{\partial\psi^{(1)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(1)}}+\frac{\partial}{\partial\psi^{(2)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(2)}}+\frac{\partial}{\partial\psi^{(3)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(3)}}\right)$$

$$\times\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int\psi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}$$

$$-\Big[\frac{\partial}{\partial v_\alpha^{(1)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(1)}}\Big(\frac{1}{\left|\bar{x}^{(4)}-\bar{x}^{(1)}\right|}\Big)\Big\}+\frac{\partial}{\partial v_\alpha^{(2)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(2)}}\Big(\frac{1}{\left|\bar{x}^{(4)}-\bar{x}^{(2)}\right|}\Big)\Big\}$$

$$+\frac{\partial}{\partial v_\alpha^{(3)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(3)}}\Big(\frac{1}{\left|\bar{x}^{(4)}-\bar{x}^{(3)}\right|}\Big)\Big\}\times\Big(\frac{\partial v_\alpha^{(4)}}{\partial x_\beta^{(4)}}\frac{\partial v_\beta^{(4)}}{\partial x_\alpha^{(4)}}-\frac{\partial g_\alpha^{(4)}}{\partial x_\beta^{(4)}}\frac{\partial g_\beta^{(4)}}{\partial x_\alpha^{(4)}}\Big)F_4^{(1,2,3,4)}$$

$$\times dx^{(4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}\Big]+6\in_{m\alpha\beta}\Omega_m F_3^{(1,2,3)}$$

$$+\Big[f\big(u_\alpha^{(1)}-v_\alpha^{(1)}\big)\frac{\partial}{\partial v_\alpha^{(1)}}+f\big(u_\alpha^{(2)}-v_\alpha^{(2)}\big)\frac{\partial}{\partial v_\alpha^{(2)}}+f\big(u_\alpha^{(3)}-v_\alpha^{(3)}\big)\frac{\partial}{\partial v_\alpha^{(3)}}\Big]F_3^{(1,2,3)}=0 \tag{4.8.44}$$

Continuing this way, we can derive the equations for evolution of $F_4^{(1,2,3,4)}$, $F_5^{(1,2,3,4,5)}$ and so on. Logically it is possible to have an equation for every $F_n$ (n is an integer) but the system of equations so obtained is not closed. Certain approximations will be required thus obtained.

## 4.9 Results and Discussion

If the fluid is clean and the system is non rotating then f=0 and $\Omega_m$=0, the transport equation for one point distribution function in MHD turbulent flow (4.8.44) becomes

$$\frac{\partial F_3^{(1,2,3)}}{\partial t}+\Big(v_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}+v_\beta^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}v_\beta^{(3)}\frac{\partial}{\partial x_\beta^{(3)}}\Big)F_3^{(1,2,3)}+\Big[g_\beta^{(1)}\Big(\frac{\partial g_\alpha^{(1)}}{\partial v_\alpha^{(1)}}+\frac{\partial v_\alpha^{(1)}}{\partial g_\alpha^{(1)}}\Big)\frac{\partial}{\partial x_\beta^{(1)}}$$

$$+g_\beta^{(2)}\Big(\frac{\partial g_\alpha^{(2)}}{\partial v_\alpha^{(2)}}+\frac{\partial v_\alpha^{(2)}}{\partial g_\alpha^{(2)}}\Big)\frac{\partial}{\partial x_\beta^{(2)}}+g_\beta^{(3)}\Big(\frac{\partial g_\alpha^{(3)}}{\partial v_\alpha^{(3)}}+\frac{\partial v_\alpha^{(3)}}{\partial g_\alpha^{(3)}}\Big)\frac{\partial}{\partial x_\beta^{(3)}}\Big]F_3^{(1,2,3)}$$

$$+\nu\left(\frac{\partial}{\partial v_\alpha^{(1)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(1)}}+\frac{\partial}{\partial v_\alpha^{(2)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(2)}}+\frac{\partial}{\partial v_\alpha^{(3)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(3)}}\right)$$

$$\times\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int v_\alpha^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}$$

$$+\lambda\Big(\frac{\partial}{\partial g_\alpha^{(1)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(1)}}+\frac{\partial}{\partial g_\alpha^{(2)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(2)}}+\frac{\partial}{\partial g_\alpha^{(3)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(3)}}\Big)$$

$$\times\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int g_\alpha^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}$$

$$+\gamma\Big(\frac{\partial}{\partial \phi^{(1)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(1)}}+\frac{\partial}{\partial \phi^{(2)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(2)}}+\frac{\partial}{\partial \phi^{(3)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(3)}}\Big)$$

$$\times\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int \phi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}$$

$$+D\Big(\frac{\partial}{\partial \psi^{(1)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(1)}}+\frac{\partial}{\partial \psi^{(2)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(2)}}+\frac{\partial}{\partial \psi^{(3)}}\lim_{\bar{x}^{(4)}\to\bar{x}^{(3)}}\Big)$$

$$\times\frac{\partial^2}{\partial x_\beta^{(4)}\partial x_\beta^{(4)}}\int \psi^{(4)}F_4^{(1,2,3,4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}$$

$$-\Big[\frac{\partial}{\partial v_\alpha^{(1)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(1)}}\Big(\frac{1}{|\bar{x}^{(4)}-\bar{x}^{(1)}|}\Big)\Big\}+\frac{\partial}{\partial v_\alpha^{(2)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(2)}}\Big(\frac{1}{|\bar{x}^{(4)}-\bar{x}^{(2)}|}\Big)\Big\}$$

$$+\frac{\partial}{\partial v_\alpha^{(3)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(3)}}\Big(\frac{1}{|\bar{x}^{(4)}-\bar{x}^{(3)}|}\Big)\Big\}\times\Big(\frac{\partial v_\alpha^{(4)}}{\partial x_\beta^{(4)}}\frac{\partial v_\beta^{(4)}}{\partial x_\alpha^{(4)}}-\frac{\partial g_\alpha^{(4)}}{\partial x_\beta^{(4)}}\frac{\partial g_\beta^{(4)}}{\partial x_\alpha^{(4)}}\Big)F_4^{(1,2,3,4)}$$

$$\times dx^{(4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}\Big]=0 \qquad (4.9.1)$$

which was obtained earlier by Azad et al [14].

If we drop the viscous, magnetic and thermal diffusive and concentration terms from the three point evolution equation (4.8.44), we have

$$\frac{\partial F_3^{(1,2,3)}}{\partial t}+\Big(v_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}+v_\beta^{(2)}\frac{\partial}{\partial x_\beta^{(2)}}v_\beta^{(3)}\frac{\partial}{\partial x_\beta^{(3)}}\Big)F_3^{(1,2,3)}+\Big[g_\beta^{(1)}\Big(\frac{\partial g_\alpha^{(1)}}{\partial v_\alpha^{(1)}}+\frac{\partial v_\alpha^{(1)}}{\partial g_\alpha^{(1)}}\Big)\frac{\partial}{\partial x_\beta^{(1)}}$$

$$+g_\beta^{(2)}\Big(\frac{\partial g_\alpha^{(2)}}{\partial v_\alpha^{(2)}}+\frac{\partial v_\alpha^{(2)}}{\partial g_\alpha^{(2)}}\Big)\frac{\partial}{\partial x_\beta^{(2)}}+g_\beta^{(3)}\Big(\frac{\partial g_\alpha^{(3)}}{\partial v_\alpha^{(3)}}+\frac{\partial v_\alpha^{(3)}}{\partial g_\alpha^{(3)}}\Big)\frac{\partial}{\partial x_\beta^{(3)}}\Big]F_3^{(1,2,3)}$$

$$-\Big[\frac{\partial}{\partial v_\alpha^{(1)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(1)}}\Big(\frac{1}{|\bar{x}^{(4)}-\bar{x}^{(1)}|}\Big)\Big\}+\frac{\partial}{\partial v_\alpha^{(2)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(2)}}\Big(\frac{1}{|\bar{x}^{(4)}-\bar{x}^{(2)}|}\Big)\Big\}$$

$$+\frac{\partial}{\partial v_\alpha^{(3)}}\Big\{\frac{1}{4\pi}\int\frac{\partial}{\partial x_\alpha^{(3)}}\Big(\frac{1}{|\bar{x}^{(4)}-\bar{x}^{(3)}|}\Big)\Big\}\times\Big(\frac{\partial v_\alpha^{(4)}}{\partial x_\beta^{(4)}}\frac{\partial v_\beta^{(4)}}{\partial x_\alpha^{(4)}}-\frac{\partial g_\alpha^{(4)}}{\partial x_\beta^{(4)}}\frac{\partial g_\beta^{(4)}}{\partial x_\alpha^{(4)}}\Big)F_4^{(1,2,3,4)}$$

$$\times dx^{(4)}dv^{(4)}dg^{(4)}d\phi^{(4)}d\psi^{(4)}\Big]=0 \qquad (4.9.2)$$

The existence of the term

$$\left(\frac{\partial g_\alpha^{(1)}}{\partial v_\alpha^{(1)}}+\frac{\partial v_\alpha^{(1)}}{\partial g_\alpha^{(1)}}\right),\ \left(\frac{\partial g_\alpha^{(2)}}{\partial v_\alpha^{(2)}}+\frac{\partial v_\alpha^{(2)}}{\partial g_\alpha^{(2)}}\right) \text{ and } \left(\frac{\partial g_\alpha^{(3)}}{\partial v_\alpha^{(3)}}+\frac{\partial v_\alpha^{(3)}}{\partial g_\alpha^{(3)}}\right)$$

can be explained on the basis that two characteristics of the flow field are related to each other and describe the interaction between the two modes (velocity and magnetic) at point $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$.

We can exhibit an analogy of this equation with the 1st equation in BBGKY hierarchy in the kinetic theory of gases. The first equation of BBGKY hierarchy is given [107] as

$$\frac{\partial F_1^{(1)}}{\partial t}+\frac{1}{m}v_\beta^{(1)}\frac{\partial}{\partial x_\beta^{(1)}}F_1^{(1)}=n\iint\frac{\partial \psi_{1,2}}{\partial x_\alpha^{(1)}}\frac{\partial F_2^{(1,2)}}{\partial v_\alpha^{(1)}}d\overline{x}^{(2)}d\overline{v}^{(2)} \tag{4.9.3}$$

where $\psi_{1,2}=\psi\left|v_\alpha^{(2)}-v_\alpha^{(1)}\right|$ is the inter molecular potential.

In order to close the system of equations for the distribution functions, some approximations are required. If we consider the collection of ionized particles, i.e. in plasma turbulence case, it can be provided closure form easily by decomposing $F_2^{(1,2)}$ as $F_1^{(1)}F_1^{(2)}$. But such type of approximations can be possible if there is no interaction or correlation between two particles. If we decompose $F_2^{(1,2)}$ as

$F_2^{(1,2)}=(1+\in)F_1^{(1)}F_1^{(2)}$

and

$F_3^{(1,2,3)}=(1+\in)^2F_1^{(1)}F_1^{(2)}F_1^{(3)}$

also

$F_4^{(1,2,3,4)}=(1+\in)^3F_1^{(1)}F_1^{(2)}F_1^{(3)}F_1^{(4)}$

where $\in$ is the correlation coefficient between the particles. If there is no correlation between the particles, $\in$ will be zero and distribution function can be decomposed in usual way. Here we are considering such type of approximation only to provide closed form of the equation.

# CHAPTER-V

# CONCLUSION

In the thesis mainly turbulent and Magneto-hydrodynamic turbulent flow in incompressible fluid has been studied. We have tried to give here a general idea of turbulence and Magneto–hydrodynamic turbulence related to this research work.

We have studied the decay of temperature fluctuations in dusty fluid homogeneous turbulence prior to the final period in section II-A. In this chapter to derive the decay law of temperature fluctuations in dusty fluid turbulence, we have considered two-point and three-point correlations between fluctuating quantities. Correlation equations between fluctuating quantities with dust particles are obtained. Converting these correlation equations to spectral form by taking their Fourier transform. By integrating energy spectrum over all wave numbers the energy decay law of temperature fluctuations in homogeneous turbulence before the final period in presence of dust particles is obtained. In this result we have seen that the energy decays more rapidly than the energy for clean fluid prior to the ultimate period. Throughout this work we have applied Deissler's method.

Applying the same method we have made an attempt to derive the energy decay law of temperature fluctuations in homogeneous dusty fluid turbulence before the final period in presence of Coriolis force in section II-B.

In section III-A, we have derived the transport equation by making use of the derivation of the constructed joint distribution function of certain variables in convective turbulent flow in presence of Coriolis force undergoing a first order reaction. We have got a partial differential equation under the deviation of the joint distribution function. The equation of motion, field equation of temperature and concentration of particles with Coriolis force have been used in above partial differential equation. Then we have simplified each term of the equation in terms of one- and two- point distribution function. By substituting the simplified terms in the above equation, the transport equation for the joint distribution functions of certain variables in convective turbulent flow in presence of Coriolis force undergoing a first order reaction is obtained. Lastly, we have compared the result with the equation for one- point distribution function in absence of the Coriolis force. We have extended the above problem for the case of dust particles due to first order reaction in section III-B. It is better to say that the system is unclosed that is why some approximations are required to close the system of equations for the joint distribution function.

In chapter-IV, we have studied the statistical theory of certain variables for three-point distribution functions in MHD turbulent flow in rotating system and their properties, e.g. reduction properties, separation properties etc. Continuity equation in term of distribution function has been considered. Equations for three-point distribution function of it have been formulated by analyzing two-point distribution functions.

# BIBLIOGRAPHY


[1] Atkinson, Simpson and Neumann. How do we understand the Coriolis force? Bulletin of the American Meteorological Society, 79, 7, July 1998, 1373-1385, 1981.

[2] Azad M. A. K. and M .S. A. Sarker. 2004. Raj. Univ. Stud. Part-B. J. Sci., 32, 193-210.

[3] Azad M. A. K. and M. S. A. Sarker. 2006. Raj. Univ. Stud., part- B. J. Sci., 34, 37-50.

[4] Azad M. A. K. and M. S. A. Sarker. 2008. J. of Appl. Sci. Res., 4(7), 793- 802.

[5] Azad M. A. K. and M. S. A. Sarker. 2009. Bangladesh J. Sci. Ind. Res., 44(4), 407-414.

[6] Azad M. A. K.**,** Aziz M. A. and Sarker M. S. A.**,** J. Mech. Contin. & Math. Sci., 4(1), 410-417(2009c).

[7] Azad M. A. K., M. A. Aziz and M. S. A. Sarker. 2009a. J. Mech. Cont. and Math. Sci., 4(1), 410-417.

[8] Azad M. A. K., M. A. Aziz and M. S. A. Sarker. 2009b. J. Phy. Sci., 13, 175-190.

[9] Azad M. A. K., M. A. Aziz and M. S. A. Sarker. 2010. Bang. J. Sci. & Ind. Res., 45(1), 39-46.

[10] Azad M. A. K., M. A. Aziz and M. S. A. Sarker. 2011. Bang. J. Sci. and Ind. Res., 46(1), 59-68.

[11] Azad M. A. K., M. S. A. Sarker and N. I. Mondal. 2006. J. Eng. and Appl. Sci., 1(2), 187-194.

[12] Azad M. A. K., S. A. Sarker and M. A. Aziz. 2007- 2008. Math Forum, 20, 32- 48.

[13] Azad M.A.K. and Mst. Mumtahinah. 2013., Res. J. Appl. Sci. Engng. and Tech., 6( 8), 1490-6.

[14] Azad M.A.K., M., Islam N, and Mumtahinah Mst**.** 2014. **Res. J. Appl. Sci. Engng. and Tech.** ( accepted for publication).

[15] Azad, M. A. K., M. H .U. Molla and M. Z. Rahman. 2012. Res. J. App. Sci. Eng. Tech., 4(20), 4150-4159.

[16] Aziz M. A., Azad M. A. K. and Sarker M. S. A. 2010c. Res. J. Math. & Stat., 2(2), 56-68

[17] Aziz M. A., M. A. K. Azad and M. S. A. Sarker. 2009. Res. J. Math. and Stat., 1(2), 35-46.

[18] Aziz M. A., M. A. K. Azad and M. S. A. Sarker. 2010. J. of Mod. Math. and Stat., 4(1), 11-21.

[19] Aziz M. A., M. A. K. Azad and M. S. A. Sarker. 2010a. Res. J. Math. and Stat., 2(2), 56-68.

[20] Aziz M. A., M. A. K. Azad and M. S. A. Sarker. 2010b. Res. J. Math. and Stat., 2(2), 37-55.

[21] Bansal R. K. 1992. Organic reaction Mechanizms, Tata Mc Graw-Hill Publishing Limited, New Delhi.

[22] Batchelor G. K. 1960. Theory of homogeneous turbulence, Cambridge Univ. Press, 1960.

[23] Batchelor G. K. and A. A. Townsend. 1948. Proc. Roy. Soc., London, A194, 527- 543.

[24] Batchelor, G. K. 1953. The Theory of Homogeneous Turbulence. New York: Cambridge University Press.

[25] Bigler R. W. 1976. Combustion Sci. and Tech., 13, 155.

[26] Bkar Pk M. A., M. A. K. Azad and Sarker M. S. A. 2013a. Res. J. Appl. Sci. Engng. & Tech., 5(2): 585-595.

[27] Bkar Pk. M. A., M. A. K. Azad, and Sarker M. S. A.. 2013d. J. Sci. Res. 5 (1),77-90.

[28] Bkar Pk. M. A., M. S. A. Sarker and Azad M. A. K.. 2013c. Res. J. Appl. Sci., Engng. & Tech. 6(15): 2789-2798.

[29] Bkar Pk. M. A., M. S. A. Sarker and Azad M. A. K.. 2013b. Res. J. Appl. Sci., Engng. & Tech. 6(10): 1749-1756.

[30] Bkar. Pk M. A., Azad M. A. K. and Sarker M. S. A. 1012. Res. J. Math. & Stat., 4(2) , 30-38.

[31] Corrsin S. 1951. J. Appl. Phy., 22, 469- 473.

[32] Deissler R. G. 1958. Phys. Fluids, 1, 111-121.

[33] Deissler R. G. 1960. Phys. Fluids, 3, 176-187.

[34] Dixit T. and B. N. Upadhyay. 1989a. Astrophysics and Space Sci., 153, 257-268.

[35] Dixit T. and B. N. Upadhyay. 1989b. Astrophysics and Space Sci., 153, 297.

[36] Edward S 1964. The statistical dynamics of homogeneous turbulence J. Fluid Mech. 18(2): 239-273.

[37] Ekman V. W. 1910. Arkiv. Mat. Astron. Fysik., 6, 12.

[38] Geankoplis C. J. 2003. Transport Processes and Separation Process Principles. Prentice Hall Professional Technical Reference. ISBN 0-13-101367-X.

[39] Ghosh K. M. 1958. Nat. Sci. India, A24, 240.

[40] Ghosh K. M. 1972. Indian. J. of Pure and Appl. Math., 3, 157.

[41] Herring J. R 1965.Self-consistent field approach to turbulence theory, Phys. Fluid 8: 2219-2225.

[42] Hinze, J. O. (1975). Turbulence (2nd ed.). New York: McGraw-Hill.

[43] Hopf E 1952. Statistical hydrodynamics and functional calculus, J. Rotational Mech. Anal. 1: 87-123.

[44] Islam M. A. and M. S. A. Sarker. 2001. Indian J. Pure Appl. Math., 32(8), 1173-1184.

[45] Islam M. A. and M. S. A. Sarker. 2007. Rajshahi University Studies. Part-B. J. Sci., Vol, 35

[46] Janicka J., W. Kolbe and W. Kollmann. 1979. Non Equilib. Thermodyn., 4, 47.

[47] Kishore N. 1978. J. Sci. Res., BHU, 28(2), 163-178.

[48] Kishore N. and M. S. A. Sarker. 1990. Astrophysics and Space Sci., 172, 279.

[49] Kishore N. and S. R. Singh. 1984. Bull. Tech. Univ. of Istambul, 37, 91-100.

[50] Kishore N. and S. R. Singh. 1985. Prog. of Maths., 9(1&2), 13-22.

[51] Kishore N. and T. Dixit. 1979. J. Scientific Research, B. H. U., 30(2), 305.

[52] Kishore N. and Y. T. Golsefid. 1988. Astrophysics and Space Sci., 150, 89.

[53] Kishore, N. and M. S. A. Sarker., 1990. I. ,J. Energy Res., 14(5), 573-577 .

[54] Kishore, N. and A. Sinha. 1988. Astrophysics and Space Sci.,146,53-58.

[55] Kishore, N. and S. R. Singh. 1984. Astrophysics and Space Scie., 104, pp.121-125.

[56] Kollmann W. and J. Janicka. 1982. Phys. Fluid., 25, 1755-1769.

[57] Kolmogorov A. N. 1941. Academia de Science de URSS XXX, No. 4.

[58] Kraichanan R. H 1959. J. Fluid Mech. 5: 497.

[59] Kumar P. and S. R. Patel. 1974. Phys. Fluid., 17, 1362.

[60] Kumar P. S. R. and S. R. Patel. 1975. Int. J. Eng. Sci., 13, 305-315.

[61] Leislie D. C. 1973. Developments in the Theory of turbulence, Clarendon Press. Oxford.

[62] Lesieur, M (1987). Turbulence in Fluid. Dordrech, Netherrlands; Martinus Nijhoff Publisherrs.

[63] Lighthill M. J. 1960. Fourier Analysis and Generalized Functions, Cambridge University Press.

[64] Loeffler A. L. and R. G. Deissler. 1961. Int. J. Heat Mass Transfer, 1, 312-324.

[65] Lundgren T. S. 1967. Phys. Fluid., 10, 967.

[66] Lundgren T. S. 1969. Phys. Fluids, 12, 485.

[67] Molla M. H. U. 2012. M. Phil. Thesis, Department of Applied Mathematics, Rajshahi University.

[68] Molla M. H. U., M. A. K. Azad and Rahman M. Z. 2012 M.Phil. Thesis, Department of Applied Mathematics, University of Rajshahi, Bangladesh, 45.

[69] Molla M. H. U., M. A. K. Azad and M. Z. Rahman. 2012**.** Res. J. Math. and Stat. (RJMS), 4(2) , 45-51.

[70] Monin A. S. and A. M. Yaglom. 1973. Stat. Fluid Mec., 1. M.I.T. Press.

[71] Monin A. S. 1967b. Prike, Mat, Mekh., 31, 1057.

[72] Noakes C. and Andrew S. 2009. "Real Fluids". An Introduction to Fluid Mechanics. University of Leeds. Retrieved 23 November 2010.

[73] Obukhov, A. 1941. On the distribution of Energy in the Spectrum of Turbulent Flow. Izv. Akad. Nauk SSSR Ser. Georg. Geofiz. 5. 453.

[74] Oertel H. (ed.), 2010. *Prandtl-Essentials of Fluid Mechanics*, 1, Applied Mathematical Sciences 158, DOI 10.1007/978-1-4419-1564-1_1, © Springer Science+Business Media, LLC.

[75] Pai S. I. 1957. Viscous Flow Theory (11Turbulent Flow), D. Van Nostrand Company Inc.

[76] Patel S.R. 1976. Int. J. Eng. Sci., 14, 75.

[77] Pedley, T. J. (2000). Blood flow in arteries and veins. In G. K. Batchelor, H. K. Moffat, & M. G. Worster (Eds.), Perspectives in Fluid Dynamics. Cambridge: Cambridge University Press.

[78] Pope S. B. 1979. Phil. Trans Roy. Soc. London, A291, 1384, 529-568.

[79] Pope S. B. 1981b. Physics of Fluid, 24, 588.

[80] Prandtl L. 1935. The Mechanics of Viscous Fluid. Vol. III of Aerodynamics Theory, div.G. W.F. Durand, ed., Julius Springer (Berlin), 1935, pp. 34-208

[81] Prengle R. S. and R. R. Rothfus. 1955. Ind. Eng. Chem., 47, 379.

[82] Rahman, M. L., 2010. J.Mech.Cont.and Math. Sci., 5(1),570-578.

[83] Reynolds O. 1883. Phil. Trans. Roy. Soc., London, 174, 935.

[84] Reynolds O. 1884. Phil. Trans. Roy. Soc., London, 186, 123.

[85] Richardson L. F. 1922. Weather Prediction by Numerical Process.

[86] Saffman P. G. 1962. J. Fluid. Mech., 13, 120-128.

[87] Sarker M. S. A. and M. A. K. Azad. 2004. Raj. Univ. Stud. Part-B, J. Sci., 32, 177-192.

[88] Sarker M. S. A and M. L. Rahman. 1998. North Bengle Uni. Review (Sci and Tech), India, 9, 91.

[89] Sarker M. S. A. 1997. Int. J. Energy Research, 21, 1399.

[90] Sarker M. S. A. 1998. Rajshahi Univ. Studies Part-B, 25.

[91] Sarker M. S. A. and M. A. Islam. 2001. Indian J. of Pure and Appl. Math., 32(7), 1065.

[92] Sarker M. S. A. and M. A. Islam. 2001. J. Math and Math. Sci., 16, 35.

[93] Sarker M. S. A. and M. A. Islam. 2001. J. of Sci., Res, 19. 153.

[94] Sarker M. S. A. and M. A. K. Azad. 2006. Bangladesh J. Sci. Ind. Res., 41(3-4), 147- 158.

[95] Sarker M. S. A. and M. L. Rahman. 1997. Rajshahi University Studies, Part-B, In Press.

[96] Sarker M. S. A. and N. Kishore. 1991a. Astrophys. Space Sci., 181, 29.

[97] Sarker M. S. A. and N. Kishore. 1991b. Int. J. Eng. Sci., 29, 1479.

[98] Sarker M. S. A. and N. Kishore. 1999. Prog. Math., BHU, India, 33(1-2), 83.

[99] Sarker M. S. A., M. A. K. Azad and M. A. Aziz. 2009. J. Phy. Sci., 13, 21-38.

[100] Sarker M.S.A, Islam M. A 2002. Raj. Univ. Stud. Part-B. J. Sci., 30:

[101] Sarker, M.S.A., M. A. Bkar. Pk and Azad M. A. K. 2012. IOSR J. Math. (IOSR-JM), 3(5), 39-46,

[102] Shimomura Y. and A. Yoshizawa. 1986. J. Physical Soc., Japan, 55, 1904-1917.

[103] Sinha, A., 1988. J. Sci. Res.,B.H.U., 38, 7.

[104] Stanisic M. M. 1985. Mathematical Theory of Turbulence, Springer-Verlag, New York.

[105] Taylor G. I. 1935a. Proc. Roy. Soc., London, A151, 421.

[106] Taylor G. I. and T. Vonkarman. 1937. J. Roy. Aeronaut. Soc., 41, 1109.

[107] Ta-You W 1966. Kinetic Theory of Gases and Plasma. Addision Wesley Phlelishing.

[108] Townsend A. A. 1976. Cambridge Uni. Press, London.