2014

New Approximate Solution of Non-Linear Differential Systems

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University of Rajshahi

NEW APPROXIMATE SOLUTION OF NON-LINEAR DIFFERENTIAL SYSTEMS

Thesis submitted in partial fulfillment of the requirements for the degree of

MASTER OF PHILOSOPHY
IN
MATHEMATICS

Submitted

BY
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DECEMBER-2014

DECLARATION

The thesis entitled ***"New Approximate Solution Of Non-Linear Differential Systems"*** *is solely written with all the endeavor and enthusiasm by me and has been submitted in partial fulfillment of the requirements for the degree of Master of Philosophy in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh. I hereby confirm that this research work is original and has never been submitted elsewhere for any degree.*

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DEDICATION

THIS THESIS WORK
IS DEDICATED
TO
MY REVERENT
PARENTS,
LOVING DAUGHTER
&
HUSBAND.

CERTIFICATE

*This is to certify that the research work entitled "**New Approximate Solution Of Non-Linear Differential Systems"** presented in this dissertation is based on the study carried out by **Mst. Razia Pervin,** Roll No.11315, Registration No.3086, Session-2011-2012 in the fulfillment of the requirements for the degree of **Master of Philosophy** in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh, has been completed under our supervision. We believe that this research work is an original one and has never been submitted elsewhere for any degree.*

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ACKNOWLEDGEMENTS

All praises and glory is to the Almighty Allah who is the Omnipotent and only Creator of all creatures that are seen or unseen in this world. All my worshiping and gratefulness bestowed only to Him Who enabled me to accomplish such a thesis task.

However, my indebtedness and gratitude to the many individuals who have helped shape this thesis work cannot adequately be conveyed in a few sentences. I would like to express my sincere admiration, appreciation and gratitude to my Supervisor, Professor Dr. Shewli Shamim Shanta for her kind guidance, invaluable suggestions and farsighted advice in implementing the thesis work.

I am really indebted to Dr. Pinakee Dey, Co-Supervisor of this thesis work whose outstanding contributions and enormous supports make this work a success.

I also feel it is important to acknowledge and thank my mentor and ex-head of Department of Mathematics of the University of Rajshahi, Rajshahi, Bangladesh Professor Dr. M. A. Sattar who taught me a lot regarding academic and research work. I think very few people get such a chance what I have gathered from his pleasant company. It is also gratefully acknowledged the supports and co-operations of my friends & colleagues.

How to express the language of indebtedness to my family members is really unknown to me. My heartfelt respects and gratitude is expressed to my reverent parents for their spontaneous support and inspiration regarding my higher education and research work.

Last but not in the least, I am overwhelmed with joys and proud that only because of them it has happened in my life to carry out such a thesis work. Thanks to my very affectionate daughter Arshiya Helal and beloved husband Bosher Al Helal. Without their co-operations I believe this thesis work may not have seen the light of this Universe. Thanks to all!

ABSTRACT

Most of the perturbation methods are developed to find periodic solutions of nonlinear systems; transients are not considered. At first, Krylov and Bogoliubov introduced a perturbation method which is well known as “asymptotic averaging method” to discuss the transients in the second order autonomous systems with small nonlinearities. Later, this method has been amplified and justified by Bogoliubov and Mitropolskii. Mitropolskii has extended the method for slowly varying coefficients to determine the steady state periodic motions and transient processes. In this dissertation, we have modified and extended the KBM method to investigate some second order nonlinear systems.

Firstly, a second order time dependent nonlinear differential system is considered. Then a new perturbation technique is developed to find an asymptotic solution of nonlinear systems in presence of an external force. Finally, this technique is used to obtain an asymptotic solution of a time dependent nonlinear differential system with slowly varying coefficients using the extended KBM method. These methods are illustrated with several examples.

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Introduction

In science and engineering, there exist many nonlinear oscillatory systems in which parameters are not small. The theory of nonlinear vibrations is an important part of modern science. Those oscillatory systems are often governed by nonlinear differential equations. To solve these problems, it is possible to replace a nonlinear differential equation with a related linear equation that approximates the original nonlinear equation closely enough to provide useful results. Often such linearization is not feasible and therefore the original nonlinear differential equation itself must be considered.

Van der Pol first paid attention to the new (self-excitation) oscillation and found that their existence is inherent in the nonlinearity of the differential systems characterizing the process. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential systems in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus, it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential systems there exist some methods. Among the methods, the method of perturbations, *i. e.*, asymptotic expansions in terms of a small parameter, are foremost. According to these techniques, the solutions are presented by the first two terms to avoid rapidly growing algebraic complexity. Although these perturbation expansions may be divergent, they can be more useful for qualitative and quantitative representations than the expansions that are uniformly convergent.

Perturbation methods are one of the fundamental tools used by all applied mathematicians and theoretical physicists and widely used in science to obtain approximate solutions based on known exact solutions to nearby problems. Such asymptotic techniques can also be used to provide initial guesses for numerical approximations, and they can now be generated through smart use of symbolic computation. An example of this occurs in boundary layer problem where the regions of rapid change in quantities are fluid velocity, temperature or concentration. This method is most effectively used to analyze problems in solid and fluid mechanics, control theory, celestial mechanics, optics, shock waves, nonlinear vibrations, nonlinear wave propagations, and reaction-diffusion systems arising in several physical and biological contexts.

In this dissertation, we shall discuss nonlinear vibrating problems that can be described by the dynamical vibrations of second and nth order time dependent nonlinear differential systems with small nonlinearities by the use of the extended Krylov-Bogoliubov-Mitropolskii (KBM) method. An important approach to study such nonlinear oscillatory problems is the small parameter expansion. Two widely spread methods are mainly used: one is averaging, particularly the KBM method and the other is the method of variation of parameters. According to the KBM technique the solution starts with the solution of linear equation, termed as generating solution, assuming that, in the nonlinear case, the amplitude and the phase of the solution of the linear differential equation are time-dependent functions rather than constants. This method introduces an additional condition on the first derivative of the generating solution for determining the solution of a second order equation. Originally, the method was developed by Krylov-Bogoliubov to obtain the periodic solutions of second order nonlinear differential systems. Now, the method is used to obtain oscillatory, damped oscillatory and non-oscillatory solutions of second, third etc. order nonlinear differential systems by imposing some restrictions to make the solutions uniformly valid.

Most of the authors found the solutions of autonomous nonlinear differential systems. Only a diminutive number of authors investigated damped forced nonlinear vibrating problems. In this dissertation, some second order time dependent nonlinear vibrating problems have been studied and their solutions are investigated.

The results may be useful to researchers working in the field of nonlinear mechanics, mathematical physics, control theory, population dynamics, etc.

Chapter 1
The Survey and the Proposal

1.1 The Survey

In the modern era, the study of nonlinear vibrating problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most physical phenomena in our real world are essentially nonlinear and are described by nonlinear equations. In the mathematical formulations many physical problems often result in differential equations that are nonlinear. However, in many cases it is possible to replace a nonlinear differential equation with a related linear differential equation that approximates the actual equation closely enough to give useful results. Often such linearization is not possible or feasible; when it is not, the original nonlinear equation itself must be tackled.

In the treatment of nonlinear oscillations by perturbation methods, e.g. Lindstedt's [28] method, Poincare's [49] method etc. only periodic oscillations have been treated; transients are not considered. For the first time, Krylov and Bogoliubov (KB) [25] have introduced a new perturbation method in order to discuss the transient state solution of the equation presented by

$$\ddot{x} + \omega^2 x = \varepsilon f\,(x, \dot{x}) \tag{1.1}$$

where ε is a small parameter. In this equation, the damping terms are small. But in the particular cases, it gives those periodic solutions obtained by Poincare [49] . Here it should be mentioned that Poincare's [49] method is well known perturbation method for determining periodic solutions of nonlinear ordinary differential equations with small nonlinearities.

When $\varepsilon = 0$, then the equation (1.1) reduces to linear equation and its solution is

$$x = a\cos(\omega t + \varphi) \tag{1.2}$$

where a and φ are arbitrary constants to be determined from the initial conditions.

Now in order to determine an approximate solution of the equation (1.1) for ε small but different from zero, Krylov and Bogoliubov assumed that the solution is still given by (1.2) with the derivative of the form

$$\dot{x} = -a\omega\sin(\omega t + \varphi) \tag{1.3}$$

where a and φ are functions of t, rather than being constants.

Differentiating (1.2) with respect to t gives

$$\dot{x} = -a\omega\sin\psi + \dot{a}\cos\psi - a\dot{\varphi}\sin\psi\,, \quad \psi = \omega t + \varphi \tag{1.4}$$

Hence

$$\dot{a}\cos\psi - a\dot{\varphi}\sin\psi = 0 \tag{1.5}$$

On account of (1.3).

Again differentiating (1.3) with respect to t gives

$$\ddot{x} = a\omega^2\cos\psi - \dot{a}\omega\sin\psi - a\omega\dot{\varphi}\cos\psi \tag{1.6}$$

Substituting (1.6) into (1.1) and utilizing (1.2) and (1.3), we obtain

$$\dot{a}\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -f(a\cos\psi, -a\omega\sin\psi) \tag{1.7}$$

Solving (1.5) and (1.7) for $\dot{a}$ and $\dot{\varphi}$ yields

$$\dot{a} = -\frac{\varepsilon}{\omega}\sin\psi\, f(a\cos\psi, -a\omega\sin\psi)\,,$$

$$\dot{\varphi} = -\frac{\varepsilon}{a\omega}\cos\psi\, f(a\cos\psi, -a\omega\sin\psi) \tag{1.8}$$

Thus according to Krylov and Bogoliubov's method, the single differential equation (1.1) of the second order for x has been replaced by the two differential equations of the first order in the unknown amplitude a and the phase φ. It is obvious that the solution is periodic with constant amplitude and period $\frac{2\pi}{\omega}$ as the limit $\varepsilon \to 0$. But one cannot tell about the amplitude and the periodicity of oscillations when ε is small, rather than sufficiently small.

Expanding $\sin\psi\ f(a\cos\psi, -a\omega\sin\psi)$ and $\cos\psi\ f(a\cos\psi, -a\omega\sin\psi)$ in Fourier series in the total phase ψ and assuming that the parameter ε is small, so that the amplitude a and the phase φ change very slowly during one period of the oscillation,

$$\text{i.e,} \quad \frac{\dot{a}}{a} << \omega, \quad \frac{\dot{\varphi}}{\varphi} << \omega, \tag{1.9}$$

The first approximate solution of (1.1) by averaging (1.8) over one period is

$$\begin{aligned} \langle \dot{a} \rangle &= -\frac{\varepsilon}{2\pi\omega} \int_0^{2\pi} \sin\psi\ f(a\cos\psi, -a\omega\sin\psi)\, d\psi \\ \langle \dot{\varphi} \rangle &= -\frac{\varepsilon}{2\pi a\omega} \int_0^{2\pi} \cos\psi\ f(a\cos\psi, -a\omega\sin\psi)\, d\psi \end{aligned} \tag{1.10}$$

where a and φ are independent of time under the integrals.

KB called their method asymptotic in the sense such that $\varepsilon \to 0$. An asymptotic series itself is not convergent, but for a fixed number of terms the approximate solution tends to the exact solution as ε tends to zero. It is noted that the term asymptotic is frequently used in the theory of oscillation, also in the sense, $\varepsilon \to \infty$. But in this case the mathematical method is quite different.

The higher order effects were obtained by Volosov [80], Musen [37] and Zabrieko [82].

The equation (1.10) is the differential equations of the first approximation in the form in which they are originally obtained by Krylov and Bogoliubov [25] and in this case they are generally used in applications.

This method, though it is restricted to differential equations of the type (1.1) has been used extensively in plasma physics, theory of oscillations and control theory. Kruskal [24] has extended this method to solve the equations of type

$$\ddot{x} = F\,(x, \dot{x}, \varepsilon) \tag{1.11}$$

The solutions of these fully nonlinear equations are based on the recurrent relations and are given in the forms of power series of the small parameter ε. Cap [18] has investigated some nonlinear systems of the type

$$\ddot{x} + \omega^2\, f(x) = \varepsilon\, F\,(x,\, \dot{x}), \tag{1.12}$$

by using elliptic functions in the sense of the Krylov and Bogoliubov method.

Later, this technique has been amplified and justified mathematically by Bogoliubov and Mitropolskii [3], and extended to a non-stationary vibrations by Mitropolskii [32]. They assumed the solution of the nonlinear differential equation (1.1) in the form

$$x = a\cos\psi + \varepsilon\, u_1\,(a,\psi) + \varepsilon^2 u_2\,(a,\psi) + \cdots\cdots + \varepsilon^n\, u_n\,(a,\, \psi) + O(\varepsilon^{n+1}) \tag{1.13}$$

where u_k, $k = 1, 2, \ldots\ldots n$ are periodic functions of ψ with a period 2π, and the quantities a and ψ are functions of time t, defined by

$$
\begin{aligned}
\dot{a} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots\cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}) \\
\dot{\psi} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \cdots\cdots + \varepsilon^n B_n(a) + O(\varepsilon^{n+1})
\end{aligned} \tag{1.14}
$$

The function u_k, A_k and B_k, $k = 1, 2, \ldots\ldots n$ are to be chosen such a way that the equation (1.13), after replacing a and ψ by the functions defined in equation (1.14), is a solution of the equation (1.1). Since there are no restrictions in choosing the functions A_k and B_k, that generate the arbitrariness in the definitions of the functions u_k. To remove this arbitrariness, the following additional conditions are imposed.

$$
\begin{aligned}
&\int_0^{2\pi} u_k(a, \psi)\cos\psi \, d\psi = 0, \\
&\int_0^{2\pi} u_k(a, \psi)\sin\psi \, d\psi = 0,
\end{aligned} \tag{1.15}
$$

These conditions guarantee the absence of secular terms in all successive approximations.

Differentiating (1.13) two times with respect to t, utilizing relations (1.14), substituting x and the derivatives $\dot{x}$, $\ddot{x}$ in the original equation (1.1), and equating the coefficients of ε^k, $k = 1, 2, \ldots\ldots n$ results a recursive system

$$
\omega^2 \left(\frac{\partial^2 u_k}{\partial \psi^2} + u_k \right) = f^{(k-1)}(a, \psi) + 2\,\omega(a\,B_k \cos\psi + A_k \sin\psi), \tag{1.16}
$$

where

$$
f^0(a, \psi) = f(a\cos\psi, -a\omega\sin\psi),
$$

$$\begin{aligned}
f^{(1)}(a,\psi) &= u_1 f_x(a\cos\psi, -a\omega\sin\psi) \\
&+\left(A_1\cos\psi - a B_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\right) \\
&\times f_{\dot{x}}(a\cos\psi, -a\omega\sin\psi) + \left(a B_1{}^2 - A_1\frac{dA_1}{da}\right)\cos\psi \\
&+\left(2A_1 B_1 - a A_1\frac{dB_1}{da}\right)\sin\psi - 2\omega\left(A_1\frac{\partial^2 u_1}{\partial a\,\partial\psi} + B_1\frac{\partial^2 u_1}{\partial\psi^2}\right).
\end{aligned} \tag{1.17}$$

It is obvious that f^{k-1} is a periodic function of the variable ψ with period 2π, which depends also on the amplitude a. Therefore, f^{k-1} as well as u_k can be expanded in a Fourier series as

$$\begin{aligned}
f^{(k-1)}(a,\psi) &= g_0{}^{(k-1)}(a) + \sum_{n=1}^{\infty} g_n{}^{(k-1)}(a)\cos n\psi + h_n{}^{(k-1)}(a)\sin n\psi \\
u_k(a,\psi) &= v_0{}^{(k-1)}(a) + \sum_{n=1}^{\infty} v_n{}^{(k-1)}(a)\cos n\psi + w_n{}^{(k-1)}(a)\sin n\psi,
\end{aligned} \tag{1.18}$$

where

$$\begin{aligned}
g_0{}^{(k-1)} &= \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi, \\
g_n{}^{(k-1)} &= \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\cos n\psi\,d\psi, \\
h_n{}^{(k-1)} &= \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, -a\omega\sin\psi)\sin n\psi\,d\psi, \qquad n\ge 1
\end{aligned} \tag{1.19}$$

Here $v_1^{(k-1)} = w_1^{(k-1)} = 0$ for all values of k, since both integrals of (1.15) vanish.

Substituting these values into the equation (1.16), it becomes

$$
\begin{aligned}
&\omega^2 v_0^{(k-1)}(a) + \sum_{n=1}^{\infty} \omega^2 (1-n^2)\left[v_n^{(k-1)}(a)\cos n\psi + w_n^{(k-1)}(a)\sin n\psi\right] \\
&= g_0^{(k-1)}(a) + \left(g_1^{(k-1)}(a) + 2a\,\omega\, B_k\right)\cos\psi + \left(h_1^{(k-1)}(a) + 2\,\omega\, B\right)\sin\psi \\
&+ \sum_{n=2}^{\infty}\left[g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi\right],
\end{aligned}
\tag{1.20}
$$

Now equating the coefficients of harmonic of the same order, we get

$$
\begin{aligned}
&g_1^{(k-1)}(a) + 2\,a\omega\, B_k = 0, \quad h_1^{(k-1)}(a) + 2\,\omega\, A_k = 0, \\
&v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2}, \quad v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \\
&w_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \ge 1
\end{aligned}
\tag{1.21}
$$

These are the sufficient conditions to obtain the desired order of approximation. For the first order approximation, we have

$$
\begin{aligned}
A_1 &= -\frac{h_1^{(1)}(a)}{2\,\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\sin\psi\, d\psi, \\
B_1 &= -\frac{g_1^{(1)}(a)}{2\,\omega\, a} = -\frac{1}{2\,\pi\, a\omega}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\, d\psi,
\end{aligned}
\tag{1.22}
$$

Therefore the variational equations in (1.14) become

$$
\begin{aligned}
\dot{a} &= -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\sin\psi\, d\psi, \\
\dot{\psi} &= \omega - \frac{\varepsilon}{2\pi a\omega}\int_0^{2\pi} f(a\cos\psi,\ -a\omega\sin\psi)\cos\psi\, d\psi,
\end{aligned}
\tag{1.23}
$$

It is noted that the equation (1.23) is similar to the equation (1.10). Thus the first order solution obtained by Bogoliubov and Mitropolskii [3] is identical with the original solution obtained by Krylov and Bogoliubov [25]. In the second case, higher order solution can be found easily. The correction term u_1 is obtained from (1.21) as

$$u_1 = \frac{g_0^{(1)}(a)}{\omega^2} + \sum_{n=2}^{\infty} \frac{g_n^{(1)}(a)\cos n\psi + h_n^{(1)}(a)\cos n\psi}{\omega^2(1-n^2)} \tag{1.24}$$

The solution (1.13) together with u_1 is known as the first order improved solution in which a and ψ are the solutions of the equation (1.23). If the value of the function A_1 and B_1 are substituted from (1.22) in the second relation of (1.17), one obtains the function $f^{(1)}$, in the similar way, one can find the unknown functions A_2, B_2 and u_2. Thus the determination of the higher order approximation is sufficiently clear.

The Krylov and Bogoliubov method has been extended by Kruskal [24] to solve the fully nonlinear differential equation

$$\ddot{x} = F\ (x,\ \dot{x},\ \varepsilon) \tag{1.25}$$

The solutions of this fully nonlinear equation are based on recurrence relations and are given in the form of power series of the small parameter ε.

Cap [18] has investigated some nonlinear systems of the form

$$\ddot{x} + \omega^2\ f(x) = \varepsilon F\ (x,\ \dot{x}) \tag{1.26}$$

He has solved this equation by using elliptical functions in the sense of the Krylov and Bogoliubov method.

Struble [78] has developed a technique for treating weakly nonlinear oscillatory systems such as those governed by

$$\ddot{x} + \omega^2 x = \varepsilon f\,(x,\ \dot{x},\ t) \tag{1.27}$$

He has expressed the asymptotic solution of this equation for small ε in the form

$$x = a\cos(\omega t - \theta) + \sum_{n=1}^{N} \varepsilon^n x_n(t) + O(\varepsilon^{Y+1}) \tag{1.28}$$

where a and θ are slowing varying functions of time.

Later the method of Krylov- Bogoliubov-Mitropolskii (KBM) has been extended by Popov [50] to damped nonlinear systems

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = \varepsilon f\,(x,\ \dot{x}), \tag{1.29}$$

where $-2k\dot{x}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the method [50] in the physical systems, involving damping force, Mendelson [29] and Bojadziev [14] rediscovered Popov's results. In the case of damped nonlinear systems the first equation of (1.14) has been replaced by

$$\dot{a} = -k\,a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \cdots\cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \tag{1.14a}$$

On the contrary, Murty, Deekshatulu and Krishna [35] have found a hyperbolic asymptotic solution of an over-damped system represented by the nonlinear differential equation (1.29) in the sense of KBM method; i. e., in the case $k > \omega$. They have used hyperbolic function, $\cosh\varphi$ or $\sinh\varphi$ instead of the harmonic function $\cos\varphi$, which have been used in [3,25,29,50]. In the case of oscillatory or damped oscillatory process $\cos\varphi$ may be used arbitrarily for all kinds of initial conditions. But in the case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions [15,35,36]. Murty, Deekshatulu [34] have developed another asymptotic method obtaining simple analytic solution of the over-damped system represented by the same equation (1.29). Shamsul [69] extended the KBM method to find the solutions of over-damped nonlinear systems, when one root becomes much smaller than the other root. Murty [36] has also presented a unified KBM method for solving the nonlinear systems represented by the equation (1.29). Bojadziev and Edwards [15] have investigated the solutions of oscillatory and non-oscillatory systems represented by (1.29) when k and ω are slowly varying functions of time t. Arya and Bojadziev [1,2] examined damped oscillatory systems and time-dependent oscillating systems with varying parameters and delay. Shamsul, Alam and Shanta [61] extended the Krylov- Bogoliubov-Mitropolskii method to certain non-oscillatory nonlinear systems with varying coefficients. Later Shamsul [70] have unified the KBM method for solving n-th order nonlinear differential equation with varying coefficients. Sattar [54] has developed an asymptotic method to solve a critically damped nonlinear system represented by (1.29). He has found the asymptotic solution of the system (1.29) in the form

$$x = a\,(1+\psi) + \varepsilon u_1(a,\psi) + \cdots\cdots + \varepsilon^n\, u_n(a,\psi) + O(\varepsilon^{n+1}) \tag{1.30}$$

where a is defined the equation (1.14a) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \cdots\cdots + \varepsilon^n\, C_n(a) + O(\varepsilon^{n+1}) \tag{1.14b}$$

Shamsul [58] has developed an asymptotic method for second–order over-damped and critically damped nonlinear systems. Shamsul [67,71] has also extended the KBM method for certain non-oscillatory nonlinear systems when the eigen-values of the unperturbed equation are real and non-positive. Shamsul [60] has presented a new perturbation method based on the work of Krylov-Bogliubov-Mitropolskii to find approximate solutions of nonlinear systems with large damping. Later, he has extended the method to n-th order nonlinear differential systems[64].

Making use of the KBM method Bojadziev [5] has investigated nonlinear damped oscillatory systems with small time lag. Bojadziev [11,12], Bojadziev and Chan [13] applied the KBM method to the problems of population dynamics. Bojadziev [14] has used the KBM method to investigate nonlinear biological and biochemical systems. Lin and Khan [27] have also used the KBM method to some biological problems. Proskurjakov [51], Bojadziev, Lardner and Arya [6] have investigated periodic solutions of nonlinear systems by KBM and Poincare method, and compared the two solutions. Bojadziev and Lardner [7,8] have investigated mono-frequent oscillations in mechanical systems including the case of internal resonance, governed by hyperbolic differential equations with small nonlinearities. Bojadziev and Lardner [9] have also investigated hyperbolic differential equations with large time delay. Freedman and Ruan [19] used the KBM method in the three-species chain models with group defense.

Most probably, Osiniskii [40], first extended the KBM method to a third nonlinear differential equation

$$\dddot{x} + k_1 \ddot{x} + k_2 \dot{x} + k_3 x = \varepsilon f\,(x, \dot{x}, \ddot{x}) \tag{1.31}$$

where ε is a small parameter and f is a nonlinear function. He has found the asymptotic solution in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \cdots\cdots + \varepsilon^n u_n(a,b,\psi) + O(\varepsilon^{n+1}), \qquad (1.32)$$

where u_k, $k = 1,2,\ldots\ldots n$ are periodic functions of ψ with period 2π and a, b and ψ are functions of time t, given by

$$\begin{aligned}
\dot{a} &= -\lambda\, a + \varepsilon\, A_1(a) + \varepsilon^2 A_2(a) + \cdots\cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}) \\
\dot{b} &= -\mu\, b + \varepsilon\, B_1(b) + \varepsilon^2 B_2(b) + \cdots\cdots + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}) \\
\dot{\psi} &= \omega + \varepsilon\, C_1(b) + \varepsilon^2 C_2(b) + \cdots\cdots + \varepsilon^n\, C_n(b) + O(\varepsilon^{n+1})
\end{aligned} \qquad (1.33)$$

where $-\lambda$, $-\mu \pm \omega$ are the characteristic roots of the equation (1.31) when $\varepsilon = 0$, and the functions u_k, A_k, B_k and C_k are chosen such that the equations (1.32) and (1.33) satisfy the differential equation (1.31). Osiniskii [41] has also extended the KBM method to a third order nonlinear partial differential equation with internal friction and relaxation. Mulholland [33] has studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [26] investigated nonlinear damped oscillations governed by a third order partial differential equation. They introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Rauch [52] has studied oscillations of a third order nonlinear autonomous system. Sattar [55] has extended the KBM asymptotic method for three-dimensional over-damped nonlinear systems. Shamsul and Sattar [56] developed a method to solve third order critically damped nonlinear systems. Shamsul [65] redeveloped the method presented in [56] to find approximate solutions of critically damped nonlinear systems in the presence of different damping forces. Shamsul and Sattar [59] have studied time dependent third order oscillating systems with damping based on an extension of the asymptotic method of Krylov-Bogoliubov-Mitropolskii. Shamsul [68] also has developed a method for obtaining non-oscillatory solution of third order nonlinear systems. Later, Shamsul and Sattar [57] have presented a unified KBM method for solving third order nonlinear systems. Shamsul [63] has also presented a unified Krylov-Bogoliubov-Mitropolskii method, which is not the formal form of the original KBM method, for solving n-th order nonlinear systems. The solution contains some unusual variables. Yet this solution is very important. Shamsul [74] has also presented a modified and compact form of Krylov-Bogoliubov-Mitropolskii unified method for solving n-th order nonlinear differential equation. The formula presented in [74] is compact, systematic and practical, and easier than that of [63].

Shamsul and Sattar [57] have extended Murty's [36] unified technique for obtaining the transient response of third order nonlinear systems. Recently, Shamsul [63] has presented a unified formula to obtain a general solution of an n-th order differential equation with constant coefficients. He considered a weakly nonlinear system as

$$\frac{d^{(n)}x}{dx^{(n)}} + k_1 \frac{d^{(n-1)}x}{dx^{(n-1)}} + \cdots\cdots + k_n\, x = \varepsilon\, f\,(x, \dot{x}, \ldots) \tag{1.34}$$

where over-dot denotes differentiation with respect to t, k_j, $j = 1, 2, \ldots\ldots n$ are constants. Shamsul [63] seeks a solution of (1.34) in the form

$$x(\varepsilon,\, t) = \sum_{j=1}^{n} a_j(t)\, e^{\lambda_j t} + \varepsilon\, w_1(a_1, a_2, \ldots\ldots, a_n,\, t) + \ldots \tag{1.35}$$

where λ_j, $j = 1, 2, \ldots\ldots n$ are the given eigen-values of the corresponding linear equation of (1.34) and each a_j satisfied a first order differential equation

$$\dot{a}_j = \varepsilon A_1(a_1, a_2, \ldots, a_n, t) + \ldots \tag{1.36}$$

Generally, in the treatment of the perturbation techniques an approximate solution is determined in terms of amplitude and phase variables. But the solution (1.35) starts with some new variables $a_1, a_2, \ldots, a_n$. Such a choice of variables is important to tackle various nonlinear problems with an easier approach. This technique greatly speeds up the KBM method to determine the asymptotic solution.

Hung and Wu [22] have presented an exact solution of a differential system in terms of Bessel's functions where the coefficients vary with time in an exponential order.

Shamsul, Hossain and Shanta [62] found an approximate solution of a time dependent nonlinear system in which a strong linear damping force acts. Shamsul [75] developed a general formula based on the extended Krylov-Bogoliubov-Mitropolskii method for obtaining asymptotic solution of an n-th order time dependent quasi-linear differential equation with damping. Nguyen Van Dinh [39] investigated stationary oscillation from a variant of the asymptotic procedure in a special case of the type

$$\ddot{x} + \omega^2 x = \varepsilon f\,(x, \dot{x}, \varphi), \qquad \omega = \varphi t \tag{1.37}$$

where x is an oscillatory variable, over dots denote derivatives with respect to time t. He has used asymptotic expansions in the following way

$$\begin{aligned} x &= a\cos\psi + \varepsilon u_1(a,\theta,\psi) + \varepsilon^2 u_2(a,\theta,\psi) + \cdots\cdots \\ \psi &= \varphi - \theta = \omega t - \theta, \end{aligned} \tag{1.38}$$

where a and θ represent amplitude and phase respectively and they satisfy the following differential systems

$$\dot{a} = \varepsilon A_1(a,\theta) + \varepsilon^2 A_2(a,\theta) + \cdots\cdots$$
$$\dot{\theta} = \varepsilon B_1(a,\theta) + \varepsilon^2 B_2(a,\theta) + \cdots\cdots \tag{1.39}$$

Bojadziev [16], Bojadziev and Hung [17] used at least two trial solutions to investigate time dependent differential systems; one is for resonant case and the other is for the non-resonant case. But Shamsul [75] used only one set of variational equations, arbitrarily for both resonant and non-resonant cases.

Shamsul [75] has investigated the solution of an n-th order time dependent quasi-linear differential equation

$$\frac{d^{(n)}x}{dx^{(n)}} + k_1 \frac{d^{(n-1)}x}{dx^{(n-1)}} + \cdots\cdots + k_n\, x = \varepsilon\, f\,(\nu t, x, \dot{x}, \ldots) \tag{1.40}$$

where $x^{(i)}$, $i = n, n-1, \ldots$ represent the i-th derivative, ε is a small parameter, k_j, $j = 1, 2, \ldots\ldots n$ are constant, f is a nonlinear function and ν is the frequency of the external acting force. Shamsul [61] seeks an asymptotic of (1.40) in the form

$$x(\varepsilon,t) = \sum_{j=1}^{n} a_j(t) e^{\lambda_j t} + \varepsilon u_1(a_1, a_2, \ldots, a_n) + \ldots + \varepsilon^m u_m(a_1, a_2, \ldots, a_n) \tag{1.41}$$

where λ_j, $j = 1, 2, \ldots\ldots n$ are the eigen-values of the unperturbed equation and each a_j satisfy first order differential equation

$$\dot{a}_j = \lambda_j a_j + \varepsilon A_j(a_1, a_2, \ldots, a_n, t) + \cdots + \varepsilon^m p_j(a_1, a_2, \ldots, a_n, t) \tag{1.42}$$

For $\varepsilon = 0$, expression Eq.(1.41) with Eq.(1.42) give the solution of the unperturbed equation

$$x(t,\, 0) = \sum_{j=1}^{n} a_{j,0}\, e^{\lambda_j t} \tag{1.43}$$

where $a_{j,0}$, $j=1,2,\ldots\ldots n$ are arbitrary constants. The proposed solution (1.41) is not chosen in a usual form of KBM method but it can be easily brought to the usual form (1.40) - (1.43) by suitable variable transformations $a_{2l-1}(t)=1/2\, b_1(t)\, e^{i\varphi_1(t)}$ and $a_{2l}(t)=1/2\, b_1(t)\, e^{-i\varphi_1(t)}$, where $b_1(t)$ and $\varphi_1(t)$, $l=1,2,\ldots\ldots n/2$ are amplitude and phase variables. It can be readily shown that solution (1.41) takes the form

$$
\begin{aligned}
x(\varepsilon,t) = {} & \sum_{l=1}^{n/2} 1/2\, b_1(t)(e^{i\varphi_1(t)} + e^{-i\varphi_1(t)}) + \varepsilon\, u_1\,(b_1, b_2, \ldots, b_{n/2}, \varphi_1, \varphi_2, \ldots, \varphi_{n/2}) \\
& + \ldots + \varepsilon^m\, u_m\,(\ldots)
\end{aligned}
\tag{1.44}
$$

and $b_1(t)$ & $\varphi_1(t)$ satisfy the equations

$$
\begin{aligned}
\dot{b}_1 &= -\mu_1 b_1 + \varepsilon\, A_1(b_1, b_2, \ldots, b_{n/2}, \varphi_1, t) + \cdots + \varepsilon^n P_n(b_1, b_2, \ldots, b_{n/2}, \varphi_1, t) \\
\dot{\varphi}_1 &= \omega_1 b_1 + \varepsilon\, B_1(b_1, b_2, \ldots, b_{n/2}, \varphi_1, t) + \cdots + \varepsilon^n Q_n(b_1, b_2, \ldots, b_{n/2}, \varphi_1, t)
\end{aligned}
\tag{1.45}
$$

where $\lambda_{2l-1} = -\mu_1 \pm i\omega_1$ are the eigen-values of the equation (1.44) when $\varepsilon = 0$.

Pinakee Dey *et al* [45] found an asymptotic solution of a second order over-damped nonlinear non-autonomous differential system in presence of an external force. Finally, the authors [46] have developed an asymptotic method for time dependent nonlinear differential systems with varying coefficients, in which the coefficients change slowly and periodically with time.

1.2 The Proposal

Herein, we propose the perturbation systems governed by second and n-th order non-linear differential equations

$$\ddot{x} + 2k\,\dot{x} + \omega^2 x = \varepsilon\, f\,(x,\, \dot{x}),$$

$$x^{(n)} + c_1\, x^{(n-1)} + c_2\, x^{(n-2)} \cdots\cdots + c_n\, x = \varepsilon\, f\,(x, \dot{x}, \ddot{x} \ldots) \tag{1.46}$$

and differential equations with varying coefficients

$$\ddot{x} + 2k\,(\tau)\dot{x} + \omega^2\,(\tau) x = \varepsilon\, f\,(x,\, \dot{x},\, \tau),$$

$$x^{(n)} + c_1\,(\tau) x^{(n-1)} + c_2\,(\tau) x^{(n-2)} \cdots\cdots + c_n\,(\tau)\, x = \varepsilon\, f\,(x, \dot{x}, \ddot{x} \ldots, \tau) \tag{1.47}$$

where $\varepsilon = 0$ is a small parameter, $\tau = \varepsilon t$ is the slowly varying time and f is a given nonlinear function.

In **Chapter 2** a perturbation technique is developed to solve approximate solution of over-damped nonlinear non-autonomous differential systems with varying coefficients.

Finally, in **Chapter 3** an asymptotic method for second order time dependent nonlinear differential systems with varying coefficients is developed.

Chapter 2

High precision numerical solution and approximate solution of over-damped nonlinear non-autonomous differential systems with varying coefficients

2.1 Introduction

There have been many analytical techniques developed for solving oscillations of nonlinear differential equations. These equations can be linearized by imposing certain restrictions and then they are solved in simple approaches. In vibrating processes many problems are solved by linearizing such differential equations when the amplitude of oscillation is small. But when the amplitude is not small enough, the linear solution is not sufficient to describe the vibration. In these cases, the Krylov-Bogoliubov-Mitropolskii (KBM) [25,3] asymptotic method is particularly convenient and extensively used methods to study nonlinear differential systems with small nonlinearities. Originally, the method was developed by Krylov and Bogoliubov [25] for obtaining periodic solution of a second order nonlinear differential equation. Latter, the method was amplified and justified mathematically by Bogoliubov and Mitropolskii [3,32]. Popov [50] extended the method to a damped oscillatory process in which a strong linear damping force acts. Arya and Bojadziev [2] have studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Arya and Bojadziev [1] have also studied a system of second order nonlinear hyperbolic differential equation with slowly varying coefficients. Murty, Deekshatulu and Krishna [35] and Shamsul [58,63,70] extended the method to over-damped nonlinear system. Recently Shamsul [63] has presented a unified method for solving an n-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes. In another recent paper, Shamsul [70] has extended the unified method [63] to similar differential system (autonomous) with slowly varying coefficient. But Murty, Deekshatulu and Krishna [35] and Shamsul [58,63,70] limited their investigations to autonomous system. The aim of this paper is to extend the result in [70] to similar nonlinear vibrating problems in which external forces act and also investigated double and high precision numerical solutions.

2.2 The method

Let us consider the nonlinear differential system

$$\ddot{x} + 2k(\tau)\dot{x} + \omega^2(\tau)x = -\varepsilon f(x, \dot{x}, \tau), \qquad \tau = \varepsilon t, \tag{2.1}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, $\tau = \varepsilon t$ is the slowly varying time, $k(\tau) \ge 0$, f is a given nonlinear function and $\omega(\tau)$ is the frequency. The coefficients in Eq. (2.1) are slowly varying in that their time derivatives are proportional to ε.

Setting $\varepsilon = 0$ and $\tau = \tau_0$ =constant, in Eq.(2.1), we obtain the unperturbed solution of the equation. Let Eq. (2.1) have two eigen-values $\lambda_j(\tau_0)$, $j = 1, 2$, where $\lambda_j(\tau_0)$ are constant, but when $\varepsilon \neq 0$, $\lambda_j(\tau)$ slowly vary with time. The unperturbed solution of Eq. (2.1) becomes

$$x(t,0) = \sum_{j=1}^{2} a_{j,0} e^{\lambda_j(\tau_0)t} \, . \tag{2.2}$$

When $\varepsilon \neq 0$ we seek a solution, in accordance with the KBM method, of the form

$$x(t,\varepsilon)=\sum_{j=1}^{2} a_{j,0}(t,\tau)+\varepsilon u_1(a_1,a_2,\tau)+\varepsilon^2 u_2(a_1,a_2,\tau)+\ldots, \tag{2.3}$$

where $a_{j,0}$, $j=1,2$ satisfy the differential equations

$$\dot{a}_j=\lambda_j(\tau)a_j+\varepsilon A_j(a_1,a_2,\tau)+\varepsilon^2\ldots, \tag{2.4}$$

The solution (2.3) together with (2.4) is not considered in a usual form of the classical KBM method. But this solution was early introduced by Murty [35] to investigate un-damped, damped and over-damped cases. Now it is being used to investigate various oscillatory and non-oscillatory problems (see [58,63,70] for details).

Confining our attention to the first few terms, $1,2,\ldots,m$ in the series expansions of (2.3) and (2.4), we evaluate the functions $u_1,\ldots,A_1,A_2\ldots$, such that a_1 and a_2 appearing in (2.3) and (2.4) satisfy (2.1) with an accuracy of ε^{m+1} [63]. In order to determine these unknown functions, it was assumed that the functions $u_1,\ldots$ do not contain the fundamental terms [58,63,70], which are included in the series expansion (2.3) of order ε^0.

Differentiating $x(t,\varepsilon)$ two times with respect to t, substituting for the derivatives $\ddot{x}$ and x in the original equation (2.1) and equating the coefficient of ε, we obtain

$$\begin{aligned}&(\Omega-\lambda_2)A_1+\lambda_1' a_1+(\Omega-\lambda_1)A_2+\lambda_2' a_2+(\Omega-\lambda_1)(\Omega-\lambda_2)u_1\\&=-f^{(0)}(a_1,a_2,\tau),\end{aligned} \tag{2.5}$$

where $\Omega\equiv\lambda_1 a_1\dfrac{\partial}{\partial a_1}+\lambda_2 a_2\dfrac{\partial}{\partial a_2}$, $\lambda_1'=\dfrac{d\lambda_1}{d\tau}$, $\lambda_2'=\dfrac{d\lambda_2}{d\tau}$, $f^{(0)}=f(x_0,\dot{x}_0,\tau)$

and $x_0 = a_1 + a_2$.

We have assumed that u_1 does not contain fundamental terms and for this reason the solution will be free from secular terms, namely $t\cos t,\ t\sin t$ and te^{-t} (see [70]).

In general the function $f^{(0)}$ can be expanded in a Taylor series as:

$$f^{(0)} = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2} a_1^{r_1} a_2^{r_2} \tag{2.6}$$

To obtain this solution (2.4), it has been proposed in [63] that u_1, u_2 exclude the terms $a_1^{r_1} a_2^{r_2}$ of $f^{(0)}$, where $r_1 - r_2 = \pm 1$. This restriction guarantees that the solution always excludes *secular*-type terms or the first harmonic terms (see [63] for details). According to our assumption, u_1 does not contain the fundamental terms, therefore equation (2.5) can be separated into three equations for unknown functions u_1 and A_1, A_2 (see [63] for details). Substituting the functional values of $f^{(0)}$ and equating the coefficients of $e^{\lambda_j t}$, $j = 1, 2$, we obtain

$$(\Omega - \lambda_2) A_1 + \lambda_1' a_1 = f^{(0)} = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2} a_1^{r_1} a_2^{r_2} \quad \text{if} \quad r_1 = r_2 + 1 \tag{2.7}$$

$$(\Omega - \lambda_1) A_2 + \lambda_2' a_2 = f^{(0)} = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2} a_1^{r_1} a_2^{r_2} \quad \text{if} \quad r_2 = r_1 + 1 \tag{2.8}$$

and

$$(\Omega - \lambda_1)(\Omega - \lambda_2) u_1 = f^{(0)} = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2} a_1^{r_1} a_2^{r_2} \tag{2.9}$$

where $f^{(0)} = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2} a_1^{r_1} a_2^{r_2}$ exclude those terms for $r_1 = r_2 \pm 1$.

Thus the particular solutions of (2.7) - (2.9) give the values of the unknown functions A_1, A_2 and u_1. We have already mentioned that equation (2.1) is not a standard form of KBM method. We shall be able to transform (2.3) to the exact form of the KBM [25,3,32] solution by substituting $a_1 = a\,e^{i\varphi}/2$ and $a_2 = a\,e^{-i\varphi}/2$. Herein, a and φ are respectively amplitude and phase variables (see [58,63,70]). Under this assumption, we shall be able to find the unknown functions u_1 and A_1, A_2 which completes the determination of the solution of a second order non-linear problem (2.1).

2.3 Example

Consider a nonlinear differential system with a non-periodic external force

$$\ddot{x} + 2\,k(\tau)\,\dot{x} + \omega^2(\tau)\,x = -\varepsilon\,x^3 + 2\,\varepsilon\,E\,e^{-\nu t}\cos\nu\,t\,, \tag{2.10}$$

The function $f^{(0)}$ becomes,

$$f^{(0)} = -\varepsilon\,(a_1^3 + 3\,a_1^2 a_2 + 3\,a_1\,a_2^2 + a_2^3) + 2\,\varepsilon\,E\,e^{-\nu t}\cos\nu\,t \tag{2.11}$$

We substitute $f^{(0)}$ in (2.5) and separate it into two parts as

$$\begin{aligned}(\Omega - \lambda_2)A_1 + \lambda_1'\,a_1 + (\Omega - \lambda_1)A_2 + \lambda_2'\,a_2 &= -a_1^3 - 3\,a_1^2 a_2 \\ &\quad + 2\,E\,e^{-\nu t}\cos\nu\,t\end{aligned} \tag{2.12}$$

and

$$(\Omega - \lambda_1)(\Omega - \lambda_2)u_1 = -(3a_1\,a_2^2 + a_2^3)\,. \tag{2.13}$$

The particular solution of (2.13) is

$$u_1 = c_1\,a_1\,a_2^2 + c_2\,a_2^3, \tag{2.14}$$

where $\quad c_1 = \dfrac{-3}{2\,\lambda_2\,(\lambda_1 + \lambda_2)},\quad c_2 = \dfrac{-1}{2\,\lambda_2\,(3\,\lambda_2 - \lambda_1)}.$

Now we have to determine two functions A_1 and A_2 from a single equation (2.12).

$$(\Omega - \lambda_2) A_1 + \lambda_1' a_1 = -a_1^3 + 2\varepsilon E e^{-\nu t} \cos \nu t, \qquad (2.15)$$

and

$$(\Omega - \lambda_1) A_2 + \lambda_2' a_2 = -3 a_1^2 a_2. \qquad (2.16)$$

The particular solution of (2.15) - (2.16) is

$$A_1 = \lambda_1' a_1 n_1 + n_2 a_1^3 + E n_3, \quad \text{and} \quad A_2 = \lambda_2' a_1 l_1 + l_2 a_1^2 a_2, \qquad (2.17)$$

where

$$n_1 = \frac{-1}{\lambda_1 - \lambda_2}, \quad n_2 = \frac{-1}{3\lambda_1 - \lambda_2}, \quad n_3 = \frac{1}{2\lambda_1 - \lambda_2},$$
$$l_1 = \frac{1}{\lambda_1 - \lambda_2}, \quad l_2 = \frac{-3}{\lambda_1 + \lambda_2},$$

Substituting the functional values of A_1 and A_2 into (2.5) and rearranging, we obtain

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon (\lambda_1' a_1 n_1 + n_2 a_1^3 + E n_3) \qquad (2.18)$$

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon (\lambda_2' a_2 l_1 + l_2 a_1^2 a_2) \qquad (2.19)$$

Therefore, the first order solution of (2.10) is

$$x(t, \varepsilon) = a_1 + a_2 + \varepsilon u_1, \qquad (2.20)$$

where a_1, a_2 are given by (2.18), (2.19) and u_1 is given by (2.14).

2.4 Results and discussions

An asymptotic solution of damped nonlinear non-autonomous vibrating system is obtained based on the extended KBM method (by Popov [50]). In order to test the accuracy of an approximate solutions obtain by a perturbation method, we compare the approximate solution to the numerical solution (consider to be exact). With regard to such a comparison concerning the presented KBM method of this paper, we refer to the works of Murty, Dekshatulu and Krishna [35] and Shamsul [58,63,70]. In this paper we have compared the perturbation solution (2.20) to those obtained by Runge-Kutta (fourth order) method for $\lambda_1 = -.05,\ \lambda_2 = -5,\ a_1 = 1,\ a_2 = 0,\ \varepsilon = 0.2,\ E = 1$ with initial condition $x(0) = 1.0,\ \dot{x} = -.050421$ and all the results are shown in Fig.2.1.

From the Fig 2.1, we observe that the approximate solutions show a good coincidence with the numerical solutions. The corresponding numerical solutions have also been computed by Runge-Kutta (fourth-order) method. From the Fig 2.2 and the Fig 2.3, the approximate solutions agree with numerical results nicely. Actually, first we compute the numerical solution in double precision. In general equation (2.20) has no exact solution. Usually a numerical procedure is used to solve it. In this paper we have used the *Runge-Kutta* (fourth order) method. Numerically, it is advantageous to solve the transformed equation (2.20) instead of the original equation (2.10) because a large step size can be used in the integration (see [38] for details).

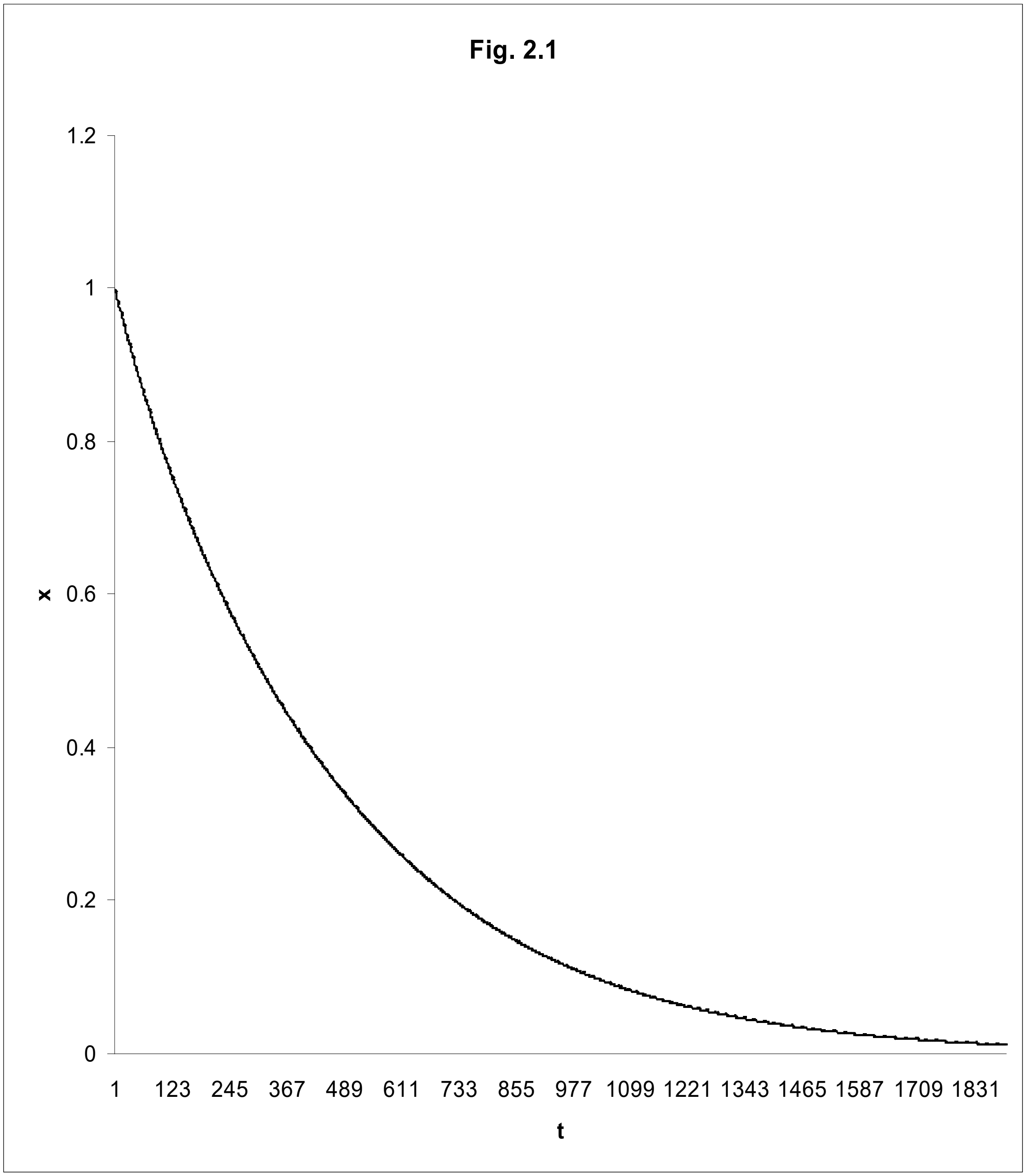


Fig 2.1: Perturbation solution with corresponding numerical solution is plotted with initial conditions $x(0)=1.0,\ \dot{x}=-.050421$ for $\lambda_1=-.05, \lambda_2=-5,\ a_1=1, a_2=0, \varepsilon=0.2, E=1$.

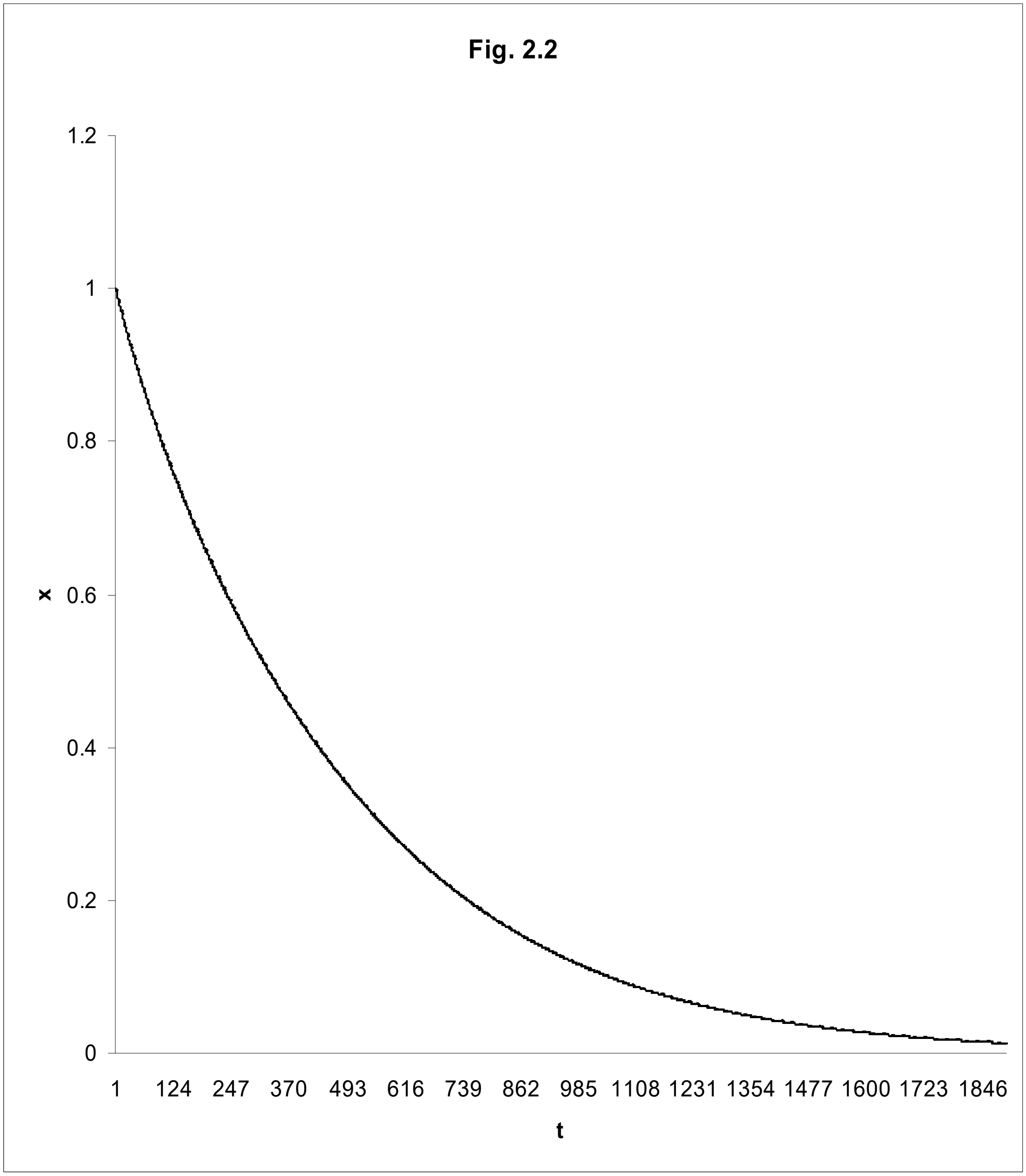


Fig 2.2: Perturbation solution with corresponding numerical solution is plotted with initial conditions $x(0)=1.0,\ \dot{x}=-.050631$ for $\lambda_1=-.05, \lambda_2=-5, a_1=1, a_2=0, \varepsilon=0.3, E=1$.

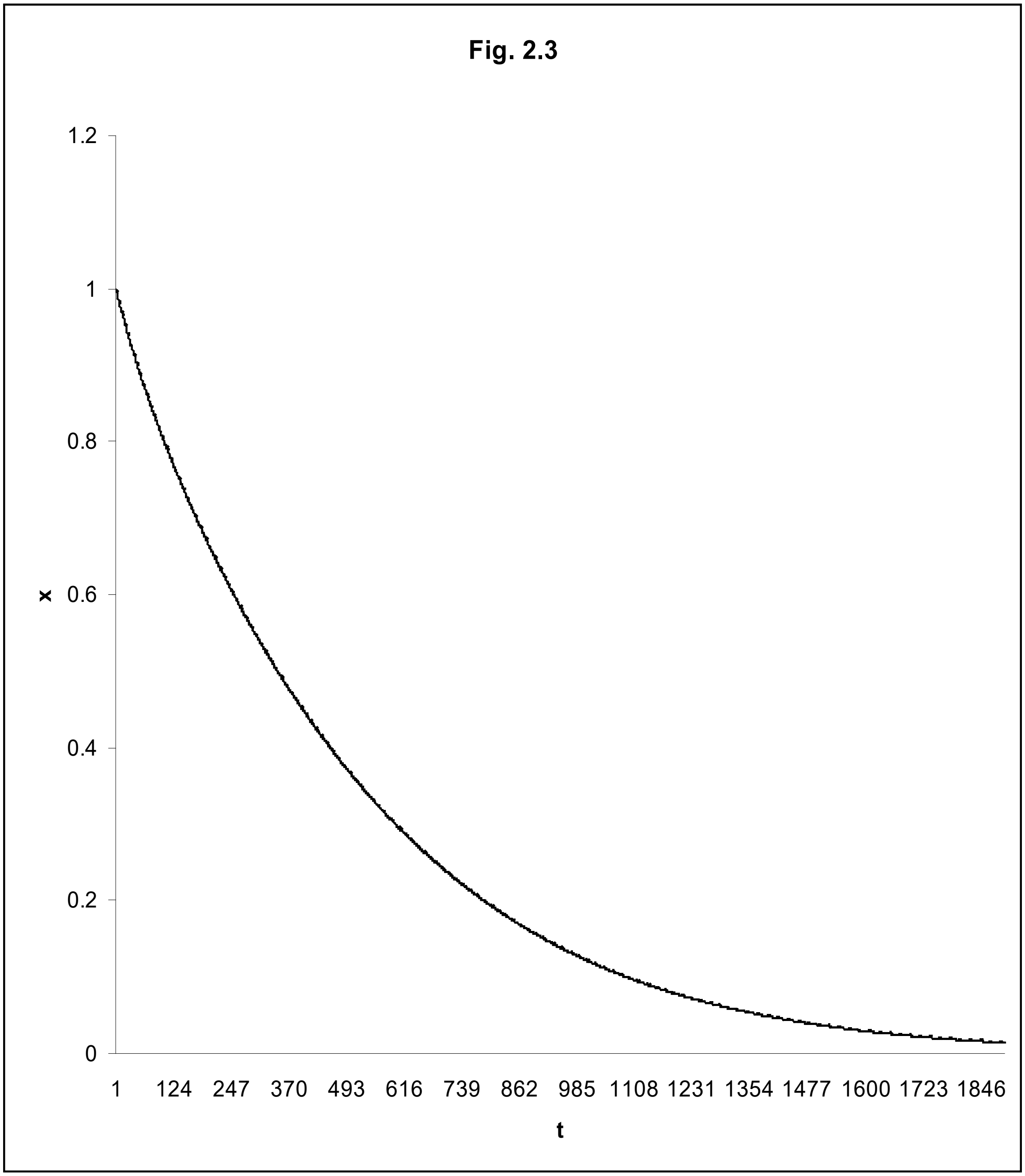


Fig 2.3: Perturbation solution with corresponding numerical solution is plotted with initial conditions $x(0) = 1.0$, $\dot{x} = -.051052$ for $\lambda_1 = -.05, \lambda_2 = -5, a_1 = 1, a_2 = 0, \varepsilon = 0.5, E = 1$.

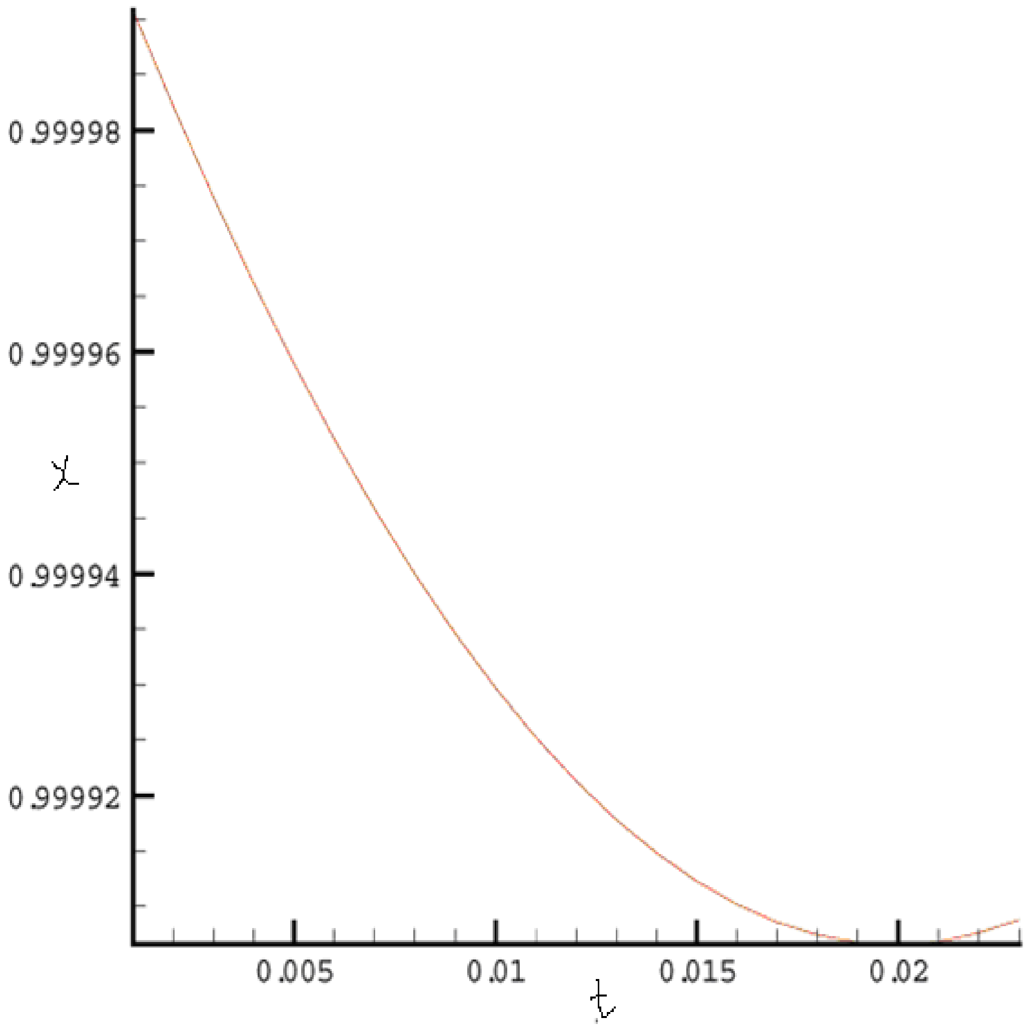
2.5 Multiple Precision (with exflib library)

Exflib (extended precision floating-point arithmetic library) is simple software for multiple-precision arithmetic in scientific numerical computation. Multiple-precision arithmetic is a method for representation and calculation of real numbers with arbitrary accuracy (see [21]).

2.6. High precision numerical results

The high precision numerical results of our problems are shown in fig.2.4. High precision numerical solutions are computed by Multiple-precision arithmetic with Exflib. Here h=.001 in Runge-Kutta method, but the above numerical solutions are obtained with h=.05.

Fig 2.4



2.7 Conclusion

An asymptotic solution has been obtained for the second order nonlinear non-autonomous differential system characterized by non-oscillatory process. The method is a generalization of extended KBM method [25,3] (by Popov [50]) and can be used to obtain desired solution for certain external forces. The solution shows a good coincident with the numerical solution. The high precision numerical results also represented. The asymptotic solutions and the high precision numerical results are of same types.

Chapter 3

Approximate solution of time dependent damped nonlinear vibrating systems with slowly varying coefficients

3.1 Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) [25,3] method is one of the most widely used methods to obtain the approximation solutions of nonlinear systems with a small non-linearity. The method, originally developed by Krylov-Bogoliubov [25] for obtaining periodic solutions, was amplified and justified by Bogoliubov and Mitropolskii [3] and latter extended by Mitropolskii [32] to similar systems with slowly varying coefficients. Popov [50] extended this method to a damped oscillation. Bojadziev and Edward [15] studied some under-damped and over-damped systems with slowly varying coefficients. Murty [36] has presented a unified KBM method for both under-damped and over-damped system with constant coefficients. Shamsul [70] has presented a unified KBM method for solving an *n*-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes with slowly varying coefficients. Hung and Wu [22] obtained an exact solution of a differential system in terms of Bessel's functions where the coefficients varying with time in an exponential order. Roy and Shamsul [53] found an asymptotic solution of a differential systems in which the coefficient changes in an exponential order of slowly varying time. Pinakee *et.al* [47] has presented extended KBM method for under-damped, damped and over-damped vibrating systems in which the coefficients change slowly and periodically with time. Recently Pinakee *et.al* [48] extended the result in [53] to similar nonlinear non-autonomous vibrating problems in which external forces act. In this article we have extended the KBM method to investigate the solution of damped forced nonlinear systems with slowly varying coefficients which measures better result for strong nonlinearities but Unified KBM method is unable to give desired results (wherein external forces act).

3.2 The Method

Let us consider the nonlinear differential system

$$\ddot{x} + 2k(\tau)\dot{x} + (c_1 + c_2 \cos\tau + c_3 \sin\tau)x = -\varepsilon f(x, \dot{x}, \tau, \nu t), \qquad \tau = \varepsilon t \tag{3.1}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, c_1, c_2 and c_3 are constants, $c_2 = c_3 = O(\varepsilon)$, $\tau = \varepsilon t$ is the slowly varying time, $k(\tau) \geq 0$, f is a given nonlinear function. Setting $\omega^2(\tau) = (c_1 + c_2 \cos\tau + c_3 \sin\tau)$, $\omega(\tau)$ is known as frequency and ν is the frequency of the external force. The coefficients in Eq. (3.1) are slowly varying in that their time derivatives are proportional to ε.

Setting $\varepsilon = 0$ and $\tau = \tau_0$ = constant, in Eq. (3.1), we obtain the unperturbed solution of (3.1) in the form

$$x(t,0) = a_{1,0} e^{\lambda_1(\tau_0)t} + a_{2,0} e^{\lambda_2(\tau_0)t}, \tag{3.2}$$

Let Eq. (3.1) have two eigen-values, $\lambda_j(\tau_0)$, $j = 1,2$, where $\lambda_j(\tau_0)$ are constants, but when $\varepsilon \neq 0$, $\lambda_j(\tau)$ vary slowly with time., When $\varepsilon \neq 0$, an approximate solution of Eq. (3.1) is chosen in the form given below

$$x(t,\varepsilon) = \sum_{j=1}^{2} a_{j,0}(t,\tau) + \varepsilon u_1(a_1, a_2, t, \tau) + \varepsilon^2 u_2(a_1, a_2, t, \tau) + \ldots, \tag{3.3}$$

where $a_{j,0}$, $j = 1,2$ satisfy the differential equations

$$\dot{a}_j = \lambda_j(\tau)a_j + \varepsilon A_j(a_1, a_2, t, \tau) + \varepsilon^2..., \tag{3.4}$$

The solution (3.3) together with (3.4) is not considered in a usual form of the classical KBM method. But this solution was early introduced by Murty [36] to investigate undamped, damped and overdamped cases. Now it is being used to investigate various oscillatory and non-oscillatory problems (see [42,48,47] for details).

Confining our attention to the first few terms, $1,2,\ldots,m$ in the series expansions of (3.3) and (3.4), we evaluate the functions $u_1,..., A_1, A_2...,$ such that a_1 and a_2 appearing in (3.3) and (3.4) satisfy (3.1) with an accuracy of ε^{m+1}. In order to determine these unknown functions, it was assumed that the functions $u_1,...$ do not contain the fundamental terms, the solution will be free from secular terms, namely $t\cos t$, $t\sin t$ and te^{-t} (see [70]), which are included in the series expansion (3.3) of order ε^0.

Differentiating $x(t,\varepsilon)$ two times with respect to *t***,** substituting for the derivatives $\ddot{x}$ and x in the original equation (3.1) and equating the coefficient of ε, we obtain

$$\begin{aligned}
&\lambda_1' a_1 + \lambda_2' a_2 - \lambda_2 A_1 - \lambda_1 A_2 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2}\right)\left(A_1 + A_2\right) \\
&+\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)u_1 \\
&= -f^{(0)}(a_1, a_2, \nu t, \tau),
\end{aligned} \tag{3.5}$$

where $\lambda_1' = \dfrac{d\lambda_1}{d\tau}$, $\lambda_2' = \dfrac{d\lambda_2}{d\tau}$, $f^{(0)} = f(x_0, \dot{x}_0, \nu t, \tau)$

and $\qquad x_0 = a_1(t,\tau) + a_2(t,\tau).$

Herein it is assumed that both $f^{(0)}$ can be expanded in Taylor's series

$$f^{(0)} = \sum_{r_1,r_2=0}^{\infty} F_{r_1,r_2}(\tau) a_1^{r_1} a_2^{r_2} \,, \tag{3.6}$$

It was early imposed by Krylov and Bogoliubov [25] that u_1 does not contain secular terms (*e.g.*, $t\cos t$ and $t\sin t$) for obtaining the periodic solution of (3.1) in which $k_1 = 0$. Popov [50] extended this method to an under-damped case in which $\sqrt{k_2} > k_1 > 0$.

Murty [36] extended the same method to the over-damped case. *i.e.*, for $k_1 > \sqrt{k_2}$.

We have already mentioned that equation (3.1) is not a standard form of KBM method. By substituting $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$, to transform (3.3) to the exact form of the KBM solution. Herein, a and φ are respectively amplitude and phase variables. Under this assumption, we shall be able to find the unknown functions A_1, A_2 and u_1.

3.3 Example:

As example of the above procedure, let us consider a nonlinear non-autonomous system with slowly varying coefficients

$$\ddot{x} + 2k(\tau)\dot{x} + (c_1 + c_2\cos\tau + c_3\sin\tau)x = -\varepsilon x^3 + \varepsilon E \sin\nu t, \tag{3.7}$$

Here over dots denote differentiation with respect to t. c_1, c_2 and c_3 are constants, $c_2 = c_3 = O(\varepsilon)$, $x_0 = a_1 + a_2$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3) + E(e^{i\nu t} - e^{-i\nu t})/2i\,. \tag{3.8}$$

Following the assumption (discussed in section 2.2) u_1 excludes the terms $3a_1^2 a_2$, $3a_1 a_2^2$ and $\varepsilon E(e^{i\nu t} - e^{-i\nu t})/2i$. We substitute in (3.8) and separate it into two parts as

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_1 + \lambda_1' a_1 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 = -(3a_1^2 a_2 + 3a_1 a_2^2) + E(e^{i\nu t} - e^{-i\nu t})/2i \tag{3.9}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) u_1 = -(a_1^3 + a_2^3) \tag{3.10}$$

The particular solution of (3.10) is

$$u_1 = -\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{3.11}$$

Now we have to solve (3.9) for two functions A_1 and A_2. According to the unified KBM method A_1 contains the term $3a_1^2 a_2$, $e^{i\nu t}/2$ and A_2 contains the term $3a_1 a_2^2$, $e^{-i\nu t}/2$ (see [48]) and thus we obtain the following equations

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_1 + \lambda_1' a_1 = -3a_1^2 a_2 + E e^{i\nu t}/2i, \tag{3.12}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 = -3a_1 a_2^2 - E e^{-i\nu t}/2i \tag{3.13}$$

The particular solutions of (3.12) and (3.13) are

$$A_1 = -\lambda_1' a_1/(\lambda_1 - \lambda_2) - 3a_1^2 a_2/2\lambda_1 + E e^{i\nu t}/2(i\nu - \lambda_2) \tag{3.14}$$

and

$$A_2 = \lambda_2' a_2 / (\lambda_1 - \lambda_2) - 3 a_1 a_2^2 / 2\lambda_2 + E\, e^{-i\nu t} / 2i(i\nu + \lambda_1) \tag{3.15}$$

Substituting the functional values of A_1, A_2 from (3.14) and (3.15) into (3.4) and rearranging, we obtain

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\lambda_1' a_1 / (\lambda_1 - \lambda_2) - 3 a_1^2 a_2 / 2\lambda_1 + E\, e^{i\nu t} / 2i(i\nu - \lambda_2)\right) \tag{3.16}$$

and

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(\lambda_2' a_2 / (\lambda_1 - \lambda_2) - 3 a_1 a_2^2 / 2\lambda_2 + E\, e^{-i\nu t} / 2i(i\nu + \lambda_1)\right) \tag{3.17}$$

The variational equations of a and φ, in the real form, transform (3.16) and (3.17) to

$$\begin{aligned}\dot{a} = -ka - \varepsilon a\omega' / 2\omega + 3\varepsilon a^3 k / 8(k^2 + \omega^2) - \varepsilon E\{k \sin\psi \\ + (\nu + \omega)\cos\psi\} / \{k^2 + (\nu + \omega)^2\}\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\dot{\varphi} = \omega + \varepsilon k' / 2\omega + 3\varepsilon a^2 \omega / 8(k^2 + \omega^2) - \varepsilon E\{-(\nu + \omega)\sin\psi \\ + k\cos\psi\} / a\{k^2 + (\nu + \omega)^2\}\end{aligned} \tag{3.19}$$

where $\omega = \sqrt{c_1 + c_2 \cos\tau + c_3 \sin\tau}$

Therefore, the first order solution of the equation (3.7) is

$$x(t, \varepsilon) = a\cos\varphi + \varepsilon u_1 \tag{3.20}$$

where a and φ are the solution of the equation (3.18) and (3.19) respectively, u_1 is given by (3.11). Substituting the values of A_1, A_2 from (3.14) and (3.15) into (3.4) and solving them, we obtain the Unified KBM solution of (3.4) similar to (3.18) and (3.19).

In this paper, we have used the *Runge-Kutta* (fourth order) method. Numerically, it is advantageous; a large step size can be used in the integration (see [38] for details).

3.4 Results and Discussions

A simple technique is presented based on the extended KBM method to determine approximate solutions of non-autonomous nonlinear vibrating systems with varying coefficients. The solution has been determined under the extended KBM method which gives better result for long time even ε is 10 times greater than existing procedures. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the function, the solution is in general confined to a low order, usually the first. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this article, we refer to the works of Murty [36], Shamsul [70] and Pinakee *et al* [48,47]. In our present paper, for different initial conditions, we have compared the perturbation solutions (3.20) of *Duffing's* equations (3.7) to those obtained by Runge-Kutta (fourth-order) procedure.

First of all, x is calculated by (3.20) with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000,\ \varphi = -.046433$ for $\varepsilon = 0.5,\ \nu = 1, \omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)},\ k = .1\sqrt{\cos\tau}$. Then corresponding numerical solutions are also computed by Runge-Kutta (fourth-order) method. The result is shown in Fig.3.1. Also we plot unified KBM solution in Fig.3.2 with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000,\ \varphi = -4.382760$ for $\varepsilon = .5,\ \omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)},\ k = .1\sqrt{\cos\tau}$. We see that in Fig.3.1 the perturbation solution nicely agrees with the numerical solution, but in this situation unified KBM solution (in Fig.3.2) does not agree. The corresponding numerical solutions have also been computed by Runge-Kutta (fourth-order) method. From Fig.3.3, Fig.3.5, Fig.3.7, Fig.3.9 and Fig.3.11, we observe that the approximate solutions agree with numerical results nicely even if $\varepsilon \geq 1.0$ but in Fig. 3.4, Fig. 3.6, Fig.3.8, Fig.3.10 and Fig.3.12 do not agree and the solution fails to give desired results.

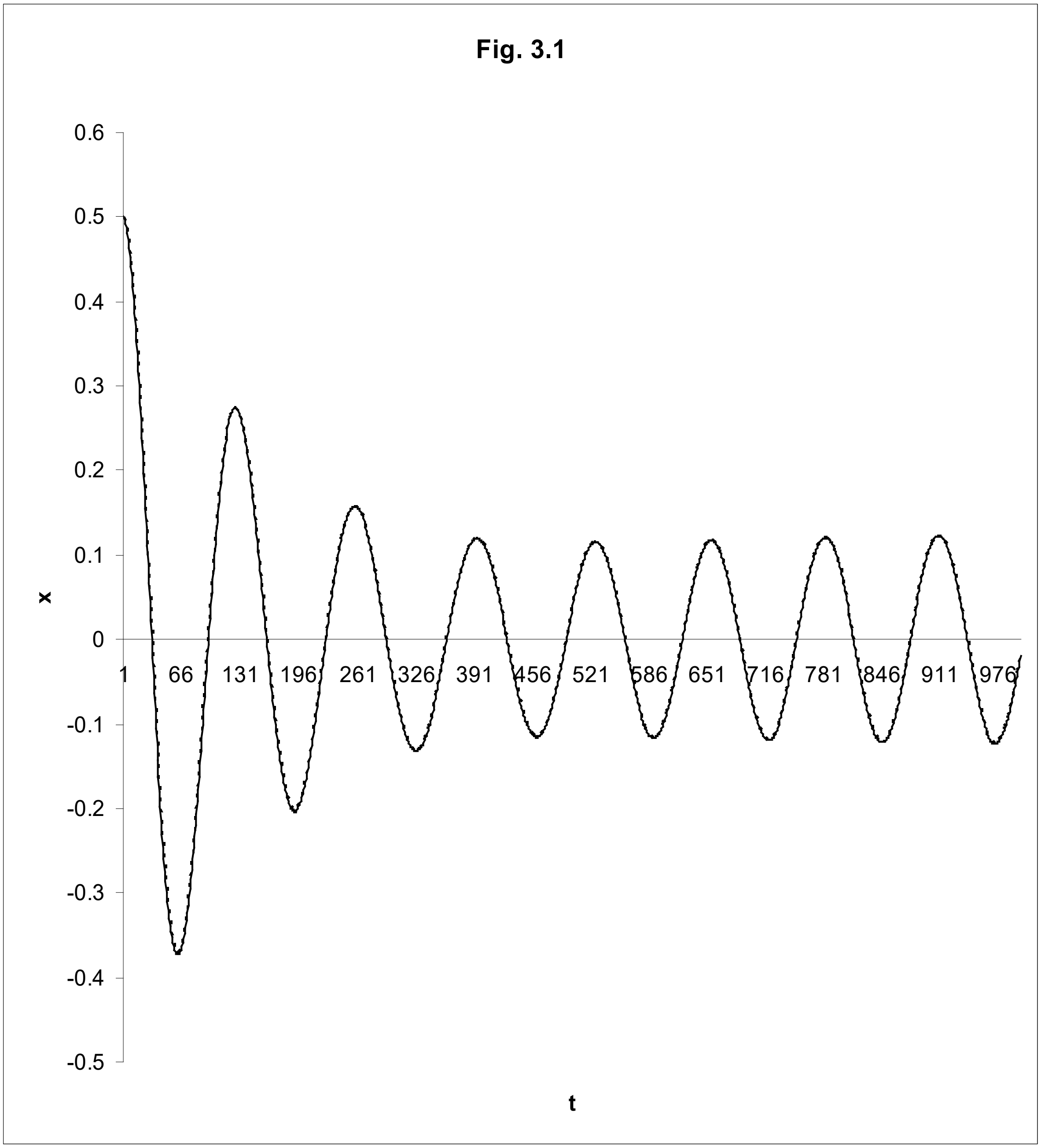


Fig 3.1: Present approximate solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000, \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -.046433$ for $\varepsilon = 0.5$, $\nu = 1.0$, $k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$

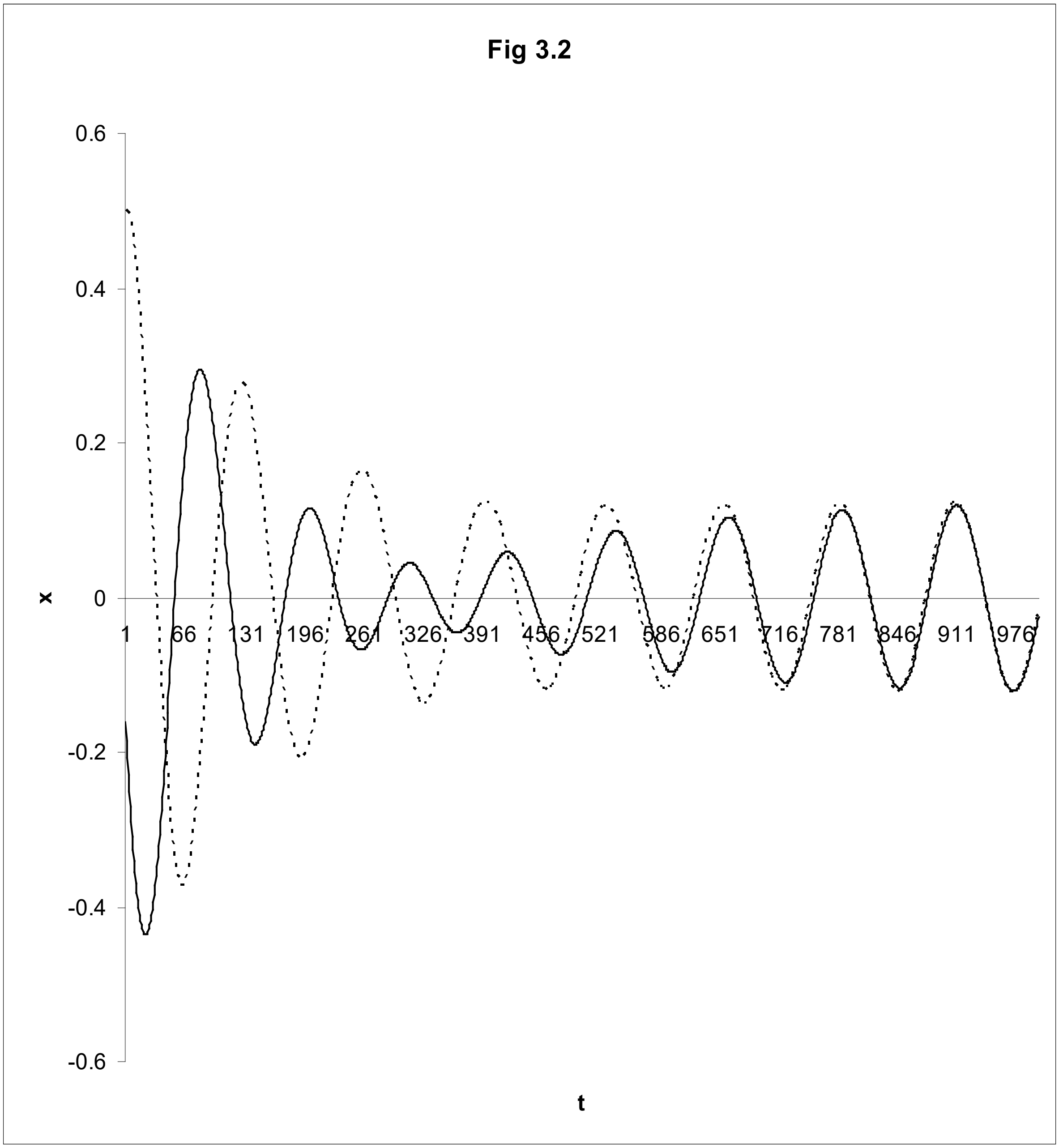


Fig 3.2: Unified KBM perturbation solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -4.382760$ for $\varepsilon = 0.5,\ \nu = 1.0,\ k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$

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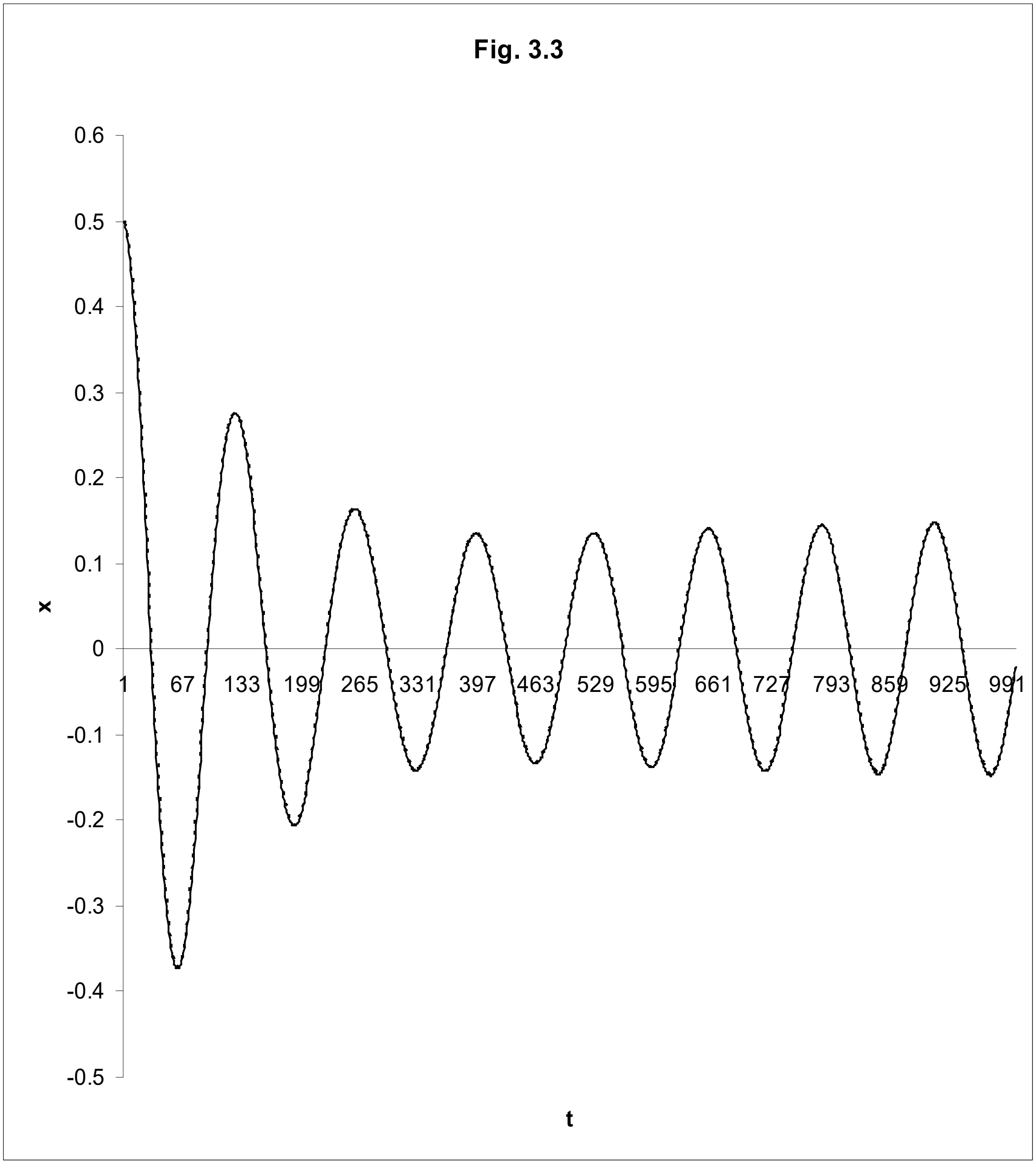

Fig 3.3: Present approximate solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -.045719$ for $\varepsilon = 0.6,\ \nu = 1.0, k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

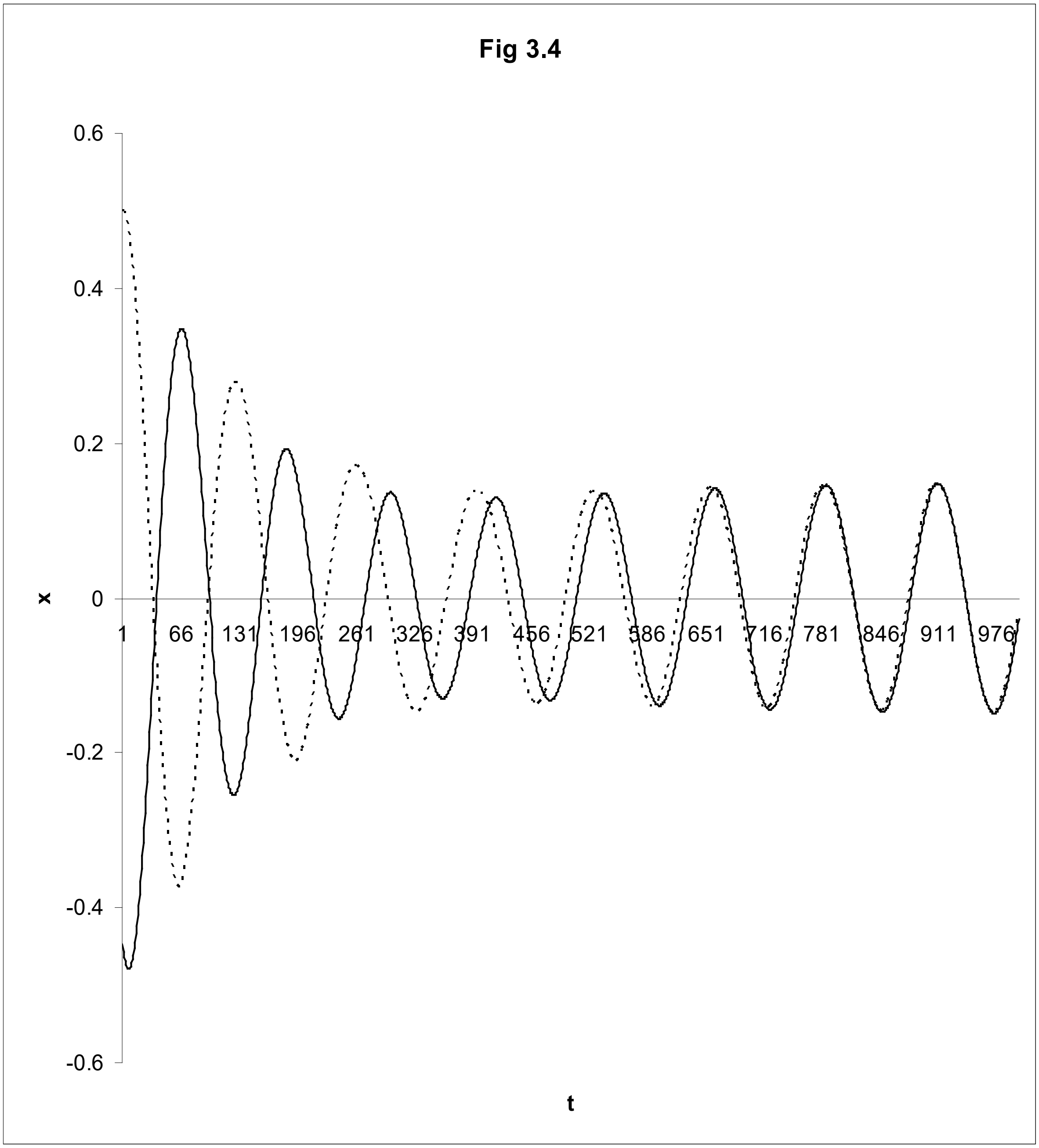

Fig 3.4: Unified KBM perturbation solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -3.6066$ for $\varepsilon = 0.6,\ \nu = 1.0,\ k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

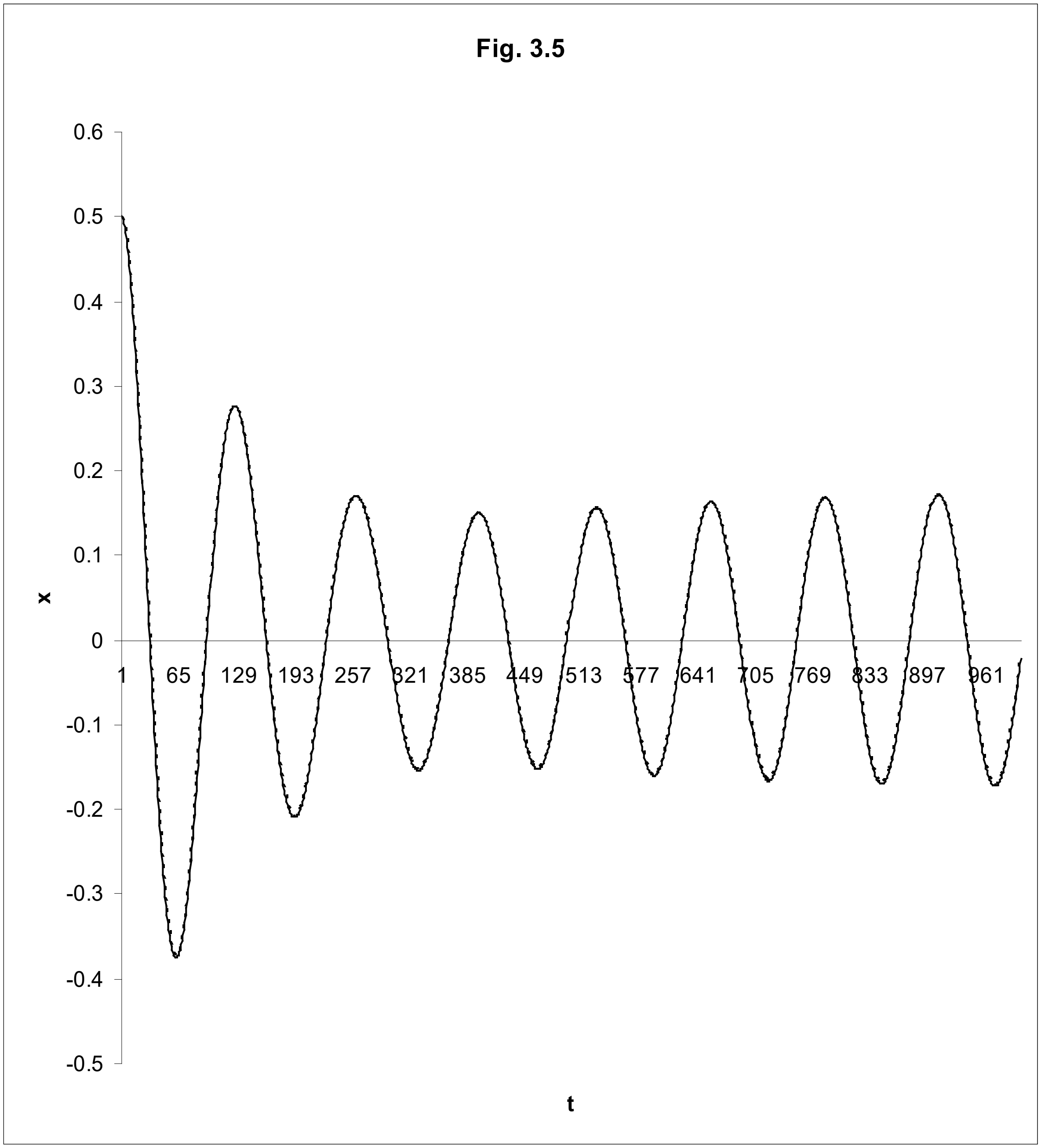

Fig 3.5: Present approximate solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -.045006$ for $\varepsilon = 0.7,\ \nu = 1.0, k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

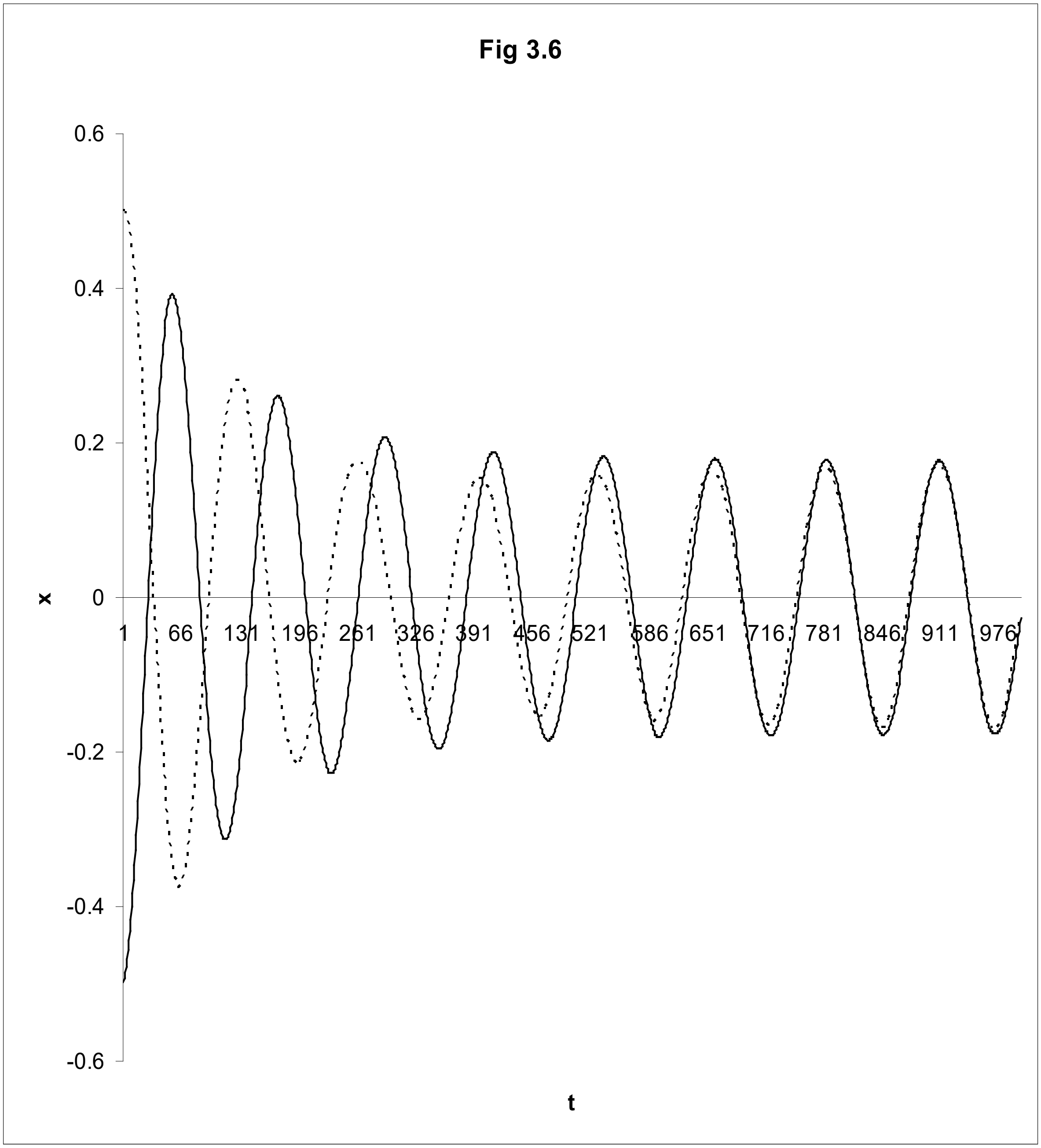

Fig 3.6: Unified KBM perturbation solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -3.0522$ for $\varepsilon = 0.7,\ \nu = 1.0,\ k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

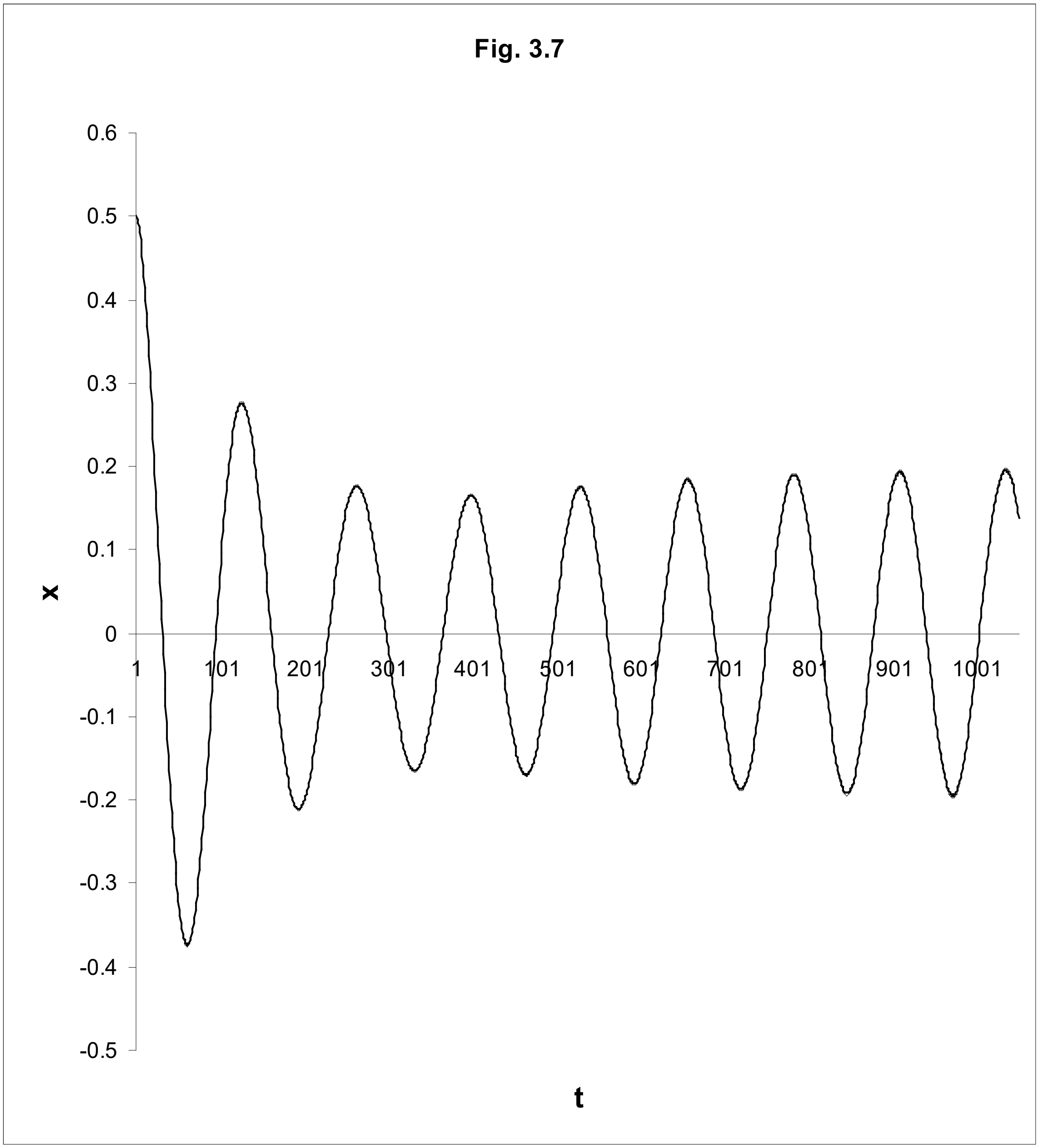

Fig 3.7: Present approximate solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -.044292$ for $\varepsilon = 0.8$, $\nu = 1.0$, $k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

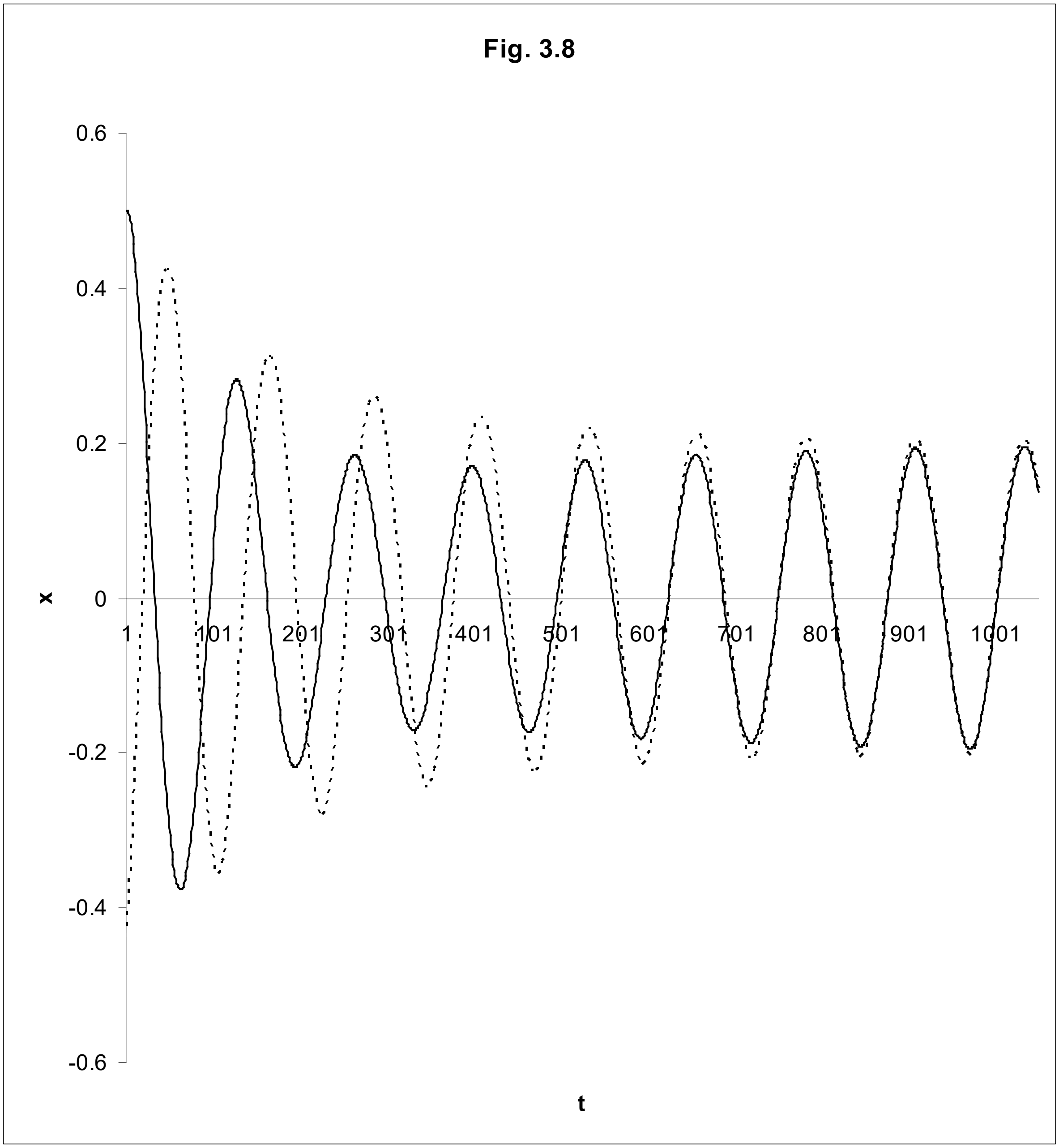

Fig 3.8: Unified KBM perturbation solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -2.6364$ for $\varepsilon = 0.8$, $\nu = 1.0, k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

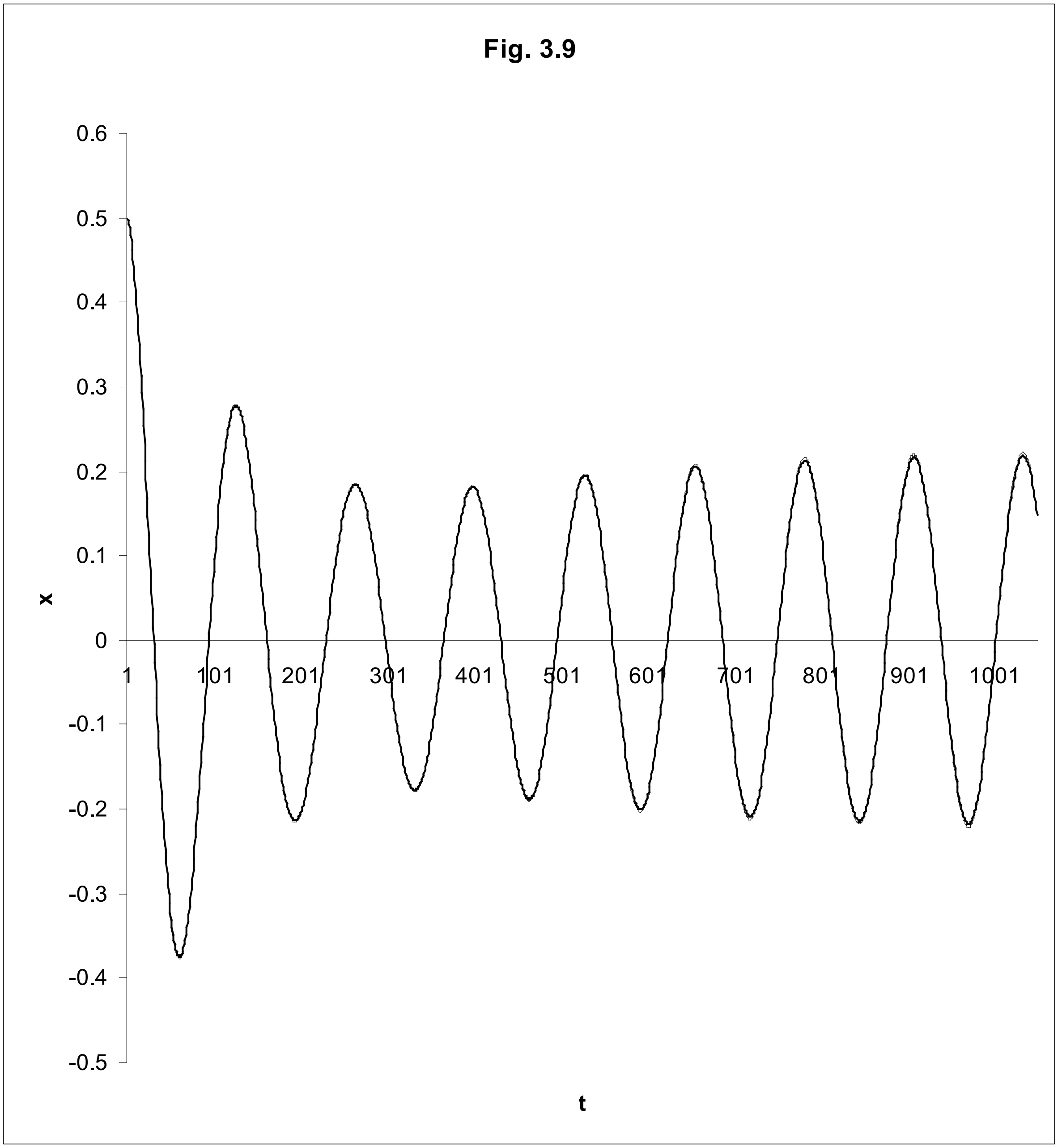

Fig 3.9: Present approximate solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -.043579$ for $\varepsilon = 0.9,\ \nu = 1.0,\ k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

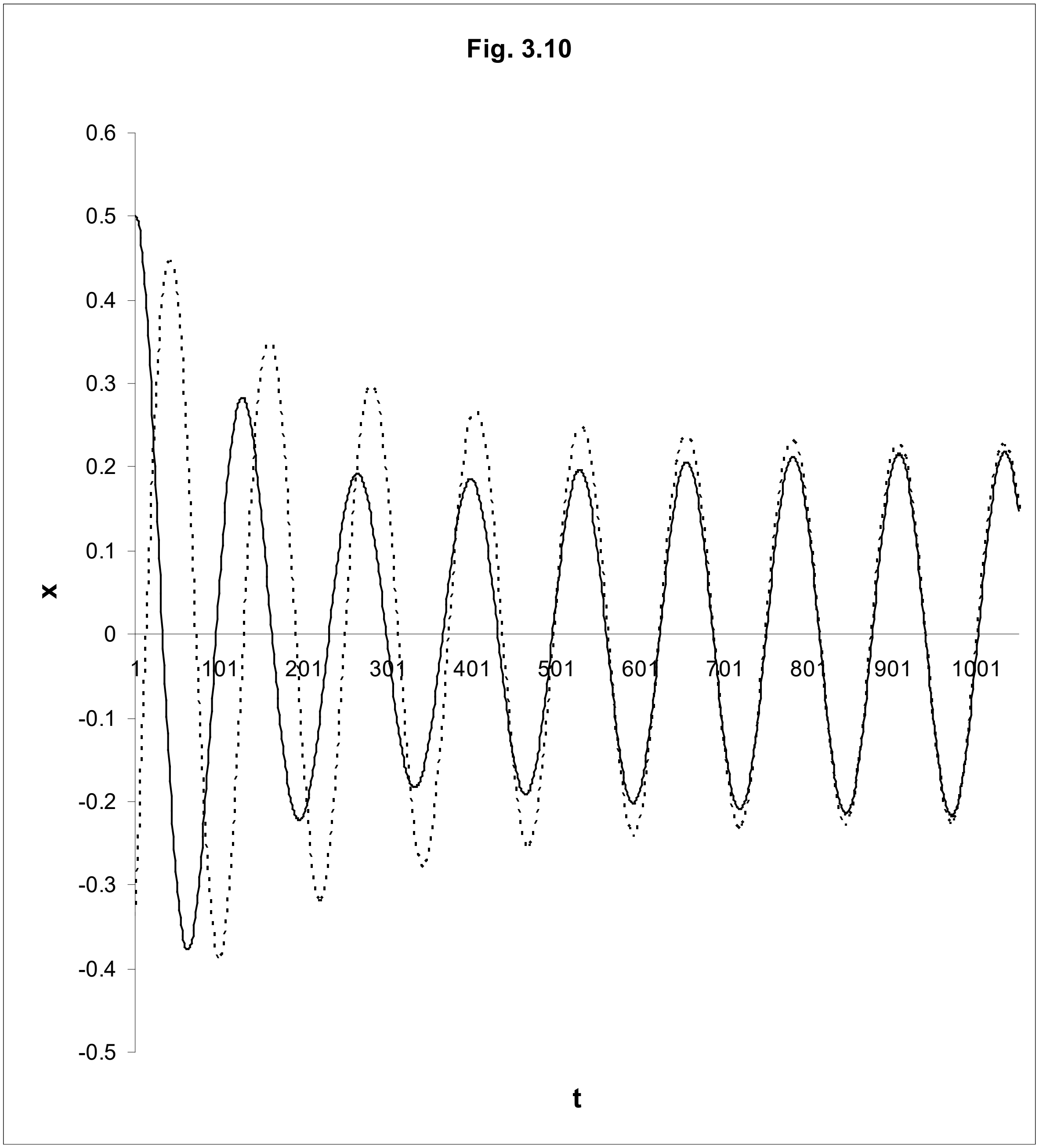

Fig 3.10: Unified KBM perturbation solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -2.313$ for $\varepsilon = 0.9,\ \nu = 1.0,\ k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

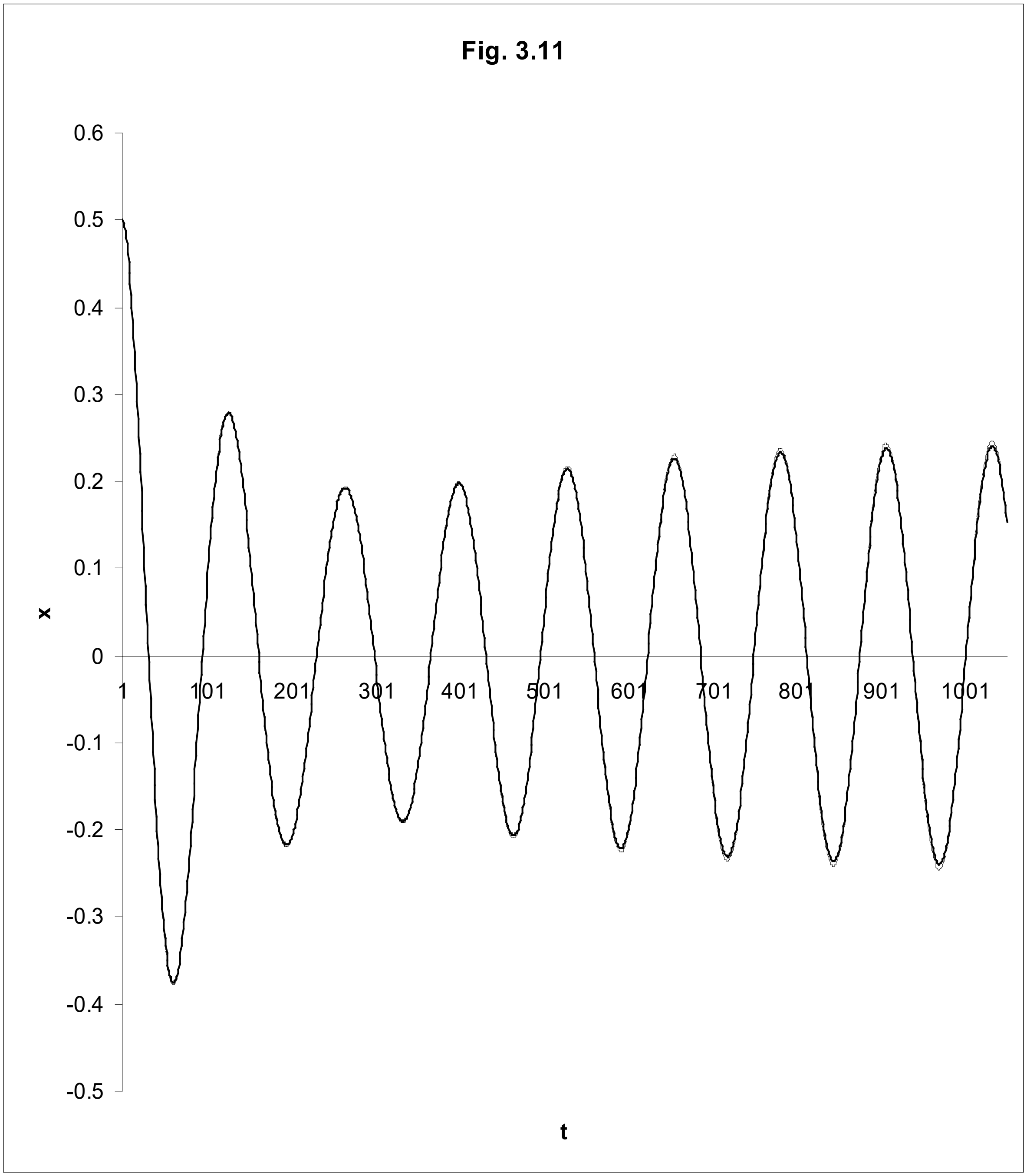

Fig 3.11: Present approximate solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -.042865$ for $\varepsilon = 1.0,\ \nu = 1.0,\ k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

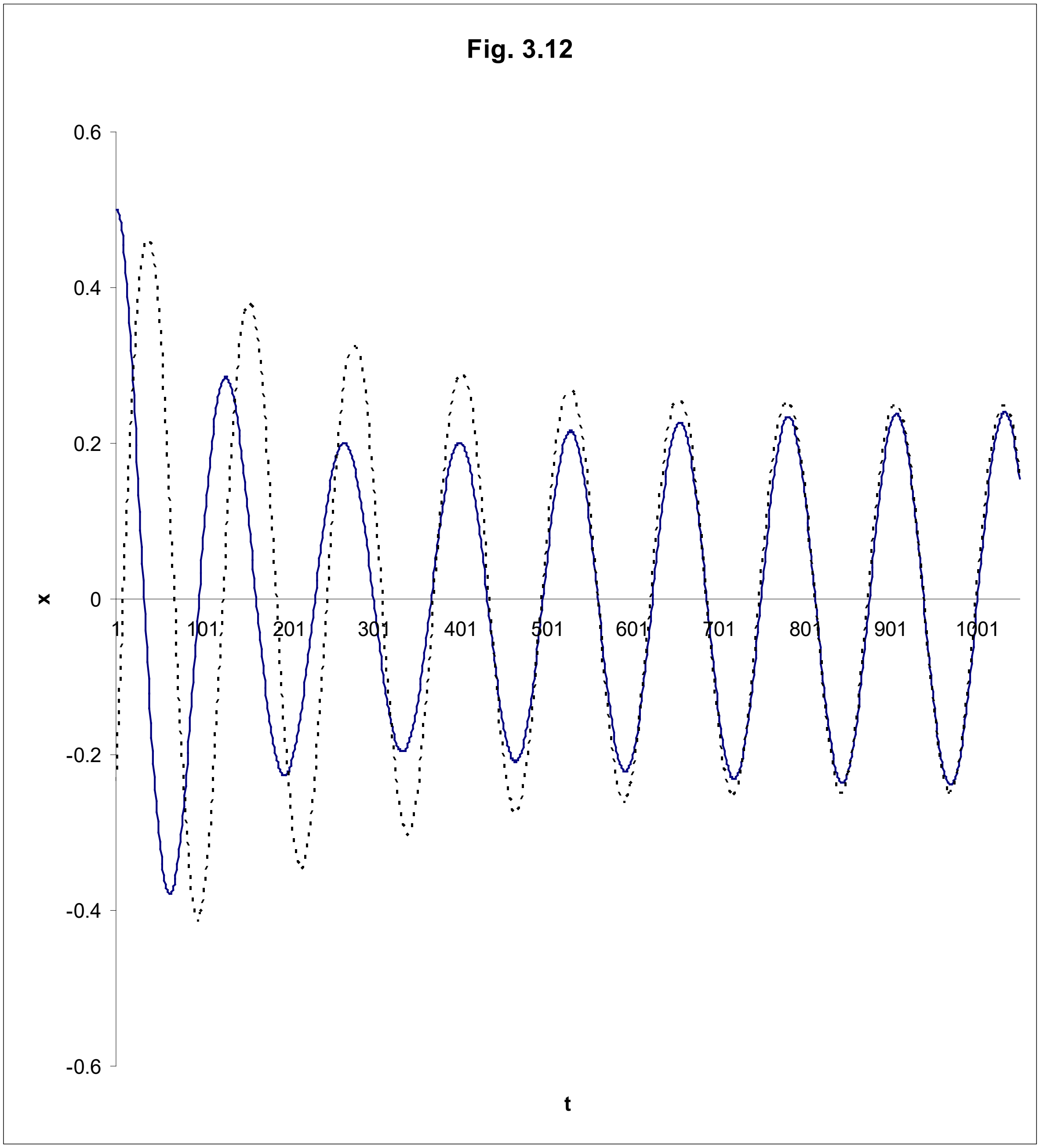

Fig 3.12: Unified KBM perturbation solution (dotted line) with corresponding numerical solution (solid line) is plotted with initial conditions $[x(0) = 0.50000,\ \dot{x}(0) = 0.00000]$ or $a = 0.50000$, $\varphi = -2.05428$ for $\varepsilon = 1.0,\ \nu = 1.0, k = .1\sqrt{\cos\tau}$, $\omega = \omega_0\sqrt{(c_1 + c_2\cos\tau + c_3\sin\tau)}$.

3.5 Conclusion

In this article we have extended the KBM method to find the approximate solution of damped forced nonlinear vibrating systems with slowly varying coefficients under the action of external force. The solutions agree with numerical results nicely even if $\varepsilon \geq 1.0$ but unified KBM solutions fail to give desire results.

References

1. Arya, J.C. and G.N. Bojadziev, Damped oscillating systems modeled by hyperbolic differential equations with slowly varying coefficients, Acta Mechanica, Vol.35, pp.215-221,1980.
2. Arya, J.C. and G.N. Bojadziev, Time dependent oscillating systems with damping, slowly varying parameters and delay, Acta Mechanica, Vol. 41, pp. 109-119, 1981.
3. Bogoliubov, N.N. and Yu. Mitropolskii, Asymptotic methods in the theory of nonlinear oscillations, Gordan and Breach, New York, 1961.
4. Bogoliubov, N.N. and Yu. Mitropolskii, Asymptotic methods in the theory of nonlinear oscillations (in Russian) State Press for Physics and Mathematical Literature, Moscow, 1963.
5. Bojadziev, G.N., On Asymptotic solutions of nonlinear differential equations with time lag, delay and functional differential equations and their applications(edited by K. Schmit), pp. 299-307, New York and London: Academic Press, 1972.
6. Bojadziev, G.N., R.W. Lardner, and J.C. Arya, On the periodic solutions of differential equations obtained by the method of Poincare and Krylov-Bogoliubov, J. Utilitas Mathematica, Vol.3,pp.49-64,1973.
7. Bojadziev, G.N. and R.W. Lardner, Mono-frequent oscillations in mechanical systems governed by hyperbolic differential equation with small nonlinearities, Int. J. Nonlinear Mech. Vol.8,pp.289-302,1973.
8. Bojadziev, G.N. and R.W. Lardner, Second order hyperbolic equations with small nonlinearities in the case of internal resonance, Int. J. Nonlinear Mech. Vol.9, pp.-397-407,1974.
9. Bojadziev, G.N. and R.W. Lardner, Asymptotic solution of a nonlinear second order hyperbolic differential equation with large time delay, J.Inst. Math. Applics Vol.14, pp.203-210, 1974.
10. Bojadziev, G.N., Damped forced nonlinear vibrations of systems with delay, J. Sound and Vibration, Vol.46, pp.113-120, 1976.
11. Bojadziev, G.N., The Krylov-Bogoliubov-Mitropolskii method applied to models of population dynamics, Bull. Math. Biol. Vol.40, pp.335-346, 1977.

12. Bojadziev, G.N.,The Krylov-Bogoliubov-Mitropolskii mehod applied to models of population dynamics, Bulletin of Mathematical Biology, Vol.40,pp.335-345,1978.
13. Bojadziev, G.N. and S.Chan, Asymptotic solutions of differential equations with time delay in population dynamics, Bull. Math. Biol. Vol.41.pp. 325-342, 1979.
14. Bojadziev, G.N.,Damped oscillating processes in biological and biochemical systems, Bull. Math.Biol. Vol.42, pp.701-717, 1980.
15. Bojadziev, G.N. and J. Edwards, on some method for non-oscillatory and oscillatory processes, J.Nonlinear Vibration Probs. Vol.20, pp.69-79, 1981.
16. Bojadziev, G.N.,Damped nonlinear oscillations modeled by a 3-dimensional differential system, Acta Mechanica, Vol.48,pp.193-201,1983.
17. Bojadziev, G.N. and C.K. Hung, Damped oscillations modeled by a 3-dimensional time dependent differential systems, Acta Mechanica, Vol.53, pp.101-114, 1984.
18. Cap, F.F., Averaging method for the solution of nonlinear differential equations with periodic non-harmonic solutions, Int. J. Nonlinear Mech. Vol.9, pp.441-450, 1974.
19. Freedman, H.I. and S. Ruan, Hopf bifurcation in three-species chain models with group defense, Math.Biosci. Vol.111, pp.73-87, 1992.
20. Feshenko S.F., Shkil N.I. and Nikolenko L.D., Asymptotic method in the theory of linear differential equation,(in Russian), Naukova Dumka, Kiev 1966 (English translation, Amer Elsevier Publishing Co., INC. New York 1967).
21. FUJIWARA Hiroshi, 2009.Multiple-Precision Arithmetic Library exflib (Fortran90/95).
22. Hung Cheng and Tai Tsun Wu., An aging sping, Studies in Applied Mathematics Vol.49, pp.183-185, 1970.
23. Kramers, H.A., Physik 39,828, 1926.
24. Kruskal, M., Asymptotic theory of Hamiltonian and other systems with all situations nearly periodic, J. Math. Phys., Vol.3, pp.806-828, 1962.
25. Krylov, N.N. and N.N. Bogoliubov, Introduction to nonlinear mechanics, Princeton University Press, New Jersey, 1947.
26. Lardner, R.W. and G.N. Bojadziev, Asymptotic solutions for third order partial differential equations with small nonlinearities, Meccanica, pp.249-256.1979.
27. Lin, J. and P. B. Khan, Averaging methods in Prey-Predator Systems and related biological models, J. Theor. Biol., Vol.57, pp.73-102, 1974.

28. Lindstedt, A., Memories del, Ac. Imper, des Science de st. Petersburg 31, 1883.
29. Mendelson, K.S., Perturbation theory for damped nonlinear oscillations, J.Math. Physics, Vol.2, pp.3413-3415, 1970.
30. Mickens, R.E., An introduction to nonlinear oscillations, Cambridge University Press, London,1980.
31. Minorsky, N., Nonlinear oscillations, Van Nostrand Co, Princeton, N.J., pp.375, 1962.
32. Mitropolskii, Yu., Problems on asymptotic methods of non-stationary oscillations (in Russian), Izdat, Nauka, Moscow,1964.
33. Mulholland, R.J., Nonlinear oscillations of third order differential equation, Int.J. Nonlinear Mechanics, Vol.6, pp.279-294, 1971.
34. Murty,I.S.N., and B.l. Deekshatulu, Method of variation of parameters for over-damped nonlinear systems, J. Control, Vol.9, No.3,pp.45-53,1971.
35. Murty,I.S.N., and B.l. Deekshatulu and G. Krishna, On asymptotic method of Krylov-Bogoliubov for over-damped nonlinear systems, J. Frank. Inst., Vol.288, pp.49-64. 1969.
36. Murty,I.S.N., A unified Krylov-Bogoliubov method for solving second order nonlinear systems, Int. J. Nonlinear Mech. Vol.6,pp.45-53,1971.
37. Museenkov, P., On the higher order effects in the methods of Krylov-Bogoliubov and Poincare, J.Astron. Sci., Vol.12, pp.129-134, 1965.
38. Nayfeh, A.H., Introduction to Perturbation Techniques, John Wiley and Sons, New York, 1981.
39. Nguyen Van Dinh, On a variant of the asymptotic procedure, Vietnum J. of Mechanics, VAST Vol.26, No.3, pp.139-147, 2004.
40. Osiniskii, Z., Longitudinal, Torsional and bending vibrations of a uniform bar with nonlinear internal friction and relaxation, Nonlinear Vibration Problems, Vol.4,pp.159-166,1962.
41. Osiniskii, Z., Vibration of a one degree freedom system with nonlinear internal friction and relaxation, Proceedings of International Symposium of Nonlinear Vibrations, Vol.111, pp.314-325, Kiev,Izadt, Akad, Nauk Ukr. SSR, 1963.

42. Pinakee Dey, M.Zulfikar Ali, M.Shamsul Alam, An Asymptotic Method for Time Dependent Non-linear Over-damped Systems, J.Bangladesh Academy of sciences., Vol.31,pp.103-108,2007.
43. Pinakee Dey, M.Zulfikar Ali, M.Shamsul Alam, An Asymptotic Method for Time Dependent Non-linear Systems with Varying Coefficient,J.Mech. Cont. & Math.Sci. Vol.3, No.2, Dec 2008.
44. Pinakee Dey, M.Zulfikar Ali, Ali Akbar and M.Shamsul Alam, Second Approximate Solutions of Second Order Damped Forced Nonlinear Systems. J.Appl.Sci.Res. Vol.4, No.6, pp.731-741, 2008.
45. Pinakee Dey, Razia Pervin, Shewli Shamim Shanta, Hitoshi Imai, Krishan Chandra Datta., High Precision Numerical Solution and Approximate Solution of Over-Damped Nonlinear NonAutonomous Differential System with Varying Coefficients. Aust.J. Basic & Appl. Sci.,Vol.8,No.1 pp.567-571, 2014.
46. Pinakee Dey, Razia Pervin, Shewli Shamim Shanta, Approximate solution of time dependent damped nonlinear vibrating systems with slowly varying coefficients. (Submitted).
47. Pinakee Dey., Harun or Rashid, M. Abul Kalam Azad and M S Uddin, Approximate Solution of Second Order Time Dependent Nonlinear Vibrating Systems with Slowly Varying Coefficients, Bull. Cal. Math. Soc, Vol.103, No.5, pp. 371-38, 2011.
48. Pinakee Dey, M. A. Sattar and M. Zulfikar Ali, Perturbation Theory for Damped Forced Vibrations with Slowly Varying Coefficients , J. Advances in Vibration Engineering, Vol. 9, No. 4,pp.375-382, 2010.
49. Poincare,H.,Les methods nouvelles de la mecanique celeste, Paris,1982.
50. Popov, I.P., A generalization of the Bogoliubov asymptotic method in the theory of nonlinear oscillations (in Russian), Dokl. Akad, Nauk. USSR, Vol.111,pp.308-310,1956.
51. Proskurjakov A.P., Comparison of the periodic solutions of quasi-linear systems constructed by the method of Poincare and Krylov-Bogoliubov(in Russian), Applied Math. And Mech., 28, 1964.

52. Rauch L.L., Oscillations of a third order nonlinear autonomous system, in Contribution to the Theory of Nonlinear Oscillations, pp.39-88, New Jersey, 1950.
53. Roy.K.C. and Shamsul Alam, M, Effect of higher approximation of Krylov-Bogliubov-Mitropolskii solution and matched asymptotic solution of a differential system with slowly varying coefficients and damping near to a turning point, Vietnum Journal of Mechanics, VAST, Vol.26,pp.182-192,2004.
54. Sattar,M.A., An asymptotic method for second order critically damped nonlinear equations, J.Frank.Inst., Vol.321,pp.109-113,1986.
55. Sattar,M.A., An asymptotic method for three-dimensional over-damped nonlinear systems, GANIT, J. Bangladesh Math. Soc. Vol.13, pp.1-8, 1993.
56. Shamsul Alam, M. and M.A. Sattar, An asymptotic method for third order Critically damped nonlinear equations, J. Mathematical and Physical Sciences, Vol. 30, pp.291-298,1996.
57. Shamsul Alam, M. and M.A. Sattar,A unified Krylov-Bogoliubov-Mitropolskii method for solving third order nonlinear systems, Indian J. pure & appl. Math. Vol.28, pp.151-167,1997.
58. Shamsul Alam, M., Asymptotic methods for second order over-damped and critically damped nonlinear systems, Soochow Journal of Math., Vol.27, pp.187-200, 2001.
59. Shamsul Alam, M. and M.A. Sattar, Time dependent third –order oscillating systems with damping, J.Acta Ciencia Indica, Vol.27,pp.463-466,2001.
60. Shamsul Alam, M., Perturbation theory for nonlinear systems with large damping, Indian J. pure & appl. Math., Vol.32, pp.1453-1461, 2001.
61. Shamsul Alam, M., M.F.Alam and S.S. Shanta, Approximate solution of non-oscillatory systems with slowly varying coefficients, GANIT, J. Bangladesh Math. Soc. Vol.21, pp.55-59, 2001.
62. Shamsul Alam, M.,B.Hossain and S.S.Shanta, Krylov-Bogoliubov-Mitropolskii method for time dependent nonlinear systems with damping, Mathematical Forum, Vol.14,pp.53-59,2001.
63. Shamsul Alam, M., A unified Krylov-Bogoliubov-Mitropolskii method for solving n-th order nonlinear systems, J.Frank.Inst. Vol.339, pp.239-248, 2002.

64. Shamsul Alam,M.,Perturbation theory for the n-th order nonlinear systems with large damping, Indian J.pure & appl. Math. Vol.33, pp.1677-1684, 2002.
65. Shamsul Alam,M., Bogoliubov's method for third order critically damped nonlinear systems, Soochow J.Math., Vol.28,pp.65-80,2002.
66. Shamsul Alam, M., Method of solution to the n-th order over-damped nonlinear systems under some special conditions, Bull.Cal.Math, Soc., Vol.96, pp.437-440.2002.
67. Shamsul Alam,M., Approximate solutions of non-oscillatory systems, Mathematical Forum, Vol.14,pp.7-16,2001-2002.
68. Shamsul Alam,M.,Asymptotic method for non-oscillatory nonlinear systems, Far East J. Appl. Math., Vol.7,pp.119-128,2002.
69. Shamsul Alam,M., On some special conditions of over-damped nonlinear systems, Soochow J.Math.,Vol.29,pp.181-190,2203.
70. Shamsul Alam,M., A unified KBM method for solving n-th order nonlinear differential equation with varying coefficients, J.Sound and Vibration, Vol.265,pp.987-1002,2003.
71. Shamsul Alam,M.,Asymptotic method for certain third–order non-oscillatory nonlinear systems, J.Bangladesh Academy of Sciences, Vol.27, pp.141-148.2003.
72. Shamsul Alam,M.,M. Bellal Hossain and S.S. Shanta, Perturbation theory for damped nonlinear systems with varying coefficients, Indian J.pure & appl. Math., Vol.34,pp.1359-1368,2003.
73. Shamsul Alam, M., On some special conditions of n-th order over-damped non-linear systems, Communication of Korean Math Soc., Vol.18, pp.755-765, 2003.
74. Shamsul Alam,M., A modified and compact form of Krylov-Bogoliubov-Mitropolskii unified method for an n-th order nonlinear differential equations, Int. J. Nonlinear Mechanics, Vol.39, pp.1343-1357,2004.
75. Shamsul Alam,M.,Damped oscillations modeled by an n-th order time dependent quasi-linear differential system, Acta Mechanica, Vol. 169, pp.112-122,2004.

76. Shamsul Alam,M., Unified KBM method under a critical condition,J.Franklin Inst., Vol.341,pp.533-542,2004.
77. Shamsul Alam,M. and M.A.Sattar, Asymptotic method of third-order Non-linear System with Varying Coefficients., Southeast Asian Bull. of Mathematics, Vol.341,pp.533-542,2004.
78. Struble, R.A., The geometry of the orbits of artificial satellites, Arch. Rational Mech. Anal., Vol.7, pp.87-104, 1961.
79. Van der pol, B., On Relaxation Oscillations, Philosophical Magazine, 7-th series, Vol.2, 1926.
80. Volosov,V.M., Higher Approximation in Averaging, Sovier Math. Dokl, Vol.2, pp.221-224, 1961.
81. Wetzel, G.Z, Physik, Vol.38, pp.518, 1926.
82. Zabrieko,P.P, Higher Approximate of the Krylov-Bogoliubov Averaging Method, Dokl, Akad,Nauk, USSR,176,pp.1453-1456,1966.

Thank You.