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A Theoretical Study on Some Aspects of Turbulent Flow

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A THEORETICAL STUDY ON SOME ASPECTS OF TURBULENT FLOW

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In

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By

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Dedicated

To

My Parents

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Firstly, I consign my limitless thanks to the Almighty Allah for giving me strength, endurance and ability to complete this course of study.

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In fine, I am alone responsible for the errors and shortcomings in this study if there be any, I am sorry for that.

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CERTIFICATE

I have the pleasure in certifying that the M. Phil. thesis entitled "A Theoretical Study on Some Aspects of Turbulent Flow" submitted by Mst. Shamima Sultana in fulfillment of the requirement for the degree of M. Phil. in Mathematics, University of Rajshahi; Rajshahi, Bangladesh has been completed under my supervision. I believe that this research work is an original one and it has not been submitted elsewhere for any degree.

I wish her a bright future and every success in life.

SAS 8/10/03
(*Professor M. Shamsul Alam Sarker*)
Supervisor

DECLARATION OF CREATIVITY

The M. Phil. thesis does not integrate without acknowledgement any substance previously submitted for a degree or diploma in any University and to the best of my knowledge and confidence, it does not contain any material previously published or written by another person except where due reference is prepared in the text.

University of Rajshahi
October 2003.

Mst. Shamima Sultana
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PREFACE

The thesis entitled "**A THEORETICAL STUDY ON SOME ASPECTS OF TURBULENT FLOW**" is being presented for the award of the degree of Master of Philosophy in Mathematics. It is the outcome of my researches conducted in the Department of Mathematics, University of Rajshahi, Bangladesh under the guidance of Dr. M. Shamsul Alam Sarker, Professor, Department of Mathematics, University of Rajshahi, Rajshahi-6205; Bangladesh.

The whole thesis has been divided into four chapters. The first is an introductory chapter and gives the general idea of turbulence and its principal concepts. Some results and theories, which are needed in the subsequent chapters, have been included in this chapter. A brief review of the past researches related to this thesis has also been given. Numbers inside brackets [ ] refer to the references which are arranged alphabetically at the end of the thesis. In the 2nd chapter we have discussed the decay of homogeneous dusty fluid turbulence before the final period for the case of three and four point correlation equations. Finally we have obtained the energy decay law of dusty fluid turbulence before the final period considering three and four point correlation equations after neglecting quintuple correlation terms.

In the third chapter, the decay of homogeneous turbulence before the final period in a rotating system has been studied using three and four point correlation equations. Three and four point correlation equations have been obtained and the set of equations is made determinate by neglecting the quintuple correlation in comparison with the third and fourth order correlation. The correlation equations have been converted into spectral forms by taking their Fourier transforms and the decay law of turbulence in a rotating system before the final period has been obtained.

In the fourth chapter, we have discussed the decay of dusty fluid turbulence before the final period in a rotating system for the case of three and four point correlation equations. In this problem we have considered three and four point correlation equations and solved these equations after neglecting the quintuple correlation term applicable at terms before the final period. Finally, the energy decay law of fluctuating velocity is obtained.

The following research papers that are extracted from this thesis have been communicated for publication in the different reputed Journals.

(i) Decay of homogeneous dusty fluid turbulence before the final period.

(ii) Decay of homogeneous turbulence before the final period in a rotating system for the case of three and four point correlation equations.

(iii) Decay of dusty fluid turbulence before the final period in a rotating system for the case of three and four point correlation equations.

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CHAPTER – I

GENERAL INTRODUCTION

1.1: Basic Concept on Turbulence

The conception of turbulent flow and the accompanying transition form laminar to turbulent flow is of fundamental importance. In everyday life, we recognized three states of matter: solid, liquid and gas. Although different in many respects, liquid and gases have a common characteristic in which they differ from solids: they are fluids, lacking the ability of solids to offer permanent resistance to a deforming force. Fluids flow under the action of such forces, deforming continuously for as long as the force applied. The fluids may be classified into different types depending upon the presence of viscosity. Osborne Reynolds shows that two entirely different types of fluids flow exist. In general words, turbulent flow is a flow, in which the inertia force is dominating over the viscosity. On the other hand, Laminar flow is a flow, in which the viscosity of the fluid is dominating over the inertia force. Osborne Reynolds demonstrated this in 1883 through an experiment. Reynold's apparatus consist of a tank, containing water and a small tank containing dye. O. Reynold's [50] was also the first to investigate in greater detail the circumstances of the transition from laminar to turbulent flow. The previously mentioned dye experiment was used by him in this connecxion, and he discovered the law of similarity which now bears his name, which states

Chapter 1

that transition from laminar to turbulent flow always occurs at nearly the same Reynold's number $\bar{v}d/\nu$, where $\bar{v} = \frac{Q}{A}$ is the mean velocity (Q = volume rate of flow, A = cross-sectional area). The numerical value of the Reynold's number at which transition occurs (critical Reynold's number) was established as being approximately

$$R_{crit} = \left(\frac{\bar{v}\,d}{\nu}\right)_{critical} = 2300\,.$$

Accordingly, flow for which the Reynold's number $R < R_{crit}$, are supposed to be laminar, and flow for which $R > R_{crit}$, are expected to be turbulent.

Turbulence is one of the most difficult open problems in physics. Turbulence is the most common, the most important and the most complicated kind of fluid motion. Applied mathematicians deal it with very carefully from the mathematical standpoint. It is curious to note that the meaning of the word "turbulent" to characterize a certain type of flow, namely, the counterpart of stream line motion. Osborne Reynolds in the study of turbulent flows, named this type of motion "sinous motion".

Turbulent means agitation, commotion, and disturbance. This definition is, however, too general and does not suffice to characterize turbulent fluid motion in the modern sense. It is common experience that the flows observed in nature such as rivers and winds usually differ from streamline flows or laminar flows of a viscous fluid.

The mean motion of such flows does not satisfy the Navier-Stokes equations for a viscous fluid. Such flows that occur at high Reynolds numbers are often termed turbulent the pressures are irregularly fluctuating. The velocity and pressure distributions in turbulent flows as well as the energy losses are determined mainly by turbulent fluctuations. The essential characteristic of turbulent flows is that the turbulent fluctuations are random in nature. In 1937, Taylor and Von Karman [67] gave the following definition: "Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquids when they flow past solid surfaces or even when neighboring streams of the same fluid flow past or over one another."

According to this definition the flow has to satisfy the condition of irregularity. Indeed, this irregularity is a very important feature; because of irregularity it is impossible to describe the motion in all details as a function of time and space coordinates. But fortunately turbulent motion is irregular in the sense that it is possible to describe by the law of probability. It appears possible to indicate distinct average values of various quantities such as velocity, pressure, temperature etc. and this is very important. Therefore it is not sufficient just to say that turbulence is an irregular motion yet we do not have clear-cut definition of turbulence. This is rather difficult. Hinze [20] suggested that "Turbulent fluid motion is an irregular condition of flow in which various quantities show random variations with time and space coordinates so that statistically distinct average values can be discerned." The addition "with time and space coordinates" is necessary; it is not sufficient to define turbulent motion as irregular in time alone. According to the definition suggested by Taylor and Von Karman [67], turbulence can be generated by fluid flow past solid surfaces or by the flow of layers of fluids at different velocities past or over one another. The definition above indicates that there are two distinct types of turbulence:

i) Turbulence generated by the viscous effect due to the presence of a solid wall is designated by wall turbulence;

ii) Turbulence, in the absence of a wall, generated by the flow of layers of fluids at different velocities is called free turbulence. Turbulent flow through conduits and past bodies are examples of wall Turbulence and turbulent jet mixing regions and wakes fall into the category of free Turbulence.

The occurrences of turbulent flows are more frequent and natural. Flows in rivers, ocean currents, natural streams, natural and artificial channels, flow in water supply pipes, flow in fluid machinery such as fans, turbines, pumps etc and air flow over land surfaces are few examples of turbulent flows occurring in every day life.

Turbulence flows always occur from instabilities of laminar motions at very high Reynolds numbers. The instabilities are closely associated with the direct interaction of the nonlinear inertia term and the viscous terms in the Navier-Stokes equation. Instability to small perturbations is also another feature of turbulent flows.

Turbulent motion is three dimensional and rotational. It is also characterized by the random distribution of velocity in which there is no unique relation between the frequency and the wave number of the Fourier modes. It is essentially diffusive and dissipative. The vorticity dynamics plays an important role in the statistical description of turbulence. From the mathematical point of view, turbulence in an incompressible fluid deals with the statistical solutions of the Navier-Stokes equation at very high Reynolds number. Physically, it is concerned with the interaction of eddies of different sizes.

From the experimental point of view the hydro-dynamical turbulence is known to consists of irregular (random) motion of fluid particles and those fluid particles move in lumps rather than individually giving rise to the concept of eddies. The eddies are not of the same size but of varying sizes and therefore turbulence can be separated by Fourier integrals. Further these eddies are not separated from each other just like molecules. In fact, small eddies are embedded in large ones, when these eddies move they affect the fluid surrounding them. These eddies and their random movements give rise to fluctuations in velocity components and pressure at any point in the flow field. The movement of these eddies in the longitudinal as well as in lateral directions imparts to the flow a greater ability for diffusion and makes the analysis of such a flow extremely complex. The origin of the idea of statistical approach to the problem of turbulence may be traced back to Taylor's paper of 1921 [65] in which he has advanced the concept of Lagrangian correlation coefficient that provides a theoretical basis for turbulent diffusion. The most important work done by Taylor [66] is that he gives up the old theories of turbulence based on the kinetic theory of gases and introduces the idea that the velocity of the fluid in turbulent motion is a random continuous function of position and time. He introduces the concept of correlation between velocities at two points. To make the turbulent motion amenable to mathematical treatment he assumes the turbulent fluid to be homogeneous and isotropic. In this support, he describes the measurements showing that the turbulence generated downstream from a regular array of rods in a wind–tunnel is approximately homogeneous and isotropic. In spite of the fact that the turbulence in nature is not always exactly homogeneous and isotropic, it is essential to study the homogeneous and isotropic turbulence as a first step to understand the more complicated phenomenon of non-homogeneous turbulence.

In this case of real viscous fluids viscous effect will result in the conversion of kinetic energy of flow into heat. Thus turbulent flow like all flow of such flows is dissipative in nature. If there is no continuous external source of energy for the continuous generation of the turbulent motion, the motion will decay. Other effects of viscosity are to make the turbulence more homogeneous and to make less dependent on direction.

In 1938 Taylor [68] took into account the non-linearity of the dynamical equations and showed that it results in the skew ness of the probability distribution of the difference between the velocity components at two points. He showed that the non-linearity of the dynamical equations is also responsible for the existence of interaction between the components of the turbulence having different fluctuations. Now instead of giving a detailed account of historical development of the subject, we shall confine to mere concepts and method of turbulence together with a few theories of turbulence that have been used in the subsequent chapters.

1.2: Method of Taking Average

In the mathematical description of turbulent flow the instantaneous velocity component u_i is generally written as

$$u_i = \overline{u_i} + u_i' \qquad (1.2.1)$$

where u_i is the ith component of the total fluid velocity , $\overline{u}_i$ is the ith mean velocity component and u_i' is the ith component of fluctuating velocity. In taking the average of a turbulent quantity, the result depends not only on the scale used but also on the method of averaging. In practice, four different methods of averaging [46] have been used to obtain the mean value of a turbulent quantity (such as velocity, density etc.). If the turbulent flow field is quasi-steady, averaging with respect to time can be used. In the case of a homogeneous turbulent flow field, averaging with respect to space can be used. If the flow field is steady and homogeneous, space-time average is used. At last, if the flow field is neither steady nor homogeneous, we assume that averaging is taken over a large number of experiments that have same initial and boundary conditions. We then speak of an ensemble average.

The methods of averaging are:

1.2(a): Time average in which we take the average at a fixed point in space over a long period of time i.e.,

$$[u(x,t)]_t = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} u(x,s)\,ds \tag{1.2.2}$$

In practice the scale used in the averaging process determines the value of the period 2T.

1.2(b): Space average in which we take the average over all the space at a given time, i.e.,

$$[u(x,t)]_s = \lim_{V_b \to \infty} \frac{1}{V_b} \int_{V_b} u(s,t)\,ds \tag{1.2.3}$$

In practice the value of space $\mathbf{V_b}$ is determined by the scale used in the averaging process.

1.2(c): Space-time average, in which we take the average over a long period of time and over the space, i.e.,

$$[u(x,t)]_{s,t} = \lim_{T \to \infty\, V_b \to \infty} \frac{1}{2TV_b} \int_{-T}^{T} \int_{V_b} u(s,y)\,ds\,dy \tag{1.2.4}$$

In practice both the values of T and of $\mathbf{V_b}$ are determined by the scale used.

1.2(d): Statistical average, in which we take the average over the whole collection of sample turbulent functions for a fixed time, i.e.,

$$[u(x,t,w)]_w = \int_{\Omega} u(x,t,w)\,d\mu(w) \tag{1.2.5}$$

over the whole Ω space of w, the random parameter. The measure is

$$\int_{\Omega} d\mu(w) = 1 \tag{1.2.6}$$

A random scalar function ***u(x,t,w)*** is a is a function of the spatial coordinates $\boldsymbol{x}$ and time $\boldsymbol{t}$, which depends on a parameter w. The parameter w is chosen at random according to some probability law in a space Ω.

1.3: Reynolds Rules of Average

At first Osborne Reynolds [50] introduced elementary statistical motions into the consideration of turbulent flow. In the theoretical investigations of turbulence, he assumed that the instantaneous fluid velocity satisfies the Navier-Stokes equations of motion for a viscous incompressible fluid and that the instantaneous velocity may be separated into a mean velocity and a turbulent fluctuating velocity. Thus the physical quantities characterizing the flow field are written as,

$$u_i = \bar{u}_i + u_i', \quad p = \bar{p} + p', \quad \rho = \bar{\rho} + \rho', \quad T = \bar{T} + T' \tag{1.3.1}$$

Here the quantities with bar denote the mean values and those with primes are fluctuations.

Furthermore, $\overline{u'} = \overline{p'} = \overline{T'} = 0$

In the study of turbulence we often have to carry out an averaging procedure not only a single quantity but also on products of quantities.

In order to develop the rule of averaging, consider three arbitrary statistically dependent physical quantities A, B, C, each consisting of a mean and fluctuating part. i.e.,

$$A = \overline{A} + a\ ,\ B = \overline{B} + b \ \text{and}\ C = \overline{C} + c \tag{1.3.2}$$

Then, $$\overline{A} = \overline{\overline{A} + a} = \overline{\overline{A}} + \overline{a} = \overline{A}\ ,\ \text{whence}\ \overline{a} = 0 \tag{1.3.3}$$

The properties used in the above relations are; the average of the sum is equal to the sum of the average, and the average of a constant time B is equal to the constant times the average of B.

Then,

$$\begin{aligned} \overline{AB} &= \overline{(\overline{A} + a)(\overline{B} + b)} = \overline{\overline{A}\,\overline{B} + \overline{A}b + \overline{B}a + ab} \\ &= \overline{\overline{A}\,\overline{B}} + \overline{\overline{A}b} + \overline{\overline{B}a} + \overline{ab} \\ &= \overline{\overline{A}}\,\overline{\overline{B}} + \overline{\overline{A}}\,\overline{b} + \overline{\overline{B}}\,\overline{a} + \overline{ab} \end{aligned} \tag{1.3.4}$$

Consequently, $$\overline{\overline{A}B} = \overline{\overline{A}}\,\overline{B} = \overline{A}\,\overline{B} \tag{1.3.5}$$

Note that the average of a product is not equal to the product of the averages; terms such as $\overline{a}\,\overline{b}$ are called "correlations".

For the product of three quantities, we have

$$\overline{ABC} = \overline{(\overline{A}+a)(\overline{B}+b)(\overline{C}+c)} = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{bc} + \overline{B}\,\overline{ac} + \overline{C}\,\overline{ab} + \overline{abc} \tag{1.3.6}$$

Also it can be shown that

$$\overline{\frac{\partial A}{\partial S}} = \frac{\partial \overline{A}}{\partial S} \tag{1.3.7}$$

and $$\overline{\int A dS} = \int \overline{A} dS \tag{1.3.8}$$

1.4: The Navier-Stokes and the Continuity Equations

The Navier-Stokes and the continuity equations for an incompressible viscous fluid flow are

$$\frac{\partial \hat{u}}{\partial t} + (u.\nabla)\hat{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\hat{u} \qquad (1.4.1)$$

$$\nabla.\hat{u} = 0 \qquad (1.4.2)$$

where $\hat{u} = \hat{u}(\hat{r}.t)$ represent the velocity field, p is the pressure, ρ is the constant density and ν is the kinematic viscosity. The Reynolds number (the ratio of inertial and viscous terms in (1.4.1)) is UL/ν where L is the characteristic length scale in which the velocity varies in magnitude U.

The use of Navier-Stokes equations for the study of the turbulence is perhaps justified since the each number of an incompressible turbulence flow is small. However, there is still a controversy for the following additional reasons. First the mathematical theory of the Navier-Stokes equations is incomplete in the sense that there are no general existence and uniqueness theorem, which ensure the posedness of the system (1.4.1) – (1.4.2). Second the closure problem of the Navier-Stokes equations is inconclusive. In view of these inherent difficulties, Ladyzhenskaya [37] and others suggest to abandon the application of the Navier-Stokes equations, especially for the study of turbulence. According to Ladyzhenskaya, if a biharmonic damping term $-\lambda\nabla^4\hat{u}$ is included in the right hand side of the Navier-Stokes equations (1.4.1), the existence and the uniqueness of solutions can be established for all $\lambda > 0$. She also formulated new equations for the description of the motion of an incompressible viscous fluid and explained the advantages of her new equations relative to the Navier-Stokes equations. It is important to make an observation form (1.4.1) – (1.4.2).

We first take the divergence of (1.4.1) and use (1.4.2) to obtain

$$\nabla^2 p_1 = -\frac{\partial^2 u_i u_j}{\partial x_i \partial x_j} \qquad (1.4.3)$$

where $p_1 = \frac{p}{\rho}$ is often referred to as the kinematic pressure.

It follows from (1.4.3) that the pressure field is determined by the velocity distribution, and satisfies the Poisson equation.

1.5: Correlation Functions

In 1935, G.I. Taylor [66] introduced new notions into the study of the statistical theory of turbulence, Taylor successfully developed a statistical theory of turbulence which is applicable to continuous movements and which satisfies the equation of motion.

The first important new notion was that of studying the correlation, or coefficient of correlation between two fluctuating quantities in turbulent flow. In his theory, Taylor makes much use of the correlation between the components of the fluctuations at neighboring points. Denote the components of the fluctuating velocity at one point p by u_1, u_2, u_3 and at another point p' by u_1', u_2', u_3'. The correlation function between any of the u_i and u_j' where i, j =1,2 or 3, are defined as

$$\rho_{ij} = \overline{u_i u_j} \qquad (1.5.1)$$

where the bar denotes the average by certain process.

Sometimes it is convenient to use the correlation coefficient such as

$$R_{ij} = \frac{\overline{u_i u_j}}{\sqrt{\overline{u_i^2}\ \overline{u_j'^2}}} \qquad (1.5.2)$$

By Cauchy inequality, we have

$$\overline{u_i u_j} - \sqrt{\overline{u_i}^2}\sqrt{\overline{u_j}^2} \leq 0 \qquad (1.5.3)$$

$$1- \leq R_{ij} \leq 1$$

If we consider $u_i u_j$ as the velocity components in a flow field, the correlation of equation (1.5.1) is a tensor of rank two. By a different process of averaging we obtain different kinds of correlations functions. If we consider u_i and u_j are the velocity components at a given point in space, u_i and u_j are the functions of time; hence, we should take the time average in equation (1.5.1) to get the correlations function ρ_{ij} .

If we consider u_i and u_j as the velocity components at a given time, u_i and u_j are functions of space co-ordinates $x(x_1, x_2, x_3)$; hence, we should take the space average in equations to get the correlations function. More generally if we consider u_i and u_j as a functions of both time t and spatial co-ordinates $x(x_1, x_2, x_3)$, we should take a space-time average in equations (1.5.1) to get the correlation function. The correlation function between the components of the fluctuating velocity at the same time at two different points of the fluid first introduced by G.I. Taylor [66] has been investigated extensively in the isotropic turbulence.

The correlation function between two the fluctuating velocity components at the same point and at the same time gives the Reynolds stress. The correlation function between two fluctuating quantities may also be defined in a manner similar to above.

1.6: Reynolds Equations and Reynolds Stresses

In turbulent flow, we usually assume that instantaneous velocity components satisfy the Navier-Stokes equations,

$$\frac{\partial u_i}{\partial t}+u_1\frac{\partial u_i}{\partial x_1}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\frac{\partial^2 u_i}{\partial x_1 \partial x_1}+f_i \tag{1.6.1}$$

Substituting the expressions for the instantaneous velocity components from (1.2.1) into Navier-Stokes equations (1.6.1) for an incompressible fluid after neglecting the body forces and taking the mean values of these equations according to Reynolds rule of averaging (1.3.1) – (1.3.5), we have the following Reynolds equation of motion for the turbulent flow of an incompressible fluid:

$$\rho\left(\frac{\partial \overline{U}_i}{\partial t}+\overline{U}_j\frac{\partial \overline{U}_i}{\partial x_j}\right)=-\frac{\partial \overline{p}}{\partial x_i}+\frac{\partial}{\partial x_j}\left(\mu\frac{\partial \overline{U}_i}{\partial x_j}-\overline{\rho u_i' u_j'}\right) \tag{1.6.2}$$

where i and j run from 1 to 3 and Einstein's summation convection is used. The bar represents the mean value and the prime, the turbulent fluctuation. Additional terms over the Navier-Stokes are due to the Reynolds stresses or eddy stresses. The eddy normal stresses are $-\overline{\rho u_i'^2}$ and the eddy shearing stresses are $-\overline{\rho u_i' u_j'}$, $(i \neq j)$ where ρ is the density of the fluid. These stresses represent the rate of transfer of momentum across the corresponding surfaces because of turbulent velocity fluctuation.

The solutions of the Reynolds equations will represent the turbulent flow, but as in the case of Navier-Stokes equations it is not at the present time possible to solve the Reynolds equations for many practical purposes.

1.7: Isotropic and Homogeneous Turbulence

The turbulence is called isotropic if its statistical features have no preference for any specific direction and minimum number of quantities and relations are required to describe its structure and behavior.

Since turbulence is a very complicated problem, in order to bring out the essential features of the turbulence problem we have to study the simplest type of turbulence. In isotropic turbulence the mean value of any function of velocity components and their space derivatives are unaltered by any rotation or reflection of axes of references. Thus, in particular

$$\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$$

and $$\overline{u_1 u_2} = \overline{u_2 u_3} = \overline{u_3 u_1} = 0$$

Isotropy introduce a great simplicity into the calculations. The study of isotropic turbulence may also be of practical importance, since far from solid boundaries it has been observed that $\overline{u_1^2}$, $\overline{u_2^2}$, $\overline{u_3^2}$ tend to become equal to one another, e.g. in the natural winds at a sufficient height above the ground and in a pipe flow near the axis.

Another simplest type of turbulence is homogeneous turbulence. It is defined as the turbulence having quantitatively the same structure in all parts of the flow field. In a homogeneous turbulent flow field the statistical characteristics are invariant for any translation in the space occupied by the fluid.

Most of the theoretical works in turbulence and MHD turbulence concern homogeneous and isotropic field in an incompressible fluid at rest. Throughout the present work, we have also assumed the homogeneity and isotropy of the turbulent flow field.

1.8: Spectral Representation of the Turbulence

Theoretical treatment of the turbulence is merely related to the solutions of the Navier-Stokes equations. These equations, however, contain more unknowns than the number of equations and therefore additional assumptions must be made. This is known as the "Closure Problem". An alternative approach is based on the spectral form of the dynamic Navier-Stokes equation. The spectral form of turbulence is still underdetermined, but it has a simple physical interpretation and is more convenient. The spectral approach is, however, almost exclusively used for the description of homogeneous turbulence [42, 43]. The principal concepts of spectral representation in the study of turbulence are described below:

If we neglect the body forces from the Navier-Stokes equation (1.6.1) and multiply the x_i component of Navier-Stokes equation written for the point p by $u_j^{/}$, and multiply the $x_j^{/}$ component of the equation written for the point $p^{/}$ by $u_i^{/}$, adding and taking the ensemble average we get

$$\frac{\partial \overline{u_i u_j^{/}}}{\partial t}+\left(\overline{u_j^{/} u_1 \frac{\partial u_i}{\partial x_1}}+\overline{u_i u_1^{/} \frac{\partial u_j^{/}}{\partial x_1^{/}}}\right)=-\frac{1}{\rho}\left(\overline{u_j^{/} \frac{\partial p}{\partial x_i}}+\overline{u_i \frac{\partial p^{/}}{\partial x_j^{/}}}\right)+\nu\left(\overline{u_j^{/} \frac{\partial^2 u_i}{\partial x_1^2}}+\overline{u_i \frac{\partial^2 u_j^{/}}{\partial x_1^2}}\right) \quad (1.8.1)$$

Since in the homogeneous turbulence, the statistical quantities are independent of position in space and considering the points p and $p^{/}$ separated by a distance vector $\bar{r}$ and applying the laws of spatial covariance, a simplified form of Equation (1.8.1) is obtained as

$$\frac{\partial \overline{u_j u_j^{/}}}{\partial t}=-\frac{\partial}{\partial r_1}\left(\overline{u_i u_j^{/} u_1}-\overline{u_i u_j^{/} u_1^{/}}\right)+\frac{1}{\rho}\left(\frac{\partial \overline{p u_j^{/}}}{\partial r_i}-\frac{\partial \overline{p^{/} u_i}}{\partial r_j}\right)+2\nu \frac{\partial^2 \overline{u_i u_j^{/}}}{\partial r_1^2} \quad (1.8.2)$$

The covariance $\overline{u_i u_j'}$ is not suitable for direct analysis of quantitative estimate of the turbulent flows and it is better to use the three-dimensional Fourier transforms of $\overline{u_i u_j'}$ with respect to $\bar{r}$. The variable that corresponds to $\bar{r}$ in the three-dimensional wave number space is vector $\bar{k} = (k_1, k_2, k_3)$. We define the wave number spectral density as

$$\phi_{ij}\left(\vec{k}\right) = \frac{1}{(2\pi)^3}\int \overline{u_i u_j'} \exp\left(-i\vec{k}.\vec{r}\right)d\vec{r}$$
$$= \frac{1}{(2\pi)^3}\iiint \overline{u_i u_j'} \exp\{-i(k_1 r_1 + k_2 r_2 + k_3 r_3)\}dr_1 dr_2 dr_3 \tag{1.8.3}$$

It can be shown that if $\overline{u_i u_j'}$ has a continuous range of wavelength, $\phi_{ij}\left(\vec{k}\right)$ has a continuous distribution in wave-number space. We can rigorously regard $\phi_{ij}\left(\vec{k}\right)dk_1 dk_2 dk_3$ as the contribution of the elementary volume $dk_1 dk_2 dk_3$ (centered at wave-number $\vec{k}$ and therefore representing a wave-number of length $\frac{2\pi}{\left|\vec{k}\right|}$ in the direction of the vector $\vec{k}$) to the value of $\overline{u_i u_j'}$ hence the name "spectral density". This is consistent with the behavior of the inverse transform

$$\overline{u_i u_j'}(\vec{r}) = \int \phi_{ij}\left(\vec{k}\right)\exp(i\vec{k}.\vec{r})d\vec{k} \tag{1.8.4}$$

The one dimensional wave-number spectrum of $\overline{u_i u_j'}$ for a wave-number component in the x_1 direction is

$$\phi_{ij}(k_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u_i u_j'(r_1)\exp(-ik_1 r_1)dk_1 \tag{1.8.5}$$

whose inverse is

$$\overline{u_i u_j'}(\vec{r}) = \int_{-\infty}^{\infty}\phi_{ij}(k_1)\exp(ik_1 r_1)dk_1 \tag{1.8.6}$$

The Equation (1.8.2) for unstrained homogeneous turbulence becomes on Fourier transforming

$$\frac{\partial \phi_{ii}(\vec{k})}{\partial t} = \Gamma_{ij}(\vec{k}) + \pi_{ij}(\vec{k}) - 2\nu k_1^2 \phi_{ij}(\vec{k}) \tag{1.8.7}$$

Where Γ and π are the transforms of the triple product and pressure terms respectively.

1.9: Fourier Transformation of the Navier-Stokes Equations

The principal reason for using Fourier transforms is that they convert differential operators into multipliers. The equations are so complicated in configuration (or coordinate) space that very little can be done with them, and the transformation to wave-number (or Fourier) space simplifies them very considerably.

Another and more Mathematical argument shows that these transforms are right method of treating a homogeneous problem. Associated with any correlation function $\phi\left(\overline{x}, \overline{x}'\right)$ is a sequence of eigen functions $\phi\left(\overline{n}, \overline{x}'\right)$ and their associated Eigen values $\lambda\left(\overline{n}\right)$. These quantities satisfy the eigen value equation

$$\int \phi\left(\vec{x}, \vec{x}'\right) \psi\left(\vec{n}, \vec{x}\right) d^3\overline{x}' = \lambda\left(\vec{n}\right) \psi\left(\vec{n}, \vec{x}\right) \tag{1.9.1}$$

and the orthonormalization relation

$$\int \psi\left(\vec{n}, \vec{x}\right) \psi^*\left(\vec{m}, \vec{x}\right) d^3\overline{x} = 1 \quad \text{if } m = n \tag{1.9.2}$$

$$= 0 \qquad \text{otherwise.}$$

These equations imply that ϕ is a scalar. Actually it is a tensor of order two, but these complete the argument without introducing anything essentially new. The index $\overline{n}$, is in general, a complex variable and ψ^* denotes the complex conjugate of ψ (strictly, ψ^* is the adjoint of ψ, but since ϕ is real and symmetric the adjoint is simply the complex conjugate). The integrations in equations (1.9.1) and (1.9.2) are over all space, which may be finite or infinite. If the space is finite, $\bar{n}$ is usually an infinite but countable sequence, while if space is infinite, $\bar{n}$ will be a continuous variable. Here the eigen functions all have real eigen values. It follows from (1.9.1) and (1.9.2) that

$$\phi\left(\vec{x},\vec{x}'\right)=\sum_{n}\lambda\left(\vec{n}\right)\psi\left(\vec{n},\vec{x}\right)\psi^*\left(\vec{n},\vec{x}'\right) \tag{1.9.3}$$

and this the diagonal representation of the correlation function in terms of its eigen functions. Evidently these functions are only defined "without a phase" that is, a factor *exp(iγ)* can be added to $\psi\left(\vec{n},\vec{x}\right)$ without altering $\phi\left(\vec{x},\vec{x}'\right)$ provided γ is real and independent of x. For a homogeneous field, ϕ is a function of $\vec{x}-\vec{x}'$ only, and the problem is to find eigen functions which are also homogeneous within a phase, in the sense that

$$\psi\left(\vec{n},\vec{x}\right)=\exp(\,i\gamma\,)\psi\left(\vec{n},\vec{x}+\vec{a}\right) \tag{1.9.4}$$

This equation is satisfied by the Fourier function,

$$\left(\vec{n},\vec{x}\right)=\exp(i\vec{n}.\vec{x})=\exp(in_jx_j) \tag{1.9.5}$$

with $\gamma=-\vec{n},\vec{a}$. In this instance, therefore, "the index" $\vec{n}$ is a wave-number Equation (1.9.3) becomes,

$$\phi\left(\vec{x},\vec{x}'\right)=\sum\lambda(n)\exp\{in(\vec{x}-\vec{x}')\} \tag{1.9.6}$$

So, that $\lambda\left(\vec{n}\right)$ may be identified with $\phi\left(\vec{n}\right)$, the Fourier transform of the correlation function.

Since we are considering homogeneous isotropic turbulence, the turbulence field must be infinite in extent. This produces mathematical difficulties, which can only be resolved by using functional calculus. This difficulty is avoided by supposing that the turbulence is confined to the inside of a large box with sides (a_1,a_2,a_3) and that it obeys cyclic boundary conditions on the sides of this box. The a_i are allowed to tend to infinity at an appropriate point in the analysis. Thus the Fourier transform is defined by

$$U_i\left(\vec{x}\right) = (2\pi)^3\left(a_1 a_2 a_3\right)^{-1}\sum_k u_i(\vec{k})\exp(i\vec{k}.\vec{x}) \qquad (1.9.7)$$

Here $\vec{k}$ is limited to wave vectors of the form

$$\frac{2n_1\pi}{a_1},\frac{2n_2\pi}{a_2},\frac{2n_3\pi}{a_3}$$

where the n_i are the integers while the a_i are, as before, the sides of the elementary box. As these sides become infinitely large. Equation (1.9.7) goes over into the standard form,

$$U_i(\vec{x}) = \int u_i\left(\vec{k}\right)\exp\left(i\vec{k}.\vec{x}\right)d^3\vec{k}. \qquad (1.9.8)$$

The inverse of (1.9.8) is

$$u_i\left(\vec{k}\right) = (2\pi)^{-3}\int_{box} U_i\left(\vec{x}\right)\exp\left(-i\vec{k}.\vec{x}\right)d^3x. \qquad (1.9.9)$$

The Fourier transforms of the Navier-Stokes equation may be written as

$$\left(\frac{d}{dt}+\nu k^2\right)u_i\left(\vec{k}\right) = M_{ijm}\left(\vec{k}\right)\overset{\Delta}{\sum} U_j(\vec{p})U_m(\vec{r}) \qquad (1.9.10)$$

Where $\overset{\Delta}{\sum}$ is a short notation for the integral operator in

$$\iint U_j(\vec{k})U_m(\vec{r})\delta(\vec{k}-\vec{p}-\vec{r})d^3\vec{p}d^3\vec{r} \tag{1.9.11}$$

where $\delta_{k,p+r}$ is the Kroneker delta symbol, which is zero unless $\vec{k}=\vec{p}+\vec{r}$

Here, $M_{ijm}(\vec{k})$ is a simple algebraic multiplier and not a differential operator. We have

$$M_{ijm}(\vec{k}) = -\frac{1}{2}ip_{ijm}(\vec{k}) \tag{1.9.12}$$

where $$P_{ijm}(\vec{k}) = k_m p_{ij}(\vec{k}) + k_j p_{im}(\vec{k})$$

and $$P_{ij} = \delta_{ij} - \frac{k_i k_j}{k^2}$$

$p_{ij}(\vec{k})$ is the Fourier transforms of $p_{ij}(\Delta)$ but $p_{ijm}(\vec{k})$ is not the transforms of $p_{ijm}(\Delta)$. As it stands, Equation (1.9.10) cannot describe stationary turbulence since it contains no input of energy to balance the dissipative effect of viscosity. In real life this input is provided by effects, such as the interaction of the mean velocity gradient with the Reynolds stress, which are incompatible with the ideas of homogeneity and isotropy. To avoid this difficulty, we introduce into the right hand side of Equation (1.9.10) a hypothetical homogeneous isotropic stirring force f_i. The equation then reads,

$$\left(\frac{d}{dt}+\nu k^2\right)u_i(\vec{k}) = M_{ijm}(k)\sum^{\Delta} U_j(\vec{p})U_m(\vec{r}) + f_i(\vec{k}) \tag{1.9.13}$$

1.10: A Brief Description of Past Researches Relevant to this Thesis Work

The main characteristic of turbulence flows is that turbulent fluctuations are random in nature and therefore, by the application of statistical laws, it has been possible to give the idea of turbulent fluctuations. The turbulent statistical flows, in the absence of external agencies always decay. Millionshtchikov [40], Batchelor and Townsend [1], Proudman and Reid [49], Deissler [12,13] and Ghosh [17,18] had given various analytical theories for the decay process of turbulence so far.

Batchelor and Townsend [1] studied the decay of turbulence in the final period. They said that the final period of a turbulent motion occurs when the effects of the inertia force in the momentum equations are negligible. Deissler [12,13] studied the decay of turbulence at times before the final period. Also Loeffler and Deissler [38] discussed the decay of temperature fluctuation in homogeneous turbulence before the final period. In their approach they considered the two and three point correlation equations and solved these equations after neglecting the fourth and higher order correlation terms in comparison to the lower order correlation terms. Using Deissler's theory Kumar and Patel [35] studied the concentration fluctuation of dilute Contaminants undergoing a first order chemical reaction before the final period of decay for the case of multipoint and single-time. Kumar and Patel [36] also extended their problem of [35] for the case of multipoint and multi-time.

Likewise the hydrodynamic turbulence, MHD turbulent fluctuations are random in nature. The statistical laws can also be applied in MHD turbulence. Sarker and Kishore [54] studied the decay of MHD turbulence. Kishore and Upadhyay [34] also studied the decay of MHD turbulence in rotating system. In both the cases they obtained the decay law for the case of multipoint and single time before the final period.

Funada, Tutiya and Ohji [16] considered the effect of coriolis force on turbulent motion in presence of strong magnetic field. Kishore and Dixit [27], Kishore and Singh [24], Dixit and Upadhyay [15], Kishore and Golsefield [29] and Kishore and Sarker [32] discussed the effect of coriolis force on acceleration and vorticity covariance in ordinary and MHD turbulent flow. Shimomura and Yoshizawa [60], Shimomura [61,62] discussed the statistical analysis of turbulent viscosity, turbulent scalar flux and turbulent shear flows respectively in a rotating system by two-scale Direct Interaction approach. Saffman [52] derived an equation that described the motion of a fluid containing small dust particles, which is applicable to laminar flows as well as turbulent flow. Using the equations given by Michael and Miller [39] discussed the motion of dusty gas occupying the semi-infinite space above a rigid plane boundary. Sinha [63], Sarker and Rahman [56] considered dust particles on their own works.

By considering the above theories, we have studied the Chapter-II, Chapter-III and Chapter-IV. In Chapter-II, we have studied the decay of dusty fluid turbulence before the final period for the case of three and four point correlation equations. In Chapter-III, we have derived the energy decay law for homogeneous turbulence before the final period in a rotating system for the case of three and four point correlation equations. In Chapter-IV, we have generalized the energy decay law of dusty fluid turbulence before the final period in a rotating system using three and four point correlation equations.

In geophysical flows, the system is usually rotating with a constant angular velocity. Such large-scale flows are generally turbulent. When the motion is referred to axes, which rotate steadily with the bulk of fluid, coriolis and centrifugal force must be supposed to act on the fluid. The coriolis force due to rotation plays an important role in a rotating system of turbulent flow, while the centrifugal force with the potential is incorporated into the pressure.

CHAPTER-II

DECAY OF HOMOGENEOUS DUSTY FLUID TURBULENCE BEFORE THE FINAL PERIOD

2.1: Introduction

The main theme is to seek a possible solution for the dynamics of decaying homogeneous turbulence. Approximately homogeneous turbulence can be produced, for instance, by passing a fluid through a grid; various stages in the decay process then occur at various distances downstream from the grid. Although a considerable amount of work has been done on the problem of homogeneous turbulence, a satisfactory solution applicable to the major portion of the lifetime of the eddies has not been obtained previously. The main difficulty lies in obtaining a determinate set of dynamical equations. One can construct, from the momentum and continuity equations, equations involving correlations between the fluctuating quantities at a number of points in the fluid. Deissler [12] developed a theory 'Decay of homogeneous turbulence for times before the final period'. Deissler [13] also generalized the theory to some extent in order to analyze the turbulence at higher Reynolds numbers. In his case, the quadruple correlation terms in the three-point correlation equation are retained. By considering the Deissler's theory, Loeffler and Deissler [38] studied the decay of temperature fluctuation in homogeneous turbulence. Saffman [52] derived an equation that described the motion of a fluid containing small dust particles, which is applicable to laminar flow as well as turbulent flow. In recent years, the motion of dusty viscous fluids has developed rapidly. The behavior of dust particles in turbulent flow depends on the concentration of the particles and the size of the particles with respect to the scale of turbulent flow. Using Deissler's theory Kumer and Patel [35] analyzed 'the first order reactant in homogeneous turbulence before the final period' for the case of multi point and single time consideration. Sinha [63] studied the effect of dust particles on the acceleration covariance of ordinary turbulence. Kishore and Sinha [23] also studied the rate of change of vorticity covariance of dust particles in hydrodynamic turbulence. Following Deissler's approach, Kishore and Sarker [33] analyzed the decay of MHD turbulence before the final period for the case of multi point and single time. The above problem [33] is extended to the case of multipoint and multi-time concentration correlation by Sarker and Islam [59].

Analyzing the Deissler's theories we have studied the decay of homogeneous turbulence before the final period in presence of dust particles using three and four point correlation equations. Here a four-point correlation equation is considered. The set of equations is made determinate by neglecting the fifth order correlation terms in comparison to the third and fourth order correlation terms. Finally, the energy decay law of dusty fluid turbulence before the final period is obtained.

In absence of dust particles the result reduces to one obtained earlier by Deissler [13].

2.2: Basic Equations

The equation of motion and continuity for turbulent flow of dusty incompressible fluid are given below:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_l)}{\partial x_l} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_l \partial x_l} + f(u_i - v_i) \tag{2.2.1}$$

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = -\frac{K}{m_s}(v_i - u_i) \tag{2.2.2}$$

and $$\frac{\partial u_i}{\partial x_i} = \frac{\partial v_i}{\partial x_i} = 0 \tag{2.2.3}$$

The subscripts can take on the values 1, 2 or 3.

Here u_i, turbulent velocity components; v_i , dust particle velocity components; ρ, fluid density; ν , kinematics viscosity; p, instantaneous pressure; $m_s = \frac{4}{3}\pi R_s^3 \rho_s$, mass of a single spherical dust particle of radius R_s ; ρ_s, constant density of the material in dust particles; K, stock's drug resistance; $f = \frac{KN}{\rho}$, dimensions of frequency; N, constant number density of dust particle.

2.3: Correlation and Spectral Equations

The equations of motion of turbulent flow in presence of dust particles for the points p, $p^{/}$ and $p^{//}$ separated by the vector $\underline{r}$ and $\underline{r}^{/}$ are

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_l)}{\partial x_l} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_l \partial x_l} + f(u_i - v_i) \tag{2.3.1}$$

$$\frac{\partial u_j'}{\partial t}+\frac{\partial(u_j'u_l')}{\partial x_l'}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_j'}+\nu\frac{\partial^2 u_j'}{\partial x_l'\partial x_l'}+f\left(u_j'-v_j'\right) \tag{2.3.2}$$

$$\frac{\partial u_k''}{\partial t}+\frac{\partial(u_k''u_l'')}{\partial x_l''}=\frac{1}{\rho}\frac{\partial p''}{\partial x_k''}+\nu\frac{\partial^2 u_k''}{\partial x_l''\partial x_l''}+f\left(u_k''-v_k''\right) \tag{2.3.3}$$

Multiplying equation (2.3.1) by $u_j'u_k''$, (2.3.2) by $u_i u_k''$ and (2.3.3) by $u_i u_j'$, adding the three equations and taking space or time averages, we obtain

$$\frac{\partial}{\partial t}\langle u_iu_j'u_k''\rangle+\frac{\partial}{\partial x_l}\langle u_iu_j'u_k''u_l\rangle+\frac{\partial}{\partial x_l'}\langle u_iu_j'u_k''u_l'\rangle+\frac{\partial}{\partial x_l''}\langle u_iu_j'u_k''u_l''\rangle$$
$$=-\frac{1}{\rho}\left(\frac{\partial}{\partial x_i}\langle pu_j'u_k'\rangle+\frac{\partial}{\partial x_j'}\langle p'u_iu_k''\rangle+\frac{\partial}{\partial x_k''}\langle p''u_iu_j'\rangle\right)$$
$$+\nu\left(\frac{\partial^2\langle u_iu_j'u_k''\rangle}{\partial x_l\partial x_l}+\frac{\partial^2\langle u_iu_j'u_k''\rangle}{\partial x_l'\partial x_l'}+\frac{\partial^2\langle u_iu_j'u_k''\rangle}{\partial x_l''\partial x_l''}\right)$$
$$+f\left(3\langle u_iu_j'u_k''\rangle-\langle v_iu_j'u_k''\rangle-\langle v_j'u_iu_k''\rangle-\langle v_k''u_iu_j'\rangle\right) \tag{2.3.4}$$

Using the transformations $\frac{\partial}{\partial x_l'}=\frac{\partial}{\partial r_l}$, $\frac{\partial}{\partial x_l''}=\frac{\partial}{\partial r_l'}$ and $\frac{\partial}{\partial x_l}=-\frac{\partial}{\partial r_l}-\frac{\partial}{\partial r_l'}$ into equations (2.3.4), we get,

$$\frac{\partial}{\partial t}\langle u_iu_j'u_k''\rangle-\frac{\partial}{\partial r_l}\langle u_iu_j'u_k''u_l\rangle-\frac{\partial}{\partial r_l'}\langle u_iu_j'u_k''u_l\rangle+\frac{\partial}{\partial r_l}\langle u_iu_j'u_k''u_l'\rangle+\frac{\partial}{\partial r_l'}\langle u_iu_j'u_k''u_l'\rangle$$
$$=-\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}\langle pu_j'u_k''\rangle-\frac{\partial}{\partial r_i'}\langle pu_j'u_k''\rangle+\frac{\partial}{\partial r_j}\langle p'u_iu_k''\rangle+\frac{\partial}{\partial r_k'}\langle p''u_iu_j'\rangle\right)$$
$$+2\nu\left(\frac{\partial^2\langle u_iu_j'u_k''\rangle}{\partial r_l\partial r_l}+\frac{\partial^2\langle u_iu_j'u_k''\rangle}{\partial r_l\partial r_l'}+\frac{\partial^2\langle u_iu_j'u_k''\rangle}{\partial r_l'\partial r_l'}\right)$$
$$+f\left(3\langle u_iu_j'u_k''\rangle-\langle v_iu_j'u_k''\rangle-\langle u_iv_j'u_k''\rangle-\langle u_iu_j'v_k''\rangle\right) \tag{2.3.5}$$

In order to convert equation (2.3.5) to spectral form, we can define the following six dimensional Fourier transforms:

$$\langle u_i u_j'(r)u_k''(r')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \beta_i \beta_j'(k)\beta_k''(k')\rangle.\exp\left[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')\right]dkdk' \qquad (2.3.6)$$

$$\langle u_i u_l u_j'(r)u_k''(r')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \beta_i \beta_l \beta_j'(k)\beta_k''(k')\rangle.\exp\left[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')\right]dkdk' \qquad (2.3.7)$$

$$\langle pu_j'(r)u_k''(r')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \alpha\beta_j'(k)\beta_k''(k')\rangle.\exp\left[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')\right]dkdk' \qquad (2.3.8)$$

and $\langle v_i u_j'(r)u_k''(r')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \gamma_i\beta_j'(k)\beta_k''(k')\rangle.\exp\left[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')\right]dkdk' \qquad (2.3.9)$

Interchanging the subscripts i and j and then interchanging the points p and $p^/$ give

$$\langle u_i u_l'(r)u_j'(r)u_k''(r')\rangle = \langle u_j u_l u_i'(-r))u_k''(r'-r)\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \beta_j \beta_l \beta_i'(-k-k')\beta_k''(k')\rangle.\exp\left[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')\right]dkdk' \qquad (2.3.10)$$

$$\langle u_i u_j'(r)u_k''(r')u_l''(r')\rangle = \langle u_k u_l u_i'(-r'))u_j''(r-r')\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \beta_k \beta_l \beta_i'(-k-k')\beta_j''(k\ \ )\rangle.\exp\left[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')\right]dkdk' \qquad (2.3.11)$$

where the points p and $p^/$ are interchanged to obtain equation (2.3.10). For equation (2.3.11), p is replaced by $p^/$, $p^/$ is replaced by $p^{//}$ and $p^{//}$ is replaced by p.

Similarly,

$$\langle u_i p'(r)u_k''(r')\rangle = \langle pu_i'(-r)u_k''(r'-r)\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \alpha\beta_i'(-k-k')\beta_k''(k')\rangle.\exp\left[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')\right]dkdk' \qquad (2.3.12)$$

$$\langle u_i u_j' p''(r')\rangle = \langle pu_i'(-r')u_j''(r-r')\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \alpha\beta_i'(-k-k')\beta_j''(k\ \ )\rangle.\exp\left[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')\right]dkdk' \qquad (2.3.13)$$

$$\langle u_i v_j' u_k''(r')\rangle = \langle v_j u_i'(-r) u_k''(r'-r)\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \gamma_j \beta_i'(-k-k')\beta_k''(k')\rangle . \exp\left[i(\underline{k}\cdot \underline{r} + \underline{k}'\cdot \underline{r}')\right] dk\,dk' \qquad (2.3.14)$$

$$\langle u_i u_j'(r) v_k''(r')\rangle = \langle v_k u_i'(-r') u_j''(r-r')\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \gamma_k \beta_i'(-k-k')\beta_j''(k\ \ )\rangle . \exp\left[i(\underline{k}\cdot \underline{r} + \underline{k}'\cdot \underline{r}')\right] dk\,dk' \qquad (2.3.15)$$

Substituting the preceding relations into equation (2.3.5), we get

$$\frac{d}{dt}\langle \beta_i \beta_j' \beta_k''\rangle + 2\gamma\left(k^2 + k_l k_l' + k'^2\right)\langle \beta_i \beta_j' \beta_k''\rangle$$
$$= \left[i\left(k_l + k_l'\right)\langle \beta_i \beta_l \beta_j' \beta_k''\rangle - ik_l\langle \beta_j \beta_l \beta_i'(-k-k')\beta_k''(k')\rangle - ik_l'\langle \beta_k \beta_l \beta_i'(-k-k')\beta_j''(k)\rangle\right]$$
$$-\frac{1}{\rho}\left[-i(k_i + k_i')\langle \alpha \beta_j' \beta_k''\rangle + ik_j\langle \alpha \beta_i'(-k-k')\beta_k''(k')\rangle + ik_k'\langle \alpha \beta_i'(-k-k')\beta_j''(k)\rangle\right] +$$
$$f\left[3\langle \beta_i \beta_j'(k)\beta_k''(k')\rangle - \langle \gamma_i \beta_j'(k)\beta_k''(k')\rangle - \langle \gamma_j \beta_i'(-k-k')\beta_k''(k')\rangle - \langle \gamma_k \beta_i'(-k-k')\beta_j''(k)\rangle\right] \qquad (2.3.16)$$

The tensor equation (2.3.16) can be converted to a scalar form by contraction of the indices i and j and inner multiplication k_k ;

$$\frac{d}{dt}\left(k_k\langle \beta_i \beta_i' \beta_k''\rangle\right) + 2\nu k_k\left(k^2 + k_l k_l' + k'^2\right)\langle \beta_i \beta_i' \beta_k''\rangle$$
$$= \left[ik_k\left(k_l + k_l'\right)\langle \beta_i \beta_l \beta_i' \beta_k''\rangle - ik_k k_l\langle \beta_i \beta_l \beta_i'(-k-k')\beta_k''(k')\rangle - ik_k k_l'\langle \beta_k \beta_l \beta_i'(-k-k')\beta_i''(k)\rangle\right]$$
$$-\frac{1}{\rho}\left[-ik_k(k_i + k_i')\langle \alpha \beta_i' \beta_k''\rangle + ik_k k_i\langle \alpha \beta_i'(-k-k')\beta_k''(k')\rangle + ik_k k_k'\langle \alpha \beta_i'(-k-k')\beta_i''(k)\rangle\right] +$$
$$fk_k\left[3\langle \beta_i \beta_i'(k)\beta_k''(k')\rangle - \langle \gamma_i \beta_i'(k)\beta_k''(k')\rangle - \langle \gamma_i \beta_i'(-k-k')\beta_k''(k')\rangle - \langle \gamma_k \beta_i'(-k-k')\beta_i''(k)\rangle\right] \qquad (2.3.17)$$

To obtain the four-point equation, we consider the equation of motion of turbulent flow in presence of dust particles of the points p, p', p'' and p''' as

$$\frac{\partial u_l}{\partial t} + \frac{\partial (u_l u_m)}{\partial x_m} = -\frac{1}{\rho}\frac{\partial p}{\partial x_l} + \nu\frac{\partial^2 u_l}{\partial x_m \partial x_m} + f(u_l - v_l) \qquad (2.3.18)$$

$$\frac{\partial u_j'}{\partial t} + \frac{\partial (u_j' u_m')}{\partial x_m'} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j'} + \nu\frac{\partial^2 u_j'}{\partial x_m' \partial x_m'} + f(u_j' - v_j') \qquad (2.3.19)$$

$$\frac{\partial u''_k}{\partial t}+\frac{\partial(u''_k u''_m)}{\partial x''_m}=-\frac{1}{\rho}\frac{\partial p''}{\partial x''_k}+\nu\frac{\partial^2 u''_k}{\partial x''_m\partial x''_m}+f(u''_k-v''_k) \qquad (2.3.20)$$

$$\frac{\partial u'''_l}{\partial t}+\frac{\partial(u'''_l u'''_m)}{\partial x'''_m}=-\frac{1}{\rho}\frac{\partial p'''}{\partial x'''_l}+\nu\frac{\partial^2 u'''_k}{\partial x'''_m\partial x'''_m}+f(u'''_l-v'''_l) \qquad (2.3.21)$$

where the repeated subscript in a term indicates a summation.

Multiplying the first equation by $u'_j u''_k u'''_l$, the second by $u_i u''_k u'''_l$, the third by $u_i u'_j u'''_l$ and the fourth by $u_i u'_j u''_k$ respectively, then adding and taking space or time averages, we get

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\langle u_i u'_j u''_k u'''_l\right\rangle+\frac{\partial}{\partial x_m}\left\langle u_i u'_j u''_k u'''_l u_m\right\rangle+\frac{\partial}{\partial x'_m}\left\langle u_i u'_j u''_k u'''_l u'_m\right\rangle\\
&+\frac{\partial}{\partial x''_m}\left\langle u_i u'_j u''_k u'''_l u''_m\right\rangle+\frac{\partial}{\partial x'''_m}\left\langle u_i u'_j u''_k u'''_l u'''_m\right\rangle\\
&=-\frac{1}{\rho}\left(\frac{\partial}{\partial x_i}\left\langle p u'_j u''_k u'''_l\right\rangle+\frac{\partial}{\partial x'_j}\left\langle p' u_i u''_k u'''_l\right\rangle+\frac{\partial}{\partial x''_k}\left\langle p'' u_i u'_j u'''_l\right\rangle+\frac{\partial}{\partial x'''_l}\left\langle p''' u_i u'_j u''_k\right\rangle\right)\\
&+\nu\left(\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial x_m\partial x_m}+\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial x'_m\partial x'_m}+\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial x''_m\partial x''_m}+\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial x'''_m\partial x'''_m}\right)+\\
&f\Big(-\left\langle v_i u'_j u''_k u'''_l\right\rangle+\left\langle u_i u'_j u''_k u'''_l\right\rangle-\left\langle u_i v'_j u''_k u'''_l\right\rangle+\left\langle u_i u'_j u''_k u'''_l\right\rangle\\
&-\left\langle u_i u'_j v''_k u'''_l\right\rangle+\left\langle u_i u'_j u''_k u'''_l\right\rangle-\left\langle u_i u'_j u''_k v'''_l\right\rangle+\left\langle u_i u'_j u''_k u'''_l\right\rangle\Big) \qquad (2.3.22)
\end{aligned}$$

Equation (2.3.22) can be written in terms of the independent variables r, r' and r'' as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\langle u_i u'_j u''_k u'''_l\right\rangle-\frac{\partial}{\partial r_m}\left\langle u_i u_m u'_j u''_k u'''_l\right\rangle-\frac{\partial}{\partial r'_m}\left\langle u_i u_m u'_j u''_k u'''_l\right\rangle-\frac{\partial}{\partial r''_m}\left\langle u_i u_m u'_j u''_k u'''_l\right\rangle\\
&+\frac{\partial}{\partial r_m}\left\langle u_i u'_j u'_m u''_k u'''_l\right\rangle+\frac{\partial}{\partial r'_m}\left\langle u_i u'_j u''_k u''_m u'''_l\right\rangle+\frac{\partial}{\partial r''_m}\left\langle u_i u'_j u''_k u'''_l u'''_m\right\rangle\\
&=-\frac{1}{\rho}\left(\frac{\partial}{\partial r_i}\left\langle p u'_j u''_k u'''_l\right\rangle-\frac{\partial}{\partial r'_i}\left\langle p\, u'_j u''_k u'''_l\right\rangle-\frac{\partial}{\partial r''_i}\left\langle p u'_j u''_k u'''_l\right\rangle+\frac{\partial}{\partial r_j}\left\langle u_i p' u''_k u''_l\right\rangle+\frac{\partial}{\partial r'_k}\left\langle u_i u'_j p'' u'''_l\right\rangle+\frac{\partial}{\partial r''_l}\left\langle u_i u'_j u''_k p'''\right\rangle\right)\\
&+2\nu\left(\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial r_m\partial r_m}+\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial r_m\partial r'_m}+\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial r_m\partial r''_m}+\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial r'_m\partial r'_m}+\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial r'_m\partial r''_m}+\frac{\partial^2\left\langle u_i u'_j u''_k u'''_l\right\rangle}{\partial r''_m\partial r''_m}\right)\\
&+f\Big(-\left\langle v_i u'_j u''_k u'''_l\right\rangle-\left\langle u_i v'_j u''_k u'''_l\right\rangle-\left\langle u_i u'_j v''_k u'''_l\right\rangle-\left\langle u_i u'_j u''_k v'''_l\right\rangle+4\left\langle u_i u'_j u''_k v'''_l\right\rangle\Big) \qquad (2.3.23)
\end{aligned}$$

where the following transformations were used:

$$\frac{\partial}{\partial x_m^{/}} = \frac{\partial}{\partial r_m}, \quad \frac{\partial}{\partial x_m^{//}} = \frac{\partial}{\partial r_m^{/}}, \quad \frac{\partial}{\partial x_m^{///}} = \frac{\partial}{\partial r_m^{//}} \quad \text{and} \quad \frac{\partial}{\partial x_m} = -\frac{\partial}{\partial r_m} - \frac{\partial}{\partial r_m^{/}} - \frac{\partial}{\partial r_m^{//}}$$

In order to convert equation (2.3.23) to spectral form, we define the following nine-dimensional Fourier transforms:

$$\langle u_i u_j^{/}(r) u_k^{//}(r^{/}) u_l^{///}(r^{//})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{ijkl}(k,k^{/},k^{//}) \exp[i(\underline{k}\cdot\underline{r} + \underline{k}^{/}\cdot\underline{r}^{/} + \underline{k}^{//}\cdot\underline{r}^{//})] dk dk^{/} dk^{//} \quad (2.3.24)$$

$$\langle u_i u_m u_j^{/}(r) u_k^{//}(r^{/}) u_l^{///}(r^{//})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{imjkl}(k,k^{/},k^{//}) \exp[i(\underline{k}\cdot\underline{r} + \underline{k}^{/}\cdot\underline{r}^{/} + \underline{k}^{//}\cdot\underline{r}^{//})] dk dk^{/} dk^{//} \quad (2.3.25)$$

$$\langle p u_j^{/}(r) u_k^{//}(r^{/}) u_l^{///}(r^{//})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_{jkl}(k,k^{/},k^{//}) \exp[i(\underline{k}\cdot\underline{r} + \underline{k}^{/}\cdot\underline{r}^{/} + \underline{k}^{//}\cdot\underline{r}^{//})] dk dk^{/} dk^{//} \quad (2.3.26)$$

$$\langle v_i u_j^{/}(r) u_k^{//}(r^{/}) u_l^{///}(r^{//})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_i \delta_{jkl}(k,k^{/},k^{//}) \exp[i(\underline{k}\cdot\underline{r} + \underline{k}^{/}\cdot\underline{r}^{/} + \underline{k}^{//}\cdot\underline{r}^{//})] dk dk^{/} dk^{//} \quad (2.3.27)$$

$$\begin{aligned}\langle u_i v_j^{/}(r) u_k^{//}(r^{/}) u_l^{///}(r^{//})\rangle &= \langle v_j u_i^{/}(-r) u_k^{//}(r^{/}-r) u_l^{///}(r^{//}-r)\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{j\delta_{ikl}}(-k-k^{/}-k^{//},k^{/},k^{//}) \exp[i(\underline{k}\cdot\underline{r} + \underline{k}^{/}\cdot\underline{r}^{/} + \underline{k}^{//}\cdot\underline{r}^{//})] dk dk^{/} dk^{//}\end{aligned} \quad (2.3.28)$$

Similarly,

$$\begin{aligned}&\langle u_i u_j^{/}(r) v_k^{//}(r^{/}) u_l^{///}(r^{//})\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_k \delta_{ijl}(-k-k^{/}-k^{//},k,k^{//}) \exp[i(\underline{k}\cdot\underline{r} + \underline{k}^{/}\cdot\underline{r}^{/} + \underline{k}^{//}\cdot\underline{r}^{//})] dk dk^{/} dk^{//}\end{aligned} \quad (2.3.29)$$

$$\begin{aligned}&\langle u_i u_j^{/}(r) u_k^{//}(r^{/}) v_l^{///}(r^{//})\rangle \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_l \delta_{ijk}(-k-k^{/}-k^{//},k,k^{/}) \exp[i(\underline{k}\cdot\underline{r} + \underline{k}^{/}\cdot\underline{r}^{/} + \underline{k}^{//}\cdot\underline{r}^{//})] dk dk^{/} dk^{//})\end{aligned} \quad (2.3.30)$$

Substituting the preceding relations into equation (2.3.23), we get

$$\frac{d}{dt}\gamma_{ijkl}+2\nu(k^2+k_m k_m^{/}+k_m k_m^{//}+k^{/^2}+k_m^{/}k_m^{//}+k^{//^2})\gamma_{ijkl}$$
$$=[i(k_m+k_m^{/}+k_m^{//})\gamma_{imjkl}(k,k^{/},k^{//})-ik_m\gamma_{jmikl}(-k-k^{/}-k^{//},k^{/},k^{//})$$
$$-ik_m^{/}\gamma_{kmijl}(-k-k^{/}-k^{//},k,k^{//})-ik_m^{//}\gamma_{lmijk}(-k-k^{/}-k^{//},k,k^{/})]$$
$$-\frac{1}{\rho}[-i(k_i+k_i^{/}+k_i^{//})\delta_{jkl}(k,k^{/},k^{//})+ik_j\delta_{ikl}(-k-k^{/}-k^{//},k^{/},k^{//})$$
$$+ik_k^{/}\delta_{ijl}(-k-k^{/}-k^{//},k\ ,k^{//})+ik_l^{//}\delta_{ijk}(-k-k^{/}-k^{//},k,k^{/})]$$
$$+f[4\gamma_{ijkl}(k,k^{/},k^{//})-\gamma_i\delta_{jkl}(k,k^{/},k^{//})-\gamma_j\delta_{ikl}(-k-k^{/}-k^{//},k^{/},k^{//})$$
$$-\gamma_k\delta_{ijl}(-k-k^{/}-k^{//},k,k^{//})-\gamma_l\delta_{ijk}(-k-k^{/}-k^{//},k,k^{/})] \qquad (2.3.31)$$

To obtain a relation between the terms on the right hand side of equation (2.3.31) derived from the quadruple correlation terms, pressure terms and the dust particle terms in equation (2.3.23), take the divergence of the equation of motion and combine with the continuity equation to give

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_m \partial x_m}=-\frac{\partial^2(u_m u_n)}{\partial x_m \partial x_m} \qquad (2.3.32)$$

Multiplying the equation (2.3.32) by $u_j^{/}u_k^{//}u_l^{///}$, taking ensemble average and writing the resulting equation in terms of the independent variables r and $r^{/}$, gives

$$\frac{1}{\rho}\left(\frac{\partial^2\langle pu_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m \partial r_m}+2\frac{\partial^2\langle pu_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m \partial r_m^{/}}+2\frac{\partial^2\langle pu_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m \partial r_m^{//}}\right.$$
$$\left.+\frac{\partial^2\langle pu_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{/} \partial r_m^{/}}+2\frac{\partial^2\langle pu_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{/} \partial r_m^{//}}+\frac{\partial^2\langle pu_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{//} \partial r_m^{//}}\right)$$
$$=-\left(\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m \partial r_n}+\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m \partial r_n^{/}}+\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m \partial r_n^{//}}\right.$$
$$+\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{/} \partial r_n}+\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{/} \partial r_n^{/}}+\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{/} \partial r_n^{//}}$$
$$\left.+\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{//} \partial r_n}+\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{//} \partial r_n^{/}}+\frac{\partial^2\langle u_m u_n u_j^{/}u_k^{//}u_l^{///}\rangle}{\partial r_m^{//} \partial r_n^{//}}\right) \qquad (2.3.33)$$

The Fourier transform of equation (2.3.33) is

$$-\frac{1}{\rho}\left(k^2+2k_mk'_m+2k_mk''_m+k'^2+2k'_mk''_m+k''^2\right)\delta_{jkl}$$
$$=(k_mk_n+k_mk'_n+k_mk''_n+k'_mk_n+k'_mk'_n+k'_mk''_n+k''_mk_n+k''_mk'_n+k''_mk''_n)\gamma_{mnjkl}$$

$$\therefore \frac{1}{\rho}\delta_{jkl}=-\frac{\left(k_mk_n+k_mk'_n+k_mk''_n+k'_mk_n+k'_mk'_n+k'_mk''_n+k''_mk_n+k''_mk'_n+k''_mk''_n\right)\gamma_{mnjkl}}{\left(k^2+2k_mk'_m+2k_mk''_m+k'^2+2k'_mk''_m+k''^2\right)} \tag{2.3.34}$$

Equations (2.3.31) and (2.3.34) are the spectral equations corresponding to the four-point correlation equations. The spectral equations corresponding to the three point correlation equations are

$$\frac{d}{dt}(k_k\beta_{iik})+2\nu(k^2+k_lk'_l+k'^2)k_k\beta_{iik}$$
$$=ik_k(k_l+k'_l)\beta_{ilik}(k,k')-ik_kk_l\beta_{ilik}(-k-k',k')-ik_kk'_l\beta_{klii}(-k-k',k\ )$$
$$-\frac{1}{\rho}\left[-ik_k(k_i+k'_i)\alpha_{ik}(k,k')+ik_kk_i\alpha_{ik}(-k-k',k')+ik_kk'_k\alpha_{ii}(-k-k',k)\right]+Rfk_k \tag{2.3.35}$$

here, $R\beta_i\beta'_i\beta''_k=3\langle\beta_i\beta'_i\beta''_k\rangle-\langle\gamma_i\beta'_i(k)\beta''_k(k')\rangle-\langle\gamma_i\beta'_i(-k-k')\beta''_k(k')\rangle-\langle\gamma_k\beta'_i(-k-k')\beta''_i(k\ )\rangle$,(say)

R is an arbitrary constant and

$$-\frac{1}{\rho}\alpha_{ik}=\frac{k_lk_m+k'_lk_m+k_lk'_m+k'_lk'_m}{k^2+2k_lk'_l+k'^2}\beta_{lmik} \tag{2.3.36}$$

Here the spectral tensors are defined by

$$\langle u_iu'_j(r)u''_k(r')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta_{ijk}(k,k')\exp[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')]dkdk' \tag{2.3.37}$$

$$\langle u_iu_lu'_j(r)u''_k(r')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta_{iljk}(k,k')\exp[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')]dkdk' \tag{2.3.38}$$

$$\langle pu'_j(r)u''_k(r')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_{jk}(k,k')\exp[i(\underline{k}\cdot\underline{r}+\underline{k}'\cdot\underline{r}')]dkdk' \tag{2.3.39}$$

A relation between β_{iljk} and γ_{ijkl} can be obtained by letting $\underline{r}^{//} = 0$ in equation (2.3.24) and comparing the result with equation (2.3.38)

$$\beta_{iljk}(k, k^{/}) = \int_{-\infty}^{\infty} \gamma_{ijkl}(k, k^{/}, k^{//}) dk^{//} \tag{2.3.40}$$

The spectral equation corresponding to the two-point correlation equations in presence of dusty fluid is

$$\frac{d}{dt}\phi_{i,i} + (2\nu k^{2} - Qf)\phi_{i,i} = ik_{k}\phi_{iki}(k) - ik_{k}\phi_{iki}(-k) \tag{2.3.41}$$

where $\phi_{i,i}$ and ϕ_{iki} are defined by

$$\langle u_{i} u_{i}^{/}(r)\rangle = \int_{-\infty}^{\infty} \phi_{ij}(k)\exp(i\underline{k}\cdot\underline{r})dk \tag{2.3.42}$$

$$\langle u_{i} u_{k} u_{j}^{/}(r)\rangle = \int_{-\infty}^{\infty} \phi_{ikj}(k)\exp(i\underline{k}\cdot\underline{r})dk \tag{2.3.43}$$

and $Q\phi_{i}\phi_{i}^{/} = 2\langle \phi_{i}\phi_{i}^{/}(k)\rangle - \langle \mu_{i}\phi_{i}^{/}(k)\rangle - \langle \mu_{i}\phi_{i}^{/}(-k)\rangle$ is an arbitrary constant.

The relation between ϕ_{ikj} and β_{ijk} obtained by letting $\underline{r}^{/} = 0$ in equation (2.3.37) and comparing the result with equation (2.3.43) is

$$\phi_{ikj}(k) = \int_{-\infty}^{\infty} \beta_{ijk}(k, k^{/})dk^{/} \tag{2.3.44}$$

2.4: Solution Neglecting Quintuple Correlations

Equation (2.3.34) shows that if the terms corresponding to the quintuple correlations are neglected, then the pressure force terms also must be neglected. Thus neglecting first and second terms on the right side of equation (2.3.31), the equation can be integrated between t_1 and t to give

$$\gamma_{ijkl} = (\gamma_{ijkl})_{1}\exp[\{-2\nu(k^{2} + k_{m}k_{m}^{/} + k_{m}k_{m}^{//} + k^{/^{2}} + k_{m}^{/}k_{m}^{//} + k^{//^{2}}) - Sf\}(t - t_{1})] \tag{2.4.1}$$

$$\frac{d}{dt}(k_k\beta_{iik}) + 2\nu(k^2 + k_l k_l' + k'^2)k_k\beta_{iik} - Rfk_k$$

$$= \frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[a]_1}{(t-t_1)^{\frac{3}{2}}}\exp\left[\left\{-2\nu\left(\frac{3}{4}k^2 + \frac{1}{2}k_l k_l' + \frac{3}{4}k'^2\right) - Sf\right\}(t-t_1)\right]$$

$$+ \frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[b]_1}{(t-t_1)^{\frac{3}{2}}}\exp\left[\left\{-2\nu\left(\frac{3}{4}k^2 + k_l k_l' + k'^2\right) - Sf\right\}(t-t_1)\right]$$

$$+ \frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[c]_1}{(t-t_1)^{\frac{3}{2}}}\exp\left[\left\{-2\nu\left(k^2 + k_l k_l' + \frac{3}{4}k'^2\right) - Sf\right\}(t-t_1)\right] \qquad (2.4.3)$$

where the bracketed quantities in equation (2.4.2) have been abbreviated as shown.

Integration of equation (2.4.3) with respect to time, results in

$$k_k\beta_{iik} = (k_k\beta_{iik})_0\exp\left[\left\{-2\nu(k^2 + k_l k_l' + k'^2) - Rf\right\}(t-t_0)\right]$$

$$+ \frac{\pi^{\frac{3}{2}}[a]_1}{\nu}\left\{-\omega^{-1}\exp\left[-\omega^2\left(\frac{3}{4}k^2 + \frac{1}{2}k_l k_l' + \frac{3}{4}k'^2\right) - Sf(t-t_1)\right]\right.$$

$$\left. + 2\left(\frac{1}{4}k^2 + \frac{1}{2}k_l k_l' + \frac{1}{4}k'^2\right)^{\frac{1}{2}}\exp\left[-\omega^2(k^2 + k_l k_l' + k'^2) - Sf(t-t_1)\right]\int_0^{\omega\left(\frac{1}{4}k^2 + \frac{1}{2}k_l k_l' + \frac{1}{4}k'^2\right)^{\frac{1}{2}}}\exp(x^2)\,dx\right\}$$

$$+ \frac{\pi^{\frac{3}{2}}[b]_1}{\nu}\left\{-\omega^{-1}\exp\left[-w^2\left(\frac{3}{4}k^2 + k_l k_l' + k'^2\right) - Sf(t-t_1)\right]\right.$$

$$\left. + k\exp\left[-\omega^2(k^2 + k_l k_l' + k'^2) - Sf(t-t_1)\right]\int_0^{\frac{1}{2}\omega k}\exp(x^2)\,dx\right\}$$

$$+ \frac{\pi^{\frac{3}{2}}[c]_1}{\nu}\left\{-\omega^{-1}\exp\left[-\omega^2\left(k^2 + k_l k_l' + \frac{3}{4}k'^2\right) - Sf(t-t_1)\right]\right.$$

$$\left. + k'\exp\left[-\omega^2(k^2 + k_l k_l' + k'^2) - Sf(t-t_1)\right]\int_0^{\frac{1}{2}\omega k'}\exp(x^2)\,dx\right\} \qquad (2.4.4)$$

where $\omega = \left[2\nu(t-t_1)\right]^{\frac{1}{2}}$

In order to simplify the calculations, we shall assume that $[a]_1 = 0$; that is, we assume that a function sufficiently general to represent the initial conditions can be obtained by considering only the terms involving $[b]_1$ and $[c]_1$

The substitution of equations (2.3.44) and (2.4.4) in equation (2.3.41) and setting

$$E = 2\pi k^2 \phi_{i,i} \text{ results in } \frac{dE}{dt} + (2\nu k^2 - Qf)E = W \tag{2.4.5}$$

where,

$$\begin{aligned} W &= k^2 \int_{-\infty}^{\infty} 2\pi i [k_k \beta_{iik}(k,k') - k_k \beta_{iik}(-k,-k')]_0 \exp[\{-2\nu(k^2 + k_l k_l' + k'^2) - Rf\}(t - t_0)] dk' \\ &+ k^2 \int_{-\infty}^{\infty} \frac{2\pi^{\frac{5}{2}} i}{\nu} [b(k,k') - b(-k,-k')]_1 \Big\{ -w^{-1} \exp[-w^2(\frac{3}{4}k^2 + k_l k_l' + k'^2) - Sf(t - t_1)] \\ &+ k \exp[-w^2(k^2 + k_l k_l' + k'^2) - Sf(t - t_1)] \int_0^{\frac{1}{2}wk} \exp(x^2) dx \Big\} dk' \\ &+ k^2 \int_{-\infty}^{\infty} \frac{2\pi^{\frac{5}{2}} i}{\nu} [c(k,k') - c(-k,-k')]_1 \Big\{ -w^{-1} \exp[-w^2(k^2 + k_l k_l' + \frac{3}{4}k'^2) - Sf(t - t_1)] \\ &+ k' \exp[-w^2(k^2 + k_l k_l' + k'^2) - Sf(t - t_1)] \int_0^{\frac{1}{2}wk'} \exp(x^2) dx \Big\} dk' \end{aligned} \tag{2.4.6}$$

The quantity E is the energy spectrum function, which represents contributions from various wave numbers or eddy sizes to the total averages. W is the energy transfer function, which is responsible for the transfer of energy between wave numbers. In order to find the solution completely and following Deissler [13], we assume that

$$(2\pi)^2 i \left[k_k \beta_{iik}(k,k') - k_k \beta_{iik}(-k,-k')\right]_0 = -\beta_0 (k^4 k'^6 - k^6 k'^4) \tag{2.4.7}$$

For the bracketed quantities in equation (2.4.6), we let

$$\frac{4\pi^{\frac{7}{2}}}{\nu} i\left[b(k,k') - b(-k,-k')\right]_1 = \frac{4\pi^{\frac{7}{2}}}{\nu} i\left[c(k,k') - c(-k,-k')\right]_1 = -2\gamma_1\left(k^6 k'^8 - k^8 k'^6\right) \tag{2.4.8}$$

where the two bracketed quantities are set equal in order to make the integrands in equation (2.4.6) anti symmetric with respect to k and k'.

By substituting equations (2.4.7) and (2.4.8) in equation (2.4.6) remembering that $dk' = 2\pi k'^2 d(\cos\theta)dk'$ and $k_i k_i' = kk' \cos\theta$ (θ is the angle between $\underline{k}$ and $\underline{k}'$), and carrying out the integration with respect to θ ,we get

$$\begin{aligned}
W = \int_0^{\infty} \Bigg[& \frac{\beta_0 (k^4 k'^6 - k^6 k'^4) kk'}{2\nu(t-t_0)} \begin{Bmatrix} \exp[-\{2\nu(k^2 + kk' + k'^2) - Rf\}(t-t_0)] \\ -\exp[-\{2\nu(k^2 - kk' + k'^2) - Rf\}(t-t_0)] \end{Bmatrix} - \gamma_1 \frac{(k^6 k'^8 - k^8 k'^6) kk'}{\nu(t-t_1)} \\
& \times \Big\{ (\omega^{-1} \exp[-\omega^2(\tfrac{3}{4}k^2 + kk' + k'^2) - Sf(t-t_1)] - \omega^{-1}\exp[-\omega^2(\tfrac{3}{4}k^2 - kk' + k'^2) - S(t-t_1)f] \\
& + \omega^{-1}\exp[-\omega^2(k^2 + kk' + \tfrac{3}{4}k'^2) - Sf(t-t_1)] - \omega^{-1}\exp[-\omega^2(k^2 - kk' + \tfrac{3}{4}k'^2) - Sf(t-t_1)] \\
& + \{k\exp[-\omega^2(k^2 - kk' + k'^2) - Sf(t-t_1)] - k\exp[-\omega^2(k^2 + kk' + k'^2) - Sf(t-t_1)]\} \int_0^{\frac{1}{2}\omega k} \exp(x^2)dx \\
& + \{k'\exp[-\omega^2(k^2 - kk' + k'^2) - Sf(t-t_1)] - k'\exp[-\omega^2(k^2 + kk' + k'^2) - Sf(t-t_1)]\} . \int_0^{\frac{1}{2}\omega k'} \exp(x^2)dx \Big\} \Bigg] dk'
\end{aligned} \tag{2.4.9}$$

where $\omega = \left[2\nu(t-t_1)\right]^{\frac{1}{2}}$

The integrand in this equation represents the contribution to the energy transfer at a wave number k, from a wave number k'. The integral is then total contribution to W at k, from all wave numbers. Carrying out the indicated integration with respect to k' in equation (2.4.9), where results in

$$W = W_\beta + W\gamma \tag{2.4.10}$$

Here

$$W_{\beta}=-\frac{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\beta_0}{256\nu^{\frac{15}{2}}(t-t_0)^{\frac{15}{2}}}\exp\left[(-\frac{3}{2}\varepsilon^2)(105\varepsilon^6+45\varepsilon^8-19\varepsilon^{10}-3\varepsilon^{12})-Rf(t-t_0)\right] \quad (2.4.11)$$

and

$$\begin{aligned}
W_{\gamma}=-\frac{\gamma_1}{\nu^{10}(t-t_1)^{10}}&\left[\frac{\pi^{\frac{1}{2}}}{16}\exp\left\{(-\eta^2)\left(\frac{3}{128}\eta^{16}+\frac{3}{8}\eta^{14}+\frac{21}{64}\eta^{12}-\frac{105}{16}\eta^{10}-\frac{945}{128}\eta^{8}\right)-Sf(t-t_1)\right\}\right.\\
&+\frac{2\pi^{\frac{1}{2}}}{\sqrt{3}}\exp\left\{\left(-\frac{4}{3}\eta^2\right)\left(\frac{160}{19683}\eta^{16}+\frac{40}{729}\eta^{14}-\frac{14}{27}\eta^{12}-\frac{455}{162}\eta^{10}-\frac{35}{18}\eta^{8}\right)-Sf(t-t_1)\right\}\\
&-\frac{(\frac{\pi}{2})^{\frac{1}{2}}}{16}\exp\left\{(-\frac{3}{2}\eta^2)\int_0^{\frac{\eta}{\sqrt{2}}}\exp(y^2)dy\left(\frac{3}{64}\eta^{17}+\frac{3}{4}\eta^{15}+\frac{21}{32}\eta^{13}-\frac{105}{8}\eta^{11}-\frac{945}{32}\eta^{9}\right)-Sf(t-t_1)\right\}\\
&+\frac{\pi^{\frac{1}{2}}}{2}\exp\left\{(-\frac{3}{2}\eta^2)(5.386\eta^8+9.118\eta^{10}+3.017\eta^{12}+0.1793\eta^{14}-0.03106\eta^{16}\right.\\
&-0.004942\eta^{18}-3.615\times10^{-4}\eta^{20}-1.890\times10^{-5}\eta^{22}-7.561\times10^{-7}\eta^{24}\\
&\left.\left.-2.447\times10^{-8}\eta^{26}-6.64\times10^{-10}\eta^{28}-1.55\times10^{-11}\eta^{30}..........\ldots)-Sf(t-t_1)\right\}\right] \qquad (2.4.12)
\end{aligned}$$

where $\eta=\nu^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}}k$ and $\varepsilon=\nu^{\frac{1}{2}}(t-t_0)k$.

The quantity W_{β} is the contribution to the energy transfer arising from consideration of the three-point correlation equation W_{γ} arises from consideration of the four-point equation. Integration of equation (2.4.10) over all wave numbers shows that

$$\int_0^{\infty} W dk = 0 \qquad (2.4.13)$$

indicating that the expression for W satisfies the conditions of continuity and homogeneity.

In order to obtain the energy spectrum function E, we integrate equation (2.4.5) with respect to time. This integration results in

$$E = E_j + E_\beta + E_\gamma \tag{2.4.14}$$

where
$$E_j = \frac{J_0 \varepsilon^4}{3\pi \nu^2 (t-t_0)^2} \exp\left\{-2\varepsilon^2 - Qf(t-t_0)\right\} \tag{2.4.15}$$

$$E_\beta = -\frac{(2\pi)^{\frac{1}{2}}\beta_0}{256\nu^{\frac{15}{2}}(t-t_0)^{\frac{13}{2}}} \exp\left[\left(-\frac{3}{2}\varepsilon^2\right)\left\{-15\varepsilon^6 - 12\varepsilon^8 + \frac{7}{3}\varepsilon^{10} + \frac{16}{3}\varepsilon^{12} - \frac{32}{3\sqrt{2}}\varepsilon^{13}\exp\left(-\frac{\varepsilon^2}{2}\right)\int_0^{\frac{\varepsilon}{\sqrt{2}}}\exp(y^2)dy]\right\} - Rf(t-t_0)\right] \tag{2.4.16}$$

and

$$\begin{aligned}
E_\gamma = -\frac{\gamma_1}{\nu^{10}(t-t_1)^9}&\left\{\frac{\pi^{\frac{1}{2}}}{32}\exp\left\{(-\eta^2)\left[\frac{189}{64}\eta^8 + \frac{1029}{256}\eta^{10} + \frac{287}{256}\eta^{12} + \frac{95}{512}\eta^{14} + \frac{71}{512}\eta^{16} - \frac{71}{512}\eta^{18}\exp(-\eta^2)\left[Ei(\eta^2) - 0.5772\right]\right] - Sf(t-t_1)\right\}\right. \\
&+ \left(\frac{\pi}{3}\right)^{\frac{1}{2}}\exp\left\{\left(-\frac{4}{3}\eta^2\right)\left[\frac{7}{9}\eta^8 + \frac{497}{324}\eta^{10} + \frac{1001}{1458}\eta^{12} + \frac{761}{4374}\eta^{14} + \frac{1963}{19683}\eta^{16} - \frac{3926}{59049}\eta^{18}\exp\left(-\frac{2}{3}\eta^2\right)\left[Ei\left(\frac{2}{3}\eta^2\right) - 0.5772\right]\right] - Sf(t-t_1)\right\} \\
&+ \frac{\pi^{\frac{1}{2}}}{2}\exp\{(-\frac{3}{2}\eta^2)[0.2307\eta^{10} + 0.3632\eta^{12} + 0.1502\eta^{14} + 0.04463\eta^{16} - 0.01326\eta^{18}\exp(-\frac{1}{2}\eta^2)[Ei(\frac{1}{2}\eta^2) - 0.5772] \\
&+ 2.459\times10^{-3}\eta^{18} + 2.935\times10^{-4}\eta^{20} + 2.846\times10^{-5}\eta^{22} + 2.52\times10^{-6}\eta^{24} + 1.69\times10^{-7}\eta^{26} + 1.25\times10^{-8}\eta^{28} \\
&+ 5.80\times10^{-10}\eta^{30} + 4.00\times10^{-11}\eta^{32}\ldots\ldots] - Sf(t-t_1)\} \\
&+ \frac{1}{2}(\pi)^{\frac{1}{2}}\exp\{(-\frac{3}{2}\eta^2)[1.077\eta^8 + 2.414\eta^{10} + 1.408\eta^{12} + 0.4416\eta^{14} + 0.1898\eta^{16} \\
&- 0.0899\eta^{18}\exp(-\frac{1}{2}\eta^2)[Ei(\frac{1}{2}\eta^2) - 0.5772] + 6.575\times10^{-4}\eta^{18} + 3.271\times10^{-5}\eta^{20} + 1.270\times10^{-6}\eta^{22} \\
&\left.+ 4.03\times10^{-8}\eta^{24} + 1.08\times10^{-9}\eta^{26} + 2.50\times10^{-11}\eta^{28} + 5.09\times10^{-13}\eta^{30} + \cdots\cdots] - Sf(t-t_1)\}\right\}
\end{aligned} \tag{2.4.17}$$

The quantity E_j is the energy spectrum function for the final period, where as E_β and E_γ are the contributions to the energy spectrum arising from consideration of the three and four point correlation equations respectively.

Integration of equation (2.4.14) can be integrated over all wave numbers to give the total turbulent energy

$$\frac{1}{2}\langle u_i u_i \rangle = \int_0^{\infty} E dk \tag{2.4.18}$$

The result carrying out the integration is, in dimensionless form,

$$\frac{\langle u_i u_i \rangle}{2} = \frac{J_0^{\frac{14}{9}} \nu^{\frac{5}{9}}}{\beta_0^{\frac{5}{9}}} \left[\frac{1}{32(2\pi)^{\frac{1}{2}}} T^{-\frac{5}{2}} \exp[Qf(t-t_0)] + 0.2296\, T^{-7} \exp[Rf(t-t_0)] \right.$$

$$\left. + 6.18 \frac{\gamma_1 \nu^{\frac{5}{9}} J_0^{\frac{5}{9}}}{\beta_0^{\frac{14}{9}}} \left(\frac{t-t_1}{t-t_0}\right)^{-\frac{19}{2}} T^{-\frac{19}{2}} \exp[Sf(t-t_1)] \right] \tag{2.4.19}$$

Thus the energy decay law of velocity fluctuations of dusty fluid turbulence may be written as

$$\langle u^2 \rangle = AT^{-\frac{5}{2}} \exp[Qf(t-t_0)] + BT^{-7} \exp[Rf(t-t_0)] + CT^{-\frac{19}{2}} \left(\frac{t-t_1}{t-t_0}\right)^{-\frac{19}{2}} \exp[Sf(t-t_1)] \tag{2.4.20}$$

where

$$\frac{t-t_1}{t-t_0} = 1 - \left(\frac{\gamma_1 \nu^{\frac{5}{9}} J_0^{\frac{5}{9}}}{\beta_0^{\frac{14}{9}}} \right)^{\frac{1}{9}} \left[\frac{(t_1 - t_0) \nu^{\frac{94}{81}} J_0^{\frac{13}{81}}}{\beta_0^{\frac{4}{81}} \gamma_1^{\frac{1}{9}}} \right] \frac{1}{T} \tag{2.4.21}$$

$$T = \frac{\nu^{\frac{11}{9}} J_0^{\frac{2}{9}} (t - t_0)}{\beta_0^{\frac{2}{9}}} \tag{2.4.22}$$

and A, B, C are arbitrary constants.

2.5: Conclusion

In equation (2.4.19) we obtain the decay law of dusty fluid turbulence before the final period considering three and four point correlation equations after neglecting quintuple correlation terms. The equation (2.4.19) shows that turbulent energy decays more rapidly in an exponential manner than the energy decay for clean fluid. This decay law contains a term in $T^{-\frac{19}{2}}$, as well as the terms in $T^{-\frac{5}{2}}$ and T^{-7}. Thus the terms associated with the higher order correlations die out faster than those associated with the lower order ones. The factor $\frac{(t-t_1)}{(t-t_0)}$ occurring in the last term in equation (2.4.19) will cause that term to decay even faster, so long as $t_1 - t_0 > 0$.

If the fluid is clean, i.e., in absence of dust particles, we put f = 0, the equation (2.4.20) becomes

$$\langle u^2 \rangle = AT^{-\frac{5}{2}} + BT^{-7} + C\left(\frac{t-t_1}{t-t_0}\right)^{-\frac{19}{2}} T^{-\frac{19}{2}} \tag{2.5.1}$$

which is obtained earlier by Deissler [13].

Considering higher order correlations in the analysis, we may also generalize more terms in higher power of T will be added to equation (2.4.19).

CHAPTER-III

DECAY OF HOMOGENEOUS TURBULENCE BEFORE THE FINAL PERIOD IN A ROTATING SYSTEM FOR THE CASE OF THREE AND FOUR POINT CORRELATION EQUATIONS

3.1: Introduction

In geophysical flows, the system is usually rotating with a constant angular velocity. Such large-scale flows are generally turbulent. When the motion is referred to axes, which rotate steadily with the bulk of the fluid, the coriolis force and centrifugal force must be supposed to act on the fluid. The coriolis force due to rotation plays an important role in a rotating system of turbulent flow while the centrifugal force with the potential is incorporated into the pressure. Batchelor and Townsend [1] studied the decay of turbulence in the final period.

Deissler [12,13] developed a theory "Decay of homogeneous turbulence for times before the final period". Kishore and Dixit [27], Kishore and Singh [25], Kishore and Golsefied [29] analyzed the effect of coriolis force on acceleration covariance in ordinary and MHD turbulent flows. Shimomura and Yoshizawa [60], Shimomura [61] & [62] also discussed the statistical analysis of turbulent viscosity, turbulent scalar flux and turbulent shear flows respectively in a rotating system by two-scale direct interaction approach. Loeffler and Deissler [38] discussed the decay of temperature fluctuations in homogeneous turbulence. In their approach they considered the two and three point correlation equations and solved these equations after neglecting the fourth and higher order correlation terms. Kishore and Upadhyay [34] studied the decay of MHD turbulence in rotating system. Islam and Sarker [21] also studied the decay of dusty fluid turbulence before the final period in a rotating system using two and three point correlation equations. It can be shown that when the system is non- rotating, the result reduces to the same as obtained by Deissler [13].

Here, we shall derive an expression for the energy decay law of homogeneous turbulence before the final period in a rotating system using three and four point correlation equations and solved these equations after neglecting the quintuple correlations in comparison to the third and fourth order correlation terms. Finally the energy decay law of homogeneous turbulence in a rotating system before the final period is obtained.

3.2: Basic Equations

The equations of motion and continuity for turbulent flow of incompressible fluid in a rotating system are given below:

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_l)}{\partial x_l} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_l \partial x_l} - 2\varepsilon_{mli}\Omega_m u_i \tag{3.2.1}$$

and

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial v_i}{\partial x_i} = 0 \tag{3.2.2}$$

where the subscripts can take on the values 1,2 or 3.

Here u_i, turbulent velocity components; v_i , dust particle velocity components; ρ , fluid density; ν , kinematics viscosity; Ω_m , constant angular velocity components; ε_{mli} , alternating tensor; p, instantaneous pressure.

3.3: Correlation and Spectral Equations

The equations of motion of homogeneous turbulence in a rotating system for the points p, $p^{/}$ and $p^{//}$ separated by the vector $\underline{r}$ and $\underline{r}^{/}$ are

$$\frac{\partial u_i}{\partial t}+\frac{\partial(u_i u_l)}{\partial x_l}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_l \partial x_l}+2\varepsilon_{mli}\Omega_m u_i \tag{3.3.1}$$

$$\frac{\partial u_j^{/}}{\partial t}+\frac{\partial(u_j^{/} u_l^{/})}{\partial x_l^{/}}=-\frac{1}{\rho}\frac{\partial p^{/}}{\partial x_j^{/}}+\nu\frac{\partial^2 u_j^{/}}{\partial x_l^{/} \partial x_l^{/}}+2\varepsilon_{nlj}\Omega_n u_j^{/} \tag{3.3.2}$$

$$\frac{\partial u_k^{//}}{\partial t}+\frac{\partial(u_k^{//} u_l^{//})}{\partial x_l^{//}}=-\frac{1}{\rho}\frac{\partial p^{//}}{\partial x_k^{//}}+\nu\frac{\partial^2 u_k^{//}}{\partial x_l^{//} \partial x_l^{//}}+2\varepsilon_{qlk}\Omega_q u_k^{//} \tag{3.3.3}$$

Multiplying equation (3.3.1) by $u_j^{/}u_k^{//}$, (3.2.2) by $u_i u_k^{//}$, and (3.3.3) $u_i u_j^{/}$, adding the three equations and taking space or time averages, we obtain

$$\frac{\partial}{\partial t}\langle u_i u_j^{/} u_k^{//}\rangle+\frac{\partial}{\partial x_l}\langle u_i u_j^{/} u_k^{//} u_l\rangle+\frac{\partial}{\partial x_l^{/}}\langle u_i u_j^{/} u_k^{//} u_l^{/}\rangle+\frac{\partial}{\partial x_l^{//}}\langle u_i u_j^{/} u_k^{//} u_l^{//}\rangle$$

$$=-\frac{1}{\rho}\left(\frac{\partial}{\partial x_i}\langle p u_j^{/} u_k^{//}\rangle+\frac{\partial}{\partial x_j^{/}}\langle p^{/} u_i u_k^{//}\rangle+\frac{\partial}{\partial x_k^{//}}\langle p^{//} u_i u_j^{/}\rangle\right)$$

$$+\nu\left(\frac{\partial^2\langle u_i u_j^{/} u_k^{//}\rangle}{\partial x_l \partial x_l}+\frac{\partial^2\langle u_i u_j^{/} u_k^{//}\rangle}{\partial x_l^{/} \partial x_l^{/}}+\frac{\partial^2\langle u_i u_j^{/} u_k^{//}\rangle}{\partial x_l^{//} \partial x_l^{//}}\right)$$

$$-2\left(\varepsilon_{mli}\Omega_m\left\langle u_i u_j^{/} u_k^{//}\right\rangle+\varepsilon_{nlj}\Omega_n\left\langle u_i u_j^{/} u_k^{//}\right\rangle+\varepsilon_{qlk}\Omega_q\left\langle u_i u_j^{/} u_k^{//}\right\rangle\right) \tag{3.3.4}$$

Using the transformations $\frac{\partial}{\partial x_l^{/}}=\frac{\partial}{\partial r_l}$, $\frac{\partial}{\partial x_l^{//}}=\frac{\partial}{\partial r_l^{/}}$ and $\frac{\partial}{\partial x_l}=-\frac{\partial}{\partial r_l}-\frac{\partial}{\partial r_l^{/}}$

into equations (3.3.4) , we get

$$\frac{\partial}{\partial t}\langle u_i u_j' u_k''\rangle-\frac{\partial}{\partial r_l}\langle u_i u_j' u_k'' u_l\rangle-\frac{\partial}{\partial r_l'}\langle u_i u_j' u_k'' u_l\rangle+\frac{\partial}{\partial r_l}\langle u_i u_j' u_k'' u_l'\rangle+\frac{\partial}{\partial r_l'}\langle u_i u_j' u_k'' u_l''\rangle$$

$$=-\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}\langle p u_j' u_k''\rangle-\frac{\partial}{\partial r_i'}\langle p u_j' u_k''\rangle+\frac{\partial}{\partial r_j}\langle p' u_i u_k''\rangle+\frac{\partial}{\partial r_k'}\langle p'' u_i u_j'\rangle\right)$$

$$+2\nu\left(\frac{\partial\langle u_i u_j' u_k''\rangle}{\partial r_l \partial r_l}+\frac{\partial\langle u_i u_j' u_k''\rangle}{\partial r_l \partial r_l'}+\frac{\partial\langle u_i u_j' u_k''\rangle}{\partial r_l' \partial r_l'}\right)$$

$$-2\left(\varepsilon_{mli}\Omega_m\langle u_i u_j' u_k''\rangle+\varepsilon_{nlj}\Omega_n\langle u_i u_j' u_k''\rangle+\varepsilon_{qlk}\Omega_q\langle u_i u_j' u_k''\rangle\right) \tag{3.3.5}$$

In order to convert equation (3.3.5) to spectral form, we can define the following six dimensional Fourier transforms:

$$\langle u_i u_j'(\underline{r})u_k''(\underline{r}')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\beta_i\beta_j'(\underline{k})\beta_k''(\underline{k}')\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}'.\underline{r}')\right]d\underline{k}\,d\underline{k}' \tag{3.3.6}$$

$$\langle u_i u_l u_j'(\underline{r})u_k''(\underline{r}')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\beta_i\beta_l\beta_j'(\underline{k})\beta_k''(\underline{k}')\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}'.\underline{r}')\right]d\underline{k}\,d\underline{k}' \tag{3.3.7}$$

$$\langle p u_j'(\underline{r})u_k''(\underline{r}')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\alpha\beta_j'(\underline{k})\beta_k''(\underline{k}')\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}'.\underline{r}')\right]d\underline{k}\,d\underline{k}' \tag{3.3.8}$$

$$\langle v_i u_j'(\underline{r})u_k''(\underline{r}')\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\gamma_i\beta_j'(\underline{k})\beta_k''(\underline{k}')\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}'.\underline{r}')\right]d\underline{k}\,d\underline{k}' \tag{3.3.9}$$

Interchanging the subscripts i and j and then interchanging the points p and p' give

$$\langle u_i u_l'(\underline{r})u_j'(\underline{r})u_k''(\underline{r}')\rangle=\langle u_j u_l u_i'(-\underline{r})u_k''(\underline{r}'-\underline{r})\rangle$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\beta_j\beta_l\beta_i'(-\underline{k}-\underline{k}')\beta_k''(\underline{k}')\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}'.\underline{r}')\right]d\underline{k}\,d\underline{k}' \tag{3.3.10}$$

$$\langle u_i u_j'(\underline{r})u_k''(\underline{r}')u_l''(\underline{r}')\rangle=\langle u_k u_l u_i'(-\underline{r}')u_l''(\underline{r}-\underline{r}')\rangle$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle\beta_k\beta_l\beta_i'(-\underline{k}-\underline{k}')\beta_j''(\underline{k})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}'.\underline{r}')\right]d\underline{k}\,d\underline{k}' \tag{3.3.11}$$

where the points p and $p^{/}$ are interchanged to obtain equation (3.3.10). For equation (3.3.11), p is replaced by $p^{/}$, $p^{/}$ is replaced by $p^{//}$ and $p^{//}$ is replaced by p.

Similarly,

$$\langle u_i p^{/}(\underline{r}) u_k^{//}(\underline{r}^{/})\rangle = \langle p u_i^{/}(-\underline{r}) u_k^{//}(\underline{r}^{/}-\underline{r})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \alpha\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_k^{//}(\underline{k}^{/})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right] d\underline{k}\, d\underline{k}^{/} \qquad (3.3.12)$$

$$\langle u_i u_j^{/} p^{//}(\underline{r}^{/})\rangle = \langle p u_i^{/}(-\underline{r}^{/}) u_j^{//}(\underline{r}-\underline{r}^{/})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \alpha\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_j^{//}(\underline{k}\ )\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right] d\underline{k}\, d\underline{k}^{/} \qquad (3.3.13)$$

$$\langle u_i v_j^{/} u_k^{//}(\underline{r}^{/})\rangle = \langle v_j u_i^{/}(-\underline{r}) u_k^{//}(\underline{r}^{/}-\underline{r})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \gamma_j\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_k^{//}(\underline{k}^{/})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right] d\underline{k}\, d\underline{k}^{/} \qquad (3.3.14)$$

$$\langle u_i u_j^{/} v_k^{//}(\underline{r}^{/})\rangle = \langle v_k u_i^{/}(-\underline{r}^{/}) u_j^{//}(\underline{r}-\underline{r}^{/})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \gamma_k\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_j^{//}(\underline{k})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right] d\underline{k}\, d\underline{k}^{/} \qquad (3.3.15)$$

Substituting the preceding relations into equation (3.3.5), we get

$$\frac{d}{dt}\left\langle \beta_i\beta_j^{/}\beta_k^{//}\right\rangle + 2\nu\left(k^2 + k_l k_l^{/} + k^{/2}\right)\left\langle \beta_i\beta_j^{/}\beta_k^{//}\right\rangle$$

$$= \left[i\left(k_l + k_l^{/}\right)\left\langle \beta_i\beta_l\beta_j^{/}\beta_k^{//}\right\rangle - ik_l\left\langle \beta_j\beta_l\,\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_k^{//}(\underline{k}^{/})\right\rangle - ik_l^{/}\left\langle \beta_k\beta_l\,\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_j^{//}(\underline{k})\right\rangle\right]$$

$$-\frac{1}{\rho}\left[-i(k_i + k_i^{/})\left\langle \alpha\beta_j^{/}\beta_k^{//}\right\rangle + ik_j\left\langle \alpha\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_k^{//}(\underline{k}^{/})\right\rangle + ik_k^{/}\left\langle \alpha\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_j^{//}(\underline{k})\right\rangle\right]$$

$$-2\left[\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q\right]\left\langle \beta_i\beta_j^{/}\beta_k^{//}\right\rangle \qquad (3.3.16)$$

The tensor equation (3.316) can be converted to a scalar form by contraction of the indices i and j and inner multiplication k_k;

$$\frac{d}{dt}\left(k_k\left\langle \beta_i\beta_i^{\prime}\beta_k^{\prime\prime}\right\rangle + 2\nu\left(k^2 + k_l k_l^{\prime} + k^{\prime 2}\right)k_k\left\langle \beta_i\beta_i^{\prime}\beta_k^{\prime\prime}\right\rangle\right)$$
$$=\left[ik_k\left(k_l + k_l^{\prime}\right)\left\langle \beta_i\,\beta_l\,\beta_i^{\prime}\beta_k^{\prime\prime}\right\rangle - ik_k k_l\left\langle \beta_i\,\beta_l\,\beta_i^{\prime}(-\underline{k}-\underline{k}^{\prime})\beta_k^{\prime\prime}(\underline{k}^{\prime})\right\rangle - ik_k k_l^{\prime}\left\langle \beta_k\beta_l\,\beta_i^{\prime}(-\underline{k}-\underline{k}^{\prime})\beta_i^{\prime\prime}(\underline{k})\right\rangle\right]$$
$$-\frac{1}{\rho}\left[-ik_k(k_i + k_i^{\prime})\left\langle \alpha\beta_i^{\prime}\beta_k^{\prime\prime}\right\rangle + ik_k k_i\left\langle \alpha\beta_i^{\prime}(-\underline{k}-\underline{k}^{\prime})\beta_k^{\prime\prime}(\underline{k}^{\prime})\right\rangle + ik_k k_k^{\prime}\left\langle \alpha\beta_i^{\prime}(-\underline{k}-\underline{k}^{\prime})\beta_i^{\prime\prime}(\underline{k})\right\rangle\right]$$
$$-2k_k\left[\varepsilon_{mli}\Omega_m + \varepsilon_{nli}\Omega_n + \varepsilon_{qlk}\Omega_q\right]\left\langle \beta_i\beta_i^{\prime}\beta_k^{\prime\prime}\right\rangle \tag{3.3.17}$$

In order to obtain the four-point equation, we consider the equation of motion of turbulence in rotating system at the points $p, p^{\prime}, p^{\prime\prime}$ and $p^{\prime\prime\prime}$ as

$$\frac{\partial u_i}{\partial t} + \frac{\partial(u_i u_m)}{\partial x_m} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_m \partial x_m} - 2\varepsilon_{nmi}\Omega_n u_i \tag{3.3.18}$$

$$\frac{\partial u_j^{\prime}}{\partial t} + \frac{\partial(u_j^{\prime} u_m^{\prime})}{\partial x_m^{\prime}} = -\frac{1}{\rho}\frac{\partial p^{\prime}}{\partial x_j^{\prime}} + \nu\frac{\partial^2 u_j^{\prime}}{\partial x_m^{\prime} \partial x_m^{\prime}} - 2\varepsilon_{pmj}\Omega_p u_j^{\prime} \tag{3.319}$$

$$\frac{\partial u_k^{\prime\prime}}{\partial t} + \frac{\partial(u_k^{\prime\prime} u_m^{\prime\prime})}{\partial x_m^{\prime\prime}} = -\frac{1}{\rho}\frac{\partial p^{\prime\prime}}{\partial x_k^{\prime\prime}} + \nu\frac{\partial^2 u_k^{\prime\prime}}{\partial x_m^{\prime\prime} \partial x_m^{\prime\prime}} - 2\varepsilon_{qmk}\Omega_q u_k^{\prime\prime} \tag{3.3.20}$$

$$\frac{\partial u_l^{\prime\prime\prime}}{\partial t} + \frac{\partial(u_l^{\prime\prime\prime} u_m^{\prime\prime\prime})}{\partial x_m^{\prime\prime\prime}} = -\frac{1}{\rho}\frac{\partial p^{\prime\prime\prime}}{\partial x_l^{\prime\prime\prime}} + \nu\frac{\partial^2 u_l^{\prime\prime\prime}}{\partial x_m^{\prime\prime\prime} \partial x_m^{\prime\prime\prime}} - 2\varepsilon_{rml}\Omega_r u_l^{\prime\prime\prime} \tag{3.3.21}$$

where the repeated subscript in a term indicates a summation.

Multiplying the first equation by $u_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}$, the second by $u_i u_k^{\prime\prime}u_l^{\prime\prime\prime}$, the third by $u_i u_j^{\prime}u_l^{\prime\prime\prime}$ and the fourth by $u_i u_j^{\prime}u_k^{\prime\prime}$ respectively, then adding and taking space or time averages, we get

$$\frac{\partial}{\partial t}\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle + \frac{\partial}{\partial x_m}\left\langle u_i u_j^{/} u_k^{//} u_l^{///} u_m\right\rangle + \frac{\partial}{\partial x_m^{/}}\left\langle u_i u_j^{/} u_k^{//} u_l^{///} u_m^{/}\right\rangle + \frac{\partial}{\partial x_m^{//}}\left\langle u_i u_j^{/} u_k^{//} u_l^{///} u_m^{//}\right\rangle + \frac{\partial}{\partial x_m^{///}}\left\langle u_i u_j^{/} u_k^{//} u_l^{///} u_m^{///}\right\rangle$$

$$= -\tfrac{1}{\rho}\left(\frac{\partial}{\partial x_i}\left\langle p u_j^{/} u_k^{//} u_l^{///}\right\rangle + \frac{\partial}{\partial x_j^{/}}\left\langle p^{/} u_i u_k^{//} u_l^{///}\right\rangle + \frac{\partial}{\partial x_k^{//}}\left\langle p^{//} u_i u_j^{/} u_l^{///}\right\rangle + \frac{\partial}{\partial x_l^{///}}\left\langle p^{///} u_i u_j^{/} u_k^{//}\right\rangle\right)$$

$$+\nu\left(\frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial x_m \partial x_m} + \frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial x_m^{/} \partial x_m^{/}} + \frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial x_m^{//} \partial x_m^{//}} + \frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial x_m^{///} \partial x_m^{///}}\right)$$

$$-2\left(\varepsilon_{nmi}\Omega_n\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle + \varepsilon_{pmj}\Omega_p\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle + \varepsilon_{qmk}\Omega_q\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle + \varepsilon_{rml}\Omega_r\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle\right) \qquad (3.3.22)$$

Equation (3.3.22) can be written in terms of the independent variables $r, r^{/} and\ r^{//}$ as

$$\frac{\partial}{\partial t}\left\langle u\ u_j^{/} u_k^{//} u_l^{///}\right\rangle - \frac{\partial}{\partial r_m}\left\langle u_i u_m u_j^{/} u_k^{//} u_l^{///}\right\rangle - \frac{\partial}{\partial r_m^{/}}\left\langle u_i u_m u_j^{/} u_k^{//} u_l^{///}\right\rangle - \frac{\partial}{\partial r_m^{//}}\left\langle u_i u_m u_j^{/} u_k^{//} u_l^{///}\right\rangle$$

$$+\frac{\partial}{\partial r_m}\left\langle u_i u_j^{/} u_m^{/} u_k^{//} u_l^{///}\right\rangle + \frac{\partial}{\partial r_m^{/}}\left\langle u_i u_j^{/} u_k^{//} u_m^{//} u_l^{///}\right\rangle + \frac{\partial}{\partial r_m^{//}}\left\langle u_i u_j^{/} u_k^{//} u_l^{///} u_m^{///}\right\rangle$$

$$= -\tfrac{1}{\rho}\left(-\frac{\partial}{\partial r_i}\left\langle p u_j^{/} u_k^{//} u_l^{///}\right\rangle - \frac{\partial}{\partial r_i^{/}}\left\langle p u_j^{/} u_k^{//} u_l^{///}\right\rangle - \frac{\partial}{\partial r_i^{//}}\left\langle p u_j^{/} u_k^{//} u_l^{///}\right\rangle\right.$$

$$\left.+\frac{\partial}{\partial r_j}\left\langle u_i p^{/} u_k^{//} u_l^{///}\right\rangle + \frac{\partial}{\partial r_k^{/}}\left\langle u_i u_j^{/} p^{//} u_l^{///}\right\rangle + \frac{\partial}{\partial r_l^{//}}\left\langle u_i u_j^{/} u_k^{//} p^{///}\right\rangle\right)$$

$$+2\nu\left(\frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial r_m \partial r_m} + \frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial r_m \partial r_m^{/}} + \frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial r_m \partial r_m^{//}}\right.$$

$$\left.+\frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial r_m^{/} \partial r_m^{/}} + \frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial r_m^{/} \partial r_m^{//}} + \frac{\partial^2\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle}{\partial r_m^{//} \partial r_m^{//}}\right)$$

$$-2\left(\varepsilon_{nmi}\Omega_n\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle + \varepsilon_{pmj}\Omega_p\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle + \varepsilon_{qmk}\Omega_q\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle + \varepsilon_{rml}\Omega_r\left\langle u_i u_j^{/} u_k^{//} u_l^{///}\right\rangle\right)$$

(3.3.23)

where the following transformations were used:

$$\frac{\partial}{\partial x_m^{/}} = \frac{\partial}{\partial r_m},\ \frac{\partial}{\partial x_m^{//}} = \frac{\partial}{\partial r_m^{/}},\ \frac{\partial}{\partial x_m^{///}} = \frac{\partial}{\partial r_m^{//}} \text{ and } \frac{\partial}{\partial x_m} = -\frac{\partial}{\partial r_m} - \frac{\partial}{\partial r_m^{/}} - \frac{\partial}{\partial r_m^{//}}$$

In order to convert equation (3.3.23) to spectral form, we define the following nine-dimensional Fourier transforms:

$$\left\langle u_i u_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'') \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{ijkl}(\underline{k}, \underline{k}', \underline{k}'') \exp[\, i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \quad (3.3.24)$$

$$\left\langle u_i u_m u_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'') \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{imjkl}(\underline{k}, \underline{k}', \underline{k}'') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \quad (3.3.25)$$

$$\left\langle p u_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'') \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_{jkl}(\underline{k}, \underline{k}', \underline{k}'') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \quad (3.3.26)$$

$$\left\langle v_i u_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'') \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_i \delta_{jkl}(\underline{k}, \underline{k}', \underline{k}'') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \quad (3.3.27)$$

Similarly,

$$\left\langle u_i v_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'') \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_j \delta_{ikl}(-\underline{k} - \underline{k}' - \underline{k}'', \underline{k}', \underline{k}'') \exp[(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \quad (3.3.28)$$

$$\left\langle u_i u_j'(\underline{r}) v_k''(\underline{r}') u_l'''(\underline{r}'') \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_k \delta_{ijl}(-\underline{k} - \underline{k}' - \underline{k}'', \underline{k}, \underline{k}'') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \quad (3.3.29)$$

$$\left\langle u_i u_j'(\underline{r}) u_k''(\underline{r}') v_l'''(\underline{r}'') \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_l \delta_{ijk}(-\underline{k} - \underline{k}' - \underline{k}'', \underline{k}, \underline{k}') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \quad (3.3.30)$$

Substituting the preceding relations into equation (3.3.23), we get

$$
\begin{aligned}
&\frac{d}{dt}(\gamma_{ijkl}) + 2\nu\left(k^2 + k_m k_m' + k_m k_m'' + k'^2 + k_m' k_m'' + k''^2\right)\gamma_{ijkl} \\
&= \left[i\left(k_m + k_m' + k_m''\right)\gamma_{imjkl}\left(\underline{k}, \underline{k}', \underline{k}''\right) - ik_m \gamma_{jmikl}\left(-\underline{k} - \underline{k}' - \underline{k}'', \underline{k}', \underline{k}''\right)\right. \\
&\left. - ik_m' \gamma_{kmijl}\left(-\underline{k} - \underline{k}' - \underline{k}'', \underline{k}, \underline{k}''\right) - ik_m'' \gamma_{lmijk}\left(-\underline{k} - \underline{k}' - \underline{k}', \underline{k}, \underline{k}'\right)\right] \\
&- \frac{1}{\rho}\left[-i\left(k_i + k_i' + k_i''\right)\delta_{jkl}\left(\underline{k}, \underline{k}', \underline{k}''\right) + ik_j \delta_{ikl}\left(-\underline{k} - \underline{k}' - \underline{k}'', \underline{k}', \underline{k}''\right)\right. \\
&\left. + ik_k' \delta_{ijl}\left(-\underline{k} - \underline{k}' - \underline{k}'', \underline{k}, \underline{k}''\right) + ik_l'' \delta_{ijk}\left(-\underline{k} - \underline{k}' - \underline{k}'', \underline{k}, \underline{k}'\right)\right] \\
&- 2\left(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r\right)\gamma_{ijkl}
\end{aligned}
\quad (3.3.31)
$$

To obtain a relation between the terms on the right hand side of equation (3.3.31) derived from the quadruple correlation terms, pressure terms and rotational terms in equation (3.3.23), take the divergence of the equation of motion and combine with the continuity equation to give

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_m \partial x_m} = -\frac{\partial^2 (u_m u_n)}{\partial x_m \partial x_n} \tag{3.3.32}$$

Multiplying the equation (3.3.32) by $u'_j u''_k u'''_l$, taking ensemble average and writing the resulting equation in terms of the independent variables r and r', gives

$$\begin{aligned}
&\frac{1}{\rho}\left(\frac{\partial^2\langle pu'_j u''_k u'''_l\rangle}{\partial r_m \partial r_m} + 2\frac{\partial^2\langle pu'_j u''_k u'''_l\rangle}{\partial r_m \partial r'_m} + 2\frac{\partial^2\langle pu'_j u''_k u'''_l\rangle}{\partial r_m \partial r''_m}\right.\\
&\left. + \frac{\partial^2\langle pu'_j u''_k u'''_l\rangle}{\partial r'_m \partial r'_m} + 2\frac{\partial^2\langle pu'_j u''_k u'''_l\rangle}{\partial r'_m \partial r''_m} + \frac{\partial^2\langle pu'_j u''_k u'''_l\rangle}{\partial r''_m \partial r''_m}\right)\\
&= -\left(\frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r_m \partial r_n} + \frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r_m \partial r'_n} + \frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r_m \partial r''_n}\right.\\
&+ \frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r'_m \partial r_n} + \frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r'_m \partial r'_n} + \frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r'_m \partial r''_n}\\
&\left.+ \frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r''_m \partial r_n} + \frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r''_m \partial r'_n} + \frac{\partial^2\langle u_m u_n u'_j u''_k u'''_l\rangle}{\partial r''_m \partial r''_n}\right)
\end{aligned} \tag{3.3.33}$$

The Fourier transform of equation (3.3.33) is

$$\begin{aligned}
&-\frac{1}{\rho}\left(k^2 + 2k_m k'_m + 2k_m k''_m + k'^2 + 2k'_m k''_m + k''^2\right)\delta_{jkl}\\
&= \left(k_m k_n + k_m k'_n + k_m k''_n + k'_m k_n + k'_m k'_n + k'_m k''_n + k''_m k_n + k''_m k'_n + k''_m k''_n\right)\gamma_{mnjkl}
\end{aligned}$$

$$-\frac{1}{\rho}\delta_{jkl} = \frac{\left(k_m k_n + k_m k'_n + k_m k''_n + k'_m k_n + k'_m k'_n + k'_m k''_n + k''_m k_n + k''_m k'_n + k''_m k''_n\right)}{\left(k^2 + 2k_m k'_m + 2k_m k''_m + k'^2 + 2k'_m k''_m + k''^2\right)}\gamma_{mnjkl} \tag{3.3.34}$$

Equations (3.3.31) and (3.3.34) are the spectral equations corresponding to the four point correlation equations. The spectral equations corresponding to the three point correlation equations are

$$\frac{d}{dt}(k_k\beta_{iik})+2\nu(k^2+k_lk_l^{\prime}+k^{\prime 2})k_k\beta_{iik}$$
$$=ik_k(k_l+k_l^{\prime})\beta_{ilik}(k,k^{\prime})-ik_kk_l\beta_{ilik}(-k-k^{\prime},k^{\prime})-ik_kk_l^{\prime}\beta_{klii}(-k-k^{\prime},k\ )$$
$$-\frac{1}{\rho}\left[-ik_k(k_i+k_i^{\prime})\alpha_{ik}(k,k^{\prime})+ik_kk_i\alpha_{ik}(-k-k^{\prime},k^{\prime})+ik_kk_k^{\prime}\alpha_{ii}(-k-k^{\prime},k)\right]$$
$$-2k_k\left[\varepsilon_{mli}\Omega_m+\varepsilon_{nli}\Omega_n+\varepsilon_{qlk}\Omega_q\right]\beta_i\beta_i^{\prime}\beta_k^{\prime\prime} \tag{3.3.35}$$

and
$$-\frac{1}{\rho}\alpha_{ik}=\frac{k_lk_m+k_l^{\prime}k_m+k_lk_m^{\prime}+k_l^{\prime}k_m^{\prime}}{k^2+2k_lk_l^{\prime}+k^{\prime 2}}\beta_{lmik} \tag{3.3.36}$$

Here the spectral tensors are defined by

$$\left\langle u_iu_j^{\prime}(\underline{r})u_k^{\prime\prime}(\underline{r}^{\prime})\right\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta_{ijk}(\underline{k},\underline{k}^{\prime})\exp[\,i(\underline{k}.\underline{r}+\underline{k}^{\prime}.\underline{r}^{\prime})]d\,\underline{k}\,d\,\underline{k}^{\prime} \tag{3.3.37}$$

$$\left\langle u_iu_lu_j^{\prime}(\underline{r})u_k^{\prime\prime}(\underline{r}^{\prime})\right\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta_{iljk}(\underline{k},\underline{k}^{\prime})\exp[\,i(\underline{k}.\underline{r}+\underline{k}^{\prime}.\underline{r}^{\prime})]d\,\underline{k}\,d\,\underline{k}^{\prime} \tag{3.3.38}$$

$$\left\langle pu_j^{\prime}(\underline{r})u_k^{\prime\prime}(\underline{r}^{\prime})\right\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_{jk}(\underline{k},\underline{k}^{\prime})\exp\left[i\left(\underline{k}.\underline{r}+\underline{k}^{\prime}.\underline{r}^{\prime}\right)\right]d\,\underline{k}\,d\,\underline{k}^{\prime} \tag{3.3.39}$$

A relation between β_{iljk} and γ_{ijkl} can be obtained by letting $\underline{r}^{\prime\prime}=0$ in equation (3.3.24) and comparing the result with equation (3.3.38)

$$\beta_{iljk}(\underline{k},\underline{k}^{\prime})=\int_{-\infty}^{\infty}\gamma_{ijkl}(\underline{k},\underline{k}^{\prime},\underline{k}^{\prime\prime})d\,\underline{k}^{\prime\prime} \tag{3.3.40}$$

The spectral equation corresponding to the two-point correlation equation in a rotating system is

$$\frac{d}{dt}\phi_{i,i}+(2\nu k^2+2\varepsilon_{mki}\Omega_m+2\varepsilon_{nki}\Omega_n)\phi_{i,i}=ik_k\phi_{iki}(\underline{k})-ik_k\phi_{iki}(-\underline{k}) \tag{3.3.41}$$

where $\phi_{i,i}$ and ϕ_{iki} are defined by

$$\left\langle u_i u_j^{\prime}(\underline{r})\right\rangle=\int_{-\infty}^{\infty}\phi_{ij}(\underline{k})\exp(i\underline{k}\cdot\underline{r})d\underline{k} \tag{3.3.42}$$

and
$$\left\langle u_i u_k u_j^{\prime}(\underline{r})\right\rangle=\int_{-\infty}^{\infty}\phi_{ikj}(\underline{k})\exp(i\underline{k}\cdot\underline{r})d\underline{k} \tag{3.3.43}$$

The relation between ϕ_{ikj} and β_{ijk} obtained by letting $\underline{r}^{\prime}=0$ in equation (3.3.37) and comparing the result with equation (3.3.43) is

$$\phi_{ikj}(\underline{k})=\int_{-\infty}^{\infty}\beta_{ijk}(\underline{k},\underline{k}^{\prime})d\underline{k}^{\prime} \tag{3.3.44}$$

3.4: Solution Neglecting Quintuple Correlations

Equation (3.3.34) shows that if the terms corresponding to the quintuple correlations are neglected, then the pressure force terms also must be neglected. Thus neglecting first and second terms on the right side of equation (3.3.31), the equation can be integrated between t_1 and t to give

$$\begin{aligned}\gamma_{ijkl}=(\gamma_{ijkl})_1\exp\big[\{&-2\nu(k^2+k_m k_m^{\prime}+k_m k_m^{\prime\prime}+k^{\prime 2}+k_m^{\prime}k_m^{\prime\prime}+k^{\prime\prime 2})\\&+2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)\}(t-t_1)\big]\end{aligned} \tag{3.4.1}$$

where $(\gamma_{ijkl})_1$ is the value of γ_{ijkl} at $t=t_1$. The quantity $(\gamma_{ijkl})_1$ can be considered also as the value of γ_{ijkl} at small values of k, $k^{\prime}$ and $k^{\prime\prime}$, at least for times when the quintuple correlations are negligible.

Equations (3.3.40) and (3.4.1) can be converted to scalar form by contracting the indices i and j, as well as k and l. Substitution of equations (3.3.36), (3.3.40) and (3.4.1) into the three point scalar equation (3.3.35) results in

$$
\begin{aligned}
&\frac{d}{dt}\left(k_k \beta_{iik}\right) + 2\nu\left(k^2 + k_l k_l^{/} + k^{/2}\right) k_k \beta_{iik} \\
&= i\left[ik_k\left(k_l + k_l^{/}\right)\gamma_{iikl}\left(\underline{k}, \underline{k}^{/}, \underline{k}^{//}\right) - ik_k\left(k_i + k_i^{/}\right)\left(\frac{k_l k_m + k_l^{/} k_m + k_l k_m^{/} + k_l^{/} k_m^{/}}{k^2 + 2k_l k_l^{/} + k^{/2}}\right)\gamma_{likm}\left(\underline{k}, \underline{k}^{/}, \underline{k}^{//}\right)\right]_1 \\
&\times \int_{-\infty}^{\infty} \exp\left[\left\{ \begin{matrix} -2\nu\left(k^2 + k_m k_m^{/} + k_m k_m^{//} + k^{/2} + k_m^{/} k_m^{//} + k^{//2}\right) \\ +2\left(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r\right) \end{matrix} \right\}(t - t_1)\right] d\underline{k}^{//} \\
&+ \left[-ik_k k_l \gamma_{iikl}\left(-\underline{k} - \underline{k}^{/}, \underline{k}^{/}, \underline{k}^{//}\right) + \left(\frac{ik_k k_i k_l k_m}{k^2}\right)\gamma_{likm}\left(-\underline{k} - \underline{k}^{/}, \underline{k}^{/}, \underline{k}^{//}\right)\right]_1 \\
&\times \int_{-\infty}^{\infty} \exp\left[\left\{ \begin{matrix} -2\nu(k^2 + k_m k_m^{/} + k^{/2} - k_m^{/} k_m^{//} + k^{//2}) \\ +2\left(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r\right) \end{matrix} \right\}(t - t_1)\right] d\underline{k}^{//} \\
&+ \left[-ik_k k_l^{/} \gamma_{kiil}\left(-\underline{k} - \underline{k}^{/}, \underline{k}, \underline{k}^{//}\right) + \left(\frac{ik_k k_k^{/} k_l^{/} k_m^{/}}{k^{/2}}\right)\gamma_{liim}\left(-\underline{k} - \underline{k}^{/}, \underline{k}, \underline{k}^{//}\right)\right]_1 \\
&\times \int_{-\infty}^{\infty} \exp\left[\left\{ \begin{matrix} -2\nu\left(k^2 + k_m k_m^{/} + k^{/2} - k_m^{/} k_m^{//} + k^{//2}\right) \\ +2\left(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r\right) \end{matrix} \right\}(t - t_1)\right] d\underline{k}^{//} \\
&- 2k_k\left(\varepsilon_{mli}\Omega_m + \varepsilon_{nli}\Omega_n + \varepsilon_{qlk}\Omega_q\right)\beta_{iik}
\end{aligned} \tag{3.4.2}
$$

where the γ 's have been assumed independent of $k^{//}$ at t.

Substituting $d\underline{k}^{//} = dk_1^{//} dk_2^{//} dk_3^{//}$ in equation (3.4.2) and carrying out the integration with respect to $k_1^{//}, k_2^{//}$ and $k_3^{//}$, we get

$$\frac{d}{dt}(k_k\beta_{iik})+2\nu\left(k^2+k_lk_l'+k'^2\right)k_k\beta_{iik}+2k_k\left(\varepsilon_{mlr}\Omega_m+\varepsilon_{nlr}\Omega_n+\varepsilon_{qlk}\Omega_q\right)\beta_{iik}$$

$$=\frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[a]_1}{(t-t_1)^{\frac{3}{2}}}\exp\left[\left\{-2\nu\left(\frac{3}{4}k^2+\frac{1}{2}k_lk_l'+\frac{3}{4}k'^2\right)+2\left(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r\right)\right\}(t-t_1)\right]$$

$$+\frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[b]_1}{(t-t_1)^{\frac{3}{2}}}\exp\left[\left\{-2\nu\left(\frac{3}{4}k^2+k_lk_l'+k'^2\right)+2\left(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r\right)\right\}(t-t_1)\right]$$

$$+\frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[c]_1}{(t-t_1)^{\frac{3}{2}}}\exp\left[\left\{-2\nu\left(k^2+k_lk_l'+\frac{3}{4}k'^2\right)+2\left(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r\right)\right\}(t-t_1)\right] \qquad (3.4.3)$$

where the bracketed quantities in equation (3.4.2) have been abbreviated as shown.

Integration of equation (3.4.3) with respect to time, results in

$$k_k\beta_{iik}=(k_k\beta_{iik})_0\exp\left[\left\{-2\nu\left(k^2+k_lk_l'+k'^2\right)+2\left(\varepsilon_{mlr}\Omega_m+\varepsilon_{nlr}\Omega_n+\varepsilon_{qlk}\Omega_q\right)\right\}(t-t_0)\right]$$

$$+\frac{\pi^{\frac{3}{2}}[a]_1}{\nu}\left\{-\omega^{-1}\exp\left[-\omega^2\left(\frac{3}{4}k^2+\frac{1}{2}k_lk_l'+\frac{3}{4}k'^2\right)+2\left(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r\right)(t-t_1)\right]\right.$$

$$+2\left(\frac{1}{4}k^2+\frac{1}{2}k_lk_l'+\frac{1}{4}k'^2\right)^{\frac{1}{2}}\exp\left[-\omega^2\left(k^2+k_lk_l'+k'^2\right)+2\left(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r\right)(t-t_1)\right]$$

$$\left.\times\int_0^{\omega\left(\frac{1}{4}k^2+\frac{1}{2}k_lk_l'+\frac{1}{2}k'^2\right)^{\frac{1}{2}}}\exp(x^2)dx\right\}$$

$$+\frac{\pi^{\frac{3}{2}}[b]_1}{\nu}\left\{-\omega^{-1}\exp\left[-\omega^2\left(\frac{3}{4}k^2+k_lk_l'+k'^2\right)+2\left(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r\right)(t-t_1)\right]\right.$$

$$\left.+k\exp\left[-\omega^2\left(k^2+k_lk_l'+k'^2\right)+2\left(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r\right)(t-t_1)\right]\int_0^{\frac{1}{2}\omega k}\exp(x^2)dx\right\}$$

$$+\frac{\pi^{\frac{3}{2}}[c]_1}{\nu}\left\{-\omega^{-1}\exp\left[-\omega^2\left(k^2+k_lk_l'+\frac{3}{4}k'^2\right)+2\left(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r\right)(t-t_1)\right]\right.$$

$$\left.+k'\exp\left[-\omega^2\left(k^2+k_lk_l'+k'^2\right)+(\varepsilon_{nmr}\Omega_n+\varepsilon_{pmr}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rmr}\Omega_r)(t-t_1)\right]\int_0^{\frac{1}{2}\omega k'}\exp(x^2)dx\right\} \qquad (3.4.4)$$

where $\omega = \left[2\nu(t - t_1) \right]^{\frac{1}{2}} k$

In order to simplify the calculations, we shall assume that $[a]_1 = 0$; that is, we assume that a function sufficiently general to represent the initial conditions can be obtained by considering only the terms involving $[b]_1$ and $[c]_1$.

The substitution of equations (3.3.44) and (3.4.4) in equation (3.3.41) and setting $E = 2\pi k^2 \phi_{i,i}$ results in

$$\frac{dE}{dt} + (2\nu k^2 + 2\varepsilon_{mki}\Omega_m + 2\varepsilon_{nki}\Omega_n)E = W \tag{3.4.5}$$

where

$$\begin{aligned}
W = {} & k^2 \int_{-\infty}^{\infty} 2\pi i \left[k_k \beta_{iik}(\underline{k}, \underline{k}') - k_k \beta_{iik}(-\underline{k}, -\underline{k}') \right]_0 \exp\left[\left\{ -2\nu(k^2 + k_l k_l' + k'^2) + 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nli}\Omega_n + \varepsilon_{qlk}\Omega_q) \right\}(t - t_0) \right] d\underline{k}' \\
& + k^2 \overset{5}{\int_{-\infty}^{\infty}} \frac{2\pi^2 i}{\nu} \left[b(\underline{k}, \underline{k}') - b(-\underline{k}, -\underline{k}') \right]_1 \left\{ -\omega^{-1} \exp\left[-\omega^2 \left(\frac{3}{4} k^2 + k_l k_l' + k'^2 \right) + 2(\varepsilon_{nml}\Omega_n + \varepsilon_{pml}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t - t_1) \right] \right. \\
& \left. + k \exp\left[-\omega^2 (k^2 + k_l k_l' + k'^2) + 2(\varepsilon_{nml}\Omega_n + \varepsilon_{pml}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t - t_1) \right] \int_0^{\frac{1}{2}\omega k} \exp(x^2)\, dx \right\} d\underline{k}' \\
& + k^2 \overset{5}{\int_{-\infty}^{\infty}} \frac{2\pi^2 i}{\nu} \left[c(\underline{k}, \underline{k}') - c(-\underline{k}, -\underline{k}') \right]_1 \left\{ -\omega^{-1} \exp\left[-\omega^2 \left(k^2 + k_l k_l' + \frac{3}{4} k'^2 \right) + 2(\varepsilon_{nml}\Omega_n + \varepsilon_{pml}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t - t_1) \right] \right. \\
& \left. + k' \exp\left[-\omega^2 (k^2 + k_l k_l' + k'^2) + 2(\varepsilon_{nml}\Omega_n + \varepsilon_{pml}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t - t_1) \right] \int_0^{\frac{1}{2}\omega k'} \exp(x^2)\, dx \right\} d\underline{k}'
\end{aligned} \tag{3.4.6}$$

The quantity E is the energy spectrum function, which represents contributions from various wave numbers or eddy sizes to the total energy. W is the energy transfer function, which is responsible for the transfer of energy between wave numbers.

In order to find the solution completely and following Deissler [13], we assume that

$$(2\pi)^2 i \left[k_k \beta_{iik}(\underline{k}, \underline{k}') - k_k \beta_{iik}(-\underline{k}, -\underline{k}') \right]_0 = -\beta_0 (k^4 k'^6 - k^6 k'^4) \tag{3.4.7}$$

For the bracketed quantities in equation (3.4.6), we let

$$\frac{4\pi^2}{\nu} i\left[b(\underline{k},\underline{k}') - b(-\underline{k},-\underline{k}')\right]_1 = \frac{4\pi^2}{\nu} i\left[c(\underline{k},\underline{k}') - c(-\underline{k},-\underline{k}')\right]_1 = -2\gamma_1 (k^6 k'^8 - k^8 k'^6) \qquad (3.4.8)$$

where the bracketed quantities are set equal in order to make the integrands in equation (3.4.6) antisymmetric with respect to k and k'.

By substituting equations (3.4.7) and (3.4.8) in equation (3.4.6) remembering that

$d\underline{k}' = 2\pi k'^2 d(\cos\theta)dk'$ and $k_l k_l' = kk' \cos\theta$, ($\theta$ is the angle between $\underline{k}$ and $\underline{k}'$), and

carrying out the integration with respect to θ, we get

$$W = \int_0^\infty \Bigg[\frac{\beta_0 (k^4 k'^6 - k^6 k'^4) kk'}{2\nu(t-t_0)} .\{\exp[\{-2\nu(k^2 + kk' + k'^2) + 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q)\}(t-t_0)]$$

$$-\exp[\{-2\nu(k^2 - kk' + k'^2) + 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q)\}(t-t_0)]\} - \gamma_1 \frac{(k^6 k'^8 - k^8 k'^6)kk'}{\nu(t-t_1)}$$

$$\times \Bigg((\omega^{-1} \exp[-\omega^2(\frac{3}{4}k^2 + kk' + k'^2) + 2(\varepsilon_{nmi}\Omega_m + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$-\omega^{-1} \exp[-\omega^2(\frac{3}{4}k^2 - kk' + k'^2) + 2(\varepsilon_{nmi}\Omega_m + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$+\omega^{-1} \exp[-\omega^2(k^2 + kk' + \frac{3}{4}k'^2) + 2(\varepsilon_{nmi}\Omega_m + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$-\omega^{-1} \exp[-\omega^2(k^2 - kk' + \frac{3}{4}k'^2) + 2(\varepsilon_{nmi}\Omega_m + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$+\{k \exp[-\omega^2(k^2 - kk' + k'^2) + 2(\varepsilon_{nmi}\Omega_m + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$-k \exp[-\omega^2(k^2 + kk' + k'^2) + 2(\varepsilon_{nmi}\Omega_m + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]\} \int_0^{\frac{1}{2}\omega k} \exp(x^2)dx$$

$$+\{k' \exp[-w^2(k^2 - kk' + k'^2) + 2(\varepsilon_{nmi}\Omega_m + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$-k' \exp[-\omega^2(k^2 + kk' + k'^2) + 2(\varepsilon_{nmi}\Omega_m + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]\}. \int_0^{\frac{1}{2}\omega k'} \exp(x^2)dx \Bigg) \Bigg] d\underline{k}'$$

$$\ldots\ldots\ldots\ldots\ldots(3.4.9)$$

where $\quad \omega = \left[2\nu(t - t_1)\right]^{\frac{1}{2}}$

The integrand in this equation represents the contribution to the energy transfer at a wave number k, from a wave number k'. The integral is the total contribution to W at k, from all wave numbers. Carrying out the indicated integration with respect to k' in equation (3.4.9), where results in

$$W = W_\beta + W\gamma \tag{3.4.10}$$

where

$$W_\beta = -\frac{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\beta_0}{256\nu^{\frac{15}{2}}(t - t_0)^{\frac{15}{2}}}$$

$$\times \exp\left[(-\frac{3}{2}\varepsilon^2)(105\varepsilon^6 + 45\varepsilon^8 - 19\varepsilon^{10} - 3\varepsilon^{12}) + 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q)(t - t_0)\right] \tag{3.4.11}$$

and

$$W_\gamma = -\frac{\gamma_1}{\nu^{10}(t-t_1)^{10}}\left[\frac{\pi^{\frac{1}{2}}}{16}\exp\left[\left(-\eta^2\right)\left(\frac{3}{128}\eta^{16} + \frac{3}{8}\eta^{14} + \frac{21}{64}\eta^{12} - \frac{105}{16}\eta^{10} - \frac{945}{128}\eta^8\right) + 2\left(\varepsilon_{nml}\Omega_n + \varepsilon_{pml}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega\right)(t-t_1)\right]\right.$$

$$+\frac{2\pi^{\frac{1}{2}}}{\sqrt{3}}\exp\left[\left(-\frac{4}{3}\eta^2\right)\left(\frac{160}{19683}\eta^{16} + \frac{40}{729}\eta^{14} - \frac{14}{27}\eta^{12} - \frac{455}{162}\eta^{10} - \frac{35}{18}\eta^8\right) + 2\left(\varepsilon_{nml}\Omega_n + \varepsilon_{pml}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r\right)(t-t_1)\right]$$

$$-\frac{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}}{16}\exp\left[\left(-\frac{3}{2}\eta^2\right)\int_0^{\eta\sqrt{2}}\exp(y^2)dy\left(\frac{3}{64}\eta^{17} + \frac{3}{4}\eta^{15} + \frac{21}{32}\eta^{13} - \frac{105}{8}\eta^{11} - \frac{945}{32}\eta^9\right)\right.$$

$$\left.+2\left(\varepsilon_{nml}\Omega_n + \varepsilon_{pml}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r\right)(t-t_1)\right]$$

$$+\frac{\pi^{\frac{1}{2}}}{2}\exp\left[(-\frac{3}{2}\eta^2)(5.3866\eta^8 + 9.1188\eta^{10} + 3.1017\eta^{12} + 0.1793\eta^{14} - 0.03106\eta^{16}\right.$$

$$-0.004942\eta^{18} - 3.615\times10^{-4}\eta^{20} - 1.890\times10^{-5}\eta^{22} - 7.561\times10^{-7}\eta^{24} - 2.447\times10^{-8}\eta^{26}$$

$$\left.\left.-6.64\times10^{-10}\eta^{28} - 1.55\times10^{-11}\eta^{30}.....) + 2(\varepsilon_{nml}\Omega_n + \varepsilon_{pml}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)\right]\right] \tag{3.4.12}$$

where $\eta = \nu^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}} k$ and $\varepsilon = \nu^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}} k$

The quantity W_β is the contribution to the energy transfer arising from consideration of the three-point correlation equation; W_γ arises from consideration of the four-point equation. Integration of equation (3.4.10) over all wave numbers shows that

$$\int_0^\infty W dk = 0 \qquad (3.4.13)$$

indicating that the expression for W satisfies the conditions of continuity and homogeneity.

In order to obtain the energy spectrum function E, we integrate equation (3.4.5) with respect to time. This integration results in

$$E = E_j + E_\beta + E_\gamma \qquad (3.4.14)$$

$$E_j = \frac{J_0 \varepsilon^4}{3\pi \nu^2 (t-t_0)^2} \exp[-2\varepsilon^2 + 2(\varepsilon_{mkl}\Omega_m + \varepsilon_{nkl}\Omega_n)(t-t_0)] \qquad (3.4.15)$$

$$E_\beta = \frac{(2\pi)^{\frac{1}{2}} \beta_0}{256 \nu^{\frac{15}{2}} (t-t_0)^{\frac{13}{2}}}$$

$$\times \exp[(-\tfrac{3}{2}\varepsilon^2)\Big(-15\varepsilon^6 - 12\varepsilon^8 + \frac{7}{3}\varepsilon^{10} + \frac{16}{3}\varepsilon^{12} - \frac{32}{3\sqrt{2}}\varepsilon^{13} \exp(-\frac{\varepsilon^2}{2}) \int_0^{\frac{\varepsilon}{\sqrt{2}}} \exp(y^2) dy\Big)$$

$$+ 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q)(t-t_0)] \qquad (3.4.16)$$

$$E_\gamma = -\frac{\gamma_1}{\nu^{10}(t-t_1)^9}\{\frac{\pi^{\frac{1}{2}}}{32}\exp[(-\eta^2)\left(\frac{189}{64}\eta^8 + \frac{1029}{256}\eta^{10} + \frac{287}{256}\eta^{12} + \frac{95}{512}\eta^{14} + \frac{71}{512}\eta^{16} - \frac{71}{512}\eta^{18}\right.$$

$$\times\exp(-\eta^2)[Ei(\eta^2) - 0.5772]) + 2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$+\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\exp[(-\frac{4}{3}\eta^2)(\frac{7}{9}\eta^8 + \frac{497}{324}\eta^{10} + \frac{1001}{1458}\eta^{12} + \frac{761}{4374}\eta^{14} + \frac{1963}{19683}\eta^{16} - \frac{3926}{59049}\eta^{18}$$

$$\times\exp(-\frac{2}{3}\eta^2)[Ei(\frac{2}{3}\eta^2) - 0.5772]) + 2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$+\frac{\pi^{\frac{1}{2}}}{2}\exp[(-\frac{3}{2}\eta^2)(0.2307\eta^{10} + 0.3632\eta^{12} + 0.1502\eta^{14} + 0.04463\eta^{16} - 0.01326\eta^{18}$$

$$\times\exp(-\frac{1}{2}\eta^2)[Ei(\frac{1}{2}\eta^2) - 0.5772] + 2.459\times10^{-3}\eta^{18} + 2.935\times10^{-4}\eta^{20} + 2.846\times10^{-5}\eta^{22}$$

$$+2.52\times10^{-6}\eta^{24} + 1.69\times10^{-7}\eta^{26} + 1.25\times10^{-8}\eta^{28} + 5.80\times10^{-10}\eta^{30} + 4.00\times10^{-11}\eta^{32}.......)$$

$$+2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]$$

$$+\frac{1}{2}\pi^{\frac{1}{2}}\exp[(-\frac{3}{2}\eta^2)(1.077\eta^8 + 2.414\eta^{10} + 1.408\eta^{12} + 0.4416\eta^{14} + 0.1898\eta^{16}$$

$$-0.0899\eta^{18}\exp(-\frac{1}{2}\eta^2)[Ei(\frac{1}{2}\eta^2) - 0.5772] + 6.575\times10^{-4}\eta^{18} + 3.271\times10^{-5}\eta^{20}$$

$$+1.270\times10^{-6}\eta^{22} + 4.03\times10^{-8}\eta^{24} + 108\times10^{-9}\eta^{26} + 2.50\times10^{-11}\eta^{28} + 5.09\times10^{-13}\eta^{30}....)$$

$$+2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)(t-t_1)]\} \qquad (3.4.17)$$

The quantity E_j is the energy spectrum function for the final period, where E_β and E_γ are the contributions to the energy spectrum arising from consideration of the three and four point correlation equations respectively.

Equation (3.4.14) can be integrated over all wave numbers to give the total turbulent energy

$$\frac{1}{2}\langle u_i u_i\rangle = \int_0^\infty E dk \qquad (3.4.18)$$

The result carrying out the integration is, in dimensionless form,

$$\frac{\langle u_i u_i\rangle}{2} = \frac{j_0^{\frac{14}{9}}\nu^{\frac{5}{9}}}{\beta_0^{\frac{5}{9}}}\Big[\frac{1}{32(2\pi)^{\frac{1}{2}}}T^{-\frac{5}{2}}\exp[-2(\varepsilon_{mki}\Omega_m+\varepsilon_{nkj}\Omega_n)](t-t_0)$$

$$+0.2296T^{-7}\exp[-2(\varepsilon_{mli}\Omega_m+\varepsilon_{nlj}\Omega_n+\varepsilon_{qlk}\Omega_q)](t-t_0)$$

$$+6.18\frac{\gamma_1\nu^{\frac{5}{9}}J_0^{\frac{5}{9}}}{\beta_0^{\frac{14}{9}}}\left(\frac{t-t_1}{t-t_0}\right)^{-\frac{19}{2}}T^{-\frac{19}{2}}\exp[-2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)](t-t_1)\Big] \quad (3.4.19)$$

$$\langle u^2\rangle = AT^{-\frac{5}{2}}\exp[-2(\varepsilon_{mki}\Omega_m+\varepsilon_{nkj}\Omega_n)](t-t_0)+BT^{-7}\exp[-2(\varepsilon_{mli}\Omega_m+\varepsilon_{nlj}\Omega_n+\varepsilon_{qlk}\Omega_q)](t-t_0)$$

$$+CT^{-\frac{19}{2}}\left(\frac{t-t_1}{t-t_0}\right)^{-\frac{19}{2}}\exp[-2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)](t-t_1) \quad (3.4.20)$$

where

$$\frac{t-t_1}{t-t_0} = 1-\left(\frac{\gamma_1\nu^{\frac{5}{9}}J_0^{\frac{5}{9}}}{\beta_0^{\frac{14}{9}}}\right)^{1/9}\left[\frac{(t_1-t_0)\nu^{\frac{94}{81}}J_0^{\frac{13}{81}}}{\beta_0^{\frac{4}{81}}\gamma_1^{\frac{1}{9}}}\right]\frac{1}{T} \quad (3.4.21)$$

$$T = \frac{\nu^{\frac{11}{9}}J_0^{\frac{2}{9}}(t-t_0)}{\beta_0^{\frac{2}{9}}} \quad (3.4.22)$$

and A, B, C are arbitrary constants.

3.5: Discussion and Conclusion

We obtain the energy decay law of turbulence in a rotating system before the final period from equation (3.4.19) and we consider here three and four point correlation equations after neglecting quintuple correlations terms. The equation (3.4.19) shows that turbulent energy decays more rapidly in an exponential manner than the energy decay for non-rotating fluid. This decay law contains a term $T^{-\frac{19}{2}}$, as well as the terms $T^{-\frac{5}{2}}$ and T^{-7} along with exponential terms, which contains only rotational terms. Thus the terms associated with the higher order correlations die out faster than those associated with the lower order ones.

If the system is non-rotating, we put Ω's = 0, the equation (3.4.20) becomes

$$\langle u^2 \rangle = AT^{-\frac{5}{2}} + BT^{-7} + C\left(\frac{t-t_1}{t-t_0}\right)^{-\frac{19}{2}} T^{-\frac{19}{2}} \qquad (3.5.1)$$

which gives the same result of homogeneous turbulence in a non-rotating system obtained earlier by Deissler[13].

If the higher order correlations were considered in the analysis, it appears that more terms in higher power of T would be added to equation (3.4.20).

CHAPTER-IV

DECAY OF DUSTY FLUID TURBULENCE BEFORE THE FINAL PERIOD IN A ROTATING SYSTEM FOR THE CASE OF THREE AND FOUR POINT CORRELATION EQUATIONS

4.1: Introduction

In recent year, the motion of dusty viscous fluids in a rotating system has developed rapidly. The motion of dusty fluid occurs in the movement of dust-laden air, in problems of fluidization, in the use of dust in a gas cooling system and in the sedimentation problem of tidal rivers. When the motion is referred to axes, which rotate steadily with the bulk of the fluid, the coriolis force and centrifugal force must be supposed to act on the fluid. The coriolis force due to rotation plays an important role in a rotating system of turbulent flow while the centrifugal force with the potential is incorporated into the pressure. Batchelor and Townsend [1] studied the decay of turbulence in the final period. Deissler [12,13] generalized a theory "Decay of homogeneous turbulence for times before the final period". Saffman [52] derived an equation that described the motion of a fluid containing small dust particles. Dixit and Upadhyay [15], Kishore and Dixit [27], Kishore and Singh [25] discussed the effect of Coriolis force on acceleration covariance in ordinary and MHD turbulent flows. Shimomura and Yoshizawa [60], Shimomura [61,62] also discussed the statistical analysis of turbulent viscosity, turbulent scalar flux and turbulent shear flows respectively in a rotating system by two-scale direct interaction approach. Kishore and Upadhyay [34] studied the decay of MHD turbulence in a rotating system. Islam and Sarker [21] also studied the decay of dusty fluid turbulence before the final period in a rotating system using two and three point correlation equations. In Chapter-II, we have studied the decay of dusty fluid turbulence before the final period for the case of three and four point correlation equations and in Chapter-III, we have derived the energy decay law for homogeneous turbulence before the final period in a rotating system for the case of three and four point correlation equations. By analyzing the above theories we have studied the decay of dusty fluid turbulence before the final period in a rotating system using three and four point correlation equations and solved these equations after neglecting the quintuple correlations in comparison to the third and fourth order correlation terms. Finally the energy decay law of homogeneous dusty fluid turbulence in a rotating system before the final period is obtained.

4.2: Basic Equations

The equations of motion and continuity for turbulent flow of dusty incompressible fluid in a rotating system are given below:

$$\frac{\partial u_i}{\partial t}+\frac{\partial(u_i u_l)}{\partial x_l}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_l \partial x_l}-2\varepsilon_{mli}\Omega_m u_i+f(u_i-v_i) \tag{4.2.1}$$

$$\frac{\partial v_i}{\partial t}+v_k\frac{\partial v_i}{\partial x_k}=-\frac{K}{m_s}(v_i-u_i) \tag{4.2.2}$$

and

$$\frac{\partial u_i}{\partial x_i}=\frac{\partial v_i}{\partial x_i}=0 \tag{4.2.3}$$

where the subscripts can take on the values 1,2 or 3.

Here u_i, turbulent velocity components; v_i, dust particle velocity components; ρ , fluid density; ν , kinematic viscosity; Ω_m, constant angular velocity components; ε_{mli} , alternating tensor; p, instantaneous pressure; $m_s = \frac{4}{3}\pi R_s^3 \rho_s$, mass of a single spherical dust particle of radius R_s; ρ_s , constant density of the material in dust particles; K, Stock's drug resistance; $f = \frac{KN}{\rho}$, dimensions of frequency; N, constant number density of dust particle.

4.3: Correlation and Spectral Equations

The equations of motion of dusty fluid turbulence in a rotating system for the points p, $p^{/}$ and $p^{//}$ separated by the vector $\underline{r}$ and $\underline{r}^{/}$ are

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_l)}{\partial x_l} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_l \partial x_l} - 2\varepsilon_{mli}\Omega_m u_i + f(u_i - v_i) \tag{4.3.1}$$

$$\frac{\partial u_j^{/}}{\partial t} + \frac{\partial (u_j^{/} u_l^{/})}{\partial x_l^{/}} = -\frac{1}{\rho}\frac{\partial p^{/}}{\partial x_j^{/}} + \nu \frac{\partial^2 u_j^{/}}{\partial x_l^{/} \partial x_l^{/}} - 2\varepsilon_{nlj}\Omega_n u_j^{/} + f(u_j^{/} - v_j^{/}) \tag{4.3.2}$$

$$\frac{\partial u_k^{//}}{\partial t} + \frac{\partial (u_k^{//} u_l^{//})}{\partial x_l^{//}} = -\frac{1}{\rho}\frac{\partial p^{//}}{\partial x_k^{//}} + \nu \frac{\partial^2 u_k^{//}}{\partial x_k^{//} \partial x_k^{//}} - 2\varepsilon_{qlk}\Omega_q u_k^{//} + f(u_k^{//} - v_k^{//}) \tag{4.3.3}$$

Multiplying equation (4.3.1) by $u_j^{/} u_k^{//}$, (4.3.2) by $u_i u_k^{//}$, and (4.3.3) $u_i u_j^{/}$, adding the three equations and taking space or time averages, we obtain

$$\frac{\partial}{\partial t}\langle u_i u_j' u_k''\rangle + \frac{\partial}{\partial x_l}\langle u_i u_j' u_k'' u_l\rangle + \frac{\partial}{\partial x_l'}\langle u_i u_j' u_k'' u_l'\rangle + \frac{\partial}{\partial x_l''}\langle u_i u_j' u_k'' u_l''\rangle$$

$$= -\frac{1}{\rho}\left(\frac{\partial}{\partial x_i}\langle p u_j' u_k'\rangle + \frac{\partial}{\partial x_j'}\langle p' u_i u_k''\rangle + \frac{\partial}{\partial x_k''}\langle p'' u_i u_j'\rangle\right)$$

$$+ \nu\left(\frac{\partial^2\langle u_i u_j' u_k''\rangle}{\partial x_l \partial x_l} + \frac{\partial^2\langle u_i u_j' u_k''\rangle}{\partial x_l' \partial x_l'} + \frac{\partial^2\langle u_i u_j' u_k''\rangle}{\partial x_l'' \partial x_l''}\right)$$

$$- 2\left(\varepsilon_{mli}\Omega_m \langle u_i u_j' u_k''\rangle + \varepsilon_{nlj}\Omega_n \langle u_i u_j' u_k''\rangle + \varepsilon_{qlk}\Omega_q \langle u_i u_j' u_k''\rangle\right)$$

$$+ f\left(3\langle u_i u_j' u_k''\rangle - \langle v_i u_j' u_k''\rangle - \langle v_j' u_i u_k''\rangle - \langle v_k'' u_i u_j'\rangle\right) \tag{4.3.4}$$

Using the transformations

$\frac{\partial}{\partial x_l'} = \frac{\partial}{\partial r_l}$, $\frac{\partial}{\partial x_l''} = \frac{\partial}{\partial r_l'}$ and $\frac{\partial}{\partial x_l} = -\frac{\partial}{\partial r_l} - \frac{\partial}{\partial r_l'}$ into equations (4.3.4),we get

$$\frac{\partial}{\partial t}\langle u_i u_j' u_k''\rangle - \frac{\partial}{\partial r_l}\langle u_i u_j' u_k'' u_l\rangle - \frac{\partial}{\partial r_l'}\langle u_i u_j' u_k'' u_l\rangle + \frac{\partial}{\partial r_l}\langle u_i u_j' u_k'' u_l'\rangle + \frac{\partial}{\partial r_l'}\langle u_i u_j' u_k'' u_l''\rangle$$

$$= -\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}\langle p u_j' u_k''\rangle - \frac{\partial}{\partial r_i'}\langle p u_j' u_k''\rangle + \frac{\partial}{\partial r_j}\langle p' u_i u_k''\rangle + \frac{\partial}{\partial r_k'}\langle p'' u_i u_j'\rangle\right)$$

$$+ 2\nu\left(\frac{\partial\langle u_i u_j' u_k''\rangle}{\partial r_l \partial r_l} + \frac{\partial\langle u_i u_j' u_k''\rangle}{\partial r_l \partial r_l'} + \frac{\partial\langle u_i u_j' u_k''\rangle}{\partial r_l' \partial r_l'}\right)$$

$$- 2\left(\varepsilon_{mli}\Omega_m \langle u_i u_j' u_k''\rangle + \varepsilon_{nlj}\Omega_n \langle u_i u_j' u_k''\rangle + \varepsilon_{qlk}\Omega_q \langle u_i u_j' u_k''\rangle\right)$$

$$+ f\left(3\langle u_i u_j' u_k''\rangle - \langle v_i u_j' u_k''\rangle - \langle u_i v_j' u_k''\rangle - \langle u_i u_j' v_k''\rangle\right) \tag{4.3.5}$$

In order to convert equation (4.3.5) to spectral form, we can define the following six dimensional Fourier transforms:

$$\langle u_i u_j'(\underline{r}) u_k''(\underline{r}')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \beta_i \beta_j'(\underline{k}) \beta_k''(\underline{k}')\rangle . \exp\left[i(\underline{k}.\underline{r} + \underline{k}'.\underline{r}')\right] d\underline{k}\, d\underline{k}' \tag{4.3.6}$$

$$\langle u_i u_l u_j'(\underline{r}) u_k''(\underline{r}')\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \beta_i \beta_l \beta_j'(\underline{k}) \beta_k''(\underline{k}')\rangle . \exp\left[i(\underline{k}.\underline{r} + \underline{k}'.\underline{r}')\right] d\underline{k}\, d\underline{k}' \tag{4.3.7}$$

$$\langle pu_j^{/}(\underline{r})u_k^{//}(\underline{r}^{/})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \alpha\beta_j^{/}(\underline{k})\beta_k^{//}(\underline{k}^{/})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right]d\underline{k}\,d\underline{k}^{/} \qquad (4.3.8)$$

$$\langle v_i u_j^{/}(\underline{r})u_k^{//}(\underline{r}^{/})\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \gamma_i\beta_j^{/}(\underline{k})\beta_k^{//}(\underline{k}^{/})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right]d\underline{k}\,d\underline{k}^{/} \qquad (4.3.9)$$

Interchanging the subscripts i and j and then interchanging the points p and $p^{/}$ give

$$\langle u_i u_l^{/}(\underline{r})u_j^{/}(\underline{r})u_k^{//}(\underline{r}^{/})\rangle = \langle u_j u_l u_i^{/}(-\underline{r})u_k^{//}(\underline{r}^{/}-\underline{r})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \beta_j\beta_l\,\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_k^{//}(\underline{k}^{/})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right]d\underline{k}\,d\underline{k}^{/} \qquad (4.3.10)$$

$$\langle u_i u_j^{/}(\underline{r})u_k^{//}(\underline{r}^{/})u_l^{//}(\underline{r}^{/})\rangle = \langle u_k u_l u_i^{/}(-\underline{r}^{/})u_j^{//}(\underline{r}-\underline{r}^{/})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \beta_k\beta_l\,\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_j^{//}(\underline{k})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right]d\underline{k}\,d\underline{k}^{/} \qquad (4.3.11)$$

where the points p and $p^{/}$ are interchanged to obtain equation (4.3.10). For equation (4.3.11), p is replaced by $p^{/}$, $p^{/}$ is replaced by $p^{//}$ and $p^{//}$ is replaced by p.

Similarly,

$$\langle u_i p^{/}(\underline{r})u_k^{//}(\underline{r}^{/})\rangle = \langle pu_i^{/}(-\underline{r})u_k^{//}(\underline{r}^{/}-\underline{r})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \alpha\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_k^{//}(\underline{k}^{/})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right]d\underline{k}\,d\underline{k}^{/} \qquad (4.3.12)$$

$$\langle u_i u_j^{/}p^{//}(\underline{r}^{/})\rangle = \langle pu_i^{/}(\underline{r}^{/})u_j^{//}(\underline{r}-\underline{r}^{/})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \alpha\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_j^{//}(\underline{k})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right]d\underline{k}\,d\underline{k}^{/} \qquad (4.3.13)$$

$$\langle u_i v_j^{/}u_k^{//}(\underline{r}^{/})\rangle = \langle v_j u_i^{/}(\underline{r})u_k^{//}(\underline{r}^{/}-\underline{r})\rangle$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle \gamma_j\beta_i^{/}(-\underline{k}-\underline{k}^{/})\beta_k^{//}(\underline{k}^{/})\rangle.\exp\left[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})\right]d\underline{k}\,d\underline{k}^{/} \qquad (4.3.14)$$

$$\langle u_i u_j' v_k''(r')\rangle = \langle v_k u_i'(-r') u_j''(r-r')\rangle$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \gamma_k \beta_i'(-k-k')\beta_j''(k)\rangle.\exp\left[i(k.r+k'.r')\right]dk\,dk' \qquad (4.3.15)$$

Substituting the preceding relations into equation (4.3.5), we get

$$\frac{d}{dt}\langle \beta_i \beta_j' \beta_k''\rangle + 2\nu(k^2 + k_l k_l' + k'^2)\langle \beta_i \beta_j' \beta_k''\rangle$$
$$= \left[i(k_l + k_l')\langle \beta_i \beta_l \beta_j' \beta_k''\rangle - ik_l\langle \beta_j \beta_l \beta_i'(-k-k')\beta_k''(k')\rangle - ik_l'\langle \beta_k \beta_l \beta_i'(-k-k')\beta_j''(k)\rangle\right]$$
$$-\frac{1}{\rho}\left[-i(k_i + k_i')\langle \alpha \beta_j' \beta_k''\rangle + ik_j\langle \alpha\beta_i'(-k-k')\beta_k''(k')\rangle + ik_k'\langle \alpha\beta_i'(-k-k')\beta_j''(k)\rangle\right]$$
$$-2\left[\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q\right]\langle \beta_i \beta_j' \beta_k''\rangle$$
$$+f\left[3\langle \beta_i \beta_j'(k)\beta_k''(k')\rangle - \langle \gamma_i \beta_j'(k')\beta_k''(k')\rangle - \langle \gamma_j \beta_i'(-k-k')\beta_k''(k')\rangle - \langle \gamma_k \beta_i'(-k-k')\beta_j''(k)\rangle\right] \qquad (4.3.16)$$

The tensor equation (4.3.16) can be converted to a scalar form by contraction of the indices i and j and inner multiplication k_k ;

$$\frac{d}{dt}\left(k_k\langle \beta_i \beta_i' \beta_k''\rangle + 2\nu(k^2 + k_l k_l' + k'^2)k_k\langle \beta_i \beta_i' \beta_k''\rangle\right)$$
$$= \left[ik_k(k_l + k_l')\langle \beta_i \beta_l \beta_i' \beta_k''\rangle - ik_k k_l\langle \beta_i \beta_l \beta_i'(-k-k')\beta_k''(k')\rangle - ik_k k_l'\langle \beta_k \beta_l \beta_i'(-k-k')\beta_i''(k)\rangle\right]$$
$$-\frac{1}{\rho}\left[-ik_k(k_i + k_i')\langle \alpha\beta_i' \beta_k''\rangle + ik_k k_i\langle \alpha\beta_i'(-k-k')\beta_k''(k')\rangle + ik_k k_k'\langle \alpha\beta_i'(-k-k')\beta_i''(k)\rangle\right]$$
$$-2k_k\left[\varepsilon_{mli}\Omega_m + \varepsilon_{nli}\Omega_n + \varepsilon_{qlk}\Omega_q\right]\langle \beta_i \beta_i' \beta_k''\rangle$$
$$+fk_k\left[3\langle \beta_i \beta_i'(k)\beta_k''(k')\rangle - \langle \gamma_i \beta_i'(k)\beta_k''(k')\rangle - \langle \gamma_i \beta_i'(-k-k')\beta_k''(k)\rangle - \langle \gamma_k \beta_i'(-k-k')\beta_i''(k)\rangle\right] \qquad (4.3.17)$$

In order to obtain the four-point equation, we consider the equations of motion of dusty fluid turbulence in a rotating system at the points p, p', p'' and p''' as

$$\frac{\partial u_i}{\partial t} + \frac{\partial(u_i u_m)}{\partial x_m} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_m \partial x_m} - 2\varepsilon_{mni}\Omega_n u_i + f(u_i - v_i) \qquad (4.3.18)$$

$$\frac{\partial u_j^{/}}{\partial t}+\frac{\partial(u_j^{/}u_m^{/})}{\partial x_m^{/}}=-\frac{1}{\rho}\frac{\partial p^{/}}{\partial x_j^{/}}+\nu\frac{\partial^2 u_j^{/}}{\partial x_m^{/}\partial x_m^{/}}-2\varepsilon_{pmj}\Omega_p u_j^{/}+f(u_j^{/}-v_j^{/}) \tag{4.3.19}$$

$$\frac{\partial u_k^{//}}{\partial t}+\frac{\partial(u_k^{//}u_m^{//})}{\partial x_m^{//}}=-\frac{1}{\rho}\frac{\partial p^{//}}{\partial x_k^{//}}+\nu\frac{\partial^2 u_k^{//}}{\partial x_m^{//}\partial x_m^{//}}-2\varepsilon_{qmk}\Omega_q u_k^{//}+f(u_k^{//}-v_k^{//}) \tag{4.3.20}$$

$$\frac{\partial u_l^{///}}{\partial t}+\frac{\partial(u_l^{///}u_m^{///})}{\partial x_m^{///}}=-\frac{1}{\rho}\frac{\partial p^{///}}{\partial x_l^{///}}+\nu\frac{\partial^2 u_l^{///}}{\partial x_m^{///}\partial x_m^{///}}-2\varepsilon_{rml}\Omega_r u_l^{///}+f(u_l^{///}-v_l^{///}) \tag{4.3.21}$$

where the repeated subscript in a term indicates a summation.

Multiplying the first equation by $u_j^{/}u_k^{//}u_l^{///}$, the second by $u_i\,u_k^{//}u_l^{///}$, the third by $u_i\,u_j^{/}u_l^{///}$ and the fourth by $u_i\,u_j^{/}u_k^{//}$ respectively, then adding and taking space or time averages, we get

$$\frac{\partial}{\partial t}\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle+\frac{\partial}{\partial x_m}\left\langle u_i u_j^{/}u_k^{//}u_l^{///}u_m\right\rangle+\frac{\partial}{\partial x_m^{/}}\left\langle u_i u_j^{/}u_k^{//}u_l^{///}u_m^{/}\right\rangle+\frac{\partial}{\partial x_m^{//}}\left\langle u_i u_j^{/}u_k^{//}u_l^{///}u_m^{//}\right\rangle+\frac{\partial}{\partial x_m^{///}}\left\langle u_i u_j^{/}u_k^{//}u_l^{///}u_m^{///}\right\rangle$$

$$=-\tfrac{1}{\rho}\left(\frac{\partial}{\partial x_i}\left\langle p u_j^{/}u_k^{//}u_l^{///}\right\rangle+\frac{\partial}{\partial x_j^{/}}\left\langle p^{/}u_i u_k^{//}u_l^{///}\right\rangle+\frac{\partial}{\partial x_k^{//}}\left\langle p^{//}u_i u_j^{/}u_l^{///}\right\rangle+\frac{\partial}{\partial x_l^{///}}\left\langle p^{///}u_i u_j^{/}u_k^{//}\right\rangle\right)$$

$$+\nu\left(\frac{\partial^2\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle}{\partial x_m\partial x_m}+\frac{\partial^2\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle}{\partial x_m^{/}\partial x_m^{/}}+\frac{\partial^2\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle}{\partial x_m^{//}\partial x_m^{//}}+\frac{\partial^2\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle}{\partial x_m^{///}\partial x_m^{///}}\right)$$

$$-2\left(\varepsilon_{nmi}\Omega_n\left\langle u_i u_j^{/}u_k^{//}u_l^{//}\right\rangle+\varepsilon_{pmj}\Omega_p\left\langle u_i u_j^{/}u_k^{//}u_l^{//}\right\rangle+\varepsilon_{qmk}\Omega_q\left\langle u_i u_j^{/}u_k^{//}u_l^{//}\right\rangle+\varepsilon_{rml}\Omega_r\left\langle u_i u_j^{/}u_k^{//}u_l^{//}\right\rangle\right)$$

$$f\left(-\left\langle v_i u_j^{/}u_k^{//}u_l^{///}\right\rangle+\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle-\left\langle u_i v_j^{/}u_k^{//}u_l^{///}\right\rangle+\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle-\left\langle u_i u_j^{/}v_k^{//}u_l^{///}\right\rangle+\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle\right.$$

$$\left.-\left\langle u_i u_j^{/}u_k^{//}v_l^{///}\right\rangle+\left\langle u_i u_j^{/}u_k^{//}u_l^{///}\right\rangle\right) \tag{4.3.22}$$

Equation (4.3.22) can be written in terms of the independent variables r, $r^{/}$ and $r^{//}$ as

$$\frac{\partial}{\partial t}\left\langle u_i u_j' u_k'' u_l'''\right\rangle - \frac{\partial}{\partial r_m}\left\langle u_i u_m u_j' u_k'' u_l'''\right\rangle - \frac{\partial}{\partial r_m'}\left\langle u_i u_m u_j' u_k'' u_l'''\right\rangle - \frac{\partial}{\partial r_m''}\left\langle u_i u_m u_j' u_k'' u_l'''\right\rangle$$

$$+\frac{\partial}{\partial r_m}\left\langle u_i u_j' u_m' u_k'' u_l'''\right\rangle + \frac{\partial}{\partial r_m'}\left\langle u_i u_j' u_k'' u_m'' u_l'''\right\rangle + \frac{\partial}{\partial r_m''}\left\langle u_i u_j' u_k'' u_l''' u_m'''\right\rangle$$

$$= -\tfrac{1}{\rho}\begin{pmatrix} -\dfrac{\partial}{\partial r_i}\left\langle p u_j' u_k'' u_l'''\right\rangle - \dfrac{\partial}{\partial r_i'}\left\langle p u_j' u_k'' u_l'''\right\rangle - \dfrac{\partial}{\partial r_i''}\left\langle p u_j' u_k'' u_l'''\right\rangle \\ +\dfrac{\partial}{\partial r_j}\left\langle u_i p' u_k'' u_l'''\right\rangle + \dfrac{\partial}{\partial r_k'}\left\langle u_i u_j' p'' u_l'''\right\rangle + \dfrac{\partial}{\partial r_l''}\left\langle u_i u_j' u_k'' p'''\right\rangle \end{pmatrix}$$

$$+2\nu\begin{pmatrix} \dfrac{\partial^2\left\langle u_i u_j' u_k'' u_l'''\right\rangle}{\partial r_m \partial r_m} + \dfrac{\partial^2\left\langle u_i u_j' u_k'' u_l'''\right\rangle}{\partial r_m \partial r_m'} + \dfrac{\partial^2\left\langle u_i u_j' u_k'' u_l'''\right\rangle}{\partial r_m \partial r_m''} \\ +\dfrac{\partial^2\left\langle u_i u_j' u_k'' u_l'''\right\rangle}{\partial r_m' \partial r_m'} + \dfrac{\partial^2\left\langle u_i u_j' u_k'' u_l'''\right\rangle}{\partial r_m' \partial r_m''} + \dfrac{\partial^2\left\langle u_i u_j' u_k'' u_l'''\right\rangle}{\partial r_m'' \partial r_m''} \end{pmatrix}$$

$$-2\left(\varepsilon_{nmi}\Omega_n\left\langle u_i u_j' u_k'' u_l'''\right\rangle + \varepsilon_{pmj}\Omega_p\left\langle u_i u_j' u_k'' u_l'''\right\rangle + \varepsilon_{qmk}\Omega_q\left\langle u_i u_j' u_k'' u_l'''\right\rangle + \varepsilon_{rml}\Omega_r\left\langle u_i u_j' u_k'' u_l'''\right\rangle\right)$$

$$+f\left(-\left\langle v_i u_j' u_k'' u_l'''\right\rangle - \left\langle u_i v_j' u_k'' u_l'''\right\rangle - \left\langle u_i u_j' v_k'' u_l'''\right\rangle - \left\langle u_i u_j' u_k'' v_l'''\right\rangle + 4\left\langle u_i u_j' u_k'' v_l'''\right\rangle\right) \qquad (4.3.23)$$

where the following transformations were used:

$$\frac{\partial}{\partial x_m'} = \frac{\partial}{\partial r_m}, \quad \frac{\partial}{\partial x_m''} = \frac{\partial}{\partial r_m'}, \quad \frac{\partial}{\partial x_m'''} = \frac{\partial}{\partial r_m''} \quad \text{and} \quad \frac{\partial}{\partial x_m} = -\frac{\partial}{\partial r_m} - \frac{\partial}{\partial r_m'} - \frac{\partial}{\partial r_m''}$$

In order to convert equation (4.3.23) to spectral form, we define the following nine-dimensional Fourier transforms:

$$\left\langle u_i u_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{ijkl}(\underline{k},\underline{k}',\underline{k}'')\exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')]d\underline{k}\,d\underline{k}'\,d\underline{k}'' \qquad (4.3.24)$$

$$\left\langle u_i u_m u_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{imjkl}(\underline{k},\underline{k}',\underline{k}'')\exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')]d\underline{k}\,d\underline{k}'\,d\underline{k}'' \qquad (4.3.25)$$

$$\left\langle p u_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta_{jkl}(\underline{k},\underline{k}',\underline{k}'')\exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')]d\underline{k}\,d\underline{k}'\,d\underline{k}'' \qquad (4.3.26)$$

$$\left\langle v_i u_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_i \delta_{jkl}(\underline{k}, \underline{k}', \underline{k}'') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \tag{4.3.27}$$

Similarly,

$$\left\langle u_i v_j'(\underline{r}) u_k''(\underline{r}') u_l'''(\underline{r}'')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_j \delta_{ikl}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}', \underline{k}'') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \tag{4.3.28}$$

$$\left\langle u_i u_j'(\underline{r}) v_k''(\underline{r}') u_l'''(\underline{r}'')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_k \delta_{ijl}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}, \underline{k}'') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \tag{4.3.29}$$

$$\left\langle u_i u_j'(\underline{r}) u_k''(\underline{r}') v_l'''(\underline{r}'')\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_l \delta_{ijk}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}, \underline{k}') \exp[i(\underline{k}\cdot\underline{r} + \underline{k}'\cdot\underline{r}' + \underline{k}''\cdot\underline{r}'')] d\underline{k}\, d\underline{k}'\, d\underline{k}'' \tag{4.3.30}$$

Substituting the preceding relations into equation (4.3.23), we get

$$\begin{aligned}
&\frac{d}{dt}(\gamma_{ijkl}) + 2\nu\left(k^2 + k_m k_m' + k_m k_m'' + k'^2 + k_m' k_m'' + k''^2\right)\gamma_{ijkl} \\
&= [i(k_m + k_m' + k_m'')\gamma_{imjkl}(\underline{k}, \underline{k}', \underline{k}'') - ik_m \gamma_{jmikl}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}', \underline{k}'') - ik_m' \gamma_{kmijl}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}, \underline{k}'') \\
&- ik_m'' \gamma_{lmijk}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}, \underline{k}')] - \frac{1}{\rho}[-i(k_i + k_i' + k_i'')\delta_{jkl}(\underline{k}, \underline{k}', \underline{k}'') + ik_j \delta_{ikl}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}', \underline{k}'') \\
&+ ik_k' \delta_{ijl}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}, \underline{k}'') + ik_l'' \delta_{ijk}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}, \underline{k}')] \\
&- 2\left(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r\right)\gamma_{ijkl} \\
&+ f\Big[4\gamma_{ijkl}(\underline{k}, \underline{k}', \underline{k}'') - \gamma_i \delta_{jkl}(\underline{k}, \underline{k}', \underline{k}'') - \gamma_j \delta_{ikl}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}', \underline{k}'') \\
&- \gamma_k \delta_{ijl}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}, \underline{k}'') - \gamma_l \delta_{ijk}(-\underline{k}-\underline{k}'-\underline{k}'', \underline{k}, \underline{k}')\Big]
\end{aligned} \tag{4.3.31}$$

To obtain a relation between the terms on the right hand side of equation (4.3.31) derived from the quadruple correlation terms, pressure terms, rotational terms and the dust particle terms in equation (4.3.23), take the divergence of the equation of motion and combine with the continuity equation to give

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial x_m \partial x_m} = -\frac{\partial^2 (u_m u_n)}{\partial x_m \partial x_n} \tag{4.3.32}$$

Multiplying the equation (4.3.32) by $u_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}$, taking ensemble average and writing the resulting equation in terms of the independent variables $\mathbf{r}$ and $\mathbf{r}^{\prime}$, gives

$$\frac{1}{\rho}\left(\frac{\partial^2\langle pu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m\partial r_m}+2\frac{\partial^2\langle pu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m\partial r_m^{\prime}}+2\frac{\partial^2\langle pu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m\partial r_m^{\prime\prime}}+\frac{\partial^2\langle pu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime}\partial r_m^{\prime}}+2\frac{\partial^2\langle pu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime}\partial r_m^{\prime\prime}}+\frac{\partial^2\langle pu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime\prime}\partial r_m^{\prime\prime}}\right)$$

$$=-\left(\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m\partial r_n}+\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m\partial r_n^{\prime}}+\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m\partial r_n^{\prime\prime}}+\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime}\partial r_n}+\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime}\partial r_n^{\prime}}\right.$$

$$\left.+\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime}\partial r_n^{\prime\prime}}+\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime\prime}\partial r_n}+\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime\prime}\partial r_n^{\prime}}+\frac{\partial^2\langle u_mu_nu_j^{\prime}u_k^{\prime\prime}u_l^{\prime\prime\prime}\rangle}{\partial r_m^{\prime\prime}\partial r_n^{\prime\prime}}\right) \qquad (4.3.33)$$

The Fourier transform of equation (4.3.33) is

$$-\frac{1}{\rho}\left(k^2+2k_mk_m^{\prime}+2k_mk_m^{\prime\prime}+k^{\prime 2}+2k_m^{\prime}k_m^{\prime\prime}+k^{\prime\prime 2}\right)\delta_{jkl}$$

$$=\left(k_mk_n+k_mk_n^{\prime}+k_mk_n^{\prime\prime}+k_m^{\prime}k_n+k_m^{\prime}k_n^{\prime}+k_m^{\prime}k_n^{\prime\prime}+k_m^{\prime\prime}k_n+k_m^{\prime\prime}k_n^{\prime}+k_m^{\prime\prime}k_n^{\prime\prime}\right)\gamma_{mnjkl}$$

$$\therefore -\frac{1}{\rho}\delta_{jkl}=\frac{\left(k_mk_n+k_mk_n^{\prime}+k_mk_n^{\prime\prime}+k_m^{\prime}k_n+k_m^{\prime}k_n^{\prime}+k_m^{\prime}k_n^{\prime\prime}+k_m^{\prime\prime}k_n+k_m^{\prime\prime}k_n^{\prime}+k_m^{\prime\prime}k_n^{\prime\prime}\right)\gamma_{mnjkl}}{\left(k^2+2k_mk_m^{\prime}+2k_mk_m^{\prime\prime}+k^{\prime 2}+2k_m^{\prime}k_m^{\prime\prime}+k^{\prime\prime 2}\right)} \qquad (4.3.34)$$

Equations (4.3.31) and (4.3.34) are the spectral equations corresponding to the four point correlation equations. The spectral equations corresponding to the three point correlation equations are

$$\frac{d}{dt}(k_k\beta_{iik})+2\nu(k^2+k_lk_l^{\prime}+k^{\prime 2})k_k\beta_{iik}$$

$$=ik_k(k_l+k_l^{\prime})\beta_{ilik}(\underline{k},\underline{k}^{\prime})-ik_kk_l\beta_{ilik}(-\underline{k}-\underline{k}^{\prime},\underline{k}^{\prime})-ik_kk_l^{\prime}\beta_{klii}(-\underline{k}-\underline{k}^{\prime},\underline{k})$$

$$-\frac{1}{\rho}\left[-ik_k(k_i+k_i^{\prime})\alpha_{ik}(\underline{k},\underline{k}^{\prime})+ik_kk_i\alpha_{ik}(-\underline{k}-\underline{k}^{\prime},\underline{k}^{\prime})+ik_kk_k^{\prime}\alpha_{ii}(-\underline{k}-\underline{k}^{\prime},\underline{k})\right]$$

$$-2k_k\left[\varepsilon_{mli}\Omega_m+\varepsilon_{nli}\Omega_n+\varepsilon_{qlk}\Omega_q\right]\beta_i\beta_i^{\prime}\beta_k^{\prime\prime}+Rfk_k \qquad (4.3.35)$$

where, $R\beta_i\beta_i^{/}\beta_k^{//} = 3\langle\beta_i\beta_i^{/}\beta_k^{//}\rangle - \langle\gamma_i\beta_i^{/}(k)\beta_k^{//}(k^{/})\rangle - \langle\gamma_i\beta_i^{/}(-k-k^{/})\beta_k^{//}(k^{/})\rangle - \langle\gamma_k\beta_i^{/}(-k-k^{/})\beta_i^{//}(k)\rangle$

(say), R is an arbitrary constant and

$$-\frac{1}{\rho}\alpha_{ik} = \frac{k_l k_m + k_l^{/} k_m + k_l k_m^{/} + k_l^{/} k_m^{/}}{k^2 + 2k_l k_l^{/} + k^{/2}}\beta_{lmik} \tag{4.3.36}$$

Here the spectral tensors are defined by

$$\left\langle u_i u_j^{/}(\underline{r}) u_k^{//}(\underline{r}^{/})\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta_{ijk}(\underline{k},\underline{k}^{/})\exp[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})]d\,\underline{k}\,d\,\underline{k}^{/} \tag{4.3.37}$$

$$\left\langle u_i u_l u_j^{/}(\underline{r}) u_k^{//}(\underline{r}^{/})\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta_{iljk}(\underline{k},\underline{k}^{/})\exp[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})]d\,\underline{k}\,d\,\underline{k}^{/} \tag{4.3.38}$$

$$\left\langle p u_j^{/}(\underline{r}) u_k^{//}(\underline{r}^{/})\right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_{jk}(\underline{k},\underline{k}^{/})\exp[i(\underline{k}.\underline{r}+\underline{k}^{/}.\underline{r}^{/})]d\,\underline{k}\,d\,\underline{k}^{/} \tag{4.3.39}$$

A relation between β_{iljk} and γ_{ijkl} can be obtained by letting $r^{//} = 0$ in equation (4.3.24) and comparing the result with equation (4.3.38)

$$\beta_{iljk}(\underline{k},\underline{k}^{/}) = \int_{-\infty}^{\infty}\gamma_{ijkl}(\underline{k},\underline{k}^{/},\underline{k}^{//})d\,\underline{k}^{//} \tag{4.3.40}$$

The spectral equation corresponding to the two-point correlation equation in presence of dusty fluid in a rotating system is

$$\frac{d}{dt}\phi_{i,j} + (2\nu k^2 - Qf + 2\varepsilon_{mki}\Omega_m + 2\varepsilon_{nki}\Omega_n)\phi_{i,j} = ik_k\phi_{iki}(\underline{k}) - ik_k\phi_{iki}(-\underline{k}) \tag{4.3.41}$$

where $\phi_{i,i}$ and ϕ_{iki} are defined by

$$\left\langle u_i u_j^{/}(\underline{r})\right\rangle = \int_{-\infty}^{\infty}\phi_{ij}(\underline{k})\exp(i\,\underline{k}\cdot\underline{r})d\,\underline{k} \tag{4.3.42}$$

and $$\left\langle u_i u_k u_j'(\underline{r})\right\rangle = \int_{-\infty}^{\infty} \phi_{ikj}(\underline{k})\exp(i\underline{k}\cdot\underline{r})d\underline{k} \tag{4.3.43}$$

The relation between ϕ_{ikj} and β_{ijk} obtained by letting $\underline{r}' = 0$ in equation (4.3.37) and comparing the result with equation (4.3.43) is

$$\phi_{ikj}(\underline{k}) = \int_{-\infty}^{\infty} \beta_{ijk}(\underline{k},\underline{k}')d\underline{k}' \tag{4.3.44}$$

4.4: Solution Neglecting Quintuple Correlations

Equation (4.3.34) shows that if the terms corresponding to the quintuple correlations are neglected, then the pressure force terms also must be neglected. Thus neglecting first and second terms on the right side of equation (4.3.31), the equation can be integrated between t_1 and t to give

$$\begin{aligned}\gamma_{ijkl} = (\gamma_{ijkl})_1 \exp[\{&-2\nu(k^2 + k_m k_m' + k_m k_m'' + k'^2 + k_m' k_m'' + k''^2) \\ &+ 2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - sf\}(t-t_1)]\end{aligned} \tag{4.4.1}$$

where

$$\begin{aligned}S\gamma_{ijkl} = 4\gamma_{ijkl}(\underline{k},\underline{k}',\underline{k}'') &- \gamma_i\delta_{jkl}(\underline{k},\underline{k}',\underline{k}'') - \gamma_j\delta_{ikl}(-\underline{k}-\underline{k}'-\underline{k}'',\underline{k}',\underline{k}'') \\ &- \gamma_k\delta_{ijl}(-\underline{k}-\underline{k}'-\underline{k}'',\underline{k},\underline{k}'') - \gamma_l\delta_{ijk}(-\underline{k}-\underline{k}'-\underline{k}'',\underline{k},\underline{k}')\end{aligned}$$

is an arbitrary constant and $(\gamma_{ijkl})_1$ is the value of γ_{ijkl} at $t = t_1$. The quantity $(\gamma_{ijkl})_1$ can be considered also as the value of γ_{ijkl} at small values of k, k' and k'', at least for times when the quintuple correlations are negligible.

Equations (4.3.40) and (4.4.1) can be converted to scalar form by contracting the indices i and j, as well as k and l. Substitution of equations (4.3.36), (4.3.40) and (4.4.1) into the three point scalar equation (4.3.35) results in

$$\frac{d}{dt}(k_k\beta_{iik})+2\nu(k^2+k_lk_l^{/}+k^{/^2})k_k\beta_{iik}$$

$$=i[ik_k(k_l+k_l^{/})\gamma_{iikl}(\underline{k},\underline{k}^{/},\underline{k}^{//})-ik_k(k_l+k_l^{/}).\frac{k_lk_m+k_l^{/}k_m+k_lk_m^{/}+k_l^{/}k_m^{/}}{k^2+2k_lk_l^{/}+k^{/^2}}\gamma_{likm}(\underline{k},\underline{k}^{/},\underline{k}^{//})]_1$$

$$\times\int_{-\infty}^{\infty}\exp[\{-2\nu(k^2+k_mk_m^{/}+k_mk_m^{//}+k^{/^2}+k_m^{/}k_m^{//}+k^{//^2})+2(\varepsilon_{mml}\Omega_n+\varepsilon_{pml}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-sf\}(t-t_1)]d\underline{k}^{//}$$

$$+\left[-ik_kk_l\gamma_{iikl}(-\underline{k}-\underline{k}^{/},\underline{k}^{/},\underline{k}^{//})+\frac{ik_kk_lk_lk_m}{k^2}\gamma_{likm}(-\underline{k}-\underline{k}^{/},\underline{k}^{/},\underline{k}^{//})\right]_1$$

$$\times\int_{-\infty}^{\infty}\exp[\{-2\nu(k^2+k_mk_m^{/}+k^{/^2}-k_m^{/}k_m^{//}+k^{//^2})+2(\varepsilon_{mml}\Omega_n+\varepsilon_{pml}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-sf\}(t-t_1)]d\underline{k}^{//}$$

$$+[-ik_kk_l^{/}\gamma_{kiil}(-\underline{k}-\underline{k}^{/},\underline{k},\underline{k}^{//})+\frac{ik_kk_k^{/}k_l^{/}k_m^{/}}{k^{/^2}}\gamma_{liim}(-\underline{k}-\underline{k}^{/},\underline{k},\underline{k}^{//})]_1$$

$$\times\int_{-\infty}^{\infty}\exp[\{-2\nu(k^2+k_mk_m^{/}+k^{/^2}-k_m^{/}k_m^{//}+k^{//^2})+2(\varepsilon_{mml}\Omega_n+\varepsilon_{pml}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-sf\}(t-t_1)]d\underline{k}^{//}$$

$$-2k_k(\varepsilon_{mli}\Omega_m+\varepsilon_{nli}\Omega_n+\varepsilon_{qlk}\Omega_q)\beta_{iik}+Rfk_k \tag{4.4.2}$$

where the γ 's have been assumed independent of $k^{//}$ at t.

Substituting $d\underline{k}^{//}=dk_1^{//}dk_2^{//}dk_3^{//}$ in equation (4.4.2) and carrying out the integration with respect to $k_1^{//}$, $k_2^{//}$ and $k_3^{//}$, we get

$$\frac{d}{dt}(k_k\beta_{iik})+2\nu(k^2+k_lk_l^{/}+k^{/^2})k_k\beta_{iik}-Rfk_k+2k_k(\varepsilon_{mli}\Omega_m+\varepsilon_{nli}\Omega_n+\varepsilon_{qlk}\Omega_q)\beta_{iik}$$

$$=\frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[a]_1}{(t-t_1)^{\frac{3}{2}}}\exp[\{-2\nu(\frac{3}{4}k^2+\frac{1}{2}k_lk_l^{/}+\frac{3}{4}k^{/^2})+2(\varepsilon_{mnl}\Omega_m+\varepsilon_{pml}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)]$$

$$+\frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[b]_1}{(t-t_1)^{\frac{3}{2}}}\exp[\{2\nu(\frac{3}{4}k^2+k_lk_l^{/}+k^{/^2})+2(\varepsilon_{nml}\Omega_n+\varepsilon_{pml}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)]$$

$$+\frac{\pi^{\frac{3}{2}}}{(2\nu)^{\frac{3}{2}}}\frac{[c]_1}{(t-t_1)^{\frac{3}{2}}}\exp[\{2\nu(k^2+k_lk_l^{/}+\frac{3}{4}k^{/^2})+2(\varepsilon_{nml}\Omega_n+\varepsilon_{pml}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)] \tag{4.4.3}$$

where the bracketed quantities in equation (4.4.2) have been abbreviated as shown.

Integration of equation (4.4.3) with respect to time, results in

$$k_k\beta_{iik} = (k_k\beta_{iik})_0 \exp[\{-2\nu(k^2 + k_lk'_l + k'^2) + 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nli}\Omega_n + \varepsilon_{qlk}\Omega_q) - Rf\}(t - t_0)]$$

$$+\frac{\pi^{\frac{3}{2}}[a]_1}{\nu}\left\{\omega^{-1}\exp\left[-\omega^2\left(\frac{3}{4}k^2 + \frac{1}{2}k_lk'_l + \frac{3}{4}k'^2\right) + \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rmi}\Omega_r) - Sf\}(t - t_1)\right]\right.$$

$$+2\left(\frac{1}{4}k^2 + \frac{1}{2}k_lk'_l + \frac{1}{4}k'^2\right)^{\frac{1}{2}}\exp[-\omega^2(k^2 + k_lk'_l + k'^2) + \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rmi}\Omega_r) - Sf\}(t - t_1)]$$

$$\times\int_0^{\left(\frac{1}{4}k^2 + \frac{1}{2}k_lk'_l + \frac{1}{2}k'^2\right)^{\frac{1}{2}}}\exp(x^2)dx\Big\} + \frac{\pi^{\frac{3}{2}}[b]_1}{\nu}\{-\omega^{-1}\exp[-\omega^2(\frac{3}{4}k^2 + k_lk'_l + k'^2)$$

$$+\{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qlk}\Omega_q + \varepsilon_{rmi}\Omega_r) - Sf\}(t - t_1)]$$

$$+k\exp[-\omega^2(k^2 + k_lk'_l + k'^2) + \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rmi}\Omega_r) - Sf\}(t - t_1)]\int_0^{\frac{1}{2}\omega k}\exp(x^2)dx\}$$

$$+\frac{\pi^{\frac{3}{2}}[c]_1}{\nu}\{-\omega^{-1}\exp[-\omega^2(k^2 + k_lk'_l + \frac{3}{4}k'^2) + \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rmi}\Omega_r) - Sf\}(t - t_1)]$$

$$+k'\exp[-\omega^2(k^2 + k_lk'_l + k'^2) + \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rmi}\Omega_r) - Sf\}(t - t_1)]\int_0^{\frac{1}{2}\omega k'}\exp(x^2)dx\} \quad (4.4.4)$$

where $\omega = [2\nu(t - t_1)]^{\frac{1}{2}}$

In order to simplify the calculations, we shall assume that $[a]_1 = 0$; that is, we assume that a function sufficiently general to represent the initial conditions can be obtained by considering only the terms involving $[b]_1$ and $[c]_1$.

The substitution of equations (4.3.44) and (4.4.4) in equation (4.3.41) and setting $E = 2\pi k^2\phi_{i,i}$ results in

$$\frac{dE}{dt} + (2\nu k^2 + 2\varepsilon_{mki}\Omega_m + 2\varepsilon_{nki}\Omega_n - Qf)E = W \quad (4.4.5)$$

where

$$W = k^2 \int_{-\infty}^{\infty} 2\pi i[k_k \beta_{iik}(k,k') - k_k \beta_{iik}(-k,-k')]_0 \exp\Big[\{-2\nu(k^2 + k_l k_l' + k'^2)$$

$$+ 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nli}\Omega_n + \varepsilon_{qlk}\Omega_q) - Rf\}(t - t_0)\Big] dk'$$

$$+ k^2 \int_{-\infty}^{\infty} \frac{2\pi^2 i}{\nu}[b(k,k') - b(-k,-k')]_1 \{-w^{-1} \exp\Big[-w^2(\frac{3}{4}k^2 + k_l k_l' + k'^2)$$

$$+ \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - Sf\}(t - t_1)\Big]$$

$$+ k \exp\Big[-w^2(k^2 + k_l k_l' + k'^2) + \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - Sf\}(t - t_1)\Big]$$

$$\times \int_0^{\frac{1}{2}wk} \exp(x^2)dx\}dk' + k^2 \int_{-\infty}^{\infty} \frac{2\pi^2 i}{\nu}[c(k,k') - c(-k,-k')]_1 \{-w^{-1} \exp\Big[-w^2(k^2 + k_l k_l' + \frac{3}{4}k'^2)$$

$$+ \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - Sf\}(t - t_1)\Big] + k' \exp[-w^2(k^2 + k_l k_l' + k'^2)$$

$$+ \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmi}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - Sf\}(t - t_1)] \int_0^{\frac{1}{2}wk'} \exp(x^2)dx\}dk' \qquad (4.4.6)$$

The quantity E is the energy spectrum function, which represents contributions from various wave numbers or eddy sizes to the total energy. W is the energy transfer function, which is responsible for the transfer of energy between wave numbers.

In order to find the solution completely and following Deissler [13], we assume that

$$(2\pi)^2 i\left[k_k \beta_{iik}(\underline{k},\underline{k'}) - k_k \beta_{iik}(-\underline{k},-\underline{k'})\right]_0 = -\beta_0(k^4 k'^6 - k^6 k'^4) \qquad (4.4.7)$$

For the bracketed quantities in equation (4.4.6), we let

$$\frac{4\pi^2}{\nu^{\frac{7}{2}}} i\left[b(\underline{k},\underline{k'}) - b(-\underline{k},-\underline{k'})\right]_1 = \frac{4\pi^2}{\nu^{\frac{7}{2}}} i\left[c(\underline{k},\underline{k'}) - c(-\underline{k},-\underline{k'})\right]_1 = -2\gamma_1(k^6 k'^8 - k^8 k'^6) \qquad (4.4.8)$$

where the bracketed quantities are set equal in order to make the integrands in equation (4.4.6) antisymmetric with respect to k and k'.

By substituting equations (4.4.7) and (4.4.8) in equation (4.4.6) remembering that

$d\underline{k}' = 2\pi k'^2 d(\cos\theta)dk'$ and $k_l k_l' = kk'\cos\theta$, (θ is the angle between k and k'), and

carrying out the integration with respect to θ ,we get

$$
\begin{aligned}
W = \int_0^\infty \Bigg[& \frac{\beta_0(k^4k'^6 - k^6k'^4)kk'}{2\nu(t-t_0)} .\{\exp[\{-2\nu(k^2+kk'+k'^2)+2(\varepsilon_{mli}\Omega_m+\varepsilon_{nlj}\Omega_n+\varepsilon_{qlk}\Omega_q)-Rf\}(t-t_0)] \\
& -\exp[\{-2\nu(k^2-kk'+k'^2)+2(\varepsilon_{mli}\Omega_m+\varepsilon_{nlj}\Omega_n+\varepsilon_{qlk}\Omega_q)-Rf\}(t-t_0)]\}-\gamma_1\frac{(k^6k'^8-k^8k'^6)kk'}{\nu(t-t_1)} \\
& \times\Bigg((\omega^{-1}\exp[-\omega^2(\frac{3}{4}k^2+kk'+k'^2)+\{2(\varepsilon_{nmi}\Omega_m+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)] \\
& -\omega^{-1}\exp[-\omega^2(\frac{3}{4}k^2-kk'+k'^2)+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)] \\
& +\omega^{-1}\exp[-\omega^2(k^2+kk'+\frac{3}{4}k'^2)+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)] \\
& -\omega^{-1}\exp[-\omega^2(k^2-kk'+\frac{3}{4}k'^2)+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)] \\
& +\{k\exp[-\omega^2(k^2-kk'+k'^2)+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)] \\
& -k\exp[-\omega^2(k^2+kk'+k'^2)+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)]\}\int_0^{\frac{1}{2}\omega k}\exp(x^2)dx \\
& +\{k'\exp[-\omega^2(k^2-kk'+k'^2)+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)] \\
& -k'\exp[-\omega^2(k^2+kk'+k'^2)+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)]\}.\int_0^{\frac{1}{2}\omega k'}\exp(x^2)dx\Bigg)\Bigg]d\underline{k}'
\end{aligned}
\tag{4.4.9}
$$

where $\omega = [2\nu(t-t_1)]^{\frac{1}{2}}$

The integrand in this equation represents the contribution to the energy transfer at a wave number k, from a wave number k'. The integral is the total contribution to W at k, from all wave numbers. Carrying out the indicated integration with respect to k' in equation (4.4.9), where results in

$$W = W_\beta + W\gamma \tag{4.4.10}$$

where

$$W_\beta = -\frac{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\beta_0}{256\nu^{\frac{15}{2}}(t-t_0)^{\frac{15}{2}}}$$

$$\times \exp\left[(-\frac{3}{2}\varepsilon^2)(105\varepsilon^6+45\varepsilon^8-19\varepsilon^{10}-3\varepsilon^{12})+\{2(\varepsilon_{mli}\Omega_m+\varepsilon_{nlj}\Omega_n+\varepsilon_{qlk}\Omega_q)-Rf\}(t-t_0)\right] \tag{4.4.11}$$

and

$$W_\gamma = -\frac{\gamma_1}{\nu^{10}(t-t_1)^{10}}\left[\frac{\pi^{\frac{1}{2}}}{16}\exp\left[\left(-\eta^2\right)\left(\frac{3}{128}\eta^{16}+\frac{3}{8}\eta^{14}+\frac{21}{64}\eta^{12}-\frac{105}{16}\eta^{10}-\frac{945}{128}\eta^8\right)\right.\right.$$

$$+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)]$$

$$+\frac{2\pi^{\frac{1}{2}}}{\sqrt{3}}\exp\left[\left(-\frac{4}{3}\eta^2\right)\left(\frac{160}{19683}\eta^{16}+\frac{40}{729}\eta^{14}-\frac{14}{27}\eta^{12}-\frac{455}{162}\eta^{10}-\frac{35}{18}\eta^8\right)\right.$$

$$+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)]$$

$$-\frac{\left(\frac{\pi}{2}\right)^{\frac{1}{2}}}{16}\exp\left[\left(-\frac{3}{2}\eta^2\right)\int_0^{\frac{\eta}{\sqrt{2}}}\exp(y^2)dy\left(\frac{3}{64}\eta^{17}+\frac{3}{4}\eta^{15}+\frac{21}{32}\eta^{13}-\frac{105}{8}\eta^{11}-\frac{945}{32}\eta^9\right)\right.$$

$$+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)]$$

$$+\frac{\pi^{\frac{1}{2}}}{2}\exp\left[\left\{(-\frac{3}{2}\eta^2)(5.386\eta^8+9.118\eta^{10}+3.1017\eta^{12}+0.1793\eta^{14}\right.\right.$$

$$-0.03106\eta^{16}-0.004942\eta^{18}-3.615\times10^{-4}\eta^{20}-1.890\times10^{-5}\eta^{22}$$

$$-7.561\times10^{-7}\eta^{24}-2.447\times10^{-8}\eta^{26}-6.64\times10^{-10}\eta^{28}-1.55\times10^{-11}\eta^{30}\text{.........})$$

$$+\{2(\varepsilon_{nmi}\Omega_n+\varepsilon_{pmj}\Omega_p+\varepsilon_{qmk}\Omega_q+\varepsilon_{rml}\Omega_r)-Sf\}(t-t_1)]] \tag{4.4.12}$$

where $\eta = \nu^{\frac{1}{2}}(t-t_1)^{\frac{1}{2}}k$ and $\varepsilon = \nu^{\frac{1}{2}}(t-t_0)^{\frac{1}{2}}k$.

The quantity W_β is the contribution to the energy transfer arising from consideration of the three-point correlation equation; W_γ arises from consideration of the four-point equation. Integration of equation (4.4.10) over all wave numbers shows that

$$\int_0^\infty W dk = 0 \qquad (4.4.13)$$

indicating that the expression for W satisfies the conditions of continuity and homogeneity.

In order to obtain the energy spectrum function E, we integrate equation (4.4.5) with respect to time. This integration results in

$$E = E_j + E_\beta + E_\gamma \qquad (4.4.14)$$

where

$$E_j = \frac{J_0 \varepsilon^4}{3\pi v^2 (t-t_0)^2} \exp[-2\varepsilon^2 + \{2(\varepsilon_{mki}\Omega_m + \varepsilon_{nkj}\Omega_n) - Qf\}(t-t_0)] \qquad (4.4.15)$$

$$E_\beta = \frac{(2\pi)^{\frac{1}{2}} \beta_0}{256 v^{\frac{15}{2}} (t-t_0)^{\frac{13}{2}}} \exp[(-\tfrac{3}{2}\varepsilon^2)(-15\varepsilon^6 - 12\varepsilon^8 + \frac{7}{3}\varepsilon^{10} + \frac{16}{3}\varepsilon^{12}$$

$$-\frac{32}{3\sqrt{2}}\varepsilon^{13} \exp(-\frac{\varepsilon^2}{2}) \int_0^{\frac{\varepsilon}{\sqrt{2}}} \exp(y^2)dy) + \{2(\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q) - Rf\}(t-t_0)] \qquad (4.4.16)$$

$$E_\gamma = -\frac{\gamma_1}{\nu^{10}(t-t_1)^9}\{\frac{\pi^{\frac{1}{2}}}{32}\exp[(-\eta^2)\left(\frac{189}{64}\eta^8 + \frac{1029}{256}\eta^{10} + \frac{287}{256}\eta^{12} + \frac{95}{512}\eta^{14}\right.$$

$$\left. + \frac{71}{512}\eta^{16} - \frac{71}{512}\eta^{18}\exp(-\eta^2)[Ei(\eta^2) - 0.5772]\right)$$

$$+\{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - Sf\}(t-t_1)]$$

$$+\left(\frac{\pi}{3}\right)^{\frac{1}{2}}\exp[(-\frac{4}{3}\eta^2)(\frac{7}{9}\eta^8 + \frac{497}{324}\eta^{10} + \frac{1001}{1458}\eta^{12} + \frac{761}{4374}\eta^{14}$$

$$\left. + \frac{1963}{19683}\eta^{16} - \frac{3926}{59049}\eta^{18}\exp(-\frac{2}{3}\eta^2)[Ei(\frac{2}{3}\eta^2) - 0.5772]\right)$$

$$+\{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - Sf\}(t-t_1)]$$

$$+\frac{\pi^{\frac{1}{2}}}{2}\exp[(-\frac{3}{2}\eta^2)\left(0.2307\eta^{10} + 0.3632\eta^{12} + 0.1502\eta^{14} + 0.04463\eta^{16}\right.$$

$$-0.01326\eta^{18}\exp(-\frac{1}{2}\eta^2)[Ei(\frac{1}{2}\eta^2) - 0.5772] + 2.459\times10^{-3}\eta^{18} + 2.935\times10^{-4}\eta^{20}$$

$$+2.846\times10^{-5}\eta^{22} + 2.52\times10^{-6}\eta^{24} + 1.69\times10^{-7}\eta^{26} + 1.25\times10^{-8}\eta^{28} + 5.80\times10^{-10}\eta^{30}$$

$$\left. +4.00\times10^{-11}\eta^{32}\ldots\ldots\ldots\ldots\right) + \{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - Sf\}(t-t_1)]$$

$$+\frac{1}{2}\pi^{\frac{1}{2}}\exp[(-\frac{3}{2}\eta^2)\left(1.077\eta^8 + 2.414\eta^{10} + 1.408\eta^{12} + 0.4416\eta^{14} + 0.1898\eta^{16}\right.$$

$$-0.0899\eta^{18}\exp(-\frac{1}{2}\eta^2)[Ei(\frac{1}{2}\eta^2) - 0.5772] + 6.575\times10^{-4}\eta^{18} + 3.271\times10^{-5}\eta^{20}$$

$$\left. +1.270\times10^{-6}\eta^{22} + 4.03\times10^{-8}\eta^{24} + 1.08\times10^{-9}\eta^{26} + 2.50\times10^{-11}\eta^{28} + 5.09\times10^{-13}\eta^{30}\cdots\right)$$

$$+\{2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r) - Sf\}(t-t_1)]\} \qquad (4.4.17)$$

The quantity E_j is the energy spectrum function for the final period, where E_β and E_γ are the contributions to the energy spectrum arising from consideration of the three and four point correlation equations respectively.

Equation (4.4.14) can be integrated over all wave numbers to give the total turbulent energy

$$\frac{1}{2}\langle u_i u_i \rangle = \int_0^{\infty} E dk \tag{4.4.18}$$

The result carrying out the integration is, in dimensionless form

$$\frac{\langle u_i u_i \rangle}{2} = \frac{j_0^{\frac{14}{9}} \nu^{\frac{5}{9}}}{\beta_0^{\frac{5}{9}}} \left[\frac{1}{32(2\pi)^{\frac{1}{2}}} T^{-\frac{5}{2}} \exp[\{Qf - 2(\varepsilon_{mki}\Omega_m + \varepsilon_{nkj}\Omega_n)\}(t - t_0)] \right.$$
$$+ 0.2296T^{-7} \exp[\{Rf - 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q)\}(t - t_0)]$$
$$\left. + 6.18 \frac{\gamma_1 \nu^{\frac{5}{9}} J_0^{\frac{5}{9}}}{\beta_0^{\frac{14}{9}}} \left(\frac{t - t_1}{t - t_0} \right)^{-\frac{19}{2}} T^{-\frac{19}{2}} \exp[\{Sf - 2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)\}(t - t_1)] \right] \tag{4.4.19}$$

$$\langle u^2 \rangle = AT^{-\frac{5}{2}} \exp[\{Qf - 2(\varepsilon_{mki}\Omega_m + \varepsilon_{nkj}\Omega_n)\}(t - t_0)]$$
$$+ BT^{-7} \exp\left[\{Rf - 2(\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q)\}(t - t_0)\right]$$
$$+ CT^{-\frac{19}{2}} \left(\frac{t - t_1}{t - t_0} \right)^{-\frac{19}{2}} \exp[\{Sf - 2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rml}\Omega_r)\}(t - t_1)] \tag{4.4.20}$$

where
$$\frac{t - t_1}{t - t_0} = 1 - \left(\frac{\gamma_1 \nu^{\frac{5}{9}} J_0^{\frac{5}{9}}}{\beta_0^{\frac{14}{9}}} \right)^{1/9} \left[\frac{(t_1 - t_0) \nu^{\frac{94}{81}} J_0^{\frac{13}{81}}}{\beta_0^{\frac{4}{81}} \gamma_1^{\frac{1}{9}}} \right] \frac{1}{T} \tag{4.4.21}$$

$$T = \frac{\nu^{\frac{11}{9}} J_0^{\frac{2}{9}} (t - t_0)}{\beta_0^{\frac{2}{9}}} \tag{4.4.22}$$

and A, B, C are arbitrary constants.

4.5: Concluding Remarks

From the above discussion we achieved the decay law of dusty fluid turbulence in a rotating system before the final period considering three and four-point correlation equations after neglecting quintuple correlation terms in equation (4.4.19). This equation shows that turbulent energy decays more rapidly in an exponential manner than the energy decay for non-rotating clean fluid. This decay law contains a term $T^{-\frac{19}{2}}$, as well as the terms $T^{-\frac{5}{2}}$ and T^{-7} along with exponential terms that also contains rotational terms in presence of dust particles. Thus the terms associated with the higher order correlations die out faster than those associated with the lower order ones. The factor $\frac{(t-t_1)}{(t-t_0)}$ occurring in the last term in equation (4.4.19) will cause that term to decay even faster, so long as $t_1 - t_0 > 0$.

If the system is non-rotating, we put Ω's = 0, the equation (4.4.20) becomes

$$\langle u^2 \rangle = AT^{-\frac{5}{2}} \exp\{Qf(t-t_0)\} + BT^{-7} \exp\{Rf(t-t_0)\} + C\left(\frac{t-t_1}{t-t_0}\right)^{-\frac{19}{2}} T^{-\frac{19}{2}} \exp\{Sf(t-t_1)\} \qquad (4.5.1)$$

which is same as in Chapter II.

Again if the fluid is clean, we put f = 0, then equation (4.4.20) becomes

$$\langle u^2 \rangle = AT^{-\frac{5}{2}} \exp[-2(\varepsilon_{mki}\Omega_m + \varepsilon_{nkj}\Omega_n)](t-t_0) + BT^{-7} \exp[-2(\varepsilon_{mli}\Omega_m + \varepsilon_{nlj}\Omega_n + \varepsilon_{qlk}\Omega_q)](t-t_0)$$
$$+ CT^{-\frac{19}{2}} \left(\frac{t-t_1}{t-t_0}\right)^{-\frac{19}{2}} \exp[-2(\varepsilon_{nmi}\Omega_n + \varepsilon_{pmj}\Omega_p + \varepsilon_{qmk}\Omega_q + \varepsilon_{rnl}\Omega_r)](t-t_1)$$

which is generalized in Chapter III.

If the system is non-rotating and the fluid is clean (Ω's, f = 0), the equation (4.4.20) becomes

$$\langle u^2 \rangle = AT^{-\frac{5}{2}} + BT^{-7} + C\left(\frac{t - t_1}{t - t_0}\right)^{-\frac{19}{2}} T^{-\frac{19}{2}} \tag{4.5.2}$$

which is obtained earlier by Deissler[13].

It appears more terms in higher power of T that would be added to equation (4.4.20) using higher order correlations in the analysis.

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