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Characterizations of Prime and Semiprime Gamma Rings with Derivations and Lie Ideals

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Characterizations of Prime and Semiprime Gamma Rings with Derivations and Lie Ideals

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Statement of Authorship

I do hereby declare that this thesis, entitled "Characterizations of Prime and Semiprime Gamma Rings with Derivations and Lie Ideals", prepared under the supervision of Professor Dr. Akhil Chandra Paul, submitted to the University of Rajshahi, Rajshahi-6205, Bangladesh, for the award of the degree of Doctor of Philosophy in Mathematics, is my own research work (except where indicated and acknowledged otherwise in the text), and that it has not at any time been previously submitted to this university or any other university/institution for the award of any degree/diploma at any level.

Signature and Date——-
(Md. Mizanor Rahman)

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This is certified that the thesis, entitled "Characterizations of Prime and Semiprime Gamma Rings with Derivations and Lie Ideals", submitted by Md. Mizanor Rahman, a record of genuine research work carried out by him under my supervision, contains the fulfillment of all the requirements for the degree of Doctor of Philosophy in Mathematics at the University of Rajshahi, Rajshahi-6205, Bangladesh.

I do believe that this research work is an original one and that it has not been submitted elsewhere for the award of any degree or diploma. Except where the proper reference is made in the text of the thesis, it contains no material published elsewhere. Nobody's work has been used in the main text of this thesis without due acknowledgement.

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This is certified that the thesis, entitled "CHARACTERIZATIONS OF PRIME AND SEMIPRIME GAMMA RINGS WITH DERIVATIONS AND LIE IDEALS", submitted by Md. Mizanor Rahman, a record of genuine research work carried out by him under my supervision, contains the fulfillment of all the requirements for the degree of Doctor of philosophy in Mathematics at the University of Rajshahi, Rajshahi-6205, Bangladesh.

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Signature and Date

(Professor Dr. Akhil Chandra Paul)

Supervisor

Dedicated

To

My Mother

Meherun Nesa *(deceased),*

who always inspired me for my higher education and enlightenment.

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Md. Mizanor Rahman
August, 2014

Abstract

Gamma ring was first introduced by N. Nobusawa as a generalization of a classical ring. W. E. Barnes generalized the definition of a gamma ring due to Nobusawa. Presently the gamma ring due to Barnes is known as a gamma ring and gamma ring due to Nobusawa is known as gamma ring in the sense of Nobusawa and is denoted by Γ_N-ring. It is clear that every ring is a gamma ring and every Γ_N-ring is also a Γ-ring. Actually, W.E.Barnes, J.Luh and S.Kyuno studied the structures of Γ-rings and obtained various generalizations analogous to the corresponding parts in ring theory. Afterwards, a number of algebraists have determined a lot of fundamental properties of Γ-rings to classify and extend numerous significant results in classical ring theory to Γ-ring theory. This thesis, entitled "Characterizations of Prime and Semiprime Gamma Rings with Derivations and Lie Ideals", aims to characterize prime and semiprime Γ-rings with various types of left derivations, derivations, generalized derivations, higher derivations, derivations on Lie ideals and (U, M)-derivations. All the necessary introductory definitions and examples of Γ-rings are discussed in considerable details in the introduction chapter.

The notions of derivation and Jordan derivation in Γ-rings have been introduced by M. Sapanci and A. Nakajima. Afterwards, in the light of some significant results due to Jordan left derivation of a classical ring obtained by K.W.Jun and B.D.Kim, some extensive results of left derivation and Jordan left derivation of a Γ-ring were determined by Y.Ceven. In classical ring theory, Joso Vukman proved that if d is a Jordan left derivation of a 2-torsion free semiprime ring R and if there exists a

positive integer n such that $D(x)^n = 0$ for all $x \in R$, then $D = 0$. He also proved that for a 2-torsion free and 3-torsion free semiprime ring R admits Jordan derivation D and $G : R \to R$ such that $D^2(x) = G(x)$ for all $x \in R$, then $D = 0$. In chapter 1, we extend this result to the Γ-ring theory in the case of Jordan left derivations. Then we construct some relevant results to prove that under a suitable condition every nonzero Jordan left derivation d of a 2-torsion free prime Γ-ring M induces the commutativity of M, and consequently, d is a left derivation of M.

Developing a number of important results on Jordan derivations of semiprime Γ-rings, we then prove under a suitable condition, every Jordan derivation of a Γ-ring M is a derivation of M, if we consider M as a 2-torsion free (i) semiprime, and (ii) completely semiprime Γ-ring, respectively. We examine all these statements in chapter 2 for the clear understanding of the concepts.

M. Asci and S. Ceran obtained some commutativity results of prime Γ-rings with left derivation. Some commutativity results in prime rings with Jordan higher left derivations were obtained by Kyuoo-Hong Park on Lie ideals and obtained some fruitful results relating this. We work on Jordan higher left derivation on a 2-torsion free prime Γ-ring and we show that under a suitable condition, the existence of a nonzero Jordan higher left derivation on a 2-torsion free prime Γ-ring M forces M commutative. For the classical ring theories, Herstein, proved a well known result that every Jordan derivation in a 2-torsion free prime ring is a derivation. Bresar proved this result in semiprime rings. Sapanci and Nakajima proved the same result in completely prime Γ-rings. C. Haetinger worked on higher derivations in prime rings and extended this result to Lie ideals in a prime ring. We introduce a higher derivation and a Jordan higher derivation in Γ-rings. Then we determine some immediate consequences due to Jordan higher derivation of Γ-rings to prove under a suitable condition every Jordan higher derivation of a 2-torsion free prime Γ-ring M is a higher derivation of M. Y. Ceven and M. A. Ozturk worked on Jordan generalized derivations in Γ-rings and they proved that every Jordan generalized derivation on some Γ-rings is a generalized derivation. A. Nakajima defined the notion of generalized higher derivations and investigated some elementary relations between generalized higher derivations and higher derivations in the usual sense. They also discussed Jordan generalized higher derivations and Lie derivations on rings. W. Cortes and C. Haetinger proved that every Jordan generalized higher derivations on a ring R is a generalized higher derivation. M. Ferrero and C. Haetinger proved that every Jordan higher derivation of a 2-torsion free semiprime ring is a higher derivation. C. Haetinger extended the above results of prime rings in Lie ideals. By the motivations of above works, we introduce a Jordan generalized higher derivations in Γ-rings. We prove that every Jordan generalized higher derivation in a 2-torsion free prime Γ-ring with the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, is a generalized higher derivation of M. Chapter 3 deals with all these important results elaborately.

The relationship between usual derivations and Lie ideals of prime rings has been extensively studied in the last 40 years. In particular, when this relationship involves the action of the derivations on Lie ideals. In 1984, R. Awtar extended a well known result proved by I. N. Herstein to Lie ideals which states that, "every Jordan derivation on a 2-torsion free prime ring is a derivation". In fact, R. Awtar proved that if $U \nsubseteq Z$ is a square closed Lie ideal of a 2-torsion free prime ring R and $d : R \rightarrow R$ is an additive mapping such that $d(u^2) = d(u)u + ud(u)$ for all $u \in U$, then $d(uv) = d(u)v + ud(v)$ for all $u, v \in U$. M. Ashraf and N. Rehman studied on Lie ideals and Jordan left derivations of prime rings. They proved that if $d : R \rightarrow R$ is an additive mapping on a 2-torsion free prime ring R satisfying $d(u^2) = 2ud(u)$ for all $u \in U$, where U is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$ then $d(uv) = d(u)v + ud(v)$ for all $u, v \in U$. A. K. Halder and A. C. Paul extended the results of Y. Ceven in Lie ideals. We generalize the Awtar's result in Γ-rings by establishing some necessary results relating to them at the beginning of the chapter 4, we then prove if U is an admissible Lie ideal of a 2-torsion free prime Γ-ring M satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$; $\alpha, \beta \in \Gamma$ and $d : M \rightarrow M$ is a Jordan derivation on U of M, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u, v \in U; \alpha \in \Gamma$ and if U is a commutative square closed Lie ideal of M, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all

$u, v \in U$ and $\alpha \in \Gamma$. Accordingly, we then define Jordan higher derivation and higher derivation on Lie ideals of Γ-rings and construct some relevant results to prove the previous results analogously in case of Jordan higher derivation on Lie ideal of a Γ-ring. Chapter 4 is devoted to a study of these materials in order to bring out the concepts defined clearly.

M. Ashraf and N. Rehman considered the question of I. N. Herstein for a Jordan generalized derivation. They showed that in a 2-torsion free ring R which has a commutator right nonzero divisor, every Jordan generalized derivation on R is a generalized derivation on R. In 2000, Nakajima defined a generalized higher derivation and gave some categorical properties. He also treated generalized higher Jordan and Lie derivations. Later, Cortes and Haetinger extended Ashraf's theorem to generalized higher derivations. They proved that if R is 2-torsion free ring which has a commutator right nonzero divisor, then every Jordan generalized higher derivation on R is a generalized higher derivation on R. Following the notions of Jordan derivation and derivation on Lie ideals of a Γ-ring in the previous chapter we then introduce the concepts of a Jordan generalized derivation and generalized derivation on Lie ideals of a Γ-ring and we extend and generalized the above mentioned result by these newly introduced concepts. Accordingly, we then define Jordan generalized higher derivation and generalized higher derivation on Lie ideals of a Γ-ring and generalized the same result by these concepts. We examine all these statements in chapter 5 for the clear understanding of the concepts.

(U, R)-derivations in rings have been introduced by A. K. Faraj, C. Haetinger and A. H. Majeed as a generalization of Jordan derivations on a Lie ideal of a ring. We introduce (U, M)-derivations in Γ-rings as a generalization of Jordan derivations on a Lie ideal of a Γ-ring. We construct some useful consequences of (U, M)-derivation of a prime Γ-ring to prove first that, $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u, v \in U, \alpha \in \Gamma$, where U is an admissible Lie ideal of M and d is a (U, M)-derivation of M. We also prove that, if $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$ then $d(u\alpha m) = d(u)\alpha m + u\alpha d(m)$ for all $u \in U, m \in M$ and $\alpha \in \Gamma$. After introducing (U, M)-derivation in

Γ-rings, we then introduce the concept of higher (U, M)-derivation in Γ-rings. We conclude the chapter 6 by proving the analogous results corresponding to the previous results considering higher (U, M)-derivations of prime Γ-rings almost similar way after developing a number of results regarding this newly introduced concept.

Following the notion of (U, M)-derivation and higher (U, M)-derivation of a Γ-ring in the previous chapter here we introduce the concept of generalized (U, M)-derivation and generalized higher (U, M)-derivation in Γ-rings. By establishing some necessary results with generalized (U, M)-derivation, we then prove the analogous results considering generalized (U, M)-derivation of prime Γ-rings corresponding to the results of (U, M)-derivation of the previous chapter. A. K. Faraj, C. Haetinger and A. H. Majeed extended Awtar's theorem to generalized higher (U, R)-derivations by proving that if R is a prime ring, char.$(R) \neq 2$, U is an admissible Lie ideal of R and $F = (f_i)_{i \in N}$ is a generalized higher (U, R)-derivations of R, then $f_n(ur) = \sum_{i+j=n} f_i(u)d_j(r)$ for all $u \in U, r \in R$ and $n \in N$. Chapter 7 also extends this result in the case of generalized higher (U, M)-derivation of prime Γ-rings. Actually, it aims to prove that if U is an admissible Lie ideal of a prime Γ-ring M and $F = (f_i)_{i \in \mathbf{N}}$ is a generalized higher (U, M)-derivation of M, then (i) $f_n(u\alpha v) = \sum_{i+j=n} f_i(u)\alpha d_j(v)$ for all $u, v \in U, \alpha \in \Gamma$ and $n \in \mathbf{N}$; and also, that (ii) $f_n(u\beta m) = \sum_{i+j=n} f_i(u)\beta d_j(m)$ for all $u \in U, m \in M, \beta \in \Gamma$ and $n \in \mathbf{N}$.

Addendum

This is to mention with due acknowledgement that some publications have resulted from certain parts of this thesis, several papers have been accepted for publications, and some more have been submitted for publications in various journals. The following is a list of the published and accepted papers.

[1] A. C. Paul and Md. Mizanor Rahman, *Jordan Left Derivations on Semiprime* Γ*-Rings*, Indian Journal of Pure and Applied Sciences and Technologies, 6(2), 2011, 131-135.

[2] M. M. Rahman and A. C. Paul, *Commutativity in Prime* Γ*-Rings with Jordan Left Derivations*, Mathematical Theory and Modeling, 2(9), 2012, 96-107.

[3] ——————————-, *Commutativity in Prime* Γ*-Rings with Jordan Higher Left Derivations*, Accepted for publication in Jagannath University Journal of Science (to be appeared in its upcoming issue).

[4] ——————————-, *Jordan Higher Derivations on Lie Ideals of Prime* Γ*-Rings*, South Asian Journal of Mathematics, 3(2), 2013, 127-132.

[5] ——————————-, *Generalized (U, M)-Derivations in Prime* Γ*-Rings*, Mathematical Theory and Modeling, 3(3), 2013, 98-104.

[6] ——————————-, *Jordan Generalized Derivations on Lie Ideals of Prime* Γ*-Rings*, South Asian Journal of Mathematics, 3(3), 2013, 148-153.

[7] ——————————-, *Higher(U, M)-Derivations in Prime* Γ*-Rings*, Journal of the Calcutta Mathematical Society, 9(2), 2013, 63-74.

[**8**] ——————————-, *Jordan Generalized Higher Derivations in Prime* Γ*-Rings*, Mathematical Theory and Modeling, 3(3), 2013, 88-97.

[**9**] ——————————-, *Jordan Generalized Higher Derivations on Lie Ideals of Prime* Γ*-Rings*, South Asian Journal of Mathematics, 3(4), 2013, 287-293.

[**10**] ——————————-, *Jordan Derivations on Lie ideals of Prime* Γ*-Rings*, Mathematical Theory and Modeling, 3(3), 2013, 128-135.

[**11**] ——————————-, *Generalized Higher(U, M)-Derivations in Prime* Γ*-Rings*, Springer Plus (a Springer Open Journal), 2014, 3:100 doi: 10.1186/2193-1801-3-100.

[**12**] ——————————-, *Jordan Derivations of Completely Semiprime* Γ*-Rings*, Accepted for publication in GANIT, Journal of Bangladesh Mathematical Society (to be appeared in its upcoming issue).

[**13**] ——————————-, *Derivations on Lie ideals of Completely Semiprime* Γ*-Rings*, Accepted for publication in Bangladesh Journal of Scientific Research (to be appeared in its upcoming issue).

[**14**] ——————————-, *Jordan Generalized Derivations on Lie ideals of Completely Semiprime* Γ*-Rings*, Accepted for publication in Bangladesh Journal of Scientific Research (to be appeared in its upcoming issue).

[**15**] ——————————-, *Jordan Derivations on 2-torsion free Semiprime* Γ*-Rings*, Accepted for publication in Journal of Bangladesh Academy of Science (JBAS) (to be appeared in its upcoming issue).

Introduction

As an immense generalization of the theory of rings, the concept of a Γ-ring was introduced. From its first advent the enlargement and generalization of many significant results in the theory of rings to the theory of Γ-rings have interested a wider observation as an emerging field of research to the modern algebraists to enhance the world of algebra. A number of renowned mathematicians have worked out on this attractive area of research to determine various basic properties of Γ-rings and have extended a lot of important results in this topic in the last four decades. There is a huge number of researchers all over the world who are recently occupied in trying to achieve more and more fruitful and inventive results of Γ-ring theory.

The notion of a Γ-ring has been introduced by N. Nobusawa [33] (which is presently known as a Γ_N-ring), as a generalization of a ring. Following W. E. Barnes [4] generalized the concept of Nobusawa's Γ-ring as a more general nature. As an immediate consequence, this generalization states that every Γ_N-ring is a Γ-ring, but the converse is not always true in general. They obtained many important basic properties of Γ-rings in various ways, while in a consecutive succession S. Kyuno [25, 26, 27, 28] , J. Luh [29], G. L. Booth [5] determined some more remarkable characterizations of Γ-rings. Nowadays, Γ-ring theory is a showpiece of mathematical unification, bringing together several branches of the subject. It is the best research area for the Mathematicians and during 40 years, many classical ring theories have been generalized in Γ-rings by many authors (mentioned hereafter within the scope of this study). Some of those are discussed in considerable detail in this chapter including all the necessary introductory definitions and examples.

0.1 Gamma Ring

We begin with the general definition of a Γ-ring. The notion of a Γ-ring was introduced by N. Nobusawa [33] and generalized by W. E. Barnes [4] as defined below.

Definition 0.1.1. Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ such that the conditions

- $(x+y)\alpha z = x\alpha z + y\alpha z,\ x(\alpha+\beta)y = x\alpha y + x\beta y,\ x\alpha(y+z) = x\alpha y + x\alpha z;$
- $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-*ring* . This concept is more general than a ring.

From the definition it is clear that every ring is a Γ-ring but the converse is not necessarily true. For example, we observe that.

Example 0.1.1. *Let R be a ring having unity element 1 and $M = M_{2,3}(R)$ be the set of all 2×3 matrices over R. If we take $\Gamma = M_{3,2}(R)$, then M is a Γ-ring under the operations of addition and multiplication of matrices.*

In general, we get the following.

Example 0.1.2. *If R is a ring with unity element 1 and $M = M_{m,n}(R)$ is the set of all $m \times n$ matrices over R. Then M is a Γ-ring under the operations of addition and multiplication of matrices if we take $\Gamma = M_{n,m}(R)$.*

The following is an example of a Γ-ring.

Example 0.1.3. *Let R be a commutative ring with characteristic 2 having unity element 1. Let $M = M_{2,2}(R)$ and* $\Gamma = \left\{ \begin{pmatrix} n_1.1 & n_3.1 \\ n_2.1 & n_4.1 \end{pmatrix} : n_i \in (Z - 2Z); n_1 = n_4, n_2 = n_3 \right\}$. *Then M is a Γ-ring.*

The following is another example of a Γ-ring given by S. Kyuno [25].

Example 0.1.4. *Let R be an ordinary associative ring, U be any ideal of R, and I be the ring of integers. Then R is a Γ-ring if we choose $\Gamma = R$ or, $\Gamma = U$ or, $\Gamma = I$. Also, U is a Γ-ring with $\Gamma = R$.*

Now we recall the initiatory definition of a Γ-ring given by N. Nobusawa [33] appeared for the first time that has been creating a new extent to enhance the theory of rings significantly.

Definition 0.1.2. Let M and Γ be additive abelian groups. If there are two mappings $M \times \Gamma \times M \to M$ and $\Gamma \times M \times \Gamma \to \Gamma$ such that

- $(x+y)\alpha z = x\alpha z + y\alpha z,\ x(\alpha+\beta)y = x\alpha y + x\beta y,\ x\alpha(y+z) = x\alpha y + x\alpha z;$
- $(\alpha+\beta)x\gamma = \alpha x\gamma + \beta x\gamma,\ \alpha(x+y)\beta = \alpha x\beta + \alpha y\beta,\ \alpha x(\beta+\gamma) = \alpha x\beta + \alpha x\gamma;$
- $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z);$
- $x\alpha y = 0$ implies $\alpha = 0$

hold for all $x, y, z \in M; \alpha, \beta \in \Gamma$, then M is called a Γ-*ring* in the sense of N. Nobusawa [33] and we express it by saying that M is a Γ_N-*ring*.

Example 0.1.5. *Let D be a division ring and $M = D_{2,3}(D)$ be the set of all 2×3 matrices over D. If we choose $\Gamma = D_{3,2}(D)$, then M is a Γ_N-ring under the operations of addition and multiplication of matrices.*

In general, we get the following:

Example 0.1.6. *If D is a division ring and $M = D_{m,n}(D)$ is the set of all $m \times n$ matrices over D. Then M is a Γ-ring under the operations of addition and multiplication of matrices if we choose $\Gamma = D_{n,m}(D)$.*

The following is another example of a Γ_N-ring given by S. Kyuno [25].

Example 0.1.7. *Let R be an ordinary associative ring with unity element 1. Then R is a Γ_N-ring if we choose $\Gamma = R$.*

The following three examples of a Γ_N-ring are given by N. Nobusawa [33].

Example 0.1.8. *Let $v_n(F)$ be a vector space of dimension n over a field F. For a, b, c are vectors in it, define $abc = (a \cdot b)c$, where $(a \cdot b)$ is the inner product of a and b. For $b, c, d \in V_n(F)$, define $(bcd)' = b(c \cdot d)$. Then $ab(cde) = (a \cdot b)(c \cdot d)e = a(bcd)'e$ i.e, $v_n(F)$ is a Γ_N-ring with $\Gamma = v_n(F)$.*

Example 0.1.9. *Let D be a division ring and $M = D_{n,m}(D)$. If $a, b, c \in M$, define $abc = ab^tc$, where b^t is the transpose of the matrix b, and the above product is well defined. For $b, c, d \in M$, define also $(bcd)' = dc^tb$. Then $ab(cde) = ab^tcd^t = a(bcd)'e$ i.e, M is a Γ_N-ring with $\Gamma = D_{m,n}(D)$.*

Example 0.1.10. *Let I be the set of all purely imaginary complex numbers. Then I is a Γ_N-ring with usual multiplication if we choose $\Gamma = I$.*

0.2 Preliminaries

We recall some important definitions which are useful for us within the scope of this study as follows.

Definition 0.2.1. An additive subgroup H of a Γ-ring M is said to be a Γ-*subring* of M if H is itself a Γ-ring. That means, it follows that an additive subgroup H of a Γ-ring M is a Γ-subring of M if $H\Gamma H \subseteq H$.

The following is an example of a Γ-subring.

Example 0.2.1. *Let R be a ring of characteristic 2 having a unity element 1. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ n.1 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$, then M is a Γ-ring. Let $N = \{(x, x) : x \in R\} \subseteq M$, then N is also Γ-ring of M, in which we can say that N is a Γ-subring of M.*

Definition 0.2.2. A subset S of the Γ-ring M is a *right ideal* (or, *left ideal*) of M if S is an additive subgroup of M and $S\Gamma M = \{s\alpha m : s \in S, \alpha \in \Gamma, m \in M\}$ (or, $M\Gamma S = \{m\alpha s : s \in S, \alpha \in \Gamma, m \in M\}$) is contained in S. If S is both a left and a right ideal of M, then S is a two sided ideal of M, or simply an *ideal of a* Γ-*ring* M.

Definition 0.2.3. Given a subset A of a Γ-ring M, the ideal generated by A is the smallest ideal of M which contains A. In fact, it is the intersection of all ideals which contain A. The elements of A are called its *generator*. In particular, if $A = \{a_1, a_2, ..., a_n\}$ then the ideal generated by A is denoted by $(a_1, a_2, ..., a_n)$. An ideal is said to be *finitely generated* if and only if it has a finite set of generators.

Remark 0.2.1. The ideal generated by a given set A can often be generated by a much smaller subset of A.

Definition 0.2.4. Let M be a Γ-ring and $m \in M$. The ideal generated by m (the intersection of all ideals of M containing m, that is, the smallest ideal of M) is called the *principal ideal* of M and it is denoted by (m). More precisely, it is the set of all finite sums of the form $\sum_i (n_i m + x_i \alpha_i m + m \beta_i y_i + u_i \gamma_i m \delta_i v_i)$, where $n_i \in \mathbf{Z}; x_i, y_i, u_i, v_i \in M$ and $\alpha_i, \beta_i, \gamma_i, \delta_i \in \Gamma$. That means, it follows that $(m) = \mathbf{Z}m + M\Gamma m + m\Gamma M + M\Gamma m \Gamma M$.

Definition 0.2.5. If I is any nonzero ideal of a Γ-ring M. Then an ideal E of a Γ-ring M is said to be an *essential ideal* of M if $E \cap I \neq 0$.

Definition 0.2.6. Let I be an ideal of a Γ-ring M, then the set $Ann_l = \{x \in M : x\Gamma I = 0\}$ is called the *left annihilator* of I of M and the set $Ann_r = \{x \in M : I\Gamma x = 0\}$ is said to be the *right annihilator* of I of M. If left annihilator and right annihilator are identical, then the set $Ann(I) = \{x \in M : x\Gamma I = I\Gamma x = 0\}$ is called the *annihilator* I of M.

Obviously, if I is an ideal of a Γ-ring M, then $Ann(I)$ is also an ideal of M.

Definition 0.2.7. Let M be a Γ-ring and $x, y \in M, \alpha \in \Gamma$, then $(x \circ y) = x\alpha y + y\alpha x$ is called the *Jordan product* of x and y with respect to α.

Definition 0.2.8. Let M be a Γ-ring ; for $x, y \in M$ and $\alpha \in \Gamma$, a new product, known as *Lie product* defined by $[x, y]_\alpha = x\alpha y - y\alpha x$ and it is called the *commutator* of x and y with respect to α.

From the definition of commutators of two elements in a Γ-ring, we make the basic commutators identities :

Lemma 0.2.1. *If M is a Γ-ring, then for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$:*

- $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha [y, z]_\beta$.
- $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha [x, z]_\beta$.

According to the condition $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, the above two identities reduces to:

- $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$.
- $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta$.

Lemma 0.2.2. *If M is a Γ-ring, then for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$:*

- $[x, y]_\alpha + [y, x]_\alpha = 0$.
- $[x, y + z]_\alpha = [x, y]_\alpha + [x, z]_\alpha$
- $[x + y, z]_\alpha = [x, z]_\alpha + [y, z]_\alpha$
- $[x, y]_{\alpha+\beta} = [x, y]_\alpha + [x, y]_\beta$.

Remark 0.2.2. A necessary and sufficient condition for a Γ-ring M to be commutative is that $[x, y]_\alpha = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 0.2.9. An additive subgroup $U \subset M$ is said to be a *Lie ideal* of M if whenever $u \in U; m \in M$ and $\alpha \in \Gamma$, then $[u, m]_\alpha \in U$. A Lie ideal is called a *square closed Lie ideal* if $u\alpha u \in U$, for all $u \in U; \alpha \in \Gamma$. Furthermore, if the Lie ideal U is square closed and $U \nsubseteq Z(M)$, where $Z(M)$ denotes the centre of M, then U is called an *admissible Lie ideal* of M.

Example 0.2.2. *Let R be a commutative ring of characteristic 2 having a unity element 1. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ n.1 \end{pmatrix} : n \in \boldsymbol{Z}, 2 \nmid n \right\}$, then M is a Γ-ring. Let $N = \{(x, x) : x \in R\} \subseteq M$, then for all $(x, x) \in N, (a, b) \in M$ and $\begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$, we have*

$$\begin{aligned}
(x, x) \begin{pmatrix} n \\ n \end{pmatrix} (a, b) - (a, b) \begin{pmatrix} n \\ n \end{pmatrix} (x, x) &= (xna - bnx, xnb - anx) \\
&= (xna - 2bnx + bnx, bnx - 2anx + xna) \\
&= (xna + bnx, bnx + xna) \in N.
\end{aligned}$$

Therefore, N is a Lie ideal of M.

The following is a very well-known result in group theory essential for us which is known as *Brauer's trick* .

Brauer's trick: If S and T are any two subgroups of a group G such that $G = S \cup T$, then either $G = S$ or $G = T$. In other words, a group cannot be a set-theoretic union of its two proper subgroups.

Definition 0.2.10. A Γ-ring M is said to be a *commutative* Γ*-ring* if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$.

The following is an example of a commutative Γ-ring.

Example 0.2.3. *Let R be a commutative ring having a unity element 1. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$. Then M is a commutative Γ-ring under the operations of matrix addition and matrix multiplication with char.$M = 2$.*

Definition 0.2.11. Let M be a Γ-ring, then the set $Z_\alpha = \{z \in M : z\alpha m = m\alpha z$ for all $m \in M\}$ is called the $\alpha -$ *centre of a* Γ*-ring* M, where α is an arbitrary but fixed element of Γ.

Definition 0.2.12. Let M be a Γ-ring, then the set $Z_\Gamma = \{z \in M : z\alpha m = m\alpha z$ for all $m \in M$ and $\alpha \in \Gamma\}$ is called the *centre of a* Γ*-ring* M and it is denoted by $Z(M)$.

It is a clear consequence that if $Z(M) = M$, then M is commutative.

Remark 0.2.3. The centre $Z(M)$ of a Γ-ring M is always a Γ-subring of M.

Definition 0.2.13. An element m of a Γ-ring M is called a right (or, left) *nonzero divisor* of M if for $x \in M, x\alpha m = 0$ (or, $m\alpha x = 0$) implies $x = 0$ for all $\alpha \in \Gamma$. If an element is both a left and a right nonzero divisor of M, then it is called a two-sided nonzero divisor of M, or simply a nonzero divisor of M.

In other words, a Γ-ring M is said to have no zero divisors if $\forall\, x, y \in M, x\alpha y = 0$ implies $x = 0$, or $y = 0$ for all $\alpha \in \Gamma$.

Definition 0.2.14. A Γ-ring M is said to be *2-torsion free* or of *characteristic not equal to 2*, denoted as *char.*$M \neq 2$, if $2x = 0$ implies $x = 0$ for all $x \in M$.

In general, we have the following.

Definition 0.2.15. A Γ-ring M is said to be *n-torsion free* or of *characteristic not equal to n* (where n is a positive integer greater than 1), denoted as *char.*$M \neq n$, if $nx = 0$ implies $x = 0$ for all $x \in M$.

Definition 0.2.16. An element m of a Γ-ring M is called a *nilpotent element* if, for any $\alpha \in \Gamma$, there exists a positive integer n (depending on α) such that $(m\alpha)^n m = (m\alpha)(m\alpha)...(m\alpha)m = 0$

Definition 0.2.17. An ideal I of a Γ-ring M is said to be a *nilpotent ideal* if there exists a positive integer n such that $(I\Gamma)^n I = ((I\Gamma)(I\Gamma)...(I\Gamma)I = 0$

Definition 0.2.18. An ideal I of a Γ-ring M is said to be a *nil ideal* if each element of I is nilpotent.

Remark 0.2.4. Every nilpotent ideal of a Γ-ring is nil.

The concepts of a prime Γ-ring and a completely prime Γ-ring were introduced by J. Luh [29] and some analogous results corresponding to the prime rings were obtained by J. Luh [29] and S. Kyuno [27].

Definition 0.2.19. A Γ-ring M is said to be a *prime* Γ*-ring* if $x\Gamma M\Gamma y = 0$ (with $x, y \in M$) implies $x = 0$ or $y = 0$.

Definition 0.2.20. A Γ-ring M is said to be a *semiprime* Γ*-ring* if $x\Gamma M\Gamma x = 0$ (with $x \in M$) implies $x = 0$.

Definition 0.2.21. A Γ-ring M is said to be a *completely prime* Γ*-ring* if $x\Gamma y = 0$ (with $x, y \in M$) implies $x = 0$ or $y = 0$.

Definition 0.2.22. A Γ-ring M is said to be a *completely semiprime* Γ*-ring* if $x\Gamma x = 0$ (with $x \in M$) implies $x = 0$.

It is obvious that every completely prime Γ-ring is prime but the converse is not necessarily true. Similarly, every completely semiprime Γ-ring is semiprime but the converse is not always true in general.

From the above definitions, it follows that every prime Γ-ring is semiprime and every completely prime Γ-ring is completely semiprime.

Example 0.2.4. *Let R be an integral domain with 1. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$, then M is a Γ-ring. Let $N = \{(a, a) : a \in R\}$, then N is a Γ-subring of M. It is easy to verify that N is a completely prime Γ-ring and therefore it is also a prime Γ-ring.*

Chapter 1

Left Derivations

We begin by explaining the introductional background behind the notions of derivation and Jordan derivation of Γ-rings. Then we recall the definitions of left derivation and right derivation of Γ-rings. We also recall the definitions of Jordan left derivation and Jordan right derivation of Γ-rings in the first section.

The second section develops some useful consequences regarding Jordan left derivations of Γ-rings which are very much needed for proving the main results of this chapter.

The result of the third section has been emanated from a theorem due to Joso Vukman which states that, if R is a 2-torsion free and 3-torsion free semiprime ring which admits Jordan derivations $D : R \to R$ and $G : R \to R$ such that $D^2(x) = G(x)$ for all $x \in R$, then $D = 0$. Here, we extend this result to the Γ-ring theory in the case of Jordan left derivation by showing that if M is a 2 and 3-torsion free semiprime Γ-ring ; $d : M \to M$ and $G : M \to M$ are Jordan left derivations such that $d^2(M) = G(M)$, then $d = 0$.

Finally, we conclude this chapter by showing that under a suitable condition every nonzero Jordan left derivation d of a 2-torsion free prime Γ-ring M induces the commutativity of M, and accordingly, d is a left derivation of M.

1.1 Introduction

The notions of derivation and Jordan derivation of a Γ-ring have been introduced by M. Sapanci and A. Nakajima [36]. Afterwards, K. W. Jun and B. D. Kim [24] obtained some significant results due to Jordan left derivation of a classical ring. Y. Ceven [10] worked on left derivations of completely prime Γ-rings and obtained some extensive results of left derivation and Jordan left derivation of a Γ-ring. M. Soyturk [37] investigated the commutativity of prime Γ-rings with the left and right derivations. He obtained some results on the commutativity of prime Γ-rings of characteristic not equal to 2 and 3. Some commutativity results of prime Γ-rings with left derivations were obtained by Asci and Ceran [1]. A. C. Paul and A. K. Halder [35] worked on Jordan left derivations of two torsion free ΓM-Modules. They proved that if M is a prime Γ-ring, then every Jordan left derivation is a left derivation.

In view of the concepts of left derivation and Jordan left derivation of classical rings developed by K. W. Jun and B. D. Kim [24], some important results due to these concepts in case of certain Γ-rings have been determined by Y.Ceven [10] after introducing the notions of left derivation and right derivation of Γ-rings as defined below.

Definition 1.1.1. In a Γ-ring M, an additive mapping $d : M \to M$ is said to be a *left derivation* if $d(a\alpha b) = a\alpha d(b) + b\alpha d(a)$ holds for all $a, b \in M; \alpha \in \Gamma$ and d is called a *right derivation* if $d(a\alpha b) = d(a)\alpha b + d(b)\alpha a$ holds for all $a, b \in M; \alpha \in \Gamma$.

Definition 1.1.2. Let M be a Γ-ring. An additive mapping $d : M \to M$ is said to be a *Jordan left derivation* if $d(a\alpha a) = 2a\alpha d(a)$ holds for all $a \in M; \alpha \in \Gamma$ and d is called a *Jordan right derivation* if $d(a\alpha a) = 2d(a)\alpha a$ holds for all $a \in M; \alpha \in \Gamma$.

In [10], Y. Ceven gave an example of a left derivation and a Jordan left derivation on a Γ-ring as follows.

Example 1.1.1. *Let R be an associative ring with 1 and $d : R \to R$ be a left derivation. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$, then M is a Γ-ring. Define a mapping $D : M \to M$ by $D((a, b)) = (d(a), d(b))$, then D is a left derivation on M. Let $N = \{(a, a) : a \in R\} \subset M$, then N is a Γ-ring. Define a mapping $D : N \to M$ by $D((a, a)) = (d(a), d(a))$, then D is a Jordan left derivation on N.*

Except otherwise mentioned, throughout this chapter, M represents a Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M; \alpha, \beta \in \Gamma$ and it is referred to as the symbol (*).

1.2 Some Consequences of Jordan Left Derivations

We recall some useful results that have already been proved earlier. We begin with the following Lemma proved by Y.Ceven [10].

Lemma 1.2.1. *Let d be a Jordan left derivation of a two torsion free Γ-ring M. For all $a, b \in M; \alpha \in \Gamma$:*

(i) $d(a\alpha b + b\alpha a) = 2a\alpha d(b) + 2b\alpha d(a)$;

In particular, if M is a 2-torsion free and satisfies the condition (*), then

(ii) $d(a\alpha b\beta a) = a\beta a\alpha d(b) + 3a\alpha b\beta d(a) - b\alpha a\beta d(a)$;

(iii) $d(a\alpha b\beta c + c\alpha b\beta a) = a\beta c\alpha d(b) + c\beta a\alpha d(b) + 3a\alpha b\beta d(c) + 3c\alpha b\beta d(a) - b\alpha c\beta d(a) - b\alpha a\beta d(c)$.

Some parts of the following lemma are essentially proved in [8, 9, 10, 24, 36]

Lemma 1.2.2. *Let M be a two torsion free Γ-ring satisfying the condition (*) and d be a Jordan left derivation on M. Then for all $a, b \in M; \alpha, \beta \in \Gamma$:*

(i) $[a, b]_\alpha \beta a\alpha d(a) = a\alpha[a, b]_\alpha \beta d(a)$;

(ii) $[a, b]_\alpha \beta(d(a\alpha b) - a\alpha d(b) - b\alpha d(a)) = 0$;

(iii) $[a, b]_\alpha \beta d([a, b]_\alpha) = 0$;

(iv) $d(a\alpha a\beta b) = a\beta a\alpha d(b) + (a\beta b + b\beta a)\alpha d(a) + a\alpha d([a, b]_\beta)$.

Proof. (i) By Lemma 1.2.1(iii), we have

$$d(a\alpha b\beta c+c\alpha b\beta a) = a\beta c\alpha d(b)+c\beta a\alpha d(b)+3a\alpha b\beta d(c)+3c\alpha b\beta d(a)-b\alpha a\beta d(c)-b\alpha c\beta d(a).$$

Replacing c by $a\alpha b$, we get

$$\begin{aligned} d((a\alpha b)\beta(a\alpha b) + (a\alpha b)\alpha b\beta a) &= a\beta(a\alpha b)\alpha d(b) + (a\alpha b)\beta a\alpha d(b) \\ &+ 3a\alpha b\beta d(a\alpha b) + 3(a\alpha b)\alpha b\beta d(a) - b\alpha a\beta d(a\alpha b) - b\alpha(a\alpha b)\beta d(a). \end{aligned}$$

This implies,

$$\begin{aligned} 2(a\alpha b)\beta d(a\alpha b) + d(a\alpha(b\alpha b)\beta a) &= a\beta a\alpha b\alpha d(b) + a\alpha b\beta a\alpha d(b) \\ &+ 3a\alpha b\beta d(a\alpha b) + 3(a\alpha b)\alpha b\beta d(a) - b\alpha a\beta d(a\alpha b) - b\alpha(a\alpha b)\beta d(a). \end{aligned}$$

Using Lemma 1.2.1(ii), we obtain

$$\begin{aligned} -a\alpha b\beta d(a\alpha b)+a\beta a\alpha d(b\alpha b)+3a\alpha b\alpha b\beta d(a)-b\alpha b\alpha a\beta d(a) &= a\beta a\alpha b\alpha d(b)+a\alpha b\beta a\alpha d(b) \\ &+ 3a\alpha b\alpha b\beta d(a) - b\alpha a\beta d(a\alpha b) - b\alpha a\alpha b\beta d(a). \end{aligned}$$

$$\Rightarrow -a\alpha b\beta d(a\alpha b) + 2a\beta a\alpha b\alpha d(b) - b\alpha b\alpha a\beta d(a) = a\beta a\alpha b\alpha d(b) + a\alpha b\beta a\alpha d(b) - b\alpha a\beta d(a\alpha b) - b\alpha a\alpha b\beta d(a).$$

This yields,

$$\begin{aligned}(a\alpha b - b\alpha a)\beta d(a\alpha b) &= a\beta a\alpha b\alpha d(b) - a\alpha b\beta a\alpha d(b) - b\alpha b\alpha a\beta d(a) + b\alpha a\alpha b\beta d(a)\\ &= a\alpha a\alpha b\beta d(b) - a\alpha b\alpha a\beta d(b) - b\alpha b\alpha a\beta d(a) + b\alpha a\alpha b\beta d(a)\\ &= a\alpha(a\alpha b - b\alpha a)\beta d(b) + b\alpha(a\alpha b - b\alpha a)\beta d(a).\end{aligned} \tag{1.1}$$

Replacing b by $a+b$ in (1.1), we get

$$(a\alpha b - b\alpha a)\beta(2a\alpha d(a) + d(a\alpha b)) = a\alpha(a\alpha b - b\alpha a)\beta d(a+b) + (a+b)\alpha(a\alpha b - b\alpha a)\beta d(a).$$

$$\Rightarrow 2(a\alpha b - b\alpha a)\beta a\alpha d(a) + (a\alpha b - b\alpha a)\beta d(a\alpha b) = 2a\alpha(a\alpha b - b\alpha a)\beta d(a) + a\alpha(a\alpha b - b\alpha a)\beta d(b) + b\alpha(a\alpha b - b\alpha a)\beta d(a).$$

Using (1.1), we obtain

$$2(a\alpha b - b\alpha a)\beta a\alpha d(a) + a\alpha(a\alpha b - b\alpha a)\beta d(b) + b\alpha(a\alpha b - b\alpha a)\beta d(a)$$
$$= 2a\alpha(a\alpha b - b\alpha a)\beta d(a) + a\alpha(a\alpha b - b\alpha a)\beta d(b) + b\alpha(a\alpha b - b\alpha a)\beta d(a).$$
$$\Rightarrow (a\alpha b - b\alpha a)\beta a\alpha d(a) = a\alpha(a\alpha b - b\alpha a)\beta d(a).$$

Therefore,

$$[a, b]_\alpha \beta a\alpha d(a) = a\alpha[a, b]_\alpha \beta d(a).$$

(ii) Replacing a by $a+b$ inLemma 1.2.2 (i)

$$((a+b)\alpha b - b\alpha(a+b))\beta(a+b)\alpha d(a+b) = (a+b)\alpha((a+b)\alpha b - b\alpha(a+b))\beta d(a+b).$$

$$\Rightarrow (a\alpha b - b\alpha a)\beta(a\alpha d(a) + b\alpha d(a) + a\alpha d(b) + b\alpha d(b)) = a\alpha(a\alpha b - b\alpha a)\beta(d(a) + d(b)) + b\alpha(a\alpha b - b\alpha a)\beta(d(a) + d(b)).$$

$$\Rightarrow (a\alpha b - b\alpha a)\beta a\alpha d(a) + (a\alpha b - b\alpha a)\beta b\alpha d(a) + (a\alpha b - b\alpha a)\beta a\alpha d(b) + (a\alpha b - b\alpha a)\beta b\alpha d(b)$$

$$= a\alpha(a\alpha b - b\alpha a)\beta d(a) + a\alpha(a\alpha b - b\alpha a)\beta d(b) + b\alpha(a\alpha b - b\alpha a)\beta d(a) + b\alpha(a\alpha b - b\alpha a)\beta d(b).$$

Now, using Lemma 1.2.2(i), we have

$$a\alpha(a\alpha b - b\alpha a)\beta d(a) + (a\alpha b - b\alpha a)\beta b\alpha d(a) + (a\alpha b - b\alpha a)\beta a\alpha d(b) - b\alpha(b\alpha a - a\alpha b)\beta d(b)$$

$$= a\alpha(a\alpha b - b\alpha a)\beta d(a) + a\alpha(a\alpha b - b\alpha a)\beta d(b) + b\alpha(a\alpha b - b\alpha a)\beta d(a) - b\alpha(b\alpha a - a\alpha b)\beta d(b).$$

Thus, using (1.1)

$$(a\alpha b - b\alpha a)\beta(b\alpha d(a) + a\alpha d(b)) = (a\alpha b - b\alpha a)\beta d(a\alpha b).$$

Therefore, we obtain

$$(a\alpha b - b\alpha a)\beta(d(a\alpha b) - a\alpha d(b) - b\alpha d(a)) = 0.$$

This implies,

$$[a, b]_\alpha \beta(d(a\alpha b) - a\alpha d(b) - b\alpha d(a)) = 0.$$

(iii) Using Lemma 1.2.1(i) in Lemma 1.2.2(ii), we get

$$(a\alpha b - b\alpha a)\beta(-d(b\alpha a) + 2a\alpha d(b) + 2b\alpha d(a) - a\alpha d(b) - b\alpha d(a)) = 0.$$

Therefore,

$$(a\alpha b - b\alpha a)\beta(d(b\alpha a) - a\alpha d(b) - b\alpha d(a)) = 0. \tag{1.2}$$

Subtracting (1.2) from Lemma 1.2.2(ii), we get

$$(a\alpha b - b\alpha a)\beta d(a\alpha b) - d(b\alpha a) = 0.$$

Therefore,

$$[a, b]_\alpha \beta d([a, b]_\alpha) = 0.$$

(iv) From Lemma 1.2.1(i), we have

$$d(a\alpha b + b\alpha a) = 2a\alpha d(b) + 2b\alpha d(a).$$

Replacing $b\beta a$ for b, we get

$$d(a\alpha b\beta a + b\beta a\alpha a) = 2a\alpha d(b\beta a) + 2b\beta a\alpha d(a). \tag{1.3}$$

Again replacing $a\beta b$ for b in Lemma 1.2.1(i)

$$d(a\alpha a\beta b + a\beta b\alpha a) = 2a\alpha d(a\beta b) + 2a\beta b\alpha d(a). \tag{1.4}$$

Subtracting (1.3) from (1.4) and using the condition (*), we get

$$d(a\alpha a\beta b + a\alpha b\beta a - a\alpha b\beta a - b\alpha a\beta a) = 2a\alpha d(a\beta b - b\beta a) + 2(a\beta b - b\beta a)\alpha d(a).$$

Therefore,

$$d(a\alpha a\beta b - b\alpha a\beta a) = 2a\alpha d(a\beta b - b\beta a) + 2(a\beta b - b\beta a)\alpha d(a). \tag{1.5}$$

Now, replacing $a\beta a$ for a in Lemma 1.2.1(i) and using the condition (*)

$$d(a\beta a\alpha b + b\alpha a\beta a) = 2a\beta a\alpha d(b) + 2b\alpha d(a\beta a) = 2a\beta a\alpha d(b) + 4b\alpha a\beta d(a)$$

$$\Rightarrow d(a\alpha a\beta b + b\alpha a\beta a) = 2a\beta a\alpha d(b) + 4b\beta a\alpha d(a). \tag{1.6}$$

Adding (1.5) and (1.6), we get

$$d(2a\alpha a\beta b) = 2a\beta a\alpha d(b) + 2a\alpha d(a\beta b - b\beta a) + 2(a\beta b + b\beta a)\alpha d(a).$$

Since M is 2-torsion free, we have

$$d(a\alpha a\beta b) = a\beta a\alpha d(b) + (a\beta b + b\beta a)\alpha d(a) + a\alpha d([a, b]_\beta).$$

$\square$

Lemma 1.2.3. *Let M be a 2-torsion free and 3-torsion free Γ-ring, and $d : M \to M$ be a Jordan left derivation. If $d([[d(x), x]_\alpha, x]_\alpha) = 0$ holds for all $x \in M; \alpha \in \Gamma$, then $[d(x), x]_\alpha \alpha d(x) = 0$ is fulfilled for all $x \in M; \alpha \in \Gamma$.*

Proof. Since $d([[d(x), x]_\alpha, x]_\alpha) = 0$. Thus

$$\begin{aligned}
0 &= d([[d(x), x]_\alpha, x]_\alpha) \\
&= d([(d(x)\alpha x - x\alpha d(x)), x]_\alpha) \\
&= d(d(x)\alpha x\alpha x - x\alpha d(x)\alpha x - x\alpha d(x)\alpha x + x\alpha x\alpha d(x)) \\
&= d(d(x)\alpha x\alpha x + x\alpha x\alpha d(x)) - 2d(x\alpha d(x)\alpha x).
\end{aligned}$$

Now, using Lemma 1.2.1(i) and Lemma 1.2.1(ii), we get

$$2(x\alpha x)\alpha d(d(x)) + 2d(x)\alpha d(x\alpha x) - 2x\alpha x\alpha d(d(x)) - 6x\alpha d(x)\alpha d(x) + 2d(x)\alpha x\alpha d(x) = 0.$$

$$\Rightarrow 2x\alpha x\alpha d^2(x) + 4d(x)\alpha x\alpha d(x) - 2x\alpha x\alpha d^2(x) - 6x\alpha d(x)\alpha d(x) + 2d(x)\alpha x\alpha d(x) = 0.$$

$$\Rightarrow 6(d(x)\alpha x - x\alpha d(x))\alpha d(x) = 0.$$

Thus, we have

$$6[d(x), x]_\alpha \alpha d(x) = 0.$$

Since M is 2 and 3-torsion free. Hence, we conclude that $[d(x), x]_\alpha \alpha d(x) = 0$ for all $x \in M, \alpha \in \Gamma$. □

1.3 Jordan Left Derivations on Semiprime Γ-Rings

In classical ring theory, Joso Vukman [38] proved that if d is a Jordan left derivation of a 2-torsion free semiprime ring R and if there exists a positive integer n such that $D(x)^n = 0$ for all $x \in R$, then $D = 0$. He also proved that for a 2-torsion free and

3-torsion free semiprime ring R admits Jordan derivations $D : R \to R$ and $G : R \to R$ such that $D^2(x) = G(x)$ for all $x \in R$, then $D = 0$.

Here, we extend the above mentioned result to the Γ-ring theory in the case of Jordan left derivation.

Theorem 1.3.1. *Let M be a 2-torsion free and 3-torsion free semiprime Γ-ring. If $d : M \to M$ and $G : M \to M$ are Jordan left derivations such that $d^2(M) = G(M)$, then $d = 0$.*

Proof. Let $x \in M$, then $x\alpha x \in M$. Putting $x\alpha x$ for x in $d^2(x) = G(x)$.

$$d(d(x\alpha x)) = G(x\alpha x).$$

$$\Rightarrow d(2x\alpha d(x)) = 2x\alpha G(x).$$

$$\Rightarrow d(x\alpha d(x)) = x\alpha G(x). \quad (1.7)$$

Now, we prove that, for all $x \in M$ and $\alpha \in \Gamma$.

$$d(d(x)\alpha x) = 2d(x)\alpha d(x) + x\alpha G(x). \quad (1.8)$$

Using Lemma 1.2.1(i), we have

$$d(d(x)\alpha x + x\alpha d(x)) = 2d(x)\alpha d(x) + 2x\alpha d^2(x).$$

Using (1.7), we obtain

$$\begin{aligned} d(d(x)\alpha x) &= 2d(x)\alpha d(x) + 2x\alpha d^2(x) - d(x\alpha d(x)) \\ &= 2d(x)\alpha d(x) + 2x\alpha G(x) - x\alpha G(x) \\ &= 2d(x)\alpha d(x) + x\alpha G(x) \\ &= 2d(x)\alpha d(x) + d(x\alpha d(x)). \end{aligned}$$

Therefore,

$$
\begin{aligned}
d(d(x)\alpha x - x\alpha d(x)) &= 2d(x)\alpha d(x). \\
\Rightarrow d[d(x), x]_\alpha &= 2d(x)\alpha d(x).
\end{aligned}
\tag{1.9}
$$

Linearize (1.9) and using (1.9), we get

$$
\begin{aligned}
&d([d(x+y), x+y]_\alpha) = 2d(x+y)\alpha d(x+y). \\
&\Rightarrow d([d(x)+d(y), x+y]_\alpha) = 2(d(x)+d(y))\alpha(d(x)+d(y)). \\
&\Rightarrow d([d(x), x]_\alpha + [d(y), x]_\alpha + [d(x), y]_\alpha + [d(y), y]_\alpha) \\
&= 2(d(x)\alpha d(x) + d(x)\alpha d(y) + d(y)\alpha d(x) + d(y)\alpha d(y)). \\
&\Rightarrow d([d(y), x]_\alpha + [d(x), y]_\alpha) = 2d(x)\alpha d(y) + 2d(y)\alpha d(x).
\end{aligned}
$$

Putting $y = x\alpha x$ in the above relation then using Lemma 1.2.1 and (1.9), we obtain

$$
\begin{aligned}
0 &= d([d(x), x\alpha x]_\alpha + [d(x\alpha x), x]_\alpha) - 2d(x)\alpha d(x\alpha x) - 2d(x\alpha x)\alpha d(x) \\
&= d([d(x), x]_\alpha \alpha x + x\alpha[d(x), x]_\alpha) + 2d(x\alpha[d(x), x]_\alpha) - 4d(x)\alpha x\alpha d(x) - 4x\alpha d(x)\alpha d(x) \\
&= 2[d(x), x]_\alpha \alpha d(x) + 2x\alpha d([d(x), x)]_\alpha) + 2d(x\alpha[d(x), x]_\alpha) - 4d(x)\alpha x\alpha d(x) - 4x\alpha d(x)\alpha d(x) \\
&= 2d(x)\alpha x\alpha d(x) - 2x\alpha d(x)\alpha d(x) + 4x\alpha d(x)\alpha d(x) + 2d(x\alpha[d(x), x]_\alpha) - 4d(x)\alpha x\alpha d(x) \\
&- 4x\alpha d(x)\alpha d(x) \\
&= -2d(x)\alpha x\alpha d(x) - 2x\alpha d(x)\alpha d(x) + 2d(x\alpha[d(x), x]_\alpha).
\end{aligned}
$$

Thus, we have

$$
d(x\alpha[d(x), x]_\alpha) = d(x)\alpha x\alpha d(x) + x\alpha d(x)\alpha d(x), \forall\, x \in M, \alpha \in \Gamma. \tag{1.10}
$$

We prove the identity

$$
d([d(x), x]_\alpha \alpha x) = d(x)\alpha x\alpha d(x) + x\alpha d(x)\alpha d(x), \forall\, x \in M, \alpha \in \Gamma. \tag{1.11}
$$

Using Lemma 1.2.1(i) and (1.9), we have

$$\begin{aligned} d([d(x),x]_\alpha\alpha x + x\alpha[d(x),x]_\alpha) &= 2[d(x),x]_\alpha\alpha d(x) + 2x\alpha d([d(x),x]_\alpha) \\ &= 2[d(x),x]_\alpha\alpha d(x) + 4x\alpha d(x)\alpha d(x). \end{aligned}$$

Now, applying (1.10), we obtain

$$\begin{aligned} d([d(x),x]_\alpha\alpha x) &= 2[d(x),x]_\alpha\alpha d(x) + 4x\alpha d(x)\alpha d(x) - d(x\alpha[d(x),x]_\alpha) \\ &= 2d(x)\alpha x\alpha d(x) - 2x\alpha d(x)\alpha d(x) + 4x\alpha d(x)\alpha d(x) \\ &- d(x)\alpha x\alpha d(x) - x\alpha d(x)\alpha d(x) \\ &= d(x)\alpha x\alpha d(x) + x\alpha d(x)\alpha d(x), \forall\, x \in M; \alpha \in \Gamma. \end{aligned}$$

Which completes the proof of (1.11). Using (1.10) and (1.11), we obtain

$$d([[d(x),x]_\alpha, x]_\alpha) = d([d(x),x]_\alpha\alpha x - x\alpha[d(x),x]_\alpha) = 0. \tag{1.12}$$

By Lemma 1.2.3, it follows

$$[d(x),x]_\alpha\alpha d(x) = 0, \forall\, x \in M; \alpha \in \Gamma. \tag{1.13}$$

Using (1.13) and Lemma 1.2.1(i) , we obtain

$$\begin{aligned} d(d(x)\alpha[d(x),x]_\alpha) &= d(d(x)\alpha[d(x),x]_\alpha + [d(x),x]_\alpha\alpha d(x)) \\ &= 2d(x)\alpha d([d(x),x]_\alpha) + 2[d(x),x]_\alpha\alpha d(d((x)). \end{aligned}$$

Using (1.9) and $d^2(x) = G(x)$, we have

$$d(d(x)\alpha[d(x),x]_\alpha) = 4d(x)\alpha d(x)\alpha d(x) + 2[d(x),x]_\alpha\alpha G(x), \forall\, x \in M; \alpha \in \Gamma. \tag{1.14}$$

Now, we prove the relation

$$d(d(x)\alpha[d(x),x]_\alpha) = -6[d(x),x]_\alpha\alpha G(x), \forall\, x \in M; \alpha \in \Gamma. \tag{1.15}$$

Using (1.13) and Lemma 1.2.1(ii), we obtain

$$\begin{aligned}0 &= d[(d(x), x]_\alpha \alpha d(x))\\ &= d(d(x)\alpha x\alpha d(x)) - d(x\alpha d(x)\alpha d(x))\\ &= d(x)\alpha d(x)\alpha d(x) + 3d(x)\alpha x\alpha d(d(x)) - x\alpha d(x)\alpha d(d(x)) - d(x\alpha d(x)\alpha d(x)).\end{aligned}$$

Thus, we have

$$d(x\alpha d(x)\alpha d(x)) = d(x)\alpha d(x)\alpha d(x) + 3d(x)\alpha x\alpha G(x) - x\alpha d(x)\alpha G(x), \forall\ x \in M, \alpha \in \Gamma.$$

Using Lemma 1.2.1(i) and $d^2(x) = G(x)$, we have

$$\begin{aligned}d((d(x)\alpha d(x))\alpha x + x\alpha(d(x)\alpha d(x))) &= 2d(x)\alpha d(x)\alpha d(x) + 2x\alpha d(d(x)\alpha d(x))\\ = 2d(x)\alpha d(x)\alpha d(x) + 2x\alpha 2d(x)\alpha d(d(x)) &= 2d(x)\alpha d(x)\alpha d(x) + 4x\alpha d(x)\alpha G(x).\end{aligned} \tag{1.16}$$

From the above relation and (1.16), it follows

$$\begin{aligned}d(d(x)\alpha d(x)\alpha x) &= 2d(x)\alpha d(x)\alpha d(x) + 4x\alpha d(x)\alpha G(x) - d(x\alpha d(x)\alpha d(x))\\ &= 2d(x)\alpha d(x)\alpha d(x) + 4x\alpha d(x)\alpha G(x) - d(x)\alpha d(x)\alpha d(x) + 3d(x)\alpha x\alpha G(x) - x\alpha d(x)\alpha G(x)\\ &= d(x)\alpha d(x)\alpha d(x) + 5x\alpha d(x)\alpha G(x) - 3d(x)\alpha x\alpha G(x).\end{aligned} \tag{1.17}$$

By the operation (1.17)-(1.16), we obtain

$$d(d(x)\alpha d(x)\alpha x) - d(x\alpha d(x)\alpha d(x)) = 6x\alpha d(x)\alpha G(x) - 6d(x)\alpha x\alpha G(x).$$

$$\Rightarrow d([d(x)\alpha d(x), x]_\alpha) = 6[x, d(x)]_\alpha \alpha G(x).$$

Thus, we have according to (1.13)

$$\begin{aligned}6[x, d(x)]_\alpha \alpha G(x) &= d([d(x)\alpha d(x), x]_\alpha)\\ &= d([d(x), x]_\alpha \alpha d(x) + d(x)\alpha[d(x), x]_\alpha)\\ &= d(d(x)\alpha[d(x), x]_\alpha).\end{aligned}$$

This implies,

$$d(d(x)\alpha[d(x),x]_\alpha) = -6[d(x),x]_\alpha\alpha G(x), \forall\, x \in M, \alpha \in \Gamma.$$

Which completes the proof of (1.15). By (1.15), (1.14) becomes

$$d(x)\alpha d(x)\alpha d(x) + 2[d(x),x]_\alpha\alpha G(x) = -6[d(x),x]_\alpha\alpha G(x).$$

$$\Rightarrow d(x)\alpha d(x)\alpha d(x) + 2[d(x),x]_\alpha\alpha G(x) = 0. \quad (1.18)$$

Now, starting from (1.13) and using Lemma 1.2.1(ii) and (1.9), we obtain

$$\begin{aligned} 0 &= d(d(x)\alpha[d(x),x]_\alpha\alpha d(x)) \\ &= d(x)\alpha d(x)\alpha d([d(x),x]_\alpha) + 3d(x)\alpha[d(x),x]_\alpha\alpha d^2(x) - [d(x),x]_\alpha\alpha d(x)\alpha d^2(x) \\ &= 2d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 3d(x)\alpha[d(x),x]_\alpha\alpha G(x) - [d(x),x]_\alpha\alpha d(x)\alpha G(x). \end{aligned}$$

Using (1.13), we have

$$2d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 3d(x)\alpha[d(x),x]_\alpha\alpha G(x) = 0.$$

$$\Rightarrow 4d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 6d(x)\alpha[d(x),x]_\alpha\alpha G(x) = 0.$$

$$\Rightarrow d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 3d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 6d(x)\alpha[d(x),x]_\alpha\alpha G(x) = 0.$$

$$\Rightarrow d(x)\alpha d(x)\alpha d(x)\alpha d(x) + 3d(x)\alpha(d(x)\alpha d(x)\alpha d(x) + 2[d(x),x]_\alpha\alpha G(x)) = 0. \quad (1.19)$$

Therefore, using (1.18), we get $(d(x)\alpha)^3 d(x) = 0$. This implies, $d(x)$ is a nilpotent element of M. Since semiprime Γ-ring does not contain any non-zero nilpotent element, consequently $d(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$. □

1.4 Commutativity in Prime Γ-Rings with Jordan Left Derivations

In this section, we obtain the commutativity result of a 2-torsion free prime Γ-rings with Jordan left derivations under the condition (*), $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, and consequently, we prove that every Jordan left derivation is a left derivation.

Theorem 1.4.1. *Let M be a 2-torsion free prime Γ-ring satisfying the assumption (*). If there exists a nonzero Jordan left derivation $d : M \to M$, then M is commutative.*

Proof. Let us assume that M is non commutative. Lemma 1.2.2(i) can be written as

$$(x\alpha(x\alpha y - y\alpha x) - (x\alpha y - y\alpha x)\alpha x)\beta d(x) = 0, \forall\, x, y \in M, \alpha, \beta \in \Gamma.$$

This gives,

$$(x\alpha x\alpha y - 2x\alpha y\alpha x + y\alpha x\alpha x)\beta d(x) = 0.$$

Replacing $[a, b]_\gamma$ for x, we have

$$[a, b]_\gamma\alpha[a, b]_\gamma\alpha y\beta d([a, b]_\gamma) - 2[a, b]_\gamma\alpha y\alpha[a, b]_\gamma\beta d([a, b]_\gamma) + y\alpha[a, b]_\gamma\alpha[a, b]_\gamma\beta d([a, b]_\gamma) = 0.$$

By Lemma 1.2.2(iii), we get

$$[a, b]_\gamma\alpha[a, b]_\gamma\alpha y\beta d([a, b]_\gamma) = 0.$$

By the primeness of M, we have $[a, b]_\gamma\alpha[a, b]_\gamma = 0$ or $d([a, b]_\gamma) = 0$. Suppose that,

$$[a, b]_\gamma\alpha[a, b]_\gamma = 0, \forall\, \alpha \in \Gamma. \tag{1.20}$$

By Lemmas 1.2.1(i) , 1.2.1(ii), 1.2.2(iii) with the use of (1.20), we get

$$
\begin{aligned}
W &= d([a,b]_\gamma\beta x\beta[a,b]_\gamma\alpha y\alpha[a,b]_\gamma + [a,b]_\gamma\alpha y\alpha[a,b]_\gamma\beta[a,b]_\gamma\beta x)\\
&= 2[a,b]_\gamma\beta x\beta d([a,b]_\gamma\alpha y\alpha[a,b]_\gamma) + 2[a,b]_\gamma\alpha y\alpha[a,b]_\gamma\beta d([a,b]_\gamma\beta x)\\
&= 2[a,b]_\gamma\beta x\beta[a,b]_\gamma\alpha[a,b]_\gamma\alpha d(y) + 6[a,b]_\gamma\beta x\beta[a,b]_\gamma\alpha y\alpha d([a,b]_\gamma)\\
&- 2[a,b]_\gamma\beta x\beta y\alpha[a,b]_\gamma\alpha d([a,b]_\gamma) + 2[a,b]_\gamma\alpha y\alpha[a,b]_\gamma\beta d([a,b]_\gamma\beta x)\\
&= 6[a,b]_\gamma\beta x\beta[a,b]_\gamma\alpha y\alpha d([a,b]_\gamma) - 2[a,b]_\gamma\beta x\beta y\alpha[a,b]_\gamma\alpha d([a,b]_\gamma)\\
&+ 2[a,b]_\gamma\alpha y\alpha[a,b]_\gamma\beta d([a,b]_\gamma\beta x)\\
&= 6[a,b]_\gamma\beta x\beta[a,b]_\gamma\alpha y\alpha d([a,b]_\gamma) + 2[a,b]_\gamma\alpha y\alpha[a,b]_\gamma\beta d([a,b]_\gamma\beta x).
\end{aligned}
$$

On the other hand, by Lemma 1.2.1(ii) with the use of (1.20)

$$
\begin{aligned}
W &= d([a,b]_\gamma\beta(x\beta[a,b]_\gamma\alpha y)\alpha[a,b]_\gamma) + d([a,b]_\gamma\alpha y\alpha[a,b]_\gamma\beta[a,b]_\gamma\beta x))\\
&= d([a,b]_\gamma\beta(x\beta[a,b]_\gamma\alpha y)\alpha[a,b]_\gamma)\\
&= [a,b]_\gamma\alpha[a,b]_\gamma\beta d(x\beta[a,b]_\gamma\alpha y) + 3[a,b]_\gamma\beta x\beta[a,b]_\gamma\alpha y\alpha d([a,b]_\gamma)\\
&- x\beta[a,b]_\gamma\alpha y\beta[a,b]_\gamma\alpha d([a,b]_\gamma)\\
&= 3[a,b]_\gamma\beta x\beta[a,b]_\gamma\alpha y\alpha d([a,b]_\gamma).
\end{aligned}
$$

Comparing these two expressions for W with the use of the condition (*), we obtain

$$3[a,b]_\gamma\beta x\alpha[a,b]_\gamma\alpha y\beta d([a,b]_\gamma) + [a,b]_\gamma\alpha y\beta 2[a,b]_\gamma\alpha d([a,b]_\gamma\beta x) = 0. \tag{1.21}$$

Also, using Lemma 1.2.1(i) and Lemma 1.2.2(iii), we obtain

$$
\begin{aligned}
V &= d([a,b]_\gamma\alpha x\beta[a,b]_\gamma + x\alpha[a,b]_\gamma\beta[a,b]_\gamma)\\
&= 2[a,b]_\gamma\alpha d(x\beta[a,b]_\gamma) + 2x\beta[a,b]_\gamma\alpha d([a,b]_\gamma)\\
&= 2[a,b]_\gamma\alpha d(x\beta[a,b]_\gamma).
\end{aligned}
$$

On the other hand, by Lemma 1.2.1(ii) with the use of (1.20)

$$\begin{aligned}
V &= d([a,b]_\gamma \alpha x \beta [a,b]_\gamma + x\alpha [a,b]_\gamma \beta [a,b]_\gamma) \\
&= d([a,b]_\gamma \alpha x \beta [a,b]_\gamma) \\
&= [a,b]_\gamma \beta [a,b]_\gamma \alpha d(x) + 3[a,b]_\gamma \alpha x \beta d([a,b]_\gamma) - x\alpha [a,b]_\gamma \beta d([a,b]_\gamma) \\
&= 3[a,b]_\gamma \alpha x \beta d([a,b]_\gamma).
\end{aligned}$$

Comparing these two expressions for V, we obtain

$$3[a,b]_\gamma \alpha x \beta d([a,b]_\gamma) = 2[a,b]_\gamma \alpha d(x\beta [a,b]_\gamma). \tag{1.22}$$

Using (1.20), we have

$$\begin{aligned}
[a,b]_\gamma \alpha d(x\beta [a,b]_\gamma + [a,b]_\gamma \beta x) &= [a,b]_\gamma \alpha (2x\beta d([a,b]_\gamma) + 2[a,b]_\gamma \beta d(x)) \\
&= 2[a,b]_\gamma \alpha x \beta d([a,b]_\gamma) + 2[a,b]_\gamma \alpha [a,b]_\gamma \beta d(x) \\
&= 2[a,b]_\gamma \alpha x \beta d([a,b]_\gamma).
\end{aligned}$$

Now, using (1.22), we get

$$\begin{aligned}
&3[a,b]_\gamma \alpha (d(x\beta [a,b]_\gamma) + d([a,b]_\gamma \beta x)) = 6[a,b]_\gamma \alpha x \beta d([a,b]_\gamma) \\
&\quad = 2.3[a,b]_\gamma \alpha x \beta d([a,b]_\gamma) = 2.2[a,b]_\gamma \alpha d(x\beta [a,b]_\gamma) = 4[a,b]_\gamma \alpha d(x\beta [a,b]_\gamma). \\
&\qquad \Rightarrow 3[a,b]_\gamma \alpha (d(x\beta [a,b]_\gamma) + 3[a,b]_\gamma \alpha d([a,b]_\gamma \beta x) = 4[a,b]_\gamma \alpha d(x\beta [a,b]_\gamma).
\end{aligned}$$

Therefore,

$$[a,b]_\gamma \alpha d(x\beta [a,b]_\gamma) = 3[a,b]_\gamma \alpha d([a,b]_\gamma \beta x). \tag{1.23}$$

Using (1.23), we have

$$\begin{aligned}
[a,b]_\gamma \alpha d(x\beta [a,b]_\gamma + [a,b]_\gamma \beta x) &= 3[a,b]_\gamma \alpha d([a,b]_\gamma \beta x) + [a,b]_\gamma \alpha (d([a,b]_\gamma \beta x) \\
&= 4[a,b]_\gamma \alpha d([a,b]_\gamma \beta x).
\end{aligned} \tag{1.24}$$

We, also obtain that

$$\begin{aligned}[a,b]_\gamma\alpha d(x\beta[a,b]_\gamma+[a,b]_\gamma\beta x) &= [a,b]_\gamma\alpha 2x\beta d([a,b]_\gamma)+[a,b]_\gamma\alpha 2[a,b]_\gamma\beta d(x)\\ &= 2[a,b]_\gamma\alpha x\beta d([a,b]_\gamma).\end{aligned} \tag{1.25}$$

From (1.24) and (1.25), we obtain

$$4[a,b]_\gamma\alpha d([a,b]_\gamma\beta x)=2[a,b]_\gamma\alpha x\beta d([a,b]_\gamma).$$

Since M is 2- torsion free, we have

$$2[a,b]_\gamma\alpha d([a,b]_\gamma\beta x)=[a,b]_\gamma\alpha x\beta d([a,b]_\gamma). \tag{1.26}$$

From (1.21)and (1.26), we get

$$3[a,b]_\gamma\beta x\alpha[a,b]_\gamma\alpha y\beta d([a,b]_\gamma)+[a,b]_\gamma\alpha y\beta[a,b]_\gamma\alpha x\beta d([a,b]_\gamma)=0. \tag{1.27}$$

Replacing $y\alpha[a,b]_\gamma\beta y$ for x in (1.26)

$$\begin{aligned}&2[a,b]_\gamma\alpha d([a,b]_\gamma\beta y\alpha[a,b]_\gamma\beta y)=[a,b]_\gamma\alpha y\beta[a,b]_\gamma\alpha y\beta d([a,b]_\gamma).\\ &\Rightarrow 2[a,b]_\gamma\alpha 2[a,b]_\gamma\beta y\alpha d([a,b]_\gamma\beta y)=[a,b]_\gamma\alpha y\beta[a,b]_\gamma\alpha y\beta d([a,b]_\gamma).\\ &\Rightarrow 4[a,b]_\gamma\alpha[a,b]_\gamma\beta y\alpha d([a,b]_\gamma\beta y)=[a,b]_\gamma\alpha y\beta[a,b]_\gamma\alpha y\beta d([a,b]_\gamma).\end{aligned}$$

Using (1.20), we get

$$[a,b]_\gamma\alpha y\beta[a,b]_\gamma\alpha y\beta d([a,b]_\gamma)=0. \tag{1.28}$$

Now, replacing y by $x+y$ in (1.28)

$$[a,b]_\gamma\alpha(x+y)\beta[a,b]_\gamma\alpha(x+y)\beta d([a,b]_\gamma)=0.$$

Using (1.28), we obtain

$$[a,b]_\gamma\alpha x\beta[a,b]_\gamma\alpha y\beta d([a,b]_\gamma)+[a,b]_\gamma\alpha y\beta[a,b]_\gamma\alpha x\beta d([a,b]_\gamma)=0. \tag{1.29}$$

Subtracting (1.29) from (1.27) and since M is 2-torsion free, we obtain

$$[a,b]_\gamma \alpha x \beta [a,b]_\gamma \alpha y \beta d([a,b]_\gamma) = 0, \forall\, y \in M.$$

Since M is prime and non commutative, so we have

$$\begin{aligned} &d([a,b]_\gamma) = 0. \\ &\Rightarrow d(a\gamma b) = d(b\gamma a), \forall\, a,b \in M; \gamma \in \Gamma. \end{aligned} \tag{1.30}$$

Using (1.30), we get

$$\begin{aligned} d((b\alpha a)\beta a + a\beta(b\alpha a)) &= d((b\alpha a)\beta a) + d(a\beta(b\alpha a)) \\ &= d((b\alpha a)\beta a) + d((b\alpha a)\beta a) \\ &= 2d((b\alpha a)\beta a). \end{aligned}$$

Using Lemma 1.2.1(ii), we obtain

$$2d((b\alpha a)\beta a) = d((b\alpha a)\beta a + a\beta(b\alpha a)) = d((b\alpha a)\beta a) + d(a\beta(b\alpha a)).$$

$$\Rightarrow 2d((b\alpha a)\beta a) - d((b\alpha a)\beta a = d(a\beta b\alpha a).$$

$$\Rightarrow d(b\alpha a\beta a) = a\alpha a\beta d(b) + 3a\beta b\alpha d(a) - b\beta a\alpha d(a). \tag{1.31}$$

On the other hand, using Lemma 1.2.1(i), we get

$$d(a\alpha(b\beta a) + (b\beta a)\alpha a) = 2a\alpha d(b\beta a) + 2(b\beta a)\alpha d(a). \tag{1.32}$$

Also, we have

$$d(a\alpha(a\beta b) + (a\beta b)\alpha a) = 2a\alpha d(a\beta b) + 2(a\beta b)\alpha d(a). \tag{1.33}$$

By the operation (1.33) -(1.32) and using the condition (*), we obtain

$$d(a\alpha a\beta b - b\beta a\alpha a) = 2a\alpha d([a,b]_\beta) + 2[a,b]_\beta \alpha d(a), \forall\, a,b \in M; \alpha,\beta \in \Gamma. \tag{1.34}$$

Now, putting $a\beta a$ for a in Lemma 1.2.1(i), we have

$$d(a\beta a\alpha b + b\alpha a\beta a) = 2a\beta a\alpha d(b) + 2b\alpha d(a\beta a) = 2(a\beta a\alpha d(b) + b\alpha 2a\beta d(a)).$$

Using the condition (*), above equation reduces to

$$d(a\alpha a\beta b + b\beta a\alpha a) = 2(a\beta a\alpha d(b) + 2b\alpha a\beta d(a)). \tag{1.35}$$

Subtracting (1.34) from (1.35) and using the condition (*), we get

$$d(2b\alpha a\beta a) = 2a\beta a\alpha d(b) + 4b\alpha a\beta d(a) - 2a\alpha d([a,b]_\beta) - 2[a,b]_\beta\alpha d(a)).$$

Therefore,

$$\begin{aligned} d(b\alpha a\beta a) &= a\beta a\alpha d(b) + 2b\alpha a\beta d(a) - a\alpha d([a,b]_\beta) - a\beta b\alpha d(a) + b\beta a\alpha d(a) \\ &= a\beta a\alpha d(b) + 3b\alpha a\beta d(a) - a\alpha d([a,b]_\beta) - a\beta b\alpha d(a). \\ \Rightarrow d(b\alpha a\beta a) &= a\alpha a\beta d(b) + 3b\beta a\alpha d(a) - a\beta b\alpha d(a). \end{aligned} \tag{1.36}$$

From (1.36) and (1.31), we obtain

$$a\alpha a\beta d(b) + 3a\beta b\alpha d(a) - b\beta a\alpha d(a) = a\alpha a\beta d(b) + 3b\beta a\alpha d(a) - a\beta b\alpha d(a).$$

$$\Rightarrow -3(b\beta a - a\beta b)\alpha d(a) - (b\beta a - a\beta b)\alpha d(a) = 0.$$

$$\Rightarrow -3[b,a]_\beta\alpha d(a) - [b,a]_\beta\alpha d(a) = 0.$$

$$\Rightarrow 4[b,a]_\beta\alpha d(a) = 0.$$

Since M is 2-torsion free, hence

$$[b,a]_\beta\alpha d(a) = 0. \tag{1.37}$$

Now, putting $b\gamma x$ for b in (1.37), we get

$$[b\gamma x, a]_\beta\alpha d(a) = 0.$$

$$\Rightarrow ([b,a]_\beta \gamma x + a[\gamma,\beta]_a x + b\gamma[x,a]\beta)\alpha d(a) = 0.$$

Since $a[\gamma,\beta]_a x = a(\gamma a\beta - \beta a\gamma)x = a\gamma a\beta x - a\beta a\gamma x = 0$, using the condition (*). Therefore, we get

$$([b,a]_\beta \gamma x + b\gamma[x,a]\beta)\alpha d(a) = 0.$$

$$\Rightarrow [b,a]_\beta \gamma x\alpha d(a) + b\gamma[x,a]\beta\alpha d(a) = 0.$$

$$\Rightarrow [b,a]_\beta \gamma x\alpha d(a) = 0, \forall\, a,b \in M; \alpha,\beta,\gamma \in \Gamma.$$

Since M is prime, thus $d(a) = 0$. Hence, we conclude that if $d \neq 0$, then M is commutative. □

Theorem 1.4.2. *If M is a 2-torsion free prime Γ-ring satisfying the assumption (*) then every Jordan left derivation on M is a left derivation.*

Proof. Since M is commutative. Thus $a\alpha b = b\alpha a$ for all $a,b \in M$ and $\alpha \in \Gamma$. By Lemma 1.2.1(i), we have

$$2d(a\alpha b) = 2a\alpha d(b) + 2b\alpha d(a).$$

Since M is 2-torsion free, we get

$$d(a\alpha b) = a\alpha d(b) + b\alpha d(a), \forall\, a,b \in M; \alpha \in \Gamma.$$

□

Chapter 2

Derivations on Semiprime Γ-Rings

This chapter deals with derivation and Jordan derivation of Γ-rings to characterize them in case of semiprime and completely semiprime Γ-rings. We recall the definitions of derivation and Jordan derivation of Γ-rings in the first section.

The second section develops some useful consequences regarding the derivation and Jordan derivation of semiprime Γ-rings which are very much needed for proving our main result in this section and to develop some needful results for the next section. Then we prove that under a suitable condition every Jordan derivation of a 2-torsion free semiprime Γ-ring is a derivation.

In the next, we develop some immediate consequences relating to the concepts of derivation and Jordan derivation of completely semiprime Γ-rings. The goal of the third section is to prove an analogous result to the previous one by showing that under a suitable condition every Jordan derivation of a 2-torsion free completely semiprime Γ-ring is a derivation.

2.1 Introduction

I. N. Herstein [19] proved a well-known result in prime rings that every Jordan derivation is a derivation. Afterwards, many Mathematicians studied extensively the derivations in prime rings. M. Bresar [6] has extended this result for semiprime rings. The concepts of derivation and Jordan derivation in Γ-rings have been introduced by M. Sapanci and A. Nakajima in [36] as below and proved the above mentioned result in completely prime Γ-rings.

Definition 2.1.1. If M is a Γ-ring, then an additive mapping $d : M \to M$ is called a *derivation* of M if $d(a\alpha b) = a\alpha d(b) + d(a)\alpha b$ is satisfied for all $a, b \in M$ and $\alpha \in \Gamma$.

Definition 2.1.2. Let M be a Γ-ring and $d : M \to M$ be an additive mapping of M. If $d(a\alpha a) = a\alpha d(a) + d(a)\alpha a$ holds for all $a \in M$ and $\alpha \in \Gamma$, then d is called a *Jordan derivation*.

In [36], M. Sapanci and A. Nakajima gave an example of a derivation and a Jordan derivation on a Γ-ring in the following way.

Example 2.1.1. *Let R be an associative ring with 1 and $d : R \to R$ be a derivation. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$, then M is a Γ-ring. Define a mapping $D : M \to M$ by $D((a, b)) = (d(a), d(b))$. Then D is clearly an additive mapping and hence it is a derivation on M. Let $N = \{(a, a) : a \in R\}$. Then N is a Γ-subring of M. Define a mapping $D : N \to N$ by $D((a, a)) = (d(a), d(a))$, then D is a Jordan derivation on N.*

2.2 Jordan Derivations on Semiprime Γ-Rings

For the sake of completeness of the study of this chapter, we prepare some useful results on Jordan derivation of Γ-rings in the following way.

Lemma 2.2.1. *Let M be a Γ-ring and d be a Jordan derivation of M. Then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:*

(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + a\alpha d(b) + b\alpha d(a)$.

(ii) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha b\beta d(a) + a\beta b\alpha d(a)$.

In particular, if M is 2-torsion free and satisfies the condition (*), then

(iii) $d(a\alpha b\beta a) = d(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a)$.

(iv) $d(a\alpha b\beta c + c\alpha b\beta a) = d(a)\alpha b\beta c + d(c)\alpha b\beta a + a\alpha d(b)\beta c + c\alpha d(b)\beta a + a\alpha b\beta d(c) + c\alpha b\beta d(a)$.

Proof. Compute $d((a+b)\alpha(a+b))$ and cancel the like terms from both sides to obtain(i). Then replace $a\beta b + b\beta a$ for b in (i) to get (ii). Using the condition (*) and since M is 2-torsion free, (iii) follows from (ii). And, finally (iv) is obtained by replacing $a + c$ for a in (iii). □

Definition 2.2.1. Let d be a Jordan derivation of a Γ-ring M. For all $a, b \in M$ and $\alpha \in \Gamma$, we define $G_\alpha(a, b) = d(a\alpha b) - d(a)\alpha b - a\alpha d(b)$. Thus, we have $G_\alpha(b, a) = d(b\alpha a) - d(b)\alpha a - b\alpha d(a)$.

Remark 2.2.1. d is a derivation of a Γ-ring M if and only if $G_\alpha(a, b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$.

Lemma 2.2.2. *Let d be a Jordan derivation of a Γ-ring M. For any $a, b, c \in M$ and*

$\alpha, \beta \in \Gamma$, *(i)* $G_\alpha(a,b) + G_\alpha(b,a) = 0$; *(ii)* $G_\alpha(a+b,c) = G_\alpha(a,c) + G_\alpha(b,c)$; *(iii)* $G_\alpha(a,b+c) = G_\alpha(a,b) + G_\alpha(a,c)$; *(iv)* $G_{\alpha+\beta}(a,b) = G_\alpha(a,b) + G_\beta(a,b)$.

Proof. (i) By the definition of $G_\alpha(a,b)$ and using Lemma 2.2.1(i), we get

$$\begin{aligned}
G_\alpha(a,b) + G_\alpha(b,a) &= d(a\alpha b) - d(a)\alpha b - a\alpha d(b) + d(b\alpha a) - d(b)\alpha a - b\alpha d(a) \\
&= d(a\alpha b + b\alpha a) - d(a)\alpha b - a\alpha d(b) - d(b)\alpha a - b\alpha d(a) \\
&= d(a)\alpha b + d(b)\alpha a + a\alpha d(b) + b\alpha d(a) - d(a)\alpha b - a\alpha d(b) \\
&\quad - d(b)\alpha a - b\alpha d(a) \\
&= 0.
\end{aligned}$$

(ii) By the definition of $G_\alpha(a,b)$, we get

$$\begin{aligned}
G_\alpha(a+b,c) &= d((a+b)\alpha c) - d(a+b)\alpha c - (a+b)\alpha d(c) \\
&= d(a\alpha c + b\alpha c) - d(a)\alpha c - d(b)\alpha c - a\alpha d(c) - b\alpha d(c) \\
&= d(a\alpha c) - d(a)\alpha c - a\alpha d(c) + d(b\alpha c) - d(b)\alpha c - b\alpha d(c) \\
&= G_\alpha(a,c) + G_\alpha(b,c).
\end{aligned}$$

(iii)-(iv): Proofs are obvious. □

Lemma 2.2.3. *Assume that M is a 2-torsion free Γ-ring satisfying the condition (*) and d is a Jordan derivation of M. Then for all $a, b, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$:*

(i) $G_\alpha(a,b)\beta m\gamma[a,b]_\alpha + [a,b]_\alpha\beta m\gamma G_\alpha(a,b) = 0$;

(ii) $G_\alpha(a,b)\alpha m\alpha[a,b]_\alpha + [a,b]_\alpha\alpha m\alpha G_\alpha(a,b) = 0$;

(iii) $G_\alpha(a,b)\beta m\beta[a,b]_\alpha + [a,b]_\alpha\beta m\beta G_\alpha(a,b) = 0$.

Proof. (i) For any $a, b, m \in M$ and $\alpha, \beta, \gamma \in \Gamma$, using Lemma 2.2.1(iv), we have

$$\begin{aligned}
W &= d(a\alpha b\beta m\gamma b\alpha a + b\alpha a\beta m\gamma a\alpha b) \\
&= d((a\alpha b)\beta m\gamma(b\alpha a) + (b\alpha a)\beta m\gamma(a\alpha b)) \\
&= d(a\alpha b)\beta m\gamma b\alpha a + a\alpha b\beta d(m)\gamma b\alpha a + a\alpha b\beta m\gamma d(b\alpha a) \\
&+ d(b\alpha a)\beta m\gamma a\alpha b + b\alpha a\beta d(m)\gamma a\alpha b + b\alpha a\beta m\gamma d(a\alpha b).
\end{aligned}$$

On the other hand, using Lemma 2.2.1(iii)

$$\begin{aligned}
W &= d(a\alpha(b\beta m\gamma b)\alpha a + b\alpha(a\beta m\gamma a)\alpha b) \\
&= d(a\alpha(b\beta m\gamma b)\alpha a) + d(b\alpha(a\beta m\gamma a)\alpha b) \\
&= d(a)\alpha b\beta m\gamma b\alpha a + a\alpha d(b\beta m\gamma b)\alpha a + a\alpha b\beta m\gamma b\alpha d(a) + d(b)\alpha a\beta m\gamma a\alpha b \\
&+ b\alpha d(a\beta m\gamma a)\alpha b + b\alpha a\beta m\gamma a\alpha d(b) \\
&= d(a)\alpha b\beta m\gamma b\alpha a + a\alpha d(b)\beta m\gamma b\alpha a + a\alpha b\beta d(m)\gamma b\alpha a + a\alpha b\beta m\gamma d(b)\alpha a \\
&+ a\alpha b\beta m\gamma b\alpha d(a) + d(b)\alpha a\beta m\gamma a\alpha b + b\alpha d(a)\beta m\gamma a\alpha b + b\alpha a\beta d(m)\gamma a\alpha b \\
&+ b\alpha a\beta m\gamma d(a)\alpha b + b\alpha a\beta m\gamma a\alpha d(b).
\end{aligned}$$

Equating two expressions for W and cancelling the like terms from both sides, we get

$$\begin{aligned}
&d(a\alpha b)\beta m\gamma b\alpha a + a\alpha b\beta m\gamma d(b\alpha a) + d(b\alpha a)\beta m\gamma a\alpha b + b\alpha a\beta m\gamma d(a\alpha b) \\
&\quad = d(a)\alpha b\beta m\gamma b\alpha a + a\alpha d(b)\beta m\gamma b\alpha a + a\alpha b\beta m\gamma d(b)\alpha a + a\alpha b\beta m\gamma b\alpha d(a) \\
&\qquad + d(b)\alpha a\beta m\gamma a\alpha b + b\alpha d(a)\beta m\gamma a\alpha b + b\alpha a\beta m\gamma d(a)\alpha b + b\alpha a\beta m\gamma a\alpha d(b).
\end{aligned}$$

This gives,

$$\begin{aligned}
&d(a\alpha b)\beta m\gamma b\alpha a - d(a)\alpha b\beta m\gamma b\alpha a - a\alpha d(b)\beta m\gamma b\alpha a + d(b\alpha a)\beta m\gamma a\alpha b \\
&\quad - d(b)\alpha a\beta m\gamma a\alpha b - b\alpha d(a)\beta m\gamma a\alpha b + a\alpha b\beta m\gamma d(b\alpha a) - a\alpha b\beta m\gamma d(b)\alpha a \\
&\quad - a\alpha b\beta m\gamma b\alpha d(a) + b\alpha a\beta m\gamma d(a\alpha b) - b\alpha a\beta m\gamma d(a)\alpha b - b\alpha a\beta m\gamma a\alpha d(b) = 0.
\end{aligned}$$

This implies,

$$(d(a\alpha b) - d(a)\alpha b - a\alpha d(b))\beta m\gamma b\alpha a + (d(b\alpha a) - d(b)\alpha a - b\alpha d(a))\beta m\gamma a\alpha b$$
$$+ a\alpha b\beta m\gamma(d(b\alpha a) - d(b)\alpha a - b\alpha d(a)) + b\alpha a\beta m\gamma(d(a\alpha b) - d(a)\alpha b - a\alpha d(b)) = 0.$$

Now, using the Definition 2.2.1 and Lemma 2.2.1(i), we obtain

$$G_\alpha(a,b)\beta m\gamma b\alpha a + G_\alpha(b,a)\beta m\gamma a\alpha b + a\alpha b\beta m\gamma G_\alpha(b,a) + b\alpha a\beta m\gamma G_\alpha(a,b) = 0.$$
$$\Rightarrow G_\alpha(a,b)\beta m\gamma b\alpha a - G_\alpha(a,b)\beta m\gamma a\alpha b - a\alpha b\beta m\gamma G_\alpha(a,b) + b\alpha a\beta m\gamma G_\alpha(a,b) = 0.$$
$$\Rightarrow -G_\alpha(a,b)\beta m\gamma(a\alpha b - b\alpha a) - (a\alpha b - b\alpha a)\beta m\gamma G_\alpha(a,b) = 0.$$

This implies,

$$G_\alpha(a,b)\beta m\gamma[a,b]_\alpha + [a,b]_\alpha\beta m\gamma G_\alpha(a,b) = 0, \forall\ a,b,m \in M, \alpha,\beta,\gamma \in \Gamma.$$

If we consider $W = d(a\alpha b\alpha m\alpha b\alpha a + b\alpha a\alpha m\alpha a\alpha b)$ and $W = d(a\alpha b\beta m\beta b\alpha a + b\alpha a\beta m\beta a\alpha b)$ for (ii) and (iii) respectively and proceeding in the same way as in the proof of (i) by the similar arguments, we get (ii) and (iii). □

In the rest of this section, M represents a semiprime Γ-ring.

Lemma 2.2.4. *Suppose $a,b,m \in M$, if $a\alpha m\beta b + b\alpha m\beta a = 0$ for all $m \in M$ and $\alpha,\beta \in \Gamma$, then $a\alpha m\beta b = 0 = b\alpha m\beta a$.*

Proof. Let $x \in M$ and $\gamma,\delta \in \Gamma$ be any elements. Using the relation $a\alpha m\beta b + b\alpha m\beta a = 0$ for all $m \in M, \alpha,\beta \in \Gamma$ repeatedly, we get

$$(a\alpha m\beta b)\gamma x\delta(a\alpha m\beta b) = -(b\alpha m\beta a)\gamma x\delta(a\alpha m\beta b)$$
$$= -(b\alpha(m\beta a\gamma x)\delta a)\alpha m\beta b = (a\alpha(m\beta a\gamma x)\delta b)\alpha m\beta b$$
$$= a\alpha m\beta(a\gamma x\delta b)\alpha m\beta b = -a\alpha m\beta(b\gamma x\delta a)\alpha m\beta b$$
$$= -(a\alpha m\beta b)\gamma x\delta(a\alpha m\beta b).$$

$$\Rightarrow 2((a\alpha m\beta b)\gamma x\delta(a\alpha m\beta b)) = 0.$$

Since M is 2-torsion free, we have

$$(a\alpha m\beta b)\gamma x\delta(a\alpha m\beta b) = 0, \forall\ a, b, x, m \in M; \alpha, \beta, \gamma, \delta \in \Gamma.$$

Therefore,

$$(a\alpha m\beta b)\Gamma M\Gamma(a\alpha m\beta b) = 0, \forall\ a, b, m \in M; \alpha, \beta \in \Gamma.$$

By the semiprimeness of M, we get $a\alpha m\beta b = 0$. Similarly, it can be shown that $b\alpha m\beta a = 0$. □

Corollary 2.2.5. *Let d be a Jordan derivation of M, and let $a, b, m \in M; \alpha, \beta, \gamma \in \Gamma$ be any elements, then (i) $G_\alpha(a, b)\beta m\gamma[a, b]_\alpha = 0; (ii)$ $[a, b]_\alpha\beta m\gamma G_\alpha(a, b) = 0;$ (iii) $G_\alpha(a, b)\alpha m\alpha[a, b]_\alpha = 0; (iv)$ $[a, b]_\alpha\alpha m\alpha G_\alpha(a, b) = 0;$ (v) $G_\alpha(a, b)\beta m\beta[a, b]_\alpha = 0; (vi)$ $[a, b]_\alpha\beta m\beta G_\alpha(a, b) = 0.$*

Proof. Applying the result of Lemma 2.2.4 in that of Lemma 2.2.3, we obtain these results. □

Lemma 2.2.6. *Suppose d is a Jordan derivation of M, then for any $a, b, x, y, m \in M; \alpha, \beta, \gamma \in \Gamma, (i)$ $G_\alpha(a, b)\beta m\beta[x, y]_\alpha = 0; (ii)$ $[x, y]_\alpha\beta m\beta G_\alpha(a, b) = 0;$ (iii) $G_\alpha(a, b)\beta m\beta[x, y]_\gamma = 0; (iv)$ $[x, y]_\gamma\beta m\beta G_\alpha(a, b) = 0.$*

Proof. (i) If we substitute $a + x$ for a in the Corollary 2.2.5(v), we get

$$G_\alpha(a + x, b)\beta m\beta[a + x, b]_\alpha = 0$$

This implies,

$$G_\alpha(a, b)\beta m\beta[a, b]_\alpha + G_\alpha(a, b)\beta m\beta[x, b]_\alpha + G_\alpha(x, b)\beta m\beta[a, b]_\alpha + G_\alpha(x, b)\beta m\beta[x, b]_\alpha = 0.$$

Using Corollary 2.2.5(v), we have

$$G_\alpha(a,b)\beta m\beta[x,b]_\alpha + G_\alpha(x,b)\beta m\beta[a,b]_\alpha = 0.$$

$$\Rightarrow G_\alpha(a,b)\beta m\beta[x,b]_\alpha = -G_\alpha(x,b)\beta m\beta[a,b]_\alpha.$$

Thus, we obtain

$$\begin{aligned}(G_\alpha(a,b)\beta m\beta[x,b]_\alpha)\beta m\beta(G_\alpha(a,b)\beta m\beta[x,b]_\alpha) &= -G_\alpha(a,b)\beta m\beta[x,b]_\alpha\beta m\beta G_\alpha(x,b)\beta m\beta[a,b]_\alpha\\ &= 0.\end{aligned}$$

By the semiprimeness of M, we get

$$G_\alpha(a,b)\beta m\beta[x,b]_\alpha = 0.$$

Similarly, by replacing $b+y$ for b in this result, we get

$$G_\alpha(a,b)\beta m\beta[x,y]_\alpha = 0.$$

(ii) Proceeding in the same way as described above by the similar replacements successively in Corollary 2.2.5(vi), we obtain

$$[x,y]_\gamma\beta m\beta G_\alpha(a,b) = 0, \forall\, a,b,x,y,m \in M, \alpha,\beta \in \Gamma.$$

(iii) Replacing $\alpha+\gamma$ for α in (i), we get

$$G_{\alpha+\gamma}(a,b)\beta m\beta[x,y]_{\alpha+\gamma} = 0.$$

Using Lemma 2.2.2(iv), we have

$$(G_\alpha(a,b) + G_\gamma(a,b))\beta m\beta([x,y]_\alpha + [x,y]_\gamma) = 0.$$

Therefore,

$$G_\alpha(a,b)\beta m\beta[x,y]_\alpha + G_\alpha(a,b)\beta m\beta[x,y]_\gamma + G_\gamma(a,b)\beta m\beta[x,y]_\alpha + G_\gamma(a,b)\beta m\beta[x,y]_\gamma = 0.$$

Using Lemma 2.2.6(i), we get

$$G_\alpha(a,b)\beta m\beta[x,y]_\gamma + G_\gamma(a,b)\beta m\beta[x,y]_\alpha = 0.$$

$$\Rightarrow G_\alpha(a,b)\beta m\beta[x,y]_\gamma = -G_\gamma(a,b)\beta m\beta[x,y]_\alpha.$$

Thus, we obtain

$$\begin{aligned}(G_\alpha(a,b)\beta m\beta[x,y]_\gamma)\beta m\beta(G_\alpha(a,b)\beta m\beta[x,y]_\gamma) &= -G_\alpha(a,b)\beta m\beta[x,y]_\gamma\beta m\beta G_\gamma(a,b)\beta m\beta[x,y]_\alpha \\ &= 0.\end{aligned}$$

Hence, by the semiprimeness of M, we obtain

$$G_\alpha(a,b)\beta m\beta[x,y]_\gamma = 0.$$

(iv) As in the proof of (iii), the similar replacement in (ii) produces (iv). □

Lemma 2.2.7. *([13], Lemma 3.6.1) Every semiprime Γ-ring contains no nonzero nilpotent ideal.*

Corollary 2.2.8. *([13], Corollary 3.6.2) Semiprime Γ-rings have no nonzero nilpotent element.*

Lemma 2.2.9. *([13], Lemma 3.6.2) The centre of a semiprime Γ-ring does not contain any nonzero nilpotent element.*

Theorem 2.2.10. *Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*), and let d be a Jordan derivation of M. Then d is a derivation of M.*

Proof. Let d be a Jordan derivation of a 2-torsion free semiprime Γ-ring M and let $a, b, y, m \in M$ and $\alpha, \beta \in \Gamma$. Then by Lemma 2.2.6(iii), we get

$$\begin{aligned}[G_\alpha(a,b),y]_\beta\beta m\beta[G_\alpha(a,b),y]_\beta &= (G_\alpha(a,b)\beta y - y\beta G_\alpha(a,b))\beta m\beta[G_\alpha(a,b),y]_\beta \\ &= G_\alpha(a,b)\beta y\beta m\beta[G_\alpha(a,b),y]_\beta - y\beta G_\alpha(a,b)\beta m\beta[G_\alpha(a,b),y]_\beta \\ &= 0.\end{aligned}$$

Since $y\beta m \in M$ and $G_\alpha(a,b) \in M$, for all $a,b,y,m \in M$ and $\alpha,\beta \in \Gamma$. Hence, by the semiprimeness of M, $[G_\alpha(a,b),y]_\beta = 0$, where $G_\alpha(a,b) \in M$, for all $a,b,y \in M$ and $\alpha,\beta \in \Gamma$. Therefore, $G_\alpha(a,b) \in Z(M)$, the centre of M. Now, let $\gamma,\delta \in \Gamma$. By Lemma 2.2.6(ii), we have

$$G_\alpha(a,b)\gamma[x,y]_\alpha \delta m \delta G_\alpha(a,b)\gamma[x,y]_\alpha = 0.$$

But M is semiprime, we get

$$G_\alpha(a,b)\gamma[x,y]_\alpha = 0. \tag{2.1}$$

Also, by Lemma 2.2.6(i), we have

$$[x,y]_\alpha \gamma G_\alpha(a,b)\delta m \delta [x,y]_\alpha \gamma G_\alpha(a,b) = 0.$$

Hence by the semiprimeness of M, we get

$$[x,y]_\alpha \gamma G_\alpha(a,b) = 0. \tag{2.2}$$

Similarly, by Lemma 2.2.6(iv), we have

$$G_\alpha(a,b)\gamma[x,y]_\beta \delta m \delta G_\alpha(a,b)\gamma[x,y]_\beta = 0.$$

Since M is semiprime, it follows that

$$G_\alpha(a,b)\gamma[x,y]_\beta = 0. \tag{2.3}$$

Also, by Lemma 2.2.6(iii), we have

$$[x,y]_\beta \gamma G_\alpha(a,b)\delta m \delta [x,y]_\beta \gamma G_\alpha(a,b) = 0.$$

Hence, by the semiprimeness of M, we get

$$[x,y]_\beta \gamma G_\alpha(a,b) = 0. \tag{2.4}$$

Thus, we have

$$\begin{aligned}
2G_\alpha(a,b)\gamma G_\alpha(a,b) &= G_\alpha(a,b)\gamma(G_\alpha(a,b)+G_\alpha(a,b))\\
&= G_\alpha(a,b)\gamma(G_\alpha(a,b)-G_\alpha(b,a))\\
&= G_\alpha(a,b)\gamma(d(a\alpha b)-d(a)\alpha b-a\alpha d(b)-d(b\alpha a)+d(b)\alpha a+b\alpha d(a))\\
&= G_\alpha(a,b)\gamma(d(a\alpha b-b\alpha a)+(b\alpha d(a)-d(a)\alpha b)+(d(b)\alpha a-a\alpha d(b)))\\
&= G_\alpha(a,b)\gamma(d([a,b]_\alpha)+[b,d(a)]_\alpha+[d(b),a]_\alpha)\\
&= G_\alpha(a,b)\gamma d([a,b]_\alpha)-G_\alpha(a,b)\gamma[d(a),b]_\alpha-G_\alpha(a,b)\gamma[a,d(b)]_\alpha.
\end{aligned}$$

Since $d(a), d(b) \in M$, using (2.1), we get

$$G_\alpha(a,b)\gamma[d(a),b]_\alpha = G_\alpha(a,b)\gamma[a,d(b)]_\alpha = 0.$$

Therefore,

$$2G_\alpha(a,b)\gamma G_\alpha(a,b) = G_\alpha(a,b)\gamma d([a,b]_\alpha). \tag{2.5}$$

Adding (2.3) and (2.4), we obtain

$$G_\alpha(a,b)\gamma[x,y]_\beta + [x,y]_\beta\gamma G_\alpha(a,b) = 0.$$

Then by Lemma 2.2.1(i) with the use of (2.3), we have

$$\begin{aligned}
0 &= d(G_\alpha(a,b)\gamma[x,y]_\beta + [x,y]_\beta\gamma G_\alpha(a,b))\\
&= d(G_\alpha(a,b))\gamma[x,y]_\beta + d([x,y]_\beta)\gamma G_\alpha(a,b) + G_\alpha(a,b)\gamma d([x,y]_\beta) + [x,y]_\beta\gamma d(G_\alpha(a,b))\\
&= d(G_\alpha(a,b))\gamma[x,y]_\beta + 2G_\alpha(a,b)\gamma d([x,y]_\beta) + [x,y]_\beta\gamma d(G_\alpha(a,b)).
\end{aligned}$$

Since $G_\alpha(a,b) \in Z(M)$ implies $d([x,y]_\beta)\gamma G_\alpha(a,b) = G_\alpha(a,b)\gamma d([x,y]_\beta)$. Hence, we get

$$2G_\alpha(a,b)\gamma d([x,y]_\beta) = -d(G_\alpha(a,b))\gamma[x,y]_\beta - [x,y]_\beta\gamma d(G_\alpha(a,b)). \tag{2.6}$$

Then, from (2.5) and (2.6), we have

$$
\begin{aligned}
4G_\alpha(a,b)\gamma G_\alpha(a,b) &= 2G_\alpha(a,b)\gamma d([a,b]_\alpha)\\
&= -d(G_\alpha(a,b))\gamma[a,b]_\alpha - [a,b]_\alpha\gamma d(G_\alpha(a,b)).
\end{aligned}
$$

Thus, we obtain

$$4G_\alpha(a,b)\gamma G_\alpha(a,b)\gamma G_\alpha(a,b) = -d(G_\alpha(a,b))\gamma[a,b]_\alpha\gamma G_\alpha(a,b) - [a,b]_\alpha\gamma d(G_\alpha(a,b))\gamma G_\alpha(a,b)$$

Here, we have by using (2.4)

$$d(G_\alpha(a,b))\gamma[a,b]_\alpha\gamma G_\alpha(a,b) = 0$$

and also, by Corollary 2.2.5(vi)

$$[a,b]_\alpha\gamma d(G_\alpha(a,b))\gamma G_\alpha(a,b) = 0.$$

Since $d(G_\alpha(a,b)) \in M$, for all $a, b \in M$ and $\alpha \in \Gamma$. So, we get

$$4G_\alpha(a,b)\gamma G_\alpha(a,b)\gamma G_\alpha(a,b) = 0.$$

Therefore,

$$4(G_\alpha(a,b)\gamma)^2 G_\alpha(a,b) = 0.$$

Since M is 2-torsion free, so we have

$$(G_\alpha(a,b)\gamma)^2 G_\alpha(a,b) = 0.$$

But, it follows that $G_\alpha(a,b)$ is a nilpotent element of the Γ-ring M. Since by Lemma 2.2.9, the centre of a semiprime Γ-ring does not contain any nonzero nilpotent element, so we get $G_\alpha(a,b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$. It means that, d is a derivation of M. □

2.3 Jordan Derivations on Completely Semiprime Γ-Rings

In sequel to the last result, now we prove it analogously in case of a 2-torsion free completely semiprime Γ-ring with the condition (*) $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. That is, here we have to show that under the above condition every Jordan derivation of a 2-torsion free completely semiprime Γ-ring is a derivation of M. To reach our goal in this section, we develop some useful results in the following way.

Lemma 2.3.1. *Suppose d is a Jordan derivation of a 2-torsion free Γ-ring M satisfying the condition (*), then $G_\alpha(a,b)\beta[a,b]_\alpha + [a,b]_\alpha\beta G_\alpha(a,b) = 0$ for all $a, b \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. For any $a, b \in M$ and $\alpha, \beta \in \Gamma$, we have, using Lemma 2.2.1(i)

$$\begin{aligned} W &= d(a\alpha b\beta b\alpha a + b\alpha a\beta a\alpha b) \\ &= d((a\alpha b)\beta(b\alpha a) + (b\alpha a)\beta(a\alpha b)) \\ &= d(a\alpha b)\beta(b\alpha a) + (a\alpha b)\beta d(b\alpha a) + d(b\alpha a)\beta(a\alpha b) + (b\alpha a)\beta d(a\alpha b). \end{aligned}$$

On the other hand, using Lemma 2.2.1(iii)

$$\begin{aligned} W &= d(a\alpha(b\beta b)\alpha a + b\alpha(a\beta a)\alpha b) \\ &= d(a)\alpha(b\beta b)\alpha a + a\alpha d(b\beta b)\alpha a + a\alpha(b\beta b)\alpha d(a) + d(b)\alpha(a\beta a)\alpha b \\ &\quad + b\alpha d(a\beta a)\alpha b + b\alpha(a\beta a)\alpha d(b) \\ &= d(a)\alpha b\beta b\alpha a + a\alpha d(b)\beta b\alpha a + a\alpha b\beta d(b)\alpha a + a\alpha b\beta b\alpha d(a) \\ &\quad + d(b)\alpha a\beta a\alpha b + b\alpha d(a)\beta a\alpha b + b\alpha a\beta d(a)\alpha b + b\alpha a\beta a\alpha d(b). \end{aligned}$$

Equating the two expressions for W, we get

$$(d(a\alpha b)-d(a)\alpha b-a\alpha d(b))\beta b\alpha a+(d(b\alpha a)-d(b)\alpha a-b\alpha d(a))\beta a\alpha b$$
$$+a\alpha b\beta(d(b\alpha a)-d(b)\alpha a-b\alpha d(a))+b\alpha a\beta(d(a\alpha b)-d(a)\alpha b-a\alpha d(b))=0.$$

Now, using the Definition 2.2.1, we obtain

$$G_\alpha(a,b)\beta b\alpha a+G_\alpha(b,a)\beta a\alpha b+a\alpha b\beta G_\alpha(b,a)+b\alpha a\beta G_\alpha(a,b)=0.$$

Using Lemma 2.2.2(i), we have

$$G_\alpha(a,b)\beta b\alpha a-G_\alpha(a,b)\beta a\alpha b-a\alpha b\beta G_\alpha(a,b)+b\alpha a\beta G_\alpha(a,b)=0.$$

This implies,

$$G_\alpha(a,b)\beta[a,b]_\alpha+[a,b]_\alpha\beta G_\alpha(a,b)=0,\forall\ a,b\in M,\alpha,\beta\in\Gamma$$

. □

In the rest of this section, M represents a completely semiprime Γ-ring.

Lemma 2.3.2. *Let $a,b\in M$ and $\alpha\in\Gamma$ be any elements. If $a\alpha b+b\alpha a=0$, then $a\alpha b=0=b\alpha a$.*

Proof. Let $\delta\in\Gamma$ be any element. Suppose $a,b\in M$ and $\alpha\in\Gamma$ such that $a\alpha b+b\alpha a=0$. Using the relation $a\alpha b=-b\alpha a$ repeatedly, we get

$$(a\alpha b)\delta(a\alpha b)=-(b\alpha a)\delta(a\alpha b)=-(b(\alpha a\delta)a)\alpha b$$
$$=(a(\alpha a\delta)b)\alpha b=a\alpha(a\delta b)\alpha b=-a\alpha(b\delta a)\alpha b=-(a\alpha b)\delta(a\alpha b).$$

This implies,

$$2((a\alpha b)\delta(a\alpha b))=0.$$

Since M is 2-torsion free, thus

$$(a\alpha b)\delta(a\alpha b) = 0.$$

Therefore, $(a\alpha b)\Gamma(a\alpha b) = 0$. By the complete semiprimeness of M, we get $a\alpha b = 0$. Similarly, it can be shown that, $b\alpha a = 0$. □

Corollary 2.3.3. *If d is a Jordan derivation of M, then for all $a, b \in M$ and $\alpha, \beta \in \Gamma$, (i) $G_\alpha(a, b)\beta[a, b]_\alpha = 0$; (ii) $[a, b]_\alpha\beta G_\alpha(a, b) = 0$.*

Proof. Applying the result of Lemma 2.3.2 in that of Lemma 2.3.1, we obtain these results. □

Lemma 2.3.4. *For every $a, b, x, y \in M$, and $\alpha, \beta, \gamma \in \Gamma$, the following statements are true: (i) $G_\alpha(a, b)\beta[x, y]_\alpha = 0$; (ii) $[x, y]_\alpha\beta G_\alpha(a, b) = 0$ (iii) $G_\alpha(a, b)\beta[x, y]_\gamma = 0$; (iv) $[x, y]_\gamma\beta G_\alpha(a, b) = 0$.*

Proof. (i) If we substitute $a + x$ for a in the Corollary 2.3.3(i), then we get

$$G_\alpha(a + x, b)\beta[a + x, b]_\alpha = 0.$$

Using Lemma 2.2.2(ii), we have

$$G_\alpha(a, b)\beta[a, b]_\alpha + G_\alpha(a, b)\beta[x, b]_\alpha + G_\alpha(x, b)\beta[a, b]_\alpha + G_\alpha(x, b)\beta[x, b]_\alpha = 0.$$

Now, using Corollary 2.3.3(i), we obtain

$$G_\alpha(a, b)\beta[x, b]_\alpha + G_\alpha(x, b)\beta[a, b]_\alpha = 0.$$

$$\Rightarrow G_\alpha(a, b)\beta[x, b]_\alpha = -G_\alpha(x, b)\beta[a, b]_\alpha.$$

Therefore,

$$(G_\alpha(a, b)\beta[x, b]_\alpha)\beta(G_\alpha(a, b)\beta[x, b]_\alpha) = -G_\alpha(a, b)\beta[x, b]_\alpha\beta G_\alpha(x, b)\beta[a, b]_\alpha = 0.$$

By the complete semiprimeness of M, we obtain

$$G_\alpha(a,b)\beta[x,b]_\alpha = 0.$$

Similarly, by replacing $b+y$ for b in this result, we get

$$G_\alpha(a,b)\beta[x,y]_\alpha = 0.$$

(ii) Proceeding in the same way as described above by the similar replacements successively in Corollary 2.3.3(ii), we obtain

$$[x,y]_\alpha\beta G_\alpha(a,b) = 0, \forall\, a,b,x,y \in M, \alpha,\beta \in \Gamma.$$

(iii) Replacing $\alpha+\gamma$ for α in (i), we get

$$G_{\alpha+\gamma}(a,b)\beta[x,y]_{\alpha+\gamma} = 0.$$

Using Lemma 2.2.2(iv), we have

$$(G_\alpha(a,b) + G_\gamma(a,b))\beta([x,y]_\alpha + [x,y]_\gamma) = 0.$$
$$\Rightarrow G_\alpha(a,b)\beta[x,y]_\alpha + G_\alpha(a,b)\beta[x,y]_\gamma + G_\gamma(a,b)\beta[x,y]_\alpha + G_\gamma(a,b)\beta[x,y]_\gamma = 0.$$

Thus using (i), we get

$$G_\alpha(a,b)\beta[x,y]_\gamma + G_\gamma(a,b)\beta[x,y]_\alpha = 0.$$
$$\Rightarrow G_\alpha(a,b)\beta[x,y]_\gamma = -G_\gamma(a,b)\beta[x,y]_\alpha.$$

Thus, we have

$$(G_\alpha(a,b)\beta[x,y]_\gamma)\beta(G_\alpha(a,b)\beta[x,y]_\gamma) = -G_\alpha(a,b)\beta[x,y]_\gamma\beta G_\gamma(a,b)\beta[x,y]_\alpha = 0.$$

Hence, by the complete semiprimeness of M, we obtain

$$G_\alpha(a,b)\beta[x,y]_\gamma = 0.$$

(iv) By performing the similar replacement in (ii)(as in the proof of (iii)), we get this result. □

Lemma 2.3.5. *([13], Lemma 3.7.1) Every completely semiprime* Γ*-ring has no nonzero nilpotent ideal.*

Corollary 2.3.6. *([13], Corollary 3.7.2) Completely semiprime* Γ*-rings have no nonzero nilpotent element.*

Lemma 2.3.7. *([13], Lemma 3.7.2) The centre of a completely semiprime* Γ*-ring does not contain any nonzero nilpotent element.*

We are now ready to prove our main result as follows.

Theorem 2.3.8. *Every Jordan derivation of a 2-torsion free completely semiprime* Γ*-ring* M *satisfying the condition (*) is a derivation of* M*.*

Proof. Let d be a Jordan derivation of a 2-torsion free completely semiprime Γ-ring M, and let $a, b, y \in M; \alpha, \beta \in \Gamma$. By Lemma 2.3.4(iii), we have

$$\begin{aligned}
[G_\alpha(a,b), y]_\beta \gamma [G_\alpha(a,b), y]_\beta &= (G_\alpha(a,b)\beta y - y\beta G_\alpha(a,b))\gamma [G_\alpha(a,b), y]_\beta \\
&= G_\alpha(a,b)\beta y\gamma [G_\alpha(a,b), y]_\beta - y\beta G_\alpha(a,b)\gamma [G_\alpha(a,b), y]_\beta \\
&= 0.
\end{aligned}$$

Since $\beta y\gamma \in \Gamma$ and $G_\alpha(a,b) \in M$ for all $a, b, y \in M$ and $\alpha, \beta \in \Gamma$. Hence, by the complete semiprimeness of M, we get

$$[G_\alpha(a,b), y]_\beta = 0, \forall\, a, b, y \in M; \alpha, \beta \in \Gamma.$$

$$\Rightarrow G_\alpha(a,b) \in Z(M), \forall\, a, b \in M; \alpha \in \Gamma.$$

Now, from Lemma 2.3.4(iii)

$$G_\alpha(a,b)\gamma[x,y]_\beta = 0. \tag{2.7}$$

Also, by Lemma 2.3.4(iv)

$$[x,y]_\beta\gamma G_\alpha(a,b) = 0. \tag{2.8}$$

Thus, we obtain

$$\begin{aligned}
2G_\alpha(a,b)\gamma G_\alpha(a,b) &= G_\alpha(a,b)\gamma(G_\alpha(a,b) + G_\alpha(a,b)) \\
&= G_\alpha(a,b)\gamma(G_\alpha(a,b) - G_\alpha(b,a)) \\
&= G_\alpha(a,b)\gamma(d(a\alpha b) - d(a)\alpha b - a\alpha d(b) - d(b\alpha a) + d(b)\alpha a + b\alpha d(a)) \\
&= G_\alpha(a,b)\gamma(d(a\alpha b - b\alpha a) + (b\alpha d(a) - d(a)\alpha b) + (d(b)\alpha a - a\alpha d(b))) \\
&= G_\alpha(a,b)\gamma(d([a,b]_\alpha) + [b,d(a)]_\alpha + [d(b),a]_\alpha) \\
&= G_\alpha(a,b)\gamma d([a,b]_\alpha) - G_\alpha(a,b)\gamma[d(a),b]_\alpha - G_\alpha(a,b)\gamma[a,d(b)]_\alpha.
\end{aligned}$$

Since $d(a), d(b) \in M$, using Lemma 2.3.4(i) , we get

$$G_\alpha(a,b)\gamma[d(a),b]_\alpha = 0 = G_\alpha(a,b)\gamma[a,d(b)]_\alpha.$$

Therefore,

$$2G_\alpha(a,b)\gamma G_\alpha(a,b) = G_\alpha(a,b)\gamma d([a,b]_\alpha). \tag{2.9}$$

Adding (2.7) and (2.8), we obtain

$$G_\alpha(a,b)\gamma[x,y]_\beta + [x,y]_\beta\gamma G_\alpha(a,b) = 0.$$

Then by Lemma 2.2.1(i), we have

$$\begin{aligned}
0 &= d(G_\alpha(a,b)\gamma[x,y]_\beta + [x,y]_\beta\gamma G_\alpha(a,b)) \\
&= d(G_\alpha(a,b))\gamma[x,y]_\beta + d([x,y]_\beta)\gamma G_\alpha(a,b) + G_\alpha(a,b)\gamma d([x,y]_\beta) + [x,y]_\beta\gamma d(G_\alpha(a,b)) \\
&= d(G_\alpha(a,b))\gamma[x,y]_\beta + 2G_\alpha(a,b)\gamma d([x,y]_\beta) + [x,y]_\beta\gamma d(G_\alpha(a,b)).
\end{aligned}$$

Since $G_\alpha(a,b) \in Z(M)$ and therefore

$$d([x,y]_\beta)\gamma G_\alpha(a,b) = G_\alpha(a,b)\gamma d([x,y]_\beta).$$

Hence, we get

$$2G_\alpha(a,b)\gamma d([x,y]_\beta) = -d(G_\alpha(a,b))\gamma[x,y]_\beta - [x,y]_\beta\gamma d(G_\alpha(a,b)). \tag{2.10}$$

Then from (2.9) and (2.10), we have

$$\begin{aligned} G_\alpha(a,b)\gamma G_\alpha(a,b) &= 2G_\alpha(a,b)\gamma d([a,b]_\alpha) \\ &= -d(G_\alpha(a,b))\gamma[a,b]_\alpha - [a,b]_\alpha\gamma d(G_\alpha(a,b)). \end{aligned}$$

Thus, we obtain

$$4G_\alpha(a,b)\gamma G_\alpha(a,b)\gamma G_\alpha(a,b) = -d(G_\alpha(a,b))\gamma[a,b]_\alpha\gamma G_\alpha(a,b) - [a,b]_\alpha\gamma d(G_\alpha(a,b))\gamma G_\alpha(a,b).$$

Here, we have by Corollary 2.3.3(ii), $d(G_\alpha(a,b))\gamma[a,b]_\alpha\gamma G_\alpha(a,b) = 0$, and also by Lemma 2.3.4(iv), $[a,b]_\alpha\gamma d(G_\alpha(a,b))\gamma G_\alpha(a,b) = 0$, since $d(G_\alpha(a,b)) \in M$, and $\gamma d(G_\alpha(a,b))\gamma \in \Gamma$ for all $a,b \in M, \alpha \in \Gamma$. Therefore, we get

$$\begin{aligned} &4G_\alpha(a,b)\gamma G_\alpha(a,b)\gamma G_\alpha(a,b) = 0. \\ &\Rightarrow 4(G_\alpha(a,b)\gamma)^2 G_\alpha(a,b) = 0. \end{aligned}$$

Since M is 2-torsion free, so we have

$$(G_\alpha(a,b)\gamma)^2 G_\alpha(a,b) = 0.$$

But, it follows that $G_\alpha(a,b)$ is a nilpotent element of the Γ-ring M. Since by Lemma 2.3.7, the centre of a completely semiprime Γ-ring does not contain any nonzero nilpotent element, so we get $G_\alpha(a,b) = 0$ for all $a,b \in M$ and $\alpha \in \Gamma$. Thus we conclude that, d is a derivation of M. $\square$

Chapter 3

Higher Derivations

In view of the notions of derivation and Jordan derivation of Γ-rings, here we introduce the concepts of higher derivation and Jordan higher derivation of Γ-rings. Following the notions of higher derivation and Jordan higher derivation of Γ-rings we then introduce the concepts of higher left derivation and Jordan higher left derivation of Γ-rings. Introductory discussions concerning these concepts are described in the first section. Finally, we introduce the concepts of generalized higher derivation and Jordan generalized higher derivation of Γ-rings in the first section.

In the second section, we show that under a suitable condition, the existence of a nonzero Jordan higher left derivation on a 2-torsion free prime Γ-ring M forces M commutative.

We use the concept of Jordan derivation and derivation of a Γ-ring introduced by M. Sapanci and A.Nakajima in the third section to develop a number of important results on Jordan derivations of a 2-torsion free Γ-ring. Here, we show that every Jordan derivation of a 2-torsion free prime Γ-ring is a derivation which is very much needed for proving the remaining result of this chapter.

Our main result in the fourth section aims to prove that, every Jordan higher derivation of a 2-torsion free prime Γ-ring with the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, is a higher derivation of M.

Then we prove the analogous result corresponding to the above mentioned result considering Jordan generalized higher derivation of a prime Γ-ring instead of Jordan higher derivation of a prime Γ-ring almost similar way which states that every Jordan generalized higher derivation of a 2-torsion free prime Γ-ring is a generalized higher derivation.

3.1 Introduction

In classical ring theory, I. N. Herstien [21], introduced the notions of derivation and Jordan derivation of rings; he proved in [19] that, every Jordan derivation in a 2-torsion free prime ring is a derivation. M. Ferrero and C. Haetinger [15] extended Herstein's theorem to higher derivations, by using Jordan triple higher derivations. Also, Haetinger in [17] worked on higher derivations on prime rings and extended Awtar's result [3] to higher derivations in Lie ideals of prime rings.

By the motivations of the above mentioned works, in this chapter we work on higher derivation, Jordan higher derivation, higher left derivation, Jordan higher left derivation, generalized higher derivation and Jordan generalized higher derivation of Γ-rings.

The notions of derivation and Jordan derivation of Γ-rings have been introduced by M. Sapanci and A. Nakajima [36] as follows.

Definition 3.1.1. For a Γ-ring M, if $d : M \to M$ is an additive mapping such that $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$, then d is called a *derivation* of M ; d is called a *Jordan derivation* of M if $d(a\alpha a) = d(a)\alpha a + a\alpha d(a)$ holds for all $a \in M$ and $\alpha \in \Gamma$.

Following these derivations, here we introduce higher derivation and Jordan higher derivation of Γ-rings in the following way.

Definition 3.1.2. Let $D = (d_i)_{i\in N_0}$ be a family of additive mappings of a $\Gamma-$ring M such that $d_0 = id_M$, where id_M is an identity mapping on M and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Then D is a *higher derivation* of M if for each $n \in \mathbf{N}_0$ and $i, j \in \mathbf{N}_0$,

$$d_n(a\alpha b) = \sum_{i+j=n} d_i(a)\alpha d_j(b), \forall\, a, b \in M; \alpha \in \Gamma,$$

and D is a *Jordan higher derivation* of M if

$$d_n(a\alpha a) = \sum_{i+j=n} d_i(a)\alpha d_j(a), \forall\, a \in M; \alpha \in \Gamma.$$

Example 3.1.1. *Let R be an associative ring with 1. Let us consider $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$, then M is a Γ-ring. Let $f_n : R \to R$ be a higher derivation for each $n \in N_0$. For $n \in N_0$, we define additive mappings $d_n : M \to M$ by $d_n((a, b)) = (f_n(a), f_n(b))$. Then an easy verifications leads to us that d_n is a higher derivation of M. Let $P = \{(a, a) : a \in R\}$, then P is a Γ-ring contained in M. In fact, P is a sub Γ-ring. Define $d_n((a, a)) = (f_n(a), f_n(a))$, then d_n is a Jordan higher derivation of P.*

Continuing in the similar way as that has been done by the earlier prominent algebraists we then introduce higher left derivation and Jordan higher left derivation of Γ-rings in the following way.

Definition 3.1.3. Suppose $D = (d_i)_{i \in N_0}$ is a family of additive mappings of a Γ-ring M such that $d_0 = id_M$, where id_M is an identity mapping on M and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Then D is called a *higher left derivation* of M if for each $n \in \mathbf{N}_0$ and $i, j \in \mathbf{N}_0$,

$$d_n(a\alpha b) = \sum_{i+j=n, i \leq j} (d_i(a)\alpha d_j(b) + d_i(b)\alpha d_j(a)), \forall\, a, b \in M; \alpha \in \Gamma,$$

and D is called a *Jordan higher left derivation* of M if

$$d_n(a\alpha a) = \sum_{i+j=n, i \leq j} d_i(a)\alpha d_j(a), \forall\, a \in M; \alpha \in \Gamma.$$

Example 3.1.2. *Let the Γ-ring M as in Example 3.1.1 Suppose $N = \{(a, a) : a \in R\}$, then N is a Γ-subring of M. If $d_n : R \to R$ is a higher left derivation for each $n \in \boldsymbol{N}_0$. Then for $n \in \boldsymbol{N}_0$, we define the additive mappings $D_n : M \to M$ by $D_n((a, b)) = (d_n(a), d_n(b))$. Then it is clear that D_n is a higher left derivation on M. If we define a mapping $D_n : N \to N$ by $D_n((a, a)) = (d_n(a), d_n(a))$, then it is obvious that D_n is an additive mapping and therefore D_n is a Jordan higher left derivation on M.*

The notions of generalized derivation and Jordan generalized derivation of a Γ-ring have been introduced by Y. Ceven and M. A. Ozturk [11] as below.

Definition 3.1.4. Assume that M is a Γ-ring and $f : M \to M$ be an additive mapping. Then f is called a *generalized derivation* of M if there exists a derivation $d : M \to M$ such that $f(a\alpha b) = f(a)\alpha b + a\alpha d(b)$ holds for all $a, b \in M$ and $\alpha \in \Gamma$; f is a *Jordan generalized derivation* of M if there exists a derivation $d : M \to M$ such that $f(a\alpha a) = f(a)\alpha a + a\alpha d(a)$ holds for all $a \in M$ and $\alpha \in \Gamma$.

Finally, we introduce generalized higher derivation and Jordan generalized higher derivation of Γ-rings as follows.

Definition 3.1.5. If $F = (f_i)_{i\in N_0}$ is a family of additive mappings of a Γ-ring M such that $f_0 = id_M$, where id_M is an identity mapping on M and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Then F is said to be a *generalized higher derivation* of M if there exists a higher derivation $D = (d_i)_{i\in N_0}$ of M such that for each $n \in \mathbf{N}_0$ and $i, j \in \mathbf{N}_0$,

$$f_n(a\alpha b) = \sum_{i+j=n} f_i(a)\alpha d_j(b), \forall\, a, b \in M; \alpha \in \Gamma,$$

and F is said to be a *Jordan generalized higher derivation* of M if there exists a higher derivation $D = (d_i)_{i\in N_0}$ of M such that

$$f_n(a\alpha a) = \sum_{i+j=n} f_i(a)\alpha d_j(a), \forall\, a \in M; \alpha \in \Gamma.$$

Example 3.1.3. *Let R be an associative ring with 1 and let $F = (f_i)_{i\in N}$ be a generalized higher derivation on R. Then there exists a higher derivation $D = (d_i)_{i\in N}$ of R such that $f_n(xy) = \sum_{i+j=n} f_i(x)d_j(y)$, for all $x, y \in M$. Now if we us consider the Γ-ring M as in Example 3.1.1, and define the mapping $K = (k_i)_{i\in \boldsymbol{N}}$ of M by $k_n((x, y)) = (d_n(x), d_n(y))$, then K is a derivation of M. Let $G = (g_i)_{i\in \boldsymbol{N}}$ be an additive mapping of M defined by $g_n((x, y)) = (f_n(x), f_n(y))$. Then it is clear that G is a generalized higher derivation on M with the associated derivation K. Let us define $N = \{(x, x) : x \in R\}$ of M, then N is a Γ-ring contained in M. We define the mapping $G : N \to N$ by $g_n((x, x)) = (f_n(x), f_n(x))$ and $k_n((x, x)) = (d_n(x), d_n(x))$, then we have seen that G is a Jordan generalized higher derivation on N with the associated generalized higher derivation K.*

Throughout this chapter (unless otherwise stated), M is a 2-torsion free prime Γ-ring which satisfies $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$; $\alpha, \beta \in \Gamma$ and we use the symbol (*) corresponding to this assumption.

3.2 Commutativity in Prime Γ-Rings with Jordan Higher Left Derivations

M. Soyturk [37] investigated the commutativity of prime Γ-rings with left and right derivations. He obtained some significant results on the commutativity of prime Γ-rings of characteristic not equal to 2 and 3. M. Asci and S. Ceran [1] obtained some commutativity results of prime Γ-rings with left derivations. Some commutativity results in prime rings with Jordan higher left derivations were obtained by Kyuoo-Hong Park [34] on Lie ideals and obtained some fruitful results relating this.

In sequel to the result of the first chapter, here we prove it analogously in case of higher left derivation. That is, we show that under a suitable condition, the existence of a nonzero Jordan higher left derivation on a 2-torsion free prime Γ-ring M forces M commutative.

Theorem 3.2.1. *Let $\Delta = (d_n)_{n \in N}$ be a Jordan higher left derivation on M. If $\Delta \neq 0$, that is, if there exists $n \in N$ such that $d_n \neq 0$, then M is commutative.*

Proof. We use the method of induction. If $n = 1$, that is, if d_1 is a Jordan left derivation on M, then we assume that M is non commutative. By the proof of Theorem 1.4.1, we have $d_1(a) = 0$, for all $a \in M$. Assume that $n \geq 2$ and $d_m = 0$ for all $m < n$. Then d_n is a Jordan left derivation on M and from the above argument, it follows that $d_n = 0$. Hence we conclude that $\Delta = 0$. This completes the proof. $\square$

3.3 Jordan Derivations in Prime Γ-Rings

In this section, we show that every Jordan derivation of a 2-torsion free prime Γ-ring is a derivation. For this we prepare the following Lemma.

Lemma 3.3.1. *Let $a, b \in M$ be any elements. If $a\alpha m\beta b + b\alpha m\beta a = 0$ for all $m \in M; \alpha, \beta \in \Gamma$, then $a = 0$ or $b = 0$.*

Proof. Replacing m by $s\delta a\mu t$ in $a\alpha m\beta b + b\alpha m\beta a = 0$, where $s, t \in M; \delta, \mu \in \Gamma$, we get $a\alpha s\delta a\mu t\beta b + b\alpha s\delta a\mu t\beta a = 0$. Since $b\alpha s\delta a = -a\alpha s\delta b$ and $a\mu t\beta b = -b\mu t\beta a$. Substituting these, we obtain

$$- a\alpha s\delta b\mu t\beta a - a\alpha s\delta b\mu t\beta a = 0.$$

$$\Rightarrow 2a\alpha s\delta b\mu t\beta a = 0.$$

Since M is 2-torsion free, so

$$a\alpha s\delta b\mu t\beta a = 0, \forall\, t \in M; \beta, \mu \in \Gamma.$$

Therefore, $(a\alpha s\delta b)\Gamma M\Gamma a = 0, \forall\, a, b, s \in M; \alpha, \delta \in \Gamma$. Since M is prime, thus we have $a\alpha s\delta b = 0$ or $a = 0$. Suppose $a\alpha s\delta b = 0$. Again applying the primeness of M, we get $a = 0$ or $b = 0$. □

Theorem 3.3.2. *If d is a Jordan derivation of a 2-torsion free prime Γ-ring M satisfying the condition (*), then d is a derivation of M.*

Proof. By Lemma 2.2.3 and Lemma 3.3.1; M being prime, we have $G_\alpha(a, b) = 0$ or $[a, b]_\alpha = 0$. For all $a \in M$, let $A = \{b \in M : G_\alpha(a, b) = 0\}$ and $B = \{b \in M : [a, b]_\alpha = 0.\}$ Then A and B are two additive subgroups of M such that $A \cup B = M$. Then by the Brauer's trick, either $A = M$ or $B = M$.

By the similar argument, let $M = \{a \in M : M = A\}$ or $M = \{a \in M : M = B\}$. For the later case, M is commutative. That is, $a\alpha b = b\alpha a$. In view of Lemma 2.2.1(i), we have $2d(a\alpha b) = 2d(a)\alpha b + 2a\alpha d(b)$. Since M is 2-torsion free, we obtain that d is a derivation of M. For the former case, $G_\alpha(a, b) = 0$ and it follows that d is a derivation on M. $\square$

3.4 Jordan Higher Derivations in Prime Γ-Rings

The objective of this section is to study Jordan higher derivations of prime Γ-rings. Higher derivations have been studied by many authors [14, 15, 16, 17, 32, 34] in classical rings. We extend some of these results in prime Γ-rings by the concept of Jordan higher derivations.

Here, we extend the result of W. Cortes and C. Haetinger [14] considering Jordan higher derivations in prime Γ-rings.

Lemma 3.4.1. *Assume that $D = (d_i)_{i\in N}$ is a Jordan higher derivation of M. Then for all $a, b, c \in M; \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$,*

(i) $d_n(a\alpha b + b\alpha a) = \sum_{i+j=n}[d_i(a)\alpha d_j(b) + d_i(b)\alpha d_j(a)]$;

(ii) $d_n(a\alpha b\beta a) = \sum_{i+j+k=n}[d_i(a)\alpha d_j(b)\beta d_k(a)]$;

(iii) $d_n(a\alpha b\beta c + c\alpha b\beta a) = \sum_{i+j+k=n}[d_i(a)\alpha d_j(b)\beta d_k(c) + d_i(c)\alpha d_j(b)\beta d_k(a)]$.

Proof. The proofs of (i) and (ii) are similar to the corresponding proofs of Lemma 2.2.1(i) and Lemma 2.2.1(iii). Replacing a by $a + c$ in (ii) and using (ii), we obtain

$$W = d_n((a + c)\alpha b\beta(a + c)) = \sum_{i+j+k=n} d_i(a + c)\alpha d_j(b)\beta d_k(a + c)$$

$$= \sum_{i+j+k=n} (d_i(a) + d_i(c))\alpha d_j(b)\beta(d_k(a) + d_k(c)) = \sum_{i+j+k=n} d_i(a)\alpha d_j(b)\beta d_k(a)$$

$$+ \sum_{i+j+k=n} d_i(a)\alpha d_j(b)\beta d_k(c) + \sum_{i+j+k=n} d_i(c)\alpha d_j(b)\beta d_k(a) + \sum_{i+j+k=n} d_i(c)\alpha d_j(b)\beta d_k(c).$$

Also, we have

$$W = d_n(a\alpha b\beta a + a\alpha b\beta c + c\alpha b\beta a + c\alpha b\beta c)$$

$$= d_n(a\alpha b\beta a) + d_n(c\alpha b\beta c) + d_n(a\alpha b\beta c + c\alpha b\beta a)$$

$$= \sum_{i+j+k=n} d_i(a)\alpha d_j(b)\beta d_k(a) + \sum_{i+j+k=n} d_i(c)\alpha d_j(b)\beta d_k(c) + d_n(a\alpha b\beta c + c\alpha b\beta a).$$

By comparing the two results for W, we obtain (iii). $\square$

Definition 3.4.1. For every Jordan higher derivation $D = (d_i)_{i\in\mathbf{N}}$ of M, we define $\phi_n^\alpha(a, b) = d_n(a\alpha b) - \sum_{i+j=n} d_i(a)\alpha d_j(b)$ for all $a, b \in M; \alpha \in \Gamma$ and $n \in \mathbf{N}$.

Remark 3.4.1. D is a higher derivation of M if and only if $\phi_n^\alpha(a, b) = 0$ for all $a, b \in M; \alpha \in \Gamma$ and $n \in \mathbf{N}$.

Lemma 3.4.2. *For every $a, b, c \in M; \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$,*
(i) $\phi_n^\alpha(a, b) + \phi_n^\alpha(b, a) = 0$; (ii) $\phi_n^\alpha(a + b, c) = \phi_n^\alpha(a, c) + \phi_n^\alpha(b, c)$;
(iii) $\phi_n^\alpha(a, b + c) = \phi_n^\alpha(a, b) + \phi_n^\alpha(a, c)$; (iv) $\phi_n^{\alpha+\beta}(a, b) = \phi_n^\alpha(a, b) + \phi_n^\beta(a, b)$.

Proof. (i) By the Definition 3.4.1 and using the Lemma 3.4.1(i), we obtain

$$\phi_n^\alpha(a, b) + \phi_n^\alpha(b, a) = d_n(a\alpha b) - \sum_{i+j=n} d_i(a)\alpha d_j(b) + d_n(b\alpha a) - \sum_{i+j=n} d_i(b)\alpha d_j(a)$$

$$= d_n(a\alpha b + b\alpha a) - \sum_{i+j=n} d_i(a)\alpha d_j(b) - \sum_{i+j=n} d_i(b)\alpha d_j(a)$$

$$= \sum_{i+j=n} d_i(a)\alpha d_j(b) + \sum_{i+j=n} d_i(b)\alpha d_j(a) - \sum_{i+j=n} d_i(a)\alpha d_j(b)$$

$$- \sum_{i+j=n} d_i(b)\alpha d_j(a) = 0.$$

(ii) By the Definition 3.4.1, we get

$$\begin{aligned}\phi_n^\alpha(a+b,c) &= d_n((a+b)\alpha c) - \sum_{i+j=n} d_i(a+b)\alpha d_j(c)\\ &= d_n(a\alpha c + b\alpha c) - \sum_{i+j=n} d_i(a)\alpha d_j(c) - \sum_{i+j=n} d_i(b)\alpha d_j(c)\\ &= d_n(a\alpha c) - \sum_{i+j=n} d_i(a)\alpha d_j(c) + d_n(b\alpha c) - \sum_{i+j=n} d_i(b)\alpha d_j(c)\\ &= \phi_n^\alpha(a,c) + \phi_n^\alpha(b,c).\end{aligned}$$

(iii)-(iv): These are also easy to proof. $\square$

Lemma 3.4.3. *Suppose $D = (d_i)_{i\in \boldsymbol{N}}$ is a Jordan higher derivation of a Γ-ring M. Let $n \in \boldsymbol{N}$ and assume that $a, b \in M; \alpha, \beta, \gamma \in \Gamma$. If $\phi_m^\alpha(a,b) = 0$, for every $m < n$, then $\phi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha + [a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b) = 0$, for every $w \in M$.*

Proof. We consider $G = d_n(a\alpha b\beta w\gamma b\alpha a + b\alpha a\beta w\gamma a\alpha b)$. First, we compute

$$G = d_n(a\alpha(b\beta w\gamma b)\alpha a) + d_n(b\alpha(a\beta w\gamma a)\alpha b).$$

Using Lemma 3.4.1(ii), we have

$$\begin{aligned}G &= \sum_{i+p+l=n} d_i(a)\alpha d_p(b\beta w\gamma b)\alpha d_l(a) + \sum_{i+p+l=n} d_i(b)\alpha d_p(a\beta w\gamma a)\alpha d_l(b)\\ &= \sum_{i+j+k+h+l=n} d_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) + \sum_{i+j+k+h+l=n} d_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b).\end{aligned}$$

On the other hand

$$G = d_n((a\alpha b)\beta w\gamma(b\alpha a) + (b\alpha a)\beta w\gamma(a\alpha b)).$$

Using Lemma 3.4.1(iii), we obtain

$$\begin{aligned}G &= \sum_{r+s+t=n} (d_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a) + d_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b))\\ &= \sum_{r+s+t=n} d_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a) + \sum_{r+s+t=n} d_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b).\end{aligned}$$

Thus comparing both expressions for G, we obtain

$$\sum_{i+j+k+h+l=n} d_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) - \sum_{r+s+t=n} d_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)$$
$$+ \sum_{i+j+k+h+l=n} d_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b) - \sum_{r+s+t=n} d_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b) = 0. \tag{3.1}$$

By the inductive assumption we can put $d_r(x\alpha y)$ for $\sum_{i+j=r} d_i(x)\alpha d_j(y)$, when $r < n$. Therefore,

$$\sum_{i+j+k+h+l=n} d_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) - \sum_{r+s+t=n} d_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)$$
$$= (\sum_{i+j=n} d_i(a)\alpha d_j(b))\beta w\gamma b\alpha a + a\alpha b\beta w\gamma(\sum_{h+l=n} d_h(b)\alpha d_l(a))$$
$$+ \sum_{i+j+k+h+l=n}^{i+j<n,h+l<n} d_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) - d_n((a\alpha b)\beta w\gamma(b\alpha a)$$
$$- (a\alpha b)\beta w\gamma d_n(b\alpha a) - \sum_{r+s+t=n}^{i+j=r<n,p+q=t<n} d_i(a)\alpha d_j(b)\beta d_s(w)\gamma d_p(b)\alpha d_q(a)$$
$$= -(d_n((a\alpha b) - \sum_{i+j=n} d_i(a)\alpha d_j(b))\beta(w\gamma b\alpha a) - (a\alpha b\beta w)\gamma(d_n(b\alpha a) - \sum_{h+l=n} d_h(b)\alpha d_l(a))$$
$$= -(\phi_n^\alpha(a,b)\beta w\gamma b\alpha a + a\alpha b\beta w\gamma\phi_n^\alpha(b,a)). \tag{3.2}$$

Similarly,

$$\sum_{i+j+k+h+l=n} d_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b) - \sum_{r+s+t=n} d_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b)$$
$$= -(\phi_n^\alpha(b,a)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^\alpha(a,b)). \tag{3.3}$$

Hence, by using (3.2) and (3.3) in (3.1), we get

$$\phi_n^\alpha(a,b)\beta w\gamma b\alpha a + a\alpha b\beta w\gamma\phi_n^\alpha(b,a) + \phi_n^\alpha(b,a)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^\alpha(a,b) = 0.$$

By Lemma 3.4.2(i), we have

$$\phi_n^\alpha(a,b)\beta w\gamma b\alpha a \;-\; a\alpha b\beta w\gamma\phi_n^\alpha(a,b) \;-\; \phi_n^\alpha(a,b)\beta w\gamma a\alpha b \;+\; b\alpha a\beta w\gamma\phi_n^\alpha(a,b) \;=\; 0.$$

This implies,

$$\phi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha + [a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b) = 0, \forall\, w \in M.$$

□

Now, we prove the main result.

Theorem 3.4.4. *Let M be a 2-torsion free prime Γ-ring satisfying the condition (*). Then every Jordan higher derivation of M is a higher derivation of M.*

Proof. By definition, we have

$$\phi_0^\alpha(a,b) = 0, \forall\, a,b \in M, \alpha \in \Gamma.$$

Also, by Theorem 3.3.5,

$$\phi_1^\alpha(a,b) = 0, \forall\, a,b \in M, \alpha \in \Gamma.$$

Now, we proceed by induction. Suppose that, $\phi_m^\alpha(a,b) = 0$. This implies , $d_m(a\alpha b) = \sum_{i+j=m} d_i(a)\alpha d_j(b)$ for all $a,b \in M; \alpha \in \Gamma$ and $m < n$. Taking $a,b \in M$, by Lemma 3.4.3, we get

$$\phi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha + [a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b) = 0, \forall\, w \in M, \alpha,\beta,\gamma \in \Gamma.$$

Since M is prime, so by Lemma 3.3.1 $\phi_n^\alpha(a,b) = 0$, or $[a,b]_\alpha = 0$. Using the similar arguments as used in the proof of Theorem 3.3.2, we obtain that every Jordan higher derivation of M is a higher derivation of M. □

3.5 Jordan Generalized Higher Derivations in Prime Γ-Rings

The notion of generalized derivation was introduced by B. Hvala [23] and M. Bresar [7]. Afterwards, many authors have investigated comparable results on prime and semiprime rings with generalized derivations. The notions of generalized derivation and Jordan generalized derivation of Γ-rings have been introduced by Y. Ceven and M. A. Ozturk [11]. A. Nakajima [32] defined the notion of generalized higher derivations and investigated some elementary relations between generalized higher derivations and higher derivations in the usual sense. W. Cortes and C. Haetinger [14] proved that every Jordan generalized higher derivations of a ring is a generalized higher derivation.

We extend the above mentioned result in prime Γ-rings considering Jordan generalized higher derivations of prime Γ-rings. We need the following Lemmas for proving this result.

Lemma 3.5.1. *Let $F = (f_i)_{i\in N}$ be a Jordan generalized higher derivation of a Γ-ring M with the associated higher derivation $D = (d_i)_{i\in N}$. Then for each fixed $n \in N$; for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:*
(i) $f_n(a\alpha b + b\alpha a) = \sum_{i+j=n}[f_i(a)\alpha d_j(b) + f_i(b)\alpha d_j(a)]$;
(ii) $f_n(a\alpha b\beta a) = \sum_{i+j+k=n}[f_i(a)\alpha d_j(b)\beta d_k(a)]$;
(iii) $f_n(a\alpha b\beta c + c\alpha b\beta a) = \sum_{i+j+k=n}[f_i(a)\alpha d_j(b)\beta d_k(c) + f_i(c)\alpha d_j(b)\beta d_k(a)]$.

Proof. Since $F = (f_i)_{i\in N_0}$ is a Jordan generalized higher derivation of M, we have

$$f_n(a\alpha a) = \sum_{i+j=n} f_i(a)d_j(a).$$

Now, replacing $a+b$ for a and simplifying, we obtain (i). Then replacing b by $a\beta b+b\beta a$ in (i) and using the condition (*), we obtain (ii). For (iii), we replace a by $a+c$ in (ii), we obtain

$$\begin{aligned} w &= (a+c)\alpha b\beta(a+c) \\ \Rightarrow f_n(w) &= \sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(c) + \sum_{i+j+k=n} f_i(c)\alpha d_j(b)\beta d_k(a) \\ &\quad + \sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(a) + \sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(c). \end{aligned}$$

On the other hand, using (ii)

$$f_n(w) = f_n(a\alpha b\beta c + c\alpha b\beta a) + \sum_{i+j+k=n} (f_i(a)\alpha d_j(b)\beta d_k(a) + f_i(c)\alpha d_j(b)\beta d_k(c)).$$

Comparing the above two expressions for $f_n(w)$, we obtain (iii). $\square$

Definition 3.5.1. For every Jordan generalized higher derivation $F = (f_i)_{i\in\mathbf{N}}$ of M, we define $\psi_n^\alpha(a,b) = f_n(a\alpha b) - \sum_{i+j=n} f_i(a)\alpha d_j(b)$, for all $a, b \in M$; $\alpha \in \Gamma$ and $n \in \mathbf{N}$.

Lemma 3.5.2. *With our notations as above, the following are true:*
(i) $\psi_n^\alpha(b,a) = -\psi_n^\alpha(a,b)$; *(ii)* $\psi_n^\alpha(a+b,c) = \psi_n^\alpha(a,c) + \psi_n^\alpha(b,c)$;
(iii) $\psi_n^\alpha(a,b+c) = \psi_n^\alpha(a,b) + \psi_n^\alpha(a,c)$; *(iv)* $\psi_n^{\alpha+\beta}(a,b) = \psi_n^\alpha(a,b) + \psi_n^\beta(a,b)$.

Proof. (i) By the definition of $\psi_n^\alpha(a,b)$ and using Lemma 3.5.1(i), we obtain

$$\begin{aligned} \psi_n^\alpha(a,b) + \psi_n^\alpha(b,a) &= f_n(a\alpha b) - \sum_{i+j=n} f_i(a)\alpha d_j(b) + f_n(b\alpha a) - \sum_{i+j=n} f_i(b)\alpha d_j(a) \\ &= f_n(a\alpha b + b\alpha a) - \sum_{i+j=n} f_i(a)\alpha d_j(b) - \sum_{i+j=n} f_i(b)\alpha d_j(a) \\ &= \sum_{i+j=n} f_i(a)\alpha d_j(b) + \sum_{i+j=n} f_i(b)\alpha d_j(a) - \sum_{i+j=n} f_i(a)\alpha d_j(b) \\ &\quad - \sum_{i+j=n} f_i(b)\alpha d_j(a) = 0. \end{aligned}$$

(ii)-(iv): The proofs are obvious. □

Remark 3.5.1. It is clear that, F is a generalized higher derivation of M if and only if $\psi_n^\alpha(a,b)=0$ for all $a,b\in M,\alpha\in\Gamma$ and $n\in\mathbf{N}$.

Lemma 3.5.3. *If $F=(f_i)_{i\in N}$ is a Jordan generalized higher derivation with the associated higher derivation $D=(d_i)_{i\in N}$ of a Γ-ring M. Assume that $a,b\in M;\alpha,\beta,\gamma\in\Gamma$ and $n\in\boldsymbol{N}$. If $\psi_m^\alpha(a,b)=0$ for every $m<n$, then $\psi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha=0$, for every $w\in M$.*

Proof. Let $G=f_n(a\alpha b\beta w\gamma b\alpha a+b\alpha a\beta w\gamma a\alpha b)$. Using Lemma 3.5.1(ii) and Lemma 3.4.1(ii), we obtain

$$\begin{aligned}
G &= f_n(a\alpha(b\beta w\gamma b)\alpha a)+f_n(b\alpha(a\beta w\gamma a)\alpha b)\\
&= \sum_{i+p+l=n} f_i(a)\alpha d_p(b\beta w\gamma b)\alpha d_l(a)+\sum_{i+p+l=n} f_i(b)\alpha d_p(a\beta w\gamma a)\alpha d_l(b)\\
&= \sum_{i+j+k+h+l=n} f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a)+\sum_{i+j+k+h+l=n} f_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b)
\end{aligned}$$

On the other hand, using Lemma 3.5.1(iii), we get

$$\begin{aligned}
G &= f_n((a\alpha b)\beta w\gamma(b\alpha a)+(b\alpha a)\beta w\gamma(a\alpha b))\\
&= \sum_{r+s+t=n}(f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)+f_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b))\\
&= \sum_{r+s+t=n} f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)+\sum_{r+s+t=n} f_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b)
\end{aligned}$$

Comparing the two expressions for G, we have

$$\sum_{i+j+k+h+l=n} f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) - \sum_{r+s+t=n} f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)$$
$$= \sum_{r+s+t=n} f_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b) - \sum_{i+j+k+h+l=n} (f_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b). \tag{3.4}$$

Since $f_m(x\alpha y) = \sum_{i+j=m} f_i(x)\alpha d_j(y)$, when $m < n$ and $D = (d_i)_{i\in N}$ is a higher derivation of M. Therefore,

$$\sum_{i+j+k+h+l=n} f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) - \sum_{r+s+t=n} f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)$$
$$= (\sum_{i+j=n} f_i(a)\alpha d_j(b))\beta w\gamma b\alpha a + a\alpha b\beta w\gamma(\sum_{h+l=n} d_h(b)\alpha d_l(a))$$
$$+ \sum_{i+j+k+h+l=n}^{i+j<n,h+l<n} (f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a)) - f_n((a\alpha b)\beta w\gamma(b\alpha a)$$
$$- (a\alpha b)\beta w\gamma d_n(b\alpha a) - \sum_{r+s+t=n}^{i+j=r<n,p+q=t<n} (f_i(a)\alpha d_j(b)\beta d_s(w)\gamma d_p(b)\alpha d_q(a))$$
$$= -(f_n((a\alpha b) - \sum_{i+j=n} f_i(a)\alpha d_j(b))\beta(w\gamma b\alpha a) - (a\alpha b\beta w)\gamma(d_n(b\alpha a) - \sum_{h+l=n} d_h(b)\alpha d_l(a))$$
$$= -(\psi_n^\alpha(a,b)\beta w\gamma b\alpha a + a\alpha b\beta w\gamma\phi_n^\alpha(b,a)). \tag{3.5}$$

Similarly,

$$\sum_{i+j+k+h+l=n} (f_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b)) - \sum_{r+s+t=n} (d_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b))$$
$$= -(\psi_n^\alpha(b,a)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^\alpha(a,b)). \tag{3.6}$$

Hence, by using (3.5) and (3.6) in (3.4), we get

$$\psi_n^\alpha(a,b)\beta w\gamma b\alpha a + a\alpha b\beta w\gamma\phi_n^\alpha(b,a) + \psi_n^\alpha(b,a)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^\alpha(a,b) = 0.$$

By Lemma 3.5.2(i), we have

$$\psi_n^\alpha(a,b)\beta w\gamma b\alpha a \; - \; a\alpha b\beta w\gamma\phi_n^\alpha(a,b) \; - \; \psi_n^\alpha(a,b)\beta w\gamma a\alpha b \; + \; b\alpha a\beta w\gamma\phi_n^\alpha(a,b) \;\; = \;\; 0.$$

This implies,

$$\psi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha + [a,b]_\alpha\beta w\gamma\phi_n^\alpha(a,b) = 0, \forall\, w \in M.$$

Since $D = (d_i)_{i\in N}$ is a higher derivation of M. Thus, by Theorem 3.4.4, we have $\phi_n^\alpha(a,b) = 0$ for all $a, b \in M; \alpha \in \Gamma; n \in \mathbf{N}$ and hence accomplishes the proof. □

Lemma 3.5.4. *For all $a, b, c, d \in M$ and $\alpha, \beta, \gamma \in \Gamma$, $\psi_n^\alpha(a,b)\beta w\gamma[c,d]_\alpha = 0$.*

Lemma 3.5.5. *If $a, b, c, d \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ are any elements, then $\psi_n^\alpha(a,b)\beta w\gamma[c,d]_\delta = 0$.*

The proofs of the above two lemmas are similar to the proof of Lemma 2.19 and Lemma 2.20 in [12].

Lemma 3.5.6. *Let M be a commutative Γ-ring, and let $F = (f_i)_{i\in N}$ be a Jordan generalized higher derivation of M with an associated higher derivation $D = (d_i)_{i\in N}$. Assume that $a, b \in M; \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$. If $\psi_m^\alpha(a,b) = 0$ for all $m < n$, then $\psi_n^\beta(a,b)\alpha a = 0$.*

Proof. Since M is a commutative Γ-ring and $D = (d_i)_{i\in N}$ is a higher derivation of M. Let $W = f_n(a\alpha a\beta b + b\beta a\alpha a) = f_n(a\alpha(a\beta b) + (a\beta b)\alpha a)$. Then using Lemma 3.5.1(i), we have

$$W = \sum_{i+j=n} (f_i(a)\alpha d_j(a\beta b) + f_i(a\beta b)\alpha d_j(a))$$

$$= \sum_{i+k+l=n} (f_i(a)\alpha d_k(a)\beta d_l(b)) + \sum_{i+j=n} (f_i(a\beta b)\alpha d_j(a)).$$

On the other hand, using Lemma 3.5.1(ii) and commutativity of M, we get

$$W = f_n(a\alpha a\beta b + b\beta a\alpha a) = f_n(a\alpha b\beta a) + f_n(a\beta b\alpha a)$$
$$= \sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(a) + \sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(a)$$
$$= \sum_{i+j+k=n} f_i(a)\alpha d_j(a)\beta d_k(b) + \sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(a).$$

Comparing both expressions for W and cancelling the like terms from both sides, we obtain

$$\sum_{i+j=n} f_i(a\beta b)\alpha d_j(a) = \sum_{i+j+k=n} f_i(a)\beta d_j(b)\alpha d_k(a).$$

By assumption, we can put $f_m(a\beta b) = \sum_{i+j=m} f_i(a)\beta d_j(b)$, when $m < n$. Therefore,

$$\sum_{i+j+k=n}^{i+j<n} f_i(a)\beta d_j(b)\alpha d_k(a) + f_n(a\beta b)\alpha a - (\sum_{i+j=n} f_i(a)\beta d_j(b))\alpha a - \sum_{i+j=n}^{r+s=i<n} f_r(a)\beta d_s(b)\alpha d_j(a).$$
$$\Rightarrow (f_n(a\beta b) - \sum_{i+j=n} f_i(a)\beta d_j(b))\alpha a = 0.$$
$$\Rightarrow \psi_n^{\beta}(a,b)\alpha a = 0, \forall\, a, b \in M; \alpha, \beta \in \Gamma; n \in \mathbf{N}.$$

□

We are now concluding this chapter by proving our main result of this section as follows.

Theorem 3.5.7. *Every Jordan generalized higher derivation of a prime Γ-ring is a generalized higher derivation.*

Proof. Let $F = (f_i)_{i\in N}$ be a Jordan generalized higher derivation of a prime Γ-ring M. By definition $\psi_0^{\alpha}(a,b) = 0$ for all $a, b \in M$ and $\alpha \in \Gamma$. Also, by Theorem 2.4 in [11], we have

$$\psi_1^{\alpha}(a,b) = 0,\ \forall\, a, b \in M; \alpha \in \Gamma.$$

Now, we proceed by induction. Suppose that $\psi^{\alpha}_{m}(a,b) = 0$, this implies, $d_m(a\alpha b) = \sum_{i+j=m} f_i(a)\alpha d_j(b)$, for all $a, b \in M$ and $\alpha \in \Gamma$ and $m < n$. Taking $a, b, c, d \in M$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. By Lemma 3.5.5, we get $\psi^{\alpha}_{n}(a,b)\beta w\gamma[c,d]_{\delta} = 0$, for every $w \in M$. Since M is prime, $\psi^{\alpha}_{n}(a,b) = 0$, or $[c,d]_{\delta} = 0$. If $[c,d]_{\delta} \neq 0$, then $\psi^{\alpha}_{n}(a,b) = 0$. If $[c,d]_{\delta} = 0$, that is, $c\delta d - d\delta c = 0, \forall\ c, d \in M; \delta \in \Gamma$, then M is commutative. Now, by Lemma 3.5.6 and commutativity of M, we obtain

$$\psi^{\beta}_{n}(a,b)\alpha a = 0.$$

By linearizing of the above expression with respect to a yields

$$\begin{aligned}
&\psi^{\beta}_{n}(a+c,b)\alpha(a+c) = 0.\\
&\Rightarrow \psi^{\beta}_{n}(a,b)\alpha a + \psi^{\beta}_{n}(c,b)\alpha a + \psi^{\beta}_{n}(a,b)\alpha c + \psi^{\beta}_{n}(c,b)\alpha c = 0.\\
&\Rightarrow \psi^{\beta}_{n}(c,b)\alpha a + \psi^{\beta}_{n}(a,b)\alpha c = 0.\\
&\Rightarrow \psi^{\beta}_{n}(a,b)\alpha c = -\psi^{\beta}_{n}(c,b)\alpha a, \forall\ a, b, c \in M; \alpha, \beta \in \Gamma.
\end{aligned}$$

Since M is a commutative. Thus, for all $m \in M; \alpha, \gamma, \delta \in \Gamma$, we have

$$\begin{aligned}
(\psi^{\beta}_{n}(a,b)\alpha c)\gamma m\delta(\psi^{\beta}_{n}(a,b)\alpha c) &= -(\psi^{\beta}_{n}(c,b)\alpha a)\gamma m\delta(\psi^{\beta}_{n}(a,b)\alpha c)\\
&= -(\psi^{\beta}_{n}(c,b)\alpha c)\gamma m\delta(\psi^{\beta}_{n}(a,b)\alpha a)\\
&= 0.
\end{aligned}$$

Since M is prime, we have $\psi^{\beta}_{n}(a,b)\alpha c = 0$. Therefore, $\psi^{\beta}_{n}(a,b)\alpha c\gamma\psi^{\beta}_{n}(a,b) = 0$, for $c \in M$ and $\gamma \in \Gamma$. By the primeness of M, we get $\psi^{\beta}_{n}(a,b) = 0, \forall\ a, b \in M; \beta \in \Gamma$. $\square$

Chapter 4

Derivations on Lie Ideals

We start the discussion with the introductory definitions of derivation and Jordan derivation on Lie ideals in Γ-rings. Then we introduce the concepts of higher derivation and Jordan higher derivation on Lie ideals in Γ-rings.

In the next, we develop some consequences relating to the concept of Jordan derivations on Lie ideals of Γ-rings. Then we prove that if $d : M \to M$ is a Jordan derivation on an admissible Lie ideal U of a 2-torsion free prime Γ-ring M, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u, v \in U; \alpha \in \Gamma$, and if U is a commutative Lie ideal of M such that $u\alpha u \in U$ for all $u \in U; \alpha \in \Gamma$, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u, v \in U$ and $\alpha \in \Gamma$.

In the third section of this chapter, we prove the analogous results corresponding to the above mentioned results with the same conditions considering Jordan higher derivations on Lie ideals of prime Γ-rings instead of Jordan derivations on Lie ideals of prime Γ-rings almost similar way.

4.1 Introduction

I. N. Herstein [19] proved a well-known result in prime rings that every Jordan derivation is a derivation. Afterwards many Mathematicians studied extensively the derivations in prime rings. Awtar [3] extended this result in Lie ideals.

We introduce the concept of derivation and Jordan derivation on Lie ideals of Γ-rings to extend the above mentioned results in the following way.

Definition 4.1.1. If U is a Lie ideal of a Γ-ring M. An additive mapping $d : M \to M$ is said to be a *derivation on a Lie ideal* U if

$$d(u\alpha v) = d(u)\alpha v + u\alpha d(v), \forall\; u, v \in U; \alpha \in \Gamma,$$

and $d : M \to M$ is said to be a *Jordan derivation on a Lie ideal* U if

$$d(u\alpha u) = d(u)\alpha u + u\alpha d(u), \forall\; u \in U; \alpha \in \Gamma.$$

We now give examples of a Jordan derivation and a derivation on a Lie ideal U of a Γ-ring M, where M satisfies $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M; \alpha, \beta \in \Gamma$.

Example 4.1.1. *Suppose $a \in M$ and $\alpha \in \Gamma$ are fixed elements. Define $d : M \to M$ by $d(x) = a\alpha x - x\alpha a \forall\; x \in U$. Then for all $y \in U$ and $\beta \in \Gamma$,*

$$\begin{aligned}
d(y\beta y) &= a\alpha(y\beta y) - (y\beta y)\alpha a \\
&= a\alpha y\beta y - y\alpha a\beta y + y\alpha a\beta y - y\beta y\alpha a \\
&= (a\alpha y - y\alpha a)\beta y + y\beta a\alpha y - y\beta y\alpha a \\
&= (a\alpha y - y\alpha a)\beta y + y\beta(a\alpha y - y\alpha a) \\
&= d(y)\beta y + y\beta d(y), \forall\; y \in U; \beta \in \Gamma.
\end{aligned}$$

Therefore, d is a Jordan derivation on U. Again for all $x, y \in U$ and $\beta \in \Gamma$, we have

$$\begin{aligned}
d(x\beta y) &= a\alpha(x\beta y) - (x\beta y)\alpha a \\
&= a\alpha x\beta y - x\alpha a\beta y + x\alpha a\beta y - x\beta y\alpha a \\
&= (a\alpha x - x\alpha a)\beta y + x\beta a\alpha y - x\beta y\alpha a \\
&= (a\alpha x - x\alpha a)\beta y + x\beta(a\alpha y - y\alpha a) \\
&= d(x)\beta y + x\beta d(y), \forall\, x, y \in U; \beta \in \Gamma.
\end{aligned}$$

Hence d is a derivation on U of M.

We now give an example of a Jordan derivation on a Lie ideal of a Γ-ring which is not a derivation on a Lie ideal of a Γ-ring.

Example 4.1.2. *If U is a Lie ideal of a Γ-ring M, and Let $d : M \to M$ be a derivation on U. Suppose $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. Define addition and multiplication on M_1 by*

$$(x, x) + (y, y) = (x + y, x + y) \text{ and } (x, x)(\alpha, \alpha)(y, y) = (x\alpha y, x\alpha y)$$

then M_1 is a Γ_1-ring. Define $U_1 = \{(u, u) : u \in U\}$, for $u\alpha x - x\alpha u \in U$,

$$\begin{aligned}
(u, u)(\alpha, \alpha)(x, x) - (x, x)(\alpha, \alpha)(u, u) &= (u\alpha x, u\alpha x) - (x\alpha u, x\alpha u) \\
&= (u\alpha x - x\alpha u, u\alpha x - x\alpha u) \in U_1
\end{aligned}$$

Hence U_1 is a Lie ideal of M_1. Define a mapping $D : M_1 \to M_1$ by $D((x, x)) = (d(x), d(x))$. Then it is clear that D is an additive mapping on U_1 of M_1. For all

$(u,u) \in U_1; (\alpha,\alpha) \in \Gamma_1$, we get

$$\begin{aligned}
D((u,u)(\alpha,\alpha)(u,u)) &= D((u\alpha u, u\alpha u)) \\
&= (d(u\alpha u), d(u\alpha u)) \\
&= (d(u)\alpha u + u\alpha d(u), d(u)\alpha u + u\alpha d(u)) \\
&= (d(u)\alpha u, d(u)\alpha u) + (u\alpha d(u), u\alpha d(u)) \\
&= (d(u), d(u))(\alpha,\alpha)(u,u) + (u,u)(\alpha,\alpha)(d(u), d(u)) \\
&= D((u,u))(\alpha,\alpha)(u,u) + (u,u)(\alpha,\alpha)D((u,u)).
\end{aligned}$$

Therefore, D is a Jordan derivation on U_1 of M_1 which is not a derivation on U_1 of M_1.

Now, we introduce the concepts of higher derivation and Jordan higher derivation on Lie ideals in Γ-rings as follows.

Definition 4.1.2. If $D = (d_i)_{i \in \mathbf{N}_0}$ is a family of additive mappings on a Lie ideal U of a Γ-ring M such that $d_0 = id_M$, where id_M is an identity mapping on M and $\mathbf{N}_0$ denotes the set of natural numbers including 0. Then D is called a *higher derivation on a Lie ideal* if for each $n \in \mathbf{N}_0$ and $i, j \in \mathbf{N}_0$,

$$d_n(a\alpha b) = \sum_{i+j=n} d_i(a)\alpha d_j(b), \forall\ a, b \in U; \alpha \in \Gamma,$$

and D is called a *Jordan higher derivation on a Lie ideal* if

$$d_n(a\alpha a) = \sum_{i+j=n} d_i(a)\alpha d_j(a), \forall\ a \in U; \alpha \in \Gamma.$$

Example 4.1.3. *Let R be a commutative ring with characteristic 2 having unity element 1. Consider $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ n.1 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$, then M is a Γ-ring. Let $N = \{(x,x) : x \in R\}$, then N is a Lie ideal of M. Let $d_n : R \to R$ be a* *higher derivation for* $n \in \boldsymbol{N}_0$. *Define* $d_n : M \to M$ *by* $D_n((a,b)) = (d_n(a), d_n(b))$. *Then for each* $(a,a),(b,b) \in N$ *and* $\begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$, *we have*

$$\begin{aligned}
D_n((a,a)\begin{pmatrix} n \\ n \end{pmatrix}(b,b)) &= D_n((an+an)(b,b)) \\
&= D_n(anb+anb, anb+anb) \\
&= (d_n(anb+anb), d_n(anb+anb)) \\
&= (\sum_{i+j=n}(d_i(a)nd_j(b)+d_i(a)nd_j(b)), \sum_{i+j=n}(d_i(a)nd_j(b)+d_i(a)nd_j(b))) \\
&= \sum_{i+j=n}(d_i(a), d_i(a))\begin{pmatrix} n \\ n \end{pmatrix}(d_j(b), d_j(b)) \\
&= \sum_{i+j=n} D_i((a,a))\begin{pmatrix} n \\ n \end{pmatrix} D_j((b,b))
\end{aligned}$$

Therefore, D_n *is a higher derivation on a Lie ideal* N *of* M.

Example 4.1.4. *Suppose* U *is a Lie ideal of a* Γ*-ring* M. *Let* $d_n : M \to M$ *be a higher derivation on* U *of* M, *then for each* $n \in \boldsymbol{N}_0$,

$$d_n(u\alpha v) = \sum_{i+j=n} d_i(u)\alpha d_j(v),\ \forall\, u,v \in U;\ \alpha \in \Gamma.$$

If we consider $M_1 = \{(x,x) : x \in M\}$ *and* $\Gamma_1 = \{(\alpha,\alpha) : \alpha \in \Gamma\}$. *Define addition and multiplication on* M_1 *by*

$$(x,x)+(y,y) = (x+y, x+y) \text{ and } (x,x)(\alpha,\alpha)(y,y) = (x\alpha y, x\alpha y)$$

then M_1 is a Γ_1-ring. Also, define $U_1 = \{(u,u) : u \in U\}$, then for $u\alpha x - x\alpha u \in U$,

$$\begin{aligned}(u,u)(\alpha,\alpha)(x,x) - (x,x)(\alpha,\alpha)(u,u) &= (u\alpha x, u\alpha x) - (x\alpha u, x\alpha u) \\ &= (u\alpha x - x\alpha u, u\alpha x - x\alpha u) \in U_1.\end{aligned}$$

Hence U_1 is a Lie ideal of M_1. Now, define a mapping $D_n : M_1 \to M_1$ by $D_n((x,x)) = (d_n(x), d_n(x))$. Then for all $(u,u) \in U_1$ and $(\alpha,\alpha) \in \Gamma_1$, we have

$$\begin{aligned}D_n((u,u)(\alpha,\alpha)(u,u)) &= D_n(u\alpha u, u\alpha u) \\ &= (d_n(u\alpha u), d_n(u\alpha u)) \\ &= (\sum_{i+j=n} d_i(u)\alpha d_j(u), \sum_{i+j=n} d_i(u)\alpha d_j(u)) \\ &= \sum_{i+j=n} (d_i(u), d_i(u))(\alpha,\alpha)(d_j(u), d_j(u)) \\ &= \sum_{i+j=n} D_i((u,u))(\alpha,\alpha)D_j((u,u)).\end{aligned}$$

Therefore, D_n is a Jordan higher derivation on a Lie ideal U_1 of M_1. Also, we have seen that it is not a higher derivation on a Lie ideal U_1 of M_1.

Throughout this chapter (until otherwise stated), M is a 2-torsion free Γ-ring satisfying the condition (*), $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$; $\alpha, \beta \in \Gamma$, and U is a Lie ideal of M.

4.2 Jordan Derivations on Lie Ideals of Prime Γ-Rings

Our purpose in this section is to prove our main results stated at the beginning of this chapter. In order to prove these results we have to determine some essential Lemmas as bellow.

Lemma 4.2.1. *Let U be a Lie ideal of a Γ-ring M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If d is a Jordan derivation on U of M, then for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$, the following statements hold:*
(i)$d(u\alpha v + v\alpha u) = d(u)\alpha v + d(v)\alpha u + u\alpha d(v) + v\alpha d(u)$;
(ii)$d(u\alpha v\beta u + u\beta v\alpha u) = d(u)\alpha v\beta u + d(u)\beta v\alpha u + u\alpha d(v)\beta u + u\beta d(v)\alpha u + u\alpha v\beta d(u) + u\beta v\alpha d(u)$.
In particular, if M is 2-torsion free and satisfies the condition (*), then
(iii) $d(u\alpha v\beta u) = d(u)\alpha v\beta u + u\alpha d(v)\beta u + u\alpha v\beta d(u)$;
(iv) $d(u\alpha v\beta w + w\alpha v\beta u) = d(u)\alpha v\beta w + d(w)\alpha v\beta u + u\alpha d(v)\beta w + w\alpha d(v)\beta u + u\alpha v\beta d(w) + w\alpha v\beta d(u)$.

Proof. (i) Since U is a Lie ideal satisfying the condition $u\alpha u \in U$ for all $u \in U; \alpha \in \Gamma$. For $u, v \in U; \alpha \in \Gamma$, we have $(u\alpha v + v\alpha u) = (u+v)\alpha(u+v) - (u\alpha u + v\alpha v)$ this implies $(u\alpha v + v\alpha u) \in U$. Also, $[u, v]_\alpha = u\alpha v - v\alpha u \in U$ and it follows that $2u\alpha v \in U$. Hence $4u\alpha v\beta w = 2(2u\alpha v)\beta w \in U$ for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.

Therefore,

$$\begin{aligned}
d(u\alpha v + v\alpha u) &= d((u+v)\alpha(u+v) - (u\alpha u + v\alpha v)) \\
&= d(u+v)\alpha(u+v) + (u+v)\alpha d(u+v) - d(u)\alpha u - u\alpha d(u) - d(v)\alpha v - v\alpha d(v) \\
&= d(u)\alpha u + d(u)\alpha v + d(v)\alpha u + d(v)\alpha v + u\alpha d(u) + u\alpha d(v) + v\alpha d(u) + v\alpha d(v) \\
&- d(u)\alpha u - u\alpha d(u) - d(v)\alpha v - v\alpha d(v) \\
&= d(u)\alpha v + u\alpha d(v) + d(v)\alpha u + v\alpha d(u).
\end{aligned}$$

(ii) Replacing $u\beta v + v\beta u$ for v in (i) and using this, we get

$$\begin{aligned}
d(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u) &= d(u)\alpha(u\beta v + v\beta u) + u\alpha d(u\beta v + v\beta u) \\
&+ d(u\beta v + v\beta u)\alpha u + (u\beta v + v\beta u)\alpha d(u).
\end{aligned}$$

$$\Rightarrow d((u\alpha u)\beta v + v\beta(u\alpha u)) + d(u\alpha v\beta u + u\beta v\alpha u) = d(u)\alpha(u\beta v + v\beta u) + u\alpha(d(u)\beta v + u\beta d(v)$$
$$+ d(v)\beta u + v\beta d(u)) + (d(u)\beta v + u\beta d(v) + d(v)\beta u + v\beta d(u))\alpha u + u\beta v\alpha d(u) + v\beta u\alpha d(u).$$

This implies,

$$\begin{aligned}
&d(u\alpha u)\beta v + (u\alpha u)\beta d(v) + d(v)\beta(u\alpha u) + v\beta d(u\alpha u) + d(u\alpha v\beta u + u\beta v\alpha u) \\
&= d(u)\alpha u\beta v + d(u)\alpha v\beta u + u\alpha d(u)\beta v + u\alpha u\beta d(v) + u\alpha d(v)\beta u + u\alpha v\beta d(u) \\
&+ d(u)\beta v\alpha u + u\beta d(v)\alpha u + d(v)\beta u\alpha u + v\beta d(u)\alpha u + u\beta v\alpha d(u) + v\beta u\alpha d(u).
\end{aligned}$$

This implies,

$$\begin{aligned}
&d(u)\alpha u\beta v + u\alpha d(u)\beta v + u\alpha u\beta d(v) + d(v)\beta u\alpha u + v\beta d(u)\alpha u + v\beta u\alpha d(u) + d(u\alpha v\beta u + u\beta v\alpha u) \\
&= d(u)\alpha u\beta v + d(u)\alpha v\beta u + u\alpha d(u)\beta v + u\alpha u\beta d(v) + u\alpha d(v)\beta u + u\alpha v\beta d(u) + d(u)\beta v\alpha u \\
&+ u\beta d(v)\alpha u + d(v)\beta u\alpha u + v\beta d(u)\alpha u + u\beta v\alpha d(u) + v\beta u\alpha d(u).
\end{aligned}$$

Now, cancelling the like terms from both sides we get the required result. Using the condition (*) and since M is 2-torsion free, (iii) follows from (ii). And finally (iv) is obtained by replacing $u+w$ for u in (iii). □

Definition 4.2.1. Let U be a Lie ideal of a Γ-ring M, and d be a Jordan derivation on U of M. We define $\phi_\alpha(u,v) = d(u\alpha v) - d(u)\alpha v - u\alpha d(v)$ for all $u,v \in U$ and $\alpha \in \Gamma$.

Remark 4.2.1. d is a derivation on U of M if and only if $\phi_\alpha(u,v) = 0$ for all $u,v \in U$ and $\alpha \in \Gamma$.

As immediate consequences emanated from the definition of $\phi_\alpha(u,v)$, we have

Lemma 4.2.2. *If d is a Jordan derivation on U of M, then for all $u,v,w \in U$ and $\alpha,\beta \in \Gamma$, (i) $\phi_\alpha(u,v) = -\phi_\alpha(v,u)$; (ii) $\phi_\alpha(u+w,v) = \phi_\alpha(u,v) + \phi_\alpha(w,v)$; (iii) $\phi_\alpha(u,v+w) = \phi_\alpha(u,v) + \phi_\alpha(u,w)$; (iv) $\phi_{\alpha+\beta}(u,v) = \phi_\alpha(u,v) + \phi_\beta(u,v)$.*

Proof. The proof of this lemma is immediate from Lemma 2.2.2 if in the statement of the lemma we replace G_α by ϕ_α and $a,b,c \in M$ by $u,v,w \in U$. □

Lemma 4.2.3. *Let U be an admissible Lie ideal of M. If d is a Jordan derivation on U of M, then $\phi_\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\phi_\alpha(u,v) = 0$, for all $u,v,w \in U$ and $\alpha,\beta,\gamma \in \Gamma$.*

Proof. U is an admissible Lie ideal of M. Thus, for all $u \in U; \alpha \in \Gamma, u\alpha u \in U$. If $u,v \in U$ and $\alpha \in \Gamma$, then $(u\alpha v + v\alpha u) = (u+v)\alpha(u+v) - (u\alpha u + v\alpha v)$ this implies, $(u\alpha v + v\alpha u) \in U$. In addition, $[u,v]_\alpha = u\alpha v - v\alpha u \in U$ and therefore $2u\alpha v \in U$ and $4u\alpha v\beta w = 2(2u\alpha v)\beta w \in U$, for all $u,v,w \in U, \alpha,\beta \in \Gamma$. Now, let

$x = 4(u\alpha v\beta w\gamma v\alpha u + v\alpha u\beta w\gamma u\alpha v)$. Then using Lemma 4.2.1(iv), we have

$$d(x) = d((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v))$$

$$= d(2u\alpha v)\beta w\gamma(2v\alpha u) + 2u\alpha v\beta d(w)\gamma 2v\alpha u + 2u\alpha v\beta w\gamma d(2v\alpha u) + d(2v\alpha u)\beta w\gamma(2u\alpha v)$$
$$+ 2v\alpha u\beta d(w)\gamma 2u\alpha v + 2v\alpha u\beta w\gamma d(2u\alpha v).$$

On the other hand, using Lemma 4.2.1(iii), we have

$$d(x) = d(u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v)$$
$$= d(u)\alpha 4v\beta w\gamma v\alpha u + u\alpha d(4v\beta w\gamma v)\alpha u + u\alpha 4v\beta w\gamma v\alpha d(u) + d(v)\alpha 4u\beta w\gamma u\alpha v$$
$$+ v\alpha d(4u\beta w\gamma u)\alpha v + v\alpha 4u\beta w\gamma u\alpha d(v)$$
$$= 4d(u)\alpha v\beta w\gamma v\alpha u + 4u\alpha d(v)\beta w\gamma v\alpha u + 4u\alpha v\beta d(w)\gamma v\alpha u + 4u\alpha v\beta w\gamma d(v)\alpha u$$
$$+ 4u\alpha v\beta w\gamma v\alpha d(u) + 4d(v)\alpha u\beta w\gamma u\alpha v + 4v\alpha d(u)\beta w\gamma u\alpha v + 4v\alpha u\beta d(w)\gamma u\alpha v$$
$$+ 4v\alpha u\beta w\gamma d(u)\alpha v + 4v\alpha u\beta w\gamma u\alpha d(v).$$

Comparing the two right sides of $d(x)$, we obtain

$$4(d(u\alpha v)\beta w\gamma v\alpha u + d(v\alpha u)\beta w\gamma u\alpha v + u\alpha v\beta w\gamma d(v\alpha u) + v\alpha u\beta w\gamma d(u\alpha v))$$
$$= 4(d(u)\alpha v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + d(v)\alpha u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v$$
$$+ u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma v\alpha d(u) + v\alpha u\beta w\gamma d(u)\alpha v + v\alpha u\beta w\gamma u\alpha d(v)).$$

This yields,

$$4((d(u\alpha v) - d(u)\alpha v - u\alpha d(v))\beta w\gamma v\alpha u + (d(v\alpha u) - d(v)\alpha u - v\alpha d(u))\beta w\gamma u\alpha v +$$
$$u\alpha v\beta w\gamma(d(v\alpha u) - d(v)\alpha u - v\alpha d(u)) + v\alpha u\beta w\gamma(d(u\alpha v) - d(u)\alpha v - u\alpha d(v))) = 0.$$

Using the Definition 4.2.1, this implies

$$4(\phi_\alpha(u,v)\beta w\gamma v\alpha u + \phi_\alpha(v,u)\beta w\gamma u\alpha v + u\alpha v\beta w\gamma\phi_\alpha(v,u) + v\alpha u\beta w\gamma\phi_\alpha(u,v)) = 0.$$

Using Lemma 4.2.2(i), we get

$$4(\phi_\alpha(u,v)\beta w\gamma v\alpha u - \phi_\alpha(u,v)\beta w\gamma u\alpha v - u\alpha v\beta w\gamma\phi_\alpha(u,v) + v\alpha u\beta w\gamma\phi_\alpha(u,v)) = 0.$$

$\Rightarrow -4(\phi_\alpha(u,v)\beta w\gamma(u\alpha v - v\alpha u) + (u\alpha v - v\alpha u)\beta w\gamma\phi_\alpha(u,v)) = 0.$

$\Rightarrow 4(\phi_\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\phi_\alpha(u,v)) = 0.$

Since M is 2-torsion free, we get $\phi_\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\phi_\alpha(u,v) = 0$, for all $u,v,w \in U; \alpha,\beta,\gamma \in \Gamma$. □

In the rest of this section, M represents a prime Γ-ring.

Lemma 4.2.4. *([18], Lemma 1) Assume that U is a Lie ideal of M with $U \nsubseteq Z(M)$. Then there exists an ideal I of M such that $[I,M]_\Gamma \subseteq U$ but $[I,M]_\Gamma$ is not contained in $Z(M)$.*

Proof. Since M is 2-torsion free and U is not contained in $Z(M)$, it follows from the result in [1] that $[U,U]_\Gamma \neq 0$ and $[I,M]_\Gamma \subseteq U$, where $I = I\Gamma[U,U]_\Gamma\Gamma M \neq 0$ is an ideal of M generated by $[U,U]_\Gamma$. Now, U is not contained in $Z(M)$ implies that $[I,M]_\Gamma$ is not contained in $Z(M)$; for if $[I,M]_\Gamma \subseteq Z(M)$, then $[I,[I,M]_\Gamma]_\Gamma = 0$, which implies that $I \subseteq Z(M)$ and hence $I \neq 0$ is an ideal of M, so $M = Z(M)$. □

Lemma 4.2.5. *([18], Lemma 2) Let U be a Lie ideal of M such that $U \nsubseteq Z(M)$. If $a,b \in M$ (resp.$b \in U$ and $a \in M$) and $a\alpha U\beta b = 0$, for all $\alpha,\beta \in \Gamma$, then $a = 0$ or $b = 0$.*

Proof. By Lemma 4.2.4, there exists an ideal I of M such that $[I,M]_\Gamma \subseteq U$ and $[I,M]_\Gamma$ is not contained in $Z(M)$. Now take $u \in U, c \in I, m \in M$ and $\alpha,\beta,\gamma \in \Gamma$, we have $[c\alpha a\beta u, m]_\Gamma \in [I,M]_\Gamma \subseteq U$ and so

$0 = a\delta[c\alpha a\beta u, m]_\gamma\mu b$, for all $\delta,\mu \in \Gamma$.

$= a\delta[c\alpha a, m]_\gamma \beta u\mu b + a\delta c\alpha a\beta[u, m]_\gamma \mu b$, by using (*)

$= a\delta[c\alpha a, m]_\gamma \beta u\mu b$, since $a\beta[u, m]_\gamma \mu b \in a\beta U\mu b$

$= a\delta(c\alpha a\gamma m - m\gamma c\alpha a)\beta u\mu b$

$= a\delta c\alpha a\gamma m\beta u\mu b - a\delta m\gamma c\alpha a\beta u\mu b$

$= a\delta c\alpha a\gamma m\beta u\mu b$, by using assumption $a\beta u\mu b = 0$. Thus $a\delta I\alpha a\gamma M\beta U\mu b = 0$.

If $a \neq 0$, then by the primeness of $M, U\mu b = 0$. Now, if $u \in U$ and $m \in M$, then $[u, m]_\alpha \in U$ for all $\alpha \in \Gamma$. Hence $[u, m]_\alpha \beta b = 0$ for all $\beta \in \Gamma$. Since $m\alpha u\beta b = 0$, so $u\alpha m\beta b = 0$. But $U \neq 0$, we must have $b = 0$. Similarly, it can be shown that if $b \neq 0$, then $a = 0$. □

Lemma 4.2.6. *Assume that U is an admissible Lie ideal of M. If $a, b \in M$ (resp. $a \in M$ and $b \in U$) such that $a\alpha x\beta b + b\alpha x\beta a = 0$ for all $x \in U$ and $\alpha, \beta \in \Gamma$, then $a\alpha x\beta b = b\alpha x\beta a = 0$.*

Proof. For $x, y \in U$ and using the relation $a\alpha x\beta b = -b\alpha x\beta a$ three times, we obtain

$$\begin{aligned} 4a\alpha x\beta b\gamma y\delta a\alpha x\beta b &= -4b\alpha x\beta a\gamma y\delta a\alpha x\beta b = -b\alpha(4x\beta a\gamma y)\delta a\alpha x\beta b \\ &= a\alpha(4x\beta a\gamma y)\delta b\alpha x\beta b = 4a\alpha x\beta(a\gamma y\delta b)\alpha x\beta b = -4a\alpha x\beta b\gamma y\delta a\alpha x\beta b. \end{aligned}$$

Therefore,

$$8a\alpha x\beta b\gamma y\delta a\alpha x\beta b = 0.$$

By the 2-torsion freeness of M, $(a\alpha x\beta b)\gamma y\delta(a\alpha x\beta b) = 0$. By Lemma 4.2.5, we get $a\alpha x\beta b = 0$. Similarly, it can be shown that $b\alpha x\beta a = 0$. □

Lemma 4.2.7. *Let U be an admissible Lie ideal of M ; $G_1, G_2, ..., G_n$ be additive groups; $S : G_1 \times G_2 \times ... \times G_n \to M$ and $T : G_1 \times G_2 \times ... \times G_n \to M$ be mappings which are additive in each argument. If $S_\alpha(a_1, ..., a_n)\beta x\gamma T_\alpha(a_1, ..., a_n) = 0$ for every*

$x \in U; a_i \in G_i, i = 1, 2, ..., n; \alpha, \beta, \gamma \in \Gamma$, then $S_\alpha(a_1, ..., a_n)\beta x\gamma T_\delta(b_1, ..., b_n) = 0$ for every $b_i \in G_i, i = 1, 2, ..., n$.

Proof. It suffices to prove the case $n = 1$. The general proof is obtained by induction on n. If $S_\alpha(a)\beta x\gamma T_\alpha(a) = 0$ for every $u \in U, a \in G_1$, we get

$$(T_\alpha(a)\beta x\gamma S_\alpha(a))\mu y\nu(T_\alpha(a)\beta x\gamma S_\alpha(a)) = 0, \forall\, x, y \in U, \mu, \nu \in \Gamma.$$

Then by Lemma 4.2.5, $T_\alpha(a)\beta x\gamma S_\alpha(a) = 0$ for every $x \in U, a \in G_1$ and $\alpha, \beta, \gamma \in \Gamma$. Now, linearizing $S_\alpha(a)\beta x\gamma T_\alpha(a) = 0$, we obtain

$$S_\alpha(a)\beta x\gamma T_\alpha(b) + S_\alpha(b)\beta x\gamma T_\alpha(a) = 0, \forall\, x \in U, a, b \in G_1.$$

So, for all $x, y \in U$,

$$(S_\alpha(a)\beta x\gamma T_\alpha(b))\mu y\nu((S_\alpha(a)\beta x\gamma T_\alpha(b)) = -S_\alpha(a)\beta x\gamma T_\alpha(b)\mu y\nu S_\alpha(b)\beta x\gamma T_\alpha(a) = 0.$$

Thus, by Lemma 4.2.5, $S_\alpha(a)\beta x\gamma T_\alpha(b) = 0$. Similarly, we can prove that $T_\alpha(b)\beta x\gamma S_\alpha(a) = 0$ for all $a, b \in G_1$ and $\alpha, \beta, \gamma \in \Gamma$. Putting $\alpha+\delta$ for α in the equation $S_\alpha(a)\beta x\gamma T_\alpha(b) = 0$ and using Lemma 4.2.2(iv) $S_\alpha(a)\beta x\gamma T_\delta(b) + S_\delta(a)\beta x\gamma T_\alpha(b) = 0$. Therefore, we have

$$(S_\alpha(a)\beta x\gamma T_\delta(b))\mu y\nu(S_\alpha(a)\beta x\gamma T_\delta(b)) = -S_\alpha(a)\beta x\gamma T_\delta(b)\mu y\nu S_\delta(a)\beta x\gamma T_\alpha(b) = 0.$$

Hence, by Lemma 4.2.5, $S_\alpha(a)\beta x\gamma T_\delta(b) = 0$. □

We have all ideas for the proof at hand now.

Theorem 4.2.8. *Assume that U is an admissible Lie ideal of M. If $d : M \to M$ is a Jordan derivation on U of M, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u, v \in U, \alpha \in \Gamma$.*

Proof. By Lemma 4.2.3, we have $\phi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + [u, v]_\alpha\beta w\gamma\phi_\alpha(u, v) = 0$ for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$. Using Lemmas 4.2.6 and 4.2.7, we have

$\phi_\alpha(u,v)\beta w\gamma[x,y]_\delta = 0$ for all $u,v,w,x,y \in U$ and $\alpha,\beta,\gamma,\delta \in \Gamma$. Since U is an admissible Lie ideal of M, so, $[x,y]_\delta \neq 0$. Hence by Lemma 4.2.5, $\phi_\alpha(u,v) = 0$. □

Theorem 4.2.9. *Let U be a commutative Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. Then every Jordan derivation on U of M is a derivation on U of M.*

Proof. Suppose U is a commutative Lie ideal of M. Then for every $u \in U, x \in M$ and $\alpha \in \Gamma$, we have $[u,[u,x]_\alpha]_\alpha = 0$. For every $z \in M$, we have $x\beta z \in M$ for every $\beta \in \Gamma$. Replacing x by $x\beta z$, we obtain

$$\begin{aligned}
0 &= [u,[u,x\beta z]_\alpha]_\alpha \\
&= [u, x\beta[u,z]_\alpha + [u,x]_\alpha\beta z]_\alpha \\
&= [u, x\beta[u,z]_\alpha]_\alpha + [u,[u,x]_\alpha\beta z]_\alpha \\
&= x\beta[u,[u,z]_\alpha]_\alpha + [u,x]_\alpha\beta[u,z]_\alpha + [u,[u,x]_\alpha]_\alpha\beta z + [u,x]_\alpha\beta[u,z]_\alpha \\
&= 2[u,x]_\alpha\beta[u,z]_\alpha.
\end{aligned}$$

By the 2-torsion freeness of M, we obtain $[u,x]_\alpha\beta[u,z]_\alpha = 0$. Now, replacing z by $z\gamma m$ for every $m \in M, \gamma \in \Gamma$, we obtain

$$\begin{aligned}
0 &= [u,x]_\alpha\beta[u,z\gamma m]_\alpha \\
&= [u,x]_\alpha\beta z\gamma[u,m]_\alpha + [u,x]_\alpha\beta[u,z]_\alpha\gamma m \\
&= [u,x]_\alpha\beta z\gamma[u,m]_\alpha.
\end{aligned}$$

Since M is prime, $[u,x]_\alpha = 0$, or $[u,m]_\alpha = 0$, for all $x,m \in M$. In either case, $U \subseteq Z(M)$. Hence by Lemma 4.2.1(i), we have $2d(a\alpha b) = 2(d(a)\alpha b + a\alpha d(b))$ for all $a,b \in U$ and $\alpha \in \Gamma$. By the 2-torsion freeness of M, we get $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ for all $a,b \in U$ and $\alpha \in \Gamma$. □

4.3 Jordan Higher Derivations on Lie Ideals of Prime Γ-Rings

Here, we determine a number of very significant consequences relating to the concept of Jordan higher derivations on Lie ideals of a prime Γ-ring to extend the results stated at the beginning of this chapter following [15, 17] in classical ring theory to Γ-ring theory.

Lemma 4.3.1. *Let U be a Lie ideal of a Γ-ring M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $D = (d_i)_{i\in N}$ is a Jordan higher derivation on U. Then for all $u, v, w \in U; \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$:*
(i) $d_n(u\alpha v + v\alpha u) = \sum_{i+j=n}[d_i(u)\alpha d_j(v) + d_i(v)\alpha d_j(u)]$
(ii) $d_n(u\alpha v\beta u) = \sum_{i+j+k=n}[d_i(u)\alpha d_j(v)\beta d_k(u)]$
(iii) $d_n(u\alpha v\beta w + w\alpha v\beta u) = \sum_{i+j+k=n}[d_i(u)\alpha d_j(v)\beta d_k(w) + d_i(w)\alpha d_j(v)\beta d_k(u)]$.

Proof. The proofs of (i) and (ii) are similar to the corresponding proofs of Lemma 4.2.1(i) and Lemma 4.2.1(iii). Replacing u by $u + w$ in (ii), we obtain $S = d_n((u + w)\alpha v\beta(u + w))$ and compute this, using (ii). It follows that

$$
\begin{aligned}
S &= d_n((u + w)\alpha v\beta(u + w)) \\
&= \sum_{i+j+k=n} d_i(u)\alpha d_j(v)\beta d_k(u) + \sum_{i+j+k=n} d_i(u)\alpha d_j(v)\beta d_k(w) + \sum_{i+j+k=n} d_i(w)\alpha d_j(v)\beta d_k(u) \\
&\quad + \sum_{i+j+k=n} d_i(w)\alpha d_j(v)\beta d_k(w).
\end{aligned}
$$

On the other hand

$$S = d_n(u\alpha v\beta u) + d_n(w\alpha v\beta w) + d_n(u\alpha v\beta w + w\alpha v\beta u)$$
$$= \sum_{i+j+k=n} d_i(u)\alpha d_j(v)\beta d_k(u) + \sum_{i+j+k=n} d_i(w)\alpha d_j(v)\beta d_k(w) + d_n(u\alpha v\beta w + w\alpha v\beta u).$$

By comparing the two results for S, we obtain (iii). □

Definition 4.3.1. If U is a Lie ideal of a Γ-ring M. For every Jordan higher derivation $D = (d_i)_{i\in\mathbf{N}}$ on U of M, we define $\phi_n^\alpha(u,v) = d_n(u\alpha v) - \sum_{i+j=n} d_i(u)\alpha d_j(v)$, for all $u, v \in U; \alpha \in \Gamma$ and $n \in \mathbf{N}$.

Remark 4.3.1. $\phi_n^\alpha(u,v) = 0$ for all $u, v \in U; \alpha \in \Gamma$ and $n \in \mathbf{N}$ if and only if D is a higher derivation on U of M.

The following results will be used in the next lemma.

Lemma 4.3.2. *For every $u, v, w \in U; \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$,*

(*i*) $\phi_n^\alpha(u,v) + \phi_n^\alpha(v,u) = 0$; (*ii*) $\phi_n^\alpha(u+v,w) = \phi_n^\alpha(u,w) + \phi_n^\alpha(v,w)$;
(*iii*) $\phi_n^\alpha(u,v+w) = \phi_n^\alpha(u,v) + \phi_n^\alpha(u,w)$; (*iv*) $\phi_n^{\alpha+\beta}(u,v) = \phi_n^\alpha(u,v) + \phi_n^\beta(u,v)$.

The proofs are obvious from the Definition 4.3.1

Lemma 4.3.3. *Let U be an admissible Lie ideal of a 2-torsion free Γ-ring M, and let $D = (d_i)_{i\in\boldsymbol{N}}$ be a Jordan higher derivation on U of M. Let $n \in \boldsymbol{N}$ and assume that $u, v \in U; \alpha, \beta, \gamma \in \Gamma$. If $\phi_m^\alpha(u,v) = 0$, for every $m < n$, then $\phi_n^\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\phi_n^\alpha(u,v) = 0$, for every $w \in U$.*

Proof. Since U is an admissible Lie ideal of M. Hence, for all $u, v \in U$ and $\alpha \in \Gamma$, $(u\alpha v + v\alpha u) = (u+v)\alpha(u+v) - (u\alpha u + v\alpha v)$ so, $(u\alpha v + v\alpha u) \in U$. Now $[u,v]_\alpha =$

$u\alpha v - v\alpha u \in U$, and therefore $2u\alpha v \in U$ and $4u\alpha v\beta w = 2(2u\alpha v)\beta w \in U$ for all $u, v, w \in U; \alpha, \beta \in \Gamma$. Let $A = d_n(4u\alpha v\beta w\gamma v\alpha u + 4v\alpha u\beta w\gamma u\alpha v)$. First, using Lemma 4.3.1(ii), we obtain

$$\begin{aligned}
A &= d_n(u\alpha(4v\beta w\gamma v)\alpha u) + d_n(v\alpha(4u\beta w\gamma u)\alpha v) \\
&= 4\sum_{i+p+l=n}(d_i(u)\alpha d_p(v\beta w\gamma v)\alpha d_l(u)) + 4\sum_{i+p+l=n}(d_i(v)\alpha d_p(u\beta w\gamma u)\alpha d_l(v)) \\
&= 4\sum_{i+j+k+h+l=n} d_i(u)\alpha d_j(v)\beta d_k(w)\gamma d_h(v)\alpha d_l(u) + 4\sum_{i+j+k+h+l=n} d_i(v)\alpha d_j(u)\beta d_k(w)\gamma d_h(u)\alpha d_l(v)
\end{aligned}$$

Now, using Lemma 4.3.1(iii), we get

$$\begin{aligned}
A &= d_n((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)) \\
&= \sum_{r+s+t=n}(d_r(2u\alpha v)\beta d_s(w)\gamma d_t(2v\alpha u) + d_r(2v\alpha u)\beta d_s(w)\gamma d_t(2u\alpha v)) \\
&= 4\sum_{r+s+t=n}(d_r(u\alpha v)\beta d_s(w)\gamma d_t(v\alpha u)) + 4\sum_{r+s+t=n}(d_r(v\alpha u)\beta d_s(w)\gamma d_t(u\alpha v)).
\end{aligned}$$

Comparing both the expressions for A, we obtain

$$\begin{aligned}
&\sum_{i+j+k+h+l=n}(d_i(u)\alpha d_j(v)\beta d_k(w)\gamma d_h(v)\alpha d_l(u)) - \sum_{r+s+t=n}(d_r(u\alpha v)\beta d_s(w)\gamma d_t(v\alpha u)) \\
&+ \sum_{i+j+k+h+l=n}(d_i(v)\alpha d_j(u)\beta d_k(w)\gamma d_h(u)\alpha d_l(v)) - \sum_{r+s+t=n}(d_r(v\alpha u)\beta d_s(w)\gamma d_t(u\alpha v)) = 0.
\end{aligned} \tag{4.1}$$

By the assumption, we have $d_m(x\alpha y) = \sum_{i+j=m} d_i(x)\alpha d_j(y)$, when $m < n$, for $x =$

u, v and $y = v, u$. Consequently,

$$\sum_{i+j+k+h+l=n} (d_i(u)\alpha d_j(v)\beta d_k(w)\gamma d_h(v)\alpha d_l(u)) - \sum_{r+s+t=n} (d_r(u\alpha v)\beta d_s(w)\gamma d_t(v\alpha u))$$

$$= (\sum_{i+j=n} d_i(u)\alpha d_j(v))\beta w\gamma v\alpha u + u\alpha v\beta w\gamma(\sum_{h+l=n} d_h(v)\alpha d_l(u))$$

$$+ \sum_{i+j+k+h+l=n}^{i+j<n, h+l<n} (d_i(u)\alpha d_j(v)\beta d_k(w)\gamma d_h(v)\alpha d_l(u)) - d_n((u\alpha v)\beta w\gamma(v\alpha u)$$

$$- (u\alpha v)\beta w\gamma d_n(v\alpha u) - \sum_{r+s+t=n}^{i+j=r<n, p+q=t<n} (d_i(u)\alpha d_j(v)\beta d_s(w)\gamma d_p(v)\alpha d_q(u))$$

$$= -(d_n((u\alpha v) - \sum_{i+j=n} d_i(u)\alpha d_j(v))\beta(w\gamma v\alpha u) - (u\alpha v\beta w)\gamma(d_n(v\alpha u) - \sum_{h+l=n} d_h(v)\alpha d_l(u))$$

$$= -(\phi_n^\alpha(u, v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma\phi_n^\alpha(v, u)). \quad (4.2)$$

Similarly,

$$\sum_{i+j+k+h+l=n} (d_i(v)\alpha d_j(u)\beta d_k(w)\gamma d_h(u)\alpha d_l(v)) - \sum_{r+s+t=n} (d_r(v\alpha u)\beta d_s(w)\gamma d_t(u\alpha v))$$

$$= -(\phi_n^\alpha(v, u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma\phi_n^\alpha(u, v)). \quad (4.3)$$

Hence, by using (4.2) and (4.3) in (4.1), we get

$$\phi_n^\alpha(u, v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma\phi_n^\alpha(v, u) + \phi_n^\alpha(v, u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma\phi_n^\alpha(u, v) = 0.$$

By Lemma 4.3.2(i), we have

$$\phi_n^\alpha(u, v)\beta w\gamma v\alpha u - u\alpha v\beta w\gamma\phi_n^\alpha(u, v) - \phi_n^\alpha(u, v)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma\phi_n^\alpha(u, v) = 0.$$

This implies,

$$\phi_n^\alpha(u, v)\beta w\gamma[u, v]_\alpha + [u, v]_\alpha\beta w\gamma\phi_n^\alpha(u, v) = 0, \forall\, w \in U.$$

$\square$

Now, we are ready to prove our main results.

Theorem 4.3.4. *Let U be an admissible Lie ideal of a 2-torsion free prime Γ-ring M. Then every Jordan higher derivation on U of M is a higher derivation on U of M.*

Proof. By definition $\phi_0^\alpha(u,v)=0$ for all $u,v\in U$ and $\alpha\in\Gamma$. Also, by Theorems 4.2.8 and 4.2.9, $\phi_1^\alpha(u,v)=0$ for all $u,v\in U$ and $\alpha\in\Gamma$. Now, we proceed by induction. Suppose that $\phi_m^\alpha(u,v)=0$. This implies,

$$d_m(u\alpha v)=\sum_{i+j=m} d_i(u)\alpha d_j(v), \forall\, u,v\in U,\alpha\in\Gamma, m<n.$$

Taking $u,v\in U$, by Lemma 4.3.3, we have

$$\phi_n^\alpha(u,v)\beta w\gamma[u,v]_\alpha+[u,v]_\alpha\beta w\gamma\phi_n^\alpha(u,v)=0, \forall\, u,v,w\in U;\alpha,\beta,\gamma\in\Gamma.$$

Since M is prime, by Lemma 4.2.6 and Lemma 4.2.7 $\phi_n^\alpha(u,v)\beta w\gamma[x,y]_\delta=0$ for all $u,v,x,y\in U$ and $\alpha,\beta,\gamma\in\Gamma$. But $U\nsubseteq Z(M)$, so we have $[x,y]_\delta\neq 0$. Therefore, by Lemma 4.2.5 we obtain $\phi_n^\alpha(u,v)=0$. □

Theorem 4.3.5. *Assume that U is a commutative Lie ideal of a 2-torsion free prime Γ-ring M such that $u\alpha u\in M$ for all $u\in U$ and $\alpha\in\Gamma$. Then every Jordan higher derivation on U of M is a higher derivation on the same.*

Proof. By the similar arguments which are used in the proof of the Theorem 4.2.9, we obtain that $U\subseteq Z(M)$. Using this in Lemma 4.3.1(i), we get

$$2d_n(u\alpha v)=2\sum_{i+j=n} d_i(u)\alpha d_j(v).$$

Since M is 2-torsion free, so we have

$$d_n(u\alpha v)=\sum_{i+j=n} d_i(u)\alpha d_j(v), \forall u,v\in U;\alpha\in\Gamma.$$

□

Chapter 5

Generalized Derivations on Lie Ideals

In this chapter, we continue our study on Lie ideals of Γ-rings. This chapter makes a study of generalized derivations and generalized higher derivations on Lie ideals of Γ-rings analogous to the study of derivations and higher derivations on Lie ideals of Γ-rings in the preceding chapter. We define various derivations on Lie ideals of Γ-rings which are generalized derivation, Jordan generalized derivation, generalized higher derivation and Jordan generalized higher derivation.

In the next, we construct some more significant results due to the defined concept of Jordan generalized derivations on Lie ideals of Γ-rings in sequel to the previous results of the foregoing chapter. Here we consider $f : M \to M$ is a Jordan generalized derivation on a Lie ideal U of a 2-torsion free prime Γ-ring M with an associated Jordan derivation $d : M \to M$ on U. We prove that if U is an admissible Lie ideal of M, then $f(u\alpha v) = f(u)\alpha v + u\alpha d(v)$ for all $u, v \in U; \alpha \in \Gamma$, and if U is a commutative Lie ideal of M, then every Jordan generalized derivation on U of M is a generalized derivation on the same.

In the third section of this chapter, we develop some consequences relating to the concept of Jordan generalized higher derivations on Lie ideals of Γ-rings to prove the analogous results corresponding to the above mentioned results considering this derivation.

5.1 Introduction

M. Ashraf and N. Rehman [2] considered the question of I. N. Herstein [21] for a Jordan generalized derivation. They showed that in a 2-torsion free ring R which has a commutator right nonzero divisor, every Jordan generalized derivation on R is a generalized derivation on R. In 2000, Nakajima defined a generalized higher derivation in [31] and gave some categorical properties which are related to [30]. He also treated generalized higher Jordan and Lie derivations. Later, Cortes and Haetinger [14] extended the theorem of M. Ashraf and N. Rehman [2] to generalized higher derivations. They proved that if R is 2-torsion free ring which has a commutator right nonzero divisor, then every Jordan generalized higher derivation on R is a generalized higher derivation on R.

First, we introduce the concepts of generalized derivation and Jordan generalized derivation on Lie ideals of Γ-rings in the following way.

Definition 5.1.1. If U is a Lie ideal of a Γ-ring M. Then an additive mapping $f : M \to M$ is said to be a *generalized derivation on a Lie ideal* U if there exists a derivation $d : M \to M$ such that $f(u\alpha v) = f(u)\alpha v + u\alpha d(v)$ for all $u, v \in U; \alpha \in \Gamma$ and $f : M \to M$ is said to be a *Jordan generalized derivation on a Lie ideal* U if there exists a derivation $d : M \to M$ on U such that $f(u\alpha u) = f(u)\alpha u + u\alpha d(u)$ for all $u \in U; \alpha \in \Gamma$.

We now give examples of a Jordan generalized derivation and a generalized derivation on a Lie ideal U of a Γ-ring M, where M satisfies $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M; \alpha, \beta \in \Gamma$.

Example 5.1.1. *If $a \in M$ and $\alpha \in \Gamma$ are fixed elements. Define $f : M \to M$ by $f(x) = a\alpha x + x\alpha a$ and $d(x) = x\alpha a - a\alpha x$. For all $x, y \in U$ and $\beta \in \Gamma$, we obtain*

$$\begin{aligned} f(y\beta y) &= a\alpha(y\beta y) + (y\beta y)\alpha a \\ &= a\alpha y\beta y + y\alpha a\beta y - y\alpha a\beta y + y\beta y\alpha a \\ &= (a\alpha y + y\alpha a)\beta y + y\beta(y\alpha a - a\alpha y) \\ &= f(y)\beta y + y\beta d(y), \forall\, y \in U; \beta \in \Gamma. \end{aligned}$$

Therefore, f is a Jordan generalized derivation on U.
Also, for all $x, y \in U$ and $\beta \in \Gamma$, we get

$$\begin{aligned} f(x\beta y) &= a\alpha(x\beta y) + (x\beta y)\alpha a \\ &= a\alpha x\beta y + x\alpha a\beta y - x\alpha a\beta y + x\beta y\alpha a \\ &= (a\alpha x + x\alpha a)\beta y + x\beta(y\alpha a - a\alpha y) \\ &= f(x)\beta y + x\beta d(y), \forall\, x, y \in U; \beta \in \Gamma. \end{aligned}$$

Therefore, f is a generalized derivation on U of M.

The following example shows that every Jordan generalized derivation on a Lie ideal of a Γ-ring need not be a generalized derivation on the same.

Example 5.1.2. *Suppose $f : M \to M$ is a generalized derivation with an associated derivation d on U of M. Let $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. If we define addition and multiplication on M_1 by $(x, x) + (y, y) = (x + y, x + y)$ and*

$(x,x)(\alpha,\alpha)(y,y) = (x\alpha y, x\alpha y)$. Then M_1 is a Γ_1-ring. Define $U_1 = \{(u,u) : u \in U\}$. Then for $u\alpha x - x\alpha u \in U$,

$$\begin{aligned}(u,u)(\alpha,\alpha)(x,x) - (x,x)(\alpha,\alpha)(u,u) &= (u\alpha x, u\alpha x) - (x\alpha u, x\alpha u)\\ &= (u\alpha x - x\alpha u, u\alpha x - x\alpha u) \in U_1.\end{aligned}$$

Hence U_1 is a Lie ideal of M_1. Now, we define a mapping $F : M_1 \to M_1$ by $F((u,u)) = (f(u), f(u))$ and $D((u,u)) = (d(u), d(u))$. Then it is clear that F is a Jordan generalized derivation on U_1 of M_1 with an associated derivation D on U_1 of M_1. Obviously, F is not a generalized derivation on U_1 of M_1.

Now, we introduce the concepts of generalized higher derivation and Jordan generalized higher derivation on Lie ideals of a Γ-ring in the following way.

Definition 5.1.2. Suppose U is a Lie ideal of a Γ-ring M. Let $F = (f_i)_{i\in\mathbf{N}_0}$ be a family of additive mappings on U such that $f_0 = id_M$, where id_M is an identity mapping on U of M and $\mathbf{N}_0$ denotes the set of natural numbers including 0. F is a *generalized higher derivation on a Lie ideal U* if there exists a higher derivation $D = (d_i)_{i\in N_0}$ on U such that for each $n \in \mathbf{N}_0; i,j \in \mathbf{N}_0$,

$$f_n(a\alpha b) = \sum_{i+j=n} f_i(a)\alpha d_j(b), \forall\, a,b \in U; \alpha \in \Gamma,$$

and F is a *Jordan generalized higher derivation on a Lie ideal U* if there exists a higher derivation $D = (d_i)_{i\in N_0}$ on U such that for each $n \in \mathbf{N}_0; i,j \in \mathbf{N}_0$,

$$f_n(a\alpha a) = \sum_{i+j=n} f_i(a)\alpha d_j(a), \forall\, a \in U; \alpha \in \Gamma.$$

The following are examples of a Jordan generalized higher derivation and a generalized higher derivation on a Lie ideal of a Γ-ring.

Example 5.1.3. *Suppose N is a Lie ideal of a Γ-ring M as in Example 4.1.3. Let U be a Lie ideal of R and $f_n : R \to R$ be a generalized higher derivation on U of R with an associated higher derivation $d_n : R \to R$ on U, where $n \in \boldsymbol{N}_0$. So, we have*

$$f_n(u\alpha v) = \sum_{i+j=n} f_i(u)\alpha d_j(v),\ \forall\, u, v \in U\ ;\ \alpha \in \Gamma.$$

Define mappings

$$F_n : M \to M \text{ by } F_n((a,b)) = (f_n(a), f_n(b));$$

$$D_n : M \to M \text{ by } D_n((a,b)) = (d_n(a), d_n(b)).$$

For each $(a,a),(b,b) \in N$ and $\begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$, we have

$$\begin{aligned}
F_n((a,a)\begin{pmatrix} n \\ n \end{pmatrix}(b,b)) &= F_n(anb + anb, anb + anb) \\
&= (f_n(anb + anb), f_n(anb + anb)) \\
&= (\sum_{i+j=n}(f_i(a)nd_j(b) + f_i(a)nd_j(b)), \sum_{i+j=n}(f_i(a)nd_j(b) + f_i(a)nd_j(b))) \\
&= \sum_{i+j=n}(f_i(a), f_i(a))\begin{pmatrix} n \\ n \end{pmatrix}(d_j(b), d_j(b)) \\
&= \sum_{i+j=n} F_i((a,a))\begin{pmatrix} n \\ n \end{pmatrix} D_j((b,b)).
\end{aligned}$$

Therefore, F_n is a generalized higher derivation on a Lie ideal N of M.

The next example gives an application of Jordan generalized higher derivation and a generalized higher derivation on a Lie ideal of a Γ-ring which shows that every Jordan generalized higher derivation on a Lie ideal of a Γ-ring need not be a generalized higher derivation.

Example 5.1.4. *Let U be a Lie ideal of a Γ-ring M, and $f_n : M \to M$ be a generalized higher derivation on U with an associated higher derivation $d_n : M \to M$ on U. Thus, we have*

$$f_n(u\alpha v) = \sum_{i+j=n} f_i(u)\alpha d_j(v), \ ;\forall\, u, v \in U;\ \alpha \in \Gamma.$$

Consider M_1, Γ_1 and U_1 as in Example 4.1.4 . Now, define mappings

$$F_n : M_1 \to M_1 \text{ by } F_n((x,x)) = (f_n(x), f_n(x));$$

$$D_n : M_1 \to M_1 \text{ by } D_n((x,x)) = (d_n(x), d_n(x)).$$

Then for all $(u,u) \in U_1$ and $(\alpha,\alpha) \in \Gamma_1$, we have

$$\begin{aligned} F_n((u,u)(\alpha,\alpha)(u,u)) &= F_n(u\alpha u, u\alpha u) \\ &= (f_n(u\alpha u), f_n(u\alpha u)) \\ &= (\sum_{i+j=n} f_i(u)\alpha d_j(u), \sum_{i+j=n} f_i(u)\alpha d_j(u)) \\ &= \sum_{i+j=n} (f_i(u), f_i(u))(\alpha,\alpha)(d_j(u), d_j(u)) \\ &= \sum_{i+j=n} F_i((u,u))(\alpha,\alpha)D_j((u,u)). \end{aligned}$$

Therefore, F_n is a Jordan generalized higher derivation on a Lie ideal U_1 of M_1 with an associated higher derivation D_n on U_1 of M_1. Also we have seen that it is not a generalized higher derivation on U_1 of M_1.

Except otherwise mentioned, throughout this chapter hereafter, U represents a Lie ideal of a 2-torsion free Γ-ring M and M satisfies $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M; \alpha, \beta \in \Gamma$ which is marked by (*).

5.2 Jordan Generalized Derivations on Lie Ideals of Prime Γ-Rings

The following lemma is the basic tool for our work in this section.

Lemma 5.2.1. *If $a\alpha a \in U$ for all $a \in U$ and $\alpha \in \Gamma$. Assume that f is a Jordan generalized derivation on U of M with an associated derivation d. Then for all $a, b, c \in U$ and $\alpha, \beta \in \Gamma$, the following statements hold:*
(i) $f(a\alpha b + b\alpha a) = f(a)\alpha b + a\alpha d(b) + f(b)\alpha a + b\alpha d(a)$;
(ii) $f(a\alpha b\beta a + a\beta b\alpha a) = f(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a) + f(a)\beta b\alpha a + a\beta d(b)\alpha a + a\beta b\alpha d(a)$.
In particular, if M is 2-torsion free and satisfies the condition (*), then
(iii) $f(a\alpha b\beta a) = f(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a)$;
(iv) $f(a\alpha b\beta c + c\alpha b\beta a) = f(a)\alpha b\beta c + a\alpha d(b)\beta c + a\alpha b\beta d(c) + f(c)\alpha b\beta a + c\alpha d(b)\beta a + c\alpha b\beta d(a)$.

Proof. Since U is a Lie ideal satisfying the condition $a\alpha a \in U$, for all $a \in U, \alpha \in \Gamma$. For $a, b \in U, \alpha \in \Gamma, (a\alpha b + b\alpha a) = (a + b)\alpha(a + b) - (a\alpha a + b\alpha b)$ and therefore $(a\alpha b + b\alpha a) \in U$. Then,

$$\begin{aligned}
f(a\alpha b + b\alpha a) &= f((a + b)\alpha(a + b) - (a\alpha a + b\alpha b)) \\
&= f(a + b)\alpha(a + b) + (a + b)\alpha d(a + b) - f(a)\alpha a - a\alpha d(a) - f(b)\alpha b - b\alpha d(b) \\
&= f(a)\alpha a + f(a)\alpha b + f(b)\alpha a + f(b)\alpha b + a\alpha d(a) + a\alpha d(b) + b\alpha d(a) + b\alpha d(b) \\
&\quad - f(a)\alpha a - a\alpha d(a) - f(b)\alpha b - b\alpha d(b) \\
&= f(a)\alpha b + a\alpha d(b) + f(b)\alpha a + b\alpha d(a).
\end{aligned}$$

Replacing $a\beta b + b\beta a$ for b in (i), we get

$$f(a\alpha(a\beta b+b\beta a)+(a\beta b+b\beta a)\alpha a) = f(a)\alpha(a\beta b+b\beta a)+a\alpha d(a\beta b+b\beta a)+f(a\beta b+b\beta a)\alpha a$$
$$+ (a\beta b + b\beta a)\alpha d(a).$$

$$\Rightarrow f((a\alpha a)\beta b+b\beta(a\alpha a))+f(a\alpha b\beta a+a\beta b\alpha a) = f(a)\alpha(a\beta b+b\beta a)+a\alpha(d(a)\beta b+a\beta d(b)$$
$$+d(b)\beta a+b\beta d(a))+(f(a)\beta b+a\beta d(b)+f(b)\beta a+b\beta d(a))\alpha a+a\beta b\alpha d(a)+b\beta a\alpha d(a).$$

This implies,

$$f(a\alpha a)\beta b + (a\alpha a)\beta d(b) + f(b)\beta(a\alpha a) + b\beta d(a\alpha a) + f(a\alpha b\beta a + a\beta b\alpha a)$$
$$= f(a)\alpha a\beta b + f(a)\alpha b\beta a + a\alpha d(a)\beta b + a\alpha a\beta d(b) + a\alpha d(b)\beta a + a\alpha b\beta d(a)$$
$$+ f(a)\beta b\alpha a + a\beta d(b)\alpha a + f(b)\beta a\alpha a + b\beta d(a)\alpha a + a\beta b\alpha d(a) + b\beta a\alpha d(a).$$

This yields,

$$f(a)\alpha a\beta b+a\alpha d(a)\beta b+a\alpha a\beta d(b)+f(b)\beta a\alpha a+b\beta d(a)\alpha a+b\beta a\alpha d(a)+f(a\alpha b\beta a+a\beta b\alpha a)$$
$$= f(a)\alpha a\beta b + f(a)\alpha b\beta a + a\alpha d(a)\beta b + a\alpha a\beta d(b) + a\alpha d(b)\beta a + a\alpha b\beta d(a)$$
$$+ f(a)\beta b\alpha a + a\beta d(b)\alpha a + f(b)\beta a\alpha a + b\beta d(a)\alpha a + a\beta b\alpha d(a) + b\beta a\alpha d(a).$$

Cancelling the like terms from both sides, we get the required result. Using the condition (*) and since M is 2-torsion free, (iii) follows from (ii). □

And, finally (iv) is obtained by replacing $a + c$ for a in (iii).

Definition 5.2.1. Let f be a Jordan generalized derivation on U of M with an associated derivation d. We define $\psi_\alpha(u, v) = f(u\alpha v) - f(u)\alpha v - u\alpha d(v)$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Remark 5.2.1. $\psi_\alpha(u,v) = 0$ if and only if f is a generalized derivation on U of M.

Lemma 5.2.2. *With the notations as above. For all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$,*

$$(i)\ \psi_\alpha(u,v) = -\psi_\alpha(v,u);\ (ii)\ \psi_\alpha(u+w,v) = \psi_\alpha(u,v) + \psi_\alpha(w,v);$$
$$(iii)\ \psi_\alpha(u,v+w) = \psi_\alpha(u,v) + \psi_\alpha(u,w);\ (iv)\ \psi_{\alpha+\beta}(u,v) = \psi_\alpha(u,v) + \psi_\beta(u,v).$$

Proof. This follows by a repetition of the argument used in the proof of Lemma 2.2.2. □

Next, we go through the following results.

Lemma 5.2.3. *If U is an admissible Lie ideal of a prime Γ-ring M and f is a Jordan generalized derivation on U of M with an associated derivation d, then*

$$\psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha = 0, \forall\ u, v, w \in U; \alpha, \beta, \gamma \in \Gamma.$$

Proof. Since U is an admissible Lie ideal of M, so it satisfies the condition $u\alpha u \in U$ for all $u \in U, \alpha \in \Gamma$. Now, for $u, v \in U, \alpha \in \Gamma$, we have $(u\alpha v + v\alpha u) = (u+v)\alpha(u+v) - (u\alpha u + v\alpha v)$ and therefore $(u\alpha v + v\alpha u) \in U$. Also, $[u,v]_\alpha = u\alpha v - v\alpha a \in U$ and it follows that $2u\alpha v \in U$. Hence, $4u\alpha v\beta w = 2(2u\alpha v)\beta w \in U$ for all $u, v, w \in U, \alpha, \beta \in \Gamma$. Let $x = 4(u\alpha v\beta w\gamma v\alpha u + v\alpha u\beta w\gamma u\alpha v)$. First, using Lemma 5.2.1(iv), we have

$$\begin{aligned} f(x) &= f((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)) \\ &= f(2u\alpha v)\beta w\gamma(2v\alpha u) + 2u\alpha v\beta d(w)\gamma 2v\alpha u + 2u\alpha v\beta w\gamma d(2v\alpha u) + \\ &\qquad f(2v\alpha u)\beta w\gamma(2u\alpha v) + 2v\alpha u\beta d(w)\gamma 2u\alpha v + 2v\alpha u\beta w\gamma d(2u\alpha v). \end{aligned}$$

Now, using Lemma 5.2.1(iii) and Lemma 4.2.1(iii), we obtain

$$f(x) = f(u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v)$$
$$= 4f(u)\alpha v\beta w\gamma v\alpha u + u\alpha d(4v\beta w\gamma v)\alpha u + 4u\alpha v\beta w\gamma v\alpha d(u) + 4f(v)\alpha u\beta w\gamma u\alpha v$$
$$+ v\alpha d(4u\beta w\gamma u)\alpha v + 4v\alpha u\beta w\gamma u\alpha d(v)$$
$$= 4f(u)\alpha v\beta w\gamma v\alpha u+4u\alpha d(v)\beta w\gamma v\alpha u+4u\alpha v\beta d(w)\gamma v\alpha u+4u\alpha v\beta w\gamma d(v)\alpha u+4u\alpha v\beta w\gamma v\alpha d(u)$$
$$+4f(v)\alpha u\beta w\gamma u\alpha v+4v\alpha d(u)\beta w\gamma u\alpha v+4v\alpha u\beta d(w)\gamma u\alpha v+4v\alpha u\beta w\gamma d(u)\alpha v+4v\alpha u\beta w\gamma u\alpha d(v).$$

Comparing the two right sides of $f(x)$, we obtain

$$4(f(u\alpha v)\beta w\gamma v\alpha u + f(v\alpha u)\beta w\gamma u\alpha v + u\alpha v\beta w\gamma d(v\alpha u) + v\alpha u\beta w\gamma d(u\alpha v))$$
$$= 4(f(u)\alpha v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + f(v)\alpha u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v$$
$$+ u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma v\alpha d(u) + v\alpha u\beta w\gamma d(u)\alpha v + v\alpha u\beta w\gamma u\alpha d(v)).$$

This implies,

$$4((f(u\alpha v) - f(u)\alpha v - u\alpha d(v))\beta w\gamma v\alpha u + (f(v\alpha u) - f(v)\alpha u - v\alpha d(u))\beta w\gamma u\alpha v$$
$$+ u\alpha v\beta w\gamma(d(v\alpha u) - d(v)\alpha u - v\alpha d(u)) + v\alpha u\beta w\gamma(d(u\alpha v) - d(u)\alpha v - u\alpha d(v))) = 0.$$

Using the Definition 5.2.1, we obtain

$$4(\psi_\alpha(u,v)\beta w\gamma v\alpha u + \psi_\alpha(v,u)\beta w\gamma u\alpha v + u\alpha v\beta w\gamma\phi_\alpha(v,u) + v\alpha u\beta w\gamma\phi_\alpha(u,v)) = 0.$$

Since d is a derivation on U of M, hence by Theorem 4.2.8, $\phi_\alpha(u,v)) = 0$ and $\phi_\alpha(v,u)) = 0, \forall\, u,v \in U; \alpha \in \Gamma$, consequently

$$4(\psi_\alpha(u,v)\beta w\gamma v\alpha u + \psi_\alpha(v,u)\beta w\gamma u\alpha v) = 0.$$

Using Lemma 5.2.2(i), and since M is 2-torsion free, so we get

$$\psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha = 0, \forall\, u,v,w \in U; \alpha,\beta,\gamma \in \Gamma.$$

□

We are now concluding this section by proving the following two theorems.

Theorem 5.2.4. *Let U be an admissible Lie ideal of a prime Γ-ring M. If $f : M \to M$ is a Jordan generalized derivation on U, then $f(u\alpha v) = f(u)\alpha v + u\alpha d(v)$ for all $u, v \in U; \alpha \in \Gamma$.*

Proof. By Lemma 5.2.3, we have

$$\psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0, \forall\, u, v, w \in U; \alpha, \beta, \gamma \in \Gamma.$$

Using Lemma 4.2.7, we get

$$\psi_\alpha(u, v)\beta w\gamma[x, y]_\delta = 0, \forall\, u, v, w, x, y \in U; \alpha, \beta, \gamma, \delta \in \Gamma.$$

Since U is an admissible Lie ideal of M, consequently $[x, y]_\delta \neq 0$. Therefore, by Lemma 4.2.5, we get $\psi_\alpha(u, v) = 0$. □

Theorem 5.2.5. *Let U be a commutative Lie ideal of a prime Γ-ring M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. Then every Jordan generalized derivation on U of M is a generalized derivation on the same.*

Proof. Since U is a commutative Lie ideal of M, $[u, v]_\alpha = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$. So, by Lemma 4.2.9, $U \subseteq Z(M)$. Now, by Lemma 5.2.1(iv), we obtain

$$f(u\alpha v\beta w+w\alpha v\beta u) = f(u)\alpha v\beta w+f(w)\alpha v\beta u+u\alpha d(v)\beta w+w\alpha d(v)\beta u+u\alpha v\beta d(w)+w\alpha v\beta d(u) \tag{5.1}$$

Since $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$, we find that

$$u\alpha v + v\alpha u \in U, \forall\, u, v \in U; \alpha \in \Gamma.$$

Also, we have

$$u\alpha v - v\alpha u \in U, \forall\, u \in U; \alpha \in \Gamma.$$

These two relations yield that $2u\alpha v \in U$. Since U is commutative, we get $u\alpha v = v\alpha u$ for all $u, v \in U$ and $\alpha \in \Gamma$. So, in view of Lemma 5.2.1(i) and using the condition (*), we get

$$\begin{aligned} 2f(u\alpha v\beta w + w\alpha v\beta u) &= f((2u\alpha v)\beta w + w\beta(2u\alpha v)) \\ &= f(2u\alpha v)\beta w + f(w)\beta 2u\alpha v + 2u\alpha v\beta d(w) + w\beta d(2u\alpha v) \\ &= 2(f(u\alpha v)\beta w + u\alpha v\beta d(w) + f(w)\beta u\alpha v + w\beta d(u)\alpha v + w\beta u\alpha d(v)). \end{aligned}$$

Using the 2-torsion freeness of M, we obtain

$$f(u\alpha v\beta w + w\alpha v\beta u) = f(u\alpha v)\beta w + u\alpha v\beta d(w) + f(w)\beta u\alpha v + w\beta d(u)\alpha v + w\beta u\alpha d(v). \tag{5.2}$$

Combining (5.1) and (5.2), using the fact that $u\alpha v = v\alpha u, U \subseteq Z(M)$ and the condition (*)

$$(f(u\alpha v) - f(u)\alpha v - u\alpha d(v))\beta w = 0, \forall\, u, v, w \in U; \alpha, \beta \in \Gamma.$$

This implies,

$$\psi_\alpha(u, v)\beta w = 0, \forall\, u, v, w \in U; \alpha, \beta \in \Gamma.$$

Now, putting $[w, m]_\gamma$ for w, for every $m \in M$ and $\gamma \in \Gamma$, we get

$$\psi_\alpha(u, v)\beta[w, m]_\gamma = 0.$$

$$\Rightarrow \psi_\alpha(u, v)\beta w\gamma m - \psi_\alpha(u, v)\beta m\gamma w = 0.$$

Since $\psi_\alpha(u, v)\beta w = 0$, we have $\psi_\alpha(u, v)\beta M\gamma w = 0$. Since $U \neq 0$ and M is prime, we find that $\psi_\alpha(u, v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$. Therefore, we get the required result. □

5.3 Jordan Generalized Higher Derivations on Lie Ideals of Prime Γ-Rings

First, we develop some consequences relating to the concept of Jordan generalized higher derivations on Lie ideals of a Γ-ring to extend the results stated at the beginning of this chapter following [14, 15, 16, 17] classical ring theory to Γ-ring theory.

Lemma 5.3.1. *Let U be a Lie ideal of M such that $a\alpha a \in U$, for all $a \in U$ and $\alpha \in \Gamma$. If $F = (f_i)_{i\in N}$ is a Jordan generalized higher derivation on U of M with an associated higher derivation $D = (d_i)_{i\in N_0}$. Then $\forall\ a, b, c \in U; \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$,*
(i) $f_n(a\alpha b + b\alpha a) = \sum_{i+j=n}[f_i(a)\alpha d_j(b) + f_i(b)\alpha d_j(a)]$;
(ii) $f_n(a\alpha b\beta a) = \sum_{i+j+k=n}[f_i(a)\alpha d_j(b)\beta d_k(a)]$;
(iii) $f_n(a\alpha b\beta c + c\alpha b\beta a) = \sum_{i+j+k=n}[f_i(a)\alpha d_j(b)\beta d_k(c) + f_i(c)\alpha d_j(b)\beta d_k(a)]$.

Proof. The proofs of (i) and (ii) are similar to the corresponding proofs of Lemma 5.2.1(i) and Lemma 5.2.1(iii). Replacing a by $a + c$ in (ii), and compute it using (ii)

$$\begin{aligned} X &= (a+c)\alpha b\beta(a+c) \\ &\Rightarrow f_n(X) = \sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(c) + \sum_{i+j+k=n} f_i(c)\alpha d_j(b)\beta d_k(a) \\ &\qquad + \sum_{i+j+k=n} f_i(a)\alpha d_j(b)\beta d_k(a) + \sum_{i+j+k=n} f_i(c)\alpha d_j(b)\beta d_k(c). \end{aligned}$$

On the other hand, using (ii)

$$f_n(X) = f_n(a\alpha b\beta c + c\alpha b\beta a) + \sum_{i+j+k=n}(f_i(a)\alpha d_j(b)\beta d_k(a) + f_i(c)\alpha d_j(b)\beta d_k(c)).$$

By comparing the two results for $f_n(X)$, we obtain (iii). □

Definition 5.3.1. For every Jordan generalized higher derivation $F = (f_i)_{i\in\mathbf{N}}$ on U of M, we define $\psi_n^\alpha(a,b) = f_n(a\alpha b) - \sum_{i+j=n} f_i(a)\alpha d_j(b)$, for all $a, b \in U; \alpha \in \Gamma$ and $n \in \mathbf{N}$.

Remark 5.3.1. F is a generalized higher derivation on U of M if and only if $\psi_n^\alpha(a,b) = 0$, for all $a, b \in U; \alpha \in \Gamma$ and $n \in \mathbf{N}$.

Lemma 5.3.2. *The following are true for all $a, b, c \in U; \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$,*
(i) $\psi_n^\alpha(a,b) + \psi_n^\alpha(b,a) = 0$; *(ii)* $\psi_n^\alpha(a+b,c) = \psi_n^\alpha(a,c) + \psi_n^\alpha(b,c)$;
(iii) $\psi_n^\alpha(a,b+c) = \psi_n^\alpha(a,b) + \psi_n^\alpha(a,c)$; *(iv)* $\psi_n^{\alpha+\beta}(a,b) = \psi_n^\alpha(a,b) + \psi_n^\beta(a,b)$.

Proof. This follows by a repetition of the argument used in the proof of Lemma 3.4.2. □

Lemma 5.3.3. *Let M be a 2-torsion free Γ-ring and U be a Lie ideal of M such that $a\alpha a \in U$ for all $a \in U$ and $\alpha \in \Gamma$. If $F = (f_i)_{i\in\boldsymbol{N}}$ is a Jordan generalized higher derivation on U of M with an associated higher derivation $D = (d_i)_{i\in N_0}$. Suppose that $n \in \boldsymbol{N}; a, b \in U; \alpha, \beta, \gamma \in \Gamma$ and $\psi_m^\alpha(a,b) = 0$, for every $m < n$, then $\psi_n^\alpha(a,b)\beta w\gamma[a,b]_\alpha = 0$, for every $w \in U$.*

Proof. Since U is a Lie ideal satisfying the condition $a\alpha a \in U$, for all $a \in U, \alpha \in \Gamma$. For $a, b \in U, \alpha \in \Gamma, (a\alpha b + b\alpha a) = (a+b)\alpha(a+b) - (a\alpha a + b\alpha b)$ and therefore $(a\alpha b + b\alpha a) \in U$. Also, $[a,b]_\alpha = a\alpha b - b\alpha a \in U$ and it follows that $2a\alpha b \in U$. Hence, $4a\alpha b\beta c = 2(2a\alpha b)\beta c \in U$, for all $a, b, c \in U, \alpha, \beta \in \Gamma$. Suppose $H = 4f_n(a\alpha b\beta w\gamma b\alpha a + b\alpha a\beta w\gamma a\alpha b)$. First, by using Lemma 5.3.1(iii), we get

$$
\begin{aligned}
H &= f_n((2a\alpha b)\beta w\gamma(2b\alpha a) + (2b\alpha a)\beta w\gamma(2a\alpha b)) \\
&= 4\sum_{r+s+t=n} (f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a) + f_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b))
\end{aligned}
$$

$$= \quad 4\sum_{r+s+t=n} f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a) \quad + \quad 4\sum_{r+s+t=n} f_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b).$$

Again, by using Lemma 5.3.1(ii) and Lemma 4.3.1(ii), we obtain

$$H = f_n(a\alpha(4b\beta w\gamma b)\alpha a) + f_n(b\alpha(4a\beta w\gamma a)\alpha b)$$

$$= 4\sum_{i+p+l=n} f_i(a)\alpha d_p(b\beta w\gamma b)\alpha d_l(a) + 4\sum_{i+p+l=n} f_i(b)\alpha d_p(a\beta w\gamma a)\alpha d_l(b)$$

$$= 4\sum_{i+j+k+h+l=n} f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) + 4\sum_{i+j+k+h+l=n} f_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b)$$

Comparing the two expressions for H, we have

$$\sum_{i+j+k+h+l=n} f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) - \sum_{r+s+t=n} f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)$$

$$+ \sum_{i+j+k+h+l=n} f_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b) - \sum_{r+s+t=n} f_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b) = 0.$$

Since $D = (d_i)_{i\in N}$ is a higher derivation on U of M and $f_m(u\alpha v) = \sum_{i+j=m} f_i(u)\alpha d_j(v)$, when $m < n$. Therefore,

$$\sum_{i+j+k+h+l=n} f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a) - \sum_{r+s+t=n} f_r(a\alpha b)\beta d_s(w)\gamma d_t(b\alpha a)$$

$$= (\sum_{i+j=n} f_i(a)\alpha d_j(b))\beta w\gamma b\alpha a + a\alpha b\beta w\gamma(\sum_{h+l=n} d_h(b)\alpha d_l(a))$$

$$+ \sum_{i+j+k+h+l=n}^{i+j<n,h+l<n} (f_i(a)\alpha d_j(b)\beta d_k(w)\gamma d_h(b)\alpha d_l(a)) - f_n((a\alpha b)\beta w\gamma(b\alpha a)$$

$$- (a\alpha b)\beta w\gamma d_n(b\alpha a) - \sum_{r+s+t=n}^{i+j=r<n,p+q=t<n} (f_i(a)\alpha d_j(b)\beta d_s(w)\gamma d_p(b)\alpha d_q(a))$$

$$= -(f_n((a\alpha b) - \sum_{i+j=n} f_i(a)\alpha d_j(b))\beta(w\gamma b\alpha a) - (a\alpha b\beta w)\gamma(d_n(b\alpha a) - \sum_{h+l=n} d_h(b)\alpha d_l(a))$$

$$= -(\psi_n^\alpha(a,b)\beta w\gamma b\alpha a + a\alpha b\beta w\gamma\phi_n^\alpha(b,a)).$$

Similarly,

$$\sum_{i+j+k+h+l=n} f_i(b)\alpha d_j(a)\beta d_k(w)\gamma d_h(a)\alpha d_l(b) - \sum_{r+s+t=n} d_r(b\alpha a)\beta d_s(w)\gamma d_t(a\alpha b)$$
$$= -(\psi_n^{\alpha}(b,a)\beta w\gamma a\alpha b + b\alpha a\beta w\gamma\phi_n^{\alpha}(a,b)).$$

Hence, we get

$$\psi_n^{\alpha}(a,b)\beta w\gamma b\alpha a \;+\; a\alpha b\beta w\gamma\phi_n^{\alpha}(b,a) \;+\; \psi_n^{\alpha}(b,a)\beta w\gamma a\alpha b \;+\; b\alpha a\beta w\gamma\phi_n^{\alpha}(a,b) \;=\; 0.$$

By Lemma 5.3.2(i), we have

$$\psi_n^{\alpha}(a,b)\beta w\gamma b\alpha a \;-\; a\alpha b\beta w\gamma\phi_n^{\alpha}(a,b) \;-\; \psi_n^{\alpha}(a,b)\beta w\gamma a\alpha b \;+\; b\alpha a\beta w\gamma\phi_n^{\alpha}(a,b) \;=\; 0.$$

This implies,

$$\psi_n^{\alpha}(a,b)\beta w\gamma[a,b]_{\alpha} + [a,b]_{\alpha}\beta w\gamma\phi_n^{\alpha}(a,b) = 0, \forall\, w \in U.$$

Since $D = (d_i)_{i\in N}$ is a higher derivation on U of M. Thus, by Theorem 4.3.4, we have $\phi_n^{\alpha}(a,b) = 0, \forall\, a,b \in U; \alpha \in \Gamma; n \in \mathbf{N}$ and hence accomplishes the proof. □

We conclude this chapter by proving following two theorems.

Theorem 5.3.4. *Assume that M is a prime Γ-ring and U is an admissible Lie ideal of M. Then every Jordan generalized higher derivation on U of M is a generalized higher derivation on the same.*

Proof. By definition $\psi_0^{\alpha}(a,b) = 0$, for all $a,b \in U$ and $\alpha \in \Gamma$. Also, by Theorem 5.2.4, $\psi_1^{\alpha}(a,b) = 0$, for all $a,b \in U$ and $\alpha \in \Gamma$. Now, we proceed by induction. Suppose that $\psi_m^{\alpha}(a,b) = 0$, when $m < n$. This implies, $f_m(a\alpha b) = \sum_{i+j=m} f_i(a)\alpha d_j(b)$, for all $a,b \in U, \alpha \in \Gamma$ and $m < n$. Taking $a,b \in U$, by Lemma 5.3.3, we get

$$\psi_n^{\alpha}(a,b)\beta w\gamma[a,b]_{\alpha} = 0, \forall\, w \in U; \alpha,\beta,\gamma \in \Gamma.$$

In view of Lemma 4.2.7, we obtain

$$\psi_n^{\alpha}(a,b)\beta w\gamma[x,y]_{\delta}=0,\forall\, a,b,x,y,w\in U;\alpha,\beta,\gamma,\delta\in\Gamma.$$

Since $U\nsubseteq Z(M)$, $[x,y]_{\delta}\neq 0$. So, by Lemma 4.2.5 $\psi_n^{\alpha}(a,b)=0$. □

Lemma 5.3.5. *Let M be a prime Γ-ring and U be a commutative Lie ideal of M such that $u\alpha u\in U$ for all $u\in U$ and $\alpha\in\Gamma$. Let $F=(f_i)_{i\in\boldsymbol{N}}$ be a Jordan generalized higher derivation on U of M with the associated higher derivation $D=(d_i)_{i\in\boldsymbol{N}}$. If $\psi_m^{\alpha}(u,v)=0$, for every $m<n; u,v\in U$ and $\alpha\in\Gamma$, then $\psi_n^{\alpha}(u,v)\beta w=0$ for all $w\in U,\beta\in\Gamma$.*

Proof. Since U is a commutative Lie ideal of M such that $u\alpha u\in U$ for all $u\in U$ and $\alpha\in\Gamma$. Hence, by Lemma 4.2.9, $U\subseteq Z(M)$. Now, in view of Lemma 5.2.1(iii), we obtain

$$f_n(u\alpha v\beta w+w\alpha v\beta u)=\sum_{i+j+k=n}(f_i(u)\alpha d_j(v)\beta d_k(w)+f_i(w)\alpha d_j(v)\beta d_k(u)).\tag{5.3}$$

By Lemma 5.3.1(i), using $u\alpha v=v\alpha u$ and the condition (*), we have

$$\begin{aligned}2f_n(u\alpha v\beta w+w\alpha v\beta u)&=f_n((2u\alpha v)\beta w+w\beta(2u\alpha v))\\&=2\sum_{i+j=n}(f_i(u\alpha v)\beta d_j(w)+f_i(w)\beta d_j(u\alpha v))\\&=2\sum_{i+j=n}f_i(u\alpha v)\beta d_j(w)+2\sum_{i+j=n}f_i(w)\beta d_j(v\alpha u)\\&=2\sum_{i+j=n}(f_i(u\alpha v)\beta d_j(w)+2\sum_{i+p+q=n}f_i(w)\beta d_p(v)\alpha d_q(u)).\end{aligned}$$

This follows that,

$$f_n(u\alpha v\beta w+w\alpha v\beta u)=\sum_{i+j=n}(f_i(u\alpha v)\beta d_j(w)+\sum_{i+p+q=n}f_i(w)\beta d_p(v)\alpha d_q(u)).\tag{5.4}$$

Comparing (5.3) and (5.4), we get

$$\sum_{i+j=n}(f_i(u\alpha v)\beta d_j(w) = \sum_{i+j+k=n}(f_i(u)\alpha d_j(v)\beta d_k(w)$$

$$\Rightarrow (f_n(u\alpha v)\beta w + \sum_{p+q+j=n}^{p+q=i<n} f_p(u)\alpha d_q(v)\beta d_j(w) = (\sum_{i+j=n} f_i(u)\alpha d_j(v))\beta w + \sum_{i+j+k=n}^{i+j<n} f_i(u)\alpha d_j(v)\beta d_k(w).$$

This implies,

$$(f_n(u\alpha v) - \sum_{i+j=n} f_i(u)\alpha d_j(v))\beta w = 0.$$

Therefore, $\psi_n^\alpha(u,v)\beta w = 0$, for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$. □

Theorem 5.3.6. *Let M be a prime Γ-ring and U be a commutative Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. Then every Jordan generalized higher derivation on U of M is a generalized higher derivation on the same.*

Proof. By definition we have $\psi_0^\alpha(u,v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$. Now we prove the theorem by induction. If $n = 1$, then by Theorem 5.2.5 we obtain $\psi_1^\alpha(u,v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$. Now we assume that $n \geq 2$ and $\psi_m^\alpha(u,v) = 0$ for all $m < n$. Then by Lemma 5.3.5, we have

$$\psi_n^\alpha(u,v)\beta w = 0, \forall\, u, v, w \in U; \alpha, \beta \in \Gamma. \tag{5.5}$$

Since $w \in U$, we have $[w,m]_\gamma \in U$ for all $m \in M$ and $\gamma \in \Gamma$. Replacing w by $[w,m]_\gamma$ and using (5.5), we obtain $\psi_n^\alpha(u,v)\beta m\gamma w = 0$. Since $U \neq 0$, the primeness of M implies that, $\psi_n^\alpha(u,v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$. This is the required result. □

Chapter 6

(U, M)-Derivations

This chapter deals with (U, M)-derivations and higher (U, M)-derivations of Γ-rings. Here we introduce the concepts of (U, M)-derivation and higher (U, M)-derivation in Γ-rings. Introductory discussions concerning these concepts are described in the first section.

The second section develops some relevant important results due to the newly introduced concept of (U, M)-derivation in Γ-rings. Then we generalized the results of A. K. Faraj, C. Haetinger and A. H. Majeed [16] in Γ-rings by the new concept of (U, M)-derivation. Here we prove that, if U is an admissible Lie ideal of a prime Γ-ring M, and d is a (U, M)-derivation of M then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u, v \in U$ and $\alpha \in \Gamma$. After that, we prove $d(u\alpha m) = d(u)\alpha m + u\alpha d(m)$ for all $u \in U, m \in M$ and $\alpha \in \Gamma$, when $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$.

In the next, we develop some consequences relating to the concept of higher (U, M)-derivations of Γ-rings. We conclude this chapter by showing that if U is an admissible Lie ideal of a prime Γ-ring M, and $D = (d_i)_{i\in\mathbf{N}}$ is a higher (U, M)-derivation of M then (i) $d_n(u\alpha v) = \sum_{i+j=n} d_i(u)\alpha d_j(v)$ for all $u, v \in U; \alpha \in \Gamma; n \in \mathbf{N}$ and (ii) $d_n(u\alpha m) = \sum_{i+j=n} d_i(u)\alpha d_j(m)$ for all $u \in U, m \in M, \alpha \in \Gamma$ and $n \in \mathbf{N}$.

6.1 Introduction

We begin by explaining the introductional background behind the notions of the (U, M)-derivation and higher (U, M)-derivation in Γ-rings. In 1950, I. N. Herstein [20, 21, 22] initiated the study of Lie and Jordan structure of associative rings. The relationship between usual derivations and Lie ideals of prime rings has been extensively studied in the last 40 years, in particular, when this relationship involves the action of the derivations on Lie ideals. R. Awtar [3] extended the Herstein's theorem to Lie ideals. He proved that if U is a Lie ideal of a 2-torsion free prime ring R such that $u^2 \in U$ for all $u \in U$ and $d : R \to R$ is an additive mapping such that d is derivation on U of R, then d is a derivation on U of R. Also, C. Haetinger in [17] extended Awtar's result to higher derivations. A. K. Faraj, C. Haetinger and A. H. Majeed [16] introduced $(U, R)-$ derivations in rings as a generalization of Jordan derivations on Lie ideals of rings. The notion of (U, R)-derivation extends the concept given of R. Awtar [3]. A. K. Faraj, C. Haetinger and A. H. Majeed [16] proved that if R is a prime ring, char$(R) \neq 2$, U a square closed Lie ideal of R and d a (U, R)-derivation of R then $d(ur) = d(u)r + ud(r)$ for all $u \in U, r \in R$. This result is a generalization of the result of R. Awtar [3].

Continuing in the similar way as that has been done earlier by the above mentioned prominent algebraists we then introduce the concepts of (U, M)-derivation and higher (U, M)-derivation of Γ-rings in the following way.

Definition 6.1.1. Suppose U is a Lie ideal of a Γ-ring M. An additive mapping $d : M \to M$ is a (U, M)- *derivation* of M if $d(u\alpha m + s\alpha u) = d(u)\alpha m + u\alpha d(m) + d(s)\alpha u + s\alpha d(u)$ holds for all $u \in U; m, s \in M$ and $\alpha \in \Gamma$.

The existence of a Lie ideal of a Γ-ring and a (U, M)-derivation of a Γ-ring are confirmed by the following example.

Example 6.1.1. *If R is an associative ring with 1, and U is a Lie ideal of R. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$, then M is a Γ-ring. Let $N = \{(x, x) : x \in R\} \subseteq M$, then N is a sub Γ-ring. Let $U_1 = \{(u, u) : u \in U\}$, then for $una - anu \in U$ for all $u \in U$ and $a \in M$, we get*

$$(u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a) - (a,a)\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u) = (una, una) - (anu, anu)$$

$$= (una - anu, una - anu) \in U.$$

Thus, U_1 is a Lie ideal of N. Let $d : R \to R$ be a (U, R)-derivation. Now, we define a mapping $D : N \to N$ by $D((x, x)) = (d(x), d(x))$. Then

$$D((u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a) + (b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)) = D((una, una) + (bnu, bnu))$$

$$= D((una + bnu, una + bnu)) = (d(una + bnu), d(una + bnu))$$

$$= (d(u)na + und(a) + d(b)nu + bnd(u), d(u)na + und(a) + d(b)nu + bnd(u))$$

$$= (d(u)na + und(a), d(u)na + und(a)) + (d(b)nu + bnd(u), d(b)nu + bnd(u))$$

$$= (d(u)na, d(u)na) + (und(a), und(a)) + (d(b)nu, d(b)nu) + (bnd(u), bnd(u))$$

$$= (d(u), d(u))\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a) + (u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(a), d(a)) + (d(b), d(b))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)$$

$$+ (b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(u), d(u))$$

$$= D((u,u))\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a) + (u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(D((a,a)) + D((b,b))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)$$
$$+ (b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}D((u,u)) = D(u_1)\alpha x + u_1\alpha D(x) + D(y)\alpha u_1 + y\alpha D(u_1),$$

where $u_1 = (u,u), \alpha = \begin{pmatrix} n \\ 0 \end{pmatrix}, x = (a,a), y = (b,b)$. *Therefore,*

$$D(u_1\alpha x + y\alpha u_1) = D(u_1)\alpha x + u_1\alpha D(x) + D(y)\alpha u_1 + y\alpha D(u_1).$$

Hence D *is a* $(U_1, N)-$*derivation of* N.

Definition 6.1.2. Let U be a Lie ideal of a Γ-ring M, and let $D = (d_i)_{i\in N_0}$ be a family of additive mappings of M into itself such that $d_0 = id_M$, where id_M is an identity mapping on M. Then D is a *higher* (U, M)*-derivation* of M if for each $n \in \mathbf{N}$,

$$d_n(u\alpha m + s\alpha u) = \sum_{i+j=n}(d_i(u)\alpha d_j(m) + d_i(s)\alpha d_j(u))$$

holds for all $u \in U; m, s \in M$ and $\alpha, \beta \in \Gamma$.

Example 6.1.2. *Suppose* N *and* U_1 *are as in Example 6.1.1. Let* $d_n : R \to R$ *be a higher* (U, R)*-derivation. If we define a mapping* $D_n : N \to N$ *by* $D_n((x,x)) = (d_n(x), d_n(x))$. *Then by the similar calculation as in Example 6.1.1, we can show that,* D_n *is a higher* (U_1, N)*-derivation of* N.

Throughout this chapter (unless otherwise stated), U represents a Lie ideal of a 2-torsion free Γ-ring M, and M satisfies the assumption $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$; $\alpha, \beta \in \Gamma$, it is denoted by (*).

6.2 (U, M)-Derivations in Prime Γ-Rings

This section is to develop the necessary results in order to reach the goal of the next section. All these results are due to the concept of (U, M)-derivations of a Γ-ring.

Lemma 6.2.1. *Let d be a (U, M)-derivation of M. Then*
(i) $d(u\alpha m\beta u) = d(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u)$ for all $u \in U, m \in M$ and $\alpha, \beta \in \Gamma$;
(ii) $d(u\alpha m\beta v + v\alpha m\beta u) = d(u)\alpha m\beta v + u\alpha d(m)\beta v + u\alpha m\beta d(v) + d(v)\alpha m\beta u + v\alpha d(m)\beta u + v\alpha m\beta d(u)$ for all $u, v \in U; m \in M$ and $\alpha, \beta \in \Gamma$.

Proof. By the definition of (U, M)-derivation of M, we have

$$d(u\alpha m + s\alpha u) = d(u)\alpha m + u\alpha d(m) + d(s)\alpha u + s\alpha d(u), \forall\ u \in U; m, s \in M; \alpha \in \Gamma.$$

Replacing m and s by $(2u)\beta m + m\beta(2u)$ and let $w = u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\beta m + m\beta(2u))\alpha u$. Then using the definition of (U, M)-derivation and the condition (*), we get

$$\begin{aligned} d(w) &= 2(d(u)\alpha(u\beta m + m\beta u) + u\alpha d(u\beta m + m\beta u) + d(u\beta m + m\beta u)\alpha u + (u\beta m + m\beta u)\alpha d(u)) \\ &= 2(d(u)\alpha u\beta m + d(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) \\ &+ d(u)\beta m\alpha u + u\beta d(m)\alpha u + d(m)\beta u\alpha u + m\beta d(u)\alpha u + u\beta m\alpha d(u) + m\beta u\alpha d(u)) \\ &= 2(d(u)\alpha u\beta m + d(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) \\ &+ d(u)\alpha m\beta u + u\alpha d(m)\beta u + d(m)\alpha u\beta u + m\alpha d(u)\beta u + u\alpha m\beta d(u) + m\alpha u\beta d(u)). \end{aligned} \tag{6.1}$$

Also, we have

$$
\begin{aligned}
d(w) &= d((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2d(u\alpha m\beta u) + 2d(u\beta m\alpha u)\\
&= 2(d(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + d(m)\beta u\alpha u + m\beta d(u)\alpha u + m\beta u\alpha d(u)\\
&+ 2d(u\alpha m\beta u) + 2d(u\alpha m\beta u)\\
&= 2(d(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + d(m)\alpha u\beta u + m\alpha d(u)\beta u + m\alpha u\beta d(u))\\
&+ 4d(u\alpha m\beta u).
\end{aligned}
\tag{6.2}
$$

By comparing (6.1) and (6.2), and since M is 2-torsion free, we obtain

$$d(u\alpha m\beta u) = d(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u), \forall\ u \in U; m \in M; \alpha, \beta \in \Gamma. \tag{6.3}$$

If we linearize (6.3) on u, then (ii) is obtained. □

Definition 6.2.1. For a (U, M)-derivation d, we define $\phi_\alpha(u, m) = d(u\alpha m) - d(u)\alpha m - u\alpha d(m)$ for all $u \in U, m \in M$ and $\alpha \in \Gamma$.

Lemma 6.2.2. *Let d be a (U, M)-derivation of a Γ-ring M. For all $u, v \in U; m, n \in M$ and $\alpha \in \Gamma$, the following statements are true:*
(i) $\phi_\alpha(m, u) = -\phi_\alpha(u, m)$; (ii) $\phi_\alpha(u + v, m) = \phi_\alpha(u, m) + \phi_\alpha(v, m)$;
(iii) $\phi_\alpha(u, m + n) = \phi_\alpha(u, m) + \phi_\alpha(u, n)$; (iv) $\phi_{\alpha+\beta}(u, m) = \phi_\alpha(u, m) + \phi_\beta(u, m)$.

Proof. (i) Using Definition 6.2.1, we get

$$
\begin{aligned}
\phi_\alpha(u, m) + \phi_\alpha(m, u) &= d(u\alpha m) - d(u)\alpha m - u\alpha d(m) + d(m\alpha u) - d(m)\alpha a - m\alpha d(u)\\
&= d(u\alpha m + m\alpha u) - d(u)\alpha m - u\alpha d(m) - d(m)\alpha u - m\alpha d(u)\\
&= d(u)\alpha m + d(m)\alpha a + u\alpha d(m) + m\alpha d(u) - d(u)\alpha m - u\alpha d(m)\\
&- d(m)\alpha u - m\alpha d(u) = 0.
\end{aligned}
$$

$\Rightarrow \phi_\alpha(m, u) = -\phi_\alpha(u, m)$.

(ii) By the definition of (U, M)-derivation of M, we obtain

$$\begin{aligned}
\phi_\alpha(u+v,m) &= d((u+v)\alpha m) - d(u+v)\alpha m - (u+v)\alpha d(m) \\
&= d(u\alpha m + v\alpha m) - d(u)\alpha m - d(v)\alpha m - u\alpha d(m) - v\alpha d(m) \\
&= d(u\alpha m) - d(u)\alpha m - u\alpha d(m) + d(v\alpha m) - d(v)\alpha m - v\alpha d(m) \\
&= \phi_\alpha(u,m) + \phi_\alpha(v,m).
\end{aligned}$$

(iii) and (iv): The proofs are too obvious to perform. □

Lemma 6.2.3. *Let U be a nonzero admissible Lie ideal of a 2-torsion free prime Γ-ring M. Then U contains a nonzero ideal of M.*

Proof. Since U is a noncentral Lie ideal of M, if $x, y \in U$ are any two elements, then $x\alpha y - y\alpha x \neq 0$ for every $\alpha \in \Gamma$. For any $m \in M$, using the condition (*)

$$\begin{aligned}
x\alpha(y\beta m) - (y\beta m)\alpha x &= x\alpha(y\beta m) - y\alpha x\beta m + y\alpha x\beta m - (y\beta m)\alpha x \\
&= (x\alpha y - y\alpha x)\beta m + y\beta x\alpha m - y\beta m\alpha x \\
&= (x\alpha y - y\alpha x)\beta m + y\beta(x\alpha m - m\alpha x) \in U.
\end{aligned}$$

Since U is a square closed Lie ideal of $M, 2y\beta(x\alpha m - m\alpha x) \in U$ this leads us, $2(x\alpha y - y\alpha x)\beta m \in U$ for all $m \in M$. Now for any $m, s \in M$, we have

$$(2(x\alpha y - y\alpha x)\beta m)\alpha s - s\alpha(2(x\alpha y - y\alpha x)\beta m) \in U; (2(x\alpha y - y\alpha x)\beta m)\alpha s \in U.$$

This implies,

$$s\alpha(2(x\alpha y - y\alpha x))\beta m \in U, \forall\, m, s \in M; \alpha, \beta \in \Gamma.$$

Let $I = M\Gamma 2(x\alpha y - y\alpha x)\Gamma M$. Then it is clear that I is an ideal contained in U. Now, we have to show that I is nonzero. Suppose that $I = 0$. By the 2-torsion freeness of M, $x\alpha y = y\alpha x$ and this a contradiction. Therefore, I is a nonzero ideal of M. □

Lemma 6.2.4. *Let U be a Lie ideal of a prime Γ-ring M such that $U \nsubseteq Z(M)$. Then there exist elements $a, b \in U$ such that $[a, b]_\alpha = a\alpha b - b\alpha a \neq 0$.*

Proof. Assume that $[x, y]_\alpha = 0$ for every $x, y \in U$ and $\alpha \in \Gamma$. This gives $[U, U]_\Gamma = 0$, a contradiction to our assumption. So, there exist elements $a, b \in U$ such that $[a, b]_\alpha = a\alpha b - b\alpha a \neq 0$. □

Lemma 6.2.5. *Assume that U is an admissible Lie ideal of a 2-torsion free prime Γ-ring M. If $t\alpha v\beta v + v\beta v\alpha t = 0$, for any $t \in M; v \in U$ and $\alpha, \beta \in \Gamma$, then $t = 0$.*

Proof. Since $t\alpha v\beta v + v\beta v\alpha t = 0$ for all $v \in U, t \in M$ and $\alpha, \beta \in \Gamma$. Linearize on v, where $u \in U$

$$\begin{aligned} 0 &= t\alpha(u+v)\beta(u+v) + (u+v)\beta(u+v)\alpha t \\ &= t\alpha(u\beta u + u\beta v + v\beta u + v\beta v) + (u\beta u + u\beta v + v\beta u + v\beta v)\alpha t \\ &= t\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha t. \end{aligned}$$

Replacing v by $v\alpha v$, we get

$$t\alpha(u\beta v\alpha v + v\alpha v\beta u) + (u\beta v\alpha v + v\alpha v\beta u)\alpha t = 0. \tag{6.4}$$

Applying $t\alpha v\beta v + v\beta v\alpha t = 0$ in (6.4), and using the condition(*)

$$t\alpha u\beta v\alpha v - v\alpha v\beta t\alpha u - u\alpha t\beta v\alpha v + v\alpha v\beta u\alpha t = 0.$$

$$\Rightarrow (t\alpha u - u\alpha t)\beta v\alpha v - v\alpha v\beta(t\alpha u - u\alpha t) = 0.$$

Therefore,

$$[t, u]_\alpha \beta v\alpha v - v\alpha v\beta[t, u]_\alpha = 0. \tag{6.5}$$

Again applying $t\alpha v\beta v + v\beta v\alpha t = 0$ in (6.5), we get

$$[t, u]_\alpha \beta v\alpha v - (-[t, u]_\alpha \beta v\alpha v) = 0.$$

$$\Rightarrow 2[t,u]_\alpha \beta v\alpha v = 0.$$

By the 2-torsion freeness of M,

$$[t,u]_\alpha \beta v\alpha v = 0, \forall\, u,v \in U; t \in M; \alpha,\beta \in \Gamma.$$

This implies,

$$[M,U]_\Gamma \Gamma(v\alpha v) = 0.$$

By Lemma 6.2.3, U contains a nonzero ideal I of M and this gives us, $[M,U]_\Gamma \Gamma I \Gamma(v\alpha v) = 0$. Therefore, $[M,U]_\Gamma \Gamma M \Gamma I \Gamma(v\alpha v) \subseteq [M,U]_\Gamma \Gamma I \Gamma(v\alpha v) = 0$. Since M is prime, so $I\Gamma(v\alpha v) = 0$ or $[M,U]_\Gamma = 0$. If $I\Gamma(v\alpha v) = 0$, then for $I \neq 0$ and by Lemma 6.2.4, we get $U = 0$, which is a contradiction. Therefore, $[M,U]_\Gamma = 0$, that is $t\beta v - v\beta t = 0$ for all $v \in U, t \in M, \beta \in \Gamma$. Since $t\alpha v\beta v + v\beta v\alpha t = 0$, and applying $t\beta v = v\beta t$

$$\begin{aligned} 0 &= t\alpha v\beta v + v\alpha v\beta t \\ &= t\alpha v\beta v + v\alpha t\beta v \\ &= t\alpha v\beta v + t\alpha v\beta v \\ &= 2t\alpha v\beta v. \end{aligned}$$

By the 2-torsion freeness of M, $t\alpha v\beta v = 0$ for all $v \in U, t \in M, \beta \in \Gamma$. Linearize $t\alpha v\beta v = 0$ on v, where $u \in U$

$$\begin{aligned} 0 &= t\alpha(u+v)\beta(u+v) \\ &= t\alpha(u\beta v + v\beta u). \end{aligned}$$

This implies,

$$t\alpha(u\beta v + v\beta u)\gamma u\alpha t = 0.$$

$$\Rightarrow t\alpha u\beta v\gamma u\alpha t + t\alpha v\beta u\gamma u\alpha t = 0.$$

Since $u\gamma u\alpha t = 0$ and $t\alpha u = u\alpha t$. Therefore,

$$(t\alpha u)\beta v\gamma(t\alpha u) = 0.$$

By the primeness of M, $t\alpha u = 0$. Since $u\beta m - m\beta u \in U$ for all $u \in U, m \in M, \beta \in \Gamma$. Therefore, $t\alpha(u\beta m - m\beta u) = 0$, that is, $t\alpha u\beta m - t\alpha m\beta u = 0$. This implies, $t\alpha m\beta u = 0$. But $u \neq 0$ and M is prime, consequently, $t = 0$. □

As explained earlier, the goal of this section is to prove the following theorem.

Theorem 6.2.6. *Let U be an admissible Lie ideal of a 2-torsion free prime Γ-ring M, and let d be a (U, M)-derivation of M. Then $\phi_\alpha(u, v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$.*

Proof. Let $x = 4(u\alpha v\beta[u, v]_\alpha\gamma v\alpha u + v\alpha u\beta[u, v]_\alpha\gamma u\alpha v)$. Then using Lemma 6.2.1(ii), we get

$$\begin{aligned} d(x) &= d((2u\alpha v)\beta[u, v]_\alpha\gamma(2v\alpha u) + (2v\alpha u)\beta[u, v]_\alpha\gamma(2u\alpha v)) \\ &= d(2u\alpha v)\beta[u, v]_\alpha\gamma(2v\alpha u) + 2u\alpha v d(\beta[u, v]_\alpha)\gamma 2v\alpha u + 2u\alpha v\beta[u, v]_\alpha\gamma d(2v\alpha u) \\ &\quad + d(2v\alpha u)\beta[u, v]_\alpha\gamma(2u\alpha v) + 2v\alpha u d(\beta[u, v]_\alpha)\gamma 2u\alpha v + 2v\alpha u\beta[u, v]_\alpha\gamma d(2u\alpha v). \end{aligned}$$

On the other hand using Lemma 6.2.1(i), we get

$$\begin{aligned} d(x) &= d(u\alpha(4v\beta[u, v]_\alpha\gamma v)\alpha u + v\alpha(4u\beta[u, v]_\alpha\gamma u)\alpha v) \\ &= d(u)\alpha 4v\beta[u, v]_\alpha\gamma v\alpha u + u\alpha d(4v\beta[u, v]_\alpha\gamma v)\alpha u + u\alpha 4v\beta[u, v]_\alpha\gamma v\alpha d(u) \\ &\quad + d(v)\alpha 4u\beta[u, v]_\alpha\gamma u\alpha v + v\alpha d(4u\beta[u, v]_\alpha\gamma u)\alpha v + v\alpha 4u\beta[u, v]_\alpha\gamma u\alpha d(v) \\ &= 4d(u)\alpha v\beta[u, v]_\alpha\gamma v\alpha u + 4u\alpha d(v)\beta[u, v]_\alpha\gamma v\alpha u + 4u\alpha v d(\beta[u, v]_\alpha)\gamma v\alpha u \\ &\quad + 4u\alpha v\beta[u, v]_\alpha\gamma d(v)\alpha u + 4u\alpha v\beta[u, v]_\alpha\gamma v\alpha d(u) + 4d(v)\alpha u\beta[u, v]_\alpha\gamma u\alpha v \\ &\quad + 4v\alpha d(u)\beta[u, v]_\alpha\gamma u\alpha v + 4v\alpha u d(\beta[u, v]_\alpha)\gamma u\alpha v + 4v\alpha u\beta[u, v]_\alpha\gamma d(u)\alpha v \\ &\quad + 4v\alpha u\beta[u, v]_\alpha\gamma u\alpha d(v). \end{aligned}$$

Equating these two expressions for $d(x)$ and using the Definition 6.2.1, we obtain

$$4(d(u\alpha v)-d(u)\alpha v-u\alpha d(v))\beta[u,v]_\alpha\gamma v\alpha u+4(d(v\alpha u)-d(v)\alpha u-v\alpha d(u))\beta[u,v]_\alpha\gamma u\alpha v$$
$$+4u\alpha v\beta[u,v]_\alpha\gamma(d(v\alpha u)-d(v)\alpha u-v\alpha d(u))+4v\alpha u\beta[u,v]_\alpha\gamma(d(u\alpha v)-d(u)\alpha v-u\alpha d(v))=0.$$

$$\Rightarrow 4(\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma v\alpha u+\phi_\alpha(v,u)\beta[u,v]_\alpha\gamma u\alpha v+u\alpha v\beta[u,v]_\alpha\gamma\phi_\alpha(v,u)$$
$$+v\alpha u\beta[u,v]_\alpha\gamma\phi_\alpha(u,v))=0.$$

Using Lemma 6.2.2(i), we get

$$4(\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma v\alpha u-\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma u\alpha v-u\alpha v\beta[u,v]_\alpha\gamma\phi_\alpha(u,v)+v\alpha u\beta[u,v]_\alpha\gamma\phi_\alpha(u,v))=0.$$

$$\Rightarrow 4(\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma[u,v]_\alpha+[u,v]_\alpha\beta[u,v]_\alpha\gamma\phi_\alpha(u,v))=0.$$

Using the condition (*) and 2-torsion freeness of M,

$$\phi_\alpha(u,v)\gamma[u,v]_\alpha\beta[u,v]_\alpha+[u,v]_\alpha\beta[u,v]_\alpha\gamma\phi_\alpha(u,v)=0, \forall\, u,v\in U,\alpha,\beta,\gamma\in\Gamma.$$

Since $U\not\subseteq Z(M)$, and therefore, $[u,v]_\alpha\neq 0$ for all $u,v\in U$ and $\alpha\in\Gamma$. Hence by Lemma 6.2.5, we obtain $\phi_\alpha(u,v)=0$ for all $u,v\in U$ and $\alpha\in\Gamma$. □

Lemma 6.2.7. *Let U be an admissible Lie ideal of a 2-torsion free prime Γ-ring M, and d be a (U,M)-derivation of M. Then $\phi_\beta(u\alpha u,m)=0$ for all $u\in U,m\in M$ and $\alpha,\beta\in\Gamma$.*

Proof. By Theorem 6.2.6, we have $\phi_\alpha(u,v)=0$ for all $u,v\in U;\alpha\in\Gamma$. Thus for all $u\in U,m\in M$ and $\alpha,\beta\in\Gamma$, we obtain

$$0=\phi_\alpha(u,u\beta m-m\beta u)$$
$$=d(u\alpha(u\beta m-m\beta u))-d(u)\alpha(u\beta m-m\beta u)-u\alpha d(u\beta m-m\beta u)$$
$$=d(u\alpha u\beta m-u\alpha m\beta u)-d(u)\alpha(u\beta m-m\beta u)-u\alpha d(u\beta m-m\beta u)$$
$$=d(u\alpha u\beta m)-d(u\alpha m\beta u)-d(u)\alpha u\beta m+d(u)\alpha m\beta u-u\alpha(d(u)\beta m+u\beta d(m)-d(m)\beta u-m\beta d(u))$$

$$= d(u\alpha u\beta m) - d(u)\alpha m\beta u - u\alpha d(m)\beta u - u\alpha m\beta d(u) - d(u)\alpha u\beta m + d(u)\alpha m\beta u$$

$$- u\alpha d(u)\beta m - u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u)$$

$$= d(u\alpha u\beta m) - d(u)\alpha u\beta m - u\alpha d(u)\beta m - u\alpha u\beta d(m)$$

$$= d((u\alpha u)\beta m) - d(u\alpha u)\beta m - -(u\alpha u)\beta d(m) = \phi_\beta(u\alpha u, m).$$

□

Now, we prove the other result as follows:

Theorem 6.2.8. *Let U be a square closed Lie ideal of a 2-torsion free prime Γ-ring M, and d be a (U, M)-derivation of M. Then $d(u\alpha m) = d(u)\alpha m + u\alpha d(m)$ for all $u \in U, m \in M$ and $\alpha \in \Gamma$.*

Proof. Since d is a (U, M)-derivation of a prime Γ-ring M, so for all $u \in U, m \in M$ and $\alpha, \beta \in \Gamma$, we have

$$d(u\alpha(u\beta m) + (u\beta m)\alpha u) = d(u)\alpha u\beta m + u\alpha d(u\beta m) + d(u\beta m)\alpha u + u\beta m\alpha d(u). \quad (6.6)$$

On the other hand

$$d(u\alpha u\beta m + u\beta m\alpha u) = d(u\alpha u\beta m) + d(u)\beta m\alpha u + u\beta d(m)\alpha u + u\beta m\alpha d(u). \quad (6.7)$$

From Lemma 6.2.7, we have

$$\phi_\beta(u\alpha u, m) = 0, \forall\ u \in U; m \in M; \alpha, \beta \in \Gamma.$$

$$\Rightarrow d(u\alpha u\beta m) - d(u)\alpha u\beta m - u\alpha d(u)\beta m - u\alpha u\beta d(m) = 0.$$

$$\Rightarrow d(u\alpha u\beta m) = d(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m). \quad (6.8)$$

Now, using (6.8) in (6.7), we get

$$d(u\alpha u\beta m + u\beta m\alpha u) = d(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + d(u)\beta m\alpha u + u\beta d(m)\alpha u + u\beta m\alpha d(u). \quad (6.9)$$

Comparing (6.6) and (6.9), we get

$$u\alpha d(u\beta m) + d(u\beta m)\alpha u = u\alpha d(u)\beta m + u\alpha u\beta d(m) + d(u)\beta m\alpha u + u\beta d(m)\alpha u.$$

Using Definition 6.2.1, we obtain

$$u\alpha\phi_\beta(u,m) + \phi_\beta(u,m)\alpha u = 0, \forall\, u \in U, m \in M; \alpha, \beta \in \Gamma. \quad (6.10)$$

Linearizing (6.10) on u and using (6.10)

$$\begin{aligned}
&(u+v)\alpha\phi_\beta(u+v,m) + \phi_\beta(u+v,m)\alpha(u+v) = 0.\\
&\Rightarrow u\alpha\phi_\beta(u,m) + u\alpha\phi_\beta(v,m) + v\alpha\phi_\beta(u,m) + v\alpha\phi_\beta(v,m)\\
&+ \phi_\beta(u,m)\alpha u + \phi_\beta(u,m)\alpha v + \phi_\beta(v,m)\alpha u + \phi_\beta(v,m)\alpha v = 0.\\
&\Rightarrow u\alpha\phi_\beta(v,m) + v\alpha\phi_\beta(u,m) + \phi_\beta(u,m)\alpha v + \phi_\beta(v,m)\alpha u = 0. \quad (6.11)
\end{aligned}$$

Replacing v by $v\gamma v$ in (6.11) and using Lemma 6.2.7, we get

$$(v\gamma v)\alpha\phi_\beta(u,m) + \phi_\beta(u,m)\alpha(v\gamma v) = 0.$$

If $U \nsubseteq Z(M)$, using Lemma 6.2.5, $\phi_\beta(u,m) = 0$ for all $u \in U, m \in M$ and $\beta \in \Gamma$.

If $U \subseteq Z(M)$, by the 2-torsion freeness of M, $(v\gamma v)\alpha\phi_\beta(u,m) = 0$. Therefore, $0 = c\delta(v\gamma v)\alpha\phi_\beta(u,m) = (v\gamma v)\delta c\alpha\phi_\beta(u,m)$, where $c \in M$ and $\delta \in \Gamma$. Since M is prime, so $v\gamma v = 0$ or $\phi_\beta(u,m) = 0$. But $v \neq 0$, hence $\phi_\beta(u,m) = 0$ for all $u \in U, m \in M$ and $\beta \in \Gamma$. This completes the proof of the theorem. □

6.3 Higher (U, M)-Derivations in Prime Γ-Rings

Ferrero and Haetinger [15] extended Herstein's [19] theorem to higher derivations using Jordan triple higher derivations. Also, Haetinger extended Awtar's [3] result to higher derivations. We generalized a result of A. K. Faraj, C. Haetinger and A. H. Majeed [16] in Γ-rings by the new concept of higher (U, M)-derivations. In order to prove the desired result stated at the beginning of the chapter, we have to determine some important results in the following way.

Lemma 6.3.1. *Let $D = (d_i)_{i\in \boldsymbol{N}}$ be a higher (U, M)-derivation of a 2-torsion free Γ-ring M. Then $d_n(u\alpha m\beta u) = \sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(u)$ for all $u \in U, m \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. Let $x = u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\beta m + m\beta(2u))\alpha u$. Replacing m and s by $(2u)\beta m + m\beta(2u)$ in Definition 6.1.2, and using the condition (*)

$$\begin{aligned}
d_n(x) &= \sum_{i+j=n} (d_i(u)\alpha d_j((2u)\beta m + m\beta(2u)) + d_i((2u)\beta m + m\beta(2u))\alpha d_j(u)) \\
&= 2\sum_{i+j=n} d_i(u)\alpha \sum_{l+t=j} ((d_l(u)\beta d_t(m) + d_l(m)\beta d_t(u)) + 2\sum_{i+j=n}\sum_{p+q=i} (d_p(u)\beta d_q(m) \\
&\qquad + d_p(m)\beta d_q(u))\alpha d_j(u)) \\
&= 2\sum_{i+l+t=n} (d_i(u)\alpha d_l(u)\beta d_t(m) + d_i(u)\alpha d_l(m)\beta d_t(u)) + 2\sum_{p+q+j=n} (d_p(u)\beta d_q(m)\alpha d_j(u) \\
&\qquad + d_p(m)\beta d_q(u)\alpha d_j(u)) \\
&= 2\sum_{i+l+t=n} d_i(u)\alpha d_l(u)\beta d_t(m) + 2\sum_{i+l+t=n} d_i(u)\alpha d_l(m)\beta d_t(u) \\
&\quad + 2\sum_{p+q+j=n} d_p(u)\alpha d_q(m)\beta d_j(u) + 2\sum_{p+q+j=n} d_p(m)\alpha d_q(u)\beta d_j(u). \quad (6.12)
\end{aligned}$$

On the other hand

$$\begin{aligned}
d_n(x) &= d_n((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2d_n(u\alpha m\beta u) + 2d_n(u\beta m\alpha u)\\
&= d_n((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2d_n(u\alpha m\beta u) + 2d_n(u\alpha m\beta u))\\
&= 2\sum_{i+j=n}(d_i(u\alpha u)\beta d_j(m) + d_i(m)\beta d_j(u\alpha u)) + 4d_n(u\alpha m\beta u)\\
&= 2\sum_{i+j=n}\sum_{r+s=i} d_r(u)\alpha d_s(u)\beta d_j(m) + 2\sum_{i+j=n}\sum_{t+k=j} d_i(m)\alpha d_t(u)\beta d_k(u) + 4d_n(u\alpha m\beta u)\\
&= 2\sum_{r+s+j=n} d_r(u)\alpha d_s(u)\beta d_j(m) + 2\sum_{i+t+k=n} d_i(m)\alpha d_t(u)\beta d_k(u) + 4d_n(u\alpha m\beta u).
\end{aligned} \tag{6.13}$$

Now, comparing (6.12) and (6.13), we get

$$4d_n(u\alpha m\beta u) = 4\sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(u), \forall\ u \in U, m \in M; \alpha, \beta \in \Gamma.$$

Using 2-torsion freeness of M, we get the desired result. $\square$

Lemma 6.3.2. *If $D = (d_i)_{i\in \boldsymbol{N}}$ is a higher (U, M)-derivation of M. Then for all $u, v \in U; m \in M$ and $\alpha, \beta \in \Gamma$, $d_n(u\alpha m\beta v+v\alpha m\beta u) = \sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(v)+ d_i(v)\alpha d_j(m)\beta d_k(u)$.*

Proof. Linearizing of $d_n(u\alpha m\beta u) = \sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(u)$ with respect to u, we obtain

$$\begin{aligned}
d_n((u+v)\alpha m\beta(u+v)) &= \sum_{i+j+k=n} d_i(u+v)\alpha d_j(m)\beta d_k(u+v)\\
&= \sum_{i+j+k=n}(d_i(u)\alpha d_j(m)\beta d_k(u) + d_i(u)\alpha d_j(m)\beta d_k(v) + d_i(v)\alpha d_j(m)\beta d_k(u)\\
&\qquad + d_i(v)\alpha d_j(m)\beta d_k(v)).
\end{aligned} \tag{6.14}$$

On the other hand

$$d_n((u+v)\alpha m\beta(u+v)) = d_n(u\alpha m\beta u) + d_n(u\alpha m\beta v + v\alpha m\beta u) + d_n(v\alpha m\beta v)$$
$$= \sum_{i+j+k=n} (d_i(u)\alpha d_j(m)\beta d_k(u) + d_n(u\alpha m\beta v + v\alpha m\beta u)$$
$$+ \sum_{i+j+k=n} (d_i(v)\alpha d_j(m)\beta d_k(v). \quad (6.15)$$

Now, comparing (6.14) and (6.15)

$$d_n(u\alpha m\beta v \ + \ v\alpha m\beta u) \quad = \quad \sum_{i+j+k=n} d_i(u)\alpha d_j(m)\beta d_k(v) \ + \ d_i(v)\alpha d_j(m)\beta d_k(u).$$

□

Definition 6.3.1. For every higher (U, M)-derivation $D = (d_i)_{i\in\mathbf{N}}$ of M, we define $\phi_n^\alpha(u, m) = d_n(u\alpha m) - \sum_{i+j=n} d_i(u)\alpha d_j(m)$ for all $u \in U, m \in M, \alpha \in \Gamma$ and $n \in \mathbf{N}$.

Remark 6.3.1. $\phi_n^\alpha(u, m) = 0$ for all $u \in U, m \in M, \alpha \in \Gamma$ and $n \in \mathbf{N}$ if and only if $d_n(u\alpha m) = \sum_{i+j=n} d_i(u)\alpha d_j(m)$ for all $u \in U, m \in M, \alpha \in \Gamma$ and $n \in \mathbf{N}$.

Lemma 6.3.3. *Let $D = (d_i)_{i\in\boldsymbol{N}}$ be a higher (U, M)-derivation of M. Then for every $u, v \in U; m, p \in M; \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$*
(i)$\phi_n^\alpha(u, m) + \phi_n^\alpha(m, u) = 0$; *(ii)*$\phi_n^\alpha(u + v, m) = \phi_n^\alpha(u, m) + \phi_n^\alpha(v, m)$;
(iii)$\phi_n^\alpha(u, m + p) = \phi_n^\alpha(u, m) + \phi_n^\alpha(u, p)$; *(iv)*$\phi_n^{\alpha+\beta}(u, m) = \phi_n^\alpha(u, m) + \phi_n^\beta(u, m)$.

Proof. (i) By the Definition 6.3.1 and using the definition of higher (U, M)-derivation of M, we obtain

$$\begin{aligned}
\phi_n^\alpha(u,m)+\phi_n^\alpha(m,u) &= d_n(u\alpha m)-\sum_{i+j=n} d_i(u)\alpha d_j(m)+d_n(m\alpha u)-\sum_{i+j=n} d_i(m)\alpha d_j(u)\\
&= d_n(u\alpha m+m\alpha u)-\sum_{i+j=n} d_i(u)\alpha d_j(m)-\sum_{i+j=n} d_i(m)\alpha d_j(u)\\
&= \sum_{i+j=n} d_i(u)\alpha d_j(m)+\sum_{i+j=n} d_i(m)\alpha d_j(u)-\sum_{i+j=n} d_i(u)\alpha d_j(m)\\
&\quad -\sum_{i+j=n} d_i(m)\alpha d_j(u)=0.
\end{aligned}$$

(ii) By the definition of $\phi_n^\alpha(u,m)$, we get

$$\begin{aligned}
\phi_n^\alpha(u+v,m) &= d_n((u+v)\alpha m)-\sum_{i+j=n} d_i(u+v)\alpha d_j(m)\\
&= d_n(u\alpha m+v\alpha m)-\sum_{i+j=n} d_i(u)\alpha d_j(m)-\sum_{i+j=n} d_i(v)\alpha d_j(m)\\
&= d_n(u\alpha m)-\sum_{i+j=n} d_i(u)\alpha d_j(m)+d_n(v\alpha m)-\sum_{i+j=n} d_i(v)\alpha d_j(m)\\
&= \phi_n^\alpha(u,m)+\phi_n^\alpha(v,m).
\end{aligned}$$

(iii) and (iv) are very easy to proof. □

Theorem 6.3.4. *If U is an admissible Lie ideal of a 2-torsion free prime Γ-ring M, and $D=(d_i)_{i\in \boldsymbol{N}}$ is a higher (U,M)-derivation of M. Let $n\in \boldsymbol{N}; u,v\in U;\alpha,\beta,\gamma\in\Gamma$ and $\phi_p^\alpha(u,v)=0$ for every $p<n$, then $\phi_n^\alpha(u,v)=0$ for all $u,v\in U;\alpha\in\Gamma$ and $n\in \boldsymbol{N}$.*

Proof. Let $T=d_n(4u\alpha v\beta w\gamma v\alpha u+4v\alpha u\beta w\gamma u\alpha v)$. By Lemma 6.3.1, we get

$$\begin{aligned}
T &= d_n(u\alpha(4v\beta w\gamma v)\alpha u)+d_n(v\alpha(4u\beta w\gamma u)\alpha v)\\
&= 4\sum_{i+p+l=n}(d_i(u)\alpha d_p(v\beta w\gamma v)\alpha d_l(u))+4\sum_{i+p+l=n}(d_i(v)\alpha d_p(u\beta w\gamma u)\alpha d_l(v))\\
&= 4\sum_{i+j+k+h+l=n} d_i(u)\alpha d_j(v)\beta d_k(w)\gamma d_h(v)\alpha d_l(u)+4\sum_{i+j+k+h+l=n} d_i(v)\alpha d_j(u)\beta d_k(w)\gamma d_h(u)\alpha d_l(v).
\end{aligned}$$

On the other hand by Lemma 6.3.2, we obtain

$$
\begin{aligned}
T &= d_n((2u\alpha v)\beta w\gamma(2v\alpha u)+(2v\alpha u)\beta w\gamma(2u\alpha v)) \\
&= \sum_{r+s+t=n}(d_r(2u\alpha v)\beta d_s(w)\gamma d_t(2v\alpha u)+d_r(2v\alpha u)\beta d_s(w)\gamma d_t(2u\alpha v)) \\
&= 4\sum_{r+s+t=n} d_r(u\alpha v)\beta d_s(w)\gamma d_t(v\alpha u)+4\sum_{r+s+t=n} d_r(v\alpha u)\beta d_s(w)\gamma d_t(u\alpha v).
\end{aligned}
$$

Comparing above two expressions for T, we obtain

$$
\begin{aligned}
&\sum_{i+j+k+h+l=n} d_i(u)\alpha d_j(v)\beta d_k(w)\gamma d_h(v)\alpha d_l(u)-\sum_{r+s+t=n} d_r(u\alpha v)\beta d_s(w)\gamma d_t(v\alpha u) \\
&+\sum_{i+j+k+h+l=n} d_i(v)\alpha d_j(u)\beta d_k(w)\gamma d_h(u)\alpha d_l(v)-\sum_{r+s+t=n} d_r(v\alpha u)\beta d_s(w)\gamma d_t(u\alpha v)=0.
\end{aligned}
$$

Since $\phi_p^\alpha(u,v)=0$, for every $p<n$, that is $d_p(u\alpha v)=\sum_{i+j=p} d_i(u)\alpha d_j(v)$. Therefore,

$$
\begin{aligned}
&\sum_{i+j+k+h+l=n}(d_i(u)\alpha d_j(v)\beta d_k(w)\gamma d_h(v)\alpha d_l(u))-\sum_{r+s+t=n}(d_r(u\alpha v)\beta d_s(w)\gamma d_t(v\alpha u)) \\
&\quad=(\sum_{i+j=n} d_i(u)\alpha d_j(v))\beta w\gamma v\alpha u+u\alpha v\beta w\gamma(\sum_{h+l=n} d_h(v)\alpha d_l(u)) \\
&\quad+\sum_{i+j+k+h+l=n}^{i+j<n,h+l<n}(d_i(u)\alpha d_j(v)\beta d_k(w)\gamma d_h(v)\alpha d_l(u))-d_n((u\alpha v)\beta w\gamma(v\alpha u) \\
&\quad-(u\alpha v)\beta w\gamma d_n(v\alpha u)-\sum_{r+s+t=n}^{i+j=r<n,p+q=t<n}(d_i(u)\alpha d_j(v)\beta d_s(w)\gamma d_p(v)\alpha d_q(u)) \\
&=-(d_n((u\alpha v)-\sum_{i+j=n} d_i(u)\alpha d_j(v))\beta(w\gamma v\alpha u)-(u\alpha v\beta w)\gamma(d_n(v\alpha u)-\sum_{h+l=n} d_h(v)\alpha d_l(u)) \\
&\quad=-(\phi_n^\alpha(u,v)\beta w\gamma v\alpha u+u\alpha v\beta w\gamma\phi_n^\alpha(v,u)).
\end{aligned}
$$

Proceeding as above, we also have

$$
\begin{aligned}
&\sum_{i+j+k+h+l=n} d_i(v)\alpha d_j(u)\beta d_k(w)\gamma d_h(u)\alpha d_l(v)-\sum_{r+s+t=n} d_r(v\alpha u)\beta d_s(w)\gamma d_t(u\alpha v) \\
&\quad=-(\phi_n^\alpha(v,u)\beta w\gamma u\alpha v+v\alpha u\beta w\gamma\phi_n^\alpha(u,v)).
\end{aligned}
$$

Therefore,

$$\phi_n^\alpha(u,v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma\phi_n^\alpha(v,u) + \phi_n^\alpha(v,u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma\phi_n^\alpha(u,v) = 0.$$

By Lemma 6.3.3(i), we obtain

$$\phi_n^\alpha(u,v)\beta w\gamma v\alpha u - u\alpha v\beta w\gamma\phi_n^\alpha(u,v) - \phi_n^\alpha(u,v)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma\phi_n^\alpha(u,v) = 0.$$

This implies,

$$\phi_n^\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\phi_n^\alpha(u,v) = 0,\ \forall\ w \in U.$$

In view of Lemma 4.2.6 and Lemma 4.2.7, we obtain $\phi_n^\alpha(u,v)\beta w\gamma[x,y]_\delta = 0$ for all $u,v,w,x,y \in U$ and $\alpha,\beta,\gamma,\delta \in \Gamma$. Since $[x,y]_\delta \neq 0$ as $U \nsubseteq Z(M)$. Hence by Lemma 4.2.5, we obtain $\phi_n^\alpha(u,v) = 0$ for all $u,v \in U; \alpha \in \Gamma$ and $n \in \mathbf{N}$. □

Lemma 6.3.5. *Let U be an admissible Lie ideal of a prime Γ-ring M, and $D = (d_i)_{i\in \boldsymbol{N}}$ be a higher (U,M)-derivation of M. Then $\phi_n^\alpha(u\beta u, m) = 0$ for all $u \in U, m \in M; \alpha,\beta \in \Gamma$ and $n \in \boldsymbol{N}$.*

Proof. By Theorem 6.3.4, we have $\phi_n^\alpha(u,v) = 0$ for all $u,v \in U$ and $\alpha \in \Gamma$. Now for any $m \in M$, replacing v by $u\beta m - m\beta u$, we get

$$\begin{aligned}
0 &= \phi_n^\alpha(u, u\beta m - m\beta u)\\
&= d_n(u\alpha(u\beta m - m\beta u)) - \sum_{i+j=n} d_i(u)\alpha d_j(u\beta m - m\beta u)\\
&= d_n(u\alpha u\beta m) - d_n(u\alpha m\beta u) - \sum_{i+j=n} d_i(u)\alpha \sum_{p+q=j}(d_p(u)\beta d_q(m) - d_p(m)\beta d_q(u))\\
&= d_n(u\alpha u\beta m) - d_n(u\alpha m\beta u) - \sum_{i+p+q=n} d_i(u)\alpha d_p(u)\beta d_q(m) + \sum_{i+p+q=n} d_i(u)\alpha d_p(m)\beta d_q(u)\\
&= d_n(u\alpha u\beta m) - d_n(u\alpha m\beta u) - \sum_{i+p+q=n} d_i(u)\alpha d_p(u)\beta d_q(m) + d_n(u\alpha m\beta u)\\
&= d_n(u\beta u\alpha m) - \sum_{s+q=n}(\sum_{i+p=s} d_i(u)\beta d_p(u))\alpha d_q(m) = \phi_n^\alpha(u\beta u, m).
\end{aligned}$$

$\square$

Theorem 6.3.6. *Let U be an admissible Lie ideal of a 2-torsion free prime Γ-ring M, and $D = (d_i)_{i\in \boldsymbol{N}}$ be a higher (U, M)-derivation of M. Then $d_n(u\beta m) = \sum_{i+j=n} d_i(u)\beta d_j(m)$ for all $u \in U, m \in M, \beta \in \Gamma$ and $n \in \boldsymbol{N}$.*

Proof. By Definition 6.3.1, we have

$$\phi_0^\alpha(u, m) = 0, \forall\ u \in U, m \in M, \alpha \in \Gamma.$$

Also, by Theorem 6.2.8,

$$\phi_1^\alpha(u, m) = 0, \forall\ u \in U, m \in M, \alpha \in \Gamma.$$

Now, we proceed by induction. Suppose $\phi_p^\alpha(u, m) = 0$ for all $u \in U, m \in M, \alpha \in \Gamma$ and $p \in \mathbf{N}$. This implies, $d_p(u\alpha m) = \sum_{i+j=p} d_i(u)\alpha d_j(m), u \in U, m \in M$ and $\alpha \in \Gamma$ and $p < n$, where $p, n \in \mathbf{N}$. Since $D = (d_i)_{i\in\mathbf{N}}$ is a higher (U, M)-derivation of M. Therefore,

$$\begin{aligned}
d_n(u\alpha(u\beta m) + (u\beta m)\alpha u) &= \sum_{i+j=n} (d_i(u)\alpha d_j(u\beta m) + d_i(u\beta m)\alpha d_j(u)) \\
&= u\alpha d_n(u\beta m) + d_n(u)\alpha(u\beta m) + \sum_{i+j=n}^{i,j<n} d_i(u)\alpha d_j(u\beta m) \\
&\quad + (u\beta m)\alpha d_n(u) + d_n(u\beta m)\alpha u + \sum_{i+j=n}^{i,j<n} d_i(u\beta m)\alpha d_j(u) \\
&= u\alpha d_n(u\beta m) + d_n(u)\alpha(u\beta m) + \sum_{i+j=n}^{i,j<n} d_i(u)\alpha \sum_{s+t=j} d_s(u)\beta d_t(m) \\
&\quad + (u\beta m)\alpha d_n(u) + d_n(u\beta m)\alpha u + \sum_{i+j=n}^{i,j<n} (\sum_{l+q=i} d_l(u)\beta d_q(m))\alpha d_j(u)
\end{aligned}$$

$$= u\alpha d_n(u\beta m) + d_n(u)\alpha(u\beta m) + \sum_{i+s+t=n}^{i,s+t<n} d_i(u)\alpha d_s(u)\beta d_t(m)$$

$$+ (u\beta m)\alpha d_n(u) + d_n(u\beta m)\alpha u + \sum_{l+q+j=n}^{l+q,j<n} d_l(u)\beta d_q(m)\alpha d_j(u). \quad (6.16)$$

On the other hand, using Lemma 6.3.1 and Lemma 6.3.5, we get

$$d_n(u\alpha(u\beta m) + (u\beta m)\alpha u) = d_n(u\alpha u\beta m) + d_n(u\beta m\alpha u)$$

$$= \sum_{p+q+j=n} d_p(u)\alpha d_q(u)\beta d_j(m) + \sum_{i+j+k=n} d_i(u)\beta d_j(m)\alpha d_k(u)$$

$$= d_n(u)\alpha u\beta m + u\alpha \sum_{q+j=n} d_q(u)\beta d_j(m) + \sum_{p+q+j=n}^{p,q+j<n} d_p(u)\alpha d_q(u)\beta d_j(m)$$

$$+ u\beta m\alpha d_n(u) + \sum_{i+j=n} d_i(u)\beta d_j(m)\alpha u + \sum_{i+j+k=n}^{i+j,k<n} d_i(u)\beta d_j(u)\alpha d_k(m). \quad (6.17)$$

By comparing (6.16) and (6.17), and using the condition (*)

$$u\alpha d_n(u\beta m) + d_n(u\beta m)\alpha u = u\alpha \sum_{q+j=n} d_q(u)\beta d_j(m) + \sum_{i+j=n} d_i(u)\beta d_j(m)\alpha u.$$

$$\Rightarrow u\alpha(d_n(u\beta m) - \sum_{q+j=n} d_q(u)\beta d_j(m)) + (d_n(u\beta m) - \sum_{i+j=n} d_i(u)\beta d_j(m))\alpha u = 0.$$

$$\Rightarrow u\alpha\phi_n^\beta(u,m) + \phi_n^\beta(u,m)\alpha u = 0. \quad (6.18)$$

Linearizing of (6.18) with respect to u, gives us

$$\phi_n^\beta(u,m)\alpha v + \phi_n^\beta(v,m)\alpha u + u\alpha\phi_n^\beta(v,m) + v\alpha\phi_n^\beta(u,m) = 0.$$

Replacing v by $v\alpha v$, then using Lemma 6.2.5 and Lemma 6.3.5, we get

$$\phi_n^\alpha(u,m) = 0, \forall\, u \in U, m \in M, \alpha \in \Gamma, n \in \mathbf{N}.$$

Hence $d_n(u\beta m) = \sum_{i+j=n} d_i(u)\beta d_j(m)$ for all $u \in U, m \in M, \beta \in \Gamma$ and $n \in \mathbf{N}$. $\square$

Chapter 7

Generalized (U, M)-Derivations

This chapter makes a study of generalized (U, M)-derivations and generalized higher (U, M)-derivations of Γ-rings analogous to the study of (U, M)-derivations and higher (U, M)-derivations of Γ-rings in Chapter 6. In view of the notions of (U, M)-derivation and higher (U, M)-derivation of Γ-rings here we introduce the concepts of generalized (U, M)-derivation and generalized higher (U, M)-derivation of Γ-rings. We start the discussion with the introductory definitions of generalized (U, M)-derivation and generalized higher (U, M)-derivation of Γ-rings.

In the next, associated with some important consequences due to (U, M)-derivations of Γ-rings developed in the previous chapter, here we determine some useful significant results on generalized (U, M)-derivations of Γ-rings. Then we extend the results of A. K. Faraj, C. Haetinger and A. H. Majeed [16] in Γ-rings by the new concept of generalized (U, M)-derivations of Γ-rings.

Finally, we conclude this chapter by proving the analogous results corresponding to the results of previous chapter considering generalized higher (U, M)-derivations of prime Γ-rings instead of higher (U, M)-derivations of prime Γ-rings almost similar way after developing a number of results regarding this derivation.

7.1 Introduction

A. K. Faraj, C. Haetinger and A. H. Majeed [16] introduced generalized (U, R)-derivations in rings as a generalization of Jordan derivations on Lie ideals of a ring. They extended Awtar's [3] theorem to generalized higher (U, R)-derivations by proving that if R is a prime ring, char$(R) \neq 2$, U is an admissible Lie ideal of R and $F = (f_i)_{i \in N}$ is a generalized (U, R)-derivations of R, then $f_n(ur) = \sum_{i+j=n} f_i(u)d_j(r)$ for all $u \in U, r \in R, n \in N$.

Following the notions of (U, M)-derivation and higher (U, M)-derivation of a Γ-ring in the previous chapter here we introduce the concepts of generalized (U, M)-derivation and generalized higher (U, M)-derivation of Γ-rings in the following way.

Definition 7.1.1. Let U be a Lie ideal of a Γ-ring M. An additive mapping $f : M \to M$ is a *generalized* (U, M)*-derivation* of M if there exists a (U, M)-derivation d of M such that $f(u\alpha m + s\alpha u) = f(u)\alpha m + u\alpha d(m) + f(s)\alpha u + s\alpha d(u)$ is satisfied for all $u \in U; m, s \in M$ and $\alpha \in \Gamma$.

The following are examples of (U, M)-derivation and generalized (U, M)-derivation of a Γ-ring.

Example 7.1.1. *Let R be an associative ring with 1, and let U be a Lie ideal of R. Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \boldsymbol{Z} \right\}$, then M is a Γ-ring. Let $N = \{(x, x) : x \in R\} \subseteq M$, then N is a sub Γ-ring of M. Let $U_1 = \{(u, u) : u \in U\}$, then U_1 is a Lie ideal of N. Let $f : R \to R$ be a generalized (U, R)-derivation . Then there exists a (U, R)-derivation $d : R \to R$ such that $f(ux + su) = f(u)x + ud(x) + f(s)u + sd(u), \forall\ u \in U, x, s \in R$. If we define a mapping $D : N \to N$ by*

$D((x,x)) = (d(x), d(x))$, then we have

$$D((u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(x,x) + (y,y)\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)) = D((unx, unx) + (ynu, ynu))$$

$$= D((unx + ynu, unx + ynu)) = (d(unx + ynu), d(unx + ynu)).$$

After calculation as in Example 6.1.1, we have

$$D(u_1\alpha x_1 + y_1\alpha u_1) = D(u_1)\alpha x_1 + u_1\alpha D(x_1) + D(y_1)\alpha u_1 + y_1\alpha D(u_1),$$

where $u_1 = (u,u), \alpha = \begin{pmatrix} n \\ 0 \end{pmatrix}, x_1 = (x,x)$ and $y_1 = (y,y)$. Hence D is a $(U_1, N)-$ derivation on N. Let $F : N \to N$ be the additive mapping defined by $F((x,x)) = (f(x), f(x))$, then considering $u_1 = (u,u) \in U_1, \alpha = \begin{pmatrix} n \\ 0 \end{pmatrix} \in \Gamma$ and $x_1 = (x,x), y_1 = (y,y) \in N$, we have

$$F(u_1\alpha x_1 + y_1\alpha u_1) = F((unx + ynu, unx + ynu)) = (f(unx + ynu), f(unx + ynu))$$

$$= (f(u)nx + und(x) + f(y)nu + ynd(u), f(u)nx + und(x) + f(y)nu + ynd(u))$$

$$= (f(u)nx + und(x), f(u)nx + und(x)) + (f(y)nu + ynd(u), f(y)nu + ynd(u))$$

$$= (f(u)nx, f(u)nx) + (und(x), und(x)) + (f(y)nu, f(y)nu) + (ynd(u), ynd(u))$$

$$= (f(u), f(u))\begin{pmatrix} n \\ 0 \end{pmatrix}(x,x) + (u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(x), d(x)) + (f(y), f(y))$$

$$\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u) + (y,y)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(u), d(u)) = F((u,u))\begin{pmatrix} n \\ 0 \end{pmatrix}(x,x)$$

$$+ (u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(D((x,x)) + F((y,y))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u) + (y,y)\begin{pmatrix} n \\ 0 \end{pmatrix}D((u,u)).$$

$$\Rightarrow \quad F(u_1\alpha x_1 \, + \, y_1\alpha u_1) \;\; = \;\; F(u_1)\alpha x_1 \, + \, u_1\alpha D(x_1) \, + \, F(y_1)\alpha u_1 \, + \, y_1\alpha D(u_1).$$

Hence F is a generalized $(U_1, N)-$derivation on N.

Definition 7.1.2. If U is a Lie ideal of a Γ-ring M and $F = (f_i)_{i\in N_0}$ is a family of additive mappings of M into itself, where $f_0 = id_M$ then F is a *generalized higher (U, M)-derivation* of M if for each $n \in \mathbf{N}$ there exists an higher (U, M)-derivation $D = (d_i)_{i\in\mathbf{N}}$ of M such that

$$f_n(u\alpha m + s\alpha u) = \sum_{i+j=n} (f_i(u)\alpha d_j(m) + f_i(s)\alpha d_j(u))$$

holds for all $u \in U, m, s \in M$ and $\alpha, \beta \in \Gamma$.

Example 7.1.2. *Let U be a Lie ideal of an associative ring R with 1, and let $f_n : R \to R$ be a generalized higher (U, R)-derivation . Then there exists a higher (U, R) derivation $d_n : R \to R$ such that*

$$f_n(ux + yu) = \sum_{i+j=n} (f_i(u)d_j(x) + f_i(y)d_j(u)), \forall\, u \in U, x, y \in R.$$

Suppose N and U_1 are as in Example 7.1.1. If we define a mapping $D_n : N \to N$ by $D_n((x, x)) = (d_n(x), d_n(x))$. Then D_n is a higher (U_1, N)-derivation on N. Let $F_n : N \to N$ be the additive mapping defined by $F_n((x, x)) = (f_n(x), f_n(x))$. Then by the similar calculation as in Example 7.1.1, we can show that, F_n is a generalized higher (U_1, N)-derivation on N.

Except otherwise mentioned, throughout this chapter, M is a 2-torsion free Γ-ring which satisfies the assumption $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$; $\alpha, \beta \in \Gamma$ and it is denoted by (*); U is a Lie ideal of M.

7.2 Generalized (U, M)-Derivations in Prime Γ-Rings

To determine some important results of prime Γ-rings with generalized (U, M)-derivations, we have to develop some needful results proceeding as follows.

Lemma 7.2.1. *If f is a generalized $(U, M)-$derivation of M for which d is the associated $(U, M)-$derivation of M. Then for all $u, v \in U$; $m \in M$ and $\alpha, \beta \in \Gamma$,*
(i) $f(u\alpha m\beta u) = f(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u)$;
(ii) $f(u\alpha m\beta v + v\alpha m\beta u) = f(u)\alpha m\beta v + u\alpha d(m)\beta v + u\alpha m\beta d(v) + f(v)\alpha m\beta u + v\alpha d(m)\beta u + v\alpha m\beta d(u)$.

Proof. By the definition of generalized (U, M)-derivation of M, we have $f(u\alpha m + s\alpha u) = f(u)\alpha m + u\alpha d(m) + f(s)\alpha u + s\alpha d(u)$ for all $u \in U; m, s \in M$ and $\alpha \in \Gamma$. Replacing m and s by $(2u)\beta m + m\beta(2u)$ and let $w = u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\beta m + m\beta(2u))\alpha u$.
On the one hand

$$f(w) = 2(f(u)\alpha(u\beta m+m\beta u)+u\alpha d(u\beta m+m\beta u)+f(u\beta m+m\beta u)\alpha u+(u\beta m+m\beta u)\alpha d(u))$$

$$= 2(f(u)\alpha u\beta m + f(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u)$$

$$+ f(u)\beta m\alpha u + u\beta d(m)\alpha u + f(m)\beta u\alpha u + m\beta d(u)\alpha u + u\beta m\alpha d(u) + m\beta u\alpha d(u))$$

$$= 2(f(u)\alpha u\beta m + f(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u)$$

$$+ f(u)\alpha m\beta u + u\alpha d(m)\beta u + f(m)\alpha u\beta u + m\alpha d(u)\beta u + u\alpha m\beta d(u) + m\alpha u\beta d(u)). \tag{7.1}$$

On the other hand

$$f(w) = f((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2f(u\alpha m\beta u) + 2f(u\beta m\alpha u)$$

$$= 2(f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(m)\beta u\alpha u + m\beta d(u)\alpha u + m\beta u\alpha d(u)) + 4f(u\alpha m\beta u)$$

$$= 2(f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(m)\alpha u\beta u + m\alpha d(u)\beta u + m\alpha u\beta d(u)) + 4f(u\alpha m\beta u). \tag{7.2}$$

Comparing (7.1) and (7.2), and since M is 2-torsion free

$$f(u\alpha m\beta u) = f(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u), \forall\ u \in U; m \in M; \alpha, \beta \in \Gamma. \tag{7.3}$$

If we linearize (7.3) on u, then (ii) is obtained. □

Definition 7.2.1. Let f be a generalized (U, M)-derivation with the associated (U, M)-derivation d of M. We define $\Psi_\alpha(u, m) = f(u\alpha m) - f(u)\alpha m - u\alpha d(m)$ and $\Phi_\alpha(u, m) = d(u\alpha m) - d(u)\alpha m - u\alpha d(m)$ for all $u \in U; m \in M$ and $\alpha \in \Gamma$.

Directly from the definition, the following properties follow at once.

Lemma 7.2.2. *If f is a generalized (U, M)-derivation of M, then for all $u, v \in U;\ m, n \in M$ and $\alpha, \beta \in \Gamma$,*
(i) $\Phi_\alpha(u, m) = -\Phi_\alpha(m, u)$; *(ii)* $\Phi_\alpha(u + v, m) = \Phi_\alpha(u, m) + \Phi_\alpha(v, m)$;
(iii) $\Phi_\alpha(u, m + n) = \Phi_\alpha(u, m) + \Phi_\alpha(u, n)$; *(iv)* $\Phi_{\alpha+\beta}(u, m) = \Phi_\alpha(u, m) + \Phi_\beta(u, m)$.

Proof. (i) By the definition of $\Phi_\alpha(u, m)$, we have $\Phi_\alpha(u, m) = f(u\alpha m) - f(u)\alpha m - u\alpha d(m)$, using the Definition 7.1.1

$$\begin{aligned}
\Phi_\alpha(u, m) + \Phi_\alpha(m, u) &= f(u\alpha m) - f(u)\alpha m - u\alpha d(m) + f(m\alpha u) - f(m)\alpha a - m\alpha d(u) \\
&= f(u\alpha m + m\alpha u) - f(u)\alpha m - u\alpha d(m) - f(m)\alpha u - m\alpha d(u) \\
&= f(u)\alpha m + f(m)\alpha a + u\alpha d(m) + m\alpha d(u) - f(u)\alpha m - u\alpha d(m) \\
&\quad - f(m)\alpha u - m\alpha d(u) = 0.
\end{aligned}$$

$\Rightarrow \Phi_\alpha(u, m) = -\Phi_\alpha(m, u)$.

(ii) By the definition of $\Phi_\alpha(u,m)$, we get

$$\begin{aligned}\Phi_\alpha(u+v,m) &= f((u+v)\alpha m) - f(u+v)\alpha m - (u+v)\alpha d(m)\\ &= f(u\alpha m + v\alpha m) - f(u)\alpha m - f(v)\alpha m - u\alpha d(m) - v\alpha d(m)\\ &= f(u\alpha m) - f(u)\alpha m - u\alpha d(m) + f(v\alpha m) - f(v)\alpha m - v\alpha d(m)\\ &= \Phi_\alpha(u,m) + \Phi_\alpha(v,m).\end{aligned}$$

(iii)- (iv): These are too easy to prove. □

Lemma 7.2.3. *With our notations as above, for any $u, v \in U$; $m \in M$ and $\alpha, \beta \in \Gamma$, the following are true:*

(i) $\Psi_\alpha(u,m) = -\Psi_\alpha(m,u)$; *(ii)* $\Psi_\alpha(u+v,m) = \Psi_\alpha(u,m) + \Psi_\alpha(v,m)$;

(iii) $\Psi_\alpha(u,m+n) = \Psi_\alpha(u,m) + \Psi_\alpha(u,n)$; *(iv)* $\Psi_{\alpha+\beta}(u,m) = \Psi_\alpha(u,m) + \Psi_\beta(u,m)$.

Proof. Proceeding in the same way of the proof of above lemma. □

In obtaining our main result of this section, the following Lemma plays an important role.

Lemma 7.2.4. *If U is an admissible Lie ideal of a 2-torsion free prime Γ-ring M and f is a generalized $(U,M)-$derivation of M for which d is the associated $(U,M)-$derivation of M, then $\Psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha = 0$ for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$.*

Proof. Let $x = 4(u\alpha v\beta w\gamma v\alpha u + v\alpha u\beta w\gamma u\alpha v)$. Using Lemma 7.2.1(ii), we have

$$\begin{aligned}f(x) &= f((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v))\\ &= 4f(u\alpha v)\beta w\gamma v\alpha u + 4u\alpha v\beta d(w)\gamma v\alpha u + 4u\alpha v\beta w\gamma d(v\alpha u) + 4f(v\alpha u)\beta w\gamma u\alpha v\\ &\qquad + 4v\alpha u\beta d(w)\gamma u\alpha v + 4v\alpha u\beta w\gamma d(u\alpha v).\end{aligned}$$

On the other hand, using Lemma 7.2.1(i), we have

$$\begin{aligned}
f(x) &= f(u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v) \\
&= f(u)\alpha 4v\beta w\gamma v\alpha u + u\alpha d(4v\beta w\gamma v)\alpha u + u\alpha 4v\beta w\gamma v\alpha d(u) + f(v)\alpha 4u\beta w\gamma u\alpha v \\
&\qquad + v\alpha d(4u\beta w\gamma u)\alpha v + v\alpha 4u\beta w\gamma u\alpha d(v) \\
&= 4f(u)\alpha v\beta w\gamma v\alpha u + 4u\alpha d(v)\beta w\gamma v\alpha u + 4u\alpha v\beta d(w)\gamma v\alpha u + 4u\alpha v\beta w\gamma d(v)\alpha u \\
&\quad + 4u\alpha v\beta w\gamma v\alpha d(u) + 4f(v)\alpha u\beta w\gamma u\alpha v + 4v\alpha d(u)\beta w\gamma u\alpha v + 4v\alpha u\beta d(w)\gamma u\alpha v \\
&\qquad + 4v\alpha u\beta w\gamma d(u)\alpha v + 4v\alpha u\beta w\gamma u\alpha d(v).
\end{aligned}$$

Comparing the right side of $f(x)$ and using the 2-torsion freeness of M

$$\begin{aligned}
&f(u\alpha v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma d(v\alpha u) + f(v\alpha u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma d(u\alpha v) \\
&\quad = f(u)\alpha v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma v\alpha d(u) \\
&\qquad + f(v)\alpha u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma d(u)\alpha v + v\alpha u\beta w\gamma u\alpha d(v).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(f(u\alpha v) - f(u)\alpha v - u\alpha d(v))\beta w\gamma v\alpha u + (f(v\alpha u) - f(v)\alpha u - v\alpha d(u))\beta w\gamma u\alpha v \\
&+ u\alpha v\beta w\gamma(d(v\alpha u) - d(v)\alpha u - v\alpha d(u)) + v\alpha u\beta w\gamma(d(u\alpha v) - d(u)\alpha v - u\alpha d(v)) = 0.
\end{aligned}$$

Using the Definition 7.2.1, we obtain

$$\Psi_\alpha(u,v)\beta w\gamma v\alpha u + \Psi_\alpha(v,u)\beta w\gamma u\alpha v + u\alpha v\beta w\gamma\Phi_\alpha(v,u) + v\alpha u\beta w\gamma\Phi_\alpha(u,v) = 0.$$

Now, using Lemma 7.2.2(i) and 7.2.3(i), we have

$$\Psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\Phi_\alpha(u,v) = 0, \forall\, u,v,w \in U; \alpha,\beta,\gamma \in \Gamma.$$

Since d is a (U,M)-derivation, we have $\Phi_\alpha(u,v) = 0$ for all $u,v \in U$ and $\alpha \in \Gamma$, by Lemma 6.2.6. Using this we obtain the desired result. $\square$

Now, we prove the following two theorems with generalized (U, M)-derivation of a prime Γ-ring M.

Theorem 7.2.5. *Assume that U is an admissible Lie ideal of a 2-torsion free prime Γ-ring M, and f is a generalized (U, M)-derivation of M, then $\Psi_\alpha(u, v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$.*

Proof. By Lemma 7.2.4, we have

$$\Psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0, \forall\, u, v, w \in U; \alpha, \beta, \gamma \in \Gamma.$$

Using the Lemma 4.2.7 in the above relation, we obtain

$$\Psi_\alpha(u, v)\beta w\gamma[x, y]_\delta = 0, \forall\, u, v, w, x, y \in U; \alpha, \beta, \gamma, \delta \in \Gamma.$$

Since U is not contained in $Z(M)$, so $[x, y]_\delta \neq 0$. Thus, by Lemma 4.2.5, we get $\Psi_\alpha(u, v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$. □

Remark 7.2.1. If we replace U by a square closed Lie ideal in the Theorem 7.2.5, then the theorem is also true.

Theorem 7.2.6. *Let U be a square closed Lie ideal of a 2-torsion free prime Γ-ring M, then $f(u\alpha m) = f(u)\alpha m + u\alpha d(m)$ for all $u \in U; m \in M$ and $\alpha \in \Gamma$.*

Proof. From Theorem 7.2.5 and Remark 7.2.1, we have

$$\Psi_\alpha(u, v) = 0, \forall\, u, v \in U; \alpha \in \Gamma \tag{7.4}$$

Replacing v by $u\beta m - m\beta u$ in (7.4), we get $\Psi_\alpha(u, u\beta m - m\beta u) = 0$. Since $u\beta m -$

$m\beta u \in U$ for all $u \in U$, $m \in M$ and $\alpha, \beta \in \Gamma$. Therefore,

$$\begin{aligned}
0 &= \Psi_\alpha(u, u\beta m - m\beta u) \\
&= f(u\alpha(u\beta m - m\beta u)) - f(u)\alpha(u\beta m - m\beta u) - u\alpha d(u\beta m - m\beta u) \\
&= f(u\alpha u\beta m) - f(u\alpha m\beta u) - f(u)\alpha u\beta m + f(u)\alpha m\beta u - u\alpha d(u)\beta m \\
&- u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) \\
&= f(u\alpha u\beta m) - f(u)\alpha m\beta u - u\alpha d(m)\beta u - u\alpha m\beta d(u) - f(u)\alpha u\beta m \\
&+ f(u)\alpha m\beta u - u\alpha d(u)\beta m - u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) \\
&= f(u\alpha u\beta m) - f(u)\alpha u\beta m - u\alpha d(u)\beta m - u\alpha u\beta d(m).
\end{aligned}$$

This implies,

$$f(u\alpha u\beta m) = f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m).$$

$$\Rightarrow f((u\alpha u)\beta m) - f(u\alpha u)\beta m - (u\alpha u)\beta d(m) = 0.$$

$$\Rightarrow \Psi_\beta(u\alpha u, m) = 0, \forall\ u \in U; m \in M; \alpha, \beta \in \Gamma. \quad (7.5)$$

Now, let $x = u\alpha u\beta m + u\beta m\alpha u$. Then by the definition of generalized (U, M)-derivation, we have

$$\begin{aligned}
f(x) &= f(u)\alpha u\beta m + u\alpha d(u\beta m) + f(u\beta m)\alpha u + u\beta m\alpha d(u) \\
&= f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(u\beta m)\alpha u + u\beta m\alpha d(u).
\end{aligned} \quad (7.6)$$

On the other hand, using (7.5) and Lemma 7.2.1(i)

$$\begin{aligned}
f(x) &= f(u\alpha u\beta m) + f(u\beta m\alpha u) \\
&= f(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + f(u)\beta m\alpha u + u\beta d(m)\alpha u + u\beta m\alpha d(u).
\end{aligned} \quad (7.7)$$

Comparing (7.6) and (7.7), we get

$$(f(u\beta m) - f(u)\beta m - u\beta d(m))\alpha u = 0.$$

This yields,

$$\Psi_\beta(u,m)\alpha u = 0, \forall\ u \in U; m \in M; \alpha, \beta \in \Gamma. \tag{7.8}$$

Linearize (7.8) on u and using equation (7.8), we get

$$\Psi_\beta(u,m)\alpha v + \Psi_\beta(v,m)\alpha u = 0. \tag{7.9}$$

Replacing v by $v\gamma v$ in equation (7.9), we obtain

$$\Psi_\beta(u,m)\alpha v\gamma v + \Psi_\beta(v\gamma v,m)\alpha u = 0.$$

Since $\Psi_\beta(v\gamma v,m) = 0$ for all $v \in U, m \in M$ and $\beta,\gamma \in \Gamma$. This is seen in the equation (7.5) for $v\gamma v$ in place of $u\alpha u$. Therefore, we have

$$\Psi_\beta(u,m)\alpha v\gamma v = 0, \forall\ u,v \in U; m \in M; \alpha,\beta,\gamma \in \Gamma. \tag{7.10}$$

Replacing v by $u+v$ in (7.10) and using (7.5), we obtain

$$\Psi_\beta(u,m)\alpha(u+v)\gamma(u+v) = 0.$$

$$\Rightarrow \Psi_\beta(u,m)\alpha(u\gamma u + u\gamma v + v\gamma u + v\gamma v) = 0.$$

$$\Rightarrow \Psi_\beta(u,m)\alpha u\gamma v + \Psi_\beta(u,m)\alpha v\gamma u = 0.$$

Now using (7.8), this implies $\Psi_\beta(u,m)\alpha v\gamma u = 0$ for all $u,v \in U; m \in M$ and $\alpha,\beta,\gamma \in \Gamma$. Since U is noncentral, by Lemma 4.2.5, $\Psi_\beta(u,m) = 0$ for all $u \in U, m \in M$ and $\beta \in \Gamma$. □

7.3 Generalized Higher (U, M)-Derivations in Prime Γ-Rings

Here, we determine a number of important consequences relating to the concept of generalized higher (U, M)-derivations of a Γ-ring to extend the results stated at the beginning of this chapter following [14, 15, 16, 17] classical ring theory to Γ-ring theory.

Lemma 7.3.1. *Let $F = (f_i)_{i \in \boldsymbol{N}}$ be a generalized higher (U, M)-derivation of M. Then $f_n(u\alpha m\beta u) = \sum_{i+j+k=n} f_i(u)\alpha d_j(m)\beta d_k(u)$ for all $u \in U; m \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. Let $x = u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\beta m + m\beta(2u))\alpha u$. Replacing m and s by $(2u)\beta m + m\beta(2u)$ in $f_n(u\alpha m + s\alpha u) = \sum_{i+j=n} f_i(u)\alpha d_j(m) + f_i(s)\alpha d_j(u)$ and using the condition (*), we get

$$\begin{aligned}
f_n(x) &= \sum_{i+j=n} f_i(u)\alpha d_j((2u)\beta m + m\beta(2u)) + f_i((2u)\beta m + m\beta(2u))\alpha d_j(u) \\
&= 2\sum_{i+j=n} f_i(u)\alpha \sum_{l+t=j}(d_l(u)\beta d_t(m) + d_l(m)\beta d_t(u)) + 2\sum_{i+j=n}\sum_{p+q=i}(f_p(u)\beta d_q(m) \\
&\qquad + f_p(m)\beta d_q(u))\alpha d_j(u) \\
&= 2\sum_{i+l+t=n}(f_i(u)\alpha d_l(u)\beta d_t(m) + f_i(u)\alpha d_l(m)\beta d_t(u)) + 2\sum_{p+q+j=n}(f_p(u)\beta d_q(m)\alpha d_j(u) \\
&\qquad + f_p(m)\beta d_q(u)\alpha d_j(u)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_n(x) &= 2\sum_{i+l+t=n} f_i(u)\alpha d_l(u)\beta d_t(m) + 2\sum_{i+l+t=n} f_i(u)\alpha d_l(m)\beta d_t(u) \\
&\quad + 2\sum_{p+q+j=n} f_p(u)\alpha d_q(m)\beta d_j(u) + 2\sum_{p+q+j=n} f_p(m)\alpha d_q(u)\beta d_j(u). \quad (7.11)
\end{aligned}$$

Again, by the definition of higher (U, M)-derivation and using the condition (*)

$$
\begin{aligned}
f_n(x) &= f_n((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2f_n(u\alpha m\beta u) + 2f_n(u\beta m\alpha u) \\
&= f_n((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2f_n(u\alpha m\beta u) + 2f_n(u\alpha m\beta u) \\
&= \sum_{i+j=n} (f_i(2u\alpha u)\beta d_j(m) + f_i(m)\beta d_j(2u\alpha u)) + 4f_n(u\alpha m\beta u) \\
&= 2\sum_{i+j=n} (\sum_{r+s=i} f_r(u)\alpha d_s(u))\beta d_j(m) + 2\sum_{i+j=n} f_i(m)\alpha(\sum_{h+k=j} d_h(u)\beta d_k(u)) + 4f_n(u\alpha m\beta u) \\
&= 2\sum_{r+s+j=n} f_r(u)\alpha d_s(u)\beta d_j(m) + 2\sum_{i+h+k=n} f_i(m)\beta d_h(u)\alpha d_k(u) + 4f_n(u\alpha m\beta u).
\end{aligned}
\tag{7.12}
$$

Now, comparing (7.11) and (7.12), we get

$$4f_n(u\alpha m\beta u) = 4\sum_{i+j+k=n} f_i(u)\alpha d_j(m)\beta d_k(u), \forall\ u \in U; m \in M; \alpha, \beta \in \Gamma.$$

Using 2-torsion freeness of M, we get the desired result. $\square$

Lemma 7.3.2. *Let $F = (f_i)_{i\in\boldsymbol{N}}$ be a generalized higher (U, M)-derivation of M. Then $f_n(u\alpha m\beta v + v\alpha m\beta u) = \sum_{i+j+k=n} f_i(u)\alpha d_j(m)d_k(v) + f_i(v)\alpha d_j(m)\beta d_k(u)$, for all $u, v \in U; m \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. Linearizing of $f_n(u\alpha m\beta u) = \sum_{i+j+k=n} f_i(u)\alpha d_j(m)\beta d_k(u)$ with respect to u,

$$
\begin{aligned}
f_n((u+v)\alpha m\beta(u+v)) &= \sum_{i+j+k=n} f_i(u+v)\alpha d_j(m)\beta d_k(u+v) \\
&= \sum_{i+j+k=n} f_i(u)\alpha d_j(m)\beta d_k(u) + \sum_{i+j+k=n} f_i(u)\alpha d_j(m)\beta d_k(v) \\
&\quad + \sum_{i+j+k=n} f_i(v)\alpha d_j(m)\beta d_k(u) + \sum_{i+j+k=n} f_i(v)\alpha d_j(m)\beta d_k(v).
\end{aligned}
\tag{7.13}
$$

On the other hand

$$f_n((u+v)\alpha m\beta(u+v)) = f_n(u\alpha m\beta u) + f_n(u\alpha m\beta v + v\alpha m\beta u) + f_n(v\alpha m\beta v)$$
$$= \sum_{i+j+k=n} f_i(u)\alpha d_j(m)\beta d_k(u) + f_n(u\alpha m\beta v + v\alpha m\beta u) + \sum_{i+j+k=n} f_i(v)\alpha d_j(m)\beta d_k(v). \tag{7.14}$$

By comparing (7.13) and (7.14), we have the required result. □

Definition 7.3.1. Let $F = (f_i)_{i\in\mathbf{N}}$ be a generalized higher (U, M)-derivation of M. For every fixed $n \in \mathbf{N}$, we define $\psi_n^\alpha(u, m) = f_n(u\alpha m) - \sum_{i+j=n} f_i(u)\alpha d_j(m)$, for all $u \in U; m \in M; \alpha \in \Gamma$. Also, let $D = (d_i)_{i\in\mathbf{N}}$ be a higher (U, M)-derivation of M. For every fixed $n \in \mathbf{N}$, we define $\phi_n^\alpha(u, m) = d_n(u\alpha m) - \sum_{i+j=n} d_i(u)\alpha d_j(m)$ for all $u \in U; m \in M; \alpha \in \Gamma$.

Remark 7.3.1. $\psi_n^\alpha(u, m) = 0$, for all $u \in U; m \in M; \alpha \in \Gamma$ and $n \in \mathbf{N}$ if and only if $f_n(u\alpha m) = \sum_{i+j=n} f_i(u)\alpha d_j(m)$, for all $u \in U; m \in M; \alpha \in \Gamma$ and $n \in \mathbf{N}$. Also $\phi_n^\alpha(u, m) = 0$, for all $u \in U; m \in M; \alpha \in \Gamma$ and $n \in \mathbf{N}$ if and only if $d_n(u\alpha m) = \sum_{i+j=n} d_i(u)\alpha d_j(m)$, for all $u \in U; m \in M; \alpha \in \Gamma$ and $n \in \mathbf{N}$.

Lemma 7.3.3. *If $F = (f_i)_{i\in\boldsymbol{N}}$ is a generalized higher (U, M)-derivation of M, then for every $u, v \in U;\ m, p \in M;\ \alpha, \beta \in \Gamma$ and $n \in \boldsymbol{N}$:*
$(i)\psi_n^\alpha(u, m) + \psi_n^\alpha(m, u) = 0; (ii)\psi_n^\alpha(u + v, m) = \psi_n^\alpha(u, m) + \psi_n^\alpha(v, m);$
$(iii)\psi_n^\alpha(u, m + p) = \psi_n^\alpha(u, m) + \psi_n^\alpha(u, p); (iv)\psi_n^{\alpha+\beta}(u, m) = \psi_n^\alpha(u, m) + \psi_n^\beta(u, m).$

Proof. (i) By Definition 7.3.1, and using generalized higher (U, M)-derivation

$$\psi_n^\alpha(u, m) + \psi_n^\alpha(m, u) = f_n(u\alpha m) - \sum_{i+j=n} f_i(u)\alpha d_j(m) + f_n(m\alpha u) - \sum_{i+j=n} f_i(m)\alpha d_j(u)$$

$$= f_n(u\alpha m + m\alpha u) - \sum_{i+j=n} f_i(u)\alpha d_j(m) - \sum_{i+j=n} f_i(m)\alpha d_j(u)$$

$$= \sum_{i+j=n} f_i(u)\alpha d_j(m) + \sum_{i+j=n} f_i(m)\alpha d_j(u) - \sum_{i+j=n} f_i(u)\alpha d_j(m) - \sum_{i+j=n} f_i(m)\alpha d_j(u) = 0.$$

(ii) By the definition of $\psi_n^\alpha(u,m)$, we get

$$\begin{aligned}\psi_n^\alpha(u+v,m) &= f_n((u+v)\alpha m) - \sum_{i+j=n} f_i(u+v)\alpha d_j(m)\\ &= f_n(u\alpha m + v\alpha m) - \sum_{i+j=n} f_i(u)\alpha d_j(m) - \sum_{i+j=n} f_i(v)\alpha d_j(m)\\ &= f_n(u\alpha m) - \sum_{i+j=n} f_i(u)\alpha d_j(m) + f_n(v\alpha m) - \sum_{i+j=n} f_i(v)\alpha d_j(m)\\ &= \psi_n^\alpha(u,m) + \psi_n^\alpha(v,m).\end{aligned}$$

(iii) and (iv) are also obvious. $\square$

Now, we prove our main results as below.

Theorem 7.3.4. *Let U be an admissible Lie ideal of a prime Γ-ring M and $F = (f_i)_{i\in \boldsymbol{N}}$ be a generalized higher (U,M)-derivation of M. Then $\psi_n^\alpha(u,v) = 0$, for all $u,v \in U$; $\alpha \in \Gamma$ and $n \in \boldsymbol{N}$.*

Proof. **Induction Beginning:** We know by Theorem 7.2.8, $\psi_1^\alpha(u,v) = 0$, for all $u,v \in U; \alpha \in \Gamma$; so $\psi_n^\alpha(u,v) = 0$, holds when $n = 1$.

Induction Hypothesis: Assume $\psi_m^\alpha(u,v) = 0$, holds for all $u,v \in U; \alpha \in \Gamma$ such that $m \in \mathbf{N}$ and $m < n$.

Let $x = 4(u\alpha v\beta w\gamma v\alpha u + v\alpha u\beta w\gamma u\alpha v)$. Then, using Lemma 7.3.2, we have

$$\begin{aligned}
f_n(x) &= f_n((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)) \\
&= 4\sum_{i+j+k=n} f_i(u\alpha v)\beta d_j(w)\gamma d_k(v\alpha u) + 4\sum_{i+j+k=n} f_i(v\alpha u)\beta d_j(w)\gamma d_k(u\alpha v) \\
&= 4f_n(u\alpha v)\beta w\gamma v\alpha u + 4u\alpha v\beta w\gamma d_n(v\alpha u) + 4\sum_{i+j+k=n}^{i,k<n} f_i(u\alpha v)\beta d_j(w)\gamma d_k(v\alpha u) \\
&\quad + 4f_n(v\alpha u)\beta w\gamma u\alpha v + 4v\alpha u\beta w\gamma d_n(u\alpha v) + 4\sum_{i+j+k=n}^{i,k<n} f_i(v\alpha u)\beta d_j(w)\gamma d_k(u\alpha v).
\end{aligned}$$

Also, by Lemma 7.3.1 and since $D = (d_i)_{i\in\mathbf{N}}$ is a higher (U, M)-derivation of M.

$$\begin{aligned}
f_n(x) &= f_n((2u\alpha v)\beta w\gamma(2v\alpha u)) + f_n((2v\alpha u)\beta w\gamma(2u\alpha v)) \\
&= \sum_{i+j+q=n} 2f_i(u\alpha v)\beta d_j(w)\gamma 2d_q(v\alpha u) + \sum_{i+j+k=n} 2f_i(v\alpha u)\beta d_j(w)\gamma 2d_k(u\alpha v) \\
&= 4u\alpha v\beta w\gamma \sum_{s+k=n} d_s(v)\alpha d_k(u) + 4\sum_{i+p=n} f_i(u)\alpha d_p(v)\beta w\gamma v\alpha u \\
&\quad + \sum_{i+p+q+s+k=n}^{s+k,i+p<n} f_i(u)\alpha d_p(v)\beta d_q(w)\gamma d_s(v)\alpha d_k(u) + 4v\alpha u\beta w\gamma \sum_{r+k=n} d_r(u)\alpha d_k(v) \\
&\quad + 4\sum_{i+l=n} f_i(v)\alpha d_l(u)\beta w\gamma u\alpha v + \sum_{i+l+t+r+k=n}^{i+l,r+k<n} f_i(v)\alpha d_l(u)\beta d_t(w)\gamma d_r(u)\alpha d_k(v).
\end{aligned}$$

Comparing the two expressions of $f_n(x)$ and using $\psi_m^\alpha(u, v) = 0$, for all $u, v \in U; \alpha \in \Gamma; m < n$, we get

$$\begin{aligned}
&4(f_n(u\alpha v) - \sum_{i+p=n} f_i(u)\alpha d_p(v))\beta w\gamma v\alpha u + 4(f_n(v\alpha u) - \sum_{i+l=n} f_i(u)\alpha d_l(v))\beta w\gamma u\alpha v \\
&+4v\alpha u\beta w\gamma(d_n(u\alpha v) - \sum_{r+k=n} f_r(u)\alpha d_k(v)) + 4u\alpha v\beta w\gamma(d_n(v\alpha u) - \sum_{s+k=n} f_s(u)\alpha d_k(v)) = 0.
\end{aligned}$$

$$\Rightarrow 4\psi_n^\alpha(u, v)\beta w\gamma v\alpha u + 4\psi_n^\alpha(v, u)\beta w\gamma u\alpha v + 4u\alpha v\beta w\gamma\phi_n^\alpha(v, u) + 4v\alpha u\beta w\gamma\phi_n^\alpha(u, v) = 0.$$

Using Lemma 7.3.3 (i) and 2-torsion freeness of M, we get

$$\psi_n^\alpha(u,v)\beta w\gamma[u,v]_\alpha + [u,v]_\alpha\beta w\gamma\phi_n^\alpha(u,v) = 0.$$

Since $D = (d_i)_{i\in\mathbf{N}}$ is a higher (U,M)-derivation of M, thus we have $\phi_n^\alpha(u,v) = 0$. By Lemma 4.2.5 and since U is noncentral, thus $\psi_n^\alpha(u,v) = 0$, for all $u,v \in U; \alpha \in \Gamma$ and $n \in \mathbf{N}$. □

Theorem 7.3.5. *Let U be an admissible Lie ideal of a prime Γ-ring M and $F = (f_i)_{i\in\boldsymbol{N}}$ be a generalized higher (U,M)-derivation of M. Then $f_n(u\beta m) = \sum_{i+j=n} f_i(u)\beta d_j(m)$ for all $u \in U; m \in M; \beta \in \Gamma$ and $n \in \boldsymbol{N}$.*

Proof. $\psi_1^\alpha(u,m) = 0$, for all $u \in U; m \in M; \alpha \in \Gamma$ (by Theorem 7.2.8). Now, we assume by induction on $n \in \mathbf{N}$, that $\psi_m^\alpha(u,m) = 0$, for all $u \in U; m \in M; \alpha \in \Gamma; m \in \mathbf{N}$ such that $m < n$.

Since $F = (f_i)_{i\in\mathbf{N}}$ is a generalized higher (U,M)-derivation and $D = (d_i)_{i\in\mathbf{N}}$ is a higher (U,M)-derivation of M, so we have

$$\begin{aligned}
0 &= \psi_n^\alpha(u, u\beta m - m\beta u)\\
&= f_n(u\alpha(u\beta m - m\beta u)) - \sum_{i+j=n} f_i(u)\alpha d_j(u\beta m - m\beta u)\\
&= f_n(u\alpha u\beta m) - f_n(u\alpha m\beta u) - \sum_{i+l+t=n} f_i(u)\alpha d_l(u)\beta d_t(m) + \sum_{i+j+k=n} f_i(u)\alpha d_j(m)\beta d_k(u)\\
&= f_n(u\alpha u\beta m) - f_n(u\alpha m\beta u) - \sum_{i+l+t=n} f_i(u)\alpha d_l(u)\beta d_t(m) + f_n(u\alpha m\beta u).
\end{aligned}$$

Therefore, we have

$$f_n(u\alpha u\beta m) = \sum_{i+l+t=n} f_i(u)\alpha d_l(u)\beta d_t(m). \tag{7.15}$$

Since $F=(f_i)_{i\in\mathbf{N}}$ is a generalized higher (U,M)-derivation of M, thus we have

$$\begin{aligned}
f_n(u\alpha(u\beta m)+(u\beta m)\alpha u) &= \sum_{i+j=n}(f_i(u)\alpha d_j(u\beta m)+f_i(u\beta m)\alpha d_j(u))\\
&= f_n(u)\alpha u\beta m+u\alpha d_n(u\beta m)+\sum_{i+j=n}^{i,j<n} f_i(u)\alpha d_j(u\beta m)\\
&\quad + f_n(u\beta m)\alpha u+u\beta m\alpha d_n(u)+\sum_{i+j=n}^{i,j<n} f_i(u\beta m)\alpha d_j(u). \quad (7.16)
\end{aligned}$$

Since $\psi_m^\alpha(u,m)=0$ for all $u\in U; m\in M; \alpha\in\Gamma; m<n$.

Therefore,

$$\begin{aligned}
f_n(u\alpha(u\beta m)+u\beta m\alpha u) &= f_n(u)\alpha(u\beta m)+u\alpha d_n(u\beta m)+\sum_{i+l+t=n}^{i,l+t<n} f_i(u)\alpha d_l(u)\beta d_t(m)\\
&\quad + f_n(u\beta m)\alpha u+u\beta m\alpha d_n(u)+\sum_{p+q+j=n}^{p+q,j<n} f_p(u)\beta d_q(m)\alpha d_j(u). \quad (7.17)
\end{aligned}$$

On the other hand, using Lemma 7.3.1 and equation (7.15), we get

$$\begin{aligned}
f_n(u\alpha(u\beta m)+(u\beta m)\alpha u) &= f_n(u\alpha u\beta m)+f_n(u\alpha m\beta u)\\
&= \sum_{i+l+t=n} f_i(u)\alpha d_l(u)\beta d_t(m)+\sum_{i+j+k=n} f_i(u)\alpha d_j(m)\beta d_k(u)\\
&= f_n(u)\alpha u\beta m+u\alpha\sum_{l+t=n} d_l(u)\beta d_t(m)+\sum_{i+l+t=n}^{i,l+t<n} f_i(u)\alpha d_l(u)\beta d_t(m)\\
&\quad + u\alpha m\beta d_n(u)+(\sum_{i+j=n} f_i(u)\alpha d_j(m))\beta u+\sum_{i+j+k=n}^{i+j,k<n} f_i(u)\alpha d_j(m)\beta d_k(u). \quad (7.18)
\end{aligned}$$

Comparing (7.17) and (7.18) and using the condition (*), we get

$$u\alpha(d_n(u\beta m)-\sum_{l+t=n} d_l(u)\beta d_t(m))+(f_n(u\beta m)-\sum_{i+j=n} f_i(u)\beta d_j(m))\alpha u=0.$$

$$\Rightarrow u\alpha\phi_n^\beta(u,m)+\psi_n^\beta(u,m)\alpha u=0, \forall\, u\in U; m\in M; \alpha,\beta\in\Gamma; n\in\mathbf{N}.$$

By Theorem 6.3.6, $\phi_n^\beta(u,m) = 0, \forall\ u \in U; m \in M; \beta \in \Gamma; n \in \mathbf{N}$, hence

$$\psi_n^\beta(u,m)\alpha u = 0, \forall\ u \in U; m \in M; \alpha, \beta \in \Gamma; n \in \mathbf{N}. \tag{7.19}$$

Linearizing of (7.19) with respect to u, gives us

$$\psi_n^\beta(u,m)\alpha v + \psi_n^\beta(v,m)\alpha u = 0, \forall\ u, v \in U; m \in M; \alpha, \beta \in \Gamma; n \in \mathbf{N}. \tag{7.20}$$

Replacing v by $v\beta v$ in (7.20) and since $\psi_n^\beta(v\beta v, m) = 0$, thus

$$\psi_n^\beta(u,m)\alpha v\beta v = 0, \forall\ u, v \in U; m \in M; \alpha, \beta \in \Gamma; n \in \mathbf{N}. \tag{7.21}$$

Again, replacing v by $u + v$ in (7.21), then using (7.19) and $\psi_n^\beta(v\beta v, m) = 0$,

$$\psi_n^\beta(u,m)\alpha v\beta u = 0, \forall\ u, v \in U; m \in M; \alpha, \beta \in \Gamma; n \in \mathbf{N}.$$

Since $U \neq 0$, hence by Lemma 4.2.5

$$\psi_n^\beta(u,m) = 0, \forall\ u \in U; m \in M; \beta \in \Gamma; n \in \mathbf{N}.$$

This proves the claim.

$\square$

Bibliography

[1] M. Asci and S. Ceran, *The commutativity in prime gamma rings with left derivations*, International Mathematical Forum **3** (2007), no. 2, 103–108.

[2] Mohammad Ashraf and Nadeem-Ur-Rehman, *On Lie ideals and Jordan left derivations of prime rings*, Archivum Mathematicum (Brno) **36** (2000), 201–206.

[3] R. Awtar, *Lie ideal and Jordan derivations of prime rings*, Proceedings of the American Mathemtatical Society **90** (1984), 9–14.

[4] W. E. Barnes, *On the Γ-rings of Nobusawa*, Pacific J. Math. **18** (1966), no. 3, 411–422.

[5] G. L. Booth, *On the radicals of Γ_N-rings*, Math. Japanica **32** (1987), no. 3, 357–372.

[6] M. Bresar, *Jordan derivations on semiprime rings*, Proceedings of the American Mathemtatical Society **104** (1988), no. 4, 1003–1004.

[7] ______, *On the distance of the composition of two derivations to the generalized derivations*, Glasgow Mathematics **33** (1991), 89–93.

[8] M. Bresar and J. Vukman, *Jordan derivations on prime rings*, Bulletin of Australian Math. Soc. **37** (1988), 321–322.

[9] ———, *On the left derivations and related mappings*, Proceedings of the American Mathemtatical Society **110** (1990), no. 1, 7–16.

[10] Y. Ceven, *Jordan left derivations on completely prime* Γ*-rings*, C. U. Fen-Edebiyat Fakultesi Fen Bilimleri Dergisi **23** (2002), 39–43.

[11] Y. Ceven and M. A. Ozturk, *On* J*ordan generalized derivations in* Γ*-rings*, Hacettepi Journal of Mathematics and Statistics **33** (2004), 11–14.

[12] S. Chakraborty and A. C. Paul, *On* J*ordan generalized k-derivations of semi-prime* Γ_N*-rings*, Bulletin of the Iranian Math. Society **36** (2010), no. 1, 41–53.

[13] ———, *k-derivations and k-homomorphisms of* Γ*-rings*, Lap Lambert Academic Publishing GmbH & Co. KG, Germany (2012).

[14] W. Cortes and C. Haetinger, *On* J*ordan generalized higher derivations in rings*, Turk. Journal of Mathematics **29** (2005), 1–10.

[15] M. Ferrero and C. Haetinger, *Higher derivations and a theorem by* H*erstein*, Qaeestiones Mathematcae **25** (2002), 1–9.

[16] A. K. Faraj; C. Haetinger and A. H. Majeed, *Generalized higher* (U,R)*-derivations*, JP Journal of Algebra, Number Theory and Applications **16** (2010), no. 2, 119–142.

[17] C. Haetinger, *Higher derivations on* L*ie ideals*, Tendencias em Matematica Aplicada e Computacional **3** (2002), no. 1, 141–145.

[18] A. K. Halder and A. C. Paul, *Jordan left derivations on* L*ie ideals*, Punjab University Journal of Mathematics **44** (2012), 23–29.

[19] I. N. Herstein, *Jordan derivations on prime rings*, Proceedings of the American Mathemtatical Society **8** (1957), 1104–1110.

[20] ———, *Lie and* J*ordan structures in simple associative rings*, Bulletin of the American Mathemtatical Society **67** (1961), no. 6, 517–531.

[21] ———, *Topics in* R*ing* T*heory*, The University of Chicago Press, Chicago (1969).

[22] ———, *On the lie structure of an associative ring*, J. Algebra **14** (1970), 561–571.

[23] B. Hvala, *Generalized derivations in rings*, Comm. Algebra **26** (1998), no. 4, 1147–1166.

[24] K. W. Jun and B. D. Kim, *A note on* J*ordan left derivations*, Bulletin of the Korean Mathemtatical Society **33** (1996), no. 2, 221–228.

[25] S. Kyuno, *Gamma* R*ings and* R*adicals*, Contributions to General Algebra 4, Proc. Krems Conf., August 16-23, 1985, Verlag Holder-Pichler-Tempsky, Wein 1987-Verlag B.G. Teubner,Stuttgart.

[26] ———, *On the semi-simple gamma rings*, Tohoku Math. J. **29** (1977), 217–225.

[27] ———, *On prime gamma rings*, Pacific J. Math. **75** (1978), 185–190.

[28] ———, *A gamma ring with minimum conditions*, Tsukuba J. Math. **5** (1981), no. 1, 47–65.

[29] J. Luh, *On the theory of simple* Γ*-rings*, Michigan Math. J. **16** (1969), 65–75.

[30] A. Nakajima, *On categorical properties of generalized derivations*, Science Mathematics **2** (1999), no. 3, 345–352.

[31] ______, *Generalized* J*ordan derivations*, Proceedings of the third Korea-China-Japan International Symposium on Ring Theory (2000), 235–243.

[32] ______, *On generalized higher derivations*, Turk. Journal of Mathematics **24** (2000), no. 3, 295–311.

[33] N. Nobusawa, *On a generalization of the ring theory*, Osaka Journal of Mathematics **1** (1964), 81–89.

[34] Kyuoo-Hong Park, *Jordan higher left derivations and commutativity in prime rings*, Journal of the Chungcheong Mathematical Society **23** (2010), no. 4, 741–748.

[35] A. C. Paul and Amitabh Kumer Halder, *Jordan left derivations of two torsion free* Γ*M-modules*, Journal of Physical Sciences **13** (2009), no. 2, 13–19.

[36] M. Sapanci and A. Nakajima, *Jordan derivations on completly prime* Γ*-rings*, Math Japonica **46** (1997), no. 1, 47–51.

[37] M. Soyturk, *The commutativity in prime gamma rings with derivations*, Turk. J. Math. **18** (1994), 149–155.

[38] Joso Vukman, *Jordan left derivations on semiprime rings*, Math. J. Okayama University **39** (1997), 1–6.

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