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# Separation Axioms on L-Topological Spaces

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# SEPARATION AXIOMS ON L-TOPOLOGICAL SPACES

**A THESIS**

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**IN MATHEMATICS**

**BY**

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Dedicated

To My Parents

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No ................................ Dated ..........................

# Declaration

*Certified that the thesis entitled "SEPARATION AXIOMS ON L-TOPOLOGICAL SPACES" submitted by Rafiqul Islam in fulfillment of the requirements for the degree of Master of Philosophy in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi, Bangladesh has been completed under my supervision. I believe that this research work is an original one and that it has not been submitted elsewhere for any degree.*

*I wish him a bright future and every success in life.*

Supervisor

( *Dr. Md. Sahadat Hossain*)

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## *STATEMENT OF ORIGINALITY*

*I declare that the content in my M.Phil. thesis entitled "SEPARATION AXIOMS ON L-TOPOLOGICAL SPACES" is original and accurate to the best of my knowledge. I also declare that the materials contained in my research work have not been previously published or written by any person for any degree or diploma.*

*(Rafiqul Islam)*

*Author*

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# Contents

# Introduction

The notion of a fuzzy set, as proposed by L.A. Zadeh[90] in 1965 to provide a foundation for the evolution of many areas of knowledge. After then in quick succession, L-fuzzy sets were introduced by Goguen[24] in 1967. As a result, this provides a natural frame work for generalizing many algebraic and topological concepts in various directions such as L-fuzzy sets, fuzzy logics, fuzzy control, fuzzy groups, fuzzy rings, fuzzy vector spaces, fuzzy topology, fuzzy bitopology, L-topology etc. Many other branches of mathematics have been developed all over the world during the last five decades. Chang [12] first introduced and studied the concept of a fuzzy topological space by using the fuzzy set in 1968. Hutton[31-34] ,Reilly[33-34],Wong [83-84], Lowen[47-48], Srivastava[73-79], Dude[16-17], Cutler[14], Ying[88], Ali[2-9], Hossain[26-30], Pu and Liu[53-54], etc., discussed and developed various aspects of fuzzy topological spaces. Ying [88] introduced fuzzifying topology and developed this in a new direction with the semantic methods of continuous valued logic. With the help of fuzzifying topology, Sinha[69-70] introduced and studied $T_0$, $T_1$, $T_{2}$-(Hausdorff), $T_{3}$-(regular), $T_{4}$-(normal), separation axioms. Mashhour et al. [51-52] introduced and studied the concepts of the family of fuzzifying semi-open sets, fuzzifying neighbourhood structure of a point and fuzzifying semi-closure of a fuzzy set. Also in fuzzifying topology they introduced and studied semi-$T_0$-, semi-$R_0$-, semi-$T_1$-, semi-$R_1$-, semi-$T_2$(semi Hausdorff)-, semi-$T_3$(semi regular)-, semi-$T_4$(semi normal)-, separation axioms. Ali and Hossain [28] developed the $R_0$ and $R_1$ separation axioms and studied their relations with the $T_1$ and $T_2$-separation axioms respectively. In 1993, Warner and Mclean [80] introduced on compact Hausdorff L-fuzzy spaces. Later Jin-xuan, Ren Bai-lin[35] introduced and studies a set of new separation axioms in L-fuzzy topological spaces. After wards, Kudri[41-42], Li[43-44], Song[71], Xu[86], ZHAO Bin[93], introduced the L-fuzzy topological spaces and studied the strong Hausdorff Separation property in L-fuzzy topological spaces.

In this present thesis, we are going to introduce some new definitions of separation axioms in L-topological spaces using the ideas of Jin-xuan and Ren Bai-lin[35]. Some of their equivalent formulations along with various new characterizations and results concerning the existing ones are presented here. Our criterion for definitions has been preserving as much as possible the relation between the corresponding separation properties for L-topological spaces. Moreover, it will be seen that the definitions of these axioms are 'good extensions' in the sense of Lowen [47-48].

We aim to develop theories of L-$T_0$, L-$T_1$, L-$T_2$ (Hausdorff), L-$R_0$ and L-$R_1$-separation axioms analogous to its counterpart in ordinary topology. The materials of this thesis have been divided into six chapters, a brief scenario of which we present as follows.

The first one is to incorporate some of the basic definitions and results of fuzzy set, fuzzy topology, fuzzy mapping, L-topology and its mapping. These results are ready references for the work in the subsequent chapter. Results are stated without proof and can be seen in the papers referred to.

Our work starts from chapter two. In this chapter, we have introduced and studied $T_0$ properties in L-topological spaces. Here we add eight more definitions to this list and we have established relationship among them. All these eight definitions are 'good extensions' of the corresponding concept $T_0$ in a topological space. We prove that all these definitions satisfy property of hereditary, productive and projective. Also we have studied some other properties of these concepts.

In chapter three, we have developed and studied $T_1$ properties in L-topological spaces. Here, we include eight more definitions to this chapter and we have established relations among them. All these eight notions are 'good extensions' of the corresponding concept of $T_1$ in a topological space. We prove that all these notions are hereditary, productive and projective. We have discussed some other properties of these concepts.

We have introduced and studied $T_2$(Hausdorff) properties in L-topological spaces, in chapter four. We have developed here seven more definitions and we established relations among them. All these properties are 'good extensions' of the corresponding concept $T_2$(Hausdorff) in a topological space. We have observed that all the definitions satisfy property of hereditary, productive and projective.

Chapter five is based on the $R_0$ properties in L-topological spaces. Here we have obtained seven more definitions. We see that all these properties are 'good extensions' of the corresponding notions in topological spaces. We have discussed that all the properties are hereditary, productive and projective. We have also studied several other properties of these concepts.

The $R_1$ properties in L-topological spaces are to be studied in chapter six. We have given here seven more definitions and we have established relations among them. All these are 'good extensions' of their corresponding concept in a topological space. Some other pleasant properties of these concepts have been studied here.

# CHAPTER-1

## Preliminaries

### 1.1 Introduction:

We have discussed the fuzzy sets, fuzzy topological spaces and L-topological spaces in this chapter. We have incorporated some of the basic definitions and results of the fuzzy sets, Grade of membership, L-fuzzy sets, complement of L-fuzzy sets, some laws of L-fuzzy sets, different mapping on L-fuzzy sets, Fuzzy topological spaces, L- topological spaces, L-$T_0$, L-$T_1$, L-$T_2$-spaces, fuzzy product topological spaces and L-product topological spaces which are to be used as ready references for understanding the subsequent chapters. Most of the results are quoted from various research papers. We make use the following general notations in this thesis paper.

$\Lambda$ : Index set.

L : Complete distributive lattice with 0 and 1.

$I = [0, 1]$ : Closed unit interval.

$I_1 = [0, 1)$ : Right open unit interval.

$I_0 = (0, 1]$ : Left open unit interval.

$\mu, \alpha, \beta, \gamma,..$ : L-Fuzzy sets.

(X, t) : Fuzzy topological space.

$(X, \tau)$ : L-topological space.

(X, T) : General topological space.

$\Pi_{i \in \Lambda} X_i$ : Usual product of $X_i$ .

$(X, \tau_1 \times \tau_2)$ : Product of L-topologies $\tau_1$ and $\tau_2$ on the set X.

$I_\alpha(\tau) = \{ u^{-1}(\alpha , 1 ] : u \in \tau^* \}, \alpha \in I_1$ : General topology on X.

**1.2 Fuzzy Set**

**1.2.1 Definition [90]:** Let X be a non-empty set and I= [0, 1]. A fuzzy set in X is a function $u: X \to I$ which assign to each element $x \in X$, a degree of membership, $u(x) \in I$.Thus a usual subset of X, is a special type of a fuzzy set in which the range of the function is [0,1].

**1.2.2 Definition:** Let X be a nonempty set and A be a subset of X. The function $1_A : X \to \{0, 1\}$ defined by

$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$ is called the characteristic function of A. We also write $1_x$ for the characteristic function of X.

**1.2.3 Definition:** The characteristic functions of subsets of a set X are referred to as the crisp sets in X.

**1.2.4 Example**: Suppose X is real number R and the fuzzy set of real numbers much greater than 5 in X that could be defined by the continuous function

$U : X \to [0,1]$ such that $u(x) = \begin{cases} 0 \ \ if \ \ x \le 5 \\ \dfrac{x-5}{50} \ \ if \ \ 5 < x < 55 \\ 1 \ \ if \ \ x \ge 55 \end{cases}$.

**1.2.5 Definition [47]:** A fuzzy subset is empty if and only if grade of membership is identically zero in X. It is denoted by 0.

**1.2.6 Definition [47]:** A fuzzy subset is whole if and only if its grade of membership is identically one in X. It is denoted by 1.

**1.2.7 Definition [82]:** Let u and v be two fuzzy subsets of a set X. Then u is said to be subset of v, i.e., $u \subseteq v$ if and only if $u(x) \le v(x)$ for every $x \in X$.

**1.2.8 Definition [82]:** Let u and v be two fuzzy subsets of a set X. Then u is said to be equal to v, i.e., u = v if and only if u(x) = v(x) for every $x \in X$.

**1.2.9 Definition [82]:** Let u and v be two fuzzy subsets of a set X. Then u is said to be the complement of v, i.e., $u = v^c$ if and only if $u(x) = 1 - v(x)$, for every $x \in X$. Obviously $(v^c)^c = v$.

**1.2.10 Definition [11]:** Let u and v be two fuzzy subsets of a set X. Then the union w of u and v, i.e., $w = u \cup v$ if and only if $w(x) = (u \cup v)(x) =$

max{ u(x), v(x) }, for every $x \in X$. The union w is a fuzzy subset of X.

In general, if $\Lambda$ be an index set and A = $\{u_i : i \in \Lambda\}$ be a family of fuzzy sets of X then the union $\cup u_i$ is defined by

$(\cup u_i)(x) = \sup \{u_i(x) : i \in \Lambda\}, x \in X$.

**1.2.11 Definition [11]:** Let u and v be two fuzzy subsets of a set X. Then the intersection m of u and v, i.e., $m = u \cap v$ is a fuzzy subset of X if and only if $m(x) = (u \cap v)(x) = \min \{u(x), v(x)\}, \forall x \in X$, and $(\cap u_{i)})(x) = \inf \{u_i(x) : i \in \Lambda\}, x \in X$, where $\{u_i, i \in \Lambda\}$.

**1.2.12 Definition:** Let u and v be two fuzzy subsets of a set X. Then the difference of u and v is defined by $u - v = u \cap v^c$.

**1.2.13 Definition:** If $\alpha \in I$ and $u \in I^X$ define by $u(x) = \alpha$, for all $x \in X$, we refer to u as a constant fuzzy set and denote it by $\alpha$ itself. In particular, we have the constant fuzzy sets 0 and 1.

**1.2.14 Example:** Let X= {x, y, z} and $u, v \in I^X$ are defined by u(x) =.6, u(y) =.7, u (z) =.5 and v(x) =.7, v(y) =.5, v (z) =.4. Then $(u \cup v)(x)$ = max {u(x), v(x)} =.7, $(u \cup v)(y)$ =max {u(y), v(y)} =.7, $(u \cup v)(y)$ = max {u(y), v(y)} =.5, $(u \cap v)(x)$ =min {u(x), v(x)} =.6, $(u \cap v)(y)$ = min {u(y), v(y)} =.5, $(u \cap v)(z)$ =min {u(z), v(z)} =.4., $u^c(x)$= 1-u(x)=.4, $u^c(y)$= 1-u(y)=.3, $u^c(z)$= 1-u(z)=.5.

**1.2.15 Laws of the algebra of fuzzy sets:**

As in ordinary set theory, idempotent laws, associative law, commutative law, distributive laws, identity law, demorgan's laws hold in the case of fuzzy sets also. But the complement laws are not necessarily true. For example, if

X= {a, b, c} and u is a fuzzy subset of X where is defined by

u ={ (a , .2) , ( b , .7 ) , ( c , 1 ) },

then $u^c$ ={ ( a , .8 ) , ( b , .3 ) , ( c , 0 ) }

so $u \cup u^c$ = { ( a , .8 ) , ( b , .7) , ( c , 1 ) }≠ 1,

$u \cap u^c$ = { ( a , .2) , ( b , .3 ), ( c , 0 ) } ≠ 0 .

Also in ordinary set theory $U \cap V = \phi$ if and only if $U \subset V^c$. But in fuzzy subsets reverse is not necessary true. For example if

v = { ( a , .6 ) , ( b , .2 ) , ( c , 0 ) } then $u \subset v^c$ ,

$u \cap v$ = { ( a , .2 ) , ( b , .2 ) , ( c , 0 ) } ≠ 0.

**1.3 Fuzzy Topology**

**1.3.1 Definition [12]:** Let $I = [0,1]$, $X$ be a non-empty set and $I^X$ be the collection of all mappings from $X$ into $I$ , i. e. the class of all fuzzy sets in $X$. A fuzzy topology on $X$ is defined as a family $t$ of members of $I^X$, satisfying the following conditions:

(i) $1, 0 \in t$

(ii) if $u_i \in t$ for each $i \in \Lambda$ then $\bigcup_{i \in \Lambda} u_i \in t$

(iii) if $u_1, u_2$ then $u_1 \cap u_2 \in t$ .

The pair $(X, t)$ is called a fuzzy topological space (fts, in short) and the members of $t$ are called $t$-open (or simply open) fuzzy sets. A fuzzy set $v$ is called a $t$-closed (or simply closed) fuzzy set if $1 - v \in t$.

**1.3.2 Example :** Let X = { a , b , c , d } , t = { 0 , 1 , u , v},

where 1 = { (a , 1 ) , ( b , 1 ) , ( c , 1 ) , ( d , 1 ) }

0 = { ( a , 0 ) , ( b , 0 ) , ( c , 0 ) , ( d , 0 ) }

u = { ( a , .2 ) , ( b , .5 ) , ( c , .7 ) , ( d , .9 ) }

v = { ( a , .3 ) , ( b , .5 ) , ( c , .8 ) , ( d , .95 ) }

Then (X, t) is a fuzzy topological space.

**1.3.3 Definition:** A fuzzy topological space (X, t) is said to be fuzzy regular if and only if for each $x \in X$ and closed fuzzy set u with u(x) =0, there exists open fuzzy sets $v, w \in t$ such that v(x) =1, $u \subseteq w$ and $v \subseteq 1 - w$.

**1.3.4 Definition:** A fuzzy topological space (X, t) is said to be fuzzy normal if and only if for each close fuzzy set m and open fuzzy set u with $m \subseteq u$, there exists a fuzzy set v such that $m \subseteq v^{\circ} \subseteq \bar{v} \subseteq u$ .

## 1.4 L-Fuzzy Set

**1.4.1 Definition [24]:** Let X be a non-empty set and L be a complete distributive lattice with 0 and 1. An L-fuzzy set in X is a function $\alpha: X \to L$ which assign to each element $x \in X$, a degree of membership, $\alpha(x) \in L$.

**1.4.2 Definition [24]:** Let $\alpha$ be an L-fuzzy set in $X$ .Then $1 - \alpha = \alpha'$ is called the complement of $\alpha$ in $X$.

**1.4.3 Definition [24]:** An L-fuzzy subset is empty if and only if grade of membership is identically zero in X. It is denoted by $0^*$.

**1.4.4 Definition [24]:** An L-fuzzy subset is whole if and only if its grade of membership is identically one in X. It is denoted by $1^*$.

**1.4.5 Definition [24]:** Let $\alpha$ and $\beta$ be two L-fuzzy subsets of a set X. Then $\alpha$ is said to be subset of $\beta$, i.e., $\alpha \subseteq \beta$ if and only if $\alpha(x) \leq \beta(x)$ for every x ∈ X.

**1.4.6 Definition [24]:** Let $\alpha$ and $\beta$ be two L-fuzzy subsets of a set X. Then $\alpha$ is said to be equal to $\beta$, i.e., $\alpha = \beta$if and only if $\alpha(x) = \beta(x)$ for every x ∈X.

**1.4.7 Definition [24]:** Let $\alpha$ and $\beta$ be two L-fuzzy subsets of a set X. Then $\alpha$ is said to be the complement of $\beta$, i.e., $\alpha = \beta^c$ if and only if $\alpha(x) = 1 - \beta(x)$, for every x ∈X. Obviously $(\beta^c)^c = \beta$.

**1.4.8 Definition [24]:** Let $\alpha$ and $\beta$ be two L-fuzzy subsets of a set X. Then the union $\gamma$ of $\alpha$ and $\beta$, i.e., $\gamma = \alpha \cup \beta$ if and only if $\gamma(x) = (\alpha \cup \beta)(x) = max\{\alpha(x), \beta(x)\}$, for every x $\in$X. The union $\gamma$ is an L-fuzzy subset of X.

In general, if $\Lambda$ be an index set and $A = \{\alpha_i : i \in \Lambda\}$ be a family of L-fuzzy sets of X then the union $\cup \alpha_i$ is defined by $(\cup \alpha_i)(x) = sup\{\alpha_i(x): i \in \Lambda\}, x \in X$.

**1.4.9 Definition [24]:** Let $\alpha$ and $\beta$ be two L-fuzzy subsets of a set X. Then the intersection $m$ of $\alpha$ and $\beta$, i.e., $m = \alpha \cap \beta$ is an L-fuzzy subset of X if and only if $m(x) = (\alpha \cap \beta)(x) = min\{\alpha(x), \beta(x)\}, \forall\, x \in X$,and$(\cap \alpha_i)(x) = inf\{\alpha_i(x) : i \in \Lambda\}, x \in X$,where $\{\alpha_i, i \in \Lambda\}$.

**1.4.10 Definition [24]:** Let $\alpha$ and $\beta$ be two L-fuzzy subsets of a set X. Then the difference of $\alpha$ and $\beta$ is defined by $\alpha - \beta = \alpha \cap \beta^c$.

**1.4.11Example:** Let $X = \{x, y, z\}$, $L = \{0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1\}$ and $\alpha, \beta \in L^X$ are defined by $\alpha(x) = .6, \alpha(y) = .7, \alpha(z) = .5$ and $\beta(x) = .7, \beta(y) = .5, \beta(z) = .4$.Then $(\alpha \cup \beta)(x) = max\{\alpha(x), \beta(x)\} = .7$, $(\alpha \cup \beta)(y) = max\{\alpha(y), \beta(y)\} = .7$, $(\alpha \cup \beta)(z) = max\{\alpha(z), \beta(z)\} = .5$, $(\alpha \cap \beta)(x) = min\{\alpha(x), \beta(x)\} = .6$, $(\alpha \cap \beta)(y) = min\{\alpha(y), \beta(y)\} = .5$, $(\alpha \cap \beta)(z) = min\{\alpha(z), \beta(z)\} = .4$, $\alpha^c(x) = 1 - \alpha(x) = .4$, $\alpha^c(y) = 1 - \alpha(y) = .3$, $\alpha^c(z) = 1 - \alpha(z) = .5$.

**1.4.12Definition [45]:** Two L-fuzzy sets $\alpha$ and $\beta$ in $X$ are said to be intersected if and only if there exist a point $x \in X$ such that $(\alpha \cap \beta)(x) \neq 0$ . In this case we say that $\alpha$ and $\beta$ intersect at x.

**1.4.13 Definition [46]:** Let $X$ be a non empty set and $\mu$ be an L-fuzzy set in $X$. A α- cut of $\mu$ is defined by $\alpha_\mu = \{x: \mu(x) \geq \alpha, \forall \alpha \in L\}$.

**1.4.14 Definition [46]:** Let X be a non empty set and $\mu$ be an L-fuzzy set in X. A strong α- cut of $\mu$ is defined by ${\alpha_\mu}^+ = \{x: \mu(x) > \alpha, \forall \alpha \in L\}$. We see that α-cut and strong α-cut are crisp subsets of $X$. The 1-cut of $\mu$ is called the core of $\mu$.

**1.4.15 Definition [46]:** Let $X$ be a non empty set and $\mu$ be an L-fuzzy set in $X$. The support of $\mu$ in $X$ is the crisp subset of $X$ that contains all the elements of X that have none zero membership grads in $\mu$, i.e., $\text{supp}\mu = \{x: \mu(x) > 0\}$.

**1.4.16 Definition [46]:** The height $h(\mu)$ of an L-fuzzy set $\mu$ is the largest membership grade obtained by any element in that set, i.e.,$h(\mu) = \sup_{x \in X} \mu(x)$.

**1.4.17 Definition [46]:** For a finite L-fuzzy set, the cardinality $|\alpha|$ defined as $|\alpha| = \sum_{x \in X} \alpha(x)$. $\|\alpha\| = \frac{|\alpha|}{|x|}$ is called the relative cardinality of $\alpha$.

**1.4.18 Definition [46]:** An L-fuzzy set $\mu$ is called normal when $h(\mu) = 1$; it is called subnormal when $h(\mu) < 1$. The height of $\mu$ may also be viewed as the supremum of α for which $\alpha_\mu \neq \emptyset$.

**1.4.19 Definition [90]:** If $r \in L$ and $\alpha$ is an L-fuzzy set in $X$ defined by $\alpha(x) = r, \ \forall\, x \in X$ then we refer to $\alpha$ as a constant L-fuzzy set and denoted it by $r$ itself.

In particular, we have the constant L-fuzzy sets 0 and 1.

**1.4.20 Definition [46]:** An L-fuzzy point $p$ in $X$ is a special L-fuzzy set with membership function $p(x) = r \quad if\ x = x_0$

$= 0 \ \ otherwise\ x \neq x_0$ where $r \in L$ .

**1.4.21 Definition [46]:** An L-fuzzy point $p$ is said to belong to an L-fuzzy set $\alpha$ in $X$ $(p \in \alpha)$ if and only if $p(x) < \alpha(x)$ and $p(y) \leq \alpha(y)$

$(y \neq x) i.e. x_r \in \alpha \Rightarrow r < \alpha(x)$ .

**1.4.22 Definition [90]:** An L-fuzzy singleton in $X$ is an L-fuzzy set in $X$ which is zero everywhere except at one point say $x$, where it takes a value say $r$ with $0 < r \leq 1$ and $r \in L$ . We denote it by $x_r$ and $x_r \in \alpha$ iff $r \leq \alpha(x)$ .

**1.4.23 Definition [46]:** An L-fuzzy singleton $x_r$ is said to be quasi-coincident (q-coincident, in short) with an L-fuzzy set $\alpha$ in $X$, denoted by $x_r q\alpha$ iff $r + \alpha(x) > 1$ . Similarly, an L-fuzzy set $\alpha$ in $X$ is said to be q-coincident with an L-fuzzy set $\beta$ in $X$, denoted by $\alpha q\beta$ if and only if $\alpha(x) + \beta(x) > 1$ for some $x \in X$ . Therefore $\alpha\bar{q}\,\beta$ iff $\alpha(x) + \beta(x) \leq 1$ for all $x \in X$ , where $\alpha\bar{q}\,\beta$ denotes that an L-fuzzy set $\alpha$ in $X$ is not q-coincident with an L-fuzzy set $\beta$ in $X$.

**1.4.24 Laws of the algebra of L-fuzzy sets:**

As in ordinary set theory, idempotent laws, associative laws, commutative laws, distributive laws, identity laws, Demorgan's laws hold in the case of L-fuzzy sets also. But the complement laws are not necessarily true. For example, if $X = \{a, b, c\}$, $L = \{0{,}0.1{,}0.2{,}0.3{,}0.4{,}0.5{,}0.6{,}0.7{,}0.8{,}0.9{,}1\}$ and $\alpha$ is an L-fuzzy subset of $X$ where is defined by

$\alpha = \{ (a, .2), (b, .7), (c, 1) \}$,

then $\alpha^c = \{ (a, .8), (b, .3), (c, 0) \}$

so$\alpha \cup \alpha^c = \{ (a, .8), (b, .7), (c, 1) \} \neq 1$,

$\alpha \cap \alpha^c = \{ (a, .2), (b, .3), (c, 0) \} \neq 0$ .

Also in ordinary set theory $U \cap V = \emptyset$ if and only if $U \subset V^c$. But in L-fuzzy subsets reverse is not necessary true. For example if $\beta = \{ ( a ,.6 ), ( b ,.2 ), ( c , 0 )\}$ then $\alpha \subset \beta^c$,

$\alpha \cap \beta = \{ ( a ,.2 ) , ( b ,.2 ) , ( c , 0 ) \} \neq 0$.

**1.5 Mapping in L-fuzzy set**

**1.5.1 Definition [23 ]:** Let $f$ be a mapping from a set $X$ into a set $Y$ and $\alpha$ is an L-fuzzy subset on $X$.Then $f$ and $\alpha$ induced an L-fuzzy subset $\beta = f(\alpha)$ of $Y$ whose membership function is defined by

$$\beta(y) = f(\alpha)(y) = \begin{cases} \sup\{\alpha(x)\} \; if \; f^{-1}[\{y\}] \neq \emptyset, x \in X \\ \qquad 0, \qquad \text{otherwise} \end{cases}$$

**1.5.2 Definition [11]:** Let $f$ be a mapping from a set $X$ into $Y$ and $\beta$ is an L-fuzzy subset of $Y$. Then the inverse of $\beta$ written as $\alpha = f^{-1}(\beta)$ is an L-fuzzy subset of X and is defined by $\alpha(x) = (f^{-1}(\beta))\,(x) = \beta(f\,(x))$, for $x \in X$.

**1.5.3Example:** Suppose that $X = \{x, y, z, w\}$ and $Y = \{a, b, c\}$. Define $f: X \longrightarrow Y$ by $f(x) = b, f(y) = c, f(z) = a, f(w) = a$. Let $\alpha \in L^X$ be given by $\alpha(x) = .2, \alpha(y) = .3, \alpha(z) = .5$ and $\alpha(w) = .4$. Then $(f\,(\alpha))(a) = sup\,\{\alpha(z), \alpha(w)\} = .5$.Similarly,$(f(\alpha))(b) = sup\; \alpha(x) = .2$, $and\; (f\,(\alpha))(c) = sup\; \alpha(y) = .3$.

On the other hand, if $\beta$ is an L-fuzzy set in $Y$ given by $\beta(a) = .6, \beta(b) = .8, \beta(c) = .7$. Then $(f^{-1}(\beta))(x) = \beta(f(x)) = \beta(b) = .8$, $(f^{-1}(\beta))(y) =$

$\beta(f(y)) = \beta(c) = .7,\ (f^{-1}(\beta))(z) = \beta(f(z)) = \beta(a) = .6,\ (f^{-1}(\beta))(w) = \beta(f(w)) = \beta(a) = .6.$

We now mention some properties of L-fuzzy subsets induced by mappings.

**1.5.4 Definition [11]:** Let $f$ be a mapping from X into Y, $\alpha$ be an L-fuzzy subset of $X$ and $\beta$ be an L-fuzzy subset of $Y$. Then the following properties are true.

(a) $f^{-1}(\beta^c) = \left( f^{-1}(\beta)\right)^c$ for any L-fuzzy subset $\beta$ of $Y$.

(b) $f\ (\alpha^c)\ = \left(f(\alpha)\right)^c$ for any L-fuzzy subset $\alpha$ of $X$.

(c) $\beta_1 \subset \beta_2 \Longrightarrow f^{-1}(\beta_1) \subset f^{-1}(\beta_2)$, where $\beta_1$ and $\beta_2$ are two L-fuzzy subsets of $Y$.

(d) $\alpha_1 \subset \alpha_2 \Longrightarrow f(\alpha_1) \subset f(\alpha_2)$, where $\alpha_1$ and $\alpha_2$ are two L-fuzzy subsets of X.

(e) $\beta \supset f\ (f^{-1}(\beta))$, for any L-fuzzy subset $\beta$ of Y.

(f) $\alpha \subset f^{-1}(f(\alpha))$, for any L-fuzzy subset $\alpha$ of X.

(g) Let $f$ be a function from $X$ into $Y$ and $g$ be a function from $Y$ into $Z$.

Then $(\,g\ o\ f\,)^{-1}(\gamma)\ =\ f^{-1}(\,g^{-1}(\gamma)\,)$, for any L-fuzzy subset $\gamma$ in $Z$, where $(\,g\ o\ f\,)$ is the composition of $g$ and $f$.

### 1.6 L-topology

**1.6.1 Definition [46]:** Let X be a non-empty set and L be a complete distributive lattice with 0 and 1. Again suppose that $\tau$ be the sub collection of all mappings from $X$ to $L$ $i.e. \tau \subseteq L^X$. Then $\tau$ is called L-topology on $X$ if it satisfies the following conditions:

(i) $0^*, 1^* \in \tau$

(ii) If $u_1, u_2 \in \tau$ then $u_1 \cap u_2 \in \tau$

(iii) If $u_i \in \tau$ for each $i \in \Lambda$ then $\cup_{i\in\Lambda} u_i \in \tau$ .

Then the pair $(X, \tau)$ is called the L-topological space (lts, in short) and the members of $\tau$ are called open L-fuzzy sets. An L-fuzzy set $v$ is called a closed L-fuzzy set if $1 - v \in \tau$.

**1.6.2Example:**Let$X = \{a, b, c\}$,$\tau = \{0^*, u, v, 1^*\}$and $L = \{0,0.05,0.1,0.15, \dots \dots \dots 0.95,1\}$.Where $0^* = \{(a, 0), (b, 0), (c, 0)\}$,$1^* = \{(a, 1), (b, 1), (c, 1)\}$, $u = \{(a, 0.1), (b, 0.3), (c, 0.5)\}$and $v = \{(a, 0.2), (b, 0.4), (c, 0.6)\}$. Then $\tau$ is an $L$ topology on $X$ and the pair $(X, \tau)$ is called L-topological space.

**1.6.3 Definition [46]:** Let $\lambda$ be an L-fuzzy set in lts $(X, \tau)$. Then the closure of $\lambda$ is denoted by $\bar{\lambda}$ or $cl\lambda$ and defined as $\bar{\lambda} = \cap \{\mu : \lambda \subseteq \mu, \mu \in \tau^c\}$.

The interior of $\lambda$ written $\lambda^0$ or int$\lambda$ is defined by$\lambda^0 = \cup \{\mu : \mu \subseteq \lambda , \mu \in \tau\}$.

**1.6.4 Definition [46]:** If $(X, \tau)$ is an lts and $A \subseteq X$ then $\tau_A = \{u|A : u \in \tau\}$ is calld the sub space L-topology on $A$ and $(A, \tau_A)$ is referred to as an L-sub space of $(X, \tau)$.

**1.6.5 Definition [46]:** Let (X, τ) be an L-topological space. A subfamily B of τ is a base for τ if and only if each member of τ can be express as the union of some members of B.

**1.6.6 Definition [46]**: Let (X, τ) be an L-topological space. A subfamily S of τ is a sub-base for τ if and only if the family of finite intersection of members of S forms a base for τ.

**1.6.7 Definition [46]:** Let $x_r$ be an L-fuzzy point in an lts $(X, \tau)$. An L-fuzzy set $\alpha$ in $X$ is called a neighborhood (in short, nhd) of $x_r$ if and only if there exists an open L-fuzzy set $\beta$ in $X$ such that $x_r \in \beta \subseteq \alpha$ .

**1.6.8 Definition [46]:** An L-fuzzy set u in an L-topological space $(X, \tau)$ is called a neighborhood of an L-fuzzy point $x_r$ if and only if there exist an L-fuzzy set $u_1 \in \tau$ such that $x_r \in u_1 \subseteq u$. A neighborhood u is called an open neighborhood if u is open. The family consisting of all the neighborhoods of $x_r$ is called the system of $x_r$.

**1.6.9 Definition [46]:** An L-topological space (X, $\tau$) is said to be L-$T_0$ if and only if

a) for all distinct elements x, y $\in$X, there exists u$\in\tau$ such that u(x) =1, u(y) =0 or u(x) =0, u(y) =1.

b) for all distinct elements x, y $\in$X, there exists u$\in\tau$ such that u(x) < u(y) or u(y) < u(x).

c) for all distinct elements x, y $\in$X, $\overline{x}_1 \cap \overline{y}_1 < 1$.

**1.6.10 Definition [46]:** An L-topological space (X, $\tau$) is said to be $T_1$ if and only if

(a) for all distinct elements x, y $\in$X, there exist u, v $\in\tau$ such that u(x) =1, u(y) =0 and v(x) =0, u(y) =1;

(b) for all distinct elements x, y $\in$X, there exist u, v $\in\tau$ such that u(x) >0, u(y) =0 and v(x) =0, v(y) >0;

(c) for all distinct elements x, y $\in$X, there exist u, v $\in\tau$ such that u(x) > u(y) and v(y) > v(x).

**1.6.11 Definition [46]:** An L-topological space (X, $\tau$) is said to be L-fuzzy Hausdorff or L-$T_2$ if and only if

(a) for all distinct elements x, y $\in$X, there exist u, v$\in$ $\tau$ such that u(x) =1=u(y) and $u \cap v = 0$;

(b) for all pair of distinct L-fuzzy points $x_r$, $y_s \in S(X)$, there exist u, v$\in \tau$ such that $x_r \in u$, $y_s \in v$ and $u \cap v = 0$;

(c) for all distinct elements x, y $\in X$, there exist u, v$\in \tau$ such that u(x) > 0, v(y) >0 and $u \cap v = 0$.

**1.7 Continuous map Open map and closed map**

**1.7.1 Definition [46]:** Let $f$ be a real-valued function on an L-topological space. If $\{x: f(x) > \alpha\}$ is open for every real $\alpha$, then $f$ is called lower-semi continuous function (lsc, in short).

**1.7.2 Definition [46]:** Let $(X, \tau)$ and $(Y, s)$ be two L-topological spaces and $f$ be a mapping from $(X, \tau)$ into $(Y, s)$ $i.e.$ $f: (X, \tau) \to (Y, s)$. Then $f$ is called-

(i) L-Continuous iff for each open L-fuzzy set $u \in s \Longrightarrow f^{-1}(u) \in \tau$

(ii) L-Open iff $f(\mu) \in s$ for each open L-fuzzy set $\mu \in \tau$ .

(iii) L-Closed iff $f(\lambda)$ is s-closed for each $\lambda \in \tau^c$ ie. $\lambda$ is a closed L-fuzzy set in $X$.

(iv) L-Homeomorphism iff $f$ is bijective and both $f$ and $f^{-1}$ is L-continuous.

**1.7.3 Proposition[46]:** Let f : ( X , τ ) ⟶ ( Y , s ) be an L-fuzzy continuous function, then the following properties hold :

(i) For every s – closed v, $f^{-1}(v)$ is τ – closed.

(ii) For each L-fuzzy point p in X and each neighborhood u of f (u), then there exist a neighborhood v of p such that f (v) = u.

(iii) For any L-fuzzy set u in X, $f\ (u) \subset (f\ (u))$.

(iv) For any L-fuzzy set v in Y, $\left(f^{-1}(\ v\ )\right) \subset\ f^{-1}(\ v\ )$.

**1.7.4 Proposition[46]:** Let f : ( X , τ ) ⟶ ( Y , s ) be a L-fuzzy open function , then the following properties hold:

**( i )** $f\ (\ u^0\ ) \subseteq \left(\ f\ (u)\right)^0$ , for each L-fuzzy set u in X .

**(ii)** $\left(f^{-1}(v)\right)^0 \subseteq f^{-1}(v^0)$, for each L-fuzzy set v in Y.

**1.7.5 Proposition[46]:** Let f : ( X , t ) ⟶ ( Y , s ) be a function. Then f is closed if and only if $f\ (u) \subseteq f\ (u)$ for each fuzzy set u in X.

**1.8 "Good extension" and Product in L-topology**

**1.8.1 Definition [46]:** Let $X$ be a none empty set and $T$ be a topology on $X$. Let $\tau = \omega(T)$ be the set of all lower semi continuous (lsc) functions from

$(X, T)$ to $L$ (with usual topology). Thus $\omega(T) = \{u \in L^X : u^{-1}(\alpha, 1] \in T\}$ for each $\alpha \in L$ . It can be shown that $\omega(T)$ is an L-topology on $X$.

Let "P" be the property of a topological space $(X, T)$ and LFP be its L-topological analogue. Then LFP is called a "good extension" of P "if the statement $(X, T)$ has P iff $(X, \omega(T))$ has LFP" holds good for every topological space$(X, T)$.

**1.8.2 Definition [91]:** Let $\{(X_i, \tau_i): i \in \Lambda\}$ be a family of L-topological space. Then the space $(\Pi X_i, \Pi\, \tau_i)$ is called the product lts of the family $\{(X_i, \tau_i): i \in \Lambda\}$ where $\Pi\tau_i$ denote the usual product L-topologies of the families $\{\tau_i : i \in \Lambda\}$ of L-topologies on $X$ .

**1.8.3 Definition [91 ] :** If $u_1$ and $u_2$ are two L-fuzzy subsets of X and Y respectively then the Cartesian product $u_1 \times u_2$ of two L-fuzzy subsets $u_1$ and $u_2$ is a fuzzy subsets of $X \times Y$ defined by $(u_1 \times u_2)(x, y) = \min\{u_1(x), u_2(y)\}$, for each pair $(x, y) \in X \times Y$.

**1.8.4 Definition [46] :** Let $\{ X_i , i \in \Lambda \}$, be any class of sets and let X denoted the Cartesian product of these sets, i.e., $X = \Pi_{i\in\Lambda} X_i$ . Note that X consists of all points $p = < a_i , i \in \Lambda >$, where $a_i \in X_i$ . Recall that , for each $j_o \in \Lambda$ , we define the projection $\pi_{jo}$ from the product set X to the coordinate space $X_{jo}$ . ie $\pi_{jo} : X \longrightarrow X_{jo}$ by $\pi_{jo} ( < a_i : i \in \Lambda > ) = a_{jo}$ .These projections are used to define the product L-topology.

**1.8.5 Definition [ 46]:** If $( X_1 , \tau_1 )$ and $( X_2 , \tau_2 )$ be two L- topological spaces and $X = X_1 \times X_2$ be the usual product and $\tau$ be the coarsest L-topology on X , then each projection $\pi_i : X \longrightarrow X_i$ , $i = 1 , 2.$, is L-fuzzy continuous. The pair $(X, \tau)$ is called the product space of the L-topological spaces $(X_1, \tau_1)$ and $(X_2, \tau_2)$.

# Chapter-2

## On $T_0$ Space in L-Topological Spaces

### 2.1Introduction

Fuzzy $T_0$ spaces have been defined and studied by Hutton and Reilly[33,34], Pu and Liu [53,54].After then, in quick succession, a large number of seemingly different definitions of fuzzy $T_0$ spaces were developed and studied by several workers, e.g. Ali [2], Hossain [30] ,Srivastava [77,78] and Choubey[13] etc. In this chapter we define possible eight definitions of $T_0$ space in L-topological spaces. We established all these definitions satisfied "good extension" property; also we show that these spaces possess many nice properties and that they are hereditary, productive and projective.

### 2.2 $T_0$-property in L-Topological Spaces

We now give the following definitions of $T_0$-property in L-topological spaces.

**2.2.1Definition:** An lts $(X , \tau)$ is called-

(a) $L - T_0(i)$ if $\forall\, x , y \in X, x \neq y$ then $\exists\, u \in \tau$ such that $u(x) \neq u(y)$.

(b) $L - T_0(ii)$ if $\forall\, x , y \in X, x \neq y$ then $\exists\, u \in \tau$ such that $u(x) = 1, u(y) = 0$ or $u(x) = 0, u(y) = 1$.

(c) $L - T_0(iii)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then $\exists\, u \in \tau$ such that $x_r \in u, y_s \notin u$ or $x_r \notin u, y_s \in u$.

(d) $L - T_0(iv)$ if for all pairs of distinct L-fuzzy singletons $x_r, y_s \in S(X)$ with $x_r \bar{q} y_s$ then $\exists\, u \in \tau$ such that $x_r \subseteq u, y_s \bar{q} u$ or $y_s \subseteq u, x_r \bar{q} u$.

(e) $L - T_0(v)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then $\exists\, u \in \tau$ such that $x_r \in u, u\bar{q} y_s$ or $y_s \in u\,, u\bar{q} x_r$.

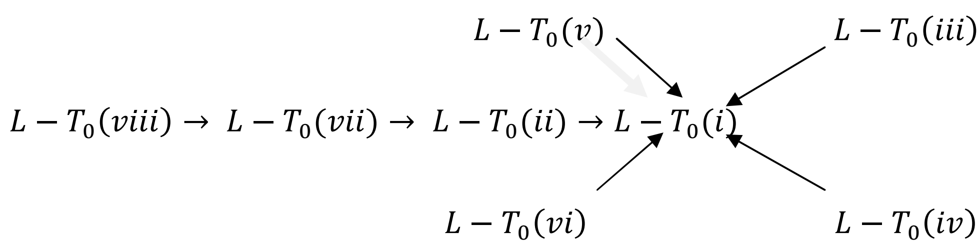
(f) $L - T_0(vi)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then $\exists\, u \in \tau$ such that $x_r \in u, y_s \cap u = 0$ or $y_s \in u, x_r \cap u = 0$.

(g) $L - T_0(vii)$ if $\forall\, x\,, y \in X, x \neq y$ then $\exists\, u \in \tau$ such that $u(x) > 0, u(y) = 0$ or $u(x) = 0, u(y) > 0$.

(h) $L - T_0(viii)$ if $\forall\, x\,, y \in X, x \neq y$ then $\exists\, u \in \tau$ such that $u(x) > u(y)$ or $u(y) > u(x)$.

A complete comparison of the definitions $L - T_0(ii), L - T_0(iii)$, $L - T_0(iv), L - T_0(v),\ L - T_0(vi), L - T_0(vii)\ and\ L - T_0(viii)$with $L - T_0(i)$ are given below:

**2.2.2Theorem:** Let $(X,\tau)$ be an lts. Then we have the following implications:




The reverse implications are not true in general.

**<u>Proof</u>:** $L - T_0(ii) \Rightarrow L - T_0(i)$ , $L - T_0(iii) \Rightarrow L - T_0(i)$ and

$L - T_0(iv) \Rightarrow L - T_0(i)$ can be proved . Now $L - T_0(v) \Rightarrow L - T_0(i)$ since $L - T_0(v) \Leftrightarrow L - T_0(iii)$ .

$L - T_0(vi) \Rightarrow L - T_0(i)$, since $L - T_0(vi) \Rightarrow L - T_0(v)$. $L - T_0(vii)$ and $L - T_0(viii) \Rightarrow L - T_0(i)$ since $L - T_0(viii) \Rightarrow L - T_0(vii)$ and

$L - T_0(vii) \Rightarrow L - T_0(ii)$ .

None of the reverse implications are true, it can be seen through the following counter example: Let $X = \{x, y\}$ , $\tau$ be the L-topology on $X$ generated by $\{\alpha : \alpha \in L\} \cup \{u\}$ where $u(x) = 0.5$ , $u(y) = 0.3$ and $L = \{0, 0.05, 0.1, 0.15, \dots \dots \dots 0.95, 1\}$ .

**<u>Proof</u>:** $L - T_0(i) \nRightarrow L - T_0(ii)$: Here the lts $(X,\tau)$ is clearly $L - T_0(i)$ but it is not $L - T_0(ii)$. Since there is no none empty L-fuzzy set in $\tau$ which takes zero value at $x$ or $y$ .

$L-T_0(i) \nRightarrow L-T_0(iii)$: For if we take the distinct L-fuzzy points $x_{3/4}, y_{4/5}$, there does not exist $u \in \tau$ such that $x_{3/4} \in u, y_{4/5} \notin u$ or $x_{3/4} \notin u, y_{4/5} \in u$.

$L-T_0(i) \nRightarrow L-T_0(iv)$: As for the distinct L-fuzzy singletons $x_1, y_1$ in $\tau$ there does not exist $u \in \tau$ such that $x_1 \subseteq u, y_1 \bar{q} u$ or $y_1 \subseteq u, x_1 \bar{q} u$.

$L-T_0(i) \nRightarrow L-T_0(v)$: This follows automatically from the fact that

$L-T_0(v) \Leftrightarrow L-T_0(iii)$ and it has already been shown that $L-T_0(i) \nRightarrow L-T_0(iii)$.

$L-T_0(i) \nRightarrow L-T_0(vi)$: Since for any two distinct L-fuzzy points $x_{3/4}, y_{4/5}$ in $X$, there does not exist $u \in \tau$ which is disjoint with $x_{3/4}$ or $y_{4/5}$.

$L-T_0(i) \nRightarrow L-T_0(vii)$ and $L-T_0(i) \nRightarrow L-T_0(viii)$: It is obvious because $L-T_0(vii) \Rightarrow L-T_0(ii)$ and $L-T_0(viii) \Rightarrow L-T_0(ii)$ and it has already been shown that $L-T_0(i) \nRightarrow L-T_0(ii)$:

### 2.3 "Good extension", Hereditary, Productive and Projective Properties in L-Topology

Now all the definitions $L-T_0(i)$,$L-T_0(ii)$, $L-T_0(iii)$, $L-T_0(iv)$, $L-T_0(v)$, $L-T_0(vi)$, $L-T_0(vii)$, $L-T_0(viii)$ are 'good extensions' of $T_0-$property, is shown below:

**2.3.1Theorem:** Let $(X, T)$ be a topological space. Then $(X, T)$ is $T_0$ iff $(X, \omega(T))$ is $L - T_0(i)$.

**<u>Proof</u>:** Let $(X, T)$ be $T_0$ space. Choose $x, y \in X$ with $x \neq y$.Then $\exists\, U \in T$ such that $x \in U, y \notin U$ or $y \in U, x \notin U$. Now consider the characteristics function $1_U$. Then $1_U \in \omega(T)$ such that $1_U(x) = 1, 1_U(y) = 0$ and so that $1_U(x) \neq 1_U(y)$. Thus $(X, \omega(T))$ is $L - T_0(i)$.

Conversely, let $(X, \omega(T))$ be $L - T_0(i)$. To show that $(X, T)$ is $T_0$. Choose $x, y \in X$ with $x \neq y$. Then $\exists\, u \in \omega(T)$ such that $u(x) \neq u(y)$. Let $u(x) < u(y)$. Choose $r$ such that $u(x) < r < u(y)$ and consider $u^{-1}(r, 1]$. Then $u^{-1}(r, 1] \in T$ with $x \notin u^{-1}(r, 1]$ and $y \in u^{-1}(r, 1]$. Hence $(X, T)$ is $T_0$. Similarly we can easily show that each of $L - T_0(ii)$, $L - T_0(iii)$, $L - T_0(iv), L - T_0(v), L - T_0(vi), L - T_0(vii), L - T_0(viii)$ are also holds 'good extension' property.

**2.3.2Theorem:** Let $(X, \tau)$ be an lts, $A \subseteq X$ and $\tau_A = \{u|A: u \in \tau\}$, then

(a) $(X, \tau)$ is $L - T_0(i) \Rightarrow (A, \tau_A)$ is $L - T_0(i)$.

(b) $(X, \tau)$ is $L - T_0(ii) \Rightarrow (A, \tau_A)$ is $L - T_0(ii)$.

(c) $(X, \tau)$ is $L - T_0(iii) \Rightarrow (A, \tau_A)$ is $L - T_0(iii)$.

(d) $(X, \tau)$ is $L - T_0(iv) \Rightarrow (A, \tau_A)$ is $L - T_0(iv)$.

(e) $(X, \tau)$ is $L - T_0(v) \Rightarrow (A, \tau_A)$ is $L - T_0(v)$.

(f) $(X, \tau)$ is $L - T_0(vi) \Rightarrow (A, \tau_A)$ is $L - T_0(vi)$.

(g) $(X, \tau)$ is $L - T_0(vii) \Rightarrow (A, \tau_A)$ is $L - T_0(vii)$.

(h) $(X, \tau)$ is $L - T_0(viii) \Rightarrow (A, \tau_A)$ is $L - T_0(viii)$.

**Proof:** We prove only (b). Suppose $(X, \tau)$ is L-topological space and $L - T_0(ii)$.

We shall prove that $(A, \tau_A)$ is $L - T_0(ii)$. Let $x, y \in A$ with $x \neq y$, then $x, y \in X$ with $x \neq y$ as $A \subseteq X$. Since $(X, \tau)$ is $L - T_0(ii)$, $\exists\, u \in \tau$ such that $u(x) = 1, u(y) = 0$ or $u(x) = 0,\ u(y) = 1$. For $A \subseteq X$ we find $u|A \in \tau_A$ such that $u|A(x) = 1, u|A(y) = 0$ or $u|A(x) = 0,\ u|A(y) = 1$. Hence it is clear that the subspace $(A, \tau_A)$ is $L - T_0(ii)$.

Similarly, (a), (c), (d), (e), (f), (g), (h) can be easily proved.

**2.3.3Theorem:** Given $\{(X_i, \tau_i): i \in \Lambda\}$ be a family of L-topological space. Then the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(j)$ iff each coordinate space $(X_i, \tau_i)$ is $L - T_0(j)$ where $j = i, ii, iii, iv, v, vi, vii, viii$.

**Proof:** Let each coordinate space $\{(X_i, \tau_i): i \in \Lambda\}$ be $L - T_0(ii)$. We show that the product space is $L - T_0(ii)$. Suppose $x, y \in X$ with $x \neq y$, again suppose that $x = \Pi x_i,\ y = \Pi y_i$ then $x_j \neq y_j$ for some $j \in \Lambda$. Now consider $x_j, y_j \in X_j$. Since $(X_j, \tau_j)$ is $L - T_0(ii)$, $\exists\, u_j \in \tau_j$ such that $u_j(x_j) = 1, u_j(y_j) = 0$ or $u_j(x_j) = 0, u_j(y_j) = 1$. Suppose $u_j(x_j) = 1,\ u_j(y_j) = 0$. Now take $u = \Pi u'_j$ where $u_j^{'} = u_j$ and $u_i = 1$ for $i \neq j$. Then $u$ is such that $u(x) = 1, u(y) = 0$. Hence the product L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(ii)$.

Conversely, let the product L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(ii)$. Take any coordinate space $(X_j, \tau_j)$, choose $x_j\,, y_j \in X_j\,, x_j \neq y_j$. Now construct $x, y \in X$ such that $x = \Pi x'_i,\ y = \Pi y'_i$ where $x_i^{'} = y_i^{'}$ for $i \neq j$ and $x_j^{'} = x_j, y_j^{'} = y_j$. Then $x \neq y$ and hence $\exists\, u \in \Pi\tau_i$ such that $u(x) = 1, u(y) = 0$ or $u(x) = 0, u(y) = 1$. Suppose $u(x) = 1,\ u(y) = 0$. Now $u$ must be the union of basic open L-fuzzy set say $u = \cup_{k\in K} b_k$. Thus $\cup\, b_k(x) = 1$ and $\cup\, b_k(y) = 0$ which implies that there exist at least one k such that $b_k(x) = 1, b_k(y) = 0$. Now let $b_k = \Pi v_i$ where $v_i = 1$ except for finitely many $i{\cdot}s$. So $\Pi v_i(x) = 1,\ \Pi v_i(y) = 0, i.e. \inf v_i(x_i^{'}) = 1$ and $\inf v_i(y_i^{'}) = 0$, which implies that $v_j(x_j) = 1, v_j(y_j) = 0$. Since $x_i^{'} = y_i^{'}$ for $i \neq j$, thus $(X_i, \tau_i)$ is $L - T_0(ii)$.

Moreover one can easily verify that

$(X_i, \tau_i), i \in \Lambda$ is $L - T_0(i) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(i)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_0(iii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(iii)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_0(iv) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(iv)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_0(v) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(v)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_0(vi) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(vi)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_0(vii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(vii)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_0(viii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_0(viii)$.

Hence, we see that $L-T_0(i), L-T_0(ii), L-T_0(iii), L-T_0(iv)$, $L-T_0(v), L-T_0(vi), L-T_0(vii)$ and $L-T_0(viii)$ Property is productive and projective.

**2.4 Mapping in L-topological spaces**

We show that $L-T_0(j)$ property is preserved under one-one, onto and continuous mapping for $j = i, ii, iii, iv, v, vi, vii, viii$ .

**2.4.1Theorem:** Let $(X,\tau)$ and $(Y,s)$ be two L-topological spaces and $f:(X,\tau) \to (Y,s)$ be one-one, onto and L-open map, then**-**

(a) $(X,\tau)$ is $L-T_0(i) \Rightarrow (Y,s)$ is $L-T_0(i)$.

(b) $(X,\tau)$ is $L-T_0(ii) \Rightarrow (Y,s)$ is $L-T_0(ii)$.

(c) $(X,\tau)$ is $L-T_0(iii) \Rightarrow (Y,s)$ is $L-T_0(iii)$.

(d) $(X,\tau)$ is $L-T_0(iv) \Rightarrow (Y,s)$ is $L-T_0(iv)$.

(e) $(X,\tau)$ is $L-T_0(v) \Rightarrow (Y,s)$ is $L-T_0(v)$.

(f) $(X,\tau)$ is $L-T_0(vi) \Rightarrow (Y,s)$ is $L-T_0(vi)$.

(g) $(X,\tau)$ is $L-T_0(vii) \Rightarrow (Y,s)$ is $L-T_0(vii)$.

(h) $(X,\tau)$ is $L-T_0(viii) \Rightarrow (Y,s)$ is $L-T_0(viii)$.

**Proof:** Suppose $(X,\tau)$ is $L-T_0(ii)$.We shall prove that $(Y,s)$ is $L-T_0(ii)$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $f$ is onto, $\exists\, x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$ and $x_1 \neq x_2$ as $f$ is one-one. Again since $(X,\tau)$ is $L - T_0(ii)$, $\exists\, u \in \tau$ such that $u(x_1) = 1, u(x_2) = 0$ or $u(x_1) = 0, u(x_2) = 1$.

Now

$f(u)(y_1) = \{sup u(x_1): f(x_1) = y_1\} = 1$

$f(u)(y_2) = \{sup u(x_2): f(x_2) = y_2\} = 0$

or

$f(u)(y_1) = \{sup u(x_1): f(x_1) = y_1\} = 0$

$f(u)(y_2) = \{sup u(x_2): f(x_2) = y_2\} = 1$.

Since $f$ is L-open, $f(u) \in s$. Now it is clear that $\exists\, f(u) \in s$ such that $\mathrm{f(u)(y_1) = 1}$, $\mathrm{f(u)(y_2) = 0}$ or $\mathrm{f(u)(y_1) = 0}$, $\mathrm{f(u)(y_2) = 1}$. Hence it is clear that the L-topological space $(\mathrm{Y, s})$ is $\mathrm{L - T_0(ii)}$. Similarly (a), (c), (d), (e), (f), (g), (h) can be proved.

**2.4.2 Theorem:** Let $(X,\tau)$ and $(\mathrm{Y, s})$ be two L-topological spaces and $\mathrm{f: (X, \tau) \to (Y, s)}$ be L-continuous and one-one map, then-

(a) $(\mathrm{Y, s})$ is $\mathrm{L - T_0(i)} \Rightarrow (\mathrm{X, \tau})$ is $\mathrm{L - T_0(i)}$.

(b) $(Y, s)$ is $\mathrm{L - T_0(ii)} \Rightarrow (\mathrm{X, \tau})$ is $\mathrm{L - T_0(ii)}$.

(c) $(\mathrm{Y, s})$ is $\mathrm{L - T_0(iii)} \Rightarrow (\mathrm{X, \tau})$ is $\mathrm{L - T_0(iii)}$.

(d) $(Y, s)$ is $L - T_0(iv) \Rightarrow (X,\tau)$ is $L - T_0(iv)$.

(e) $(\mathrm{Y, s})$ is $L - T_0(v) \Rightarrow (X,\tau)$ is $L - T_0(v)$.

(f) $(Y, s)$ is $L - T_0(vi) \Rightarrow (X,\tau)$ is $L - T_0(vi)$.

(g) $(Y, s)$ is $L - T_0(vii) \Rightarrow (X, \tau)$ is $L - T_0(vii)$.

(h) $(Y, s)$ is $L - T_0(viii) \Rightarrow (X, \tau)$ is $L - T_0(viii)$.

**<u>Proof</u>:** Suppose $(Y, s)$ is $L - T_0(ii)$. We shall prove that $(X, \tau)$ is $L - T_0(ii)$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2, \Rightarrow f(x_1) \neq f(x_2)$ as f is one-one. Since $(Y, s)$ is $L - T_0(ii)$, then $\exists\, u \in s$ such that $u(f(x_1)) = 1, u(f(x_2)) = 0$ or $u(f(x_1)) = 0, u(f(x_2)) = 1$. Suppose $u(f(x_1)) = 1$, $u(f(x_2)) = 0$. This implies that $f^{-1}(u)(x_1) = 1, f^{-1}(u)(x_2) = 0$ and $f^{-1}(u) \in \tau$ as f is

L-continuous and $u \in s$. Now it is clear that $f^{-1}(u) \in \tau$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) = 0$. Hence the L-topological space $(X, \tau)$ is $L - T_0(ii)$.

Similarly (a), (c), (d), (e), (f), (g), (h) can be proved.

# Chapter-3

## On $T_1$ Space in L-Topological Spaces

### 3.1Introduction

The concept of fuzzy $T_1$ space was introduced Ali [2, 3], Hossain[26], Srivastava and Lal[79], Sinha[69,70], Malghan[49,50] and other mathematician have contributed to the development of the theory. In this chapter, we discuss possible eight definitions of $T_1$ space in L-topological spaces; all these notions satisfy "good extension" property. We show that these notions possess many nice properties which are hereditary, productive and projective.

### 3.2 $T_1$-property in L-Topological Spaces

Here, we define the following definitions of $T_1$-property in L-topological spaces.

**3.2.1 Definition:** An lts $(X, \tau)$ is called-

(a) $L - T_1(i)$ if $\forall\, x, y \in X, x \neq y$ then $\exists\, u, v \in \tau$ such that $u(x) \neq u(y)$ and $v(x) \neq v(y)$.

(b) $L - T_1(ii)$ if $\forall\, x\,, y \in X, x \neq y$ then $\exists\, u, v \in \tau$ such that $u(x) = 1, u(y) = 0$ and $v(x) = 0,\; v(y) = 1$.

(c) $L - T_1(iii)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then $\exists\, u, v \in \tau$ such that $x_r \in u, y_s \notin u$ and $x_r \notin v, y_s \in v$.

(d) $L - T_1(iv)$ if for all pairs of distinct L-fuzzy singletons $x_r, y_s \in S(X)$with $x_r \bar{q} y_s$ then $\exists\, u, v \in \tau$ such that $x_r \subseteq u, y_s \bar{q} u$ and $y_s \subseteq v, x_r \bar{q} v$.

(e) $L - T_1(v)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then $\exists\, u, v \in \tau$ such that $x_r \in u, u \bar{q} y_s$ and $y_s \in v\,, v \bar{q} x_r$.

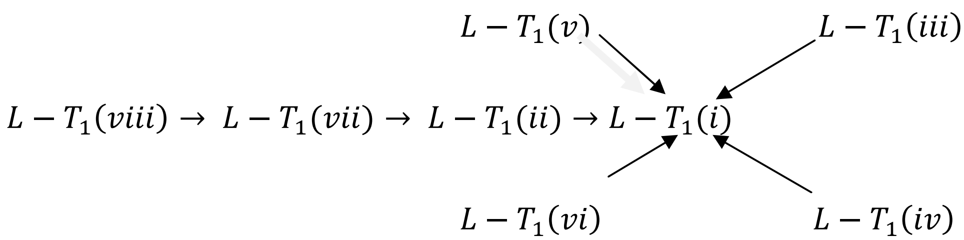
(f) $L - T_1(vi)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then $\exists\, u, v \in \tau$ such that $x_r \in u, y_s \cap u = 0$ and $y_s \in v, x_r \cap v = 0$.

(g) $L - T_1(vii)$ if $\forall\, x\,, y \in X, x \neq y$ then$\exists\, u, v \in \tau$ such that $u(x) > 0, u(y) = 0$ and $v(x) = 0,\; v(y) > 0$.

(h) $L - T_1(viii)$ if $\forall\, x\,, y \in X, x \neq y$ then $\exists\, u, v \in \tau$ such that $u(x) > u(y)$ and $v(y) > v(x)$.

Here, we established a comparison of the definitions $L - T_1(ii)$, $L - T_1(iii), L - T_1(iv),\; L - T_1(v),\; L - T_1(vi), L - T_1(vii), L - T_1(viii)$with $L - T_1(i)$ is given below:

**3.2.2Theorem:** Let $(X, \tau)$ be an lts. Then we have the following implications:




The reverse implications are not true in general.

**<u>Proof</u>:** $L-T_1(ii) \Rightarrow L-T_1(i), L-T_1(iii) \Rightarrow L-T_1(i)$ and $L-T_1(iv) \Rightarrow$ $L-T_1(i)$ can be proved easily. Now $L-T_1(v) \Rightarrow L-T_1(i)$ since

$L-T_1(v) \Leftrightarrow L-T_1(iii)$.

$L-T_1(vi) \Rightarrow L-T_1(i)$, since $L-T_1(vi) \Rightarrow L-T_1(v)$. $L-T_1(vii)$and $L-T_1(viii) \Rightarrow L-T_1(i)$ since $L-T_1(viii) \Rightarrow L-T_1(vii)$ and $L-T_1(vii) \Rightarrow$ $L-T_1(ii)$.

None of the reverse implications are true; it can be seen through the following example:

Let $X = \{x, y\}$, $\tau$ be the L-topology on $X$ generated by $\{\alpha: \alpha \in L\} \cup \{u, v\}$ where $u(x) = 0.5, u(y) = 0.6$ and $v(x) = 0.7, v(y) = 0.4$ and

$L = \{0, 0.05, 0.1, 0.15, \dots \dots \dots 0.95, 1\}$.

**<u>Proof</u>:** $L-T_1(i) \nRightarrow L-T_1(ii)$: Here the lts $(X, \tau)$ is clearly $L-T_1(i)$ but it is not $L-T_1(ii)$ . Since there is no none empty L-fuzzy set in $\tau$ which takes zero value at $x$ or $y$.

$L-T_1(i) \nRightarrow L-T_1(iii)$: For if we take the distinct L-fuzzy points $x_{3/5}, y_{1/2} \in S(X)$,then there does not exist $u, v \in \tau$ such that $x_{3/5} \in u, y_{1/2} \notin u$ and $x_{3/5} \notin v, y_{1/2} \in v$.

$L-T_1(i) \nRightarrow L-T_1(iv)$: As for the distinct L-fuzzy singletons $x_1, y_1$ in $\tau$ there does not exist $u, v \in \tau$ such that $x_1 \subseteq u, y_1 \bar{q} u$ and $y_1 \subseteq v, x_1 \bar{q} v$.

$L-T_1(i) \nRightarrow L-T_1(v)$: This follows automatically from the fact that

$L-T_1(v) \Leftrightarrow L-T_1(iii)$ and it has already been shown that $L-T_1(i) \nRightarrow$

$L-T_1(iii)$.

$L-T_1(i) \nRightarrow L-T_1(vi)$: Since for any two distinct L-fuzzy points $x_{3/5}, y_{1/2}$ in $S(X)$, then there does not exist $u, v \in \tau$ which is disjoint with $x_{3/5}$ and $y_{1/2}$.

$L-T_1(i) \nRightarrow L-T_1(vii)$ and $L-T_1(i) \nRightarrow L-T_1(viii)$: It is obvious because $L-T_1(vii) \Rightarrow L-T_1(ii)$ and $L-T_1(viii) \Rightarrow L-T_1(ii)$ and it has already been shown that $L-T_1(i) \nRightarrow L-T_1(ii)$.

## 3.3 "Good extension", Hereditary, Productive and Projective Properties in L-Topology

Here, we show that all the definitions $L-T_1(i), L-T_1(ii), L-T_1(iii)$, $L-T_1(iv), L-T_1(v)$, $L-T_1(vi), L-T_1(vii)$ and $L-T_1(viii)$ are 'good extensions' of $T_1-$ property, is shown below:

**3.3.1Theorem:** Let $(X, T)$ be a topological space. Then $(X, T)$ is $T_1$ iff $(X, \omega(T))$ is $L - T_1(i)$.

**Proof:** Let $(X, T)$ be $T_1$. Choose $x, y \in X$ with $x \neq y$. Then $\exists\, U, V \in T$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Now consider the lower semi continuous functions $1_U, 1_V$. Then $1_U, 1_V \in \omega(T)$ with $1_U(x) = 1,\ 1_U(y) = 0$ and $1_V(x) = 0, 1_V(y) = 1$ and so that $1_U(x) \neq 1_U(y)$ and $1_V(x) \neq 1_V(y)$. Thus $(X, \omega(T))$ is $L - T_1(i)$.

Conversely, let $(X, \omega(T))$ be $L - T_1(i)$. To show that $(X, T)$ is $T_1$. Choose $x, y \in X$ with$x \neq y$. Then $\exists\, u, v \in \omega(T)$ such that $u(x) \neq u(y)$and $v(x) \neq v(y)$. Let $u(x) < u(y)$ and $v(y) < v(x)$. Choose $r$ and $s$ such that $u(x) < r < u(y)$ and $v(y) < s < v(x)$ and consider $u^{-1}(r, 1]$ and $v^{-1}(s, 1]$. Then $u^{-1}(r, 1], v^{-1}(s, 1] \in T$ and is $x \notin u^{-1}(r, 1]$, $y \in u^{-1}(r, 1]$ and $x \in v^{-1}(s, 1]$, $y \notin v^{-1}(s, 1]$. Hence $(X, T)$ is $T_1$.

Similarly we can show that $L - T_1(ii), L - T_1(iii), L - T_1(iv),\ L - T_1(v),$ $L - T_1(vi),\ L - T_1(vii), L - T_1(viii)$ are also hold 'good extension' property.

**3.3.2Theorem:** Let $(X, \tau)$ be an lts, $A \subseteq X$ and $\tau_A = \{u|A : u \in \tau\}$, then

(a) $(X, \tau)$ is $L - T_1(i) \Rightarrow (A, \tau_A)$ is $L - T_1(i)$.

(b) $(X, \tau)$ is $L - T_1(ii) \Rightarrow (A, \tau_A)$ is $L - T_1(ii)$.

(c) $(X, \tau)$ is $L - T_1(iii) \Rightarrow (A, \tau_A)$ is $L - T_1(iii)$.

(d) $(X, \tau)$ is $L - T_1(iv) \Rightarrow (A, \tau_A)$ is $L - T_1(iv)$.

(e) $(X, \tau)$ is $L - T_1(v) \Rightarrow (A, \tau_A)$ is $L - T_1(v)$.

(f) $(X, \tau)$ is $L - T_1(vi) \Rightarrow (A, \tau_A)$ is $L - T_1(vi)$.

(g) $(X, \tau)$ is $L - T_1(vii) \Rightarrow (A, \tau_A)$ is $L - T_1(vii)$.

(h) $(X, \tau)$ is $L - T_1(viii) \Rightarrow (A, \tau_A)$ is $L - T_1(viii)$.

**Proof:** We prove only (b). Suppose $(X, \tau)$ is L-topological space and $L - T_1(ii)$. We shall prove $(A, \tau_A)$ is $L - T_1(ii)$. Let $x, y \in A$ with $x \neq y$, then $x, y \in X$ with $x \neq y$ as $A \subseteq X$. Since $(X, \tau)$ is $L - T_1(ii)$, $\exists\, u, v \in \tau$ such that $u(x) = 1, u(y) = 0$ and $v(x) = 0,\ v(y) = 1$. For $A \subseteq X$ we find $u|A, v|A \in \tau_A$ and $u|A(x) = 1, u|A(y) = 0$ and $v|A(x) = 0, v|A(y) = 1$ as $x, y \in A$. Hence it is clear that the subspace $(A, \tau_A)$ is $L - T_1(ii)$.

Similarly, (a), (c), (d), (e), (f), (g), (h) can be easily proved.

**3.3.3Theorem:** Given $\{(X_i, \tau_i) : i \in \Lambda\}$ be a family of L-topological space. Then the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(j)$ iff each coordinate space $(X_i, \tau_i)$ is $L - T_1(j)$ where $j = i, ii, iii, iv, v, vi, vii, viii$.

**Proof:** Let each coordinate space $\{(X_i, \tau_i) : i \in \Lambda\}$ be $L - T_1(ii)$. Then we show that the product space is $L - T_1(ii)$. Suppose $x, y \in X$ with $x \neq y$, again suppose $x = \Pi x_i, y = \Pi y_i$ then $x_j \neq y_j$ for some $j \in \Lambda$. Now consider

$x_j, y_j \in X_j$. Since $(X_j, \tau_j)$ is $L - T_1(ii)$, $\exists\, u_j, v_j \in \tau_j$ such that $u_j(x_j) = 1, u_j(y_j) = 0$ and $v_j(x_j) = 0, v_j(y_j) = 1$. Now take $u = \Pi u'_j, v = \Pi v'_j$ where $u'_j = u_j, v'_j = v_j$ and $u_i = v_i = 1$ for $i \neq j$. Then $u, v \in \Pi\tau_i$ such that $u(x) =$

$1, u(y) = 0$ and $v(x) = 0, v(y) = 1$. Hence the product L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(ii)$.

Conversely, let the product L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(ii)$. Take any coordinate space $\left(X_j, \tau_j\right)$, choose $x_j, y_j \in X_j, x_j \neq y_j$. Now construct $x, y \in X$ such that $x = \Pi x'_i, y = \Pi y'_i$ where $x'_i = y'_i$ for $i \neq j$ and $x'_j = x_j, y'_j = y_j$. Then $x \neq y$ and using the product space $L - T_1(ii)$ $\exists\, u, v \in \Pi\tau_i$ such that $u(x) = 1, u(y) = 0$ and $v(x) = 0,\ v(y) = 1$. Now choose any L-fuzzy point $x_r$ in $u$. Then $\exists$ a basic open L-fuzzy set $\Pi u_j^r \in \Pi\tau_j$ such that $x_r \in \Pi u_j^r \subseteq u$ which implies that $r < \Pi u_j^r(x)$ or that $r < inf_j u_j^r\left(x'_j\right)$ and hence $r < \Pi u_j^r\left(x'_j\right) \forall\, j \in \Lambda \ldots\ldots (i)$ and $u(y) = 0 \Rightarrow$ $\Pi u_j(y) = 0\ \ldots\ldots (ii)$. Similarly, corresponding to a fuzzy point $y_s \in v$ there exists a basic open L-fuzzy set $\Pi v_j^s \in \Pi\tau_j$ that $y_s \in \Pi v_j^s \subseteq v$ which implies that $s < v_j^s(j) \forall\, j \in \Lambda \ldots\ldots (iii)$ and $v_j^s(y) = 0 \ldots\ldots (iv)$. Further, $\Pi u_j^r(y) = 0 \Rightarrow u_i^r(y_i) = 0$, since for $j \neq i, x'_j = y'_j$ and hence from $(i)$, $u_j^r\left(y_j\right) = u_j^r\left(x_j\right) > r$. Similarly, $\Pi v_j^s(x) = 0 \Rightarrow v_i^s(x_i) = 0$ using $(iii)$. Thus we have $u_i^r(x_i) > r, u_i^r(y_i) = 0$ and $v_i^s(y_i) > s, v_i^s(x_i) = 0$. Now consider $sup_r u_i^r = u_i, sup_s v_i^s = v_i \in \tau_i$ then $u_i(x_i) = 1, u_i(y_i) = 0$ and $v_i(x_i) = 0, v_i(y_i) = 1$ showing that $(X_i, \tau_i)$ is $L - T_1(ii)$.

Moreover one can easily verify that

$(X_i, \tau_i), i \in \Lambda$ is $L - T_1(i) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(i)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_1(iii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(iii)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_1(iv) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(iv)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_1(v) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(v)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_1(vi) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(vi)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_1(vii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(vii)$

$(X_i, \tau_i), i \in \Lambda$ is $L - T_1(viii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_1(viii)$ .

Hence we see that $L - T_1(i), L - T_1(ii), L - T_1(iii), L - T_1(iv),$ $L - T_1(v), L - T_1(vi), L - T_1(vii), L - T_1(viii)$ Properties are productive and projective.

**3.4 Mapping in L-topological spaces**

We show that $L - T_1(j)$ property is preserved under one-one, onto and continuous mapping for $j = i, ii, iii, iv, v, vi, vii, viii$.

**3.4.1Theorem:** Let $(X, \tau)$ and $(Y, s)$ be two L-topological spaces and $f: (X, \tau) \to (Y, s)$ be one-one, onto and L-open map, then-

(a) $(X\ , \tau)$ is $L - T_1(i) \Rightarrow (Y, s)$ is $L - T_1(i)$.

(b) $(X\ , \tau)$ is $L - T_1(ii) \Rightarrow (Y, s)$ is $L - T_1(ii)$.

(c) $(X\ , \tau)$ is $L - T_1(iii) \Rightarrow (Y, s)$ is $L - T_1(iii)$.

(d) $(X\ , \tau)$ is $L - T_1(iv) \Rightarrow (Y, s)$ is $L - T_1(iv)$.

(e) $(X\ , \tau)$ is $L - T_1(v) \Rightarrow (Y, s)$ is $L - T_1(v)$.

(f) $(X,\tau)$ is $L-T_1(vi) \Rightarrow (Y,s)$ is $L-T_1(vi)$.

(g) $(X,\tau)$ is $L-T_1(vii) \Rightarrow (Y,s)$ is $L-T_1(vii)$.

(h) $(X,\tau)$ is $L-T_1(viii) \Rightarrow (Y,s)$ is $L-T_1(viii)$.

**<u>Proof</u>:** Suppose $(X,\tau)$ is $L-T_1(ii)$. We shall prove that $(Y,s)$ is $L-T_1(ii)$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $f$ is onto, $\exists\, x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$ and $x_1 \neq x_2$ as $f$ is one-one. Again since $(X,\tau)$ is $L-T_1(ii)$ $\exists\, u, v \in \tau$ such that $u(x_1) = 1,\ u(x_2) = 0$ and $v(x_1) = 0, v(x_2) = 1$. Now

$f(u)(y_1) = \{sup u(x_1): f(x_1) = y_1\} = 1$

$f(u)(y_2) = \{sup u(x_2): f(x_2) = y_2\} = 0$ and

$f(v)(y_1) = \{sup v(x_1): f(x_1) = y_1\} = 0$

$f(v)(y_2) = \{sup v(x_2): f(x_2) = y_2\} = 1$.

Since $f$ is L-open, $f(u), f(v) \in s$. Now it is clear that $\exists\, f(u), f(v) \in s$ such that $f(u)(y_1) = 1, f(u)(y_2) = 0$ and $f(v)(y_1) = 0, f(v)(y_2) = 1$. Hence it is clear that the L-topological space $(Y,s)$ is $L-T_1(ii)$.

Similarly (a), (c), (d), (e), (f), (g), (h) can be proved.

**3.4.2Theorem:** Let $(X,\tau)$ and $(Y,s)$ be two L-topological spaces and $f:(X,\tau) \to (Y,s)$ be L-continuous and one-one map, then-

(a) $(Y,s)$ is $L-T_1(i) \Rightarrow (X,\tau)$ is $L-T_1(i)$.

(b) $(Y,s)$ is $L-T_1(ii) \Rightarrow (X,\tau)$ is $L-T_1(ii)$.

(c) $(Y,s)$ is $L-T_1(iii) \Rightarrow (X,\tau)$ is $L-T_1(iii)$.

(d) $(Y,s)$ is $L-T_1(iv) \Rightarrow (X,\tau)$ is $L-T_1(iv)$.

(e) $(Y,s)$ is $L-T_1(v) \Rightarrow (X,\tau)$ is $L-T_1(v)$.

(f) $(Y,s)$ is $L-T_1(vi) \Rightarrow (X,\tau)$ is $L-T_1(vi)$.

(g) $(Y,s)$ is $L-T_1(vii) \Rightarrow (X,\tau)$ is $L-T_1(vii)$.

(h) $(Y,s)$ is $L-T_1(viii) \Rightarrow (X,\tau)$ is $L-T_1(viii)$.

**<u>Proof</u>:** Suppose $(Y,s)$ is $L-T_1(ii)$. We shall prove that $(X,\tau)$ is $L-T_1(ii)$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2, \Rightarrow f(x_1) \neq f(x_2)$ as $f$ is one-one. Since $(Y,s)$ is $L-T_1(ii)$, $\exists\, u,v \in s$ such that $u(f(x_1)) = 1, u(f(x_2)) = 0$ and $v(f(x_1)) = 0, v(f(x_2)) = 1$. This implies that $f^{-1}(u)(x_1) = 1, f^{-1}(u)(x_2) = 0$ and $f^{-1}(v)(x_1) = 0, f^{-1}(v)(x_2) = 1$ and hence $f^{-1}(u), f^{-1}(v) \in \tau$ as $f$ is L-continuous and $u,v \in s$ . Now it is clear that $f^{-1}(u), f^{-1}(v) \in \tau$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) = 0$ and $f^{-1}(v)(x_1) = 0$, $f^{-1}(v)(x_2) = 1$. Hence the L-topological space $(X,\tau)$ is $-T_1(ii)$.

Similarly (a), (c), (d), (e), (f), (g), (h) can be proved.

# Chapter-4

## On $T_2$ Space in L-Topological Spaces

### 4.1Introduction:

Hausdorff, [25] introduced the fundamental concept of $T_2$ space in general topology. $T_2$ space in fuzzy topology was introduced by Ghanim et.al. [23], Ganguly [22], Shinha [70] and Fora[1] etc. Later $FT_2$ space has been developed by Ali [2, 6], Cutler [14], Reilly [34] and Hossain [27]. Seven concepts of $T_2$ space in L-topological spaces are introduced and studied in this chapter. We showed that all these concepts satisfy "good extension" property. We also establish some relationships among them and study some other properties of these spaces.

### 4.2 $T_2$-property in L-Topological Spaces

We now give the following definitions of $T_2$-property in L-topological spaces.

**4.2.1Definition:** An lts $(X, \tau)$ is called-

(a) $L-T_2(i)$ if $\forall x, y \in X, x \neq y$ then $\exists\, u, v \in \tau$ such that $u(x) = 1, u(y) = 0, v(x) = 0, v(y) = 1$ and $u \cap v = 0$ .

(b) $L - T_2(ii)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then $\exists\, u, v \in \tau$ such that $x_r \in u, y_s \notin u$ and $x_r \notin v, y_s \in v$ and $u \cap v = 0$.

(c) $L - T_2(iii)$ if for all pairs of distinct L-fuzzy singletons $x_r, y_s \in S(X)$ with $x_r \bar{q} y_s$ then $\exists\, u, v \in \tau$ such that $x_r \subseteq u, y_s \bar{q} u$ and $y_s \subseteq v, x_r \bar{q} v$ and $u \cap v = 0$.

(d) $L - T_2(iv)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then $\exists\, u, v \in \tau$ such that $x_r \in u, u \bar{q} y_s$ and $y_s \in v$ , $v \bar{q} x_r$ and $u \cap v = 0$.

(e) $L - T_2(v)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ then$\exists\, u, v \in \tau$ such that $x_r \in u \subseteq co y_s$, $y_s \in v \subseteq co x_r$ and $u \subseteq cov$.

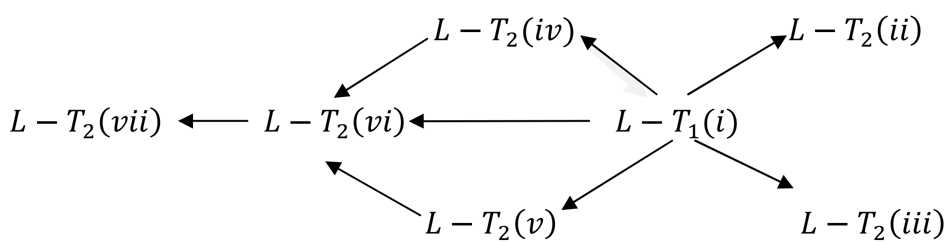
(f) $L - T_2(vi)$ if $\forall\, x , y \in X, x \neq y$ then $\exists\, u, v \in \tau$ such that $u(x) > 0, u(y) = 0$ and $v(x) = 0, v(y) > 0$ .

(g) $L - T_2(vii)$ if $\forall\, x , y \in X, x \neq y$ then $\exists\, u, v \in \tau$ such that $u(x) > u(y)$ and $v(y) > v(x)$.

Here, we established a complete comparison of the definitions

$L - T_2(ii), L - T_2(iii),\ L - T_2(iv), L - T_2(v), L - T_2(vi)$ and $L - T_2(vii)$ with $L - T_2(i)$.

**4.2.2Theorem:** Let $(X\,,\tau)$ be an lts. Then we have the following implications:




The reverse implications are not true in general, except $L-T_2(vi)$ and $L-T_2(vii)$.

**Proof:** $L-T_2(i)\Rightarrow L-T_2(ii)\,, L-T_2(i)\Rightarrow L-T_2(iii)$can be proved easily. Now $L-T_2(i)\Rightarrow L-T_2(iv)$and $L-T_2(i)\Rightarrow L-T_2(v)$, since $L-T_2(ii)\Leftrightarrow L-T_2(iv)$and $L-T_2(iv)\Leftrightarrow L-T_2(v)$. $L-T_2(i)\Rightarrow$

$L-T_2(vi)$; It is obvious. $L-T_2(i)\Rightarrow L-T_2(vii)$, since $L-T_2(vi)\Rightarrow L-T_2(vii)$.

The reverse implications are not true in general, except $L-T_2(vi)$ and $L-T_2(vii)$, as can be seen through the following counter-examples:

Example-1: Let $X=\{x,y\}$, $\tau$ be the L-topology on $X$ generated by $\{\alpha{:}\,\alpha\in L\}\cup\{u,v\}$ where $u(x)=0.5, u(y)=0$ $v(x)=0, v(y)=0.6$, $L=\{0{,}0.05{,}0.1{,}0.15,\ldots{,}0.95{,}1\}$ and $r=0.4, s=0.3$.

Example-2: Let $X=\{x,y\}$, $\tau$ be the L-topology on $X$ generated by $\{\alpha{:}\,\alpha\in L\}\cup\{u,v\}$ where $u(x)=0.5, u(y)=0$ $v(x)=0, v(y)=0.4$, $L=\{0{,}0.05{,}0.1{,}0.15,\ldots{,}0.95{,}1\}$ and $r=0.5, s=0.4$.

**Proof:** $L-T_2(ii) \nRightarrow L-T_2(i)$: From example-1, we see that the lts $(X,\tau)$ is clearly $L-T_2(ii)$ but it is not $L-T_2(i)$ . Since there is no L-fuzzy set in $\tau$ which grade of membership is 1.

$L-T_2(iii) \nRightarrow L-T_2(i)$: From example-2, we see the lts $(X,\tau)$ is clearly $L-T_2(iii)$ but it is not $L-T_2(ii)$. Since $L-T_2(iii) \nRightarrow L-T_2(ii)$ and $L-T_2(ii) \nRightarrow L-T_2(i)$ so $L-T_2(iii) \nRightarrow L-T_2(i)$.

$L-T_1(i) \nRightarrow L-T_1(iv)$: As for the distinct L-fuzzy singletons $x_1, y_1$ in $\tau$ there does not exist $u, v \in \tau$ such that $x_1 \subseteq u, y_1\bar{q}u$ and $y_1 \subseteq v, x_1\bar{q}v$ .

$L-T_2(iv) \nRightarrow L-T_2(i)$: This follows from the fact that

$L-T_2(ii) \Leftrightarrow L-T_2(iv)$ and it has already been shown that $L-T_2(ii) \nRightarrow L-T_2(i)$so $L-T_2(iv) \nRightarrow L-T_2(i)$ .

$L-T_2(v) \nRightarrow L-T_2(i)$ : Since $L-T_2(iv) \Leftrightarrow L-T_2(v)$and $L-T_2(iv) \nRightarrow L-T_2(i)$ so $L-T_2(v) \nRightarrow L-T_2(i)$. But $L-T_2(vii) \Rightarrow L-T_2(vi) \Rightarrow T_2(i)$ is obvious.

## 4.3 "Good extension", Hereditary, Productive and Projective Properties in L-Topology

Here we showed that all definitions $L-T_2(i)$, $L-T_2(ii), L-T_2(iii), L-T_2(iv), L-T_2(v)$, $L-T_2(vi)$ and $L-T_2(vii)$ are 'good extensions' of $T_2-$property, is shown below:

**4.3.1Theorem:** Let $(X, T)$ be a topological space. Then $(X, T)$ is $T_2$ iff $(X, \omega(T))$ is $L - T_2(j)$, where $j = i, ii, iii, iv, v, vi, vii$.

**Proof:** Let $(X, T)$be $T_2$. Choose $x, y \in X$ with $x \neq y$.Then $\exists\, U, V \in T$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$ and $U \cap V = \emptyset$. Now consider the lower semi continuous functions $1_U, 1_V$. Then $1_U, 1_V \in \omega(T)$ such that $1_U(x) = 1, 1_U(y) = 0$ and $1_V(x) = 0, 1_V(y) = 1$ and so that $1_U \cap 1_V = 0$. Thus $(X, \omega(T))$ is $L - T_2(i)$.

Conversely, let $(X, \omega(T))$ be $L - T_2(i)$. To show that $(X, T)$ is $T_2$. Choose $x, y \in X$ with $x \neq y$. Then $\exists\, u, v \in \omega(T)$ such that $u(x) = 1, u(y) = 0$,$v(x) = 0, v(y) = 1$ and $u \cap v = 0$, i.e.,$u(y) < u(x)$ and $v(x) < v(y)$. Choose $r$ and $s$ such that $u(y) < r < u(x)$ and $v(x) < s < v(y)$ and consider $u^{-1}(r, 1]$ and $v^{-1}(s, 1]$. Then $u^{-1}(r, 1], v^{-1}(s, 1] \in T$ and is $x \notin u^{-1}(r, 1]$, $y \in u^{-1}(r, 1]$, $x \in v^{-1}(s, 1]$, $y \notin v^{-1}(s, 1]$ and $u^{-1}(r, 1] \cap v^{-1}(s, 1] = \emptyset$ as $u \cap v = 0$ . Hence $(X, T)$ is $T_2$.

Similarly, we can show that $L - T_2(ii), L - T_2(iii), L - T_2(iv)$, $L - T_2(v), L - T_2(vi)$, $L - T_2(vii)$ are also hold 'good extension' property.

**4.3.2Theorem:** Let $(X, \tau)$ be an lts, $A \subseteq X$ and $\tau_A = \{u|A : u \in \tau\}$, then

(a) $(X, \tau)$ is $L - T_2(i) \Rightarrow (A, \tau_A)$ is $L - T_2(i)$.

(b) $(X, \tau)$ is $L - T_2(ii) \Rightarrow (A, \tau_A)$ is $L - T_2(ii)$.

(c) $(X,\tau)$ is $L-T_2(iii) \Rightarrow (A,\tau_A)$ is $L-T_2(iii)$.

(d) $(X,\tau)$ is $L-T_2(iv) \Rightarrow (A,\tau_A)$ is $L-T_2(iv)$.

(e) $(X,\tau)$ is $L-T_2(v) \Rightarrow (A,\tau_A)$ is $L-T_2(v)$.

(f) $(X,\tau)$ is $L-T_2(vi) \Rightarrow (A,\tau_A)$ is $L-T_2(vi)$.

(g) $(X,\tau)$ is $L-T_2(vii) \Rightarrow (A,\tau_A)$ is $L-T_2(vii)$.

**Proof:** The author proved only (a). Suppose $(X,\tau)$ is L-topological space and $L-T_2(i)$. We shall prove $(A,\tau_A)$ is $L-T_2(i)$. Let $x,y \in A$ with $x \neq y$, then $x,y \in X$ with $x \neq y$ as $A \subseteq X$. Since $(X,\tau)$ is $L-T_2(i)$, $\exists\, u,v \in \tau$ such that $u(x)=1, u(y)=0, v(x)=0, v(y)=1$ and $u \cap v = 0$. For $A \subseteq X$ we find $u|A, v|A \in \tau_A$ and $u|A(x)=1, u|A(y)=0$ and $v|A(x)=0, v|A(y)=1$ and $u|A \cap v|A = (u \cap v)|A = 0$ as $x,y \in A$. Hence it is clear that the subspace $(A,\tau_A)$ is $L-T_2(i)$.

Similarly, (b), (c), (d), (e), (f), (g) can be proved.

**4.3.3 Theorem:** Given $\{(X_i,\tau_i): i \in \Lambda\}$ be a family of L-topological space. Then the product of L-topological space $(\Pi X_i, \Pi\,\tau_i)$ is $L-T_2(j)$ iff each coordinate space $(X_i,\tau_i)$ is $L-T_2(j)$ where $j = i, ii, iii, iv, v, vi, vii$.

**Proof:** Let each coordinate space $\{(X_i,\tau_i): i \in \Lambda\}$ be $L-T_2(i)$. We showed that the product space is $L-T_2(i)$. Suppose $x,y \in X$ with $x \neq y$, again suppose $x = \Pi x_i$, $y = \Pi y_i$ then $x_j \neq y_j$ for some $j \in \Lambda$. Now consider $x_j, y_j \in X_j$. Since $(X_j,\tau_j)$ is $L-T_2(i)$, $\exists\, u_j, v_j \in \tau_j$ such that $u_j(x_j) = 1, u_j(y_j) = 0, v_j(x_j) = 0, v_j(y_j) = 1$ and $u_j \cap v_j = 0$.

Now take $u = \Pi u'_j, v = \Pi v'_j$ where $u'_j = u_j, v'_j = v_j$ and $u_i = v_i = 1$ for $i \neq j$. Then $u, v \in \Pi\tau_i$ such that $u(x) = 1, u(y) = 0, v(x) = 0, v(y) = 1$ and $u \cap v = 0$. Hence the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - T_2(i)$.

Conversely, let the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - T_2(i)$. Take any coordinate space $(X_j, \tau_j)$, choose $x_j, y_j \in X_j, x_j \neq y_j$. Now construct $x, y \in X$ such that $x = \Pi x'_i, y = \Pi y'_i$ where $x'_i = y'_i$ for $i \neq j$ and $x'_j = x_j, y'_j = y_j$. Then $x \neq y$ and using the product space $L - T_2(i)$ $\exists\, u, v \in \Pi\tau_i$ such that $u(x) = 1, u(y) = 0, v(x) = 0,\ v(y) = 1$ and $u \cap v = 0$. Now choose any L-fuzzy point $x_r$ in $u$. Then $\exists$ a basic open L-fuzzy set $\Pi u_j^r \in \Pi\tau_j$ such that $x_r \in \Pi u_j^r \subseteq u$ which implies that

$r < \Pi u_j^r(x)$ or that $r < inf_j u_j^r(x'_j)$ and hence $r < \Pi u_j^r(x'_j) \forall\, j \in \Lambda \ldots \ldots (i)$

and $u(y) = 0 \Rightarrow\ \Pi u_j(y) = 0\ \ldots \ldots (ii)$.

Similarly, corresponding to a fuzzy point $y_s \in v$ there exists a basic fuzzy open set $\Pi v_j^s \in \Pi\tau_j$ such that $y_s \in \Pi v_j^s \subseteq v$ which implies that

$s < v_j^s(j) \forall\, j \in \Lambda \ldots \ldots (iii)$ and

$\Pi v_j^s(y) = 0 \ldots \ldots\ (iv)$. Further, $\Pi u_j^r(y) = 0 \Rightarrow u_i^r(y_i) = 0$, since for $j \neq i, x'_j = y'_j$ and hence from $(i)$, $u_j^r(y_j) = u_j^r(x_j) > r$. Similarly, $\Pi v_j^s(x) = 0 \Rightarrow v_i^s(x_i) = 0$ using $(iii)$. Thus we have $u_i^r(x_i) > r, u_i^r(y_i) = 0$ and $v_i^s(y_i) > s, v_i^s(x_i) = 0$.

Now consider $sup_r u_i^r = u_i, sup_s v_i^s = v_i \in \tau_i$ then $u_i(x_i) = 1, u_i(y_i) = 0, v_i(x_i) = 0, v_i(y_i) = 1$ and $u_i \cap v_i = 0$, showing that $(X_i, \tau_i)$ is $L - T_2(i)$.

Moreover one can verify that

$(X_i, \tau_i), i \in \Lambda$ is $L - T_2(ii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_2(ii)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - T_2(iii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_2(iii)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - T_2(iv) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_2(iv)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - T_2(v) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_2(v)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - T_2(vi) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_2(vi)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - T_2(vii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - T_2(vii)$.

Hence it is seen that $L - T_2(i), L - T_2(ii), L - T_2(iii), L - T_2(iv), L - T_2(v), L - T_2(vi), L - T_2(vii)$ Properties are productive and projective.

**4.4Mapping in L-topological spaces**

We showed that $L - T_2(j)$ property is preserved under one-one, onto and continuous mapping for $j = i, ii, iii, iv, v, vi, vii$.

**4.4.1 Theorem:** Let $(X, \tau)$ and $(Y, s)$ be two L-topological spaces and $f: (X, \tau) \rightarrow (Y, s)$ be one-one, onto and L-open map, then-

(a) $(X\,, \tau)$ is $L - T_2(i) \Rightarrow (Y, s)$ is $L - T_2(i)$.

(b) $(X, \tau)$ is $L - T_2(ii) \Rightarrow (Y, s)$ is $L - T_2(ii)$.

(c) $(X, \tau)$ is $L - T_2(iii) \Rightarrow (Y, s)$ is $L - T_2(iii)$.

(d) $(X, \tau)$ is $L - T_2(iv) \Rightarrow (Y, s)$ is $L - T_2(iv)$.

(e) $(X, \tau)$ is $L - T_2(v) \Rightarrow (Y, s)$ is $L - T_2(v)$.

(f) $(X, \tau)$ is $L - T_2(vi) \Rightarrow (Y, s)$ is $L - T_2(vi)$.

(g) $(X, \tau)$ is $L - T_2(vii) \Rightarrow (Y, s)$ is $L - T_2(vii)$.

**Proof:** Suppose $(X, \tau)$ is $L - T_2(i)$.We shall prove that $(Y, s)$ is $L - T_2(i)$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $f$ is onto, $\exists\, x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$ and $x_1 \neq x_2$ as $f$ is one-one. Again since $(X, \tau)$ is $L - T_2(i)$ $\exists\, u, v \in \tau$ such that $u(x_1) = 1$, $u(x_2) = 0$, $v(x_1) = 0$, $v(x_2) = 1$ and $u \cap v = 0$.

Now

$f(u)(y_1) = \{\sup u(x_1) : f(x_1) = y_1\} = 1$

$f(u)(y_2) = \{\sup u(x_2) : f(x_2) = y_2\} = 0$

$f(v)(y_1) = \{\sup v(x_1) : f(x_1) = y_1\} = 0$

$f(v)(y_2) = \{\sup v(x_2) : f(x_2) = y_2\} = 1$

and

$f(u \cap v)(y_1) = \{\sup(u \cap v)(x_1) : f(x_1) = y_1\}$

$f(u \cap v)(y_2) = \{\sup(u \cap v)(x_2) : f(x_2) = y_2\}$

Hence $f(u \cap v) = 0 \Rightarrow f(u) \cap f(v) = 0$

Since $f$ is L-open, $f(u), f(v) \in s$ . Now it is clear that $\exists\, f(u), f(v) \in s$ such that $f(u)(y_1) = 1$ , $f(u)(y_2) = 0$,$f(v)(y_1) = 0$, $f(v)(y_2) = 1$ and $f(u) \cap f(v) = 0$. Hence it is clear that the L-topological space $(Y, s)$ is$L - T_2(i)$. Similarly (b), (c), (d), (e), (f), (g) can be proved.

**4.4.2 Theorem:** Let $(X, \tau)$ and $(Y, s)$ be two L-topological spaces and $f: (X, \tau) \rightarrow (Y, s)$ be L-continuous and one-one map, then-

(a) $(Y\ , s)$ is $L - T_2(i) \Rightarrow (X, \tau)$ is $L - T_2(i)$.

(b) $(Y\ , s)$ is $L - T_2(ii) \Rightarrow (X, \tau)$ is $L - T_2(ii)$.

(c) $(Y\ , s)$ is $L - T_2(iii) \Rightarrow (X, \tau)$ is $L - T_2(iii)$.

(d) $(Y\ , s)$ is $L - T_2(iv) \Rightarrow (X, \tau)$ is $L - T_2(iv)$.

(e) $(Y\ , s)$ is $L - T_2(v) \Rightarrow (X, \tau)$ is $L - T_2(v)$.

(f) $(Y\ , s)$ is $L - T_2(vi) \Rightarrow (X, \tau)$ is $L - T_2(vi)$.

(g) $(Y\ , s)$ is $L - T_2(vii) \Rightarrow (X, \tau)$ is $L - T_2(vii)$.

**Proof:** Suppose $(Y, s)$ is $L - T_2(i)$.We shall prove that $(X, \tau)$ is $L - T_2(i)$. Let $x_1\ , x_2 \in X$ with $x_1 \neq x_2, \Rightarrow f(x_1) \neq f(x_2)$ as $f$ is one-one. Since $(Y, s)$ is $L - T_2(i), \exists\, u, v \in s$ such that $u(f(x_1)) = 1, u(f(x_2)) = 0$,$v(f(x_1)) = 0, v(f(x_2)) = 1$ and $u \cap v = 0$. This implies that $f^{-1}(u)(x_1) = 1, f^{-1}(u)(x_2) = 0$,$f^{-1}(v)(x_1) = 0, f^{-1}(v)(x_2) = 1$ and $f^{-1}(u \cap v) = 0 \Rightarrow f^{-1}(u) \cap f^{-1}(v) = 0$. Hence $f^{-1}(u), f^{-1}(v) \in \tau$ as $f$ is L-continuous and $u, v \in s$ . Now it is clear that $f^{-1}(u), f^{-1}(v) \in \tau$ such that $f^{-1}(u)(x_1) = 1$ ,$f^{-1}(u)(x_2) = 0$, $f^{-1}(v)(x_1) = 0, f^{-1}(v)(x_2) = 1$ and $f^{-1}(u) \cap f^{-1}(v) = 0$. Hence the L-topological space $(X, \tau)$ is $L - T_2(i)$. Similarly (b), (c), (d), (e), (f), (g) can be proved.

# Chapter-5

## On $R_0$ Space in L-Topological Spaces

### 5.1: Introduction

The concept of $R_0$-property first defined by Shanin[64] and there after Dude[16], Naimpally[57], Dorsett[15], Caldas[10], Ekici[18], as earlier Keskin[38] and Roy[62] defined many characterizations of $R_0$-properties. The concepts of fuzzy $R_0$-propertise are established and discussed by Hutton[33,34], Srivastava[78], Ali[9], Khedr[40], Zhang[92] and many other fuzzy topologist. In this chapter we define possible eight definitions of $R_0$ space in L-topological spaces and we show that this space possesses many nice properties which are hereditary, productive and projective.

### 5.2 $R_0$-property in L-Topological Spaces

We now give the following definitions of $R_0$-property in L-topological spaces.

**5.2.1Definition:** An lts $(X, \tau)$ is called-

(a) $L-R_0(i)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, u \in \tau$ with $u(x) \neq u(y)$ then $\exists\, v \in \tau$ such that $v(x) \neq v(y)$.

(b) $L-R_0(ii)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, u \in \tau$ with $u(x) = 1, u(y) = 0$ then $\exists\, v \in \tau$ such that $v(x) = 0, v(y) = 1$.

(c) $L - R_0(iii)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ whenever $\exists\, u \in \tau$ with $x_r \in u, y_s \notin u$ then $\exists\, v \in \tau$ such that $x_r \notin v, y_s \in v$.

(d) $L - R_0(iv)$ if for all pairs of distinct L-fuzzy singletons $x_r, y_s \in S(X)$ and $x_r \bar{q} y_s$ whenever $\exists\, u \in \tau$ with $x_r \subseteq u, y_s \bar{q} u$ then $\exists v \in \tau$ such that $y_s \subseteq v, x_r \bar{q} v$.

(e) $L - R_0(v)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$whenever $\exists\, u \in \tau$ with $x_r \in u, u \bar{q} y_s$ then $\exists v \in \tau$ such that $y_s \in v\,, v \bar{q} x_r$.

(f) $L - R_0(vi)$ if for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ whenever $\exists\, u \in \tau$ with $x_r \in u, y_s \cap u = 0$ then $\exists v \in \tau$ such that $y_s \in v, x_r \cap v = 0$.

(g) $L - R_0(vii)$ if $\forall\, x\,, y \in X, x \neq y$ whenever $\exists\, u \in \tau$ with $u(x) > 0, u(y) = 0$ then $\exists v \in \tau$ such that $v(x) = 0, v(y) > 0$.

(h) $L - R_0(viii)$ if $\forall\, x\,, y \in X, x \neq y$ whenever $\exists\, u \in \tau$ with $u(x) > u(y)$ then $\exists v \in \tau$ such that $v(y) > v(x)$.

## 5.3 "Good extension", Hereditary, Productive and Projective Properties in L-Topology

Now all the definitions $L - R_0(i), L - R_0(ii), L - R_0(iii), L - R_0(iv)$, $L - R_0(v), L - R_0(vi), L - R_0(vii)$and $L - R_0(viii)$ are 'good extensions' of $R_0 -$ property, is shown below:

**5.3.1Theorem:** Let $(X,T)$ be a topological space. Then $(X,T)$ is $R_0$ iff $(X,\omega(T))$ is $L-R_0(j)$, for $j=i,ii,iii,iv,v,vi,vii,viii$.

**Proof:** Let the topological space $(X,T)$ be $R_0$, we shall prove that the fuzzy topological space $(X,\omega(T))$ is $L-R_0(ii)$. Choose $x,y\in X$ with $x\neq y$. Let $u\in\omega(T)$ with $u(x)=1,\ u(y)=0$, then it is clear that $u^{-1}(r,1]\in T$, for any $r\in \mathrm{I}_1$ and $x\in u^{-1}(r,1]$, $y\notin u^{-1}(r,1]$. Since $(X,T)$ is $R_0$, then there exist $V\in T$ with $x\notin V, y\in V$. Now consider the characteristics function $1_V$. We see that $1_V\in\omega(T)$with $1_V(x)=0, 1_V(y)=1$. Thus $(X,\omega(T))$ is $L-R_0(ii)$.

Conversely, let $(X,\omega(T))$ be $L-R_0(ii)$, we shall prove that $(X,T)$ is $R_0$. Choose $x,y\in X$, $x\neq y$ and $U\in T$, with $x\in U, y\notin U$, but we know that the characteristic function $1_U\in\omega(T)$. Also it is clear that $1_U(x)=1, 1_U(y)=0$. Since $(X,\omega(T))$ is $L-R_0(ii)$, then $\exists v\in\omega(T)$ such that $v(x)=0, v(y)=1$. Again since $v$ is lower semi continuous function then $v^{-1}(0,1]\in T$ and from above, we get $x\notin v^{-1}(0,1]$, $y\in v^{-1}(0,1]$. Hence $(X,T)$ is $R_0$.

Similarly we can show that $L-R_0(i), L-R_0(iii), L-R_0(iv), L-R_0(v)$, $L-R_0(vi), L-R_0(vii), L-R_0(viii)$are also hold 'good extension' property.

**5.3.2Theorem:** Let $(X,\tau)$ be an lts, $A\subseteq X$ and $\tau_A=\{u|A: u\in\tau\}$, then

(a) $(X,\tau)$ is $L-R_0(i)\Rightarrow(A,\tau_A)$ is $L-R_0(i)$.

(b) $(X,\tau)$ is $L-R_0(ii)\Rightarrow(A,\tau_A)$ is $L-R_0(ii)$.

(c) $(X, \tau)$ is $L - R_0(iii) \Rightarrow (A, \tau_A)$ is $L - R_0(iii)$.

(d) $(X, \tau)$ is $L - R_0(iv) \Rightarrow (A, \tau_A)$ is $L - R_0(iv)$.

(e) $(X, \tau)$ is $L - R_0(v) \Rightarrow (A, \tau_A)$ is $L - R_0(v)$.

(f) $(X, \tau)$ is $L - R_0(vi) \Rightarrow (A, \tau_A)$ is $L - R_0(vi)$.

(g) $(X, \tau)$ is $L - R_0(vii) \Rightarrow (A, \tau_A)$ is $L - R_0(vii)$.

(h) $(X, \tau)$ is $L - R_0(viii) \Rightarrow (A, \tau_A)$ is $L - R_0(viii)$.

**Proof:** We prove only (b). Suppose $(X, \tau)$ is L-topological space and $L - R_0(ii)$.We shall prove $(A, \tau_A)$ is $L - R_0(ii)$. Let $x, y \in A$, $x \neq y$, and $w \in \tau_A$ with $w(x) = 1$, $w(y) = 0$. Then $x, y \in X$ with $x \neq y$ as $A \subseteq X$. Consider $u$ be the extension function of $w$ on the set X, then it is clear that $u(x) = 1, u(y) = 0$. Since $(X, \tau)$ is $L - R_0(ii)$. Then $\exists\, v \in \tau$ such that $v(x) = 0, v(y) = 1$. For $A \subseteq X$, we find $\exists\, v|A \in \tau_A$ such that $v|A(x) = 0, v|A(y) = 1$ as $x, y \in A$. Hence it is clear that the subspace $(A, \tau_A)$ is $L - R_0(ii)$. Similarly, (a), (c), (d), (e), (f), (g), (h) can be proved.

**5.3.3Theorem:** Given $\{(X_i, \tau_i): i \in \Lambda\}$ be a family of L-topological space. Then the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(j)$ iff each coordinate space $(X_i, \tau_i)$ is $L - R_0(j)$ where $j = i, ii, iii, iv, v, vi, vii, viii$.

**Proof:** Let each coordinate space $\{(X_i, \tau_i): i \in \Lambda\}$ be $L - R_0(ii)$. Then we show that the product space is $L - R_0(ii)$. Suppose $x, y \in \Pi X_i$, $x \neq y$, and $u \in \Pi\, \tau_i$ with $u(x) = 1, u(y) = 0$. Choose $x = \Pi x_i$, $y = \Pi y_i$, but we have $u(x) = \min\{u_i(x_i)$, for $i \in \Lambda$ and $u_i \in \tau_i\}$,

$u(y) = \min\{u_i(y_i)$ , for $i \in \Lambda$ and $u_i \in \tau_i\}$ , then there exist at least one $j \in \Lambda$, such that $x_j \neq y_j$ and $u_j(x_j) = 1$, $u_j(y_j) = 0$. Since $(X_j, \tau_j)$ is $L - R_0(ii)$ for each $j \in \Lambda$, then $\exists\, v_j \in \tau_j$ such that $v_j(x_j) = 0, v_j(y_j) = 1$. Now take, $v = \Pi v'_j$ where $v'_j = v_j$ and $v_i = 1$ for $i \neq j$. then $\exists v \in \Pi\tau_i$ such that $v(x) = 0, v(y) = 1$. Hence the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is

$L - R_0(ii)$.

Conversely, let the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(ii)$. We shall prove the each coordinate space $(X_j, \tau_j)$ is also $L - R_0(ii)$. Choose $x_j, y_j \in X_j, x_j \neq y_j$ and $u_j \in \tau_j$ with $u_j(x_j) = 1, u_j(y_j) = 0$. Now construct $x, y \in X$ such that $x = \Pi x'_i, y = \Pi y'_i$ where $x'_i = y'_i$ for $i \neq j$ and $x'_j = x_j, y'_j = y_j$. Then $x \neq y$. Further, let $\pi_j: X \to X_j$ be a projection map from $X$ into $X_j$. Now, we observe that $u_j\left((\pi_j)(x)\right) = u_j(x_j) = 1, u_j\left((\pi_j)(y)\right) = u_j(y_j) = 0$, i.e for $u_j o\pi_j \in \Pi\, \tau_i$, with $(u_j o\pi_j)(x) = 1, (u_j o\pi_j)(y) = 1$. Since the product space $(\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(ii)$. Then $\exists v \in \Pi\tau_i$ such that $v(x) = 0, v(y) = 1$. Now choose any L-fuzzy point $y_r$ in $v$. Then $\exists$ a basic open L-fuzzy set $\Pi v_j^r \in \Pi\tau_j$ such that $y_r \in \Pi v_j^r \subseteq v$ which implies that $r < \Pi v_j^r(y)$ or that $r < inf_j v_j^r(y'_j)$ and hence $r < \Pi v_j^r(y'_j) \forall\, j \in \Lambda \ldots \ldots (i)$ and $v(x) = 0 \Rightarrow \Pi v_j(x) = 0 \ldots \ldots (ii)$. Further, $\Pi v_j^r(x) = 0 \Rightarrow v_i^r(x_i) = 0$, since for $j \neq i, x'_j = y'_j$ and hence from $(i)$, $u_j^r(x_j) = v_j^r(y_j) > r$. Thus we have $v_i^r(y_i) > r, v_i^r(x_i) = 0$.

Now consider $sup_r v_i^r = v_i$ then $v_i \in \tau_i$ with $v_i(y_i) = 1, v_i(x_i) = 0$. This showing that $(X_i, \tau_i)$ is $L - R_0(ii)$.

Moreover one can easily verify that

$(X_i, \tau_i), i \in \Lambda$ is $L - R_0(i) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(i)$

$(X_i, \tau_i), i \in \Lambda$ is $L - R_0(iii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(iii)$

$(X_i, \tau_i), i \in \Lambda$ is $L - R_0(iv) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(iv)$

$(X_i, \tau_i), i \in \Lambda$ is $L - R_0(v) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(v)$

$(X_i, \tau_i), i \in \Lambda$ is $L - R_0(vi) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(vi)$

$(X_i, \tau_i), i \in \Lambda$ is $L - R_0(vii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(vii)$

$(X_i, \tau_i), i \in \Lambda$ is $L - R_0(viii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_0(viii)$.

Hence we see that $L - R_0(i), L - R_0(ii), L - R_0(iii), L - R_0(iv)$, $L - R_0(v), L - R_0(vi), L - R_0(vii), L - R_0(viii)$ Properties are productive and projective.

### 5.4 Mapping in L-topological spaces

We show that $L - R_0(j)$ property is preserved under one-one, onto and continuous mapping for $j = i, ii, iii, iv, v, vi, vii, viii$.

**5.4.1Theorem:** Let $(X, \tau)$ and $(Y, s)$ be two L-topological spaces and $f: (X, \tau) \to (Y, s)$ be one-one, onto, L-continuous and L-open map, then-

(a) $(X, \tau)$ is $L - R_0(i) \Rightarrow (Y, s)$ is $L - R_0(i)$.

(b) $(X, \tau)$ is $L - R_0(ii) \Rightarrow (Y, s)$ is $L - R_0(ii)$.

(c) $(X, \tau)$ is $L - R_0(iii) \Rightarrow (Y, s)$ is $L - R_0(iii)$.

(d) $(X, \tau)$ is $L - R_0(iv) \Rightarrow (Y, s)$ is $L - R_0(iv)$.

(e) $(X, \tau)$ is $L - R_0(v) \Rightarrow (Y, s)$ is $L - R_0(v)$.

(f) $(X, \tau)$ is $L - R_0(vi) \Rightarrow (Y, s)$ is $L - R_0(vi)$.

(g) $(X, \tau)$ is $L - R_0(vii) \Rightarrow (Y, s)$ is $L - R_0(vii)$.

(h) $(X, \tau)$ is $L - R_0(viii) \Rightarrow (Y, s)$ is $L - R_0(viii)$.

**Proof:** Suppose $(X, \tau)$ is $L - R_0(ii)$.We shall prove that $(Y, s)$ is $L - R_0(ii)$. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$ and $u \in s$ with $u(y_1) = 1$ , $u(y_2) = 0$. Since $f$ is onto, $\exists\, x_1, x_2 \in X$ such that $f(x_1) = y_1$ , $f(x_2) = y_2$ and $x_1 \neq x_2$ as $f$ is one-one.

Now $f^{-1}(u)(x_1) = u(f(x_1)) = u(y_1) = 1$ and

$f^{-1}(u)(x_2) = u(f(x_2)) = u(y_2) = 0$

Since $f$ is L-continuous then $f^{-1}(u) \in t$ and $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) = 0$. Since $(X, \tau)$ is $L - R_0(ii)$, then $\exists\, v \in \tau$ such that $v(x) = 0, v(y) = 1$.

Now

$f(v)(y_1) = \{sup v(x_1): f(x_1) = y_1\} = 0$

$f(v)(y_2) = \{sup v(x_2): f(x_2) = y_2\} = 1.$

Since $f$ is L-open, $f(u) \in s$. Now it is clear that $\exists\, f(v) \in s$ such that $f(v)(y_1) = 0$, $f(v)(y_2) = 1$. Hence it is clear that the L-topological space $(Y, s)$ is $L - R_0(ii)$ .

Similarly (a), (c), (d), (e), (f), (g), (h) can be proved.

**5.4.2Theorem:** Let $(X, \tau)$ and $(Y, s)$ be two L-topological spaces and $f: (X, \tau) \to (Y, s)$ be one-one, L-continuous and L-open map, then-

(a) $(Y\ , s)$ is $L - R_0(i) \Rightarrow (X, \tau)$ is $L - R_0(i)$.

(b) $(Y\ , s)$ is $L - R_0(ii) \Rightarrow (X, \tau)$ is $L - R_0(ii)$.

(c) $(Y\ , s)$ is $L - R_0(iii) \Rightarrow (X, \tau)$ is $L - R_0(iii)$.

(d) $(Y\ , s)$ is $L - R_0(iv) \Rightarrow (X, \tau)$ is $L - R_0(iv)$.

(e) $(Y\ , s)$ is $L - R_0(v) \Rightarrow (X, \tau)$ is $L - R_0(v)$.

(f) $(Y\ , s)$ is $L - R_0(vi) \Rightarrow (X, \tau)$ is $L - R_0(vi)$.

(g) $(Y\ , s)$ is $L - R_0(vii) \Rightarrow (X, \tau)$ is $L - R_0(vii)$.

(h) $(Y\ , s)$ is $L - R_0(viii) \Rightarrow (X, \tau)$ is $L - R_0(viii)$.

**Proof:** Suppose $(Y, s)$ is $L - R_0(ii)$. We shall prove that $(X, \tau)$ is

$L - R_0(ii)$. Let $x_1\, , x_2 \in X$, $x_1 \neq x_2$ and $u \in \tau$ with $u(x_1) = 1, u(x_2) = 0$.

Since $f$ is one-one map then $f(x_1) \neq f(x_2)$.

Now $f(u)\big(f(x_1)\big) = \sup\{u(x_1)\} = 1$ as $f$ is one-one

And $f(u)\big(f(x_2)\big) = \sup\{u(x_2)\} = 0$

So, we have $f(u) \in s$, with $f(u)\big(f(x_1)\big) = 1,\ f(u)\big(f(x_2)\big) = 0$, as $f$ is L-open map. Since $(Y, s)$ is $L - R_0(ii)$, then $\exists v \in s$ such that $v\big(f(x_1)\big) = 0, v(f(x_2)) = 1$. This implies that $f^{-1}(v)(x_1) = 0, f^{-1}(v)(x_2) = 1$ and $f^{-1}(v) \in \tau$ as $f$ is L-continuous and $v \in s$. Now it is clear that $\exists\ f^{-1}(v) \in \tau$ such that $f^{-1}(v)(x_1) = 0,\ f^{-1}(v)(x_2) = 1$. Hence the L-topological space $(X, \tau)$ is $L - R_0(ii)$.

Similarly (a), (c), (d), (e), (f), (g), (h) can be proved.

# Chapter-6

## On $R_1$ Space in L-Topological Spaces

### 6.1: Introduction

The concept of $R_1$-property first defined by Yang [87] and there after Murdeshwar[56], Dorset[15], Dude[17], Caldas[10], Ekici[18], as earlier Keskin[38] and Roy[62] defined many characterizations of $R_1$-properties. The concepts of fuzzy $R_1$-propertise are established and discussed by Hutton[31,32], Srivastava[76], Ali[8], Khedr[40], Kandil[36], Hossain[28] and many other fuzzy topologist.

In this chapter we define possible seven definitions of $R_1$ space in L-topological spaces. All these definitions satisfy 'good extension' property and we establish some implications among them. Finally we show that all these definitions are hereditary, productive and projective and preserved under one-one, onto and continuous maps.

### 6.2 $R_1$-property in L-Topological Spaces

We now give the following definitions of $R_1$-property in L-topological spaces.

**6.2.1 Definition:** An lts $(X, \tau)$ is called-

(a) $L-R_1(i)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then $\exists\, u, v \in \tau$ such that $u(x) = 1, u(y) = 0, v(x) = 0$, $v(y) = 1$ and $u \cap v = 0$.

(b) $L-R_1(ii)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ and $\exists\, u, v \in \tau$ such that $x_r \in u, y_s \notin u$ and $x_r \notin v, y_s \in v$ , $u \cap v = 0$.

(c) $L-R_1(iii)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then for all pairs of distinct L-fuzzy singletons $x_r, y_s \in S(X), x_r \bar{q} y_s$ and $\exists\, u, v \in \tau$ such that $x_r \subseteq u, y_s \bar{q} u$ and $y_s \subseteq v, x_r \bar{q} v$ and $u \cap v = 0$.

(d) $L-R_1(iv)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ and $\exists\, u, v \in \tau$ such that $x_r \in u, u \bar{q} y_s$ and $y_s \in v$ , $v \bar{q} x_r$ and $u \cap v = 0$.

(e) $L-R_1(v)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ and for any pair of distinct L-fuzzy points $x_r, y_s \in S(X)$ and $\exists\, u, v \in \tau$ such that $x_r \in u \subseteq co y_s$, $y_s \in v \subseteq co x_r$ and $u \subseteq co v$.

(f) $L-R_1(vi)$ if $\forall\, x, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then $\exists\, u, v \in \tau$ such that $u(x) > 0, u(y) = 0$ and $v(x) = 0, v(y) > 0$ .

(g) $L - R_1(vii)$ if $\forall\, x\,, y \in X, x \neq y$ whenever $\exists\, w \in \tau$ with $w(x) \neq w(y)$ then $\exists\, u, v \in \tau$ such that $u(x) > u(y)$ and $v(y) > v(x)$.

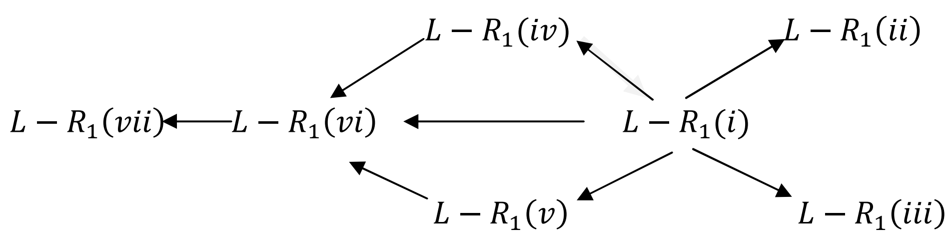
Here, we established a complete comparison of the definitions

$L - R_1(ii), L - R_1(iii),\ L - R_1(iv), L - R_1(v), L - R_1(vi)$ and

$L - R_1(vii)$ with $L - R_1(i)$.

**6.2.2Theorem:** Let $(X\,, \tau)$ be an lts. Then we have the following

implications:




The reverse implications are not true in general except $L - R_1(vi)$ and

$L - R_1(vii)$.

**<u>Proof</u>:** $L - R_1(i) \Rightarrow L - R_1(ii)\,, L - R_1(i) \Rightarrow L - R_1(iii)$can be proved easily. Now $L - R_1(i) \Rightarrow L - R_1(iv)$and $L - R_1(i) \Rightarrow L - R_1(v)$, since $L - R_1(ii) \Leftrightarrow L - R_1(iv)$and $L - R_1(iv) \Leftrightarrow L - R_1(v)$. $L - R_1(i) \Rightarrow L - R_1(vi)$; It is obvious. $L - R_1(i) \Rightarrow L - R_1(vii)$, since $L - R_1(vi) \Rightarrow L - R_1(vii)$.

The reverse implications are not true in general except $L - R_1(vi)$ and $L - R_1(vii)$, it can be seen through the following counter examples:

Example-1: Let $X = \{x, y\}$, $\tau$ be the L-topology on $X$ generated by $\{\alpha: \alpha \in L\} \cup \{u, v, w\}$ where $w(x) = 0.6, w(y) = 0.7, u(x) = 0.5, u(y) = 0$ $v(x) = 0, v(y) = 0.6$, $L = \{0,0.05,0.1,0.15, \dots \dots \dots 0.95,1\}$ and $r = 0.4, s = 0.3$.

Example-2: Let $X = \{x, y\}$, $\tau$ be the L-topology on $X$ generated by $\{\alpha: \alpha \in L\} \cup \{u, v, w\}$ where $w(x) = 0.8, w(y) = 0.9, u(x) = 0.5, u(y) = 0$ $v(x) = 0, v(y) = 0.4$, $L = \{0,0.05,0.1,0.15, \dots \dots \dots 0.95,1\}$ and $r = 0.5, s = 0.4$.

**<u>Proof</u>:** $L - R_1(ii) \nRightarrow L - R_1(i)$: From example-1, we see that the lts $(X, \tau)$ is clearly $L - R_1(ii)$ but it is not $L - R_1(i)$. Since there is no L-fuzzy set in $\tau$ which grade of membership is 1.

$L - R_1(iii) \nRightarrow L - R_1(i)$: From example-2, we see the lts $(X, \tau)$ is clearly $L - R_1(iii)$ but it is not $L - R_1(ii)$. Since $L - R_1(iii) \nRightarrow L - R_1(ii)$ and $L - R_1(ii) \nRightarrow L - R_1(i)$ so $L - R_1(iii) \nRightarrow L - R_1(i)$.

$L - R_1(iv) \nRightarrow L - R_1(i)$: This follows automatically from the fact that

$L - R_1(ii) \Leftrightarrow L - R_1(iv)$ and it has already been shown that $L - R_1(ii) \nRightarrow$

$L - R_1(i)$ so $L - R_1(iv) \nRightarrow L - R_1(i)$.

$L - R_1(v) \nRightarrow L - R_1(i)$: Since $L - R_1(iv) \Leftrightarrow L - R_1(v)$and $L - R_1(iv) \nRightarrow L - R_1(i)$ so $L - R_1(v) \nRightarrow L - R_1(i)$. But $L - R_1(vii) \Rightarrow L - R_1(vi) \Rightarrow$

$L - R_1(i)$ is obvious.

**6.3 "Good extension", Hereditary, Productive and Projective Properties in L-Topology**

We show that all definitions $L - R_1(i), L - R_1(ii), L - R_1(iii)$,

$L - R_1(iv), L - R_1(v)$, $L - R_1(vi)$ and $L - R_1(vii)$ are 'good extensions' of $R_1 -$ property, is shown below:

**6.3.1Theorem:** Let $(X, T)$ be a topological space. Then $(X, T)$ is $R_1$ iff $(X, \omega(T))$ is $L - R_1(j)$, where $j = i, ii, iii, iv, v, vi, vii$.

**Proof:** Let $(X, T)$ be $R_1$. Choose $x, y \in X$, $x \neq y$. Whenever $\exists\, W \in T$ with $x \in W, y \notin W$ or $x \notin W, y \in W$ then $\exists\, U, V \in T$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$ and $U \cap V = \emptyset$. Suppose $x \in W, y \notin W$ since $W \in T$ then $1_w \in \omega(T)$ with $1_w(x) \neq 1_w(y)$. Also consider the lower semi continuous function $1_U, 1_V$, then $1_U, 1_V \in \omega(T)$ such that $1_U(x) = 1, 1_U(y) = 0$ and $1_V(x) = 0, 1_V(y) = 1$ and so that $1_U \cap 1_V = 0$ as $U \cap V = \emptyset$. Thus $(X, \omega(T))$ is $L - R_1(i)$.

Conversely, let $(X, \omega(T))$ be $L - R_1(i)$. To show that $(X, T)$ is $R_1$. Choose $x, y \in X$ with $x \neq y$. Whenever $\exists\, w \in T$ with $w(x) \neq w(y)$ then $\exists\, u, v \in \omega(T)$ such that $u(x) = 1, u(y) = 0, v(x) = 0, v(y) = 1$ and $u \cap v = 0$. Since $w(x) \neq w(y)$, then either $w(x) < w(y)$ or $w(x) > w(y)$. Choose $w(x) < w(y)$, then $\exists\, s \in L$ such that $w(x) < s < w(y)$. So it is clear that $w^{-1}(s, 1] \in T$ and $x \notin w^{-1}(s, 1]$, $y \in w^{-1}(s, 1]$. Let $U = u^{-1}\{1\}\, and\, V =$

$v^{-1}\{1\}$, then $U,\ V \in T$ and is $x \in U, y \notin U \ \ , x \notin V,\ y \in V$, and $U \cap V = \emptyset$ as $u \cap v = 0$ . Hence $(X, T)$ is $R_1$.

Similarly, we can show that $L - R_1(ii), L - R_1(iii), L - R_1(iv)$,

$L - R_1(v), L - R_1(vi),\ L - R_1(vii)$ are also hold 'good extension' property.

**6.3.2 Theorem:** Let $(X, \tau)$ be an lts, $A \subseteq X$ and $\tau_A = \{u|A : u \in \tau\}$, then

(a) $(X, \tau)$ is $L - R_1(i) \Rightarrow (A, \tau_A)$ is $L - R_1(i)$.

(b) $(X, \tau)$ is $L - R_1(ii) \Rightarrow (A, \tau_A)$ is $L - R_1(ii)$.

(c) $(X, \tau)$ is $L - R_1(iii) \Rightarrow (A, \tau_A)$ is $L - R_1(iii)$.

(d) $(X, \tau)$ is $L - R_1(iv) \Rightarrow (A, \tau_A)$ is $L - R_1(iv)$.

(e) $(X, \tau)$ is $L - R_1(v) \Rightarrow (A, \tau_A)$ is $L - R_1(v)$.

(f) $(X, \tau)$ is $L - R_1(vi) \Rightarrow (A, \tau_A)$ is $L - R_1(vi)$.

(g) $(X, \tau)$ is $L - R_1(vii) \Rightarrow (A, \tau_A)$ is $L - R_1(vii)$.

**Proof:** We prove only (a). Suppose $(X, \tau)$ is L-topological space and is also $L - R_1(i)$.We shall prove that $(A, \tau_A)$ is $L - R_1(i)$. Let $x, y \in A$ with $x \neq y$ and $\exists\, w \in \tau_A$ such that $w(x) \neq w(y)$, then $x, y \in X$ with $x \neq y$ as $A \subseteq X$. Consider $m$ be the extension function of $w$ on X, then $m(x) \neq m(y)$, Since $(X, \tau)$ is $L - R_1(i)$, $\exists\, u, v \in \tau$ such that $u(x) = 1, u(y) = 0, v(x) = 0, v(y) = 1$ and $u \cap v = 0$. For $A \subseteq X$, we find ,$u|A, v|A \in \tau_A$ and $u|A(x) = 1, u|A(y) = 0$ and $v|A(x) = 0, v|A(y) = 1$ and $u|A \cap v|A = (u \cap v)|A = 0$ as $x, y \in A$. Hence it is clear that the subspace $(A, \tau_A)$ is $L - R_1(i)$.

Similarly, (b), (c), (d), (e), (f), (g) can be proved.

**6.3.3Theorem:** Given $\{(X_i, \tau_i): i \in \Lambda\}$ be a family of L-topological space. Then the product of L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(j)$ iff each coordinate space $(X_i, \tau_i)$ is $L - R_1(j)$, where $j = i, ii, iii, iv, v, vi, vii$.

**<u>Proof</u>:** Let each coordinate space $\{(X_i, \tau_i): i \in \Lambda\}$ be $L - R_1(i)$. Then we show that the product space is $L - R_1(i)$. Suppose $x\,, y \in X$ with $x \neq y$ and $w \in \Pi\, \tau_i$ with $w(x) \neq w(y)$, again suppose $x = \Pi x_i,\ y = \Pi y_i$ then $x_j \neq y_j$ for some $j \in \Lambda$.But we have $w(x) = \min\{w_i(x_i):\ i \in \Lambda\}$, and $w(y) = \min\{w_i(y_i):\ i \in \Lambda\}$. Hence we can find at least one $w_j \in \tau_j$ with $w_j(x_j) \neq w_j(y_j)$, since each $(X_i, \tau_i): i \in \Lambda$ be $L - R_1(i)$ then $\exists\, u_j, v_j \in \tau_j$ such that $u_j(x_j) = 1, u_j(y_j) = 0, v_j(x_j) = 0, v_j(y_j) = 1$ and $u_j \cap v_j = 0$. Now take $u = \Pi u'_j, v = \Pi v'_j$ where $u'_j = u_j\,, v'_j = v_j$ and $u_i = v_i = 1$ for $i \neq j$. Then $u, v \in \Pi\tau_i$ such that $u(x) = 1, u(y) = 0, v(x) = 0, v(y) = 1$and $u \cap v = 0$. Hence the product of

L-topological space is also L-topological space and $(\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(i)$.

Conversely, let the product L-topological space $(\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(i)$. Take any coordinate space $(X_j, \tau_j)$, choose $x_j\,, y_j \in X_j\,, x_j \neq y_j$ and $w_i \in \Pi\tau_i$ with $w_i(x_i) \neq w_i(y_i)$. Now construct $x, y \in X$ such that $x = \Pi x'_i, y = \Pi y'_i$ where $x'_i = y'_i$ for $i \neq j$ and $x'_j = x_j, y'_j = y_j$. Then $x \neq y$ and using the product space $L - R_1(i)$, $\Pi w_i \in \Pi\tau_i$ with $\Pi w_i(x_i) \neq \Pi w_i(y_i)$, since

$(\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(i)$ then $\exists\, u, v \in \Pi\tau_i$ such that $u(x) = 1, u(y) = 0, v(x) = 0,\ v(y) = 1$ and $u \cap v = 0$. Now choose any L-fuzzy point $x_r$ in $u$. Then $\exists$ a basic open L-fuzzy set $\Pi u_j^r \in \Pi\tau_j$ such that $x_r \in \Pi u_j^r \subseteq u$ which implies that $r < \Pi u_j^r(x)$ or that $r < inf_j u_j^r(x_j')$

and hence $r < \Pi u_j^r(x_j') \forall\, j \in \Lambda \ldots \ldots (i)$ and

$u(y) = 0 \Rightarrow\ \Pi u_j(y) = 0\ \ldots \ldots (ii)$.

Similarly, corresponding to a fuzzy point $y_s \in v$ there exists a basic fuzzy open set $\Pi v_j^s \in \Pi\tau_j$ such that $y_s \in \Pi v_j^s \subseteq v$ which implies that

$s < v_j^s(j) \forall\, j \in \Lambda \ldots \ldots (iii)$ and

$\Pi v_j^s(y) = 0 \ldots \ldots (iv)$. Further, $\Pi u_j^r(y) = 0 \Rightarrow u_i^r(y_i) = 0$, since for $j \neq i, x_j' = y_j'$ and hence from $(i)$, $u_j^r(y_j) = u_j^r(x_j) > r$. Similarly, $\Pi v_j^s(x) = 0 \Rightarrow v_i^s(x_i) = 0$ using $(iii)$.

Thus we have $u_i^r(x_i) > r, u_i^r(y_i) = 0$ and $v_i^s(y_i) > s, v_i^s(x_i) = 0$. Now consider $sup_r u_i^r = u_i, sup_s v_i^s = v_i$, then $u_i(x_i) = 1, u_i(y_i) = 0, v_i(x_i) = 0, v_i(y_i) = 1$ and $u_i \cap v_i = 0$, showing that $(X_i, \tau_i)$ is $L - R_1(i)$.

Moreover one can easily verify that

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(ii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(ii)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(iii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(iii)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(iv) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(iv)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(v) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(v)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(vi) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(vi)$.

$(X_i, \tau_i), i \in \Lambda$ is $L - R_1(vii) \Leftrightarrow (\Pi X_i, \Pi\, \tau_i)$ is $L - R_1(vii)$.

Hence, we see that $L - R_1(i), L - R_1(ii), L - R_1(iii), L - R_1(iv)$,

$L - R_1(v), L - R_1(vi), L - R_1(vii)$ Properties are productive and projective.

### 6.4 Mapping in L-topological spaces

We show that $L - R_1(j)$ property is preserved under one-one, onto and continuous mapping for $j = i, ii, iii, iv, v, vi, vii$.

**6.4.1Theorem:** Let $(X, \tau)$ and $(Y, s)$ be two L-topological space and $f: (X, \tau) \to (Y, s)$ be one-one, onto L-continuous and L-open map, then-

(a) $(X\,, \tau)$ is $L - R_1(i) \Rightarrow (Y, s)$ is $L - R_1(i)$.

(b) $(X\,, \tau)$ is $L - R_1(ii) \Rightarrow (Y, s)$ is $L - R_1(ii)$.

(c) $(X\,, \tau)$ is $L - R_1(iii) \Rightarrow (Y, s)$ is $L - R_1(iii)$.

(d) $(X\,, \tau)$ is $L - R_1(iv) \Rightarrow (Y, s)$ is $L - R_1(iv)$.

(e) $(X\,, \tau)$ is $L - R_1(v) \Rightarrow (Y, s)$ is $L - R_1(v)$.

(f) $(X\,, \tau)$ is $L - R_1(vi) \Rightarrow (Y, s)$ is $L - R_1(vi)$.

(g) $(X\,, \tau)$ is $L - R_1(vii) \Rightarrow (Y, s)$ is $L - R_1(vii)$.

**Proof:** Suppose $(X,\tau)$ is $L-R_1(i)$.We shall prove that $(Y,s)$ is $L-R_1(i)$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$ and $w \in s$ with $w(y_1) \neq w(y_2)$. Since $f$ is onto then $\exists\, x_1,\ x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$, also $x_1 \neq x_1$, as $f$ is one-one. Now we have $f^{-1}(w) \in \tau$, Since $f$ is L-continuous, also we have $f^{-1}(w)(x_1) = wf(x_1) = w(y_1)$ and $f^{-1}(w)(x_2) = wf(x_2) = w(y_2)$.Therefore $f^{-1}(w)(x_1) \neq f^{-1}(w)(x_2)$. Again since $(X,\tau)$ is $L-R_1(i)$ and $\exists f^{-1}(w) \in \tau$ with $f^{-1}(w)(x_1) \neq f^{-1}(w)(x_2)$ then $\exists\, u, v \in \tau$ such that $u(x_1) = 1,\ u(x_2) = 0, v(x_1) = 0, v(x_2) = 1$ and $u \cap v = 0$.

Now

$f(u)(y_1) = \{sup u(x_1): f(x_1) = y_1\} = 1$

$f(u)(y_2) = \{sup u(x_2): f(x_2) = y_2\} = 0$

$f(v)(y_1) = \{sup v(x_1): f(x_1) = y_1\} = 0$

$f(v)(y_2) = \{sup v(x_2): f(x_2) = y_2\} = 1$

And

$f(u \cap v)(y_1) = \{\sup(u \cap v)(x_1): f(x_1) = y_1$

$f(u \cap v)(y_2) = \{\sup(u \cap v)(x_2): f(x_2) = y_2$

Hence $f(u \cap v) = 0 \Rightarrow f(u) \cap f(v) = 0$

Since $f$ is L-open, $f(u), f(v) \in s$. Now it is clear that $\exists\, f(u), f(v) \in s$ such that $f(u)(y_1) = 1$ , $f(u)(y_2) = 0$,$f(v)(y_1) = 0$, $f(v)(y_2) = 1$ and

$f(u) \cap f(v) = 0$. Hence it is clear that the L-topological space $(Y, s)$ is $L - R_1(i)$.

Similarly (b), (c), (d), (e), (f), (g) can be proved.

**6.4.2Theorem:** Let $(X, \tau)$ and $(Y, s)$ be two L-topological spaces and $f: (X, \tau) \to (Y, s)$ be L-continuous and one-one map, then-

(a) $(Y\ , s)$ is $L - R_1(i) \Rightarrow (X, \tau)$ is $L - R_1(i)$.

(b) $(Y\ , s)$ is $L - R_1(ii) \Rightarrow (X, \tau)$ is $L - R_1(ii)$.

(c) $(Y\ , s)$ is $L - R_1(iii) \Rightarrow (X, \tau)$ is $L - R_1(iii)$.

(d) $(Y\ , s)$ is $L - R_1(iv) \Rightarrow (X, \tau)$ is $L - R_1(iv)$.

(e) $(Y\ , s)$ is $L - R_1(v) \Rightarrow (X, \tau)$ is $L - R_1(v)$.

(f) $(Y\ , s)$ is $L - R_1(vi) \Rightarrow (X, \tau)$ is $L - R_1(vi)$.

(g) $(Y\ , s)$ is $L - R_1(vii) \Rightarrow (X, \tau)$ is $L - R_1(vii)$.

**<u>Proof</u>:** Suppose $(Y, s)$ is $L - R_1(i)$.We shall prove that $(X, \tau)$ is $L - R_1(i)$. Let $x_1\ , x_2 \in X$ with $x_1 \neq x_2$ and $w \in \tau$ with $w(x_1) \neq w(x_2)$, $\Rightarrow f(x_1) \neq f(x_2)$ as $f$ is one-one, also $f(w) \in s$ as $f$ is L-open. We have $f(w)(f(x_1)) = \sup\{w(x_1)\}$ and $f(w)\big(f(x_2)\big) = \sup\{w(x_2)\}$ and $f(w)(f(x_1)) \neq f(w)\big(f(x_2)\big)$. Since $(Y, s)$ is $L - R_1(i), \exists\, u, v \in s$ such that $u\big(f(x_1)\big) = 1, u(f(x_2)) = 0, v\big(f(x_1)\big) = 0, v(f(x_2)) = 1$ and $u \cap v = 0$. This implies that $f^{-1}(u)(x_1) = 1, f^{-1}(u)(x_2) = 0, f^{-1}(v)(x_1) = 0, f^{-1}(v)(x_2) = 1$ and $f^{-1}(u \cap v) = 0 \Rightarrow f^{-1}(u) \cap f^{-1}(v) = 0$.

Now it is clear that $\exists\, f^{-1}(u), f^{-1}(v) \in \tau$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) = 0$, $f^{-1}(v)(x_1) = 0$, $f^{-1}(v)(x_2) = 1$ and $f^{-1}(u) \cap f^{-1}(v) = 0$.

Hence the L-topological space $(X, \tau)$ is $L - R_1(i)$.

Similarly (b), (c), (d), (e), (f), (g) can be proved.

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