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Analysis of Viscous Incompressible Fluid Flows and Heat Transfer

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Thesis Dedicated
To My
Beloved Parents

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I am pleased to certify that the thesis entitled "**ANALYSIS OF VISCOUS INCOMPRESSIBLE FLUID FLOWS AND HEAT TRANSFER**", submitted by Md. Sharif Uddin, for the award of Ph.D. (Science) degree in Mathematics of University of Rajshahi, is based absolutely on his own work. I myself and Dr. G. C. Layek, Department of Mathematics, the University of Burdwan, W.B., India have agreed that the thesis of Md. Sharif Uddin has contributed substancially to his field of research.

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Mr. Uddin has fulfilled all the conditions required for the submission of the thesis under the rules of University of Rajshahi and I certify that neither the thesis nor any part of it has been submitted for any degree/diploma or any other academic award any where before. In my opinion this work is wrothy of consideration for the award of **Doctor of Philosophy** in Science (Mathematics). I am also pleased to certify that there is nothing in his character, which may debar him from being admitted to the degree of **Doctor of Philosophy** (Science).

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DECLARATION

I do hereby declare that the thesis entitled "**ANALYSIS OF VISCOUS INCOMPRESSIBLE FLUID FLOWS AND HEAT TRANSFER**", submitted to the University of Rajshahi for the award of Ph.D. degree in Science (Mathematics) is based on my research work and has not been submitted to any University/Institute for the award of any degree or diploma.

Date: 20.08.2010

Md. Sharif Uddin

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NOMENCLATURE

a	stretching constant
B	magnetic field
B_0	magnetic field strength
C	concentration
C_w	wall concentration
C_∞	free stream concentration
c_p	specific heat
D	diffusion coefficient
Da_x	local Darcy number,
E	electric field
f	dimensionless stream function
Gr_x	local Grashof number
g	acceleration due to gravity
J	current density
k	permeability of the porous medium
k^*	permeability parameter of the porous medium
k_0	coefficient of viscoelasticity
L	heat source/sink parameter
M	magnetic parameter
Pr	Prandtl number
Q_0	heat generation or absorption coefficient
R	reaction rate of the solute
R_M	magnetic Reynolds number
Re_x	local Reynolds number
S	suction or blowing parameter
Sc	Schmidt number
T	temperature
T_w	wall temperature

T_∞	free stream temperature
u	velocity component in x-direction
u_∞	free stream velocity,
v	velocity component in y-direction
v_w	distribution of suction or blowing
β	reaction rate parameter of the solute
β_1	volumetric coefficient of thermal expansion
δ	velocity slip parameter
η	similarity variable
γ	thermal slip parameter
κ	fluid thermal conductivity
λ	buoyancy or mixed convection parameter
λ_1	viscoelastic parameter
μ	coefficient of fluid viscosity
υ	kinematic viscosity of fluid
ϕ	dimensionless concentration
ρ	density of fluid
ψ	stream function
σ	electrical conductivity of the fluid
θ	dimensionless temperature

CHAPTER I

Introduction

We first present a brief ideas and principles in "Fluid Mechanics" which may serve as the background materials of viscous, incompressible laminar Newtonian or non-Newtonian fluid flow problems considered in the present thesis. The basic equations viz, the continuity equation and the momentum equation for the motion of viscous incompressible fluid under the limits of continuum hypothesis are presented. The laws of classical mechanics apply throughout the continuous medium under consideration. The length scale of the flow is always taken to be large compared with the molecular mean-free-path, so that the fluid can be considered as a continuum. It excludes the flow of gases at very low pressures i.e., rarefied gases. Liquid flows can usually be treated as incompressible fluid. The classification of fluids, say, Newtonian and non-Newtonian, Prandtl's boundary layer concept, boundary layer equations, concept of similarity variable for analyzing the viscous flow problems, group-theoretic approach of for finding invariant solution of an incompressible viscous fluid flows, are discussed systematically. Using the similarity variable for some specific flow problems, a set of nonlinear ordinary differential equations, known as self-similar equations are derived. Analytical or closed-form solution as well as numerical solution of these nonlinear differential equations relating to particular class of flow problems are obtained and the corresponding flow quantities are shown graphically and discussed physically.

In general, matter is found to exist in four phases or states e.g. solid, liquid, gas and plasma (ionized gases). Out of these last three states of matter are termed as fluid. Fluid mechanics is the subject in which we deal with the flow problems pertaining to one of these phases e.g. liquid, gas and plasma or combination, mainly of the first two or last two phases.

Essentially, the fluid flow problems are of widely spread interest in various fields of engineering as well as in meteorology, oceanography and other subjects of physical sciences. We live in a world which is largely a fluid. Air, oceans, rivers and so on are all fluids whose behaviour is mostly described using the principles of continuum hypothesis. The constitutive equations are framed using some assumptions based on the material behaviors of fluid and the flow conditions. It's study is important to physicists or applied mathematicians whose main interest is in understanding the related physical phenomena. On the other hand fluid dynamical engineers worked out many problems of practical interest using empirical formula. Also an understanding of this subject helps us to explain a variety fascinating natural phenomena around us. The cause of natural calamities like tornadoes, hurricanes and monsoon can be understood basically through the use of the laws of fluid mechanics. The human circulatory systems are governed by the principles of fluid dynamics. The flow of air through our respiratory passages into the lungs and the flows of blood through our arteries and veins and other circulatory systems are considered important. Besides, flows of water in channels, rivers etc. or other Newtonian/non-Newtonian liquid flows occurring in technical devices are concerned to our day to day living conditions. Here we shall discuss some of the basic development in fluid mechanics which are relevant to the problems undertaken in this thesis.

A fluid is defined as a coherent material substance whose parts are readily moved past one another or, in other words, a substance which offers little resistance to change its shape in contrast to solid substances [Kaufmann, (1963)]. Understanding of the basic principle obeyed by different types of fluids is highly relevant to the protection of our natural living conditions and many technical developments. Experimental observations lead us to classify fluid flows generally into two types e.g., laminar and turbulent. Osborne Reynolds (1894) conducted an experiment on fluid flows through a pipe at different speeds by injecting dye. At low velocity he found that the flows exhibits streamline pattern of flow while at certain high velocity of the flow starts exhibiting mixing of stream lines. The former case is termed as laminar flow and the later one is turbulent motion. A non-dimensional number associated with the motion of viscous fluid flow, e.g., Reynolds number is introduced to distinguish the above two states of motion. In case of viscous fluid flows through a pipe the Reynolds number is defined by $\mathrm{Re} = Ud\rho/\mu$, where U is a characteristic mean velocity and d is the diameter of the pipe, ρ is the fluid density and μ is kinematic viscosity of the fluid. The Reynolds number is a non-dimensional number relating to the motion of viscous fluid and is defined as above. It is approximately the ratio of inertia force to viscous force of the fluid motion.

The analytical solutions of governing nonlinear equations are not possible except few ones. So we sought numerical solution using finite-difference method or shooting method.

1.1 Classification of fluids

The fluids can be classified as (i) Ideal fluid and (ii) Real fluid based on its physical properties.

1.1.1 Ideal fluids : A fluid is said to be an ideal fluid if the stress vector at a point is normal to any surface through the point and there is no tangential forces (shearing stresses) even when the fluid is in motion. This is equivalent to stating that an ideal fluid offers no internal resistance to change its shape during its motion. The pressure at every point in an ideal fluid is equal in all directions (isotropic in nature), whether the fluid is at rest or in motion. Ideal fluids are also termed as inviscid fluids or perfect fluids as frictionless fluids.

1.1.2 Real fluids : A fluid is said to be real if the stress vector at a point in the fluid across any surface through this point has both normal and tangential components. These tangential or friction forces in a real fluid motion are connected with a property, which is called the viscosity of the fluid i.e., real fluids are more or less viscous fluids. Viscosity is caused by internal friction of fluid and it plays an important role during the motion of the fluid. It offers resistance or shearing stress during the motion. So, the physical property that characterizes the flow resistance of simple fluids in the viscosity. This internal resistance, unlike solids, does not depend on the deformation itself but on the rate of deformation also. Viscosity is a physical property of fluid derived under the hypothesis of continuum. It represents the tendency of a fluid to undergo deformation when subject to a shear stress. The deformation rate is determined by the inter-molecular force, which provides the force to balance the applied shear force. In fact, viscosity is a global material constant or constants representing the propensity for a collection of fluid molecules to externally applied stress.

In case of a pure liquid consisting of simple molecules, viscosity is thought to be an intrinsic property and its value is dependent on the temperature of the fluid alone. For more complex fluids, such as solutions of macromolecules, viscosity may contain more than one constant and generally shows a complex dependence on an additional number of parameters, such as concentration, molecular weight etc of the solute. Nonetheless, even in these more complex cases, viscosity is ascribed to the intrinsic properties of the fluid itself.

Real fluids are classified into two categories Newtonian fluids and non-Newtonian fluids.

1.1.3 Newtonian fluids : The rate of deformation or rate of strain of a fluid is proportional to the applied shearing stress (force). If the shearing stress is linearly proportional to the rate of strain, the fluid is called Newtonian fluid. Newtonian fluid follows Newton's law of friction which states that the shearing stress (τ) between the layers of fluid moving past each other is proportional to the velocity gradient (du/dy). The shear stress (τ) per unit area is defined by $\tau = \mu(du/dy)$ or shear rate, where the proportionality constant μ is called co-efficient of viscosity or dynamic viscosity or shear viscosity. Because of the above linear relationship between stress and rate of strain and because of the fact that the stress or the rates of strain components don't enter through their time derivatives, this linear law provides, a great simplicity in mathematical analysis of fluid motion. Besides, it cannot be called as hypothetical constitutive equation as it provides a fairly good description of the flow properties of a very large class of real fluids, for examples water, air, mercury etc. [Batchelor (1993), White (1974)].

1.1.4 Non-Newtonian fluids : The non-Newtonian fluids are generally highly viscous fluids. These are particular classes of fluids where the shear stress of a fluid is a nonlinear function of rate of strain. It is known that a large class of fluids deviate the Newtonian's law. Many solid-liquid and liquid-liquid suspensions are considered non-Newtonian, as they comprise of solutions of macro-molecules, molten plastics, and mammalian whole blood and synovial fluid. The study of non-Newtonian fluid mechanics is therefore of wide interest to the researchers who work in biological, non-biological, chemical process engineering, material science, plastic engineering and other related fields. The typical non-biological examples of this class of fluids are pastes, plastics, molasses, molten rubber, printers ink, Coal-ter, Clay condensed milk, collides, macro/molecular materials, solution of high polymers and so on. The non-Newtonian fluids are modeled in several ways. In the following section we shall discuss different kinds of non-Newtonian fluid models and their mathematical equations.

1.1.5 Visco-elastic fluid model : These fluids possess certain degree of elasticity in addition to viscosity. When a visco-elastic fluid is in motion, a certain amount of energy is stored up in the material as strain energy while some energy is lost due to viscous dissipation. In this class of fluids unlike inelastic viscous fluids, one can not neglect the strain, however small it may be as it is responsible for the recovery to the original state and for possible reverse flow that follows the removal of the stress. During the flow the natural state of the fluid changes constantly and it tries to attain the instantaneous state or the deformed state, but it does never succeed completely. This lag is a measure of the elasticity or the so called "memory" of the fluid. But there are some fluids like soap solution, polymer solution, which have some elastic properties besides having viscous properties. Such fluids are the examples of visco-elastic fluids.

There are various models for visco-elastic fluids. Examples are second order (Rivlin-Erickson fluids) Oldroyd fluids, Walters B' fluid [Beard and Walters (1964)] and so on. There are another class of visco-elastic fluids such as second grade or third grade fluids.

Coleman and Noll (1960), originally suggested a constitutive equation for the incompressible visco-elastic second grade fluid, based on the postulate of fading memory as

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2$$

where T is the stress tensor, p is the pressure, μ is the dynamic fluid viscosity, α_1 and α_2 are the first and second normal stress coefficients. A_1 and A_2 are the kinematic tensors, expressed as:

$$A_1 = grad(V) + (grad(V))^T$$

and $$A_2 = \frac{d}{dt}A_1 + A_1(gradV) + (gradV)^T A_1$$

where V is the velocity and $\frac{d}{dt}$ is the material time derivative.

The constitutive equation for second-order visco-elastic fluids was given by Rivlin-Ericson and is written as follows.

$$\tau_{ij} = -p\delta_{ij} + vA_{ij}^{(1)} + v'A_{ij}^{(2)} + v''A_{ik}^{(1)}A_{kj}^{(1)},$$

$$A_{ij}^{(1)} = \left(u_{i,j} + u_{j,i}\right), \quad A_{ij}^{(2)} = \frac{\partial}{\partial t}A_{ij}^{(1)} + u_m A_{ij,m}^{(1)} + A_{im}^{(1)}u_{m,j} + A_{mj}^{1}u_{m,i}$$

where τ_{ij} is the stress tensor, p is pressure, δ_{ij} is Kronecker delta, v, v' and v''' are three measurable material constants. They denote respectively the viscosity, elasticity and cross-viscosity. u_i is the velocity of the fluid.

1.1.6 Power-law fluid model : The mathematical model for describing the mechanistic behaviour of a variety of commonly used non-Newtonian fluids is the power-law model which is also known as Ostwald-de Waele model. According to Ostwald-de Waele model, the constitutive equation is represented as $\tau = m\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy}$ where m denotes the flow consistency index and n is the flow behaviour index. Viscosity is the ratio of shear stress to the deformation rate. For power law fluid model, it is $m\left|\frac{du}{dy}\right|^{n-1}$, known as apparent fluid viscosity. When $n < 1$, the model is valid for pseudoplastic fluids such as gelatine, blood, milk etc. In these types of fluids, the apparent fluid viscosity decreases with increasing deformation rate (n<1) and are called shear thinning fluids. When n>1, the model is valid for dilatant fluids, such as sugar in water, aqueous suspension of rice starch, sand etc. this fluid model is also known as shear thickening fluid in which the apparent fluid viscosity in creases with increasing rate of deformation of the fluid.

1.1.7 Bingham plastic :

There are some substances which require a yield stress for the deformation rate (i.e., the flow) to be established, and hence their constitutive equations do not pass through the origin thus violating the basic definition of fluid. These are termed as Bingham plastic, the shear stress deformation rate relationship is linear, i.e., $\tau_{yx} = \tau_y + \mu_p \frac{du}{dy}$. Examples: Clay suspensions, drilling mud, and toothpaste. The study of non-Newtonian fluids is further complicated to the fact that the apparent viscosity may be time-dependent. Thixotropic fluids show a decrease in apparent viscosity with time under a constant applied shear stress; many paints are thixotropic. Rheopectic fluids show an increase in apparent viscosity with time. After deformation some fluids partially return to their originally shape when the applied stress is released [Bird et al. (1960)].

1.2 Continuity and momentum equations

Navier-Stokes equations provide a complete description of viscous fluid flow problems along with equation for continuity. Various initial and boundary conditions are properly prescribed depending on flow conditions. The Navier-Stokes equations can be solved to predict different characteristics related to the flow phenomena including flow separation and other unsteady flow phenomena. The equations are non-linear partial differential equations and no analytical closed-form solutions are obtained except few special flow problems. Even the numerical approach faces difficulties because of nonlinearity, unsteadiness and the irregular boundary.

In fluid mechanics, the law of conservation of mass is very important. This law states that mass can be neither created nor-destroyed in a specific control volume with having no source or sink. The equation of conservation of mass is obtained as

$$\frac{D\rho}{Dt} + \rho \nabla . \vec{u} = 0 \qquad (1.1)$$

where ρ be the fluid density and $\vec{u}$ be the velocity vector.

This equation is interpreted as : For any closed surface drawn in the fluid, the increase in the mass of fluid within the surface in any time interval must be equal to the excess of mass that flows into the volume through the surface over the mass that flows out in that interval [Yuan (1969), Kundu and Cohen (2008)].

For the incompressible case ρ is constant and the above equation is reduced to the form as

$$\nabla \cdot \vec{u} = 0 \tag{1.2}$$

Physically this means that the rate of expansion is every where zero.

The equation of motion of a fluid can be derived from Newton's second law of motion which states that the total force acting on a fluid mass enclosed in an arbitrary volume is equal to the time rate of change of linear momentum of the fluid enclosed in that volume. The momentum balance is expressed as

$$\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{1.3}$$

at all points of the fluid. Here F_i be the component of the body force per unit mass of the fluid, σ_{ij} is the stress tensor, ρ the density of the fluid, u_i, the i-th component of the velocity and $\frac{D}{Dt}$ is the material derivative due to the fluid acceleration.

The stress tensor σ_{ij} can be expressed as

$$\sigma_{ij} = -p\delta_{ij} + d_{ij}, \tag{1.4}$$

where p is the fluid pressure and the non-isotropic part d_{ij} is termed as the deviatoric stress tensor [Batchelor (1993)]. On choosing the coefficient of fluid viscosity μ, as the one independent scalar constant, the deviatoric stress tensor d_{ij} (considering Stokes' assumption) can be expressed as

$$d_{ij} = 2\mu\left(e_{ij} - \frac{1}{3}\Delta\delta_{ij}\right) \tag{1.5}$$

where $$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.6}$$

and $\Delta = e_{ij}$ is the rate of expansion. Substituting (1.4), (1.5) and (1.6) in (1.3), we get the momentum equation as follows.

$$\rho \frac{Du_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[2\mu\left(e_{ij} - \frac{1}{3}\Delta\delta_{ij}\right)\right] \tag{1.7}$$

This equation is usually called the Navier-Stokes equations of motion for an incompressible viscous fluid.

The coefficient of viscosity μ depends significantly on temperature for many fluids. So it is required to consider μ as a function of position when appreciable temperature differences exist in the flow field. For small temperature differences it will suffice the purpose if we take μ as uniform over the whole region of the fluid.

Finally, the equation of motion for an incompressible fluid ($e_{ii} = 0$) becomes, in vector notation

$$\rho \frac{D\bar{u}}{Dt} = \rho \vec{F} - \nabla p + \mu \nabla^2 \vec{u} \tag{1.8}$$

In the absence of external force, the unknowns are the components of velocity field and pressure. Some difficulties arise in case of incompressible flow problems since the boundary conditions only exist for the velocity field. Combining the continuity and the momentum equations, Poisson equation for pressure in the following form

$$\nabla^2 p = g(u, v, w) \tag{1.9}$$

where $g(u,v,w)$ is a function of the components of the velocity vector. This equation must be solved subject to the boundary condition $\frac{\partial p}{\partial n}$ (the normal pressure gradient) obtained from the momentum equation.

1.3 Mathematical derivation of Prandtl's boundary layer theory

In 1904, Ludwig a Prandtl little known physicist revolutionized fluid dynamics with his notion that the effects of internal friction due to fluid viscosity are experienced only very near an object moving through a fluid. This concept relating to the flow behaviour near the boundary, known as boundary layer first time introduced by Ludwig Prandtl (1904) at third International Mathematics Congress held at Heidelberg, Germany. The frictional effects were experienced only a thin region near the surface and out side this fluid layer or boundary layer, the flow was essentially the inviscid flow. It is obvious that the thickness of the boundary layer approaches zero as the viscosity goes to zero. Prandtl's hypothesis reconciled two rather contradictory facts. On one hand he supported the intuitive idea that the effects of fluid viscosity are indeed negligible in most of the flow field if the fluid viscosity is small. At the same time Prandtl was able to account for drag by insisting that the no-slip condition must be satisfied at the wall, no matter how small the fluid viscosity. This reconciliation was Prandtl's aim, which he achieved brilliantly, and in such a simple way that it now seems strange that nobody before him thought of it. Prandtl also showed how the equations of motion within the boundary layer can be simplified drastically even the nature of the equation is changed. Since the time of Prandtl, the concept of the boundary layer can be simplified. Since the time of Prandtl, the concept of the boundary layer has been generalized, and the mathematical techniques involved have been formalized, extended and applied to various other branches of physical science. The concept of the boundary layer is considered one of the cornerstones in the history of fluid mechanics.

The velocity changes over a very short distance normal to the surface. The boundary layer is a region of very large velocity gradient, the local shear stress can be very large within the boundary layer. Another marked result according to Prandtl is flow separation phenomena. With the advent of Prandtl's boundary layer concept, the Navier- Stokes equations can be reduced to a simpler form. The major mathematical break through is that the boundary layer equations exhibit a completely different mathematical behavior than the Navier-Stokes equations. The Navier-Stokes equations have an elliptic behavior. The complete flow field must be solved simultaneously, in accord with specific boundary conditions defined along the boundary of the flow. In contrast, the boundary layer equations have parabolic behavior greatly afford the analytical and computational simplification. The equation can be solved step-by-step by marching down stream from where the flow encounters a body, subjected to initial and boundary conditions. Fluid viscosity only played a role in the thin layer of flow immediately adjacent to a surface [H. Evans (1968)].

We shall now derive the differential equation for boundary layer flow which is formed in the immediate neighbourhood of the solid body. L. Prandtl (1904) made the most seminal contribution to fluid mechanics through his construction of boundary layer theory. For the sake of simplicity let us consider a two-dimensional motion of a liquid with very small fluid viscosity.

We take the x-axis along the wall and y-axis perpendicular to the wall. In absence of external forces the Navier-Stokes equations of motion can be written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{1.10}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \tag{1.11}$$

and $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ (1.12)

The boundary conditions are as follows:

The no-slip boundary conditions give us

$$u=v=0 \text{ on } y=0 \tag{1.13}$$

and $y\to\infty$, i.e., for away from the solid wall, $u\to U$, the free-stream velocity. (1.14)

We assume the fluid viscosity to be very small. According to Prandtl's boundary layer theory, the velocity component u parallel to the wall in the boundary wall rises rapidly from zero value on the wall to a free stream value U within a short distance, say δ (boundary layer thickness), [Schlichting (2000)]. We take u, x, t to be of the order 0(1), i.e., $u\approx 0(1), x\approx 0(1), t=0(1)$ and y, the distance from the wall to be of the order δ, i.e., $y\approx 0(\delta)$, $\delta<<1$.

From the equation of continuity, we get

$$\left.\begin{array}{l}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\ \frac{\partial u}{\partial x}\approx 0(1)\\ \text{It gives the term } \frac{\partial \upsilon}{\partial y}\approx 0(1)\end{array}\right\} \tag{1.15}$$

But $y\approx 0(\delta)$, it gives $v\approx 0(\delta)$ (1.16)

Using (1.14) & (1.15), we analyse the orders of each term in the momentum equation as follows.

$$\frac{\partial u}{\partial t}\approx 0(1),\ u\frac{\partial u}{\partial x}\approx 0(1)\times 0(1)\approx 0(1)$$

$$v\frac{\partial u}{\partial y}=0(\delta)\times 0(1/\delta)\approx 0(1)$$

Thus, the inertial terms in the left hand side of (1.10) is of the order 1. Next, we consider the viscous term in the equation.

$$\frac{\partial^2 u}{\partial x^2}\approx 0(1) \text{ and } \frac{\partial^2 u}{\partial y^2}\approx 0\left(\frac{1}{\delta^2}\right)$$

This indicates that the term $\frac{\partial^2 u}{\partial x^2}$ is negligible compare to the term $\frac{\partial^2 u}{\partial y^2}$. Physically, the variation of streamwise term is very small compare to the variation of cross-streamwise term. So, we can write the equation (1.10) approximately as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\upsilon\frac{\partial^2 u}{\partial y^2} \tag{1.17}$$

Now, we assume that in the equation (1.17) the inertial term and the viscous term are of the same order, i.e., 0(1). Then we have

$$\upsilon\frac{\partial^2 u}{\partial y^2}\approx 0(1)$$

Or, $\upsilon\approx 0(\delta^2)\Rightarrow \delta\approx 0(\sqrt{\upsilon})$ (1.18)

The thickness of boundary layer (δ) is the square root of υ, i.e. the thickness δ is the square root of the fluid Kinematic Viscosity. We us consider the *v*-momentum equation (1.11). We get the followings order analysis

$$\frac{\partial v}{\partial t}\approx 0(\delta),\ u\frac{\partial v}{\partial x}\approx 0(\delta),\ v\frac{\partial v}{\partial y}\approx 0(\delta)\times 0(1)\approx 0(\delta)$$

So, the inertial terms is of the order δ.

Now, $\frac{\partial^2 v}{\partial x^2}\approx 0(\delta),\ \frac{\partial^2 v}{\partial y^2}\approx 0\left(\frac{1}{\delta}\right).$

Hence, the term $\frac{\partial^2 u}{\partial x^2}$ is negligible in compare to $\left(\frac{\partial^2 u}{\partial y^2}\right)$. Also, using (1.17), we have $\upsilon\left(\frac{\partial^2 v}{\partial y^2}\right)\approx 0(\delta^2)\times 0\left(\frac{1}{\delta}\right)\approx 0(\delta)$. Both the inertial and viscous terms in equation (1.11) are of the order δ.

Hence $-\frac{1}{\rho}\frac{\partial p}{\partial y} \approx 0(\delta)$. (1.19)

Integrating (1.19) along the normal direction, we get $p \approx 0\left(\delta^2\right)$. Now, the equation (1.11) can be replaced by

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{1.20}$$

and the continuity equation remains unchanged.

Hence, the boundary layer equations for the two-dimensional motion of a viscous fluid past a flat plate with usual notations are given by

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial y^2} \\ &\qquad\qquad -\frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \end{aligned}\right\} \tag{1.21}$$

with boundary conditions

$u = v = 0$ (usual no slip conditions) at $y = 0$, $u \to U$ as $y \to \infty$ (1.22)

The pressure may be calculated by the inviscid flow outside the boundary layer since the pressure is independent in the perpendicular direction to the boundary layer. Outside the boundary layer $u=U$, $v = 0$, we may write the u-momentum equation as

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 U}{\partial y^2}\right)$$

Or, $-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{dU}{dx}$ [since U is a function of x only and $v \approx 0$ outside the boundary layer.]

Now, for steady case, the above equations becomes $U\frac{dU}{dx} = -\frac{1}{\rho}\frac{dp}{dx}$.

At the outer edge of the boundary layer, the u-component of velocity becomes equal to the outer flow $U(x, t)$. Since there is no large velocity gradient here, the viscous terms vanish for large values of flow Reynolds number, and consequently for the outer flow we obtain the following equation as [Batchelor (1993)].

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

In case of steady flow situation, the equation is simplified still further. The pressure depends only on x. We shall write this by writing the derivative as $\frac{dp}{dx}$, so that $U\frac{dU}{dx}=-\frac{1}{\rho}\frac{dp}{dx}$. This may also be written in the usual form Bernouli's equation for inviscid flows.

1.4 Importance of Prandtl's boundary layer theory in fluid mechanics

It plays a vital role in the motion of viscous fluid. Also, it is very powerful method of analyzing the complex behaviour of real fluid motion. This theory can be utilized to simplify the Navier-Stokes equations to such an extent that it becomes possible to tackle many practical problems of great importance. On the other hand, the boundary layer theory is capable of explaining the difficulties encountered by ideal fluid dynamics. It is able to predict the flow separation. It can explain the existence of wake. The pressure distribution produces a net force in the direction of motion. The analysis of heat transfer near solid wall is explained satisfactorily. The boundary conditions may be implemented appropriately.

1.5 MHD boundary layer equations

Magnetohydrodynamics (MHD) is a rapidly advancing subject since its inception. Application of MHD to natural events received a significant interest when astrophysicists came to realize how prevent throughout the universe are conducting, ionized gases (plasma) and significantly strong magnetic fields [Shercliff (1965), Cowling (1956)].

Magnetohydrodynamics (MHD) is the study of the motion of an electrically conducting fluid in the presence of a magnetic field. Electric currents, induced in the conducting fluid as a result of its motion modify the field; at the same time the magnetic field produces mechanical forces which modify the motion. Basically ,in

MHD there are two basic effects viz.(i) an induced magnetic field associated with there currents appears, perturbing the original magnetic field, (ii) an electromagnetic force due to the interaction of currents and field appears, perturbing the original motion. MHD owes its particular interest and difficulty to its interaction between the field and the motion.

The equations of MHD are the ordinary electromagnetic and hydrodynamic equations, modified to take account of the interaction between the fluid motion and the magnetic field. As in most electromagnetic problems involving conductors, other than those concerned with rapid oscillations, Maxwell's displacement currents are ignored. Accumulations of electric charge are neglected in the equation of continuity of charge. So electric currents are regarded as flowing in closed circuits. When the magnetic Reynolds number denoted by $R_M (= \mu\sigma Vd)$ is small, one may neglect the induced current. Here μ be the magnetic permeability, σ the magnetic diffusivity, V the velocity of conducting fluid and d be the characteristic length.

The modified u-momentum equation in two-dimensional motion taking into account the body force due to the electromagnetic force (known as Lorentz force) may be written as follows taking into account some assumption on the magnetic field and the restriction flow filed.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\left(\frac{\mu}{\rho}\vec{J}\times\vec{H}\right)+\upsilon\nabla^2 u,$$

Here μ is the magnetic permeability, $\vec{J}$ the current density vector, υ is the kinematic viscosity of the fluid. The body force is equal to $\vec{J}\times\vec{B}=\left(\mu\vec{J}\times\vec{H}\right)=\mu(\nabla\times\vec{H})\times\vec{H}$.

Since the induced magnetic field is negligible for the small value of magnetic Reynolds number, the current $\vec{J}$ governs from Ohm's law as given below.

$\vec{J}=\sigma\left[\vec{E}+\vec{q}\times\vec{B}\right]=\sigma\left[\vec{E}+\mu\vec{q}\times\vec{H}\right].$

If $\vec{E}\approx 0$, then $\vec{J}=\sigma\left(\mu\vec{q}\times\vec{H}\right)$, σ being conductivity of the material.

We are considering a two-dimensional motion of an electrically conducting incompressible viscous fluid. So, $\vec{q}=(u,v,0)$ and $\vec{H}=(0,H_0,0)$ because a uniform magnetic field H_0 is applied in the direction of y and there is no induced magnetic field. So, $\vec{J} = \sigma\left[\mu(u,v,0)\times(0,H_0,0)\right] = \sigma\mu H_0 u\hat{k}$, that is, the current vector acts in the z-direction, normal to the plane of flow.

Now, $\mu\vec{J}\times\vec{H} = \mu^2\sigma H_0\left[(0,0,u)\times(0,H_0,0)\right] = -\mu^2\sigma H_0^{\ 2}u\hat{i} = -\sigma B_0^{\ 2}u\hat{i}$

So, $\frac{1}{\rho}\left[\mu\vec{J}\times\vec{H}\right] = -\frac{\sigma B_0^{\ 2}}{\rho}u\hat{i}$

In this study, the displacement current is zero, since the flow velocity is small. Thus, the MHD boundary layer equations in two-dimensions (using the boundary layer approximations) under no external body forces may be written as follows.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\sigma B_0^2}{\rho}u+\upsilon\frac{\partial^2 u}{\partial y^2},$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=0.$$

1.6 Similarity variable

Prandtl (1904) pointed out the pressure is a known variable in boundary layer analysis and assumed $p(x)$ is to be impressed upon the boundary layer from the potential flow. This means that free-stream out side the boundary layer, $U=U(x)$ is related to $p(x)$ by Bernoulli's theorem for incompressible flow, as

$$-\frac{1}{\rho}\frac{dp}{dx}=U\frac{dU}{dx} \tag{1.23}$$

Accordingly, the momentum and continuity equations are now written as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial x}=U\frac{dU}{dx}+v\frac{\partial^2 u}{\partial y^2} \tag{1.24}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{1.25}$$

The boundary conditions are, given by

$$u(x,0)=v(x,0)=0; \quad u(x,\infty)=U(x) \tag{1.26}$$

Blasius (1908) forwarded a solution for boundary layer past a flat plate. He considered a special case when the displacement thickness is small (in this situation Reynolds number, $(\text{Re} >> 1)$, U=constant, $\frac{dU}{dx} = 0$ in equation (1.24). He estimated the boundary layer thickness $\delta =$ constant $\times (\nu x / U)^{1/2}$ and introduced a similarity variable

$$\eta = y\sqrt{\frac{U}{2\nu x}} \qquad (1.27)$$

Now the stream function of the flow is given by

$$\psi = \int u dy \quad \text{at} \quad x = \text{constant} \qquad (1.28)$$

which should increase with $\delta\left(\sim x^{1/2}\right)$. ψ has the non-dimensional form

$$\psi = \sqrt{2\nu U x}\, f(\eta). \qquad (1.29)$$

The velocity components u and v are now obtained from the definition of stream function, as

$$u = \frac{\partial \psi}{\partial y} = Uf'(\eta) \qquad v = -\frac{\partial \psi}{\partial x} = \sqrt{\frac{\nu U}{2x}}\left(\eta f' - f\right). \qquad (1.30)$$

Substituting the relation (1.30) in the Blasius boundary layer equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}. \qquad (1.31)$$

We obtain a third order non-linear ordinary differential equation e.g.,

$$f''' + ff'' = 0. \qquad (1.32)$$

And the boundary conditions are reduced to

$$f'(0) = f(0) = 0; \quad f'(\infty) = 1. \qquad (1.33)$$

1.7 Slip conditions in fluid flow problems

A common feature of all these analysis is the assumption that the flow field obeys the conventional no-slip condition at the solid boundary .The no-slip boundary condition is one of the central tenets of the Navier-Stokes theory. However, there are situations wherein this condition does not hold. In particular the inadequacy of the no-slip condition is evident for most non-Newtonian fluids [Andersson (2002), Ariel et al. (2006)]. For example, polymer melts often exhibit macroscopic wall slip and that in general is governed by a nonlinear and monotone relation between the slip velocity and the traction. Also, the fluids that exhibit boundary slip have important technological applications such as in the polishing of artificial heart valves and internal cavities. Navier (1827) proposed a slip boundary condition which depends linearly on the shear stress. The mathematical equation of this condition may be stated as: $u(x,y) = L\frac{\partial u}{\partial y}$ which relates the fluid velocity u to the shear rate $\frac{\partial u}{\partial y}$ at the boundary.. Here, L is the slip length, and y denotes the coordinate perpendicular to the surface. In the present work we have used the slip boundary condition for the analysis of flow over a stretching sheet and also the flow around a orthogonal stagnation point.

1.8 Heat transfer in boundary layer

In fluids flowing past heated or cooled bodies the transfer of heat takes place by conduction and convection. When the conductivity of the fluid is small, which is true in ordinary fluids, the heat transport due to conduction is comparable to that due to convection only across a thin layer near the surface of the body. This means that the temperature field which spreads from the body extends essentially, over a narrow zone in the immediate vicinity of its surface, whereas the fluid at a larger distance from the surface is not materially affected by the heated body. This narrow region (thin layer) near the surface of the body is known as thermal boundary layer analogous to the concept of velocity boundary layer. The problems of thermal boundary layers may be classified in to two categories, viz., (i) forced convection and (ii) free convection. By forced convection we mean the flow in which the velocities arising from the variable density (i.e. due to the force of buoyancy) are negligible in comparison with the velocity of the main or forced flow, whereas in free convection, also known as natural convection, the motion is essentially caused by the effect of gravity on the heated fluid of variable density. The temperature in the thermal boundary layer rises rapidly from its value at the wall to the value in the main stream within a short distance from the wall [Muralidhar and Sundararajan (1995), Bansal (1998)].

1.9 Viscous flow around stretching sheet

The study of hydrodynamic flow and heat transfer over a stretching sheet has gained considerable attention due to its vast applications in industry and its importance to several technological processes. The production of sheeting material arises in a number of industrial manufacturing processes and includes both metal and polymer sheets. The tangential velocity imparted by the sheet induces motion in the surrounding fluid that alters the convection cooling of the sheet. Knowledge of the flow properties of the fluid is desirable because the quality of the resulting sheeting material, as well as the cost of production, is affected by the speed of collection and mass transfer rate. The viscous flow of an incompressible fluid past a moving surface in otherwise quiescent surroundings has several engineering applications for examples, polymer processing. The extrusion of a polymer sheet from a dye or in the drawing of plastic films. Crane (1970) studied the steady two-dimensional flow caused by a stretching sheet whose velocity U varies linearly with the distance from a fixed point on the sheet. This flow problem has later been investigated in different contexts by several researchers viz. Chiam (1982), Andersson and Dandapat (1991) etc. In view of the above mentioned applications the rate heat transfer over the moving sheet is also very important. The desired characteristics of the final product depend on the rate of heat transfer or cooling. In recent years, a great deal of interest has been generated in the area of boundary layer mixed convection flow on a vertical stretching surface in view of its numerous and ever increasing industrial and technical applications which include aerodynamic extrusion of plastic sheets, cooling of metallic sheets in a cooling bath, crystal growing etc. In the study of horizontal heated or cooled surfaces, the effect of buoyancy force is neglected. However, for vertical or inclined surfaces, the buoyancy force modifies the flow field and hence the heat transfer rate. The importance of this phenomenon is increasing day by day due to the enhanced concern in science and technology about buoyancy induced motions in the atmosphere, the bodies in water and quasi-solid bodies such as earth. Buoyancy plays an important role where the temperature differences between land and air give rise to a complicated flow and in enclosures such as ventilated and heated rooms. The buoyancy force arising due to the heating of a stretching surface, under some circumstances, may alter significantly the flow and thermal fields and thereby the heat transfer behaviour in the manufacturing process.

1.10 Flow through porous medium

It is a material consisting of a solid matrix with an interconnected void. The solid matrix is either rigid in the usual situation or it undergoes small deformation. The interconnectedness of the void (the pores) allows the flow of one or more fluids through the material. In the simplest situation, i.e., single-phase flow, the void is saturated by a single fluid. This simple situation is considered in the present thesis. In the two-phase flow, a liquid and a gas share the void space. In a natural porous medium the distribution of pores with respect to shape and size is irregular. Examples of natural porous media are beach sand, sand stone, limestone, wood and the human lung. On the pore scale (the microscopic scale), the flow variables, viz., pressure, velocity, density, etc. have shown irregular behavious. But the flow variables are measured over areas that cross many pores. Such space average (macroscopic) quantities change in a regular manner with respect to space and time (Nield and Bejan (2003)). The porosity of a porous medium is defined as the fraction of the total volume of the medium which is occupied by void space.

1.11 Group-theoretic approach

Special group transformations are useful tool for producing similarity solutions. Our purpose here is to discuss about solution of differential equations based on finding group invariants [Hansen (1964), Bluman and Cole (1974), Pakdemirli and Yurusoy (1998)]. The scaling group of transformations are applied to find dynamical equations to get similarity variable and finally one get the self- similar equation for the fluid dynamical equations.

Let us consider a simple first order differential equation as

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \tag{1.34}$$

This equation remains invariant by the substitution $x = \lambda x^*$, $y = \lambda y^*$, as

$$\frac{dy^*}{dx^*} = f\left(\frac{y^*}{x^*}\right) \tag{1.35}$$

where λ is any real constant. Since $\frac{y}{x} \to \left(\frac{y^*}{x^*}\right)$ under the transformation, putting $v = \frac{y}{x}$, we obtain easily the solution of the equation (1.34) as

$$\int \frac{dv}{f(v) - v} = \int \frac{dx}{x} + \text{constant}. \tag{1.36}$$

If λ be considered as a continuous parameter, the set of transformations $x \to \lambda x^*$, $y \to \lambda y^*$ form a group under composition of transformations. In fact, this is an example of a Lie group.

Let us consider a subset D of R^2 on which $x \to x_1 = f(x, y, \varepsilon)$; $y \to y_1 = g(x, y, \varepsilon)$. Here x_1 and y_1 are assumed to vary continuously with the parameter λ. This set of transformations form a one parameter group or Lie group under the following rules :

(i) With $\varepsilon = 0$, $f(x; y; 0) = x$ and $g(x; y; 0) = y$, this means the transformation with $\varepsilon = 0$ is an identity transformation.

(ii) Changing ε to -ε, we have an inverse transformation such that $x_1 = f(x; y; \varepsilon)$ and $y_1 = g(x; y; \varepsilon)$ leads to $x = f(x_1; y_1; -\varepsilon)$ and $y = g(x_1; y_1; -\varepsilon)$.

(iii) Composition of two transformations is also a member of the set of transformations. For examples, if $x_1 = f(x; y; \varepsilon)$, $y_1 = g(x; y; \varepsilon)$, $x_2 = f(x_{1;} y_1; \delta)$ and $y_2 = g(x_{1;} y_1; \delta)$ then $x_2 = f(x; y; \varepsilon + \delta)$ and $y_2 = g(x; y; \varepsilon + \delta)$.

Some useful one-parameter groups are:

(iv) Horizontal translation $H(\varepsilon): x_1 = x + \varepsilon, \; y_1 = y$

(v) Vertical translation $V(\varepsilon): x_1 = x, \; y_1 = y + \varepsilon$

(vi) Magnification $M(\varepsilon): x_1 = e^{\varepsilon} x, \; y_1 = e^{\varepsilon} y$

(v) Rotation $R(\varepsilon): x_1 = x\cos\varepsilon - y\sin\varepsilon,\ y_1 = x\sin\varepsilon + y\cos\varepsilon$

We now discuss the application of group theoretic concept, following Cebeci and Bradshaw (1977), to the problem of two dimensional steady constant property laminar flow over a flat surface as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.37}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + v\frac{\partial^2 u}{\partial y^2} \tag{1.38}$$

where the symbols have their usual meanings in fluid flow problem.

We introduce simplest set of boundary conditions for a boundary layer flows in which the variation in pressure of y-direction is negligible, i.e., $\partial p / \partial y \approx 0$ as follows.

$u = v = 0$ (usual no-slip conditions) at $y = 0$ and $u \to u_e(x)$ (free-stream velocity) as

$$y \to \infty \tag{1.39}$$

In the context of the two dimensional incompressible shear layer, equations given by (1.37) and (1.38), may be written as

$$\frac{u}{u_e} = g(x, y) \tag{1.40}$$

And in the special case equation (1.40) can be written as

$$\frac{u}{u_e} = g(\eta), \tag{1.41}$$

Where η is specific function of x and y and called a similarity variable. Here the notation is clear that the number of independent variables is reduced from two (x and y) to one (η) and as a result the equations (1.37) and (1.38) would become ordinary differential equations for u and v.

To find the similarity variable η and the necessary condition under which equations (1.37) and (1.38) reduces to ordinary differential equations, we shall use the group-theoretic method as discussed by Hansen (1964).

First to introduce the linear group transformation and apply the same to equations (1.37) and (1.38):

$$x = A^{\alpha_1}\overline{x},\ y = A^{\alpha_2}\overline{y},\ u = A^{\alpha_3}\overline{u},\ v = A^{\alpha_4}\overline{v},\ u_e = A^{\alpha_5}\overline{u}_e. \tag{1.42}$$

Here $\alpha_1, \alpha_2, \ldots, \alpha_5$ are constants, and A is called the parameter of transformation. Substituting the relations of (1.42) in to equations (1.37) and (1.38), we obtain

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=A^{\alpha_3-\alpha_1}\frac{\partial \bar{u}}{\partial \bar{x}}+A^{\alpha_4-\alpha_2}\frac{\partial \bar{v}}{\partial \bar{y}}=0 \qquad (1.43)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-u_e\frac{du_e}{dx}-v\frac{\partial^2 u}{\partial y^2}=A^{2\alpha_3-\alpha_1}\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}+A^{\alpha_3+\alpha_4-\alpha_2}\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}-A^{2\alpha_5-\alpha_1}\bar{u}\frac{d\bar{u}_e}{d\bar{x}}-A^{\alpha_3-2\alpha_2}v\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}=0$$

(1.44)

Obviously, the set of equations will remain invariant if the powers of A in each term are all equal. That is, we get the following set of equations as

$$\alpha_3-\alpha_1=\alpha_4-\alpha_2 \qquad (1.45)$$

$$2\alpha_3-\alpha_1=\alpha_3+\alpha_4-\alpha_2=2\alpha_5-\alpha_1=\alpha_3-2\alpha_2 \qquad (1.46)$$

From (1.45) and (1.46) we obtain easily

$$\alpha_3=\alpha_5=\alpha_1-2\alpha_2\,;\;\; \alpha_4=-\alpha_2 \qquad (1.47)$$

Now from the first two relations (1.42) we obtain

$$A=\left(\frac{x}{\bar{x}}\right)^{\frac{1}{\alpha_1}}=\left(\frac{y}{\bar{y}}\right)^{\frac{1}{\alpha_2}}$$

Or, $$\frac{y}{x^{\alpha}}=\frac{\bar{y}}{\bar{x}^{\alpha}} \qquad (1.48)$$

Writing $\frac{\alpha_2}{\alpha_1}=\alpha$ and considering the rest of equations of (1.42) and as well as the relations (1.47) we can write

$$\frac{u}{x^{1-2\alpha}}=\frac{\bar{u}}{\bar{x}^{1-2\alpha}};\frac{v}{x^{-\alpha}}=\frac{\bar{v}}{\bar{x}^{-\alpha}};\;\frac{u_e}{x^{1-2\alpha}}=\frac{\bar{u}_e}{\bar{x}^{1-2\alpha}} \qquad (1.49)$$

Thus the combinations of variables in equations (1.48) and (1.49) are seen to be invariant under the linear group of transformations (1.42) and these are called absolute variables. These invariants are similarly variables if the boundary conditions (1.39) can be transformed and expressed independent of x.

We now put $\eta=\frac{y}{x^{\alpha}}, f^*(\eta)=\frac{u}{x^{1-2\alpha}},\; g^*(\eta)=\frac{v}{x^{-\alpha}}$ (1.50)

and $h^*(\eta)=\frac{u_{\varepsilon}}{x^{1-2\alpha}}=C$ (1.51)

Where f^* and g^* are functions of η and $h^*(\eta)=C$ is a constant, since the main stream velocity u_e is function of x only and thus can not be non-constant function of η becomes this would introduce dependence on y.

The boundary conditions (1.39) are now transformed and expressed in-terms of the possible similarity variables given in equations (1.50). Accordingly, using (1.50) and (1.51) the boundary conditions (1.39) reduces to

$$f^* = g^* = 0 \text{ at } \eta = 0 \text{ and } f^* \to C \text{ as } \eta \to \infty . \tag{1.52}$$

Now the two equations (1.37) and (1.38) can be transformed to the following ordinary differential equation in which m is written for $1-2\alpha$ to conform to usual notation and writing $u_e = Cx^m$:

$$mf^* - \frac{1-m}{2}\eta f^{*\prime} + g^{*\prime} = 0 \tag{1.53}$$

and $$mf^{*2} - \frac{1-m}{2}\eta f^* f^{*\prime} + g^* f^{*\prime} = mC^2 + \nu f^{*\prime\prime} . \tag{1.54}$$

The above equations can be solved numerically subject to boundary condition (1.52). In the present thesis we would consider the group theoretic method, as illustrated above to work out the problem.

The following numerical techniques are used to solve nonlinear ordinary differential equation corresponding to viscous in compressible flow problem in the present thesis.

1.12 Shooting method

Consider a two-point boundary value problem $y'' = f(x, y, y')$ with the boundary conditions $y(a) = y_0$ and $y(b) = y_1$. We set $y' = z$ and $z' = f(x, y, z)$ with $y(a) = y_0$. Since these equations are non-linear, we can not get the solution as super position principle. In order to integrate the above first-order system as an initial value problem we require a value for $z(a)$ but no such value is given. However, if we take a guess for $z(a)$ and use it to compute a numerical solution we can then compare the calculated value for y at $x=b$ with the given boundary condition $y(b) = y_1$ and adjust the guess value, $z(a)$, to give a better approximation for the solution. Since the derivative at $x=a$ gives the trajectory of the computed solution, this technique is called a shooting method. Basically, we seek a solution which satisfies $y(b) = y_1$. By successively refining the interval a suitable solution can be found. It is also possible to improve the solution by linear interpolation or other root finding methods. The shooting method depends on the choice of the initial slope, i.e., $y'(a)$ which is required to start the integration. Actually, the problem of boundary value problem is converted to a problem of initial value problem (IVP). We have considered only second-order linear differential equation. First, we shall discuss shooting method for linear equation which is very easy and also sure to convergence.

1.13 Shooting method for second order linear ordinary differential equations

We first consider a homogeneous following second-order boundary value problem as

$$u''(x) = p(x)u'(x) + q(x)u(x), a < x < b \tag{1.55}$$

subject to the boundary conditions

$$u(a) = r_1, \quad u(b) = r_2 \tag{1.56}$$

where $p(x)$, $q(x)$ and $r(x)$ are given functions in $[a, b]$.

Since equation (1.55) is linear, for any two linearly independent solutions $u_1(x)$ and $u_2(x)$ of (1.55), the general solution is given by

$$u(x) = c_1 u_1(x) + c_2 u_2(x) \tag{1.57}$$

where c_1, c_2 are arbitrary constants.

We take initial conditions as

$$u_1(a) = r_1, \quad u_1'(a) = d_1 \text{ (guess value)} \tag{1.58}$$

d_1 being any constant. Using the fourth order Runge-Kutta method, we integrate (1.55) with the initial conditions (1.58) and obtain $u_1(b)$.

Similarly, we shall take

$$u_2(a) = r_1, \quad u_2'(a) = d_2 \text{ (Guess value)} \tag{1.59}$$

We determine $u_2(b)$. The conditions (1.57) give us

$$r_1 = u(a) = c_1 u_1(a) + c_2 u_2(a) = c_1 r_1 + c_2 r_1 \tag{1.60}$$

i.e., $c_1 + c_2 = 1$ (1.61)

and $r_2 = u(b) = c_1 u_1(b) + c_2 u_2(b)$ (1.62)

Solving (1.61) and (1.62) we get the values of c_1 and c_2 as

$$c_2 = \frac{r_2 - u_1(b)}{u_2(b) - u_1(b)}, \quad c_1 = 1 - c_2 \tag{1.63}$$

The required solution at a point, say $x_i \in [a,b]$ is given by

$$u(x_i) = c_1 u_1(x_i) + c_2 u_2(x_i) \tag{1.64}$$

(i=1 ,2,3, …………,n-1), n being the number of equal subintervals of $[a,b]$.

1.14 Numerical Solution of non-linear ordinary differential equation using shooting method

Shooting method can be applied for the solution of non-linear ordinary differential equation. The basic strategy is to convert the boundary value problem into and initial value problem as discussed earlier. It is possible to convert higher order differential equation into a system of first order differential equations. The convergent of the method depends on the proper choice of initial slope and the solution is very sensitive for the guess of slope value particularly for highly nonlinear equation. The fluid flow equation is basically non-linear in nature. In this study boundary layer flows in different context are transformed into ordinary non-linear equations using the concept of similar flow. The problem is a boundary value problem. The shooting method may be applied efficiently for obtaining numerical solution of non-linear self-similar equation arising from boundary layer flow in fluid mechanics and other types of boundary value problems. Let us discuss the solution of the following non-linear equation using shooting method for ready reference.

We shall discuss the well known Falkner-Skan equation (self-similar equation) for a laminar boundary layer flow in fluid mechanics as represented by

$$f''' + \frac{m+1}{2} ff'' + m[1 - (f')^2] = 0 \tag{1.65}$$

It is a third-order nonlinear ordinary differential equation with a parameter m.

Here $f(\eta)$ is a dimensionless stream function and the parameter m is a (constant) dimensionless pressure gradient ranging the values in $-0.0904 < m < \infty$. The derivation and its physical significance are elaborately discussed in any standard text book in viscous fluid mechanics. It is to be solved subject to the following two-point boundary conditions:

$$\text{at } \eta = 0,\ f = 0 \text{ and } f' = 0 \tag{1.66}$$

$$\text{at } \eta = \eta_\infty,\ f' = 1. \tag{1.67}$$

Here $\eta = \eta_\infty$ corresponds to the edge of the boundary layer: for computational purposes, η_∞ is chosen arbitrarily to be larger than δ, (the boundary layer thickness). If appropriate boundary conditions are supplied at $\eta = 0$, we can integrate outward once only to obtain a solution, using fourth-order Runge-Kutta method. To satisfy Eqs. (1.66) and (1.67) for the nonlinear equation (1.65), we need to iterate using a shooting method or otherwise.

A shooting method that can be used to solve the Falkner-Skan equation or other ordinary nonlinear differential equations was developed by Keller (1968). One of the features of this method is the systematic way by which new values of $f''(0)$ are determined. The traditional trial-and-error searching technique is replaced by Newton's method (see Isaacson and Keller (1966). This generally provides quadratic convergence of the iterations and decreases the computation time.

According to Keller's shooting method, we first replace equation (1.65) by a system of three first-order ordinary differential equations. If the unknowns f, f' and f'' are denoted by f, f_1, and f_2 respectively, the system of three first-order equations can be written as

$$f' = f_1 \tag{1.68}$$

$$f_1' = f_2 \tag{1.69}$$

$$f_2' = -\frac{m+1}{2} f f_2 - m\left(1 - f_1^2\right) \tag{1.70}$$

Here f_1 is related to the x-component velocity, but f_2 is related to the velocity gradient. The boundary conditions given by equations (1.66) and (1.67) are replaced by

$$f(0) = 0,\ f_1(0) = 0 \tag{1.71}$$

$$f_1(\eta_\infty) = 1 \tag{1.72}$$

It is to be noted that $f_2(0)$ is related to wall shear stress. Now, we choose

$$f_2(0) = s,\ s \text{ being a guess value} \tag{1.73}$$

The problem is to find s such that the solution of the initial value problem, satisfies the outer boundary condition (1.72).That is, if we denote the solution of this initial value problem by $\left[f(\eta,s), f_1(\eta,s), f_2(\eta,s)\right]$, then we seek s such that

$$f_1(\eta_\infty, s) - 1 \equiv \phi(s) = 0. \tag{1.74}$$

To solve Equation (1.74), we employ Newton's method. This widely used method for finding the root of an equation by successive approximation is most simply explained by reference to Fig.1. If s^0 is a guess for a root of the equation $\phi(s) = 0$, a better guess, s^1, is (usually) obtained by extrapolation to the axis of the tangent to $y = \phi(s)$ at $s = s^0$ and so on. This yields the iterates s^v defined by

$$s^{v+1} = s^v - \frac{\phi(s^v)}{(d\phi/ds)(s^v)} \equiv s^v - \frac{f_1(\eta_\infty, s^v) - 1}{(\partial u/\partial s)(\eta_\infty, s^v)},\ v = 0,1,2,\ldots \tag{1.75}$$

Obviously, $s^{v+1} = s^v$ only if $\phi(s^v) = 0$, and then equation (1.72) is satisfied exactly. In general, this will not occur for any finite v; instead we iterate until $\left|s^{v+1} - s^v\right| \le \varepsilon$ for some sufficiently small ε. Then the condition (1.72) is also approximately satisfied.

In order to obtain the derivative of f_1 with respect to s, we take the derivatives of equations (1.68)-(1.70), (1.71), and (1.73). This leads to the following linear differential equations, known as the variational equations, for equations (1.68)-(1.70):

$$F' = U \tag{1.76}$$

$$U' = V \tag{1.77}$$

$$V' = -\frac{m+1}{2}(fV + \upsilon F) + 2muU \tag{1.78}$$

and to the initial conditions, $\eta = 0$:

$$F(0) = 0,\ U(0) = 0 \text{ and } V(0) = 1 \tag{1.79}$$

Here $F(\eta, s) \equiv \frac{\partial f}{\partial s}$, $U(\eta, s) \equiv \frac{\partial u}{\partial s}$ *and* $V(\eta, s) \equiv \frac{\partial v}{\partial s}$ (1.80)

Note that the boundary condition at η_∞ has disappeared, being replaced by the last of equation (1.79).

Once the initial-value problem given by equations (1.68)-(1.70), (1.71), and (1.76)-(1.79) is solved, $u(\eta_\infty, s^v)$ and $U(\eta_\infty, s^v)$ are known, and consequently the next approximation to $\upsilon(0)$, namely, s^{v+1}, can be computed from equation (1.75). A number of integration methods can be used to solve the initial value problem. Here, because of its simplicity, we use a fourth-order Runge-Kutta method.

The theory behind the Runge-Kutta method (Isaacson and Keller [27]) is rather complicated and is not necessary for the present discussion. Its virtue is that it is "self starting": that is, a forward step of integration, evaluating the integral by numerical approximation, requires only the initial conditions already given for the ODE. Other, non-self-starting methods require details of the solution for several previous steps before a new step can be executed by integrating polynomial fits to the previous values of the derivatives, but the Runge-Kutta method can "pull itself up by its bootstraps". As might be excepted, non-self-starting methods are faster to run, and where many calculations are to be done, it is common to start with a Runge-Kutta method and then switch over to a non-self-starting method such as the predictor-corrector method described by Isaacson and Keller (1966).

1.15 Summary of the problems worked out in the thesis

In Chapter-I, we have presented the basic developments in viscous fluid dynamics which are relevant to the problems undertaken in this research works. The different kinds of non-Newtonian fluid models, their mathematical representations and drawbacks of the models are discussed. Viscous flows are governed by coupled nonlinear partial differential equations. The analytical or closed form solutions are difficult to find. In this work we mainly seek similarity solution for a particular class of flow problems. The heat transfer in the concept of boundary layer is also discussed. The numerical solutions are sought for most flow cases under consideration. The shooting method for nonlinear equations is developed for approximate solutions of flow fields and heat transfer quantities. We shall now give the summary of the problems worked out in the thesis.

The Chapter-II is devoted to the steady boundary layer mixed convection flow of Newtonian fluid in a vertical stretching sheet in porous medium. Mixed convection in porous medium is very interesting for investigations. The partial slip condition is relevant in such flow problems and is used here. The self-similar equations governing the flow are obtained using similarity transformations and are solved numerically shooting method. From the analysis it reveals that with the increasing value of slip parameter the velocity decreases resulting the increase of the temperature. The increase in Prandtl number (Pr) decreases the velocity along the sheet as well as the velocity boundary layer thickness. In addition, the wall skin friction and heat transfer from the sheet decrease with the increase of boundary slip. The physical explanations for each of the effects are discussed.

Chapter-III deals with the solutions for MHD steady boundary layer slip flow and heat transfer over a stretching surface in presence of heat source or sink. The effect of velocity slip parameter on a viscous incompressible fluid is to suppress the velocity field which in turn causes the enhancement of the temperature field. The rate of heat transfer decreases with the increase of velocity as well as thermal slip parameter. The results pertaining to the present study indicate that due to internal heat generation thermal boundary layer increases. The boundary-layer edge is reached faster as Pr increases. The increasing Prandtl number has a suppressive effect on temperature. It is hoped that, the physics of flow over the stretching sheet can be utilized as the basis for many engineering and scientific applications with the help of our present model. The results pertaining to the present study may be useful for the different model investigations. The findings of the present problem are also of great interest in those areas where the surface layers are being stretched.

In Chapter-IV, an analysis is made to find the distribution of the chemically reactant solute in the flow of an electrically conducting viscous incompressible fluid over a stretching sheet subjected to magnetic field applied externally. The governing partial differential equations along with the appropriate boundary conditions for flow field and reactive solute are transformed into a set of non-linear ordinary differential equations by using scaling group of transformations (Lie-group of transformations). An exact analytical solution is obtained for the velocity field in such a flow. Using this velocity field, we obtain numerical solution for the reactant concentration field. It reveals that contaminate solute transfers from the plate are enhanced with the increase of the magnetic field and decreases with the increase of Schmidt number. The curve corresponding reactant solute decreases its value significantly with the increase of chemical reaction-rate parameter. The problem under consideration is relevant in chemical engineering processes.

In Chapter-V, we investigate the effect of diffusion of chemically reactive species on forced convective boundary layer flow over a porous flat plate in a porous media. The reaction-rate of the reactive species is considered such that it is inversely proportional to position along the plate. A self-similar set of equations are obtained and then solved numerically using shooting method. This analysis reveals that due to the permeability of the porous medium the velocity increases but the concentration decreases. The suction reduces the thicknesses of momentum and thermal boundary layers but blowing enhances their thicknesses. Both for the Schmidt number and the reaction rate parameter, the reactive concentration profile decrease.

In Chapter-VI, we obtained solution of the boundary layer flow and heat transfer with internal heat generation or absorption for two classes of visco-elastic fluid viz. over a stretching sheet. The governing equations are transformed into self-similar ordinary differential equations by similarity transformations. The flow equation relating to momentum equation is solved analytically and the heat equation by using the Kummer's function. The analysis reveals that for the increase in magnitude of visco-elastic parameter, both the velocity and temperature at a fixed point increase for second-grade fluid but decrease for Walter's liquid B. Due to increasing Prandtl number and heat sink parameter, the thermal boundary layer thickness reduces, whereas increasing heat source parameter increases the thickness of the thermal boundary layer.

CHAPTER II

Effects of partial slip on boundary layer mixed convection flow towards a vertical stretching sheet in porous medium

2.1 Introduction

The flow due to stretching sheet in a porous medium is a very important problem in fluid dynamics due to its significant applications in polymer processing industries, several biological processes and many others. Crane (1970) who was first investigated the flow over a linearly stretching plate, gave an exact similarity solution in closed analytical form for steady boundary layer flow of an incompressible viscous fluid. The work of Crane is extended by many researchers such as Gupta and Gupta (1977), Chen and Char (1988) taking the effects of heat and mass transfer under various physical conditions. Chen (1998) demonstrated the mixed convection laminar flow adjacent to continuously stretching vertical sheet and Ishak et al. (2008) analyzed the hydromagnetic effects to this mixed convection flow. Elbashbeshy and Bazid (2004) studied the heat transfer over a stretching surface in a porous medium, with internal heat generation and suction or injection. Ouaf (2005) obtained an exact solution of thermal radiation effects on magnetohydrodynamic (MHD) steady asymmetric flow over a porous stretching sheet. In recent past, Kumaran et al. (2009) investigated the MHD flow past a quadratically stretching permeable sheet.

A common feature is assumed in all those investigations stated above, the flow field obeys the conventional no-slip condition at the boundary. But, in certain situations the no-slip boundary condition does not hold good and is required to replace by partial slip boundary condition. Beavers and Joseph (1967) investigated the fluid flow over a permeable wall using the slip boundary condition. The effects of slip boundary condition on the flow of Newtonian fluid past a stretching sheet were studied by Andersson (2002) and Wang (2002). On the other hand, Ariel (2008) analyzed the flow of a visco-elastic fluid over a stretching sheet with partial velocity slip.

In the present investigation, we have studied the effects of partial slip boundary condition on steady Newtonian boundary layer mixed convection flow and heat transfer in a vertical stretching sheet in porous medium. The variable initial temperature distribution along the sheet is taken into account. Using similarity solution technique, the governing partial differential equations are transformed into a set of non-linear self-similar ordinary differential equations. Then the transformed self-similar equations are solved numerically by using shooting method with the approximate choice of guess value. The results are plotted in figures and discussed physically in all contexts.

2.2 Mathematical formulation of the flow problem

Let us consider the mixed convection boundary layer flow of a viscous incompressible fluid due to a vertical sheet stretching with linear velocity. The sheet embedded in a porous medium. Using the usual velocity and thermal boundary layers approximations, the equations for steady two-dimensional flow and the temperature equation may be written in usual notations as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{2.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\upsilon}{k}u\pm g\beta_1(T-T_\infty) \tag{2.2}$$

and $$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}\,, \tag{2.3}$$

where u and v are velocity components in x- and y-directions respectively, ρ is the fluid density, μ is the coefficient of fluid viscosity, υ $(=\mu/\rho)$ is the kinematic fluid viscosity, k is the permeability of the porous medium, β_1 is the volumetric coefficient of thermal expansion, g is the acceleration due to gravity, T is the temperature, κ is the fluid thermal conductivity, c_p is the specific heat. The last term on the right-hand side of equation (2.2) represents the influence of the thermal buoyancy force on the flow field, with "+" sign corresponding to the buoyancy assisting flow region and "–" sign the buoyancy opposing flow region. In Fig1, the physical description of this flow in a heated vertical stretching sheet is presented.

The appropriate boundary conditions for the velocity components and the temperature are given by

$u=ax+D(\partial u/\partial y)$(Velocity Slip Condition), $v=v_w$ at $y=0$; $u\to 0$ as $y\to\infty$ (2.4)

and $T=T_w=T_\infty+T_0x$ at $y=0$; $T\to T_\infty$ as $y\to\infty$, (2.5)

where a is stretching constant with $a>0$, D denotes the slip length, $T_w=T_\infty+T_0x$ is variable temperature distributed over sheet and T_∞ is the free stream temperature assumed to be constant with $T_w > T_\infty$. Here v_w is a prescribed distribution of suction ($v_w<0$) or blowing ($v_w>0$), applied through the porous vertical stretching sheet where the velocity of the sheet is so small so that it does not affect the porosity of the sheet as well as the medium.

We now introduce the stream function $\psi(x,y)$ of this two-dimensional steady flowas

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x}. \tag{2.6}$$

The continuity equation (2.1) is satisfied automatically and the momentum equation (2.2) and the temperature equation (2.3) take the following forms:

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \upsilon \frac{\partial^3 \psi}{\partial y^3} - \frac{\upsilon}{k}\frac{\partial \psi}{\partial y} \pm g\beta_1 (T - T_\infty), \tag{2.7}$$

$$\text{and } \frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}. \tag{2.8}$$

The boundary conditions (2.4) of the flow reduce to

$$\frac{\partial \psi}{\partial y} = ax + D\frac{\partial^2 \psi}{\partial y^2}, \frac{\partial \psi}{\partial x} = -v_w \text{ at } y = 0; \frac{\partial \psi}{\partial y} \to 0 \text{ as } y \to \infty. \tag{2.9}$$

Next, we introduce the dimensionless variables for ψ and T as given below:

$$\psi = \sqrt{a\upsilon}\, x f(\eta) \text{ and } T = T_\infty + (T_w - T_\infty)\theta(\eta), \tag{2.10}$$

where the similarity variable η is defined as $\eta = y(a/\upsilon)^{1/2}$.

Using Equation (2.10) we obtain following self-similar equations as

$$f''' + ff'' - f'^2 - k^* f' + \lambda\theta = 0 \tag{2.11}$$

$$\text{and } \theta'' + Pr(f\theta' - f'\theta) = 0, \tag{2.12}$$

where $k^*=(\upsilon/ka)$ is the permeability parameter of the porous medium, $\lambda=\pm Gr_x/Re_x^2$ [the significance of "±" is same as above] is the buoyancy or mixed convection parameter with $\lambda>0$ and $\lambda<0$ corresponding to the assisting flow and opposing flow, respectively, $Gr_x=g\beta_1(T_w-T_\infty)x^3/\upsilon^2$ is the local Grashof number, $Re_x=ax^2/\upsilon$ is the local Reynolds number and $Pr=\mu c_p/\kappa$ is the Prandtl number.

The boundary conditions (2.9) and (2.5) reduce to the following forms:

$$f(\eta)=S, f'(\eta)=1+\delta f''(\eta) \text{ at } \eta=0;\ f'(\eta)\to 0 \text{ as } \eta\to\infty \tag{2.13}$$

$$\text{and } \theta(\eta)=1 \text{ at } \eta=0;\ \theta(\eta)\to 0 \text{ as } \eta\to\infty, \tag{2.14}$$

where $S=-v_w/(a\upsilon)^{1/2}$, $S>0$ (i.e. $v_w<0$) is corresponding to suction and $S<0$ (i.e. $v_w>0$) is corresponding to blowing, $\delta=L(a/\upsilon)^{1/2}$ is the velocity slip parameter.

2.3 Numerical method

The nonlinear coupled differential equations represented by (2.11) and (2.12) along with the boundary conditions (2.13) and (2.14) form a two point boundary value problem (BVP) and are solved using shooting technique with fourth order Runge-Kutta method by transferring into an initial value problem (IVP). In this method, we have to choose a suitable finite value of $\eta\to\infty$, say η_∞. We set following first-order systems as

$$\left.\begin{aligned} f' &= z, \\ z' &= p, \\ p' &= z^2 - fp + k^* z - \lambda\theta \end{aligned}\right\} \tag{2.15}$$

and

$$\left.\begin{aligned} \theta' &= q, \\ q' &= -Pr(fq - z\theta) \end{aligned}\right\} \tag{2.16}$$

with the following boundary conditions

$$f(0)=S,\ z(0)=1+\delta p(0),\ \theta(0)=1. \tag{2.17}$$

To solve (2.15) and (2.16) as an IVP we must need values for $p(0)$ i.e. $f''(0)$ and $q(0)$ i.e. $\theta'(0)$ but no such values are given. The initial guess values for $f''(0)$ and $\theta'(0)$ are chosen and the solution procedure is carried out. We compare the calculated values of $f'(\eta)$ and $\theta(\eta)$ at $\eta_\infty(=10)$ with the given boundary conditions $f'(\eta_\infty)=0$ and $\theta(\eta_\infty)=0$ and adjust values $f''(0)$ and $\theta'(0)$ using Secant method to give better approximation for the solution. The step-size is taken as $h=0.01$. The process is repeated until we get the results correct up to the desired accuracy of 10^{-5} level.

2.4 Results and discussions

Numerical computations have been carried out for various values of the parameters viz., the permeability parameter k^*, the buoyancy or mixed convection parameter λ, the Prandtl number Pr, velocity slip parameter δ and suction/blowing parameter S. The computed values are depicted in some figures and the physical clarifications are given for all cases.

First, we concentrate on variation of velocity field and temperature distribution for different values of the permeability parameter k^*. The velocity and temperature profiles for several values of k^* are represented in Fig1 and Fig2, respectively. It is noticed that the velocity along the sheet decreases when k^* increases. On the other hand, the value of the temperature profile increases significantly with the increase of k^*.

Next, we discuss the effects of the mixed convection parameter λ on velocity and temperature. In Fig3 and Fig4, the velocity and temperature curves are depicted for different values of λ. Velocity at a point increases if λ is increased, while temperature at a fixed point decreases with increasing λ. This buoyancy effect on velocity profiles is very important in physical and practical point of views.

Since this flow is mixed convection type, the Prandtl number also affects the velocity fields in addition with the temperature and this can be seen from Fig5 and Fig6. From these figures we observe that the velocity along the sheet as well as velocity boundary layer thickness decrease with increase in Pr and also temperature at a point and the thermal boundary layer thickness rapidly decrease with increasing Pr. The Prandtl number effects, specially, on the velocity field is realistic in flow dynamics.

Fig7 and Fig8 explain the effects of non-conventional slip boundary condition on the velocity and temperature. For increase in slip parameter δ causes the decrease of velocity at a fixed point and the increase of temperature. But the boundary layer thickness of velocity increases with increase in the slip except very large values of δ, because for very large value of δ (tends to infinity) the boundary layer will disappear. Also the thermal boundary layer thickness decreases with increasing δ. In Table1 we compare our obtained results for various values of slip parameter with the published results of Andersson (2002) for force convection flow in non-porous medium and found in excellent agreement.

Finally, the effects of suction/blowing parameter S on velocity and temperature profiles are presented in Fig9 and Fig10. For the increase of applied suction, the velocity and temperature profiles at a fixed η decrease, while with the increase in blowing the velocity and temperature increase.

The variation of skin-friction coefficient $f''(0)$ and the temperature gradient at the sheet $-\theta'(0)$ which is proportional to the rate of heat transfer from the sheet, for different values of parameters k^*, λ, Pr and δ with S=0.5 are represented in Table2. The skin-friction coefficient increases with the permeability parameter and Prandtl the number, where as it decreases with the mixed convection parameter and the slip parameter. The rate of heat transfer increase with increasing values of λ and Pr and decreases with increasing k^* and δ.

2.5 Concluding remarks

The main objective of this investigation is to study the effects of partial slip on the steady boundary layer mixed convection flow and the heat transfer over a vertical stretching sheet in a porous medium. The obtained self-similar nonlinear coupled ordinary differential equations are solved numerically using shooting method. The similarity variable is used to obtain self-similar equations for the momentum and temperature equations in this boundary layer flow and heat transfer. From our study it can be concluded that all parameters viz. the permeability parameter, the mixed convection parameter, the Prandtl number and suction/blowing parameter affect the flow field as well as heat transfer. The analysis reveals that with increase of mixed convection parameter (λ), the velocity boundary layer thickness increases and the thermal boundary layer thickness decreases. When Prandtl number increases, the thicknesses of velocity as well as thermal boundary layers decrease. With the increase of velocity slip parameter, velocity along the sheet decreases and the temperature increases. The rate of heat transfer from the sheet decreases with increasing value of slip parameter. The main finding of this work is that the slip condition opposes the rate of heat transfer from the sheet. This finding may be useful in many industrial problems relating with rate of heat transfer.

Table1 The skin-friction coefficient $f''(0)$ for various values of δ with $k^*=0$, $\lambda=0$ and $S=0$.

δ	Andersson (2002)	Present study
0.0	−1.0000	−1.000000
0.1	−0.8721	−0.872083
0.2	−0.7764	−0.776377
0.5	−0.5912	−0.591195
1.0	−0.4302	−0.430160
2.0	−0.2840	−0.283980
5.0	−0.1448	−0.144840
10.0	−0.0812	−0.081242
20.0	−0.0438	−0.043789
50.0	−0.0186	−0.018597
100.0	−0.0095	−0.009551

Table2 The values of skin-friction coefficient $f''(0)$ and the temperature gradient at the sheet $-\theta'(0)$ for several values of k^*, λ, δ and Pr with $S=0.5$.

k^*	λ	δ	Pr	$f''(0)$	$-\theta'(0)$
1.0	0.07	0.1	1.0	−1.373000	2.033619
2.0	0.07	0.1	1.0	−1.496903	1.898219
0.5	−0.2	0.1	1.0	−1.388889	1.851392
0.5	0.0	0.1	1.0	−1.265100	2.136948
0.5	0.2	0.1	1.0	−1.167556	2.292551
0.5	0.07	0.0	1.0	−1.460227	2.404774
0.5	0.07	0.2	1.0	−1.066353	2.046521
0.5	0.07	0.1	0.75	−1.221917	1.431772
0.5	0.07	0.1	1.5	−1.240760	4.640295

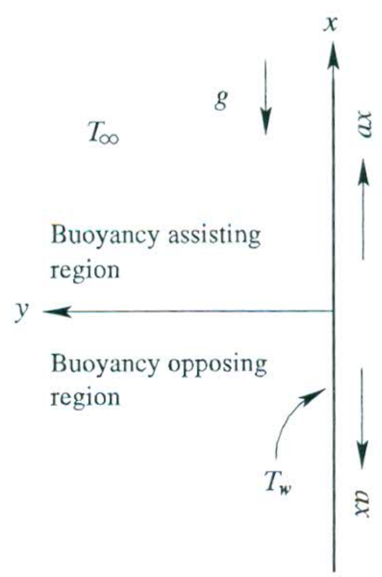

Fig1 A sketch of the physical problem.

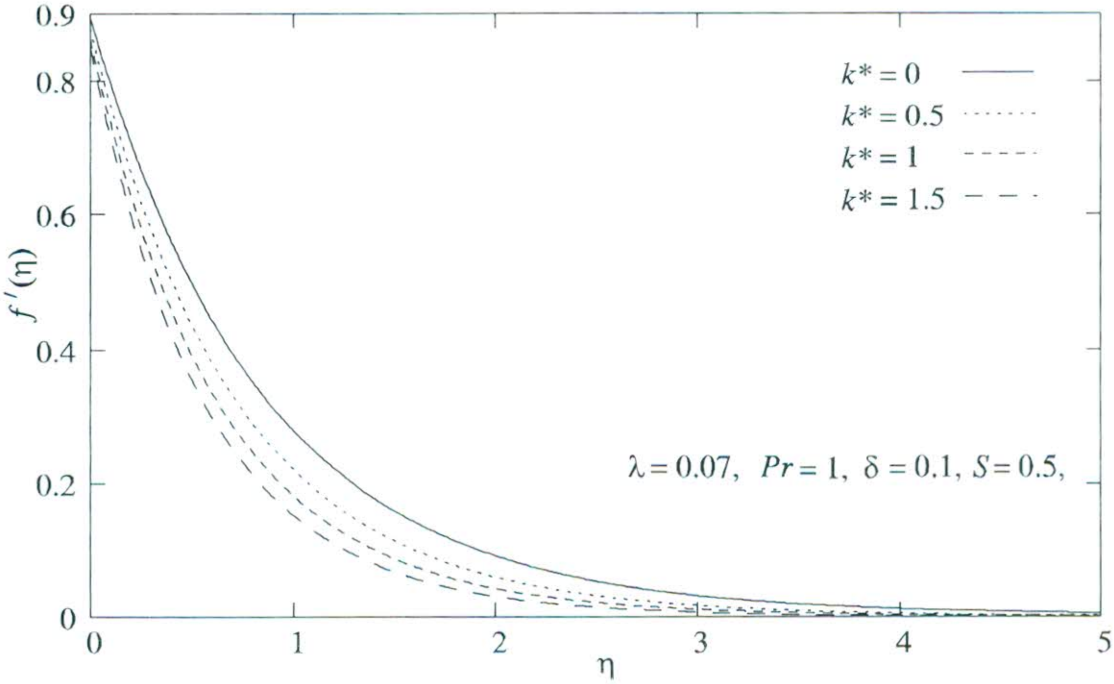

Fig2 Self-similar velocity profiles $f'(\eta)$ for various values of permeability parameter k^*.

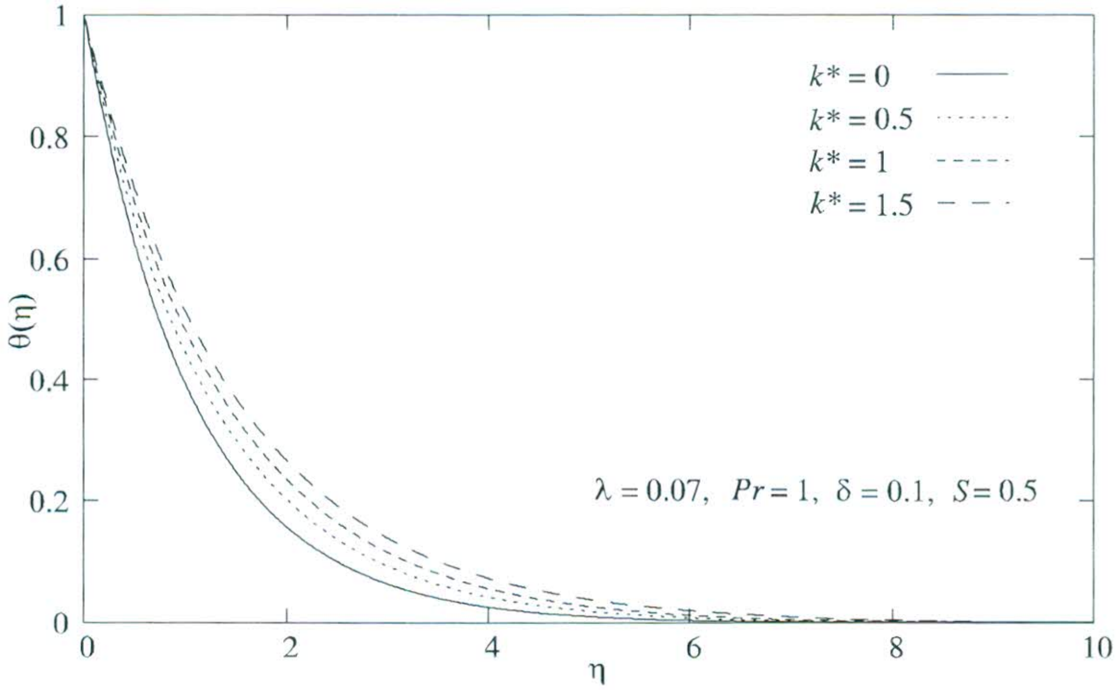

Fig3 Self-similar temperature profiles $\theta(\eta)$ for various values of k^*.

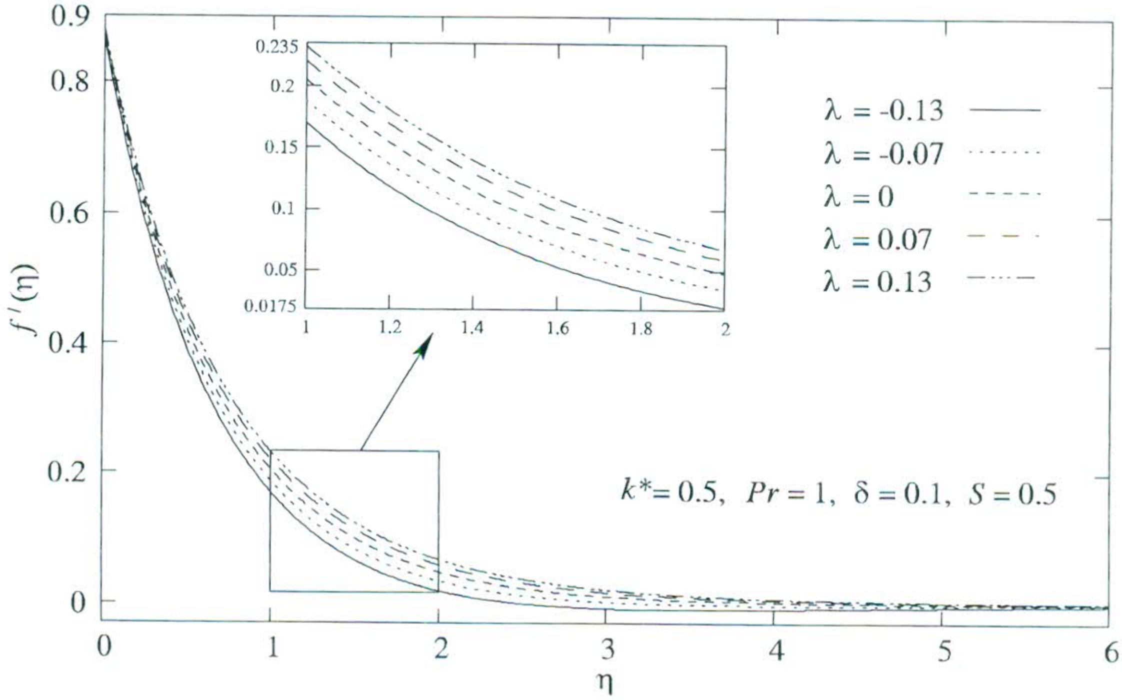

Fig4 Self-similar velocity profiles $f'(\eta)$ for various values of λ.

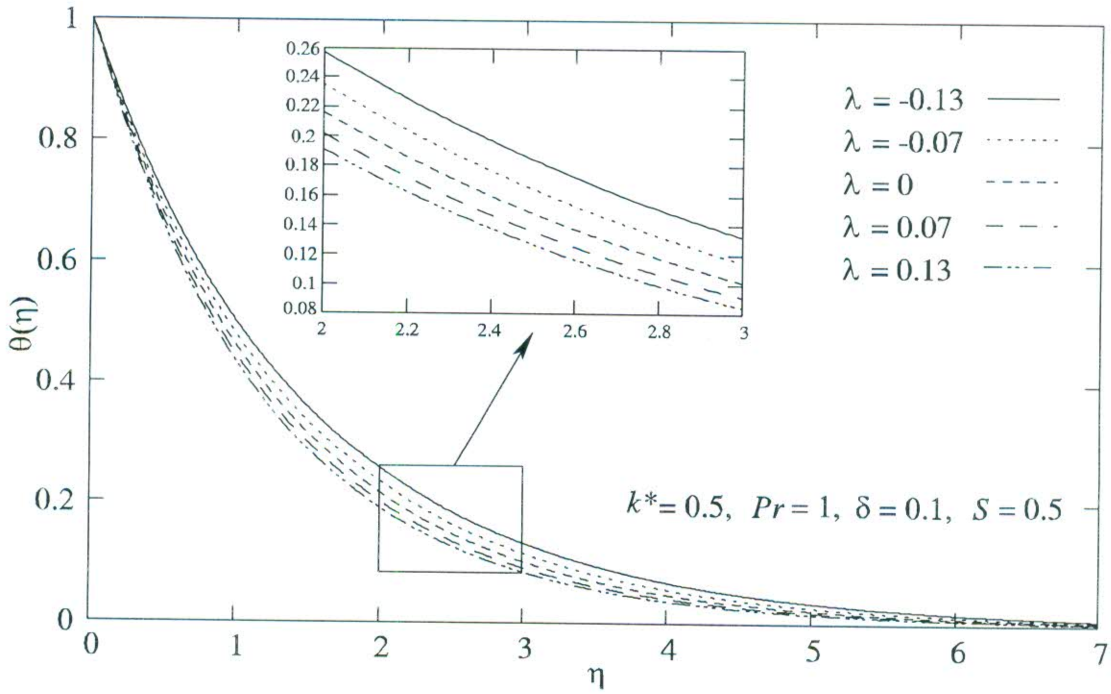

Fig5 Self-similar temperature profiles $\theta(\eta)$ for various values of λ.

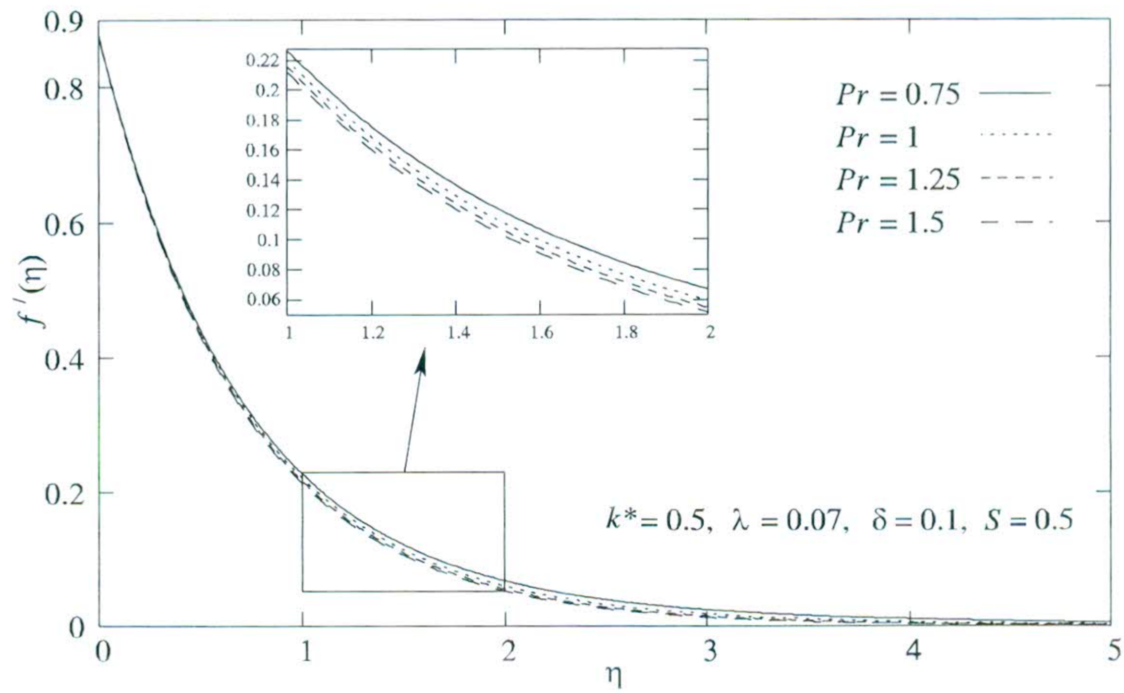

Fig6 Self-similar velocity profiles $f'(\eta)$ for various values of Prandtl numbers Pr.

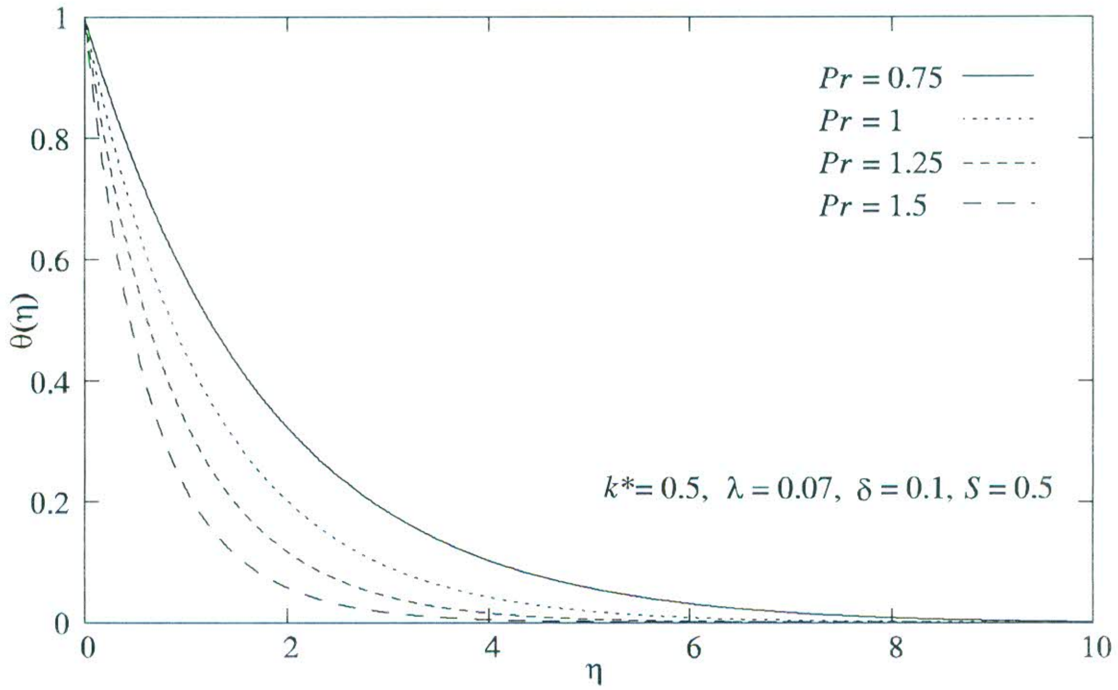

Fig7 Temperature profiles $\theta(\eta)$ for various values of Prandtl numbers *Pr*.

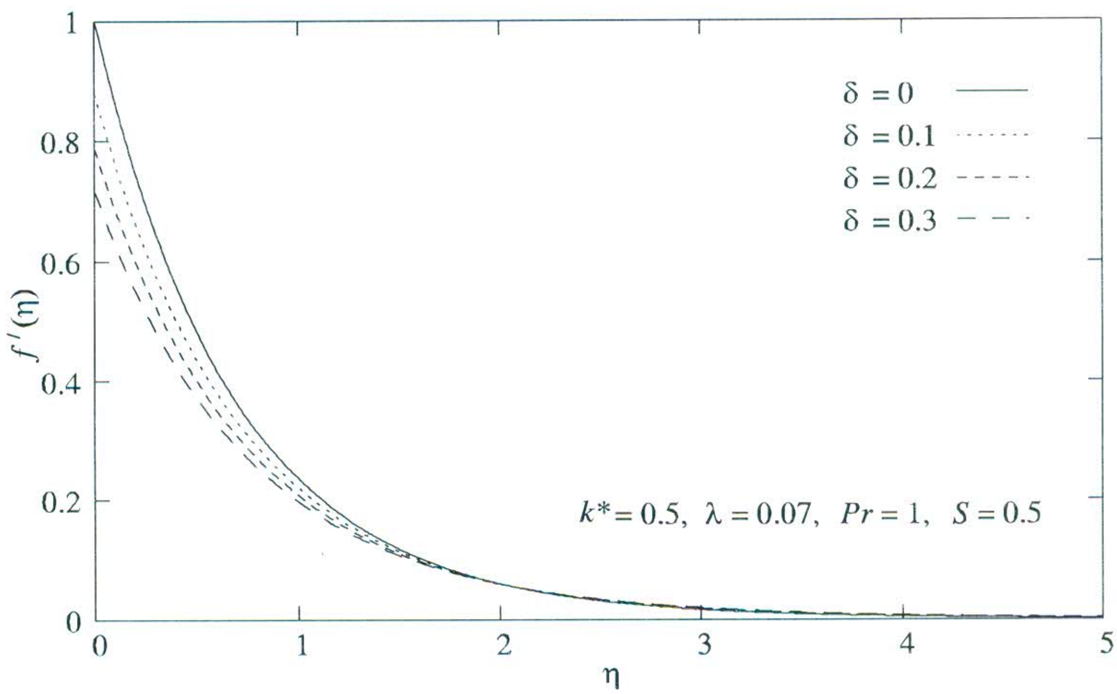

Fig8 Velocity profiles $f'(\eta)$ for various values of slip-parameter δ.

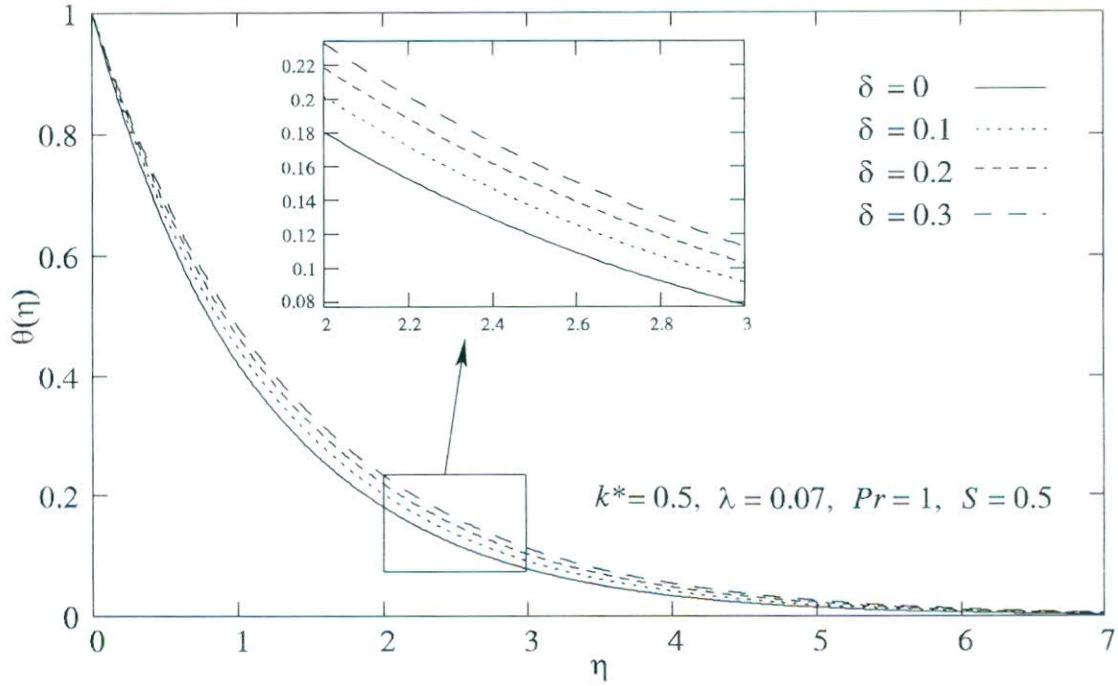

Fig9 Temperature profiles $\theta(\eta)$ for various values of slip-parameter δ.

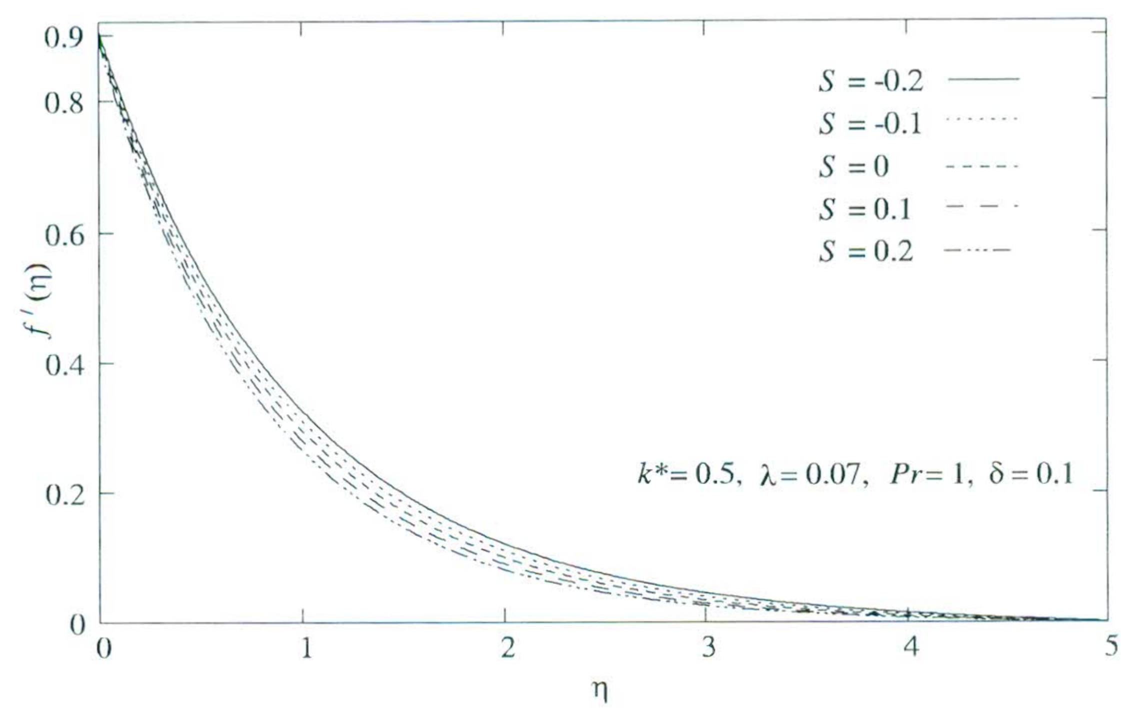

Fig10 Velocity profiles $f'(\eta)$ for various values of suction/blowing parameter S.

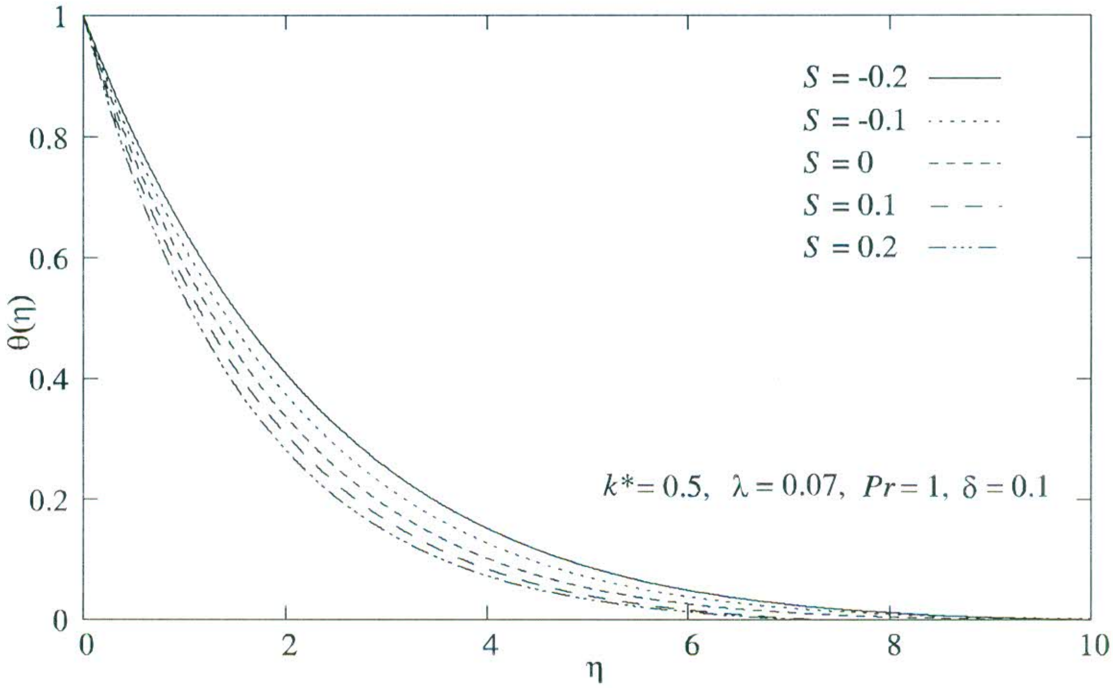

Fig11 Temperature profiles $\theta(\eta)$ for various values of suction/blowing parameter S.

CHAPTER III

Lie group analysis of MHD boundary layer equation for flow past a heated stretching sheet with velocity and thermal slips at boundary in presence of heat source/sink

3.1 Introduction

The boundary layer flow over a continuously stretching surface is an often-encountered in many engineering processes. There are lots of applications in industries such as hot rolling, wire drawing, glass-fiber production, etc. This type of study has gained considerable attention due to its applications in industries and important bearings on several technological processes. Crane (1970) investigated the flow caused by the stretching of a sheet. Many researchers such as Gupta and Gupta (1977), Chen and Char (1988), Dutta et al. (1985) extended the work of Crane (1970) by including the effect of heat and mass transfer analysis under different physical situations.

All the above mentioned studies continued their discussions by assuming the no slip boundary conditions. The non-adherence of the fluid to a solid boundary, also known as velocity slip, is a phenomenon that has been observed under certain circumstances [Yoshimura and Prudhomme (1998)]. Recently, many researchers [Wang (2002), Andersson (2002), Ariel (2008), Abbas et al. (2009) etc.] investigated the flow problems taking slip flow condition at the boundary.

The magnetohydrodynamics (MHD) of an electrically conducting fluid is encountered in many problems in geophysics, astrophysics, engineering applications and other industrial areas. Hydromagnetic free convection flows have a greet significance for the applications in the fields of steller and planetary magnetospheres, aeronautics. Engineers employ magnetohydrodynamics principles in the design of heat exchangers, pumps, in space vehicle propulsion, thermal protection, control and re-entry and in creating novel power generating systems. However, hydromagnetic flow and heat transfer problems have become more important industrially. In many metallurgical processes involve the cooling of many continuous strips or filaments by drawing them through an electrically conducting fluid subject to a magnetic field, the rate of cooling can be controlled and final product of desired characteristics can be achieved. Another important application of hydromagnetic to metallurgy lies in the purification of molten metals from nonmetallic inclusions by the application of a magnetic field. The study of magnetohydrodynamic (MHD) flow of an electrically conducting fluid is of considerable interest in modern metallurgical and metal-working processes. Moreover, control of boundary layer flow is of practical significance. Several methods have been developed for the purpose of artificially controlling the behaviour of the boundary layer. The application of MHD principle is an important method for affecting the flow field in the desired direction by altering the structure of the boundary layer. Recently Mukhopadhyay et al. (2005) investigated the MHD boundary layer flow with variable fluid viscosity over a heated stretching sheet. A new dimension is added to the above mentioned study by considering the effects of velocity and thermal slips at the wall.

In certain applications, the effects of working fluid in heat generation (source) or absorption (sink) effects are important. Representative studies dealing with these effects have been reported by authors such as Gupta and Sridhar (1985), Abel and Veena (1998).

The present work deals with fluid flow and heat transfer over a stretching sheet in presence of heat source/sink. Most of the researchers try to obtain the similarity solutions in such cases using the similarity variables. But in this chapter, a special form of Lie group transformations, known as scaling group of transformations is used to find out the full set of symmetries of the problem and then to study which of them are appropriate to provide group-invariant or more specifically similarity solutions. Because group-theoretic method is the only rigorous mathematical method to find all the symmetries of a given differential equation and no ad hoc assumptions or a prior knowledge of the equation under investigation is needed. Moreover, this method unifies almost all known exact integration techniques for both ordinary and partial differential equations. This method can be used as a tool for finding the similarity solutions for those problems where the similarity solutions can not be found easily by usual method. The system remains invariant due to some relations among the parameters of the scaling group of transformations. Using this transformation, a third order and a second order ordinary differential equations corresponding to the momentum and the energy equations are derived. These equations are solved numerically using shooting method. The effects of the magnetic parameter, velocity and thermal slip parameters, heat source / sink parameter and the influence of Prandtl number on velocity and temperature fields are investigated and analyzed with the help of their graphical representations.

3.2 Equations of motion

We consider a steady two-dimensional flow of an electrically conducting viscous incompressible fluid over a heated stretching sheet in the region $y > 0$. Keeping the origin fixed, two equal and opposite forces are applied along the x-axis which results in stretching of the sheet and a uniform magnetic field of strength B_0 is imposed along the y-axis.

The continuity, momentum and energy equations governing such type of flow are written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \upsilon\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho}u, \tag{3.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{\rho c_p}(T - T_\infty), \tag{3.3}$$

where u and v are the components of velocity respectively in the x and y-directions, T is the temperature, κ is the coefficient of thermal conductivity, $Q_0 (Js^{-1}m^{-3}K^{-1})$ is the dimensional heat generation or absorption coefficient, c_p is the specific heat, ρ is the fluid density (assumed constant), μ is the coefficient of fluid viscosity, σ is the electrical conductivity of the conducting fluid, $\upsilon(=\mu/\rho)$ is the kinematic viscosity of the fluid, B_0 is the strength of applied magnetic field.

3.2.1 Boundary conditions of the flow problem

The appropriate boundary conditions for the problem are given by

$$u = ax + D\upsilon\frac{\partial u}{\partial y} \text{ (Velocity Slip)}, \; v = 0, \; T = T_w + N\frac{\partial T}{\partial y} \text{ (Thermal Slip) at } y = 0, \tag{3.4}$$

$$u \to 0, T \to T_\infty \text{ as } y \to \infty. \tag{3.5}$$

Here $a(>0)$ is a stretching constant, T_w is the uniform wall temperature, T_∞ is the temperature far from the sheet. D and N are respectively the velocity and thermal slip factors. The no-slip case is recovered for D=N=0.

3.2.2 Method of solution

We now introduce the following relations for u, v and θ as

$$u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x} \text{ and } \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{3.6}$$

where ψ is the stream function of this two-dimensional steady flow.

Using the relations (3.6) in the boundary layer equation (3.2) and in the energy equation (3.3) we get the following equations

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \upsilon \frac{\partial^3 \psi}{\partial y^3} - \frac{\sigma B_0^2}{\rho}\frac{\partial \psi}{\partial y}, \tag{3.7}$$

and $$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{\kappa}{\rho c_p}\frac{\partial^2 \theta}{\partial y^2} + \frac{Q_0}{\rho c_p}\theta, \tag{3.8}$$

The boundary conditions (3.4) and (3.5) then become

$$\frac{\partial \psi}{\partial y} = ax + D\upsilon \frac{\partial^2 \psi}{\partial y^2}, \frac{\partial \psi}{\partial x} = 0, \theta = 1 + N\frac{\partial \theta}{\partial y} \text{ at } y = 0. \tag{3.9}$$

$$\frac{\partial \psi}{\partial y} \to 0, \theta \to 0 \text{ as } y \to \infty. \tag{3.10}$$

3.2.3 Scaling group of transformations

We now introduce the simplified form of Lie-group transformations namely, the scaling group of transformations [Mukhopadhyay et al. (2005)],

$$\Gamma : x^* = xe^{\varepsilon\alpha_1}, y^* = ye^{\varepsilon\alpha_2}, \psi^* = \psi e^{\varepsilon\alpha_3}, u^* = ue^{\varepsilon\alpha_4}, v^* = ve^{\varepsilon\alpha_5}, \theta^* = \theta e^{\varepsilon\alpha_6}. \tag{3.11}$$

Equation (3.11) may be considered as a point-transformation which transforms co-ordinates $(x, y, \psi, u, v, \theta)$ to the co-ordinates $(x^*, y^*, \psi^*, u^*, v^*, \theta^*)$.

Substituting (3.11) in (3.7) and (3.8) we get,

$$e^{\varepsilon(\alpha_1 + 2\alpha_2 - 2\alpha_3)}\left(\frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi}{\partial y^{*2}}\right) = \upsilon e^{\varepsilon(3\alpha_2 - \alpha_3)}\frac{\partial^3 \psi^*}{\partial y^{*3}} - \frac{\sigma B_0^2}{\rho}e^{\varepsilon(\alpha_2 - \alpha_3)}\frac{\partial \psi^*}{\partial y^*} \tag{3.12}$$

$$e^{\varepsilon(\alpha_1 + \alpha_2 - \alpha_3 - \alpha_6)}\left(\frac{\partial \psi^*}{\partial y^*}\frac{\partial \theta^*}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial \theta^*}{\partial y^*}\right) = \frac{\kappa}{\rho c_p}e^{\varepsilon(2\alpha_2 - \alpha_6)}\frac{\partial^2 \theta^*}{\partial y^{*2}} + \frac{Q_0}{\rho c_p}e^{-\varepsilon\alpha_6}\theta^*. \tag{3.13}$$

The system will remain invariant under the group of transformations Γ, we would have the following relations among the parameters, namely

$\alpha_1 + 2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3 = \alpha_2 - \alpha_3$ and $\alpha_1 + \alpha_2 - \alpha_3 - \alpha_6 = 2\alpha_2 - \alpha_6 = -\alpha_6$.

These relations give $\alpha_1 = \alpha_3$ and $\alpha_2 = 0 = \alpha_6$.

The boundary conditions yield $\alpha_1 = \alpha_4, \alpha_5 = 0$.

Thus the set Γ reduces to a one parameter group of transformations:

$$x^* = xe^{\varepsilon\alpha_1}, y^* = y, \psi^* = \psi e^{\varepsilon\alpha_1}, u^* = ue^{\varepsilon\alpha_1}, v^* = v, \theta = \theta^*. \quad (3.14)$$

Expanding by Taylor's series we get,

$$x^* - x = x\varepsilon\alpha_1,\ y^* - y = 0,\ \psi^* - \psi = \psi\varepsilon\alpha_1,\ u^* - u = u\varepsilon\alpha_1,\ v^* - v = 0,\ \theta^* - \theta = 0. \quad (3.15)$$

In terms of differentials, we get,

$$\frac{dx}{\alpha_1 x} = \frac{dy}{0} = \frac{d\psi}{\alpha_1\psi} = \frac{du}{\alpha_1 u} = \frac{dv}{0} = \frac{d\theta}{0}. \quad (3.16)$$

Solving the above equations we get,

$$y = \eta,\ \psi = xF(\eta),\ \theta = \theta(\eta). \quad (3.17)$$

The equations (3.12) and (2.13) become

$$F^{/2} - FF^{//} = \upsilon F^{///} - \frac{\sigma B_0^2}{\rho} F^{/} \quad (3.18)$$

$$\frac{\kappa}{\rho c_p}\theta^{//} + F\theta^{/} + \frac{Q_0}{\rho c_p}\theta = 0. \quad (3.19)$$

The boundary conditions become

$$F^{/} = a + D\upsilon F^{//}, F = 0, \theta = 1 + N\theta^{/} \text{ at } \eta = 0. \quad (3.20)$$

$$F^{/} \to 0,\ \theta \to 0 \text{ as } \eta \to \infty. \quad (3.21)$$

Introducing $\eta = \upsilon^{\alpha} a^{\beta} \eta^*$, $F = \upsilon^{\alpha'} a^{\beta'} F^*$, $\theta = \upsilon^{\alpha'''} a^{\beta''} \bar{\theta}$ in equations (3.18) and (3.19) we get

$$\alpha^{/} = \alpha = \frac{1}{2}, \alpha^{//} = 0, \beta^{/} = -\beta = \frac{1}{2}, \beta^{//} = 0.$$

The equations (3.20) and (3.21) transformed to

$$F^{*/2} - F^* F^{*//} = F^{*///} - M^2 F^{*/}, \quad (3.22)$$

and $\bar{\theta}^{//} + Pr(F^*\bar{\theta}^{/} + L\bar{\theta}) = 0$ (3.23)

where $M = \sqrt{\sigma B_0^2 / \rho a}$ is the magnetic parameter, $Pr = \mu c_p / \kappa$ is the Prandtl number and $L = Q_0 / \rho a c_p$ is the heat source/sink parameter.

Taking $F^* = f$ and $\bar{\theta} = \theta$ the equations (3.22) and (3.23) finally take the following form:

$$f''' + ff'' - f'^2 - M^2 f' = 0, \tag{3.24}$$

and $$\frac{1}{Pr}\theta'' + f\theta' + L\theta = 0. \tag{3.25}$$

The boundary conditions take the following forms

$$f' = 1 + \delta f'', \; f = 0, \; \theta = 1 + \gamma\theta' \text{ at } \eta^* = 0, \tag{3.26}$$

and $$f' \to 0, \; \theta \to 0 \text{ as } \eta^* \to \infty \tag{3.27}$$

where $\delta = D(a\upsilon)^{1/2}$ is the velocity slip parameter and $\gamma = D(c/\upsilon)^{1/2}$ is the thermal slip parameter.

3.3 Numerical method for solution

The above equations (3.24) and (3.25) along with boundary conditions are solved by converting them to an initial value problem. We set

$$f' = z, \; z' = p, \; p' = [z^2 - fp + M^2 z]. \tag{3.28}$$

$$\theta' = q, \; q' = -Pr(fq + L\theta) \tag{3.29}$$

with the boundary conditions

$$f(0) = 0, \; f'(0) = 1 + \delta f''(0), \; \theta(0) = 1 + \gamma\theta'(0). \tag{3.30}$$

In order to integrate (3.28) and (3.29) as an initial value problem we require a value for $p(0)$ i.e. $f''(0)$ and $q(0)$ i.e. $\theta'(0)$ but no such values are given in the boundary. The suitable guess values for $f''(0)$ and $\theta'(0)$ are chosen and then integration is carried out. We compare the calculated values for f' and θ at $\eta = 10$ (say) with the given boundary condition $f'(10) = 0$ and $\theta(10) = 0$ and adjust the estimated values, $f''(0)$ and $\theta'(0)$, to give a better approximation for the solution.

We take the series of values for $f''(0)$ and $\theta'(0)$, and apply the fourth order classical Runge-Kutta method with step-size h=0.01. The above procedure is repeated until we get the results up to the desired degree of accuracy, 10^{-5}.

3.4 Results and discussions

In order to analyse the results, numerical computation has been carried out using the method described in the previous section for various values of the magnetic parameter (M), velocity slip parameter (δ), thermal slip parameter (γ), heat source or sink parameter (L) and the Prandtl number (Pr). For illustrations of the results, numerical values are plotted in the figures (1) to (7). In all figures we take Pr=1.

In order to assess the accuracy of the method, the results (in case of no-slip boundary condition and in the absence of magnetic field and heat source/sink) are compared with those of Grubka and Bobba (1985) and Chen (1998) for local Nusselt number $Nu_x \mathrm{Re}_x^{-1/2} = -\theta'(0)$ for forced convection flow on a linearly stretching surface. The results (presented through Table-1) are found to agree well.

Table-1: Values of $Nu_x \mathrm{Re}_x^{-1/2}$ for several values of Pr with M=0, δ=0, γ=0, L=0

Pr	Grubka and Bobba (1985)	Chen (1998)	Present study
0.01	0.0294	0.02942	0.02944
0.72	1.0885	1.08853	1.08855
1.00	1.3333	1.33334	1.33334
3.00	2.5097	2.50972	2.50971

First, we concentrate on the effects of magnetic parameter M on velocity distribution and heat transfer in case of no-slip condition at the boundary and in the absence of any heat source/sink. In Fig1, horizontal velocity profiles are shown for different values of M (M=0.1, 0.4, 0.6). The horizontal velocity curves show that the rate of transport decreases with the increasing distance (η) of the sheet. In all cases the velocity vanishes at some large distance from the sheet (at η=6). The velocity curves show that the rate of transport is considerable reduced with increasing values of M. This is due to the fact that with the increasing M, the Lorentz force associated with the magnetic field increases and it produces more resistance to the transport phenomena. Same effect of magnetic parameter is noted in case of slip at the boundary. Fig2 exhibits the temperature profiles for the values of M. In each case, temperature is found to decrease with the increase of η until it vanishes at η=7. But the temperature is found to increase for any non-zero fixed value of η with the increase of M.

Now we concentrate in the velocity and temperature distribution for the variation of velocity slip parameter in the absence and presence of magnetic field without heat generation or absorption. Fig3(a) and Fig3(b) demonstrate the effects of velocity slip parameter (δ) in the absence (M=0) and presence (M=0.1) of magnetic field respectively. With the increasing B, the horizontal velocity is found to decrease [Fig3(a) and Fig3(b)]. When slip occurs, the flow velocity near the sheet is no longer equal to the sheet stretching velocity. With the increases in δ, such slip velocity increases and consequently fluid velocity decreases because under the slip condition, the pulling of the stretching sheet can be only partly transmitted to the fluid. It is noted that δ has a substantial effect on the solutions. Fig4(a) and Fig4(b) exhibit that the temperature $\theta(\eta)$ in boundary layer increases monotonically with the increasing values of velocity slip parameter δ in both the cases i.e. in absence (M=0) and presence (M=1) of magnetic parameter respectively. It is interesting to note that the rate of heat transfer decreases with the velocity slip parameter δ.

Fig5(a) represents the shear stress profile for variable velocity slip parameter δ in presence of magnetic field. It is seen that magnitude of shear stress decreases with increasing slip parameter δ. An interesting nature of temperature gradient is noticed from Fig5(b). With increasing δ, the magnitude of temperature gradient decreases upto $\eta \approx 1.8$, but it increases after it.

Fig6(a) and Fig6(b) show the effects of thermal slip γ on temperature and temperature gradient respectively in presence of magnetic field and velocity slip. With the increase of thermal slip parameter γ, less heat is transferred to the fluid from the sheet and so temperature and also the temperature gradient (in magnitude) are found to decrease.

In Fig7(a), effects of heat source/sink on the temperature field is shown, taking fixed values for the other parameters involved in this study. In this case, the temperature field increases with the increase of heat source/sink parameter L. This feature prevails up to certain heights and then the process is slowed down and at a far distance from the wall temperature vanishes. On the other hand, the temperature field increases with the decrease in the amount of heat absorption. Again, far away from the wall, such feature is smeared out. Fig7(b) exhibits the nature of temperature gradient for variable heat source/sink parameter L. The magnitude of temperature gradient decreases upto $\eta \approx 1.8$ with the increasing values of L. But the temperature gradient increases with increasing L for $\eta > 1.8$.

It is also observed that temperature decreases with the increasing Prandtl number because thermal boundary layer thickness decreases due to increase in Pr.

3.5 Conclusions

The present study gives the solutions for MHD steady boundary layer flow of an electrically conducting fluid and heat transfer over a stretching surface in presence of heat source or sink. The effects of velocity and thermal slips at the stretching surface are employed in the analysis. The effect of velocity slip parameter on a viscous incompressible fluid is to suppress the velocity field which in turn causes the enhancement of the temperature field. The rate of heat transfer decreases with the increase of velocity as well as thermal slip parameter. The results pertaining to the present study indicate that due to internal heat generation thermal boundary layer increases. The boundary-layer edge is reached faster as Pr increases. The increasing Prandtl number has a suppressive effect on temperature.

It is hoped that, the physics of flow over the stretching sheet can be utilized as the basis for many engineering and scientific applications with the help of our present model. The results pertaining to the present study may be useful for the different model investigations. The findings of the present problem are also of great interest in those areas where the surface layers are being stretched.

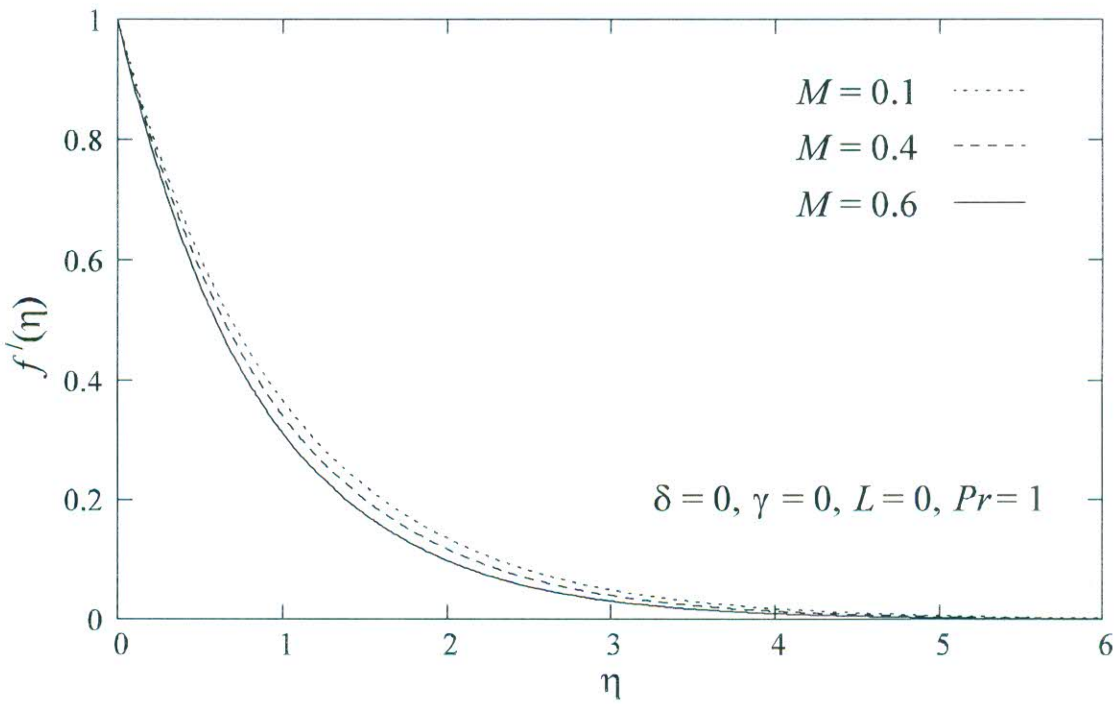

Fig1 Variation of horizontal velocity $f^{/}(\eta)$ with η for several values of magnetic parameter M.

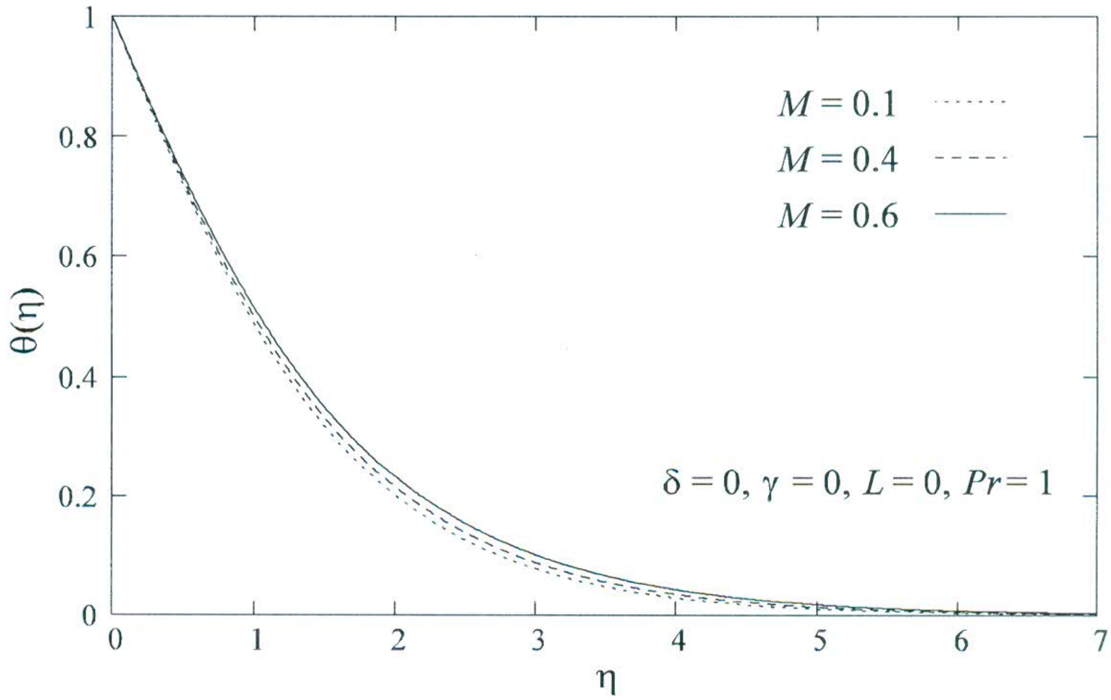

Fig2 Variation of temperature $\theta(\eta)$ with η for several values of magnetic parameter M.

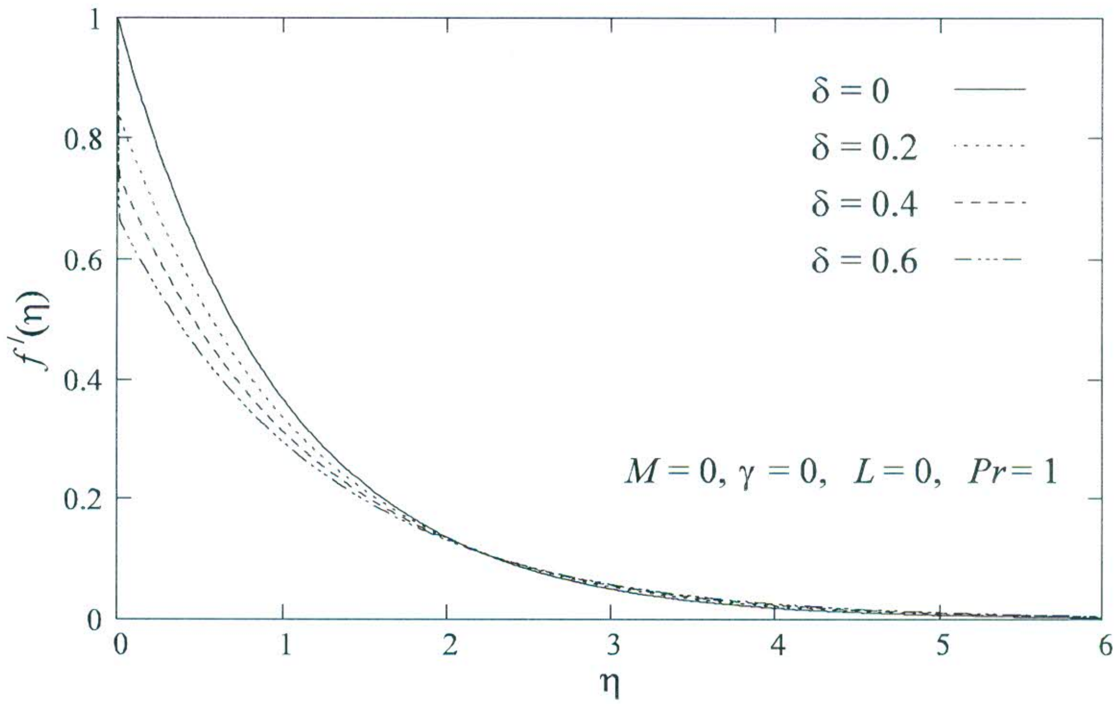

Fig3(a) Variation of horizontal velocity $f^{\prime}(\eta)$ with η for several values of velocity slip parameter δ in the absence of magnetic field.

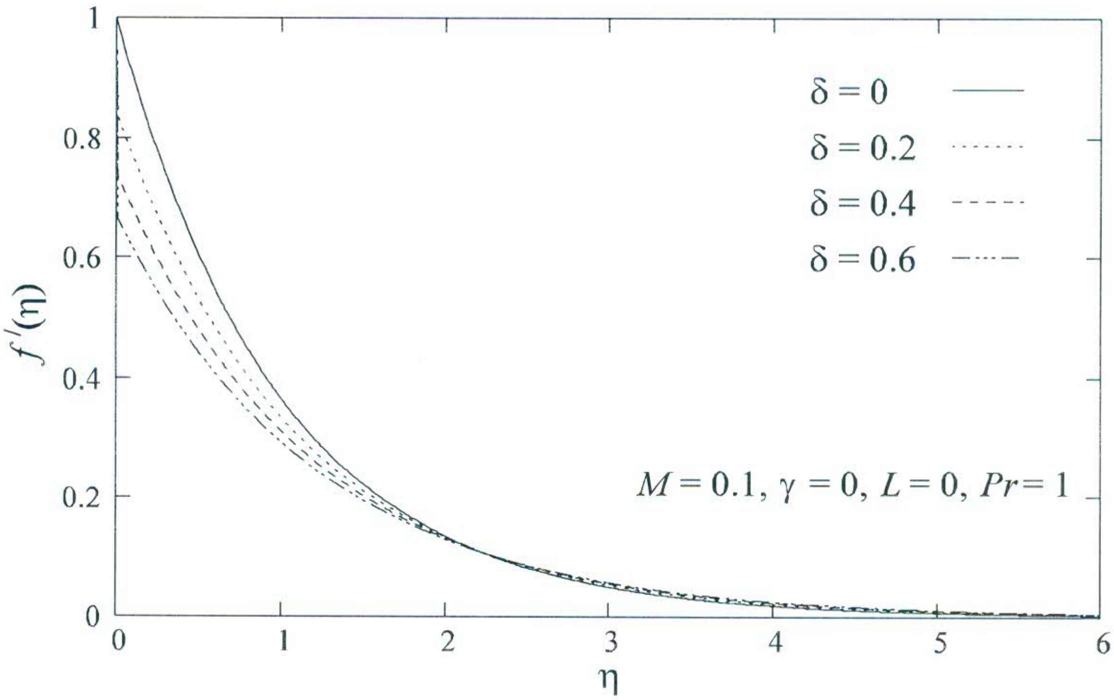

Fig3(b) Variation of horizontal velocity $f'(\eta)$ with η for several values of velocity slip parameter δ in presence of magnetic field.

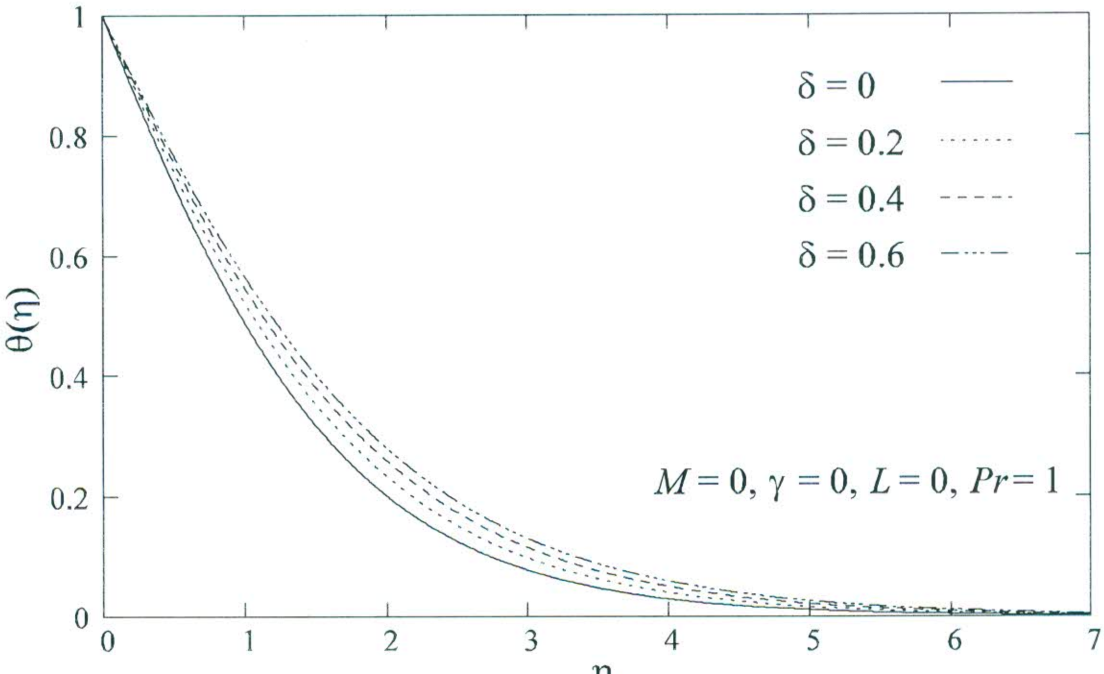

Fig4(a) Variation of temperature $\theta(\eta)$ with η for several values of velocity slip parameter δ in the absence of magnetic field.

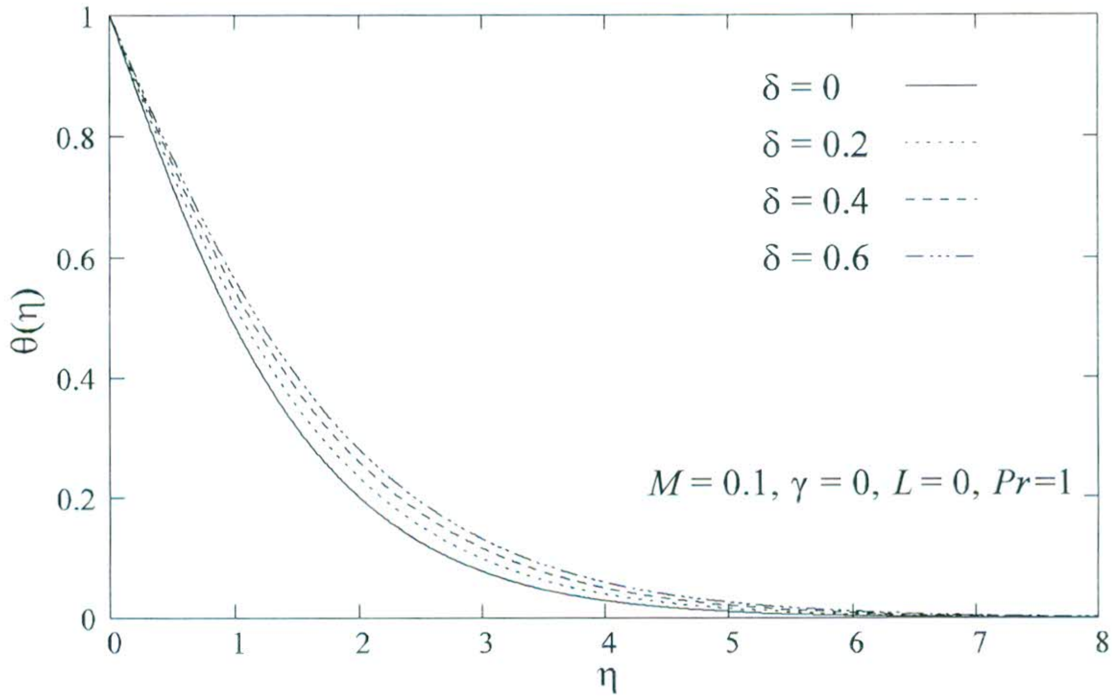

Fig4(b) Variation of temperature $\theta(\eta)$ with η for several values of velocity slip parameter δ in presence of magnetic field.

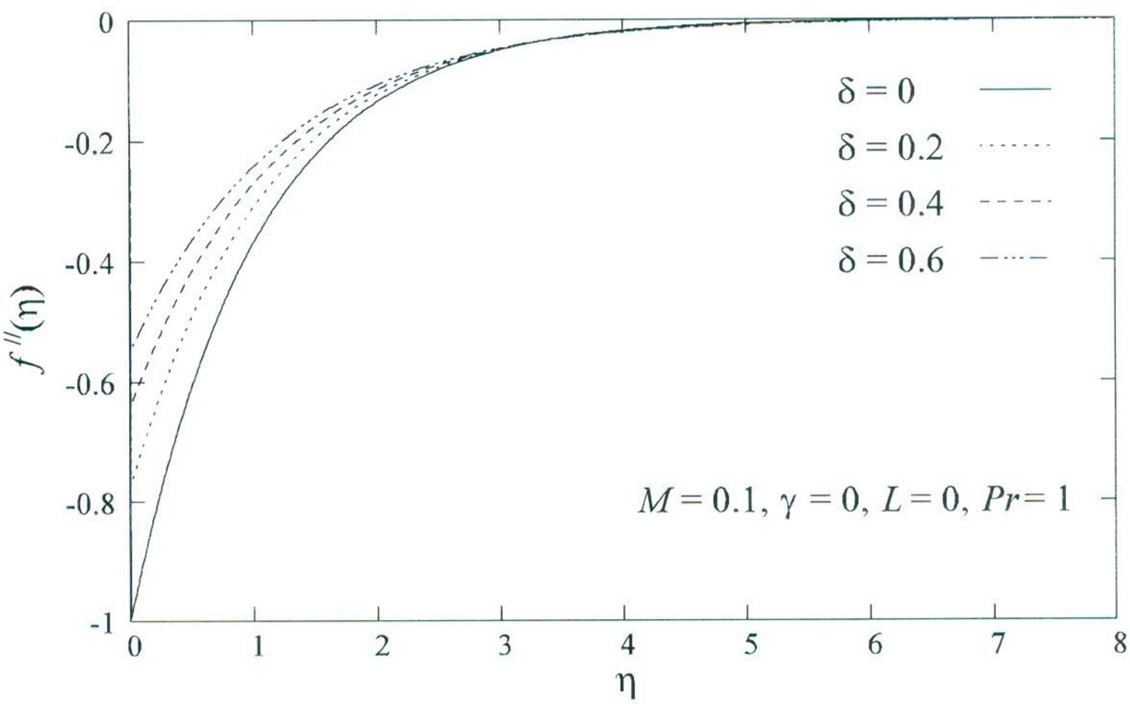

Fig5(a) Variation of shear stress $f^{//}(\eta)$ with η for several values of velocity slip parameter δ in presence of magnetic field.

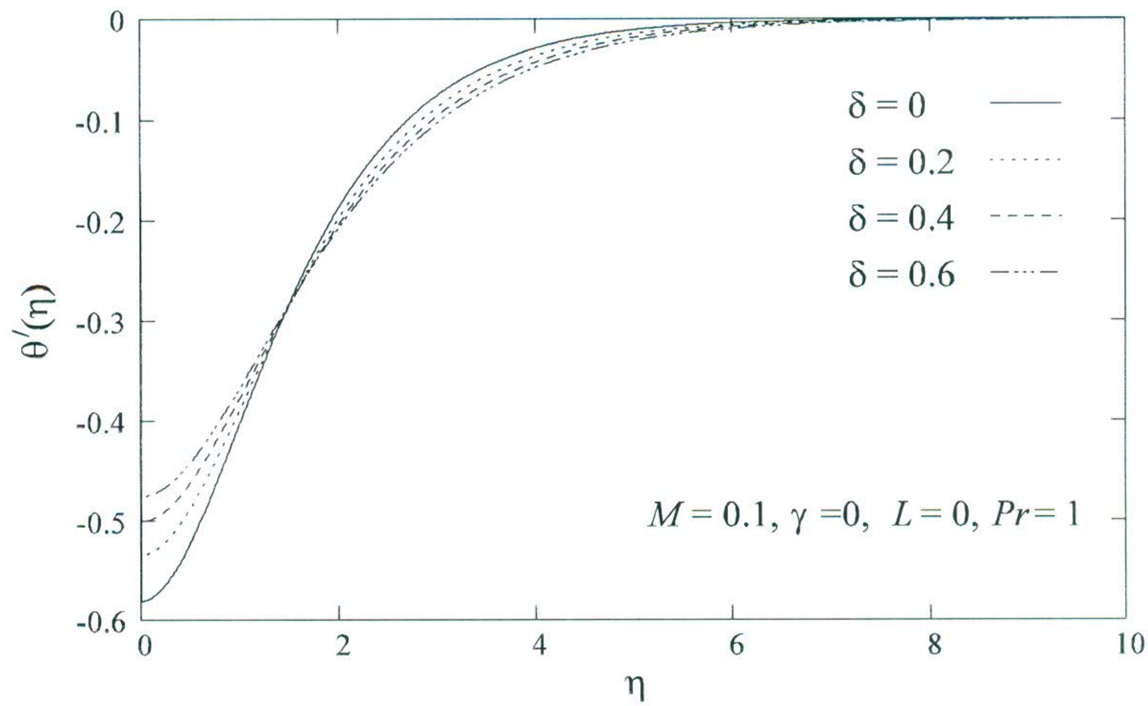

Fig5(b) Variation of temperature gradient $\theta^{/}(\eta)$ with η for several values of velocity slip parameter δ in presence of magnetic field.

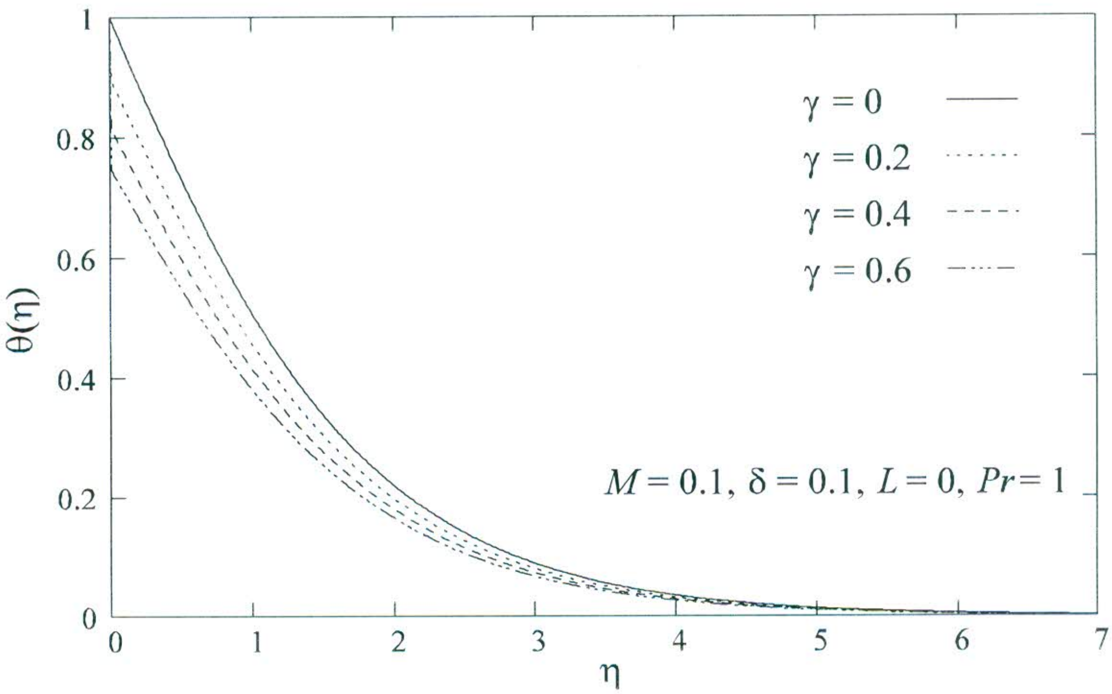

Fig6(a) Variation of temperature $\theta(\eta)$ with η for several values of thermal slip parameter γ.

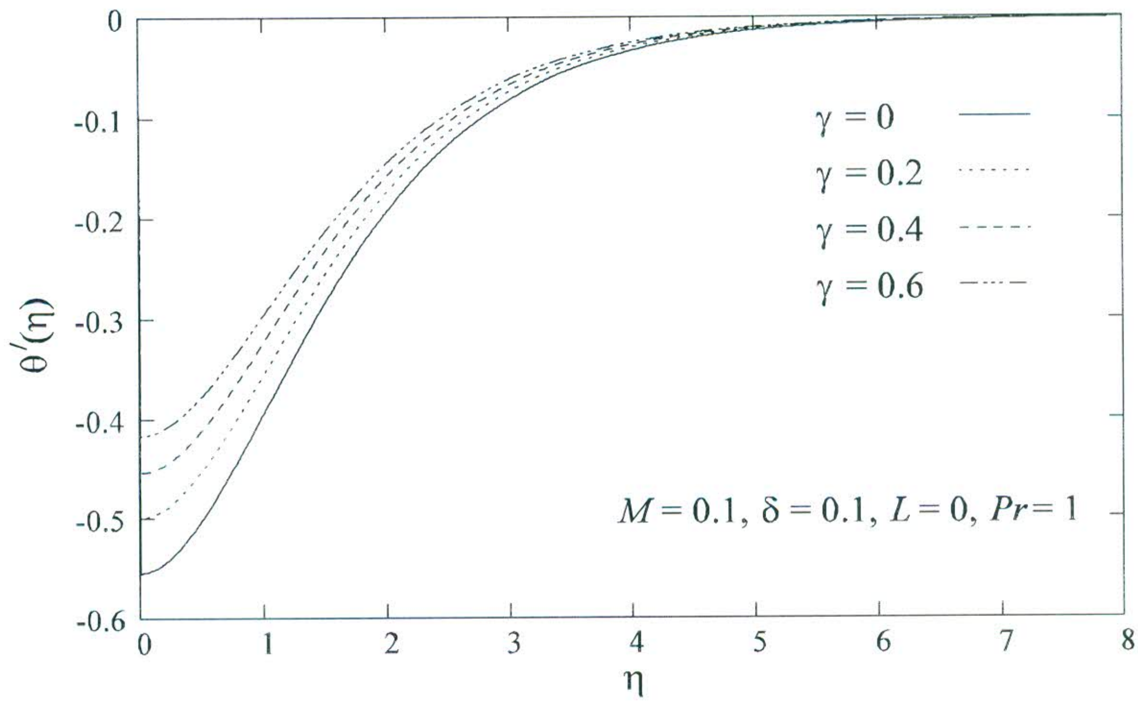

Fig6(b) Variation of temperature gradient $\theta^{/}(\eta)$ with η for several values of thermal slip parameter γ.

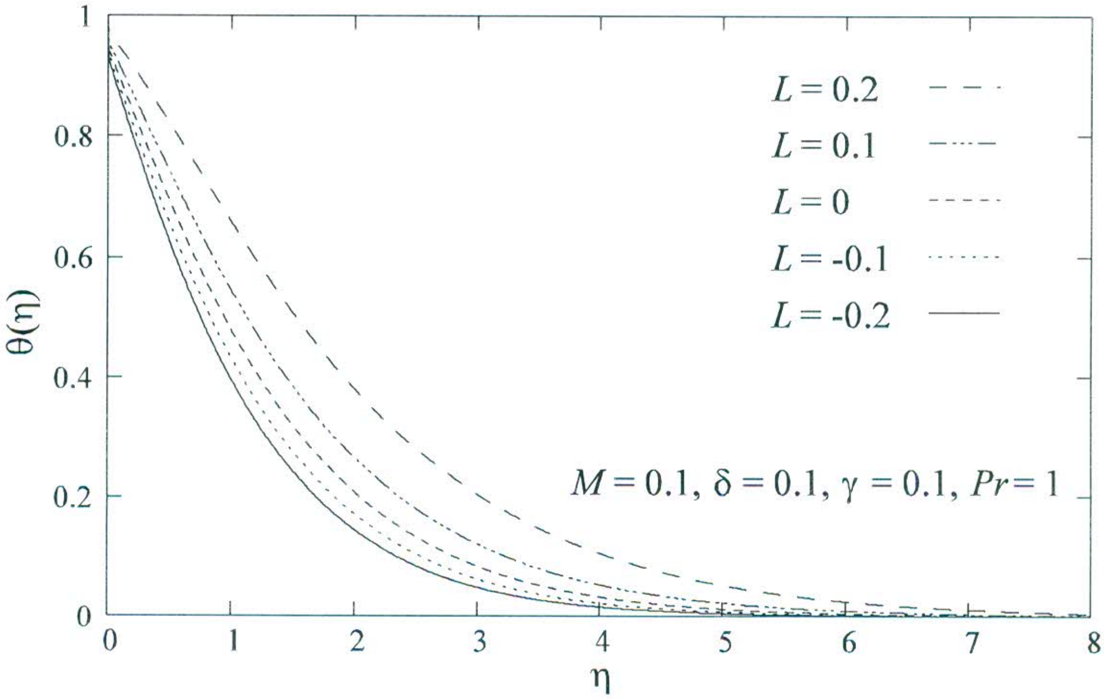

Fig7(a) Variation of temperature $\theta(\eta)$ with η for several values of heat source/sink parameter L.

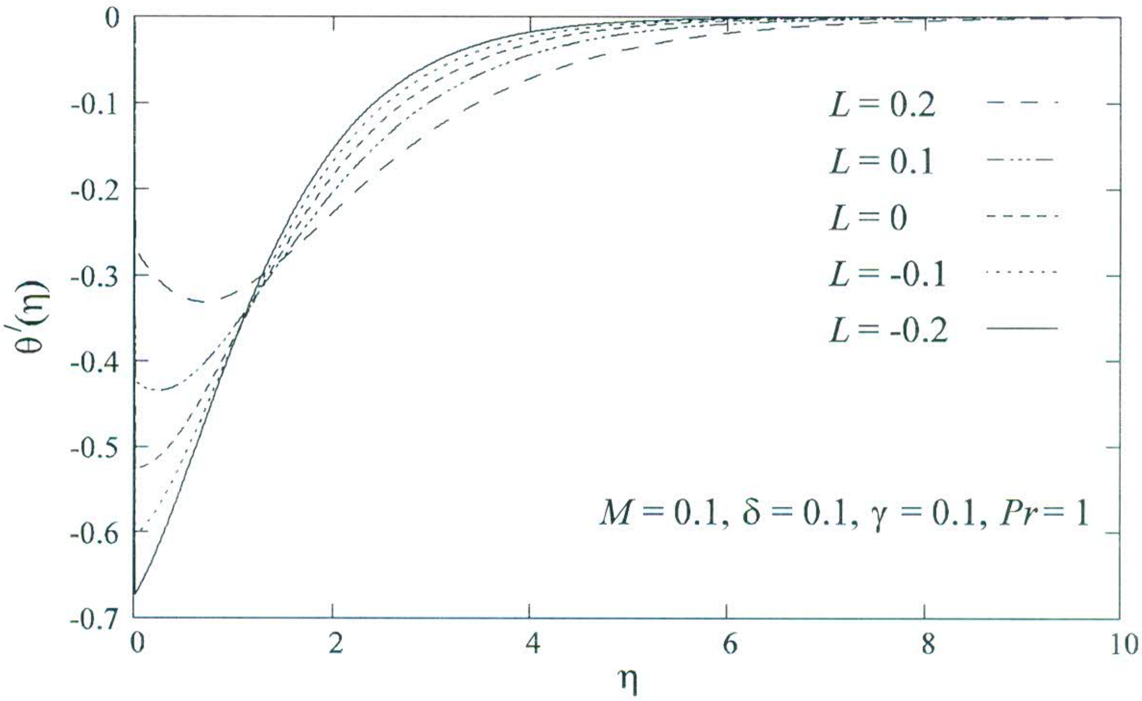

Fig7(b) Variation of temperature gradient $\theta'(\eta)$ with η for several values of heat source/sink parameter L.

CHAPTER IV

Chemically reactive solute distribution in a steady MHD boundary layer flow over a stretching surface*

* The content of this chapter is accepted for publication in Journal of Applied Fluid Mechanics

4.1 Introduction

The flow of an electrically conducting fluid is a classic problem in fluid mechanics now-a days. The distribution of solute under going chemical reaction corresponding to boundary layer flow due to moving sheet are relevant to many practical applications in the metallurgy industry, filaments drawn through a quiescent electrically conducting fluid and the purification of molten metals from non-metallic inclusions. In these situations, the boundary layer flow consideration is appropriate to understand the processes.

The boundary layer equations play a central role in many aspects of fluid mechanics because they describe the motion of a viscous fluid close to a surface. These equations are especially very important since they have the capacity to admit a large number of invariant solutions. Lie-group analysis, also called symmetry analysis was developed by Sophius Lie to find point-transformations that map a given differential equation to itself. This method unifies almost all known exact integration techniques for both ordinary and partial differential equations [Pakdemirli and Yurusoy (1990)] Group analysis is the only rigorous mathematical method to find all symmetries of a given differential equation and no adhoc assumptions or a prior knowledge of the equation under investigation is needed.

The non-linear character of the partial differential equations governing the motion of the fluid produces difficulties in solving the equations. In fluid mechanics, researchers try to obtain the similarity solutions in such cases. In case of scaling group of transformations, the group-invariant solutions are nothing but the well known similarity solutions [Mukhopadhyay et al. (2005)]. A special form of Lie-group of transformations, known as scaling group transformations, is used in this work to find out the full set of symmetries of the flow problem.

Crane (1970) extended the work of Sakiadis (1961a,b) who was the first person to study the laminar boundary layer flow caused by a rigid surface moving in its own plane. The heat and mass transfer problem associated with Newtonian boundary layer flow past a stretching sheet was studied by Gupta and Gupta (1977). Chakrabarti and Gupta (1979) analyzed the magnetohydrodynamic (MHD) flow of Newtonian fluid initially at rest, over a stretching sheet at a different values of parameter related with uniform temperature. Anjali Devi and Ganga (2010) exhibited dissipation effects on MHD nonlinear flow and heat transfer past a stretching porous surface embedded in a porous medium under a transverse magnetic field.

The effects of chemically reactive solute distribution on fluid flow due to a stretching sheet also bear equal importance in engineering researches. The chemical reaction effects were studied by many researchers on several physical aspects. The diffusion of a chemically reactive species in a laminar boundary layer flow over a flat plate was demonstrated by Chambre and Young (1958). The effect of transfer of chemically reactive species in the laminar flow over a stretching sheet explained by Andersson et al. (1994). Takhar et al. (2000) analyzed the flow and mass transfer on a stretching sheet with a magnetic field and chemically reactive species with n-th order reaction. Afify (2004) explicated the MHD free convective flow of viscous incompressible fluid and mass transfer over a stretching sheet with chemical reaction. Liu (2005) studied the momentum, heat and mass transfer of a hydromagnetic flow past a stretching sheet in the presence of a uniform transverse magnetic field. Akyildiz et al. (2006) obtained a solution for diffusion of chemically reactive species in a flow of a non-Newtonian fluid over a stretching sheet immersed in a porous medium. Cortell (2007) investigated the motion and mass transfer for two classes of viscoelastic fluid over a porous stretching sheet with chemically reactive species. Recently, Kandasamy et al. (2010) investigated the effects of temperature-dependent fluid viscosity and chemical reaction on MHD free convective heat and mass transfer with variable stream conditions.

In the present investigation, we have studied the Newtonian MHD boundary layer flow and reactive solute distribution with first order reaction past a stretching surface. The variable initial solute distribution along the surface is taken into account. The scaling group of transformation is applied into the governing equations without adopting any adhoc assumption and finally a set of self-similar ordinary differential equations are obtained. Then the transformed self-similar equations are solved. Exact analytical solution of MHD boundary layer flow is obtained and then solution of concentration is obtained numerically. The obtained results are plotted and discussed physically in various contexts.

4.2 Mathematical formulation of the flow problem and reactive solute distribution

Consider a steady MHD flow of an electrically conducting viscous incompressible fluid undergoing a first order chemical reaction over a stretching surface. The continuity, momentum and reactive concentration equations for governing the flow and concentration distribution in the boundary layer region along the stretching surface may be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}+\frac{1}{\rho}(\mathbf{J}\times\mathbf{B})_x \tag{4.2}$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial^2 y}-R(C-C_\infty), \tag{4.3}$$

where u and v are velocity components in x- and y-directions respectively, υ is the kinematic viscosity, ρ is the density of the fluid, $\mathbf{J}$ is the current density and $\mathbf{B}$ is the magnetic field. One may note that in writing equation (4.2), we have neglected the induced magnetic field since the magnetic Reynolds number R_M for the flow is assumed to be small. Here D is the diffusion coefficient and R denotes the reaction rate of the solute. In this flow problem the working fluid is poorly conducting.

The appropriate boundary conditions for the velocity components and reactant concentration, C are given by

$$\left.\begin{aligned} &u=ax,\ \ v=0 \text{ at } \ y=0\\ &u\to 0 \ \text{ as } \ y\to\infty \end{aligned}\right\} \tag{4.4}$$

and
$$\left.\begin{aligned} &C=C_w=C_\infty+C_0x^n \text{ at } \ y=0\\ &C\to C_\infty \ \text{ as } \ y\to\infty, \end{aligned}\right\} \tag{4.5}$$

where a is assumed to be stretching constant and we consider a variable solute distribution along the stretching surface i.e. $C_w=C_\infty+C_0x^n$, where C_∞ is constant solute at infinity, C_0 is a positive solute constant, n is a power-law exponent, which signifies the change of amount solute in the x-direction and $C_w>C_\infty$.

The magnetic field **B** having components $(0,B_0,0)$ with B_0 non-negative constant, the relation $\nabla\cdot\mathbf{B}=0$ is automatically satisfied. It is noted that the electric current in the flow acts parallel to z-axis (i.e. normal to the plane of the flow). Hence from Ohm's law we get the components of **J** as

$$j_x=0,\ j_y=0,\ j_z=\sigma[E_z+(\mathbf{q}\times\mathbf{B})_z]=\sigma[E_z+uB_0], \tag{4.6}$$

where σ is the constant electrical conductivity of the fluid and E_z is the component of electric field along the z-direction and **q** the velocity vector. Now as the flow is steady, Maxwell's equation gives

$$\nabla\times\mathbf{E}=0 \tag{4.7}$$

where **E** is the electric field which is along the z-axis. This gives from equation (4.7) $\partial E_z/\partial y=0$ and $\partial E_z/\partial x=0$ so that E_z is a function of z only.

Since the induced magnetic field is neglected in view of the assumption $R_M<<1$ electric current in the flow is determined from Ohm's law and not from $\nabla\times\mathbf{B}=\mu_e\mathbf{J}$, μ_e being the magnetic permeability. But the consequence $\nabla\cdot\mathbf{J}=0$ of this equation must be satisfied [Shercliff, (1965)]. This readily gives from equation (4.6), E_z=constant since E_z is independent of x and y. Thus using equation (4.6), we find from equation (4.2),

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u-\frac{\sigma B_0 E_z}{\rho} \tag{4.8}$$

In the free stream one can write

$$\frac{\sigma B_0^2}{\rho}U-\frac{\sigma B_0}{\rho}E_z=0,$$

which gives $E_z=-B_0U$.

Here U is the free-stream velocity and according to this problem $U=0$ as $y\to\infty$.

The momentum equation (4.8) becomes

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u \tag{4.9}$$

Introducing the stream function to this boundary layer flow we get the following relation as

$$u=\frac{\partial\psi}{\partial y},\quad v=-\frac{\partial\psi}{\partial x} \tag{4.10}$$

and the variable C is related by

$$C = C_\infty + \bar{C}\left(C_w - C_\infty\right) \tag{4.11}$$

The continuity equation (4.1) is satisfied clearly by the relations (4.10). In view of the relations (4.10) and (4.11), the equations (4.7) and (4.3) reduce respectively to

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \upsilon \frac{\partial^3 \psi}{\partial y^3} - \frac{\sigma B_0^2}{\rho}\frac{\partial \psi}{\partial y} \tag{4.12}$$

$$x\frac{\partial \psi}{\partial y}\frac{\partial \bar{C}}{\partial x} + n\frac{\partial \psi}{\partial y}\bar{C} - x\frac{\partial \psi}{\partial x}\frac{\partial \bar{C}}{\partial y} = xD\frac{\partial^2 \bar{C}}{\partial y^2} - xR\bar{C} \tag{4.13}$$

and the boundary conditions become

$$\left.\begin{aligned} &\frac{\partial \psi}{\partial y} = ax \text{ and } \frac{\partial \psi}{\partial x} = 0 \quad \text{at} \quad y = 0 \\ &\frac{\partial \psi}{\partial y} \to 0 \quad \text{at} \quad y \to \infty \end{aligned}\right\} \tag{4.14}$$

$$\left.\begin{aligned} &\bar{C} = 1 \quad \text{at} \quad y = 0 \\ &\bar{C} \to 0 \quad \text{at} \quad y \to \infty \end{aligned}\right\} \tag{4.15}$$

4.3 Invariant solution through scaling group of transformations

We now introduce the simplified form of Lie-group transformations, namely, the scaling group of transformations [Tapanidis et al. (2003)] as

$$\Gamma : \begin{cases} x^* = xe^{\varepsilon\alpha_1},\ y^* = ye^{\varepsilon\alpha_2},\ \psi^* = \psi e^{\varepsilon\alpha_3}, \\ u^* = ue^{\varepsilon\alpha_4},\ v^* = ve^{\varepsilon\alpha_5} \text{ and } C^* = \bar{C}e^{\varepsilon\alpha_6} \end{cases} \tag{4.16}$$

The transformation (4.16) may be considered as a point transformation, which transformed the coordinates (x, y, ψ, u, v) to the coordinates $(x^*, y^*, \psi^*, u^*, v^*)$.

Taking the relations (4.16) in to account in equations (4.12) and (4.13), we obtain respectively

$$e^{\varepsilon(\alpha_1+2\alpha_2-2\alpha_3)}\left(\frac{\partial \psi^*}{\partial y^*}\cdot\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}}\right) = e^{\varepsilon(3\alpha_2-\alpha_3)}\upsilon\frac{\partial^3 \psi^*}{\partial y^{*3}} - e^{\varepsilon(\alpha_2-\alpha_3)}\frac{\sigma B_0^2}{\rho}\frac{\partial \psi^*}{\partial y^*} \tag{4.17}$$

and $e^{\varepsilon(\alpha_2-\alpha_3-\alpha_6)}x^*\left(\frac{\partial \psi^*}{\partial y^*}\frac{\partial C^*}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial C^*}{\partial y^*}\right) + e^{\varepsilon(\alpha_2-\alpha_3-\alpha_6)}n\frac{\partial \psi^*}{\partial y^*}C^*$

$$= e^{\varepsilon(2\alpha_2-\alpha_1-\alpha_6)} x^* D \frac{\partial^2 C^*}{\partial y^{*2}} - e^{-\varepsilon(\alpha_1+\alpha_6)} R x^* C^* \quad (4.18)$$

In order that, the system will remain invariant under the group of transformation Γ we then would have the following relations among the transformation parameters

$$\left.\begin{aligned} &\alpha_1 + 2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3 = \alpha_2 - \alpha_3 \\ \text{and } &\alpha_2 - \alpha_3 - \alpha_6 = 2\alpha_2 - \alpha_1 - \alpha_6 = -\alpha_1 - \alpha_6 \end{aligned}\right\} \quad (4.19)$$

From (4.19) we can obtain easily $\alpha_2 = 0$ and $\alpha_1 = \alpha_3$. The relation $u^* = \frac{\partial \psi^*}{\partial y^*}$ and $v^* = -\frac{\partial \psi^*}{\partial x^*}$ gives us $\alpha_3 = \alpha_4$, $\alpha_5 = 0$. In view of these, the boundary conditions (2.14) and (2.15) are transformed to

$$\left.\begin{aligned} &\frac{\partial \psi^*}{\partial y^*} = a x^* \text{ and } \frac{\partial \psi^*}{\partial x^*} = 0 \text{ at } y^* = 0 \\ &\frac{\partial \psi^*}{\partial y^*} \to 0 \text{ at } y^* \to \infty \end{aligned}\right\} \quad (4.20)$$

$$\left.\begin{aligned} &C^* = 1 \text{ at } y^* = 0 \\ &C^* \to 0 \text{ at } y^* \to \infty \end{aligned}\right\} \quad (4.21)$$

where the boundary condition $C^* = 1$ gives $\alpha_6 = 0$.

Thus the set Γ finally reduces to a one-parameter group of transformation

$$\Gamma : \begin{cases} x^* = x e^{\varepsilon \alpha_1} \; y^* = y, \; \psi^* = \psi e^{\varepsilon \alpha_1}, \\ u^* = u e^{\varepsilon \alpha_1}, \; v^* = v \text{ and } C^* = \bar{C} \end{cases} \quad (4.22)$$

Firstly, we consider the absolute invariant, η which is a function of the independent variables and is taken as $\eta = y^* x^{*s}$.

Since the quantity η is absolute invariant, we get $y^* x^{*s} = y x^s$.

Now, $y^* x^{*s} = y x^s e^{\varepsilon \alpha_1 s} = y x^s$ if $s = 0$ (since α_1 cannot be zero)

Hence, we get the first absolute invariant as $\eta = y^*$.

We now find the second absolute invariant, $G = f(\eta)$ which involves the dependent variable ψ^* and assume that $G = x^{*^r}\psi^*$. Since G is an absolute invariant, we will find r such that $x^{*^r}\psi^* = x^r\psi$.

Now, $x^{*^r}\psi^* = \left(xe^{\varepsilon\alpha_1}\right)^r \psi e^{\varepsilon\alpha_1} = \left(x^r e^{\varepsilon\alpha_1 r}\right)\psi e^{\varepsilon\alpha_1} = e^{\varepsilon\alpha_1(r+1)}x^r\psi = x^r\psi$ if $r = -1$.

Putting $r = -1$, the second absolute invariant G becomes $G = x^{*^{-1}}\psi^*$ i.e. $f(\eta) = x^{*^{-1}}\psi^*$.

Lastly, we want to find the third absolute invariant, $H = \phi(\eta)$ which involves the independent variables and the dependent variable C^* and is taken as $H = x^{*^p}C^*$. H is an absolute invariant if $x^{*^p}C^* = x^p\overline{C}$.

Now, $x^{*^p}C^* = \left(xe^{\varepsilon\alpha_1}\right)^p \overline{C}e^{\varepsilon\alpha_1} = \left(x^p e^{\varepsilon\alpha_1 p}\right)\overline{C} = e^{\varepsilon\alpha_1 p}x^p\overline{C} = x^p\overline{C}$ if $p = 0$

Thus, the third absolute invariant is $H = C^*$ i.e. $\phi(\eta) = C^*$.

Finally, from three absolute invariants, we get the transformations as given below:

$$\eta = y^*,\ \psi^* = x^* f(\eta) \text{ and } C^* = \phi(\eta) \tag{4.23}$$

In view of the above relations, the equations (4.17) and (4.18) become

$$\upsilon f''' + ff'' - f'^2 - \frac{\sigma B_0^2}{\rho} f' = 0 \tag{4.24}$$

$$D\phi'' + f\phi' - n f'\phi - R\phi = 0 \tag{4.25}$$

and the boundary conditions reduced to

$$\left.\begin{aligned} &f(\eta) = 0,\ \ f'(\eta) = a \ \text{ at } \ \eta = 0 \\ &f'(\eta) \to 0 \ \text{ as } \ \eta \to \infty \end{aligned}\right\} \tag{4.26}$$

$$\text{and } \left.\begin{aligned} &\phi = 1 \ \text{ at } \ \eta = 0 \\ &\phi \to 0 \ \text{ as } \ \eta \to \infty \end{aligned}\right\} \tag{4.27}$$

Again, we introduce the following transformations for η, f and ϕ in equations (4.24)-(4.27):

$$\eta = \upsilon^{\alpha'} a^{\beta'} \overline{\eta},\ f = \upsilon^{\alpha''} a^{\beta''} \overline{f} \text{ and } \phi = \upsilon^{\alpha'''} a^{\beta'''} \overline{\phi} \tag{4.28}$$

and we obtain $\alpha' = \alpha'' = \frac{1}{2}$, $\beta' = \frac{1}{2}$, $\beta'' = -\frac{1}{2}$ and $\alpha''' = \beta''' = 0$.

Finally, in view of the above transformations and taking $\bar{\eta} = \eta$, $\bar{f} = f$ and $\bar{\phi} = \phi$, the equations (2.24) and (2.25) reduce to the following forms:

$$f''' + ff'' - f'^2 - Mf' = 0 \tag{4.29}$$

and $\phi'' + Sc\, f\phi' - Sc\left(n\, f' + \beta\right)\phi = 0$ (4.30)

where $M = \sigma B_0^2 / a\rho$ is the magnetic parameter, $Sc = \upsilon / D$ is the Schmidt number and $\beta = R/a$ is reaction rate parameter of the solute.

The boundary conditions (4.26) and (4.27) reduce to the following forms:

$$\left.\begin{aligned} & f(\eta) = 0, \quad f'(\eta) = 1 \ \text{ at } \ \eta = 0 \\ & f'(\eta) \to 0 \ \text{ as } \ \eta \to \infty \end{aligned}\right\} \tag{4.31}$$

and $\left.\begin{aligned} & \phi = 1 \ \text{ at } \ \eta = 0 \\ & \phi \to 0 \ \text{ as } \ \eta \to \infty. \end{aligned}\right\}$ (4.32)

4.4 Solution of the problem

The equation (4.29) along with the boundary condition (4.31) is solved analytically [Sarpkaya, (1961)] and the exact solution is given by

$$f(\eta) = \frac{1 - \exp\left(-\sqrt{1+M}\,\eta\right)}{\sqrt{1+M}}, \quad \eta \ge 0. \tag{4.33}$$

After substitution of the function f and using finite-difference technique for the equation (4.30) along with the boundary conditions (4.32), we obtained the numerical solution of the concentration equation. The expression for wall shear stress is given by $\left|f''(0)\right| = \sqrt{1+M}$ which increases with the increase of magnetic field M. consequently, the boundary layer thickness of the moving surface decreases and the rate of mass transfer will be enhanced from the plate.

4.5 Results and discussions

The analytic solution of velocity has presented for various values of the magnetic parameter M. The reactant solute equation is solved numerically and the results are shown graphically.

The velocity profiles for various values of the magnetic parameter M have been plotted in Fig1. From the figure it is noted that with increase of M, the velocity profile for any fixed value of η decreases. Thus it is clear that the magnetic field opposes motion. This is due to the fact that variation of M leads to the variation of Lorenz force producing more resistance to the transport process. Consequently, the momentum boundary layer thickness reduces with the increase in M.

Fig2 exhibits concentration profiles for various values of M with $Sc=1$, $\beta=1, n=1$. From the figure it reveals that the value of contaminate solute at particular value of η increases with the increase of the magnetic parameter. This implies that the magnetic field acts to enhance the distribution of the reaction solute from the moving plate in case of an electrically conducting fluid subject to magnetic field. This result may be useful, in the situation where the enhancement of solute from the plate is the prime important.

Now, we concentrate on the solute curves for variation of Schmidt number Sc keeping fixed values of $M=1$, $\beta=1$, $n=1$. The curves are drawn in the Fig3. The effect on the variation of the distribution of the solute is to decrease its value from the plate. The diffusion of the solute decreases with the increase of Sc resulting the loss of solute transfer.

Fig4 is the graphical representation of concentration profiles for various values of reaction rate parameter β with $M=1$, $Sc=1$, $n=1$. It has been found that the thickness of the concentration boundary layer decreases with increasing β. So, in case of the distribution of reactive solute, the reaction constant parameter is a decelerating agent.

Finally, Fig5 and Fig6 exhibit the concentration profiles in the boundary layer flow region for different values of power-law exponent n. It is noticed from Fig5 that for the increasing values of n with $n>0$, the curve representing the distribution of solute for specific value of η decreases. While, in Fig6 the concentration profile increases with increase in the magnitude of n with $n<0$ and for large negative values of n, the overshoot of solute is observed near the surface. Thus, the effect of increase of n when the surface concentration is $C_w=C_\infty+C_0x^n$ is completely opposite to the effect of increase n when the surface concentration is $C_w=C_\infty+C_0/x^n$. Note that, the wall concentration is constant as $n=0$.

4.6 Conclusions

An analysis is made to investigate the distribution of contaminate solute transfer in steady MHD boundary layer flow. The analytical solution is obtained and the reactant solute equation is solved numerically using the value of known velocity. The magnetic field tends to reduce the rate of flow from the wall due to Lorentz force arising due to electro-magnetic interaction near the moving wall and is broadening the solute layer. The reaction rate parameter diminishes the rate of solute transfer. Also, the solute transfer is significantly decreased with the increase of Schmidt number. The effects of initial variable solute distribution over a stretching surface is very interesting i.e. when the magnitude of n increases, the concentration decreases when $n>0$ whereas increases when $n<0$. This boundary layer flow and contaminate solute transfer play an important roles in analyzing the pollutant transfer in environmental sciences and chemical industries.

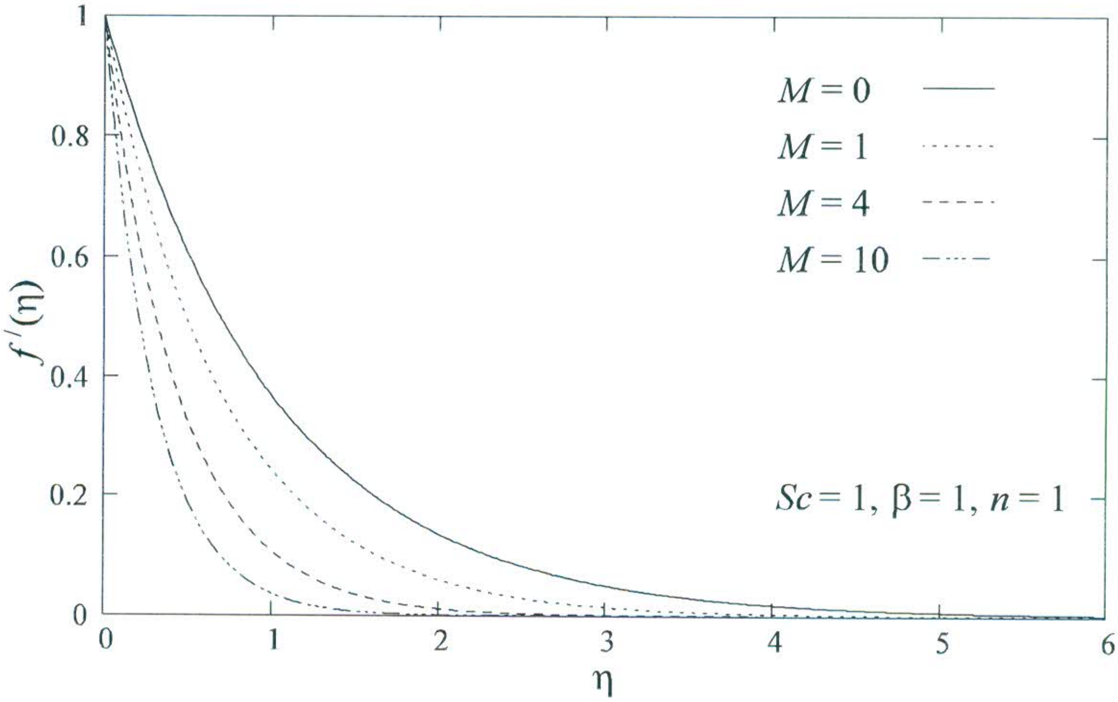

Fig1 Velocity profiles $f'(\eta)$ for various values of M

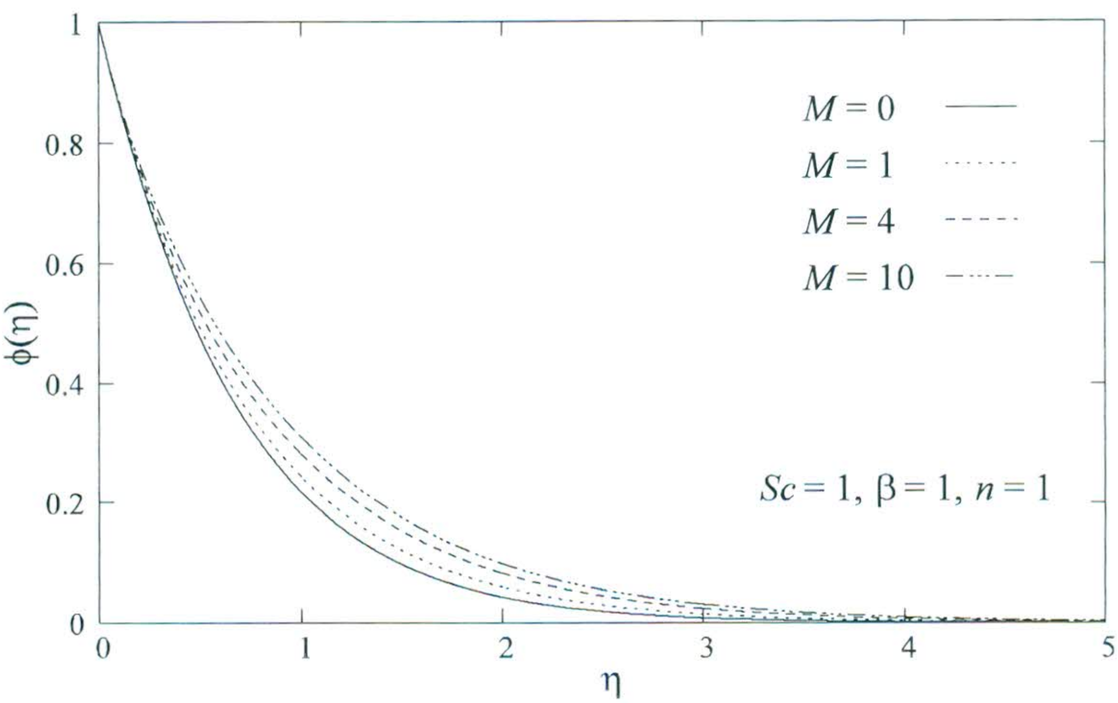

Fig2 Concentration profiles $\phi(\eta)$ for various values of *M*.

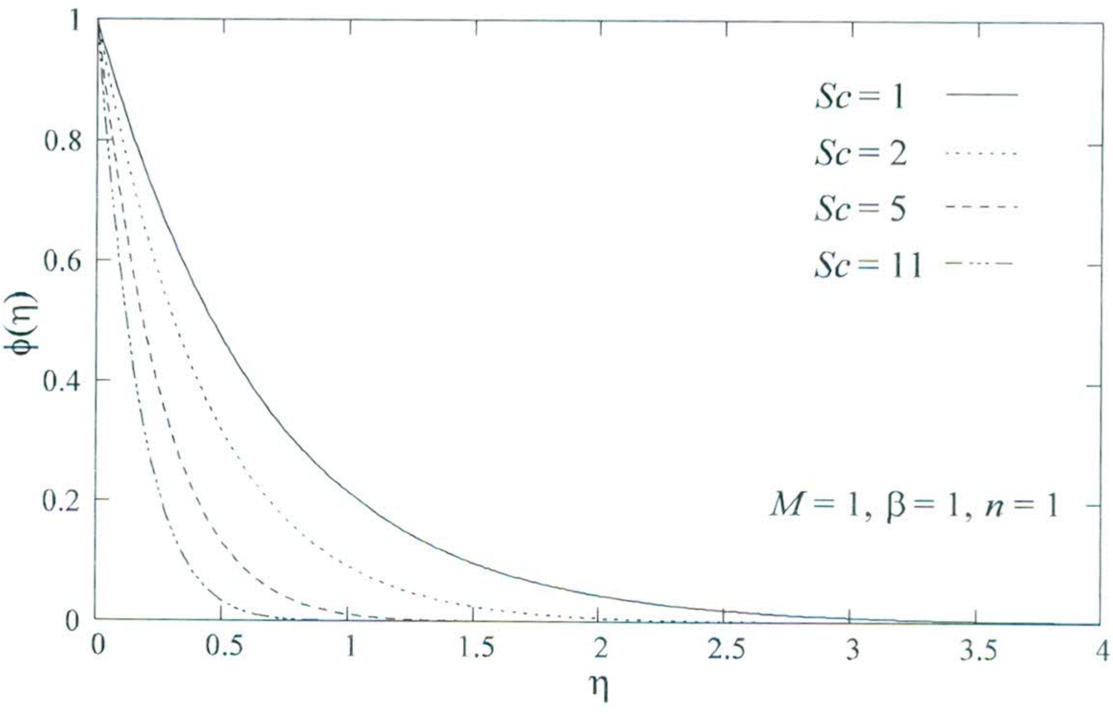

Fig3 Concentration profiles $\phi(\eta)$ for various values of *Sc*.

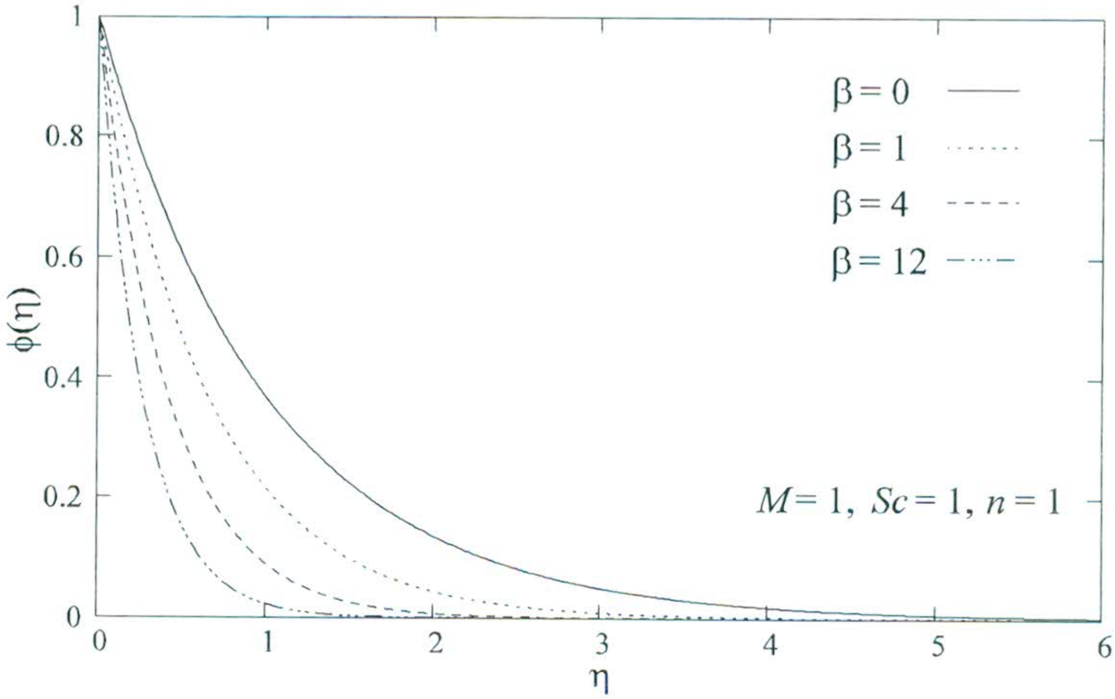

Fig4 Concentration profiles $\phi(\eta)$ for various values of β.

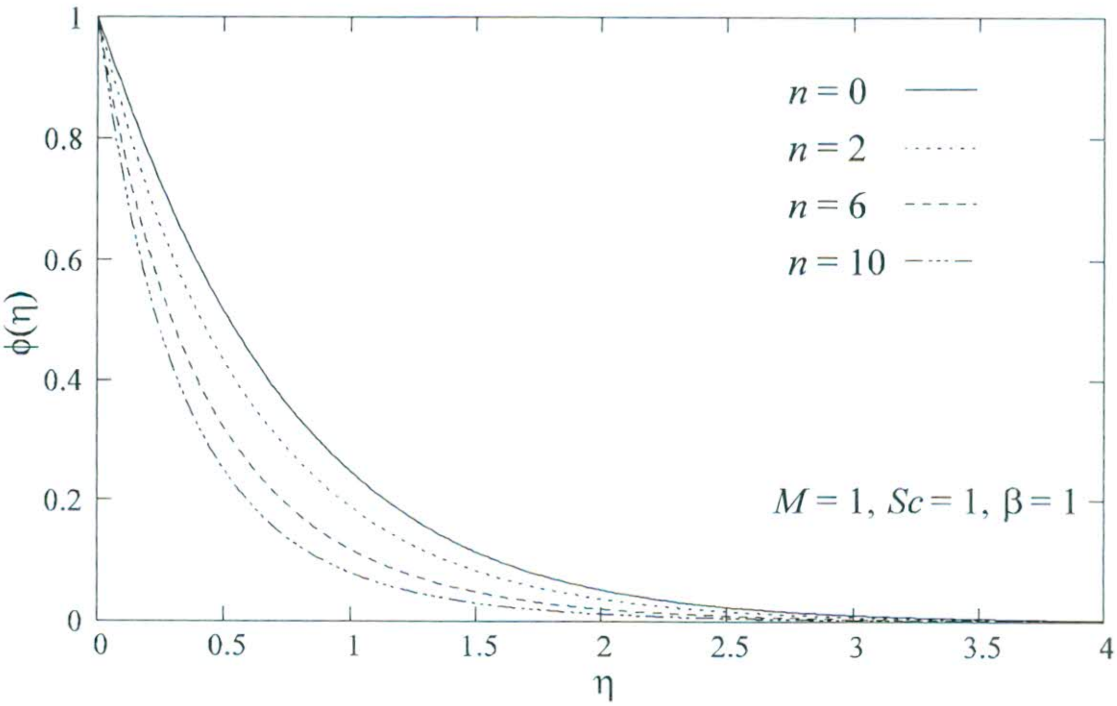

Fig5 Concentration profiles $\phi(\eta)$ for various values of $n(\geq 0)$.

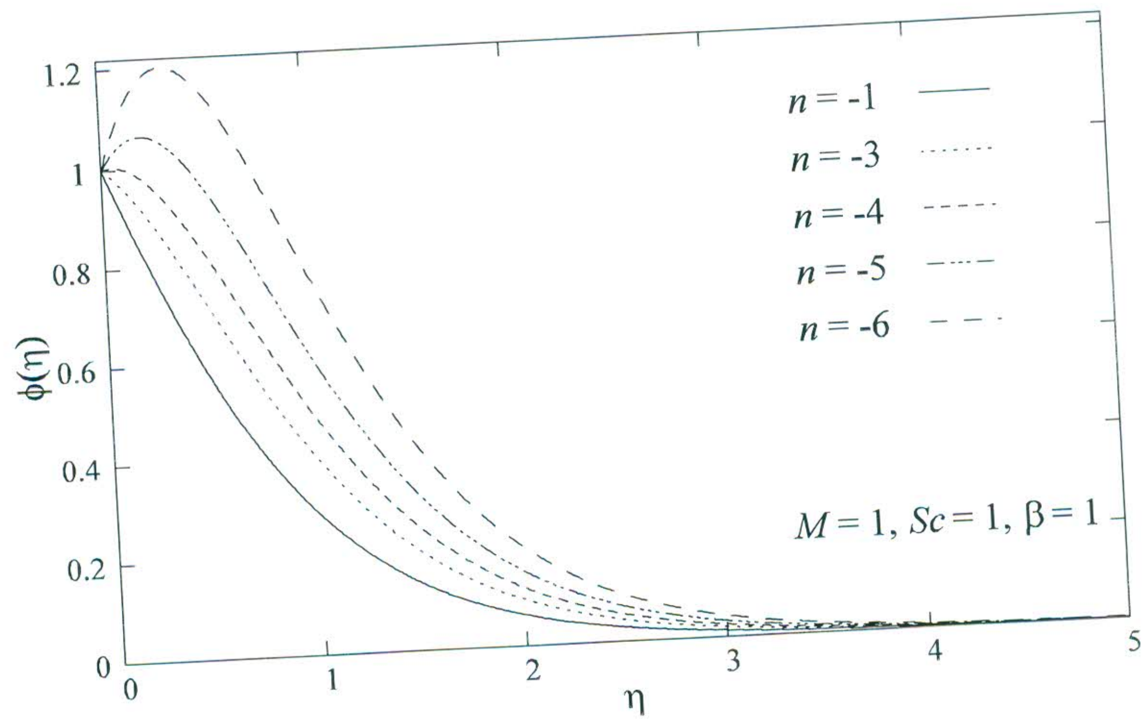

Fig6 Concentration profiles $\phi(\eta)$ for various values of $n(<0)$.

CHAPTER V

Diffusion of chemically reactive solute in a steady boundary layer flow over a porous plate in a porous medium

5.1 Introduction

The fluid flow over a flat plate is a classical problem in fluid mechanics. The boundary layer flow problems have been studied extensively under various aspects. In the year 1908, Blasius first considered the steady laminar boundary layer flow over a flat plate. The non-linear third order ordinary differential equation was obtained using similarity variable and analytic solution was also obtained. Howarth (1938) had obtained the numerical solution of this flow problem. The mass transfer analysis in boundary layer flow is of great importance in extending the theory of separation processes and chemical kinetics. The diffusion of a chemically reactive species in a laminar boundary layer flow over a flat plate was discussed by Chambre and Young (1958). After that many researcher studied the heat and mass transfer with and without chemical reaction. Gebhart and Pera (1971) studied the combined buoyancy effects of thermal and mass diffusion on vertical natural convection. The mass transfer effects on the flow past an impulsively started infinite vertical plate under several conditions were analyzed by Soundalgekar (1979), Soundalgekar et al. (1984), Das et al. (1994) and Muthucumaraswamy and Ganesan (2000,2001). The mixed convectional aspects of the flat plate flow were investigated by Afzal and Hussain (1984) and Yao (1987). Andersson et al. (1994) studied the diffusion of a chemically reactive species from a stretching sheet. Fan et al. (1998) obtained the similarity solution for the diffusion of chemically reactive species in mixed convection flow over a horizontal moving plate. Anjalidavi and Kandasamy (1999,2000) studied the effects of chemical reaction, heat and mass transfer on laminar flow along a semi infinite horizontal plate and also analyzed the effects of a chemical reaction on the flow past a semi-infinite plate in presence of a transverse magnetic field.

Mass and heat transfer phenomenon in porous medium have given significant attention of modern researchers due to its huge applications in chemical industries, reservoir engineering, environmental science and many other technological processes. Some important characteristics of the flow through the porous medium were depicted in the works of Cheng (1977), Vafai and Tien (1981) and Hsu and Cheng (1985). In recent past, Postelnicu (2007) described the influence of chemical reaction on heat and mass transfers by natural convection from vertical surface in porous media by taking into account the Soret and Dufour effects and Mukhopadhyay and Layek (2009) analyzed the radiation effects on forced convective flow and heat transfer over a porous plate in a porous medium.

In the present study, we investigate the effects of diffusion of chemically reactive solute or contaminant on forced convective laminar boundary layer flow over a porous flat plate in a porous medium. In this analysis, the reaction rate is taken inversely proportional to position along the plate. By using similarity transformation, a self-similar set of equations are obtained and then solved numerically using well known shooting method. Computed numerical results are plotted and the flow and heat transfer characteristics are thoroughly analyzed.

5.2 Mathematical formulation of the problem

Let us consider a steady two-dimensional flow of viscous incompressible fluid and solute transfer over a porous flat plate in porous medium. Using boundary layer approximations, the equations for the flow and the concentration distribution may be written in usual notation as follows.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,, \tag{5.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\upsilon}{k}(u-u_\infty) \tag{5.2}$$

and $$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-R(C-C_\infty)\,, \tag{5.3}$$

where u and v are velocity components in x- and y-directions respectively, ρ is the fluid density, μ is the coefficient of fluid viscosity, υ ($=\mu/\rho$) the kinematic fluid viscosity, u_∞ is the free stream velocity, k is the permeability of the porous medium, C the concentration, D is the diffusion coefficient, C_∞ is the value of the concentration in the free stream. $R(x)$ is the variable reaction rate and is given by $R(x)=R_0/x$, R_0 is a constant which is inversely proportional to the distance of the plate under consideration.

The appropriate boundary conditions for the velocity components and the temperature variable under boundary layer flow assumption are given by

$u=0,\ v=v_w$ at $y=0$; $u\to u_\infty$ as $y\to\infty$ (5.4)

and $C=C_w$ at $y=0$; $C\to C_\infty$ as $y\to\infty$ (5.5)

where C_w is the plate concentration and v_w is prescribed suction or blowing through the porous plate and is given by $v_w=v_0/(x)^{1/2}$, v_0 is the initial value of suction ($v_0<0$) or blowing ($v_0>0$).

We now introduce the stream function $\psi(x,y)$ as

$$u=\frac{\partial\psi}{\partial y} \text{ and } v=-\frac{\partial\psi}{\partial x}. \tag{5.6}$$

Now, using relation (5.6), the equation (5.1) is automatically satisfied and the equations (5.2) and (5.3) become

$$\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\upsilon\frac{\partial^3\psi}{\partial y^3}-\frac{\upsilon}{k}\left(\frac{\partial\psi}{\partial y}-u_\infty\right), \tag{5.7}$$

and $$\frac{\partial\psi}{\partial y}\frac{\partial C}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2}-\frac{R_0}{x}(C-C_\infty). \tag{5.8}$$

The boundary conditions (5.4) of the flow reduce to

$$\frac{\partial\psi}{\partial y}=0, \frac{\partial\psi}{\partial x}=-v_w \text{ at } y=0; \frac{\partial\psi}{\partial y}\to u_\infty \text{ as } y\to\infty. \tag{5.9}$$

Next, we shall introduce the dimensionless variables for ψ and T as given below:

$$\psi=\sqrt{u_\infty\upsilon x}\,f(\eta) \text{ and } C=C_\infty+(C_w-C_\infty)\phi(\eta) \tag{5.10}$$

where the similarity variable η is defined as $\eta=y(u_\infty/\upsilon x)^{1/2}$.

In view of relations in (5.10) we finally obtain the self-similar equations in the following form:

$$f'''+\frac{1}{2}ff''-\frac{1}{Da_x Re_x}(f'-1)=0 \tag{5.11}$$

and $$\phi''+\frac{1}{2}Sc\,f\phi'-Sc\beta\phi=0 \tag{5.12}$$

where $Da_x=k/x^2=k_0/x$ is the local Darcy number, $k=k_0/x$, k_0 is the initial value of the permeability, $Re_x=u_\infty x/\upsilon$ is the local Reynolds number, $Sc=\upsilon/D$ is the Schmidt number and $\beta=R_0/u_\infty$ is the reaction rate parameter.

Equation (5.11) can be written as

$$f''' + \frac{1}{2} ff'' - k^* (f' - 1) = 0 \tag{5.13}$$

where $k^*=1/Da_xRe_x$ is the permeability parameter of the porous medium (Mukhopadhyay and Layek [20]).

The boundary conditions (9) and (5) finally become

$$f(\eta)=S, f'(\eta)=0 \text{ at } \eta=0;\ f'(\eta)\to 1 \text{ as } \eta\to\infty \tag{5.14}$$

and $\phi(\eta)=1$ at $\eta=0$; $\phi(\eta)\to 0$ as $\eta\to\infty$ (5.15)

where $S=(-2v_w/u_\infty)(Re_x)^{1/2}=-2v_0/(u_\infty\upsilon)^{1/2}$, $S>0$ (i.e. $v_0<0$) is corresponding to suction and $S<0$ (i.e. $v_0>0$) is corresponding to blowing.

5.3 Numerical solution

The nonlinear coupled differential equations (5.13) and (5.12) along with the boundary conditions form a boundary value problem (BVP) and are solved using shooting method, by converting into an initial value problem (IVP). In this method we have to choose a suitable finite value of $\eta\to\infty$, say η_∞. We get following first-order systems

$$f' = p,\ p' = q,\ q' = -\frac{1}{2} fq + k^* (p-1) \tag{5.16}$$

and $\phi' = z,\ z' = -\frac{1}{2} Sc\ fz + Sc\beta\phi$ (5.17)

with the boundary conditions

$$f(0)=S,\ p(0)=0,\ \theta(0)=1. \tag{5.18}$$

To solve (5.15) and (5.16) as an IVP, we must need values for $q(0)$ i.e. $f''(0)$ and $z(0)$ i.e. $\theta'(0)$ but no such values are given. The initial guess values for $f''(0)$ and $\theta'(0)$ are chosen and applying fourth order Runge-Kutta method to obtained the approximate solution. We compare the calculated values of $f'(\eta)$ and $\theta(\eta)$ at $\eta_\infty(=20)$ with the given boundary conditions $f'(\eta_\infty)=1$ and $\theta(\eta_\infty)=0$ and adjust values of $f''(0)$ and $\theta'(0)$ using standerd Secant method to give better approximation for the solution. The step-size is taken as $h=0.01$. The process is repeated until we get the results correct up to the desired accuracy of 10^{-6} level.

5.4 Results and discussions

The numerical computations are carried out for different values of parameters involved in the equations viz., the permeability parameter of the porous medium (k^*), the suction or blowing parameter (S), the Schmidt number (Sc) and the reaction rate parameter (β). The computed results are explained by plotting some figures and corresponding physical reasoning are also discussed at length.

For the verification of results obtained, the results corresponding to the velocity profile for $k^*=0$ and $S=0$ (i.e. in non-porous medium and in absence of suction or blowing) with the given result of Granger (1995) shown in Fig1.

We shall now pay our attention to see the influence of the permeability parameter k^* on the velocity as well as concentration (reactant solute). In Fig2, the curves depicting various values of k^* are drawn and the reactant solute $\phi(\eta)$ curves for various values of k^* are also shown in Fig3. In both cases the effects are prominent. The increase in permeability of the porous medium enhances the velocity at a point. It is due to the fact that the momentum boundary layer thickness is reduced with increase in k^*. But opposite effect is observed in Fig3 of concentration. The concentration profile decreases with increasing values of permeability parameter k^*.

The effects of externally applied suction or blowing through the porous plate on velocity and concentration distributions play a vital role for the flow in a porous medium. For several values of the parameter S, the dimensionless velocity and concentration profiles are plotted in Fig4 and Fig5. For the increase of suction applied through the porous plate, the value of the velocity and the concentration profiles at a fixed point decrease. On the other hand, reverse nature is noticed for applied blowing (negative suction) i.e. with increasing blowing, the velocity and the concentration increase. It is noted that the thicknesses of momentum as well as thermal boundary layers decrease with suction and increase with blowing.

In Fig6 the effect of the Schmidt number on the concentration distribution is exhibited. The concentration at a fixed η promptly decreases with increasing values of Sc, because the Schmidt number acts to reduce the thermal boundary layer thickness. Moreover, after certain increment of Sc the value of concentration profiles is negative.

Finally, we discuss the variation of reactive concentration distribution for several values of the reaction rate parameter β. In Fig7, a clear view of concentration profiles for increment of β is represented. From the figure it reveals that the values of the concentration decrease with increasing β. Thus the chemical reaction opposes the diffusion of reactant concentration undergoing first-order chemical reaction.

5.5 Conclusions

The present investigation concentrates on the effects of diffusion of chemically reactive solute in forced convective boundary layer flow over a porous flat plate in a porous media. The reaction rate of the reactive species is considered such that it is inversely proportional to position along the plate. A self-similar set of equations are obtained and then solved numerically using shooting method. This analysis reveals that the permeability of the porous medium increases the velocity but decreases the concentration. The suction reduces the thicknesses of momentum and thermal boundary layers but blowing enhances the above thicknesses. It is interesting to note that the diffusion of reactant solute undergoing first-order chemical reaction decreases with increasing values of reactant-rate parameter β.

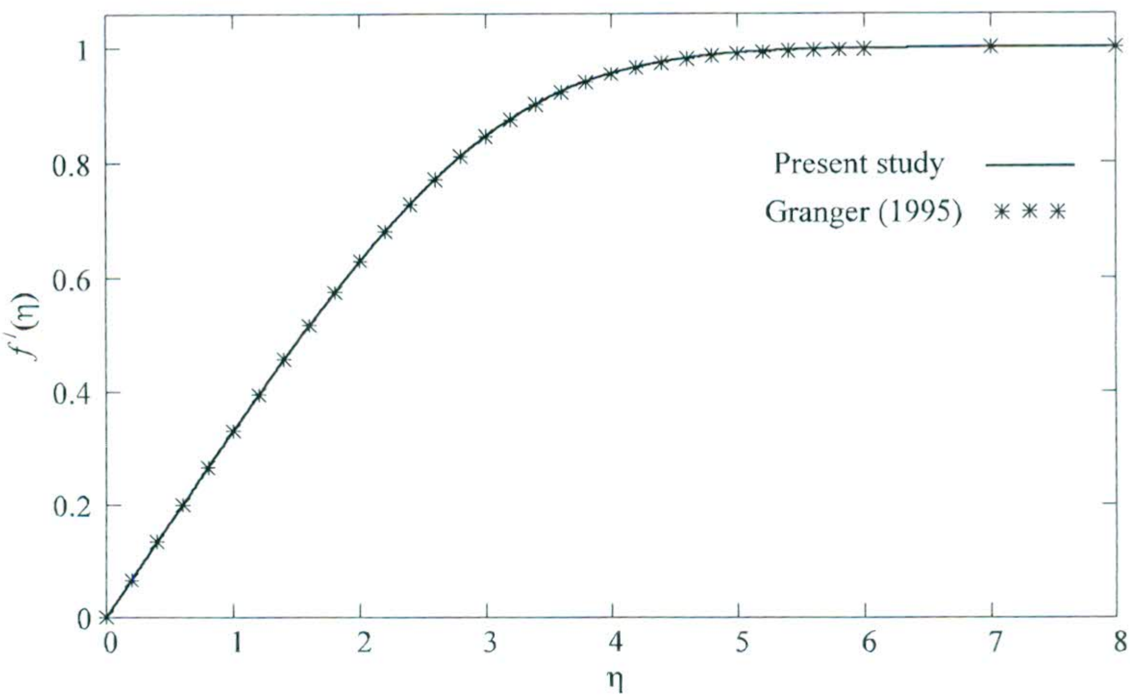

Fig1 Velocity profiles $f'(\eta)$ for k^*=0 and S=0.

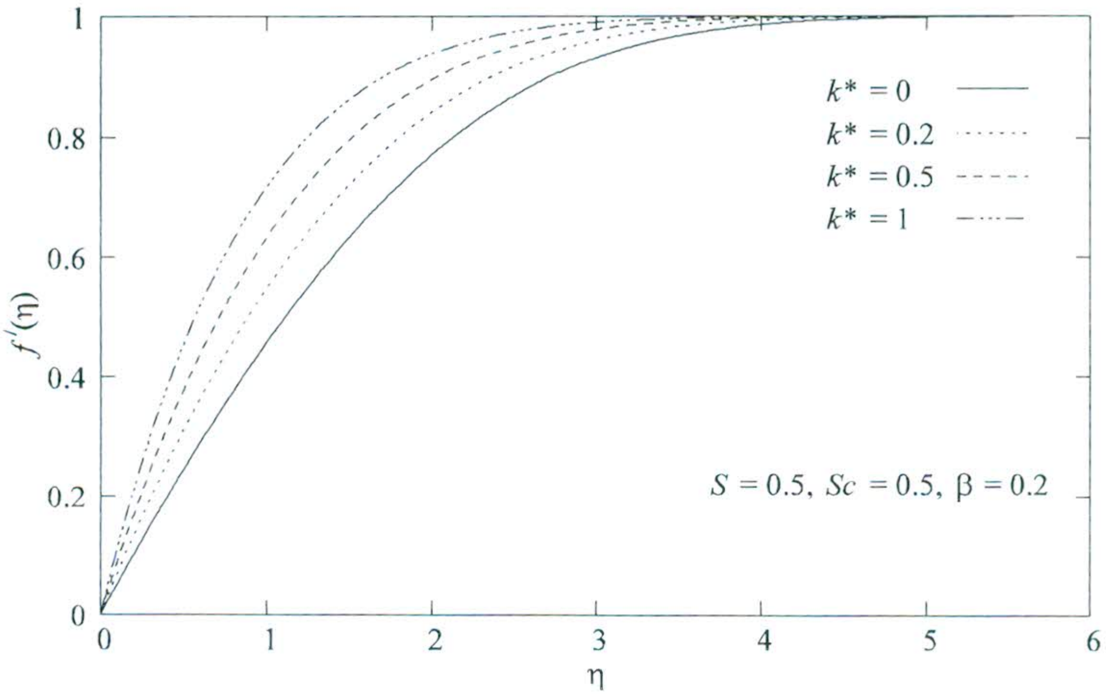

Fig2 Velocity profiles $f'(\eta)$ for different values of k^*.

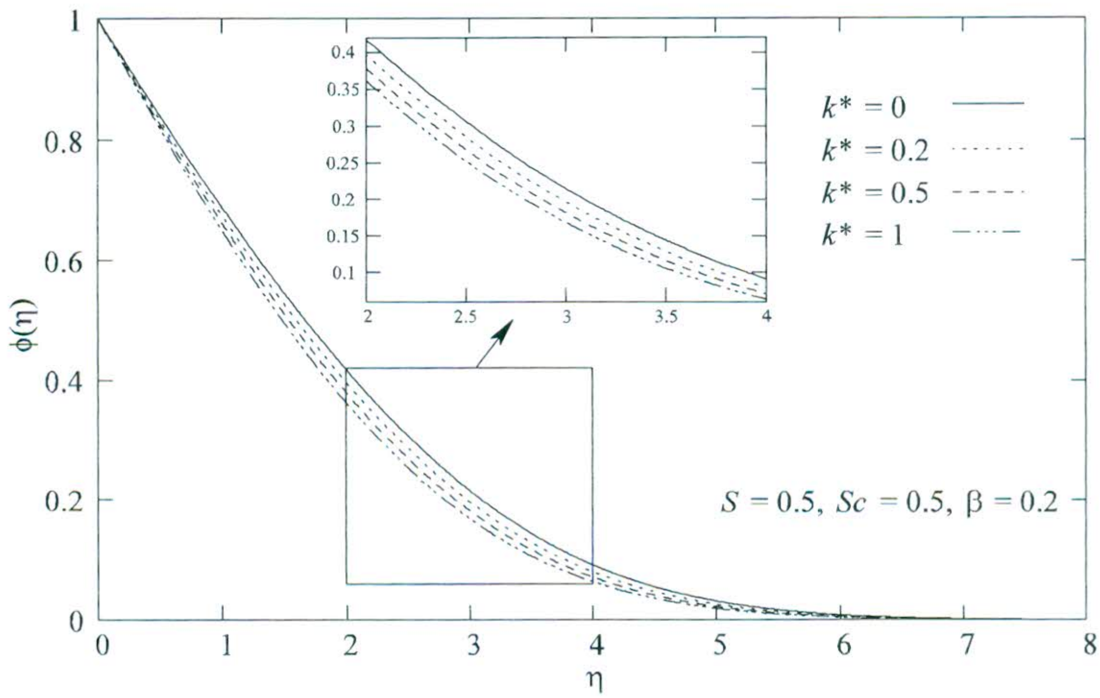

Fig3 Concentration profiles $\phi(\eta)$ for different values of k^*.

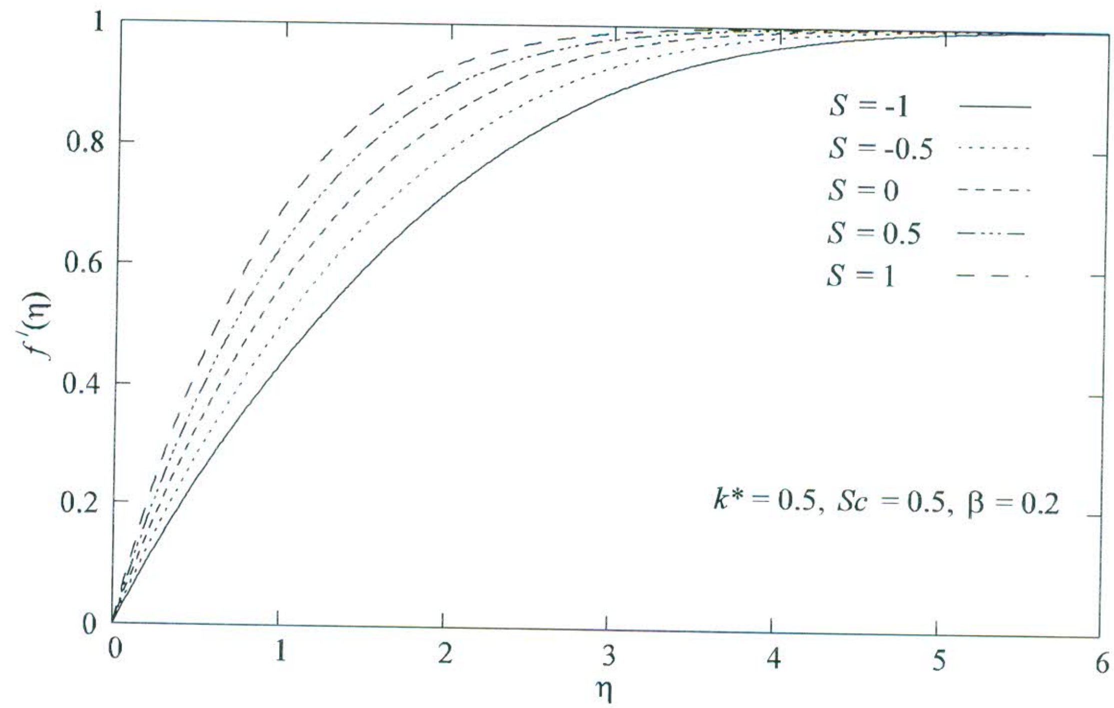

Fig4 Velocity profiles $f'(\eta)$ for different values of S.

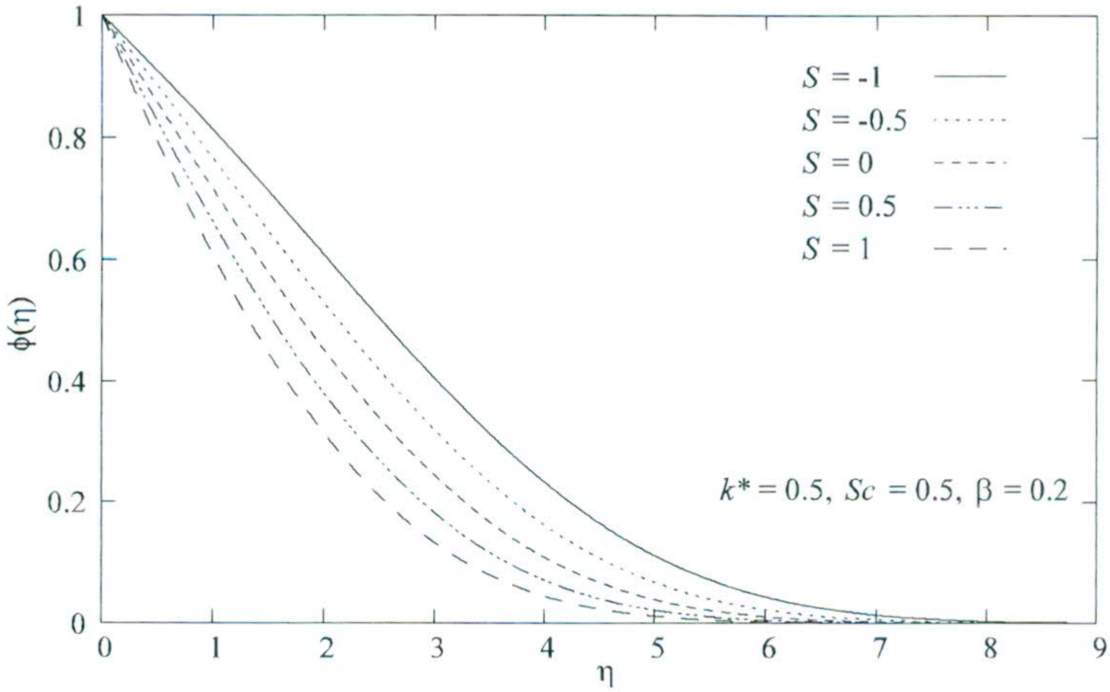

Fig5 Concentration profiles $\phi(\eta)$ for different values of S.

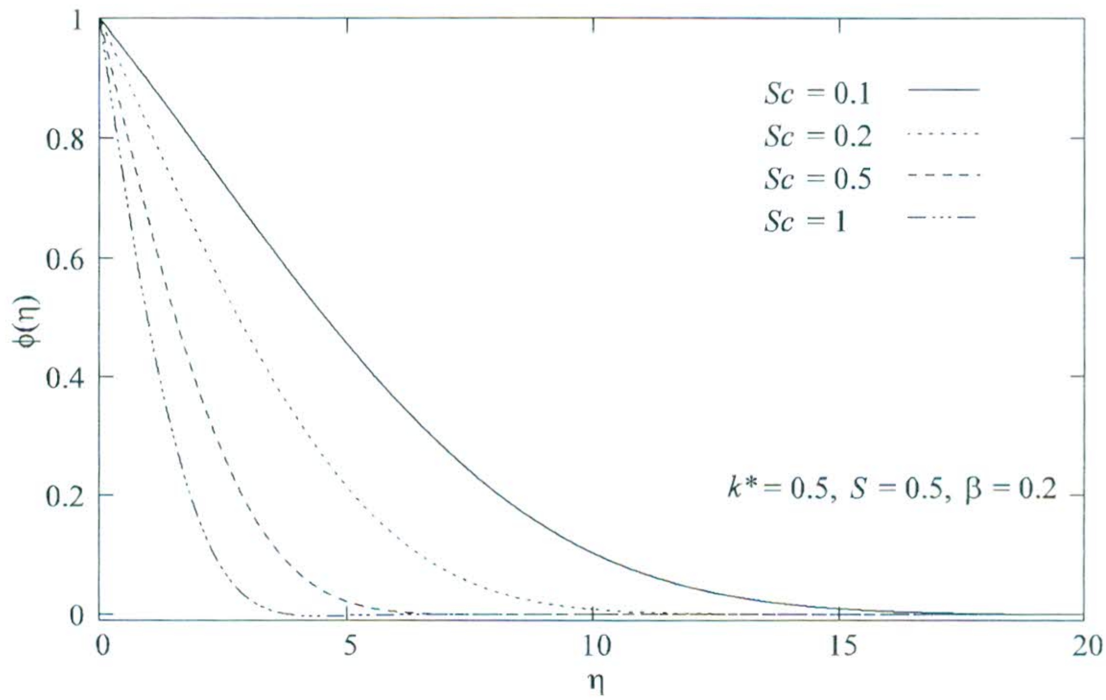

Fig6 Concentration profiles $\phi(\eta)$ for different values of Sc.

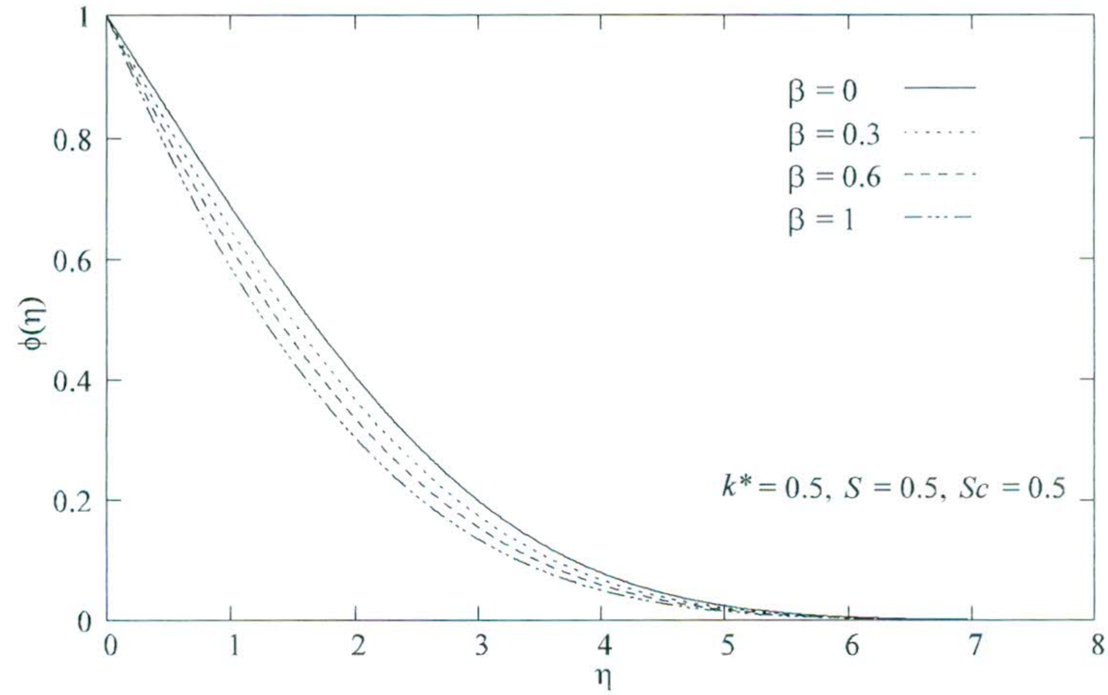

Fig7 Concentration profiles $\phi(\eta)$ for different values of β.

CHAPTER VI

Analysis of boundary layer flow and heat transfer with heat generation or absorption for two classes of visco-elastic fluid over a stretching sheet

6.1 Introduction

The stretching sheet problem relating the flows of Newtonian fluids is a classical problem in fluid mechanics in which the solution of the problem is obtained easily. Study of momentum and heat transfer in laminar boundary layer flow of a visco-elastic fluid model over a linearly stretching sheet is of great importance in polymer processing industries in particular, the manufacturing artificial fibers. The concept of viscous flow due to a linearly stretching sheet was introduced by Crane (1970). The work of Crane was extended by Rajagopal et al. (1984) by taking visco-elastic fluids. Siddappa and Abel (1985) also discussed some other important aspects of this type of flow. Troy et al. (1987) established the uniqueness of solution of the flow. But two years latter, Chang (1989) showed that the solution of the flow of visco-elastic fluid is not unique and he obtained the different forms of non-unique solutions. In 1987, Bujurke et al. studied the heat transfer analysis for the flow of second order visco-elastic fluid over a stretching sheet. Lawrence and Rao (1992) also demonstrated the heat transfer in the flow of visco-elastic fluid past a stretching sheet. Andersson (1992) showed the magnetic effects on the flow of visco-elastic Walter's liquid B over a stretching sheet. Cortell (1994) obtained the similarity solution for the flow and heat transfer of a viscoelastic fluid over a stretching sheet.

Siddheshwar and Mahabaleswar (2005) investigated the radiation effects on MHD flow of a visco-elastic fluid and heat transfer over a stretching sheet with taking into account the internal heat generation/absorption. In case of heat transfer analysis, the thermal radiation is very important physically. Khan (2006) studied the effects of radiation as well as heat source/sink and mass suction/blowing on heat transfer in visco-elastic fluid flow over a stretching surface. Cortell (2007) analyzed the mass transfer with chemically reactive species for two classes of visco-elastic fluids viz. second-grade and Walter's liquid B over a porous stretching sheet.

In our study, we investigate the effects of heat generation or absorption on the flow of two different classes of visco-elastic fluids over a stretching sheet which is being stretched linearly. We obtained the exact solutions for both momentum and heat equations. The solution of self-similar heat conducting equation the solution is in the form of Kummer's function (a confluent hypergeometric function). Actually, this work is the generalization of the work of Cortell (1994).

6.2 Formulation of the flow and heat transfer problems

We consider a steady laminar flow of an incompressible viscoelastic fluid over a plane sheet coinciding with the plane $y=0$, the flow being confined in the plane $y>0$. The motion is caused due to a linear stretching of the sheet because of simultaneous application of two equal opposite forces along the x-axis so that the sheet stretched keeping the origin fixed. The boundary layer equations representing the momentum and heat transfer, may be written as follows

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (6.1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}\pm k_0\left\{u\frac{\partial^3 u}{\partial x\partial y^2}+\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}-\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}+v\frac{\partial^3 u}{\partial y^3}\right\} \qquad (6.2)$$

and $$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{Q_0}{\rho c_p}(T-T_\infty), \qquad (6.3)$$

where u and v are velocity components in x and y directions respectively, $\upsilon(=\mu/\rho)$ the kinematic fluid viscosity, ρ is the density, μ is the coefficient of viscosity, k_0 is the coefficient of viscoelasticity. The positive sign in the right hend side of equation (6.2) corresponds to second-grade fluid [Cortell (1994, 2006)] whereas the negative sign for Walter's liquid B [Prasad and Abel (2000), Khan et al. (2003)], also termed as second-order fluid [Khan and Sanjayanand (2000)], T is the temperature, κ is the fluid thermal conductivity, c_p is the specific heat, Q_0 is the heat generation or absorption coefficient, T_∞ is the free stream temperature.

The appropriate boundary conditions for the velocity components and the temperature are given by

$u=ax$, $v=0$ at $y=0$; $u\to 0$, $\partial u/\partial y\to 0$ as $y\to\infty$ (6.4)

and $T=T_w$ at $y=0$; $T\to T_\infty$ as $y\to\infty$, (6.5)

where a is stretching constant with $a>0$, T_w is temperature of the sheet assumed to be constant. The last condition of (6.4) is the augmented condition because the flow is in an unbounded domain, which had been discussed by Garg and Rajagopal (1992).

We now introduce the stream function $\psi(x,y)$ as

$u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$. (6.6)

The mass-conservation equation (6.1) is satisfied automatically and the momentum equation (6.2) and temperature equation (6.3) take the following forms.

$$\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\upsilon\frac{\partial^3\psi}{\partial y^3}\pm k_0\left\{\frac{\partial\psi}{\partial y}\frac{\partial^4\psi}{\partial x\partial y^3}+\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial^3\psi}{\partial y^3}-\frac{\partial^2\psi}{\partial y^2}\frac{\partial^3\psi}{\partial x\partial y^2}-\frac{\partial\psi}{\partial x}\frac{\partial^4\psi}{\partial y^4}\right\} \tag{6.7}$$

and $$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\frac{\kappa}{\rho c_p}\frac{\partial^2 T}{\partial y^2}+\frac{Q_0}{\rho c_p}(T-T_\infty). \tag{6.8}$$

Accordingly, the boundary conditions (6.4) reduces to

$$\frac{\partial\psi}{\partial y}=ax,\frac{\partial\psi}{\partial x}=0\text{ at }y=0;\frac{\partial\psi}{\partial y}\to 0,\frac{\partial^2\psi}{\partial y^2}\to 0\text{ as }y\to\infty. \tag{6.9}$$

Next, we introduce the dimensionless variables for ψ and T as given below:

$$\psi=\sqrt{a\upsilon}\,xf(\eta)\text{ and }T=T_\infty+(T_w-T_\infty)\theta(\eta). \tag{6.10}$$

The similarity variable denoted by η is given by $\eta=y(a/\upsilon)^{1/2}$.

Using the dimensionless variables and similarity variable, the above equations finally have taken the following self-similar forms:

$$(f')^2-ff''=f'''+\lambda_1[2f'f'''-(f'')^2-ff^{iv}], \tag{6.11}$$

and $$\theta''+Pr[f\theta'-L\theta]=0, \tag{6.12}$$

where $\lambda_1=\pm k_0a/\upsilon$ is the visco-elastic parameter with $\lambda_1>0$ corresponding to the second-grade fluid and $\lambda_1<0$ for the Walter's liquid B, $Pr=\mu c_p/\kappa$ is the Prandtl number and $L=Q_0/\rho c_p a$ is the heat source ($L<0$) or sink ($L>0$) parameter.

The boundary conditions (6.7) and (6.5) also transform to

$f(\eta)=0, f'(\eta)=1$ at $\eta=0$; $f'(\eta)\to 0, f''(\eta)\to 0$ as $\eta\to\infty$ (6.13)

and $\theta(\eta)=1$ at $\eta=0$; $\theta(\eta)\to 0$ as $\eta\to\infty$. (6.14)

6.3 Solutions of self-similar equations

The self-similar equation (6.11) representing the momentum equation with the boundary condition (6.13) has a solution of the form

$$f(\eta) = \frac{1}{\alpha}(1 - \exp(-\alpha\eta)) \tag{6.15}$$

where $\alpha = \sqrt{1/(1+\lambda_1)}$. (6.16)

According the sign of λ_1, we can consider the fluid to be either second-grade (λ_1>0) or Walter's liquid B (λ_1<0, $|\lambda_1|$<1). Also, replacing λ_1 by $-\lambda_1$ in the equations (6.15) and (6.16) earlier analyses of Cortell (1994) and Khan (2006) in nonporous case can be obtained.

The analytic solution of equation (6.12) with respective boundary condition (6.14) can be written in form of hypergeometric function:

$$\theta(\eta) = \frac{e^{-\alpha(G+H)\eta} M(G+H, 1+2H, -(Pr/\alpha^2)e^{-\alpha\eta})}{M(G+H, 1+2H, -Pr/\alpha^2)} \tag{6.17}$$

where $G = Pr/2\alpha^2$ and $H = (Pr^2 - 4L\alpha^2 Pr)^{1/2}/2\alpha^2$. (6.18)

The confluent hypergeomertic function M is the Kummer's function (Abramowitz and Stegun (1965)) and is defined by

$$M(g_0, h_0, z) = 1 + \sum_{n=1}^{\infty} \frac{(g_0)_n z^n}{(h_0)_n n!}$$

with $(g_0)_n = g_0(g_0+1)(g_0+2)\cdots\cdots(g_0+n-1)$

and $(h_0)_n = h_0(h_0+1)(h_0+2)\cdots\cdots(h_0+n-1)$.

Also, the solution of the boundary value problem (6.12) with (6.14) is solved numerically by the standard Runge-Kutta scheme using shooting method after transferring in first order systems with appropriate choice of guess value.

6.4 Results and discussions

The solution curves for velocities are plotted and are shown in Fig1(a) and Fig1(b) for several values of λ_1 for second-grade (λ_1>0) and Walter's liquid B (λ_1<0), respectively. It is noticed that the velocity represented by $f'(\eta)$ increases with an increase in λ_1 with λ_1>0 i.e. for second order fluid and on the other hand it decreases as magnitude of λ_1 increases when λ_1<0 i.e. Walter's liquid B. So, two opposite behaviours in velocity curves for two classes of visco-elastic fluids are noticed.

Now, the solution curves for the temperature distribution for several values of λ_1 are presented in Fig2(a) and Fig2(b). In Fig2(a), for the case of second-grade fluid the similar temperature profile representing $\theta(\eta)$ increases with increasing λ_1, whereas from Fig2(b) i.e. for the case of Walter's liquid B, $\theta(\eta)$ decreases when the magnitude of λ_1 increases. The nature of the dimensionless temperature profiles is same as that of the dimensionless velocity profiles.

The deviation in the temperature profiles for the variation of the Prandtl number Pr is demonstrated in Fig3(a) and Fig3(b). For both type of visco-elastic fluids, the temperature as well as the thermal boundary layer thickness decrease rapidly with increasing values of Pr. Thus the Prandtl number affects the temperature distribution for both case in similar manner.

Finally, the effects of internal heat source or sink on the heat transfer are exhibited in the Fig4(a) and Fig4(b). Due to the increase in the heat source or sink parameter L, the thickness of the thermal boundary layer is reduced. Thus, with the increase in strength of heat source increases the thermal boundary layer thickness increases but opposite trend is shown in case of increasing values of heat sink parameter. The above facts are found in both second-grade fluid and Walter's liquid B.

6.5 Conclusions

The objective of our study is to investigate the flow behavious and heat transfer analysis of two classes of non-Newtonian visco-elastic fluid over a stretching sheet with internal heat generation or absorption. Using similarity variables, the momentum and heat transfer equations are transferred into self-similar ordinary differential equations. The nonlinear ordinary differential equations representing momentum equation is solved analytically and also the heat transfer equation using a confluent hypergeomertic function, the well known Kummer's function. The exact and numerical results are sketched in some figures and the following conclusions can be drawn:

1. For the increase in the magnitude of visco-elastic parameter, the velocity and temperature increase for second-grade fluid and decrease for Walter's liquid B.

2. For both type fluid models, increase in Prandtl number reduces the thermal boundary layer thickness.
3. The temperature at a point increases with the heat source strength and decreases with heat sink strength for second-grade as well as Walter's B fluid models.

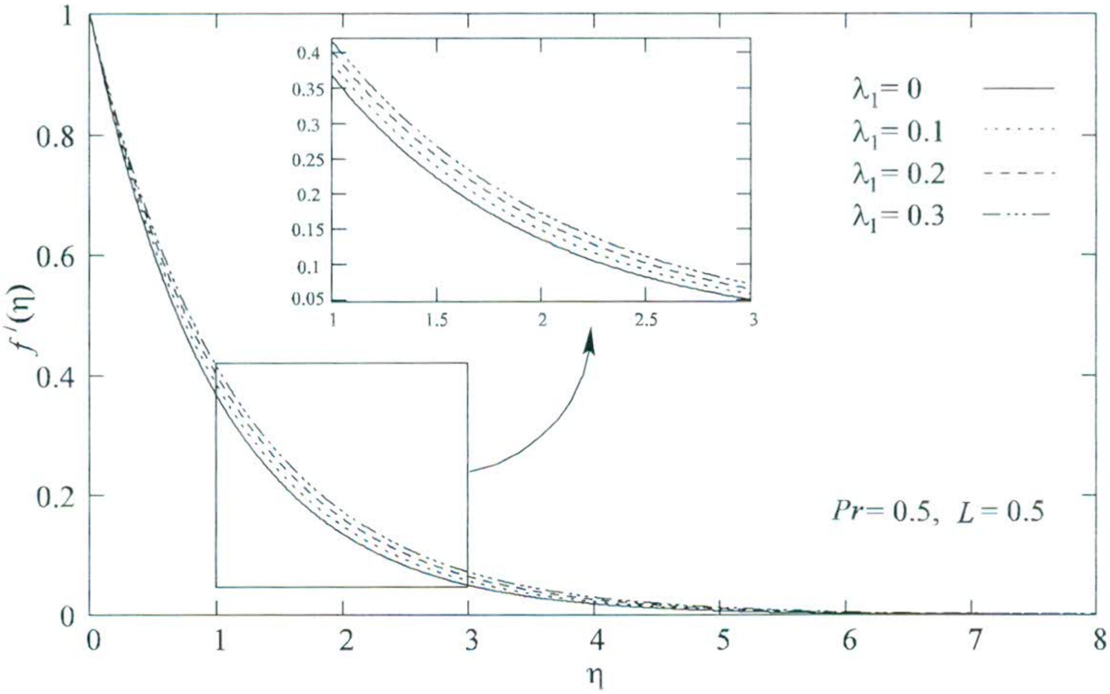

Fig1(a) Velocity profiles for several values of λ_1 for second-grade fluid (i.e. $\lambda_1 > 0$).

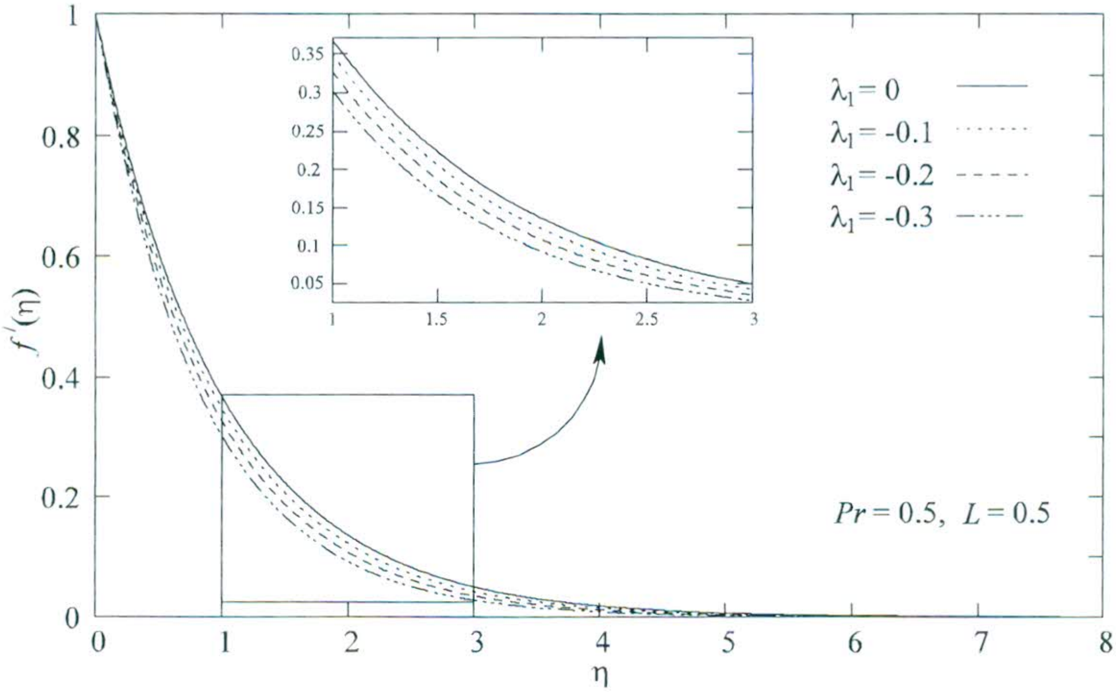

Fig1(b) Velocity profiles for several values of λ_1 Walter's liquid B (i.e. $\lambda_1 < 0$).

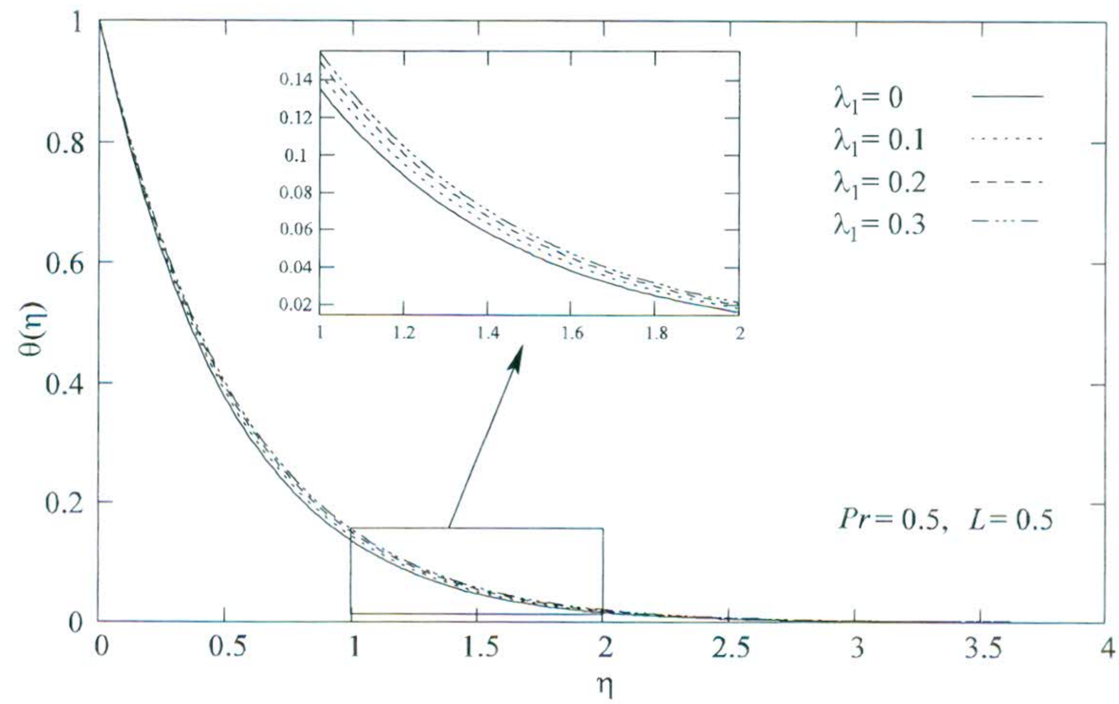

Fig2(a) Temperature profiles for several values of λ_1 for second-grade fluid (i.e. $\lambda_1 > 0$).

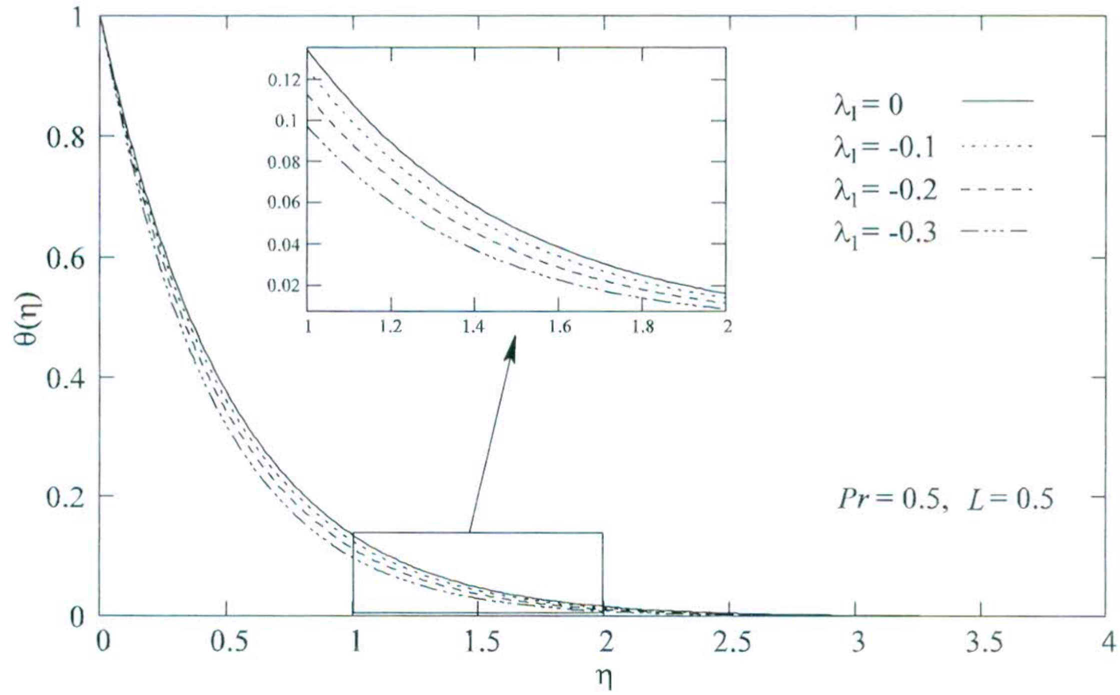

Fig2(b) Temperature profiles for several values of λ_1 Walter's liquid B (i.e. λ_1<0).

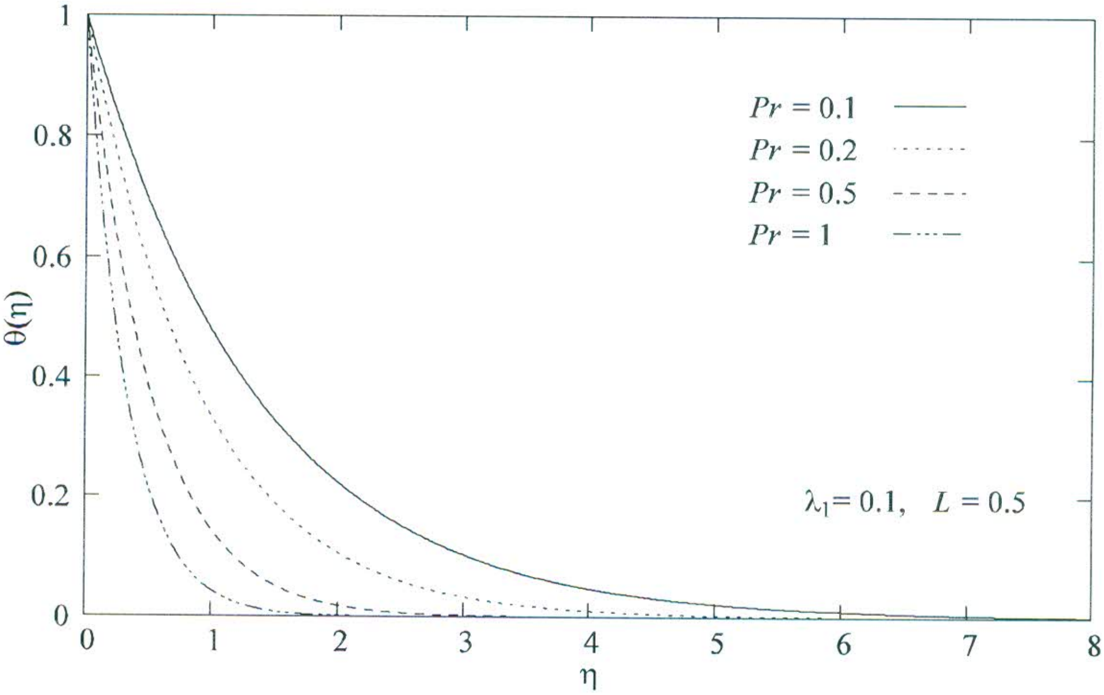

Fig3(a) Temperature profiles for several values of Pr for second-grade fluid (i.e. $\lambda_1>0$).

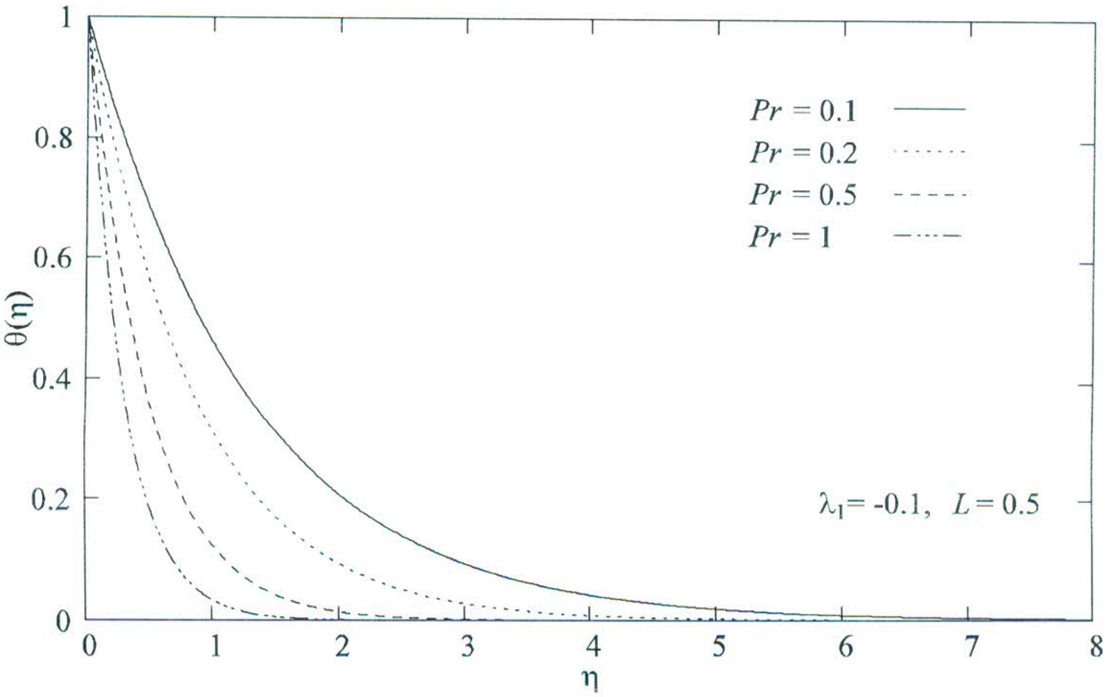

Fig3(b) Temperature profiles for several values of *Pr* for Walter's liquid B (i.e. λ_1<0).

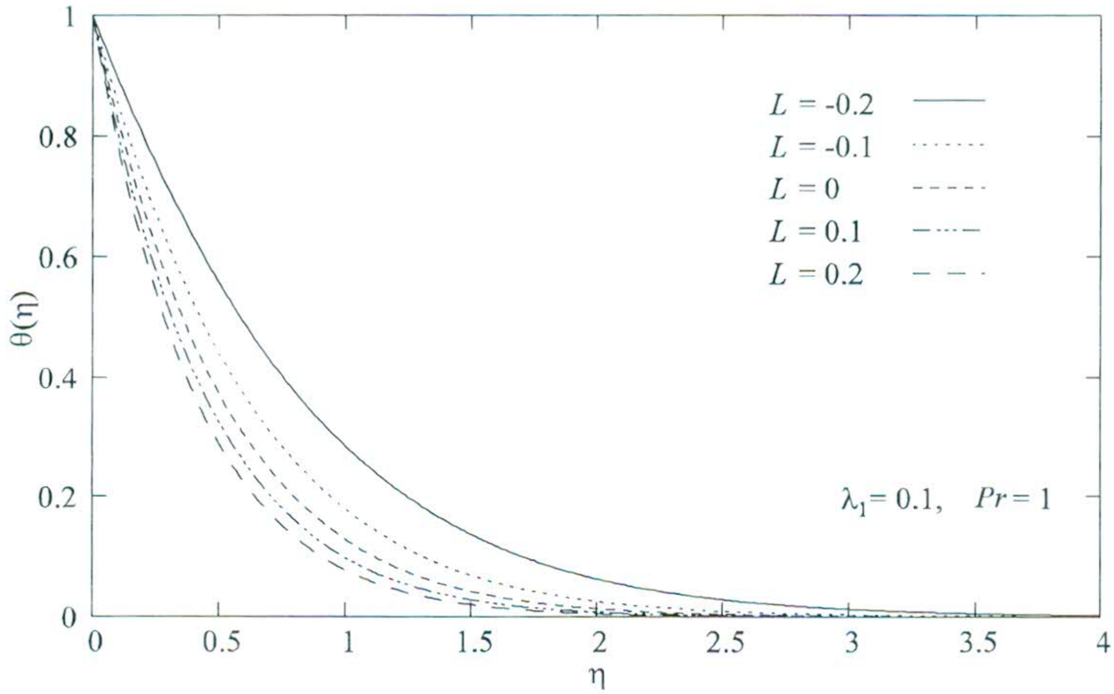

Fig4(a) Temperature profiles for several values of L for second-grade fluid (i.e. $\lambda_1>0$).

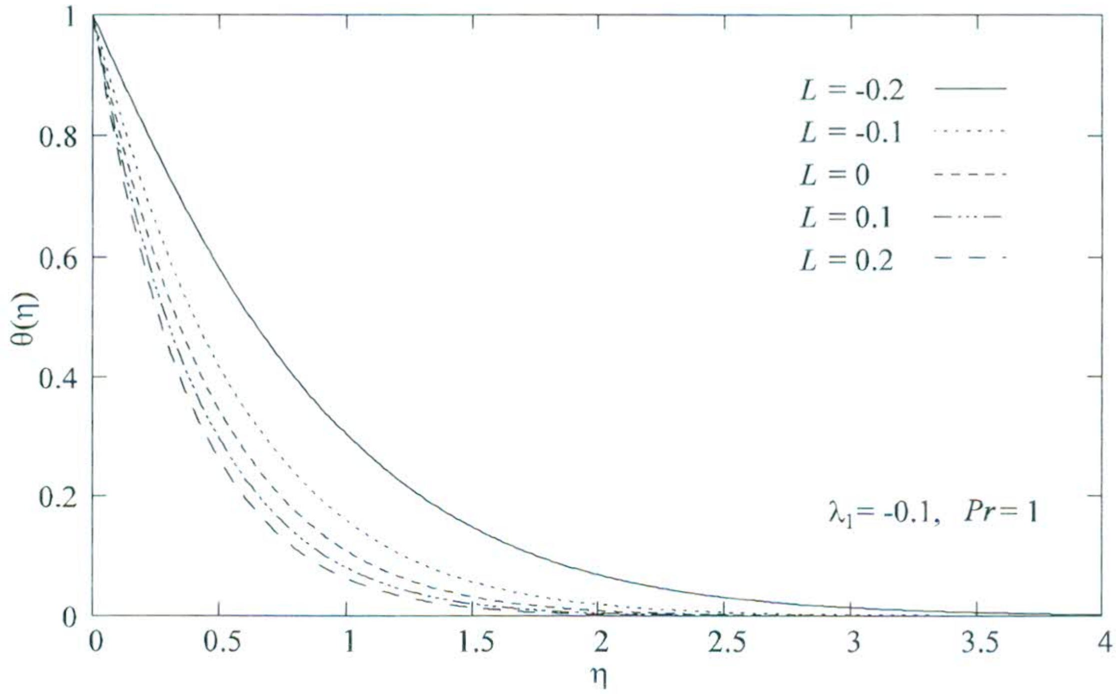

Fig4(b) Temperature profiles for several values of L for Walter's liquid B (i.e. $\lambda_1<0$).

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