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# Massive Particle Tunneling from Black Hole Spacetime

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# Massive Particle Tunneling from Black Hole Spacetime

THESIS SUBMITTED FOR THE DEGREE OF
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# Certificate From The Supervisor

This is to certify that the PhD thesis entitled "**Massive Particle Tunneling from Black Hole Spacetime"** which is being submitted by Md. Ilias Hossain, Assistant Professor, Department of Mathematics, University of Rajshahi as a fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** in **Applied Mathematics**, University of Rajshahi, Rajshahi-6205, Bangladesh. This is a record of bonafide research work done by him under my supervision. I believe that the results embodied in the thesis are new and it has not been submitted elsewhere for any degree or any other academic award anywhere before.

To the best of my knowledge Md. Ilias Hossain bears a good moral character and is mentally and physically fit to get the degree. I wish him a bright future and every success in his life.

**(Dr. M. Atiqur Rahman)**
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# Statement of Originality

I declare that the contents in my PhD thesis entitled “**Massive Particle Tunneling from Black Hole Spacetime”** is original and accurate to the best of my Knowledge and I am the sole author of this thesis. I also certify that the materials contained in my research work have not been previously published or written by others for a degree or diploma.

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# Dissertation copyright Authorization



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*DEDICATED TO MY BELOVED FATHER & MOTHER*

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# ABSTRACT

We investigate the Hawking radiation from different kind of black holes by massive particle tunneling process near the event horizon of the black hole in de Sitter and anti-de Sitter spaces. We calculate the imaginary part of the action from the relativistic Hamilton-Jacobi equation avoid by exploring the equation of motion of the radiation particle in Painleavé coordinate system in order to explore the Hawking non-thermal and purely thermal radiations.

The thesis is organized as follows:

In **chapter 1** we give a brief discussion about our work of studying of massive particle tunneling from black hole spacetime.

In **chapter 2** we review the relativistic Hamilton-Jacobi equation to perform our prime work.

In **chapter 3 to 10** we investigate the Hawking non-thermal and purely thermal radiations using massive particles tunneling process by employing Hamilton-Jacobi method for **Schwarzschild-de Sitter (SdS), Schwarzschild-anti-de Sitter (SAdS), Reissner-Nordström-de Sitter (RNdS), Reissner-Nordström-anti-de Sitter (RNAdS), Kerr-de Sitter (KdS), Kerr-anti-de Sitter (KAdS), Kerr-Newman-de Sitter (KNdS) and Kerr-Newman-anti-de Sitter (KNAdS)** black holes. We express the position of all kind of black holes in an infinite series in terms of black hole parameters so that the spacetime metric becomes dynamical and derive the new line elements. Taking into account the energy conservation, the angular momentum conservation and the unfixed background spacetime. When self-gravitation interaction is considered, the derived emission/radiation spectrums are not purely thermal and the tunneling rates are related to the change of the Bekenstein-Hawking entropy, which satisfy an underlying unitary theory. Our new process provides an interesting correction to the Hawking pure thermal radiation of the black hole and in the limiting case, the results are accordant with that obtained by Parikh and Wilczek's method of the black hole.

# Contents

# Notations, Conventions and Acronyms

- Greek indices μ, ν, ... at tensors cycle the numbers 0 to 3 and Latin indices *i*, *j*, ... cycle only spatial coordinates from 1 to 3. The temporal index is denoted by *t* and number 0.
- M or *m* → Mass of black hole
- $r_i$ → location of the event horizon before the particles emission
- $r_f$ → location of the event horizon after the particles emission
- ADM → Arnowitt, Deser, and Misner
- AdS → Anti-de Sitter
- CFT → Conformal Field Theory
- dS → de Sitter
- KdS → Kerr-de Sitter
- KAdS → Kerr-anti-de Sitter
- RN → Reissner-Nordström
- RNdS → Reissner-Nordström-de Sitter
- RNAdS → Reissner-Nordström-anti-de Sitter
- SdS → Schwarzschild-de Sitter
- SAdS → Schwarzschild-anti-de Sitter
- WKB → Wentzel–Kramers–Brillouin
- KN → Kerr-Newman
- KNdS → Kerr-Newman-de Sitter
- KNAdS → Kerr-Newman-anti-de Sitter
- HJE → Hamilton-Jacobi Equation

# Publications

(During the period of my PhD study)

Part of the contents of this thesis has already appeared in the following Papers:

## Referred Journals:

[1] **Rahman M. A. and Hossain M. I.**, "Hawking radiation of Schwarzschild-de Sitter black hole by Hamilton-Jacobi method". Journal of Physics Letters B **(Elsevier, Impact factor: 4.569, 2012). V**ol. **712**, P.1-5 (2012); [arXiv: gr-qc/1205.1216].

[2] **Rahman M. A. and Hossain M. I.**, "Hawking non-thermal and thermal radiations of Schwarzschild-anti-de Sitter black hole by Hamilton-Jacobi method". Journal of Astrophysics and Space Science **(Springer, Impact factor: 2.064, 2012). V**ol. **345,** P.325-330 (2013); [arXiv: gr-qc/1205.1390].

[3] **Hossain M. I. and Rahman M. A.**, "Hawking non-thermal and thermal radiations of Reissner-Nordström-anti-de Sitter black hole by Hamilton-Jacobi method'. Journal of Astrophysics and Space Science **(Springer, Impact factor: 2.064, 2012). V**ol. **347,** P.91-97 (2013).

## In preparation:

[1] **Hossain M. I. and Rahman M. A.**, "Hawking radiation of Reissner-Nordström-de Sitter black hole by Hamilton-Jacobi method". [arXiv: gr-qc/1309.0067].

[2] **Hossain M. I and Rahman M. A.**, "Hawking non-thermal and Purely thermal radiations of Kerr-de Sitter black hole by Hamilton-Jacobi method". [arXiv: gr-qc/1309.0505].

# Chapter 1

# Introduction

General relativity, which was published by Albert Einstein (1879-1955) in 1915 [1], was almost as epochal as Newton's theory. It is sometimes maintained that general relativity is difficult to understand. If so, the problem is not that the theory itself would be conflicting or complicated. On the contrary, it may be considered as one of the most beautiful theories ever developed. The problem is that general relativity forces us to change our classical conceptions of time and space in a very radical manner. Nevertheless, these changes are necessary if one want to achieve a deeper comprehension of Nature. In general relativity space and time are no longer separated but together constitute a four-dimensional continuum called spacetime. Einstein's ingenious idea was that matter interacts with spacetime in such a way that spacetime becomes curved. This interaction between matter and spacetime is described by Einstein's field equation. Furthermore, the paths of objects are determined by the geometry of spacetime which can be applied to spacetime of any shape: Objects with free fall velocity move along geodesics, i.e., routes of stationary length between spacetime points. Hence, matter tells spacetime how to curve, whereas the geometry of spacetime tells matter how to move. In this sense, gravitation may be considered as a manifestation of the curvature of spacetime. Thus General relativity describes the effects of curved or accelerated motion and of gravitational fields on mass, size, and time. It also states that matter and empty space influence each other in a complex fashion and that the Universe is finite in size. In classical general relativity, spacetime is considered as a curved of four dimensional manifold, whose shape is defined by Einstein's field equations. The Einstein equations are unavoidably involved in any matter where the geometry of spacetime is of consequence. One of the most successful and useful applications of Einstein's General Theory of Relativity is within the field of cosmology and it also part of the framework of the standard Big Bang model of cosmology.

The cosmological constant was first introduced into the equations of general relativity by Einstein himself, who later famously criticized this move as his 'greatest blunder'. In his paper of 1917 [2] he found the first cosmological solution of a consistent theory of gravity. In spite of its drawbacks this bold step can be regarded as the beginning of modern cosmology. The relevance of the cosmological constant in modern gravitational physics is manifest, and it is interesting to focus on the solutions of Einstein's field equations with cosmological constant, to investigate its role on different scales. For instance, the Schwarzschild-de Sitter metric, which describes a point-like mass in a spacetime with a cosmological constant, has been recently studied in [3, 4, 5, 6]. In particular, the Schwarzschild-de Sitter metric has been considered to investigate the influence of the cosmological constant on gravitational lensing in [7, 8, 9, 10]. The cosmological constant, conventionally denoted by the Greek letter $\Lambda$, is a parameter describing the energy density of the vacuum (empty space), and a potentially important contributor to the dynamical history of the universe. The value of $\Lambda$ in our present universe is not known, and may be zero, although there is some evidence for a nonzero value; a precise determination of this number will be one of the primary goals of observational cosmology in the near future. In a universe with both matter and vacuum energy, there is a competition between the tendency of $\Lambda$ to cause acceleration and the tendency of matter to cause deceleration, with the ultimate fate of the universe depending on the precise amounts of each component. To a good approximation, the cosmological constant more precisely, the conventionally defined cosmological constant $\Lambda$ is proportional to the vacuum energy density $\rho_\Lambda$ ; they are related by $\Lambda = \frac{8\pi G}{3c^2}\rho_\Lambda$, where G is Newton's constant of gravitation and $c$ is the speed of light. It was not until years after Einstein introduced $\Lambda$ as a parameter in cosmology that it was realized that the same parameter measured the energy density of the vacuum.

General relativity has developed into an essential tool in modern astrophysics and provides the foundation for the current understanding of black holes. According to general relativity, a sufficiently compact mass will deform spacetime to form a black hole. Black holes are very subtle and mysterious objects in this universe. It can be defined as: **"black holes are regions of space where the gravitational effects are so strong that even light cannot escape from those regions"**. The existence of such regions was proposed for the first time by Michell and Laplace already in the late 18th century (and probably independently of each other) [11, 12]. Their arguments, however, were based on Newton's theory of gravitation. General relativity also predicts the existence of black holes, and the first black hole solution to Einstein's field equation was found by Schwarzschild in 1916 [13]. At first, black holes were thought to be only theoretical curiosities which would not exist in Nature. However, through the works of Chandrasekhar, Oppenheimer, Volkoff, and Snyder it is also quite clear that black holes are born, in some situations, as the final states of stars [14, 15]. Therefore, one may indeed expect that there exist black holes in our universe. Classically, black holes do not emit any type of radiations and are perfect absorbers.

The topic of black hole thermodynamics has been a subject of interest since the 1970's when Bekenstein first conjectured that there was a fundamental relationship between the properties of black holes and the laws of thermodynamics [16] and is very important in this regard. It is impossible to define a temperature for black holes because, everything goes into the black hole and as a result of this, there is no any output. If this is the case, the second law of thermodynamics would be contradicted due to entry of matter having its own entropy, into the black hole which results the decrease of the total entropy of the universe and violates the second law of thermodynamics. In 1972, again Bekenstein showed that black holes possess entropy similar to its surface area, whose increase overcomes the decrease of the exterior entropy such that the second law of thermodynamics is preserved. He also related the surface gravity, which is the gravitational acceleration experienced at the surface of the black hole or any object, with temperature of the body in thermal equilibrium. Possibly, black holes are the most perfectly thermal objects in the universe, and yet their thermal properties are not fully understood. They are described very accurately by a small number of macroscopic parameters (e.g., mass, angular momentum, and charge), but the microscopic degrees of freedom that lead to their thermal behavior have not yet been adequately identified. Strong hints of the thermal properties of black holes came from the behavior of their macroscopic properties that were formalized in the (classical) four laws of black hole mechanics [17], which have analogues in the corresponding four laws of thermodynamics:

**The zeroth law of black hole mechanics** is that "the surface gravity($\kappa$) is constant over the horizon (event) [17, 18] for a stationary black hole". This is analogous to the zeroth law of thermodynamics which states that "the temperature $T$ is constant throughout a body in thermal equilibrium". So in this sense the surface gravity is analogous to the temperature.

**The first law of black hole mechanics** is that "the mass of the black hole changes in terms of its area, angular momentum, and electric charge". These are all related by the equation:
$dM = \frac{\kappa}{8\pi} dA + \Omega\delta J + \Phi Q,$
where $M$ is the mass of the black hole, $\kappa$ is the surface gravity, $A$ is the area of the horizon, $\Omega$ is the angular velocity, $J$ is the angular momentum of the black hole, $Q$ is the electric charge, and $\Phi$ is the electrostatic potential ($\Phi = Q/r$ for a point charge). This is analogous to the first law of thermodynamics: $dE = TdS+$ work terms,

where $E$ is the energy, $T$ is the temperature, and $S$ is the entropy. Notice that a change in mass is a change in energy (i.e. $E = Mc^2$ and one using $c = 1$ units so $E = M$). The terms $\Omega\delta J + \Phi Q$ are work terms. This implies that $\frac{\kappa}{8\pi}dA$ is analogous to $TdS$. So the temperature is analogous to surface gravity and black hole area is analogous to the entropy i.e. this first law is essentially the same as the first law of black hole mechanics.(This law expresses the conservation of energy i.e., the first law of thermodynamics is a statement of energy conservation.)

**The second law of black hole mechanics** is Hawking's area theorem [19], that "the area $A$ of a black hole horizon cannot decrease by any (classical) process (i.e $dA \geq 0$)". This is obviously analogous to the second law of thermodynamics which is the fact that the entropy $S$ of a closed system (or the universe) cannot decrease ($dS \geq 0$). So once again area is seen to be analogous to entropy.

**The third law of black hole mechanics** is that "the surface gravity $\kappa$ cannot be reduced to zero by any finite sequence of operations [20]". This is analogous to the weaker (Nernst) form of the third law of thermodynamics, that "the temperature $T$ of a system cannot be reduced to absolute zero in a finite number of operations". However, the classical third law of black hole mechanics is not analogous to the stronger (Planck) form of the third law of thermodynamics, that the entropy of a system goes to zero when the temperature goes to zero. The only black holes that have zero surface gravity are extremal black holes.

Thus the four laws of black hole mechanics are analogous to the four laws of thermodynamics if one makes an analogy between temperature $T$

and some multiple of the black hole surface gravity $\kappa$ and between entropy $S$ and some inversely corresponding multiple of the black hole area $A$. That is, one might say that $T = \epsilon\kappa$ and $S = \eta A$, with $8\pi\epsilon\eta = 1$ , so that the $\kappa dA/(8\pi)$ term in the first law of black hole mechanics becomes the heat transfer term $TdS$ in the first law of thermodynamics.

Even before the formulation of the four laws of black hole mechanics, Bekenstein [16, 21, 22, 23] proposed that a black hole has an entropy $S$ that is some finite multiple $\eta$ of its area $A$. He was not able to determine the exact value of $\eta$, but he gave heuristic arguments for conjecturing that it was $(ln2)/(8\pi)$ (in Planck units, $\hbar = c = G = \kappa = 4\pi\epsilon_0 = 1$).

However, for the first law of black hole mechanics to be equivalent to the first law of thermodynamics, this would logically imply that the black hole would have to have a temperature $T$ that is a corresponding nonzero multiple of the surface gravity $\kappa$. E.g., if $\eta = (ln2)/(8\pi)$ as Bekenstein proposed, then one would get $\epsilon = 1/(ln2)$, so that $T = \kappa/(ln2)$. But since it was thought then that black holes can only absorb and never emit, it seemed that black holes really would have zero temperature, or $\epsilon = 0$, which would make Bekenstein's proposal inconsistent with any finite $\eta$ [17].

A wonderful discovery [24, 25] by Hawking in 1974 that black holes can radiate thermally reconciled a serious contradiction among General Relativity, Quantum Mechanics and Thermodynamics at that time and put the first law of black hole thermodynamics on a solid fundament. At a big cost, however, this discovery also caused another controversial problem: what happen to information during the black hole evaporation? In the classical theory, the loss of information was not a serious problem since the information could be thought of as preserved inside the black hole but just not very accessible. However, taking the quantum effect into consideration, the situation is changed. With the emission of thermal radiation [24, 25], black holes could lose energy, shrink, and eventually evaporate away completely. Since the radiation with a precise thermal spectrum carries no information, the information carried by a physical system falling toward black hole singularity has no way to be recovered after a black hole has disappeared completely. This problem is now generally known as "the paradox of black hole information loss" [26, 27], which means that pure quantum states (the original matter that forms the black hole) can evolve into mixed states (the thermal spectrum at infinity). This directly violates the principle of unitarity for quantum dynamics of an isolated system and brings a serious challenge to the foundations of modern physics. In the past decades, several methods [28, 29, 30, 31] have been suggested for resolving the "information loss paradox"; none has been successful. In fact, each failed attempt for a resolution seems to have made the existence of this paradox more serious and attracted more interest, especially after the possibility that information about infallen matter may hide inside the correlations between the Hawking radiation and the internal states of a black hole was ruled out. While the information paradox can perhaps be attributed to the semi-classical nature of the investigations of Hawking radiation [32], researches in string theory indeed support the idea that Hawking radiation can be described within a manifestly unitary theory, however, it still remains a mystery how information is recovered. Although a complete resolution of the information loss paradox might be within a unitary theory of quantum gravity or string/M-theory, it is argued that information could come out if the emitted radiations were not exactly thermal but instead the radiation spectrum contains a subtle non-thermal correction [33]. On the other hand, the mechanism of black hole radiance remains shrouded in some degree of mystery. In the original derivation of black hole evaporation, Hawking described the thermal radiation as a quantum tunneling process [34] triggered by vacuum fluctuations near the event horizon. According to this scenario, a pair of particles is spontaneously generated inside the horizon. The positive energy particle tunnels out to the infinity while the negative energy one remains in the black hole. Alternatively, the positive and negative energy pair is created outside the horizon, and the negative energy particle tunnels into the black hole because its orbit exists only inside the horizon, while the positive energy one remains outside and emerges at infinity.

Indeed, the above viewpoint that regards the radiation as quantum tunneling out from inside the black hole has been proved very convenient to explore the issue of dynamics. But, actual derivation [35] of Hawking radiation did not proceed in this way at all, most of which based upon quantum field theory on a fixed background spacetime without considering the fluctuation of the spacetime geometry. Moreover, there is another fundamental issue that must necessarily be dealt with, namely, the energy conservation. It seems clear that the background geometry of a radiating black hole should be altered with the loss of energy, but this dynamical effect is often neglected in formal treatments. Due to this breakthrough in the field of black hole physics, many research works on the thermal radiation of black holes have been made [36, 37, 38, 39, 40]. This procedure provides a leading correction to the tunneling probability (emission rate) arising from the reduction of the black hole mass because of the energy carried by the emitted massless or massive quanta.

Wilczek and his collaborators have developed two universal methods to correctly recover Hawking radiation of black holes. One is the gravitational anomaly method [41] in which the Hawking radiation can be determined by anomaly canceled conditions and regularity requirement at the event horizon. Later on, this method is widely used to calculate the Hawking radiation for different black holes [42, 43, 44, 45, 46, 47, 48, 49, 50].

The another is the semi-classical tunneling method developed by Parikh and Wilczek [51] presented a greatly simplified model (based upon the previous introduced by Kraus and Wilczek [52, 53, 54]) to implement the Hawking radiation as a semi-classical tunneling process from the event horizon of the four-dimensional Schwarzschild and Reissner-Nordström black holes by treating the background geometries as dynamical and incorporating the self-gravitation correction of the radiation. The radiant spectra that they derived under the consideration of energy conservation give a leading-order correction to the emission rate arising from the loss of mass of black holes, which corresponds to the energy carried by the radiated quanta. Their result shows that the actual emission spectrum of black hole radiation deviates from strictly pure thermality, which might serve as a potential mechanism to resolve the information loss paradox. Since the semi-classical tunneling method has been successfully applied to deal with Hawking radiation of black holes, a lot of work shows its validity [38, 50, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81]. But most of them are focus on studying Hawking radiation of scalar particles tunneling from different-type black holes.

Based on semi-classical tunneling picture two new methods have been employed to calculate the imaginary part of the action, one the null geodesic method developed by Parikh and Wilczek [51, 82, 83, 84] and another method proposed by Srinivasan and Padmanabhan [85, 86, 87, 88, 89]. The later method then extended by Angheben et. al [90] and successfully presented to derive the imaginary part of the action by solving the Hamilton-Jacobi equation, which is, later called as the 'Hamilton-Jacobi method' [85, 86, 87, 88, 89]. The difference of later method from Parikh's is mainly that such method concentrates on introducing the proper spatial distance and upon calculating the relativistic Hamilton-Jacobi equation. The latter method also involves consideration of a emitted scalar particle, ignoring its self-gravitation and assumes that its action satisfies the relativistic Hamilton-Jacobi equation. An appropriate ansatz for the action can be obtained from the symmetries of the spacetime which is known as the Hamilton-Jacobi ansatz. Both the methods show that when the self-gravitational interaction and the unfixed background spacetime are taken into account, the actual Hawking radiation spectrum deviates from the purely thermal one, satisfies the underlying unitary theory and gives a leading correction to the radiation spectrum. Based on the Hamilton-Jacobi method, Banerjee and Majhi [91] developed the tunneling method beyond semi-classical approximation to include quantum corrections and many researches have been made to calculate quantum corrections of black hole entropy [92, 93, 94, 95, 96, 97]. In 2005, Zhang and Zhao have first proposed the Hawking radiation from massive uncharged particle tunneling [98] and charged particle tunneling [43, 44, 46, 81, 99, 100, 101]. All the results supported Parikh's opinion and gave a correction to the Hawking pure thermal spectrum. Exploiting this work, a few researches have been carried out as charged particle tunneling [77, 102, 103, 104, 105]. Kerner and Mann [106, 107, 108] extended Kraus and Wilczek's [53, 54] work and also developed quantum tunneling methods for analyzing the temperature of Taub-NUT black holes [109] using both the null-geodesic and Hamilton-Jacobi methods by ignoring the self-gravitation interaction and energy conservation of emitted particle. This method is also applied to higher dimensional black holes [110, 111, 112], black holes in String theory [113], black strings [114, 115, 116, 117], accelerating and rotating black holes [118, 119, 120], dilation black holes [121, 122], BTZ black holes [123], black holes with NUT parameter [109, 124] and Kerr-Newman black hole [125]. Taking the self-gravitation interaction and unfixed background spacetime into account Chen, Zu and Yang reformed Hamilton-Jacobi method for massive particle tunneling and investigate the Hawking radiation of the Taub-NUT black hole [126]. Using this method Hawking radiation of Kerr-NUT black hole [65], Kerr-de Sitter black hole [127], the charged black hole with a global monopole [99, 128] have been reviewed. In fact, a black hole can radiate all types of particles charged, massless or massive.

Recently, we have developed a new Hamilton-Jacobi method by reformulating the method of Chen et al. [125, 126], for massive particle tunneling and investigate the Hawking radiation of black holes with cosmological constant (SdS black hole [129], SAdS black hole [130], RNAdS black hole [131]) by considering the self-gravitation interaction and unfixed background spacetime. In general relativity, different black holes are characterized by mass $M$, charge $Q$ and rotation $a$ parameter. The fourth parameter is cosmological parameter $\Lambda$, which is taken to be constant since otherwise the calculations would be too complex to solve analytically. To proceed analytically, we have solved the position of the black hole as a series of infinite terms so that the spacetime metric becomes dynamical. By taking self-gravitational effect and energy conservation into account we have shown that the tunneling rate is related to the change of Bekenstein-Hawking entropy and the emission spectrum deviates from the precisely thermal one which in accordance with Parikh and Wilczek's opinion [51, 82, 83] and gives another method to study the Hawking radiation of black hole with cosmological constant. In de Sitter/anti-de Sitter spaces, very little work have been investigated either for massless/charged particle or massive particle tunneling from black hole due to tough calculation. So our present research on black holes with cosmological constant is important and meaningful.

In recent years, considerable attention has been concentrated on the study of black holes in de Sitter (dS) and anti-de Sitter (AdS) spaces. The motivation behind it is based on two aspects: first, the recent observed accelerating expansion [132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152] of our universe indicates the cosmological constant might be a positive one [132, 153, 154]; Secondly, like the AdS/CFT correspondence [26, 155, 156], an interesting proposal, the so-called dS/CFT correspondence, has been suggested that there is a dual relation between quantum gravity on a dS space and Euclidean conformal field theory (CFT) on a boundary of dS space [157, 158, 159]. The solutions of black holes in Anti-de Sitter spaces come from the Einstein equations with a negative cosmological constant. Anti-de Sitter black holes are different from de Sitter black holes. The difference consisting in them is due to minimum temperatures that occur when their sizes are of the order of the characteristic radius of the anti-de Sitter space. For larger Anti-de Sitter black holes, their red-shifted temperatures measured at infinity are greater. This implies that such black holes can be in stable equilibrium with thermal radiation at a certain temperature. Anti-de Sitter (AdS) geometry has been considered as a challenging field for quantum field theory in different frameworks, including supersymmetry and string theory. The string /M-theory have also greatly stimulated the study of black hole solutions in AdS space. So our study on different kinds of black holes in de Sitter and anti-de Sitter spaces are meaningful and significant.

The outline of this thesis is the following: the **second chapter** is a review work since it contains a review of work done by others in addition to extension relativistic Hamilton-Jacobi equation that we have done ( i.e devoted to recall with the well-known relativistic Hamilton-Jacobi equation). The remaining chapters will consist entirely of original calculations partly we have done in the papers [129, 130, 131].

In the **third chapter**, we have investigated the Hawking radiation of Schwarzschild-de Sitter (SdS) black hole [129] by massive particles tunneling method. Here we have expressed the radius of the SdS black hole in terms of mass and cosmological parameter in an infinite series and the new line element near the event horizon is derived which are totally new ideas of this research. Using Hamilton-Jacobi method, we consider the spacetime background to be dynamical, incorporate the self-gravitation effect of the emitted particles and show that the tunneling rate is related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum when energy and angular momentum are conserved. Our result is in accordance with Parikh and Wilczek's opinion [51, 82, 83] and gives a correction to the Hawking radiation of the SdS black hole.

In the **fourth chapter**, the massive particles tunneling method has been used to explore the Hawking non-thermal and purely thermal radiations of Schwarzschild-anti-de Sitter (SAdS) black hole [130]. Using the same view of chapter three we have shown that the non-thermal and purely thermal tunneling rates are related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum [51, 82, 83].

In the **fifth chapter**, we have investigated the Hawking purely thermal and non-thermal radiations of Reissner-Nordström-de Sitter (RNdS) black hole [160] by including charge parameter to SdS black hole [129]. Considering the same assumption of SdS black hole [129] we have shown that the tunneling rate is related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum. The result is in accordance with Parikh and Wilczek's opinion [51, 82, 83] and recovered the new result for Hawking radiation of RNdS black hole.

In the **sixth chapter**, we have generalized our work given in chapter 4 (SAdS black hole [130]) with charge parameter and introduce the Hawking non-thermal and purely thermal radiations of Reissner-Nordström-anti-de Sitter (RNAdS) black hole [131] by massive particles tunneling method. Like SAdS black hole here we have also shown that the non-thermal and purely thermal tunneling rates are related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum. The results for the RNAdS black hole is also in the same manner with Parikh and Wilczek's opinion [51, 82, 83] and explored the new result for Hawking radiation of RNAdS black hole.

In the **seventh chapter**, we have revised the work given in chapter 3 (SdS black hole [129]) with rotating parameter and investigate the Hawking non-thermal and purely thermal radiations of Kerr-de Sitter (KdS) black hole [161] using Hamilton-Jacobi method. The dragging coordinates transformation have been used to derive the new line element near the event horizon. Taking self-gravitation effect into account we have shown that the tunneling rate is related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum [51, 82, 83]. The explored results gives a correction to the Hawking radiation of KdS black hole.

In the **eighth chapter**, using the same opinion as chapter 4 (SAdS black hole [130]) we have explored Hawking non-thermal and purely thermal radiations of Kerr-anti-de Sitter (KAdS) black hole by including rotating parameter. Applying the dragging coordinate transformation and taking self-gravitational effect into consideration we have shown that the tunneling rates are related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum and also the obtaining results for the KAdS black hole is in accordance with Parikh and Wilczek's opinion [51, 82, 83] and gives a correction to the Hawking radiation of the KAdS black hole.

In the **ninth chapter**, we have generalized the work given in chapter 7 with charge parameter and derived Hawking non-thermal and purely thermal tunneling rates employing Hamilton-Jacobi method. Here, as KdS black hole we have shown that the tunneling rate of Kerr-Newman-de Sitter (KNdS) black hole is related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum, which is full consistent with Ref. [51, 82, 83].

In the **tenth chapter**, we have investigated the Hawking non-thermal and purely thermal tunneling rates of the Kerr-Newman-anti-de Sitter (KNAdS) black hole which is the Kerr-anti-de Sitter black hole [81] generalized with a charge parameter. As KAdS black hole here we have also shown that the tunneling rate is related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum [51, 82, 83], and gives a correction to the Hawking radiation of the KNAdS black hole.

Finally, in **chapter eleven** we give a brief description of the results of our prime work from chapter three to chapter ten .

# Chapter 2

# A Review of the Elementary Formulation of the Relativistic Hamilton-Jacobi equation

## 2.1 Introduction

The relativistic Hamilton-Jacobi equation (HJE) is a necessary condition describing extremal geometry in generalizations of problems from the calculus of variations. It is named for William Rowan Hamilton and Carl Gustav Jacob Jacobi. The Hamilton-Jacobi equation is particularly useful in identifying conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself cannot be solved completely. The resultant Hamilton-Jacobi theory and later developments are presented in several famous texts: Arnol'd (1974), Landau and Lifshitz (1969), Gantmacher (1970), Born and Wolf (1965), Lanczos (1949), Carathodory (1982), Courant and Hilbert (1962).

The equations of motion for a relativistic massive particle moving in an electromagnetic field written in a form of the second law of Newton. Which can be reduced with the help of elementary operations to the relativistic

Hamilton-Jacobi equation. In turn, the action $I$ obeys the Hamilton-Jacobi equation. The latter is a partial differential equation of the first order. A transition from Newton's second law to the Hamilton-Jacobi equation can be achieved with the help of the algorithm for transforming a system of ordinary differential equations into a partial differential equation. Although the fact that such transformation algorithm is well-known (e.g., [162]) the actual transformation of the equations of motion of a charged relativistic particle in the electromagnetic field into a respective PDE (the Hamilton-Jacobi equation) is not quoted in the physical literature to the best of our knowledge. The usual approach to the problem of derivation of the Relativistic Hamilton-Jacobi equation is to heuristically introduce classical action I and to vary it (for fixed initial and final times). The formulation is based on a possibility of transforming the equation of motion to a completely antisymmetric form.

In the next section, at once time we obtain the principle of least action and taking into account it we derive the relativistic Hamilton-Jacobi equation.

## 2.2 Formulation of the Relativistic HJ equation

By keeping in mind the momentum as a function of both temporal and spatial coordinates, we provide an elementary derivation of the Hamilton-Jacobi where the concept of action emerges in a natural way. This can be imagined by considering first a non-relativistic classical particle moving from one point $A$(say) to another point $B$(say). The particle can do that by taking any possible paths connecting these two points. Therefore for any fixed moment of time, say $t = 1$ the momentum would depend on the spatial coordinate, that is $\vec{p} = \vec{p}(\vec{x}, t)$. In a sense one have replaced watching the particle evolution in time by watching the evolution of its velocity (momentum) in space and time and this situation is in accordance to the Euler's description of motion of a fluid (an alternative to the Lagrange description). The other way, one consider a "flow" of an "elemental" path and describe its "motion" in terms of its coordinates and velocity (determined by a slope of the path at a given point). This permits us to represent Newton's second law for a particle (mass $m$) moving in a conservative field $U(\vec{x})$ as follows

$$\frac{d\vec{p}}{dt} = \frac{\partial\vec{p}}{\partial t} + \frac{1}{m}(\vec{p}\cdot\nabla)\vec{p} = -\nabla U. \tag{2.1}$$

Taking curl on both sides to the Eq.(2.1), we obtain

$$\nabla \times \frac{d\vec{p}}{dt} = \frac{\partial}{\partial t}(\nabla \times \vec{p}) + \frac{1}{m}\nabla \times (\vec{p}\cdot\nabla)\vec{p} = 0. \tag{2.2}$$

Using the vector formula

$$(\vec{a}\cdot\nabla)\vec{a} = \frac{\nabla a^2}{2} + (\nabla \times \vec{a}) \times \vec{a}. \tag{2.3}$$

Eq.(2.2) becomes

$$\frac{\partial}{\partial t}(\nabla \times \vec{p}) + \frac{1}{m}\nabla \times \{(\nabla \times \vec{p}) \times \vec{p}\} = 0. \tag{2.4}$$

The trial solution of Eq.(2.4) is

$$\nabla \times \vec{p} = 0 \tag{2.5}$$

similar to an irrotational motion in Euler's picture of a fluid motion. From Eq.(2.5), one must get

$$\vec{p} = \nabla I, \tag{2.6}$$

where $I(\bar{x}, t)$ is some scalar function. Generally speaking, one can choose the negative value of $\nabla I$. The conventional choice is connected with the fact that the corresponding value of the kinetic energy has to be positive. Inserting Eq.(2.6) into Eq.(2.1) and with the help of Eq.(2.3) one obtain the following equation

$$\nabla\left\{\frac{\partial I}{\partial t} + \frac{1}{2m}(\nabla I)^2 + U\right\} = 0. \tag{2.7}$$

In turn Eq.(2.7) means that

$$\frac{\partial I}{\partial t} + \frac{1}{2m}(\nabla I)^2 + U = f(t), \tag{2.8}$$

where $f(t)$ is some function of time. Defining a new function $I' = I - \int f(t)dt$ one get from Eq.(2.7) the Hamilton-Jacobi equation with respect to the function $I'$ (representing the classical action):

$$\frac{\partial I'}{\partial t} + \frac{1}{2m}(\nabla I')^2 + U = f(t), \tag{2.9}$$

Using the relation $\vec{p} = m\vec{v}$ in Eq.(2.6) and drop the prime at $I'$, the Hamilton-Jacobi equation can be rewritten as follows

$$\frac{\partial I}{\partial t} + \vec{v} \cdot \nabla I = \frac{mv^2}{2} - U. \tag{2.10}$$

Now using $\frac{dI}{dt} = \frac{\partial I}{\partial t} + \vec{v} \cdot \nabla I$, the expression for the action $I$ by integrating Eq.(2.10) from the point $A$ to $B$

$$I = \int_{t_A}^{t_B} \left(\frac{mv^2}{2} - U\right) dt \equiv \int_{t_A}^{t_B} L(\vec{x}, \vec{v}, t)dt, \tag{2.11}$$

where $L(\vec{x}, \vec{v}, t) = \frac{mv^2}{2} - U$ is the lagrangian of a particle of mass $m$.

Now we can arrive at the principle of least action (without postulating it a priori) directly from the Hamilton-Jacobi equation. To this end one subject the action $I$ to small perturbations $\delta I \ll I$ and (by dropping the term $(\nabla \delta I)^2$) get from Eq.(2.9) the equation with respect to $\delta I$

$$\frac{\partial \delta I}{\partial t} + \frac{1}{m}(\nabla I) \cdot (\nabla \delta I) = 0. \tag{2.12}$$

Since $\frac{\nabla I}{m} = \vec{v}$ Eq.(2.12) represents the substantial derivative of $\delta I$, so

$$\frac{d\delta I}{dt} = 0. \tag{2.13}$$

Integrating we get

$$\delta I = constant. \tag{2.14}$$

Thus for a specific function $I$ satisfying the Hamilton-Jacobi equation the respective perturbations $\delta I = constant$. On the other hand, according to Eq.(2.11) the action $I$ is defined on a set of all possible paths connecting point $A$ and point $B$. This means that perturbations $\delta I$ correspond to perturbations of all these path.

After all one of these paths $\delta I = constant$, according to Eq.(2.14). In order to determine this constant one consider into account that at the fixed points $A$ and $B$ the paths are also fixed, that is the respective perturbations $\delta I = 0$ at these points. Therefore only for the specific path determined by the Hamilton-Jacobi equation that is by the second law of Newton $\delta I = 0$, thus yielding the principle of least action:

$$\delta \int_{t_A}^{t_B} L(\vec{x}, \vec{v}, t) = 0. \tag{2.15}$$

The formulation given by Eq.(2.15) serves as a guide for a derivation of the relativistic Hamilton-Jacobi equation for a (relativistic) massive particle of charge $q$ and mass $m$ moving in the electromagnetic field. Our approach is to reduce the respective equations of motion to the form which would be analogous to an irrotational motion in Euler's picture. The very structure of the spacetime metric allows one to arrive at the required result in a natural way.

Therefore, one start with the second law of Newton for a relativistic charged particle of a charge $q$ and mass $m$ moving in the electromagnetic field:

$$\frac{dp^{\alpha}}{dt} = q[E^{\alpha} + \epsilon^{\alpha\beta\gamma}v^{\beta}B^{\gamma}], \tag{2.16}$$

where Greek indices $\alpha, \beta, \gamma...$ take the values 1, 2, 3, $\epsilon^{\alpha\beta\gamma}$ is the absolutely antisymmetric tensor of the third rank, $p^{\alpha} = \frac{mv^{\alpha}}{(1-v^{\delta}v^{\delta})^{1/2}}$ is the momentum of the particle, $E^{\alpha}$ is the electric field , $v^{\alpha} = \vec{v}$ is the velocity of the particle and $B^{\alpha}$ is the magnetic field.

For the subsequent analysis one cast Eq.(2.16) into the standard co - and contra-variant forms and to this end one use the metric $g^{ik} = g_{ik} = [1, -1, -1, -1]$ and use units where the speed of light is $c = 1$. In this metric $x^0 = x_0 = t$, $x^{\alpha} = \vec{x} = -x_{\alpha}$, the four- potential $A^i(A^0, A^{\alpha})$ whose scalar part $A^0 = \phi$ (where $\phi$ is the scalar potential) and $A^{\alpha} \equiv \vec{A}$ is the vector potential, and the roman indices $i, j, k, ...$take the values $0, 1, 2, 3$. From the Maxwell equations then follows (e.g.[163]) that the electric field $E^{\alpha}$ intensity and the magnetic induction $B^{\alpha}$ are

$$E^{\alpha} = -\left(\frac{\partial A^0}{\partial x^{\alpha}} + \frac{\partial A^{\alpha}}{\partial x^0}\right) \tag{2.17}$$

$$B^{\alpha} = \epsilon^{\alpha\beta\gamma}\frac{\partial A^{\gamma}}{\partial x^{\beta}}. \tag{2.18}$$

In terms of the vector-potential $A^\alpha \equiv \vec{A}$, using Eq.(2.18) one express the second term on the right-hand side of Eq.(2.16)

$$\epsilon^{\alpha\beta\gamma}\upsilon^\beta B^\gamma = \epsilon^{\alpha\beta\gamma}\epsilon^{\gamma\delta\lambda}\frac{\partial A^\lambda}{\partial x^\delta} = \upsilon^\beta\left(\frac{\partial A^\beta}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\beta}\right). \tag{2.19}$$

Inserting Eq.(2.18) and Eq.(2.19) into Eq.(2.16) yields

$$\frac{dp^\alpha}{dx^0} = q\Big[-\left(\frac{\partial A^0}{\partial x^\alpha} + \frac{\partial A^\alpha}{\partial x^0}\right) + \beta^\gamma\left(\frac{\partial A^\gamma}{\partial x^\alpha} - \frac{\partial A^\alpha}{\partial x^\gamma}\right)\Big], \tag{2.20}$$

where $\beta^\gamma = \upsilon^\gamma$. In refs.[163], one use in Eq.(2.20) the antisymmetric tensor $F^{ik}$ such that

$$F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k} \tag{2.21}$$

the relation between contra-($A^\alpha$) and co-variant ($A_\alpha$) vectors($A^\alpha = -A_\alpha$), introduce the spacetime interval

$$ds \equiv dt\sqrt{(1-\beta^\alpha\beta^\alpha)} \equiv dt\sqrt{(1-\beta^2)}$$

and the four-velocity

$$u^i(u^0 = 1 = \sqrt{(1-\beta^2)}, u^\alpha = -u_\alpha = \frac{\beta^\alpha}{\sqrt{(1-\beta^2)}}.$$

we get

$$\frac{dp^\alpha}{ds} = qF^{\alpha k}u_k = -qF^{k\alpha}u_k. \tag{2.22}$$

The next step is to find the zeroth components of Eq.(2.22). Using the special relativistic identity for the momentum $p_i = mu_i$, $p_ip^i = m^2$ one find

$$p_0\frac{dp^0}{ds} \equiv -p_\alpha\frac{dp^\alpha}{ds} = p^\alpha\frac{dp^\alpha}{ds}. \tag{2.23}$$

Using Eq.(2.23) into Eq.(2.22) one obtain

$$p^{\alpha}\frac{dp^{\alpha}}{ds} = qp^{\alpha}[F^{\alpha\beta}u_{\beta} + F^{00}u_0]. \tag{2.24}$$

On the other hand, since $F^{ik} = -F^{ki}(F^{00} = F_{00} = 0)$.

$$p^{\alpha}u_{\beta}F^{\alpha\beta} = 0.$$

Hence from Eq.(2.23) and Eq.(2.24) follows that

$$\frac{dp^0}{ds} = qu^0F^{\alpha 0} = qF^{0\alpha}u_{\alpha} = qF^{0i}u_i. \tag{2.25}$$

Adding Eq.(2.25) and Eq.(2.22) and using the definition of $F^{ik}$, Eq.(2.21), one arrive at the equation of motion in the contra-variant form:

$$\frac{dp^i}{ds} = qF^{ik}u_k = q\left(\frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}\right)u_k. \tag{2.26}$$

The respective co-variant form follows from raising and lowering indices in Eq.(2.26):

$$\frac{dp_i}{ds} = qF_{ik}u^k = q\left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right)u^k. \tag{2.27}$$

Reducing these equations to a form similar to the condition defining an irrotational flow in fluid mechanics and for one rewrite (2.26) and (2.27) in the following form

$$\begin{aligned} u_k\Big[\frac{\partial}{\partial x_k}(mu^i + qA^i) - \frac{\partial}{\partial x_i}(qA^k)\Big] &= 0 \\ u^k\Big[\frac{\partial}{\partial x^k}(mu_i + qA_i) - \frac{\partial}{\partial x^i}(qA_k)\Big] &= 0 \end{aligned} \tag{2.28}$$

and add to the third term the identity

$$u_k\frac{\partial u^k}{\partial x_i} = u^k\frac{\partial u_k}{\partial x^i} \equiv \frac{1}{2}\frac{\partial}{\partial x_i}(u_ku^k) = 0.$$

Therefore, one get

$$u_k\Big[\frac{\partial}{\partial x_k}(mu^i + qA^i) - \frac{\partial}{\partial x_i}(mu^k + qA^k)\Big] = 0 \tag{2.29}$$

or equivalently

$$u^k\Big[\frac{\partial}{\partial x^k}(mu_i + qA_i) - \frac{\partial}{\partial x^i}(mu_k + qA_k)\Big] = 0. \tag{2.30}$$

The expressions $mu_i + qA_i$ or $(mu^i + qA^i)$ in square brackets of the above equations represent a four-curl of the four-vector. Both equations are identically satisfied if this four-curl is 0. Once again, this can be interpreted as the fact that the respective vector field is irrotational, that is the four-vector $m\vec{u} + q\vec{A}$ (here we use notation $\vec{a}$ for a four-vector) is the four-gradient of a scalar function, say $-I$

$$mu^i + qA^i = -\frac{\partial I}{\partial x_i} \tag{2.31}$$

$$mu_i + qA_i = -\frac{\partial I}{\partial x^i}. \tag{2.32}$$

This scalar function $I$ ( a potential function) is the classical relativistic action, and choice of the sign is dictated by the consideration that expressions Eq.(2.31) must become the expressions for the momentum and energy in the non-relativistic limit. To find the explicit expression for $I$ one integrate Eq.(2.31) [ or (2.32)] and obtain

$$I = -\int_a^b (mu^i + qA^i)dx_i \equiv= -\int_a^b (m + A^i u_i)ds, \tag{2.33}$$

where $a$ and $b$ are points on the world line of the particle, $ds = (dx^i dx_i)^{1/2}$, and $u_i = \frac{dx_i}{ds}$. Here the expression Eq.(2.33) coincides with the conventional definition of the action (introduced on the basis of considerations not connected to the second law of Newton). It is interesting to note that in a conventional approach to the action, the term $A^i dx_i$ "cannot be fixed on the basis of general considerations alone" [163]. Here however this term is "fixed" by the very nature of the equations of motion.

Eqs.(2.31)and (2.32) produce the determining PDE for the function $I$ (the relativistic Hamilton-Jacobi equation for a massive charged particle in the electromagnetic field) if we eliminate $u_i$ and $u^i$ from this equations with the help of the identity $u_i u^i = 1$:

$$\left(\frac{\partial I}{\partial x_i} + qA^i\right)\left(\frac{\partial I}{\partial x^i} + qA_i\right) = m^2, i = 0, 1, 2, 3 \tag{2.34}$$

where one have to retain ( in the classical region) only one sign, either plus or minus.

Following the well-known procedure of reducing the integration of the partial differential equation of the first order to the integration of a system of the respective ordinary differential equations [162]. In particular, given the Hamilton-Jacobi equation (2.34) one derive (2.26). To this end one subject action $I$ to small perturbations $\delta I$

$$I = I_0 + \delta I \tag{2.35}$$

and find the equation governing these perturbations. Here $I_0$ must satisfy the original unperturbed Hamilton-Jacobi equation (2.34), and $\delta I \ll I_0$. Using (35) into (34) one get with accuracy to the first order in $\delta I$

$$\left(\frac{\partial I_0}{\partial x_i} + qA^i\right)\frac{\partial}{\partial x^i}(\delta I) + \left(\frac{\partial I}{\partial x^i} + qA_i\right)\frac{\partial}{\partial x_i}(\delta I) = 0 \tag{2.36}$$

or equivalently

$$\left(\frac{\partial I}{\partial x^i} + qA_i\right)\frac{\partial}{\partial x_i}(\delta I) = 0. \tag{2.37}$$

Equation (37) is a quasi-linear first-order PDE whose characteristics are given by the following equations

$$\frac{dx_0}{\partial I_0/\partial x^0 + qA_0} = \frac{dx^\alpha}{\partial I_0/\partial x_\alpha + qA^\alpha}. \tag{2.38}$$

Here the repeated indices do not represent summation, and $\alpha = 1, 2, 3$. It is immediately seen that the characteristics of linearized Hamilton-Jacobi equation (2.38) are the four- velocity $u^i$:

$$u^i = \frac{1}{m}\left(\frac{\partial I_0}{\partial x_i} + qA^i\right). \tag{2.39}$$

Inversely, these characteristics are the solutions of the equations of motion written in a form of the second law of Newton. To demonstrate that one divide both sides of (2.39) by $ds$, use Eqs. (2.31), (2.32) and the fact that $\frac{d}{ds} = u_k \frac{\partial}{\partial x_k}$ and obtain

$$\begin{aligned} &mc\frac{du^i}{ds}\frac{1}{m}\left(\frac{\partial I_0}{\partial x^k} + qA_k\right)\frac{\partial}{\partial x_k}\left(\frac{\partial I_0}{\partial x_i} + qA^i\right) \\ &\equiv \frac{1}{m}\left(\frac{\partial I_0}{\partial x^k} + qA_k\right)\left[\frac{\partial}{\partial x_k}\left(\frac{\partial I_0}{\partial x_i} + qA^i\right) + q\frac{\partial A^i}{\partial x_k} - q\frac{\partial A^k}{\partial x_i}\right] \\ &\equiv \frac{1}{m}\left(\frac{\partial I_0}{\partial x^k} + qA_k\right)\left[\frac{\partial}{\partial x_i}\left(\frac{\partial I_0}{\partial x_k} + qA^k\right) - q\left(\frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}\right)\right] \\ &= \frac{1}{2m}\frac{\partial}{\partial x_i}(u_k u^k) + \frac{1}{m}qu_k\left(\frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}\right) = qu_kF^{ik} \end{aligned} \tag{2.40}$$

that is the second law of Newton, Eq.(2.26). Now one return to the linearized equation (2.37) which one rewrite in the identical form

$$mu_i\frac{\partial}{\partial x_i}\delta I \equiv \frac{d}{ds}\delta I = 0. \tag{2.41}$$

Which implies that $\delta I = constant$ along a certain world line, singled out of a continuous set of possible world lines according to this condition. Without any loss of generality one can take the above $constant = 0$.

For a specific function $I$ satisfying the Hamilton-Jacobi equation the respective perturbations $\delta I = constant$. On the other hand, according to Eq. (2.33) the action $I$ is defined on a set of all possible world lines connecting world points $a$ and $b$. This means that perturbations $\delta I$ correspond to perturbations of all these world lines. However, only for one of these world lines $\delta I = constant$, according to (2.41). To determine this constant one take into account that at the fixed world points $a$ and $b$ the world lines are also fixed, that is the respective perturbations $\delta I = 0$ at these points. If one apply condition Eq.(2.41) to the action $I$, Eq. (2.31), the former would choose out of all possible world lines the only one satisfying that condition, that is one arrive at the classical principle of least action.

$$\delta \int_a^b (mu^i + qA^i) dx_i = 0. \tag{2.42}$$

At last, one demonstrate in an elementary fashion how the same technique of transforming the equations of motion in the Newtonian form to the Hamilton-Jacobi equation can be applied to a motion of a charged particle in general relativity. The equations of motion of a charged particle in gravitational and electromagnetic field are [164].

$$M(u^l \frac{\partial u^i}{\partial x^l} + \Gamma^i_{kl} u^k u^l) = q g^{im} F_{mk} u^k, \tag{2.43}$$

where

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

is the Ricci tensor.

The expression $\Gamma^i_{kl}u^k u^l$ is significantly simplified according to the following identities:

$$\Gamma^i_{kl}u^k u^l \equiv u^l \frac{1}{2}\Big[u^k\left(\frac{\partial g^{im}g_{mk}}{\partial x^l} - g_{mk}\frac{\partial g^{im}}{\partial x^l}\right) - u^k g_{lm}\frac{\partial g^{im}}{\partial x^k} - u^k g^{im}\frac{\partial g^{kl}}{\partial x^m}\Big]$$

$$\equiv -\frac{1}{2}\Big[u^l u_m \frac{\partial g^{im}}{\partial x^l} + u^l u^k\left(g_{lm}\frac{\partial g^{im}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x_i}\right)\Big]$$

$$\equiv -\frac{1}{2}\Big[2u^l u_m \frac{\partial g^{im}}{\partial x^l} + u^l u^k \frac{\partial g^{kl}}{\partial x_i}\Big] \equiv -u^l\left(\frac{\partial g^{im}u_m}{\partial x^l} - g^{im}\frac{\partial u_m}{\partial x^l}\right) - \frac{1}{2}u^l u^k \frac{\partial g_{kl}}{\partial x_i}$$

$$\equiv -u^l \frac{\partial u_i}{\partial x^l} + u^l g^{ik}\frac{\partial u_k}{\partial x^l} - \frac{1}{2}u^l\left(\frac{\partial g_{kl}u^k}{\partial x^i} - g_{kl}\frac{\partial u^k}{\partial x^i}\right)$$

$$\equiv -u^l \frac{\partial u_i}{\partial x^l} + u^l g^{ik}\frac{\partial u_k}{\partial x^l} - \frac{1}{2}\left(u^l\frac{\partial u_l}{\partial x^i} - u_l\frac{\partial u^l}{\partial x_i}\right)$$

$$\equiv -u^l \frac{\partial u_i}{\partial x^l} + u^l g^{ik}\frac{\partial u_k}{\partial x^l} - \frac{1}{2}\left(u^l\frac{\partial u_l}{\partial x_i} - \frac{\partial u^l u_l}{\partial x_i} + u^l\frac{\partial u_l}{\partial x_i}\right)$$

$$\equiv -u^l \frac{\partial u_i}{\partial x^l} + u^l g^{ik}\frac{\partial u_k}{\partial x^l} - u^l\frac{\partial u_l}{\partial x_i}$$

$$\equiv -u^l \frac{\partial u_i}{\partial x^l} + u^l g^{ik}\left(\frac{\partial u_k}{\partial x^l} - \frac{\partial u_l}{\partial x^k}\right).$$

Now inserting this result into (2.43) and use the expression (2.21) for $F^{ik}$, one obtain

$$g^{ik}u^l\Big[\frac{\partial}{\partial x^l}(Mu_k + qA_k) - \frac{\partial}{\partial x^k}(Mu_l + qA_l)\Big] = 0. \tag{2.44}$$

Equation (2.44) is identically satisfied if we set

$$Mu_k + qA_k = -\frac{\partial I}{\partial x^k}, \tag{2.45}$$

where $I$ is the action and the negative sign, representing a conventional choice of positive energies in classical mechanics. Raising and lowering the indices in (2.45), expressing the respective 4-velocities $u_k$ and $u^k$ in terms of $\frac{\partial I}{\partial x^k}$, and using the identity $g^{ik}u_i u_k = 1$, we arrive at the relativistic Hamilton-Jacobi equation:

$$g^{ik}\left(\frac{\partial I}{\partial x^i} + qA_i\right)\left(\frac{\partial I}{\partial x^k} + qA_k\right) = m^2. \tag{2.46}$$

Since it has to retain (in the classical region) only one sign either plus or minus in the case of relativistic mass, so as our motivation is to perform our prime work, we replace $-m^2$ in the place of $m^2$ and then Eq.(2.46) can be written as

$$g^{ik}\left(\frac{\partial I}{\partial x^i} + qA_i\right)\left(\frac{\partial I}{\partial x^k} + qA_k\right) + m^2 = 0. \tag{2.47}$$

Now if we taking into account the charge as fixed, then the electromagnetic potential $A_\mu$ can be neglected and therefore Eq.(2.47) takes on form as

$$g^{ik}\left(\frac{\partial I}{\partial x^i}\right)\left(\frac{\partial I}{\partial x^k}\right) + m^2 = 0, \tag{2.48}$$

which is the required relativistic Hamilton-Jacobi equation to perform our prime work.

# Chapter 3

## Hawking Non-thermal and Purely thermal radiations of Schwarzschild-de Sitter Black Hole by Hamilton-Jacobi method

### 3.1 Introduction

Hawking radiation is viewed as tunneling process caused by vacuum fluctuation near the event horizon of black hole [24, 25]. A method to describe Hawking radiation as tunneling process was first developed by Kraus and Wilczek [52, 53, 54] and then reinterpreted by Parikh and Wilczek [51] as quantum tunneling by considering a particle with negative energy just inside, a positive energy just outside the horizon which can be explained as a virtual particle pair spontaneously created near the horizon of black hole and materializes as a true particle. The particle with negative energy tunnels into the horizon and is absorbed, while the particle with positive energy left outside the horizon to infinite distance and forms the Hawking radiation.

From the past decade the tunneling method has been successfully applied to deal with Hawking radiation of black holes. A lot of works for various spacetimes [38, 50, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 165] show its validity and all of these are limited to massless particle. Based on the above tunneling picture, two different methods have been employed to calculate the imaginary part of the action, one by Parikh and Wilczek [51, 82, 83, 84] and other by Angheben et al. [90] named as null-geodesic and Hamilton-Jacobi methods respectively. In fact, the method of Angheben et al. [90] is an extension of the complex path analysis proposed by Padmanabhan et al. [85, 86, 87, 88, 89]. On the other hand, Hawking radiation from massive uncharged particle tunneling [98] and charged particle tunneling [100] were proposed by Zhang and Zhao. Following this work, few researches have been carried out as charged particle tunneling [77, 99, 101, 102, 103, 104, 105].

Recently, Kerner and Mann developed quantum tunneling methods for analyzing the temperature of Taub-NUT black holes [109] using both the null-geodesic and Hamilton-Jacobi methods. The latter method involve calculating the relativistic Hamilton-Jacobi equation in which the derive radiation spectrum was only a leading term due to the fact that the self-gravitation interaction and energy conservation of emitted particle were ignored. According to the Parikh and Wilczek's opinion the true radiation spectrum is not strictly thermal but satisfies the underlying unitary theory when self-gravitation interaction and energy conservation are considered. It is clear that the background geometry of a radiating black hole should be altered (unfixed) with the loss of energy. Taking the self-gravitation interaction and unfixed background spacetime into account Chen, Zu and

Yang reformed Hamilton-Jacobi method for massive particle tunneling and investigate the Hawking radiation of the Taub-NUT black hole [126]. Connecting this method Hawking radiation of Kerr-NUT black hole [65] and the charged black hole with a global monopole [99, 128] have been developed. We apply these method to investigate the Hawking radiation of Schwarzschild-de Sitter (SdS) black hole. Since our prime concern of this work is to calculate the imaginary part of action from Hamilton-Jacobi equation avoid by exploring the equation of motion of the radiation particle in Painlevé coordinate system and calculating the Hamilton equation. We need not differentiate radiation particle, although the equation of motion of massive particle is different from massless particle. After considering the self-gravitational interaction and the unfixed background spacetime, the derived radiation spectrum deviates from the purely thermal one and the tunneling rate is related to the change of Bekenstein-Hawking entropy.

Study of Hawking radiation on black holes with a positive cosmological constant become important due to the two reasons. One, the recent observed accelerating expansion of our universe indicates the cosmological constant might be a positive one [132, 153, 154], and conjecture about de Sitter/CFT correspondence [157, 166, 167]. For black hole with positive cosmological constant particles can be created at both black hole and cosmological horizon and there exists different tunneling behaviors. The outgoing and incoming particles tunnel from black hole and cosmological horizon respectively and formed Hawking radiation. For black hole horizon, the incoming particles can fall into the horizon along classically permitted trajectories but for cosmological horizon outgoing particles can fall classically out of the horizon. So our study of black hole in de Sitter space is important and meaningful.

This chapter is designed as follows: In the section 3.2 of this chapter we describe the SdS black hole spacetime with the position of event horizon and also near the event horizon the new line element of SdS black hole is derived here. Taking the unfixed background spacetime and the self-gravitational interaction into account, we review the Hawking radiation of SdS black hole from massive particle tunneling method in section 3.3. In section 3.4, the Hawking purely thermal radiation is developed and finally, in section 3.5, we present our remarks.

## 3.2 Schwarzschild-de Sitter black hole

The Schwarzschild-de Sitter black hole, which is the solution of Einstein equations with a positive $\Lambda(= 3/\ell^2)$ term corresponding to a vacuum state spherical symmetric configuration of the form

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= -\left(1 - \frac{2m}{r} - \frac{r^2}{\ell^2}\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{3.1}$$

where $m$ being the mass of the black hole and the coordinates are defined such that $-\infty \leq t \leq \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. At large $r$, the metric (3.1) tends to the dS space limit. The explicit dS case is obtained by setting $m = 0$ while the explicit Schwarzschild case is obtained by taking the limit $\ell \to \infty$. When $\ell^2$ is replaced by $-\ell^2$, the metric (3.1) describes an interesting nonrotating AdS black hole called the

Schwarzschild-Anti-de Sitter (SAdS) black hole.

The horizons of the SdS black hole are located at the real positive roots of $\frac{1}{\ell^2 r}(r - r_h)(r - r_c)(r_- - r) = 0$, and there are more than one horizon if $0 < \Xi < 1/27$ where $\Xi = M^2/\ell^2$. The black hole (event) horizon $r_h$ and the cosmological horizon $r_c$ are located, respectively, at

$$r_h = \frac{2m}{\sqrt{3\Xi}} \cos\frac{\pi + \psi}{3}, \tag{3.2}$$

$$r_c = \frac{2m}{\sqrt{3\Xi}} \cos\frac{\pi - \psi}{3}, \tag{3.3}$$

where

$$\psi = \cos^{-1}(3\sqrt{3\Xi}). \tag{3.4}$$

In the limit $\Xi \to 0$, one finds that $r_h \to 2m$ and $r_c \to \ell$, and it is obvious that $r_c > r_h$, i.e., the event horizon is the smallest positive root. The spacetime is dynamic for $r < r_h$ and $r > r_c$. The two horizons coincide: $r_h = r_c = 3m$ (extremal), when $\Xi = 1/27$, and the spacetime then becomes the well known Nariai spacetime. Expanding $r_h$ in terms of mass and cosmological parameter with $\Xi < 1/27$, we obtain

$$r_h = 2m\left(1 + \frac{4m^2}{\ell^2} + \cdots\right), \tag{3.5}$$

that is, the event horizon of the SdS black hole is greater than the Schwarzschild event horizon, $r_{Sch} = 2m$. For $\Xi > 1/27$, the spacetime is dynamic for all $r > 0$, that is, the metric (3.1) then represents not a black hole but an unphysical naked singularity at $r = 0$. For the convenient of discussion, we define $\Delta = r^2 - 2mr - \frac{r^4}{\ell^2}$ and then the line element becomes

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2(d\theta^2 + \sin^\theta d\phi^2). \tag{3.6}$$

The position of black hole horizon is same as given in Eq. (3.5). Near the black hole horizon, the line element takes of the form

$$ds^2 = -\frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}dt^2 + \frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}dr^2 + r_h^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.7)$$

where

$$\Delta_{,r}(r_h) = \frac{d\Delta}{dr}\Big|_{r=r_h} = 2(r_h - m - 2\frac{r_h^3}{\ell^2}). \quad (3.8)$$

Since the event horizon of SdS black hole coincides with the outer infinite redshift surface, here we can apply the geometrical optics limit. Within WKB approximation [168] the relationship between the tunneling rate and the action of the radiative particle is as

$$\Gamma \sim \exp(-2\mathrm{Im}I).$$

## 3.3 The Hamilton-Jacobi (HJ) Method

Here we used the method of Chen et al. [125, 126] to discuss the Hawking-Radiation from the action of radiation particles. As mention before this method is different from Parikh and Wilczek's method in which the action mainly relies on the exploration of the equation of motion in the Painlevé coordinates systems and the calculation of Hamilton equation. In the Hamilton-Jacobi method we avoid this and calculate the imaginary part of the action from the relativistic Hamilton-Jacobi equation.

The action $I$ of the outgoing particle from the black hole horizon satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\left(\frac{\partial I}{\partial x^\mu}\right)\left(\frac{\partial I}{\partial x^\nu}\right) + u^2 = 0, \quad (3.9)$$

in which $u$ and $g^{\mu\nu}$ are the mass of the particle and the inverse metric tensors derived from the line element (3.7).

For the metric (3.7), we get non-null inverse metric tensors

$$g^{00} = -\frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}, \qquad g^{11} = \frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2},$$
$$g^{22} = \frac{1}{r_h^2}, \qquad g^{33} = \frac{1}{r_h^2 \sin^2\theta}. \tag{3.10}$$

Using Eq. (3.10), we have from Eq. (3.9)

$$-\frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}\left(\frac{\partial I}{\partial t}\right)^2 + \frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}\left(\frac{\partial I}{\partial r}\right)^2$$
$$+\frac{1}{r_h^2}\left(\frac{\partial I}{\partial \theta}\right)^2 + \frac{1}{r_h^2 \sin^2\theta}\left(\frac{\partial I}{\partial \phi}\right)^2 + u^2 = 0. \tag{3.11}$$

It is very difficult to solve the action $I$ for $I(t, r, \theta, \phi)$. Considering the properties of black hole spacetime, the separation of variables can be taken as follows

$$I = -\omega t + R(r) + H(\theta) + j\phi, \tag{3.12}$$

where $\omega$ and $j$ are respectively the energy and angular momentum of the particle. Since SdS black hole is nonrotating, the angular velocity of the particle at the horizon is $\Omega_h = \left.\frac{d\phi}{dt}\right|_{r=r_h} = 0$. Using Eq. (3.12) into Eq. (3.11), we obtain

$$-\frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}(\omega)^2 + \frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}\left(\frac{\partial R(r)}{\partial r}\right)^2$$
$$+\frac{1}{r_h^2}\left(\frac{\partial H}{\partial \theta}\right)^2 + \frac{j^2}{r_h^2 \sin^2\theta} + u^2 = 0.$$

$$\Rightarrow \frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}\left(\frac{\partial R(r)}{\partial r}\right)^2 =$$

$$\frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}(\omega)^2-\frac{1}{r_h^2}\left(\frac{\partial H}{\partial\theta}\right)^2-\frac{j^2}{r_h^2\sin^2\theta}-u^2$$

$$\Rightarrow\frac{\partial R(r)}{\partial r}=\pm\frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}$$
$$\times\sqrt{\left\{\omega^2-\frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}\left[\frac{1}{r_h^2}\left(\frac{\partial H}{\partial\theta}\right)^2+\frac{j^2}{r_h^2\sin^2\theta}+u^2\right]\right\}}$$

Therefore, $R(r)$ yields

$$R(r)=\pm\frac{r_h^2}{\Delta_{,r}(r_h)}\int\frac{dr}{(r-r_h)}$$
$$\times\sqrt{\omega^2-\frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}[g^{22}(\partial_\theta H(\theta))^2+g^{33}j^2+u^2]}. \quad (3.13)$$

We consider the emitted particle as an ellipsoid shell of energy $\omega$ to tunnel across the event horizon and should not have motion in $\theta$-direction ($d\theta=0$) and therefore, finishing the above integral we get

$$R(r)=\pm\frac{\pi i r_h^2}{\Delta_{,r}(r_h)}\omega+\xi, \quad (3.14)$$

where $\pm$ sign comes from the square root and $\xi$ is the constant of integration. Inserting Eq. (3.14) into Eq. (3.12), the imaginary part of two different actions corresponding to the outgoing and incoming particles can be written as

$$\text{Im}I_\pm=\pm\frac{\pi r_h^2}{\Delta_{,r}(r_h)}\omega+\text{Im}(\xi). \quad (3.15)$$

In the classical limit [169], we ensure the incoming probability to be unity when there is no reflection i.e., every thing is absorbed by the horizon. In this situation the appropriate value of $\xi$ instead of zero or infinity can be taken as $\xi = \frac{\pi i r_h^2}{\Delta_{,r}(r_h)}\omega + \mathrm{Re}(\xi)$. Therefore, $\mathrm{Im}I_- = 0$ and $I_+$ give the imaginary part of action $I$ corresponding to the outgoing particle. Namely, we get

$$\begin{aligned} \mathrm{Im}I &= \frac{2\pi r_h^2}{\Delta_{,r}(r_h)}\omega \\ &= \frac{\pi r_h^2}{r_h - m - 2\frac{r_h^3}{\ell^2}}\omega. \end{aligned} \tag{3.16}$$

Using Eq. (3.5) into Eq. (3.16), we get the imaginary part of action as

$$\mathrm{Im}I = \frac{\pi 4m^2\left(1+\frac{4m^2}{\ell^2}+\cdots\right)^2}{2m\left(1+\frac{4m^2}{\ell^2}+\cdots\right) - m - \frac{2}{\ell^2}\{2m\left(1+\frac{4m^2}{\ell^2}+\cdots\right)\}^3}\omega. \tag{3.17}$$

Since the SdS spacetime is dynamic, we fix the Arnowitt-Deser-Misner (ADM) mass of the total spacetime and allow the SdS black hole to fluctuate. When a particle with energy $\omega$ tunnels out, the mass of the SdS black hole changed into $m-\omega$. Since the angular velocity of the particle at the horizon is zero ($\Omega_h = 0$), the angular momentum is equal to zero. Taking the self-gravitational interaction into account, the imaginary part of the true action can be calculated from Eq. (3.16) in the following integral form

$$\mathrm{Im} = \pi\int_0^\omega \frac{4m^2\left(1+\frac{4m^2}{\ell^2}+\cdots\right)^2}{2m\left(1+\frac{4m^2}{\ell^2}+\cdots\right) - m - \frac{2}{\ell^2}\{2m\left(1+\frac{4m^2}{\ell^2}+\cdots\right)\}^3}d\omega' \tag{3.18}$$

Replacing $m$ by $m-\omega$ we have

$$\mathrm{Im}I = -\pi\int_m^{(m-\omega)} \frac{4(m-\omega')^2\left(1+\frac{4(m-\omega')^2}{\ell^2}+\cdot\cdot\right)^2}{2(m-\omega')\left(1+\frac{4(m-\omega')^2}{\ell^2}+\cdot\cdot\right)+A} \times d(m-\omega'), \tag{3.19}$$

where $A = -(m-\omega') - \frac{2}{\ell^2}\{2(m-\omega')\left(1+\frac{4(m-\omega')^2}{\ell^2}+\cdot\cdot\right)\}^3$.

Within WKB approximation, we can neglect the terms $(m-\omega')^n$ for $n \geq 5$. Therefore, we rewrite Eq. (3.19) of the form

$$\begin{aligned}\mathrm{Im}I &= -4\pi\int_m^{(m-\omega)} \frac{(m-\omega')\left(1+\frac{8(m-\omega')^2}{\ell^2}\right)}{\left(1-\frac{8(m-\omega')^2}{\ell^2}\right)} \times d(m-\omega'),\\ &= -\frac{\pi}{2}\left[4(m-\omega)^2\left(1+\frac{8(m-\omega)^2}{\ell^2}\right) - 4m^2\left(1+\frac{8m^2}{\ell^2}\right)\right].\end{aligned} \tag{3.20}$$

Therefore, the tunneling rate for the SdS black hole is given by

$$\begin{aligned}\Gamma \sim \exp(-2\mathrm{Im}I) &= \exp\{\pi[4(m-\omega)^2\left(1+\frac{8(m-\omega)^2}{\ell^2}\right)\\ &\quad -4m^2\left(1+\frac{4m^2}{\ell^2}\right)]\}\\ &= \exp[\pi(r_f^2 - r_i^2)]\\ &= \exp(\Delta S_{BH}).\end{aligned} \tag{3.21}$$

Here, $r_i = 2m\left(1+\frac{4m^2}{\ell^2}\right)$ and $r_f = 2(m-\omega)\left(1+\frac{4(m-\omega)^2}{\ell^2}\right)$ are the locations of the SdS event horizon before and after the particles emission, and $\Delta S_{BH} = S_{BH}(m-\omega) - S_{BH}(m)$ is the change of Bekenstein-Hawking entropy.

## 3.4 Purely Thermal Radiation

It is clear from Eq. (3.21) that the radiation spectrum is not pure thermal although gives a correction to the Hawking radiation of SdS black hole. Expanding the tunneling rate in power of $\omega$ upto second order, the purely thermal spectrum can be derived from Eq. (3.21) as discussed by Liu et al. [65] of the form

$$\begin{aligned}\Gamma &\sim \exp(\Delta S_{BH}) \\ &= \exp\left\{-\omega\frac{\partial S_{BH}(m)}{\partial m}+\frac{\omega^2}{2}\frac{\partial^2 S_{BH}(m)}{\partial m^2}\right\}. \end{aligned} \quad (3.22)$$

The derivatives are calculated from

$$S_{BH}(m-\omega) = 4\pi(m-\omega)^2\left(1+\frac{8(m-\omega)^2}{\ell^2}\right).$$

Thus Eq. (3.22) becomes

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{-8\pi\omega\left[\left(m+\frac{16m^3}{\ell^2}\right)-\frac{\omega}{2}\left(1+\frac{48m^2}{\ell^2}\right)\right]\right\}. \quad (3.23)$$

When $\ell \to \infty$, the pure thermal spectrum can be reduced for Schwarzschild black hole as $\Gamma \sim \exp(\Delta S_{BH}) = \exp\left[-8\pi\omega\left(m-\frac{\omega}{2}\right)\right]$. Obviously our result in accordance with the result of Parikh and Wilczek [51, 82, 83] . The radiation spectrum given by Eq. (3.23) is more accurate and provides an interesting correction to Hawking pure thermal spectrum.

## 3.5 Concluding Remarks

In this chapter, we have presented the Hawking radiation as massive particle tunneling method from SdS black hole [129]. We have found that the tunneling rate at the event horizon of SdS black hole is related to the Bekenstein-Hawking entropy, and the factual radiation spectrum deviates from the precisely thermal one when energy conservation and self-gravitational interaction are taken into account. Specially, when $\ell \to \infty$, i.e., $\Lambda = 0$, the SdS black hole reduces to the Schwarzschild black hole. The positions of the event horizon of Schwarzschild black hole before and after the emission of the particles with energy $\omega$ are $r_i = 2m$ and $r_f = 2(m-\omega)$. From Eq. (3.21), the tunneling rate of Schwarzschild black hole can be written as

$$\begin{aligned}\Gamma \sim \exp(-2\mathrm{Im}I) &= \exp\left\{\pi\left[4(m-\omega)^2 - 4m^2\right]\right\} \\ &= \exp[\pi(r_f^2 - r_i^2)] \\ &= \exp(\Delta S_{BH}),\end{aligned} \tag{3.24}$$

which is fully consistent with that obtained by Parikh and Wilczek [51, 82, 83].

# Chapter 4

## Hawking Non-thermal and Purely thermal radiations of Schwarzschild-anti-de Sitter Black Hole by Hamilton-Jacobi method

### 4.1 Introduction

According to the information loss paradox [24, 25], the information carried out by a physical system falling toward black hole singularity has no way to recover after a black hole has completely disappeared because the state of the radiation is determined only by the geometry of the black hole outside the horizon, and the black hole has no hair that records any detailed information about the collapsing body. With the emission of thermal radiation [24, 25], a black hole has radiated away most of its mass and becomes smaller and smaller until evaporate away completely. In this basis, many research works on the thermal radiation of black holes have been made [36, 37, 38, 39, 40]. This procedure provides a leading correction to the tunneling probability (emission rate) arising from the reduction of the black hole mass because of the energy carried by the emitted massive quanta.

In 1974, Bekenstein first conjectured was strengthened by Hawking, who was able to show that black holes can radiate when quantum effects are taken in account [25, 34, 37, 170] and hence the situation is changed. It seems that an initially pure quantum state (original matter), by collapsing to a black hole and then evaporating completely, has evolved to a mixed states (the thermal spectrum at infinity) that violates the fundamental postulate of quantum mechanics due to prescribe a unitary time evolution of basis states. When the black hole has evaporated down to the Planck size, quantum fluctuations dominate and the semi-classical calculations would no longer be valid, as spacetime is subject to violent quantum fluctuations on this scale. Therefore, it is still mysterious how the information be recovered. Recent development of string/M theory and the AdS/CFT correspondence argued that the information could be recovered if the outgoing radiation were not exactly thermal but had subtle corrections [36]. Other possibilities include the information being contained in a Planckian remnant left over at the end of Hawking radiation or a modification of the laws of quantum mechanics to allow for non-unitary time evolution.

Wilczek and his collaborators have developed two universal methods to correctly recover Hawking radiation of black holes. One is the gravitational anomaly method [41] in which the Hawking radiation can be determined by anomaly canceled conditions and regularity requirement at the event horizon. Later on, this method is widely used to calculate the Hawking radiation for different black holes [42, 43, 44, 45, 46, 47, 48, 49, 50]. The another is the semi-classical tunneling method initiated by Kraus and Wilczek [52, 53, 54] that has been used to describe Hawking radiation successfully for various spacetimes [38, 50, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 165], where a particle moves in dynamical geometry and all of these works are limited to massless particle. This method involve calculating the imaginary part of the action for the process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for emission at the Hawking temperature. Applying this method, two different methods have been employed to calculate the imaginary part of the action, one the null geodesic method developed by Parikh and Wilczek [51, 82, 83, 84] and other by Angheben et al. [90]. In fact, the method of Angheben et al. [90] is an extension of the complex path analysis proposed by Padmanabhan et al. [85, 86, 87, 88, 89]. The latter method involves consideration of a emitted scalar particle, ignoring its self-gravitation and assumes that its action satisfies the relativistic Hamilton-Jacobi equation. An appropriate ansatz for the action can be obtained from the symmetries of the spacetime which is known as the Hamilton-Jacobi ansatz. Both the methods show that when the self-gravitational interaction and the unfixed background spacetime are taken into account, the actual Hawking radiation spectrum deviates from the purely thermal one, satisfies the underlying unitary theory and gives a leading correction to the radiation spectrum.

On the other hand, Hawking radiation from massive uncharged particle tunneling [98] and charged particle tunneling [100] from black hole was first proposed by Zhang and Zhao. Exploiting this work, a few researches have been carried out as charged particle tunneling [99, 101, 102, 103, 104]. Recently, Kerner and Mann developed quantum tunneling methods for analyzing the temperature of Taub-NUT black holes [109] using both the null-geodesic and Hamilton-Jacobi methods. In the latter method the self-gravitation interaction and energy conservation of emitted particle were ignored to calculate the thermal radiation spectrum. Parikh and Wilczek have shown that these radiation spectrum is not strictly thermal but satisfies the underlying unitary theory when self-gravitation interaction and energy conservation are considered. Considering Kerner and Mann's process Chen, Zu and Yang reformed Hamilton-Jacobi method for massive particle tunneling and investigate the Hawking radiation of the Taub-NUT black hole [126]. Using this method Hawking radiation of Kerr-NUT black hole [65], the charged black hole with a global monopole [99, 128] and Schwarzschild-de Sitter (SdS) black hole [129] have been reviewed. We apply these method to investigate the Hawking radiation of Schwarzschild-anti-de Sitter (SAdS) black hole.

The solutions of black holes in Anti-de Sitter spaces come from the Einstein equations with a negative cosmological constant. Anti-de Sitter black holes are different from de Sitter black holes. The difference consisting in them is due to minimum temperatures that occur when their sizes are of the order of the characteristic radius of the anti-de Sitter space. For larger Anti-de Sitter black holes, their red-shifted temperatures measured at infinity are greater. This implies that such black holes can be in stable equilibrium with thermal radiation at a certain temperature. Moreover, recent development in string /M-theory greatly stimulate the study of black holes in anti-de Sitter spaces. One example is the AdS/CFT correspondence [155, 156, 171] between a weakly coupled gravity system in an anti-de Sitter background and a strongly coupled conformal field theory on its boundary. So our study on the Schwarzschild-anti-de Sitter black holes is reasonable and meaningful.

The next section will outline the position of event horizon of SAdS black hole. In section 4.3, we then consider the unfixed background spacetime and the self-gravitational interaction into account, we review the Hawking non-thermal radiation of SAdS black hole from massive particle tunneling method. The new line element of SAdS black hole near the even horizon is also derived in this section. In section 4.4, we have derived the Hawking purely thermal radiation from non-thermal rate. Finally, in section 4.5, we present our remarks.

## 4.2 Schwarzschild-anti-de Sitter black hole

The Schwarzschild-anti-de Sitter black hole with mass $M$ and a negative cosmological constant $\Lambda = -3/\ell^2$ is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.1}$$

where the lapse function $f(r)$, is given by

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \tag{4.2}$$

and the coordinates are defined such that $-\infty \leq t \leq \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The lapse function vanished at the zeros of the cubic equation

$$r^3 + \ell^2 r - 2m\ell^2 = 0. \tag{4.3}$$

The only real roots of this equation is

$$r_+ = \frac{2}{3}\sqrt{3}\ell \sinh\left(\frac{1}{3}\sinh^{-1}\left(3\sqrt{3}\frac{m}{\ell}\right)\right). \quad (4.4)$$

Expanding $r_+$ in terms of $m$ and $\ell$ with $1/\ell^2 << m^2/9$, we obtain

$$r_+ = 2m\left(1 - \frac{4m^2}{\ell^2} + ....\right). \quad (4.5)$$

Therefore, we can write $r_+ = 2m\eta$, with $\eta < 1$. The event horizon of the SAdS black hole is smaller than the Schwarzschild event horizon, $r_H = 2m$.

## 4.3 The HJ Method for Non-thermal Radiation

We next consider the method of Chen et al. [125, 126] for calculating the imaginary part of the action making use of the Hamilton-Jacobi equation [90]. We assume that the action of the outgoing particle is given by the classical action $I$ satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\left(\frac{\partial I}{\partial x^\mu}\right)\left(\frac{\partial I}{\partial x^\nu}\right) + u^2 = 0, \quad (4.6)$$

in which $u$ and $g^{\mu\nu}$ are the mass of the particle and the inverse metric tensors derived from the line element (4.1). Since the event horizon of SAdS black hole coincides with the outer infinite redshift surface, here we can apply the geometrical optics limit. Using the WKB approximation [168], the tunneling probability for the classically forbidden trajectory of the s-wave coming from inside to outside of SAdS event horizon is given by

$$\Gamma \sim \exp(-2\text{Im}I). \quad (4.7)$$

As mention before, this method is different from Parikh and Wilczek's method (Null geodesic) in which the action mainly relies on the exploration of the equation of motion in the Painlevé coordinates systems and the calculation of Hamilton equation. But in the Hamilton-Jacobi method we avoid this for calculating the imaginary part of the action $I$. For the convenient of discussion, we define $\Delta = r^2 - 2mr + \frac{r^4}{\ell^2}$ and then the line element (4.1) can be written as

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2(d\theta^2 + \sin^\theta d\phi^2). \tag{4.8}$$

Near the event horizon, the above line element can be rewritten as

$$\begin{aligned} ds^2 = -\frac{\Delta_{,r}(r_+)(r - r_+)}{r_+^2}dt^2 + \frac{r_+^2}{\Delta_{,r}(r_+)(r - r_+)}dr^2 \\ +r_+^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{4.9}$$

where

$$\Delta_{,r}(r_+) = \frac{d\Delta}{dr}\bigg|_{r=r_+} = 2(r_+ - m + 2\frac{r_+^3}{\ell^2}). \tag{4.10}$$

For the metric (4.9), the non-null inverse metric tensors are

$$\begin{aligned} g^{00} &= -\frac{r_+^2}{\Delta_{,r}(r_+)(r - r_+)}, \quad g^{11} = \frac{\Delta_{,r}(r_+)(r - r_+)}{r_+^2}, \\ g^{22} &= \frac{1}{r_+^2}, \quad g^{33} = \frac{1}{r_+^2\sin^2\theta}. \end{aligned} \tag{4.11}$$

The Hamilton-Jacobi equation (4.6), with the help of Eq. (4.11) becomes

$$\begin{aligned} -\frac{r_+^2}{\Delta_{,r}(r_+)(r - r_+)}\left(\frac{\partial I}{\partial t}\right)^2 + \frac{\Delta_{,r}(r_+)(r - r_+)}{r_+^2}\left(\frac{\partial I}{\partial r}\right)^2 \\ +\frac{1}{r_+^2}\left(\frac{\partial I}{\partial \theta}\right)^2 + \frac{1}{r_+^2\sin^2\theta}\left(\frac{\partial I}{\partial \phi}\right)^2 + u^2 = 0. \end{aligned} \tag{4.12}$$

It is crux to solve the action $I$ for $I(t, r, \theta, \phi)$. Considering the properties of black hole spacetime, the separation of variables can be taken as follows

$$I = -\omega t + R(r) + H(\theta) + j\phi, \tag{4.13}$$

where $\omega$ and $j$ are respectively the energy and angular momentum of the particle. Since SAdS black hole is nonrotating, the angular velocity of the particle at the horizon is $\Omega_+ = \frac{d\phi}{dt}\Big|_{r=r_+} = 0$. Using Eq.(4.13) into Eq. (4.12) and solving $R(r)$ yields an expression of

$$\begin{aligned} R(r) &= \pm\frac{r_+^2}{\Delta_{,r}(r_+)}\int\frac{dr}{(r-r_+)} \\ &\quad\times\sqrt{\omega^2-\frac{\Delta_{,r}(r_+)(r-r_+)}{r_+^2}\left[g^{22}(\partial_\theta H(\theta))^2+g^{33}j^2+u^2\right]}. \end{aligned} \tag{4.14}$$

We consider the emitted particle as an ellipsoid shell of energy $\omega$ to tunnel across the event horizon and should not have motion in $\theta$-direction ($d\theta = 0$) and therefore, finishing the above integral we get

$$\begin{aligned} R(r) &= \pm\frac{2.\pi i r_+^2}{\Delta_{,r}(r_+)}\omega+\xi \\ &= \pm\frac{\mathrm{i}4\pi m^2}{(r_+-m+2\frac{r_+^3}{\ell^2})}\left(1-\frac{4m^2}{\ell^2}+....\right)^2\omega+\xi, \end{aligned} \tag{4.15}$$

where $\pm$ sign comes from the square root and $\xi$ is the constant of integration. Inserting Eq. (4.15) into Eq. (4.13), the imaginary part of actions corresponding to outgoing and incoming particles can be written as

$$\mathrm{Im}I_\pm = \pm\frac{4\pi m^2}{(r_+-m+2\frac{r_+^3}{\ell^2})}\left(1-\frac{4m^2}{\ell^2}+....\right)^2\omega+\xi. \tag{4.16}$$

According to the classical limit given in Ref. [169], we ensure that the incoming probability to be unity when there is no reflection i.e., every thing is absorbed by the horizon. In this situation the appropriate value of $\xi$ instead of zero or infinity can be taken as $\xi = \frac{4\pi m^2}{(r_+ - m + 2\frac{r_+^3}{\ell^2})}\left(1 - \frac{4m^2}{\ell^2} + ....\right)^2 \omega +$ Re($\xi$). Therefore, Im$I_- = 0$ and $I_+$ give the imaginary part of action $I$ corresponding to the outgoing particle with the help of Eq. (4.10) to the form

$$\text{Im}I = \frac{4\pi m^2}{(r_+ - m + 2\frac{r_+^3}{\ell^2})}\left(1 - \frac{4m^2}{\ell^2} + ....\right)^2 \omega. \tag{4.17}$$

Substituting Eq. (4.5) into Eq. (4.17), the imaginary part of action takes the form

$$\text{Im}I = \frac{4\pi m^2 \left(1 - \frac{4m^2}{\ell^2} + \cdot\cdot\right)^2 \omega}{2m\left(1 - \frac{4m^2}{\ell^2} + \cdot\cdot\right) - m + \frac{2}{\ell^2}\left\{2m\left(1 - \frac{4m^2}{\ell^2} + \cdot\cdot\right)\right\}^3}. \tag{4.18}$$

Since the SAdS spacetime is dynamic due to the presence of cosmological constant, we consider the ADM (Arnowitt-Deser-Misner) mass of the total spacetime to be fixed and permit the SAdS black hole to fluctuate. When a particle with energy $\omega$ tunnels out, the mass of the SAdS black hole changed into $m - \omega$. Since the angular velocity of the particle at the horizon is zero ($\Omega_+ = 0$), the angular momentum is equal to zero. Taking self-gravitation interaction into account it has been shown in refs. [53, 172] that the black hole radiation is no longer thermal and therefore in view of this assumption, the imaginary part of the true action can be calculated from Eq. (4.18) in the following integral form

$$\text{Im}I = 4\pi \int_0^\omega \frac{m^2\left(1 - \frac{4m^2}{\ell^2} + \cdots\right)^2}{2m\left(1 - \frac{4m^2}{\ell^2} + \cdots\right) - m + \frac{2}{\ell^2}\{2m\left(1 - \frac{4m^2}{\ell^2} + \cdots\right)\}^3} d\omega'.$$

Replacing $m$ by $m-\omega$, we have

$$\mathrm{Im}I = -4\pi \int_m^{(m-\omega)} \frac{(m-\omega')^2\left(1-\frac{4(m-\omega')^2}{\ell^2}+\cdot\cdot\right)^2}{2(m-\omega')\left(1-\frac{4(m-\omega')^2}{\ell^2}+\cdot\cdot\right)+A} \times d(m-\omega') \tag{4.19}$$

where $A=-(m-\omega')+\frac{2}{\ell^2}\{2(m-\omega')\left(1-\frac{4(m-\omega')^2}{\ell^2}+\cdot\cdot\right)\}^3$.

Employing WKB approximation, we neglect the terms $(m-\omega')^n$ for $n\geq 5$, and rewrite Eq. (4.19) as

$$\begin{aligned}\mathrm{Im}I &= -4\pi \int_m^{(m-\omega)} \frac{(m-\omega')\left(1-\frac{8(m-\omega')^2}{\ell^2}\right)}{\left(1+\frac{8(m-\omega')^2}{\ell^2}\right)} \times d(m-\omega') \\ &= -\frac{\pi}{2}\left[4(m-\omega)^2\left(1-\frac{8(m-\omega)^2}{\ell^2}\right)-4m^2\left(1-\frac{4m^2}{\ell^2}\right)\right]\end{aligned} \tag{4.20}$$

Therefore, from Eq. (4.7) the tunneling probability for the SAdS black hole is given by

$$\begin{aligned}\Gamma\sim\exp(-2\mathrm{Im}I) &= \exp\{\pi[4(m-\omega)^2\left(1-\frac{8(m-\omega)^2}{\ell^2}\right) \\ &\quad -4m^2\left(1-\frac{4m^2}{\ell^2}\right)]\} \\ &= \exp[\pi(r_f^2-r_i^2)] \\ &= \exp(\Delta S_{BH}),\end{aligned} \tag{4.21}$$

where $r_i = 2m\left(1-\frac{4m^2}{\ell^2}\right)$ and $r_f = 2(m-\omega)\left(1-\frac{4(m-\omega)^2}{\ell^2}\right)$ are the locations of the SAdS event horizon before and after the particle emission, and $\Delta S_{BH}=S_{BH}(m-\omega)-S_{BH}(m)$ is the change of Bekenstein-Hawking entropy.

## 4.4 Purely Thermal Radiation

The radiation spectrum described by Eq. (4.21) is not pure thermal although gives a correction to the Hawking radiation of SAdS black hole. The purely thermal spectrum can be derived from Eq. (4.21) by expanding the tunneling rate in power of $\omega$ upto second order as discussed by Liu et al. [65] of the form

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{-\omega\frac{\partial S_{BH}(m)}{\partial m} + \frac{\omega^2}{2}\frac{\partial^2 S_{BH}(m)}{\partial m^2}\right\}. \tag{4.22}$$

It is clear from Eq. (4.21) that

$$S_{BH}(m-\omega) = 4\pi(m-\omega)^2\left(1-\frac{8(m-\omega)^2}{\ell^2}\right), \tag{4.23}$$

which gives

$$\begin{aligned}\frac{\partial S_{BH}(m-\omega)}{\partial m} &= 8\pi(m-\omega)\left(1-\frac{16(m-\omega)^2}{\ell^2}\right),\\ \frac{\partial^2 S_{BH}(m-\omega)}{\partial m^2} &= 8\pi\left(1-\frac{48(m-\omega)^2}{\ell^2}\right),\end{aligned} \tag{4.24}$$

with $\omega = 0$, the above equation takes the following simple form

$$\begin{aligned}\frac{\partial S_{BH}(m)}{\partial m} &= 8\left(m-\frac{16m^3}{\ell^2}\right),\\ \frac{\partial^2 S_{BH}(m)}{\partial m^2} &= 8\left(1-\frac{48m^2}{\ell^2}\right).\end{aligned} \tag{4.25}$$

The purely thermal spectrum described by Eq. (4.22) can be reduced with the help of Eq. (4.25) of the form

$$\begin{aligned}&\Gamma \sim \exp(\Delta S_{BH})\\ &= \exp\left\{-8\pi\omega\left[\left(m-\frac{16m^3}{\ell^2}\right)-\frac{\omega}{2}\left(1-\frac{48m^2}{\ell^2}\right)\right]\right\}.\end{aligned} \tag{4.26}$$

If we replace $\ell^2$ with $-\ell^2$, the Hawking non-thermal spectrum and pure thermal spectrum agree with these of SdS black hole [129].

## 4.5 Concluding Remarks

In this chapter, we have presented an extension of the classical tunneling framework [58, 65, 128] for the spherically symmetric black hole cases to deal with Hawking radiation of massive particles as tunneling process through the event horizon of SAdS black hole. By treating the background spacetime as dynamical, the energy and the angular momentum as conservation, we have found the non-thermal and purely thermal tunneling probabilities of SAdS black hole when the particle's self-gravitation is taken into account. The non-thermal tunneling probability of particle emission is proportional to the phase space factor depending on the initial and final entropy of the system (the change of the Bekenstein-Hawking entropy), which implies that the emission spectrum actually deviates from perfect thermally but is in agreement with an underlying unitary theory. The similar results have been shown under the same assumption for massive particles tunneling across the event horizon of SdS [129] and Taub-NUT [126] black holes. Our motivation also indeed support the results obtained by massless or massless charged particles tunneling from different spacetimes such as charged black hole with a global monopole [99, 128], Kerr-NUT black hole [65] and Kerr and Kerr-Newman black holes [103] as well as other cases [48, 172]. We therefore come to the conclusion that the actual radiation spectrum of SAdS black hole is not precisely thermal, which provides an interesting correction to Hawking pure thermal spectrum.

In the limiting case, i.e., when $\Lambda = 0$, our results for non-thermal and purely thermal radiations are reduced to

$$\Gamma \sim \exp(-2\mathrm{Im}I) = \exp\left\{\pi\left[4(m-\omega)^2 - 4m^2\right]\right\}, \tag{4.27}$$

and

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\{-8\pi\omega(m - \frac{\omega}{2})\}. \tag{4.28}$$

These are the non-thermal and purely thermal tunneling rates of Schwarzschild black hole, where $r_i = 2m$ and $r_f = 2(m-\omega)$ are the positions of the event horizon of Schwarzschild black hole before and after the emission of the particles. Obviously, both the results are fully consistent with that obtained by Parikh and Wilczek [51, 82, 83].

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# Chapter 5

## Hawking Non-thermal and Purely thermal radiations of Reissner-Nordström-de Sitter Black Hole by Hamilton-Jacobi method

### 5.1 Introduction

A wonderful fact of black hole radiation [24, 25] have discovered by Hawking in 1975 and several works have been done to calculate this quantum effect [34]. Nowadays, the radiation of black holes is called 'Hawking radiation'. Furthermore Hawking proposed that the radiation of black holes can be shown as tunneling and the emission spectrum in light of quantum field theory in curved spacetime with the exception of following the tunneling picture. The tunneling phenomenon has been extensively studied [169, 173, 174, 175, 176, 177, 178, 179, 180] and a lot of work has already been successfully applied on various black hole spacetimes in references [38, 50, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81, 165, 181]. Here, a particle moves in dynamical geometry and all of these works are limited to massless particle and gives a correction to the emission rate arising from loss of mass of the black hole crresponding to the energy carried by radiated quantum. The method delineated Hawking radiation as tunneling process was first disclosed by Kraus and Wilczek [53, 54] and then reinterpreted by Parikh and Wilczek [51]. In this method the tunneling rate is related to the calculating of the imaginary part of the action for the process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for emission at the Hawking temperature. In general, based on semiclassical tunneling picture two universal methods are applied in references to derive the action. One method is called as the Null Geodesic method developed by Parikh and Wilczek [51, 82, 83] and another method, proposed by Angheben et al. [90] known as Hamilton-Jacobi methods and it is an extension of the complex path analysis proposed by Padmanabhan et al.[85, 86, 87, 88, 89].

In 2005, Zhang and Zhao have proposed the Hawking radiation from massive uncharged particle tunneling [98] and charged particle tunneling [100]. Following this work several researches have been carried out as charged particle tunneling [99, 101, 102, 103, 104]. Kerner and Mann have developed quantum tunneling methods for calculating the thermal radiation spectrum of Taub-NUT black holes [109] using both the null-geodesic and Hamilton-Jacobi methods by ignoring the self-gravitation interaction and energy conservation of emitted particle. However, according to the Parikh and Wilczek's opinion [51], the radiation spectrum is not strictly thermal but satisfies the underlying unitary theory when self-gravitation interaction and energy conservation are considered. Considering Kerner and Mann's process Chen, Zu and Yang reformed Hamilton-Jacobi method for massive particle tunneling and investigate the Hawking radiation of the Taub-NUT black hole [126]. Using this method Hawking radiation of Kerr-NUT black hole [65], the charged black hole with a global monopole [99] have been reviewed.

Recently, we have reformed Hamilton-Jacobi method and investigate the Hawking radiation of the SdS black hole [129] where the position of the black hole horizon is taken in a series of black hole's parameters so that the spacetime metric becomes dynamical and self-gravitation interaction are taken into account. Here, we also assume that the changed of background geometry can be treated as the loss of radiated energy of the black hole. In this chapter, the same method have been applied to investigate the Hawking radiation of Reissner-Nordström-de Sitter (RNdS) black hole [160]. In order to narrate Hawking-Radiation from the action of radiation particles the method of Chen et al. [125, 126] is used. Our chief purpose concerned of this work is to calculate the imaginary part of action from Hamilton-Jacobi equation avoid by exploring the equation of motion of the radiation particle in Painlevé coordinate system and calculating the Hamilton equation. Though the equation of motion of massive particles are different from massless particle, We no need differentiate radiation particle. Above all as the self-gravitational interaction and the unfixed background spacetime are not assumed, the derived radiation spectrum deviates from the purely thermal one and the tunneling rate is related to the change of Bekenstein-Hawking entropy.

The cosmological constant with positive sign plays a prominent role in two reasons. First, the accelerating expansion of our universe indicates the cosmological constant might be a positive one [132, 153, 154]. Secondly, conjecture about de Sitter/CFT correspondence [166, 167] has been suggested that there is a dual relation between quantum gravity on a dS space and Euclidean conformal field theory (CFT) on a boundary of dS space [157, 159]. The outgoing particles tunnel from black hole horizon and incoming particles tunnel from cosmological horizon and formed Hawking radiation and the incoming particles can fall into the horizon along classically permitted trajectories for black hole horizon, but outgoing particles can fall classically out of the horizon for cosmological horizon.

The latter section of this chapter describes the RNdS black hole spacetime with the position of event horizon. Near the event horizon the new line element of RNdS black hole is also derived here. The unfixed background spacetime and the self-gravitational interaction are taken into account, we review the Hawking radiation of RNdS black hole from massive particle tunneling method in section 5.3. In section 5.4, we have developed the Hawking purely thermal rate from non-thermal rate. Finally, in section 5.5, we present our remarks.

## 5.2 Reissner-Nordström-de Sitter black hole

The line element of Reissner-Nordström-de Sitter black hole, which is the Schwarzschild black hole generalized with a charge parameter and a positive cosmological constant $\Lambda(= 3/\ell^2)$ has the form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.1}$$

where the metric function $f(r)$ is given by

$$f(r) = 1 - \frac{2m}{r} - \frac{r^2}{\ell^2} + \frac{q^2}{r^2}.$$

Here, $m$ being the mass, $\ell$ the cosmological radius, $q$ the total charge(electric plus magnetic) with respect to the static de Sitter space are defined such that $-\infty \leq t \leq \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. At large $r$, the metric (5.1) tends to the dS space limit. The explicit dS case is obtained by setting $m = 0$ while the explicit Reissner-Nordström case is obtained by taking the limit $\ell \to \infty$. When $\ell^2$ is replaced by $-\ell^2$, the metric (5.1) describes an interesting nonrotating AdS black hole called the Reissner-Nordström-Anti-de Sitter (RNAdS) black hole.

The spacetime causal structure depends strongly on the singularities of the metric given by the zeros of $f(r)$. Depending on the black hole parameters $M$, $q$ and $\ell$, the function $f(r)$ may have three, two, or even no real positive zeros. For the RNdS black hole case we are interested in which $f(r)$ has two simple real, positive roots: $r_h$ and $r_c$. Here we indicate $r_h$ as the outer (event) horizon and $r_c$ the cosmological horizon. To get these zeros of $f(r)$, we have $r^4 - \ell^2 r^2 + 2m\ell^2 r - \ell^2 q^2 = 0$. The black hole event horizon $r_h$ and the cosmological horizon $r_c$ are located, respectively, at

$$\begin{aligned} r_h &= \frac{\ell}{\sqrt{3}}\sin\left[\frac{1}{3}\sin^{-1}\frac{3m\sqrt{3}}{\ell\sqrt{1+\frac{4q^2}{\ell^2}}}\right] \\ &\times\left(1+\sqrt{1-\frac{q^2\ell}{\sqrt{3}m}.\frac{2}{1+\delta}\mathrm{cosec}\left[\frac{1}{3}\sin^{-1}\frac{3m\sqrt{3}}{\ell\sqrt{1+\frac{4q^2}{\ell^2}}}\right]}\right), \qquad (5.2) \end{aligned}$$

$$\begin{aligned} r_c &= \frac{\ell}{\sqrt{3}}\sin\Big[\frac{1}{3}\sin^{-1}\frac{3m\sqrt{3}}{\ell\sqrt{1+\frac{4q^2}{\ell^2}}}\Big] \\ &\times\Big(\sqrt{1+\frac{(1+\delta)\ell 3m}{2\sqrt{3}}\text{cosec}^3\Big[\frac{1}{3}\sin^{-1}\frac{3m\sqrt{3}}{\ell\sqrt{1+\frac{4q^2}{\ell^2}}}\Big]}-1\Big), \end{aligned} \tag{5.3}$$

where

$$\delta = \sqrt{1-\frac{4q^2}{3m^2}\sin^2\Big[\frac{1}{3}\sin^{-1}\frac{3\sqrt{3}m}{\ell\sqrt{1+\frac{4q^2}{\ell^2}}}\Big]}. \tag{5.4}$$

Expanding $r_h$ in terms of $m$, $\ell$ and $q$ with $27\frac{m^2}{\ell^2} < 1$ as well as $\frac{3\sqrt{3}m}{\ell\alpha} < 1$ and setting $\delta = 1$, we obtain

$$r_h = \frac{m}{\alpha}\left(1+\frac{4m^2}{\ell^2\alpha^2}+\cdots\right)\left(1+\sqrt{1-\frac{q^2\alpha}{m^2}}\right), \tag{5.5}$$

which can be written as

$$r_h = \frac{1}{\alpha}\left(1+\frac{4m^2}{\ell^2\alpha^2}+\cdots\right)\left(m+\sqrt{m^2-q^2\alpha}\right), \tag{5.6}$$

where $\alpha = \sqrt{1+\frac{4q^2}{\ell^2}}$.

that is, the event horizon of the RNdS black hole is greater than the Reissner-Nordström event horizon $r_{RN} = m + \sqrt{m^2 - q^2}$.

Again it gives the Reissner-Nordström (RN) black hole [99] for $\ell \to \infty$ and Schwarzschild-de Sitter black hole [129] for $q = 0$. The metric (5.1) represents an interesting asymptotically de-Sitter extreme RN black hole for $q^2 = \alpha m^2$, while for $q^2 > \alpha m^2$ it does represent any black hole but an unphysical naked singularity at $r = 0$. We now define $\Delta = r^2+q^2-2mr-\frac{r^4}{\ell^2}$ and then the line element (5.1) becomes

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.7}$$

The position of black hole horizon is same as given in Eq. (5.6). The line element near the black hole horizon becomes

$$ds^2 = -\frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}dt^2 + \frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}dr^2 + r_h^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (5.8)$$

where

$$\Delta_{,r}(r_h) = \frac{d\Delta}{dr}\Big|_{r=r_h} = 2(r_h - m - 2\frac{r_h^3}{\ell^2}). \quad (5.9)$$

The relationship between the tunneling rate and the action of the radiative particle using the WKB approximation [168] is as

$$\Gamma \sim \exp(-2\text{Im}I).$$

## 5.3 The Hamilton-Jacobi Method

In the Hamilton-Jacobi method we avoid the exploration of the equation of motion in the Painlevé coordinates system. To calculate the imaginary part of the action from the relativistic Hamilton-Jacobi equation, the action $I$ of the outgoing particle from the black hole horizon satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\left(\frac{\partial I}{\partial x^\mu}\right)\left(\frac{\partial I}{\partial x^\nu}\right) + u^2 = 0, \quad (5.10)$$

in which $u$ and $g^{\mu\nu}$ are the mass of the particle and the inverse metric tensors derived from the line element (5.8).

The non-null inverse metric tensors for the metric (5.8) are

$$g^{00} = -\frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}, \quad g^{11} = \frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2},$$
$$g^{22} = \frac{1}{r_h^2}, \quad g^{33} = \frac{1}{r_h^2\sin^2\theta}. \quad (5.11)$$

We can write Eq. (5.10) with the help of Eq. (5.11) as

$$-\frac{r_h^2}{\Delta_{,r}(r_h)(r-r_h)}\left(\frac{\partial I}{\partial t}\right)^2+\frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}\left(\frac{\partial I}{\partial r}\right)^2$$
$$+\frac{1}{r_h^2}\left(\frac{\partial I}{\partial \theta}\right)^2+\frac{1}{r_h^2\sin^2\theta}\left(\frac{\partial I}{\partial \phi}\right)^2+u^2=0. \quad (5.12)$$

It is not easy to done to solve the action $I$ for $I(t,r,\theta,\phi)$. Considering the properties of black hole spacetime, the separation of variables can be taken as follows

$$I=-\omega t+R(r)+H(\theta)+j\phi, \quad (5.13)$$

where $\omega$ and $j$ are respectively the energy and angular momentum of the particle. Since RNdS black hole is nonrotating, the angular velocity of the particle at the horizon is $\Omega_h=\frac{d\phi}{dt}\Big|_{r=r_h}=0$. Inserting Eq. (5.13) into Eq. (5.12) and solving $R(r)$ contains an expression of

$$R(r) = \pm\frac{r_h^2}{\Delta_{,r}(r_h)}\int\frac{dr}{(r-r_h)}$$
$$\times\sqrt{\omega^2-\frac{\Delta_{,r}(r_h)(r-r_h)}{r_h^2}\left[g^{22}(\partial_\theta H(\theta))^2+g^{33}j^2+u^2\right]}. \quad (5.14)$$

We consider the emitted particle as an ellipsoid shell of energy $\omega$ to tunnel across the event horizon and should not have motion in $\theta$-direction ($d\theta=0$) and therefore, finishing the above integral we get

$$R(r)=\pm\frac{\pi i r_h^2}{\Delta_{,r}(r_h)}\omega+\xi, \quad (5.15)$$

where $\pm$ sign comes from the square root and $\xi$ is the constant of integration. Inserting Eq. (5.15) into Eq. (5.13), the imaginary part of two different actions corresponding to the outgoing and incoming particles can be written as

$$\mathrm{Im}I_{\pm} = \pm\frac{\pi r_h^2}{\Delta_{,r}(r_h)}\omega + \mathrm{Im}(\xi). \tag{5.16}$$

In accordance with classical limit [169], we make certain the incoming probability to be unity when there is no reflection i.e., everything is absorbed by the horizon. In this situation the appropriate value of $\xi$ instead of zero or infinity can be taken as $\xi = \frac{\pi i r_h^2}{\Delta_{,r}(r_h)}\omega + \mathrm{Re}(\xi)$. Therefore, $\mathrm{Im}I_- = 0$ and $I_+$ give the imaginary part of action $I$ corresponding to the outgoing particle of the form

$$\begin{aligned} \mathrm{Im}I &= \frac{2\pi r_h^2}{\Delta_{,r}(r_h)}\omega \\ &= \frac{\pi r_h^2}{r_h - m - 2\frac{r_h^3}{\ell^2}}\omega. \end{aligned} \tag{5.17}$$

Using Eq. (5.6) into Eq. (5.17), we get the imaginary part of action as

$$\mathrm{Im}I = \frac{\frac{1}{\alpha^2}\left(1 + \frac{4m^2}{\ell^2\alpha^2} + \cdot\cdot\right)^2 (m + \sqrt{m^2 - q^2\alpha})^2}{\frac{1}{\alpha}\left(1 + \frac{4m^2}{\ell^2\alpha^2} + \cdot\cdot\right)(m + \sqrt{m^2 - q^2\alpha}) - m - A}\omega, \tag{5.18}$$

where $A = \frac{2}{\ell^2\alpha^3}\left(1 + \frac{4m^2}{\ell^2\alpha^2} + \cdot\cdot\right)^3 (m + \sqrt{m^2 - q^2\alpha})^3$.

$$\mathrm{Im}I = \frac{\frac{1}{\alpha^2}(m + \sqrt{m^2 - q^2\alpha})^2}{\frac{1}{\alpha}\left[\left(1 - \frac{4m^2}{\ell^2\alpha^2} + \cdot\cdot\right)(m + \sqrt{m^2 - q^2\alpha}) - m\alpha\left(1 - \frac{8m^2}{\ell^2\alpha^2} + \cdot\cdot\right) - B\right]}\omega,$$

where $B = \frac{2}{\ell^2\alpha^2}\left(1 + \frac{4m^2}{\ell^2\alpha^2} + \cdot\cdot\right)(m + \sqrt{m^2 - q^2\alpha})^3$.

Now for the simplicity, neglecting $m^3$ and its higher order terms, we then get

$$\mathrm{Im}I = \frac{1}{\alpha}.\frac{(m + \sqrt{m^2 - q^2\alpha})^2}{(m + \sqrt{m^2 - q^2\alpha}) - m\alpha}\omega. \tag{5.19}$$

In presence of cosmological constant, RNdS spacetime is dynamic, we fix the ADM(Arnowitt-Deser-Misner) mass of the total spacetime and allow the RNdS black hole to fluctuate. When a particle with energy $\omega$ tunnels out, the mass of the RNdS black hole changed into $m-\omega$. Since the angular velocity of the particle at the horizon is zero ($\Omega_h = 0$), the angular momentum is equal to zero. Taking the self-gravitational interaction into account, the imaginary part of the true action can be calculated from Eq. (5.19) in the following integral form

$$\text{Im}I = \pi\frac{1}{\alpha}.\int_0^{\omega}\frac{(m+\sqrt{m^2-q^2\alpha})^2}{(m+\sqrt{m^2-q^2\alpha})-m\alpha}d\omega' \tag{5.20}$$

$$\text{Im}I = \pi\frac{1}{\alpha}.\int_0^{\omega}\frac{(m+\sqrt{m^2-q^2\alpha})^2}{\sqrt{m^2-q^2\alpha}+(1-\alpha)m}d\omega'. \tag{5.21}$$

For the maximum value of integration, neglecting $(1-\alpha)m$. Equation (5.21) becomes

$$\text{Im}I = \pi\frac{1}{\alpha}.\int_0^{\omega}\frac{\left(m+\sqrt{m^2-q^2\alpha}\right)^2}{\sqrt{m^2-q^2\alpha}}d\omega'. \tag{5.22}$$

Replacing $m$ by $m-\omega$ we have

$$\text{Im}I = -\pi\frac{1}{\alpha}.\int_m^{(m-\omega)}\frac{\left(m-\omega'+\sqrt{(m-\omega')^2-q^2\alpha}\right)^2}{\sqrt{(m-\omega')^2-q^2\alpha}}d(m-\omega') \tag{5.23}$$

$$\text{Im}I = -\pi\frac{1}{\alpha}.\int_m^{(m-\omega)}\frac{2(m-\omega')^2+2(m-\omega')\sqrt{(m-\omega')^2-q^2\alpha}-q^2\alpha}{\sqrt{(m-\omega')^2-q^2\alpha}} \times d(m-\omega'). \tag{5.24}$$

Finishing the integral we get

$$\mathrm{Im}I = -\pi\frac{1}{\alpha}.[(m-\omega)\sqrt{(m-\omega)^2 - q^2\alpha} + (m-\omega)^2 - m\sqrt{m^2 - q^2\alpha} - m^2]. \tag{5.25}$$

Therefore, the tunneling rate for the RNdS black hole is given by

$$\begin{aligned}\Gamma \sim \exp(-2\mathrm{Im}I) &= \exp\{\pi.\frac{1}{\alpha}[2(m-\omega)^2 \\ &\quad +2(m-\omega)\sqrt{(m-\omega)^2 - q^2\alpha} - 2m\sqrt{m^2 - q^2\alpha} - 2m^2]\} \\ &= \exp[\pi(r_f^2 - r_i^2)] \\ &= \exp(\Delta S_{BH}).\end{aligned} \tag{5.26}$$

Here, $r_i = \frac{1}{\sqrt{\alpha}}[m + \sqrt{m^2 - q^2\alpha}]$ and $r_f = \frac{1}{\sqrt{\alpha}}[(m-\omega) + \sqrt{(m-\omega)^2 - q^2\alpha}]$ are the locations of the RNdS event horizon before and after the particles emission, and $\Delta S_{BH} = S_{BH}(m-\omega) - S_{BH}(m)$ is the difference of Bekenstein-Hawking entropy.

## 5.4 Purely Thermal Radiation

The radiation spectrum is not pure thermal although gives a correction to the Hawking radiation of RNdS black hole as point out by Eq. (5.26). In the form of a thermal spectrum, using the WKB approximation the tunneling rate is also related to the energy and the Hawking temperature of the radiative particle as $\Gamma \sim \exp(-\frac{\Delta\omega}{T})$. If $\Delta\omega < 0$ is the energy of the emitted particle then due to energy conservation, the energy of the outgoing shell must be $-\Delta\omega$, then above expression becomes

$$\Gamma \sim \exp(\frac{\Delta\omega}{T}).$$

Now using the first law of thermodynamics, we can write $\Gamma \sim \exp(\Delta S)$, which is related to the change of Bekenstein-Hawking entropy as follows

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\{S_{BH}(M-\omega) - S_{BH}(M)\}. \tag{5.27}$$

We establish Eq.(5.27) as developed by Rahman et al. [129] in power of $\omega$ upto second order using Taylor's theorem of the form

$$\Gamma \sim \exp(\Delta S_{BH}) \;=\; \exp\left\{-\omega\frac{\partial S_{BH}(m)}{\partial m} + \frac{\omega^2}{2}\frac{\partial^2 S_{BH}(m)}{\partial m^2}\right\}. \tag{5.28}$$

Using Eqs. (5.26) and (5.27), we obtain from Eq. (5.28) as follows

$$\begin{aligned}&\Gamma \sim \exp(\Delta S_{BH})\\ &= \exp\Big[-\frac{2\pi\omega}{\alpha}\{(2m+\sqrt{m^2-q^2\alpha}+\frac{m^2}{\sqrt{m^2-q^2\alpha}})\\ &-\frac{\omega}{2}(2+\frac{3m}{\sqrt{m^2-q^2\alpha}}-\frac{m^3}{(m^2-q^2\alpha)^{\frac{3}{2}}})\}\Big].\end{aligned} \tag{5.29}$$

When $\ell \to \infty$, then $\alpha = 1$ the pure thermal spectrum can be reduced for the Reissner-Nordström black hole [99]. It is clear that the result in accordance with the result of Parikh and Wilczek [51, 82, 83]. The radiation spectrum given by (5.29) is more accurate and provides an interesting correction to Hawking pure thermal spectrum.

## 5.5 Concluding Remarks

Hawking radiation as massive particle tunneling method from RNdS black hole [160] have been presented in this chapter. By taking into account the self-gravitational interaction, the background spacetime as dynamical and the energy as conservation, we have recovered that the tunneling rate at the event horizon of RNdS black hole is related to the Bekenstein-Hawking entropy. Specially, when $\ell \to \infty$, then $\alpha = 1$ the RNdS black hole reduces to the Reissner-Nordström black hole [99]. The positions of the event horizon of Reissner-Nordström black hole before and after the emission of the particles with energy $\omega$ are $r_i = m + \sqrt{m^2 - q^2}$ and $r_f = (m - \omega) + \sqrt{(m - \omega)^2 - q^2}$. From Eq. (5.26), the non-thermal tunneling rate of Reissner-Nordström black hole can be written as

$$\begin{aligned}\Gamma \sim \exp(-2\text{Im}I) &= \exp\{\pi[\{(m-\omega) + \sqrt{(m-\omega)^2 - q^2}\}^2 \\ &\quad -\{m + \sqrt{m^2 - q^2}\}^2]\} \\ &= \exp[\pi(r_f^2 - r_i^2)] \\ &= \exp(\Delta S_{BH}), \end{aligned} \tag{5.30}$$

and the purely thermal rate of Reissner-Nordström black hole can be written as

$$\begin{aligned}\Gamma \sim \exp(\Delta S_{BH}) = \exp\Big[ -2\pi\omega\{(2m + \sqrt{m^2 - q^2} + \frac{m^2}{\sqrt{m^2 - q^2}}) \\ -\frac{\omega}{2}(2 + \frac{3m}{\sqrt{m^2 - q^2}} - \frac{m^3}{(m^2 - q^2)^{\frac{3}{2}}})\}\Big]. \end{aligned} \tag{5.31}$$

It is interesting that when $q = 0$, Eq. (5.26) gives the result of SdS black hole [129]. Also, when $\ell \to \infty$ and $q = 0$ our results coincide with that obtained by Parikh and Wilczek [51, 82, 83] for spherically symmetric black holes.

# Chapter 6

## Hawking Non-thermal and Purely thermal radiations of Reissner-Nordström-anti-de Sitter Black Hole by Hamilton-Jacobi method

### 6.1 Introduction

By the information loss paradox [24, 25], the information carried out by a physical system falling toward black hole singularity has no way to recover after a black hole has completely disappeared. The loss of information was considered as preserved inside the black hole and so was not a serious problem in the classical theory. In 1976 a semi-classical calculation of black hole radiance was proposed by Hawking and showed that the emitted radiation is exactly thermal. In particular, the detailed form of the radiation does not depend on the detailed structure of the body that collapsed to form the black hole. With the emission of thermal radiation [24, 25], black holes could lose energy, shrink, and eventually evaporate and becomes smaller and smaller until disappears completely. In this basis, many research works on the thermal radiation of black holes have been made [38, 51, 52, 53]. It seems that an initially pure quantum state, by collapsing to a black hole and then evaporating completely, has evolved to a mixed state and in this situation it is impossible for one to predict about certainty what the final quantum state will be even if the initial quantum state were precisely known and therefore violates the fundamental principles of quantum theory due to prescribe a unitary time evolution of basis states. When the black hole has evaporated down to the Planck size, quantum fluctuations dominate and the semi-classical calculations would no longer be valid, as spacetime is subject to violent quantum fluctuations on this scale. There are various ideas about how the paradox is solved.

Since the 1997 proposal of the AdS/CFT correspondence, the predominant belief among physicists is that information is preserved and that Hawking radiation is not precisely thermal but receives quantum corrections. Other possibilities include the information being contained in a Planckian remnant left over at the end of Hawking radiation or a modification of the laws of quantum mechanics to allow for non-unitary time evolution.

Hawking radiation from massive uncharged particle tunneling [98] and charged particle tunneling [100] from black hole was first proposed by Zhang and Zhao. Accomplishment this work, a few researches have been carried out as charged particle tunneling [99, 101, 102, 103, 104]. By the null-geodesic and Hamilton-Jacobi methods, for analyzing the temperature of Taub-NUT black holes [109], Kerner and Mann developed quantum tunneling methods and Hamilton-Jacobi method is rolled up for calculating the relativistic Hamilton-Jacobi equation. Here the radiation spectrum was only a heading term because the fact that the self-gravitation interaction and energy conservation of emitted particle were ignored. Parikh and Wilczek's opinion the true radiation spectrum is not strictly thermal but satisfies the underlying unitary theory when self-gravitation interaction and energy conservation are considered. Clearly the background geometry of a radiating black hole should be altered (unfixed) with the loss of energy. Self-gravitation interaction and unfixed background spacetime are taken into account Chen, Zu and Yang reformed Hamilton-Jacobi method for massive particle tunneling and investigate the Hawking radiation of the Taub-NUT black hole [126] and using this method Hawking radiation of Kerr-NUT black hole [65] and the charged black hole with a global monopole [99, 128] have been developed. These method have been applied to investigate the Hawking radiation of Reissner-Nordström-anti-de Sitter (RNAdS) black hole [131]. Our chief purpose concerned of this work is to calculate the imaginary part of action from Hamilton-Jacobi equation avoid by exploring the equation of motion of the radiation particle in Painlevé coordinate system and calculating the Hamilton equation. Many scientist have developed two universal methods to correctly recover Hawking radiation of black holes. One is the gravitational anomaly method [41] in which the Hawking radiation can be determined by anomaly canceled conditions and regularity requirement at the event horizon. Later on, this method is widely used to calculate the Hawking radiation for different black holes [43, 44, 45, 46, 47, 48, 49, 50]. Other is the semi-classical tunneling method initiated by Kraus and Wilczek [52, 53] that has been used to describe Hawking radiation successfully for various spacetimes

[38, 50, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81], where a particle moves in dynamical geometry and all of these works are limited to massless particle. This method involve calculating the imaginary part of the action for the process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for emission at the Hawking temperature. Two different methods have been employed to calculate the imaginary part of the action, one the null geodesic method developed by Parikh and Wilczek [51, 82, 83] and other by Angheben et al. [90]. Actually, the method of Angheben et al. [90] is an extension of the complex path analysis proposed by Padmanabhan et al. [85, 86, 87, 88, 89].

Recently, Rahman et al. [130] have developed the Hawking radiation of Schwarzschild-anti-de Sitter black hole by Hamilton-Jacobi method. In this method, the imaginary part of the action come from the relativistic Hamilton-Jacobi equation when the self-gravitational interaction and the unfixed background spacetime are taken into account, the actual Hawking radiation spectrum deviates from the purely thermal one, satisfies the underlying unitary theory and gives a leading correction to the radiation spectrum. In this chapter, we have investigated the hawking radiation of RNAdS black hole by Hamilton-Jacobi method.

In the last few years, people have growing interest to investigate hawking radiation in anti-de sitter space due to AdS/CFT correspondence [166, 167]. According to the AdS/CFT correspondence, a large static black hole in asymptotically AdS spacetime corresponds to an (approximately) thermal state in the CFT. So the time scale for the decay of the black hole perturbation, which is given by the imaginary part of its action, corresponds to the timescale to reach thermal equilibrium in the strongly coupled CFT [158]. Further, recent development in string /M-theory greatly stimulate the study of black holes in anti-de Sitter spaces. Thus our study on the Reissner-Nordström-anti-de Sitter black hole is plausible and worthwhile.

In the remainder of this chapter we describe the RNAdS black hole spacetime with the position of event horizon in section 6.2 and also near the event horizon the new line element of RNAdS black hole is derived here. In section 6.3, the unfixed background spacetime and the self-gravitational interaction are taken into account, we review the Hawking non-thermal radiation of RNAdS black hole from massive particle tunneling method. In section 6.4, we have derived the Hawking purely thermal radiation from non-thermal rate. Finally, in section 6.5, we present our remarks.

## 6.2 Reissner-Nordström-anti-de Sitter black hole

The line element of Reissner-Nordström-anti-de Sitter black hole with a negative cosmological constant $\Lambda$ term is given by

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{r^2}{\ell^2} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{r^2}{\ell^2} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (6.1)$$

where $m$ being the mass, $\ell$ is the cosmological radius, $q$ the total charge(electric plus magnetic) with respect to the static anti-de Sitter space are defined such that $-\infty \leq t \leq \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. At large $r$, the metric (6.1) tends to the AdS space limit.

The black hole parameters M, q, and $\ell$ are related to the roots of $r^4 + \ell^2 r^2 - 2m\ell^2 r + \ell^2 q^2 = 0$ . The black hole (event) horizon $r_+$ (positive real root) is located at

$$\begin{aligned} r_+ &= \frac{\ell}{\sqrt{3}}\sinh\Big[\frac{1}{3}\sinh^{-1}\frac{3m\sqrt{3}}{\ell\sqrt{1-\frac{4q^2}{\ell^2}}}\Big] \\ &\times\Bigg(1+\sqrt{1-\frac{q^2\ell}{\sqrt{3}m}\frac{2}{1+\delta}\text{cosech}\Big[\frac{1}{3}\sinh^{-1}\frac{3m\sqrt{3}}{\ell\sqrt{1-\frac{4q^2}{\ell^2}}}\Big]}\Bigg), \end{aligned} \tag{6.2}$$

where

$$\delta = \sqrt{1-\frac{4q^2}{3m^2}\sinh^2\Big[\frac{1}{3}\sinh^{-1}\frac{3\sqrt{3}m}{\ell\sqrt{1-\frac{4q^2}{\ell^2}}}\Big]}. \tag{6.3}$$

Expanding only positive real root $r_+$ in terms of mass, electric charge and cosmological parameter with $27\frac{m^2}{\ell^2} < 1$ as well as $\frac{3\sqrt{3}m}{\ell\alpha} < 1$ and setting $\delta = 1$, we obtain

$$r_+ = \frac{m}{\alpha}\left(1 - \frac{4m^2}{\ell^2\alpha^2} + \cdots\right)\left(1 + \sqrt{1 - \frac{q^2\alpha}{m^2}}\right), \tag{6.4}$$

which can be written as

$$r_+ = \frac{1}{\alpha}\left(1 - \frac{4m^2}{\ell^2\alpha^2} + \cdots\right)\left(m + \sqrt{m^2 - q^2\alpha}\right), \tag{6.5}$$

where $\alpha = \sqrt{1 - \frac{4q^2}{\ell^2}}$.

Now we can write $r_+ = (m + \sqrt{m^2 - q^2\alpha})\kappa$, with $\kappa < 1$. Therefore, the event horizon of the RNAdS black hole is smaller than the Reissner-Nordström event horizon, $r_h = m + \sqrt{m^2 - q^2}$. It gives the RN black hole [99] for $\ell \to \infty$ and Schwarzschild-anti-de Sitter black hole [130] for

$q = 0$. The metric (6.1) represents an interesting asymptotically anti-de Sitter extreme RN black hole for $q^2 = \alpha m^2$, while for $q^2 > \alpha m^2$ it does represent any black hole but an unphysical naked singularity at $r = 0$. As the event horizon of RNAdS black hole synchronizes with the outer infinite redshift surface, we can apply the geometrical optics limit. Using the WKB approximation [168], the tunneling probability for the classically forbidden trajectory of the s-wave coming from inside to outside of RNAdS event horizon is given by

$$\Gamma \sim \exp(-2\mathrm{Im}I). \tag{6.6}$$

The method different from Parikh and Wilczek method (Null geodesic) in which the action mainly depends on the exploration of the equation of motion in the Painlevé coordinates systems and the calculation of Hamilton equation. In the Hamilton-Jacobi method we avoid this for calculating the imaginary part of the action $I$. For comfortable discussion, we define $\Delta = r^2 + q^2 - 2mr + \frac{r^4}{\ell^2}$ and then the line element (6.1) can be written as

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.7}$$

The position of black hole horizon is same as given in Eq. (6.5). The line element (6.7) near the black hole horizon can be rewritten as

$$ds^2 = -\frac{\Delta_{,r}(r_+)(r-r_+)}{r_+^2}dt^2 + \frac{r_+^2}{\Delta_{,r}(r_+)(r-r_+)}dr^2 + r_+^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6.8}$$

where

$$\Delta_{,r}(r_+) = \frac{d\Delta}{dr}\Big|_{r=r_+} = 2(r_+ - m + 2\frac{r_+^3}{\ell^2}). \tag{6.9}$$

## 6.3 The Hamilton-Jacobi Method

For calculating the imaginary part of the action, we use the method of Chen et al. [125] and making use of Hamilton-Jacobi equation. The action $I$ of the outgoing particle from the black hole horizon satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\left(\frac{\partial I}{\partial x^{\mu}}\right)\left(\frac{\partial I}{\partial x^{\nu}}\right)+u^2=0, \tag{6.10}$$

in which $u$ and $g^{\mu\nu}$ are the mass of the particle and the inverse metric tensors derived from the line element (6.8).

The non-null inverse metric tensors for the metric (6.8) are

$$g^{00}=-\frac{r_+^2}{\Delta_{,r}(r_+)(r-r_+)},\quad g^{11}=\frac{\Delta_{,r}(r_+)(r-r_+)}{r_+^2},$$
$$g^{22}=\frac{1}{r_+^2},\quad g^{33}=\frac{1}{r_+^2\sin^2\theta}. \tag{6.11}$$

With the help of Eq. (6.11), the Hamilton-Jacobi Eq.(6.10) can be written as

$$-\frac{r_+^2}{\Delta_{,r}(r_+)(r-r_+)}\left(\frac{\partial I}{\partial t}\right)^2+\frac{\Delta_{,r}(r_+)(r-r_+)}{r_+^2}\left(\frac{\partial I}{\partial r}\right)^2+\frac{1}{r_+^2}\left(\frac{\partial I}{\partial \theta}\right)^2$$
$$+\frac{1}{r_+^2\sin^2\theta}\left(\frac{\partial I}{\partial \phi}\right)^2+u^2=0. \tag{6.12}$$

To solve the action $I$ conveniently for $I(t,r,\theta,\phi)$, we consider the properties of black hole spacetime, the separation of variables can be taken as follows

$$I=-\omega t+R(r)+H(\theta)+j\phi, \tag{6.13}$$

where $\omega$ and $j$ are respectively the energy and angular momentum of the particle. Since RNAdS black hole is nonrotating, the angular velocity of the particle at the horizon is $\Omega_+ = \frac{d\phi}{dt}\Big|_{r=r_+} = 0$. Inserting Eq. (6.13) into Eq. (6.12) and solving $R(r)$ holds an expression of

$$\begin{aligned} R(r) &= \pm\frac{r_+^2}{\Delta_{,r}(r_+)}\int\frac{dr}{(r-r_+)} \\ &\times\sqrt{\omega^2-\frac{\Delta_{,r}(r_+)(r-r_+)}{r_+^2}\left[g^{22}(\partial_\theta H(\theta))^2+g^{33}j^2+u^2\right]}. \end{aligned} \quad (6.14)$$

Suppose the emitted particle as an ellipsoid shell of energy $\omega$ to tunnel across the event horizon and should not have motion in $\theta$-direction ($d\theta = 0$) and therefore, finishing the above integral we get

$$R(r) = \pm\frac{\pi i r_+^2}{\Delta_{,r}(r_+)}\omega+\epsilon, \quad (6.15)$$

where $\pm$ sign comes from the square root and $\epsilon$ is the constant of integration. Inserting Eq. (6.15) into Eq. (6.13), the imaginary part of two different actions corresponding to the outgoing and incoming particles can be written as

$$\text{Im}I_\pm = \pm\frac{\pi r_+^2}{\Delta_{,r}(r_+)}\omega+\text{Im}(\epsilon). \quad (6.16)$$

In accordance with classical limit [169], we make certain the incoming probability to be unity when there is no reflection i.e., everything is absorbed by the horizon. In this situation the appropriate value of $\epsilon$ instead of zero or infinity can be taken as $\epsilon = \frac{\pi i r_+^2}{\Delta_{,r}(r_+)}\omega+\text{Re}(\epsilon)$. Therefore, $\text{Im}I_- = 0$ and using Eq. (6.9) $I_+$ give the imaginary part of action $I$ corresponding to the outgoing particle of the form

$$\begin{aligned} \text{Im}I &= \frac{2\pi r_+^2}{\Delta_{,r}(r_+)}\omega \\ &= \frac{\pi r_+^2}{r_+-m+2\frac{r_+^3}{\ell^2}}\omega. \end{aligned} \quad (6.17)$$

Using Eq. (6.5) into Eq. (6.17), we get the imaginary part of action as

$$\mathrm{Im}I = \frac{\pi.\frac{1}{\alpha^2}\left(1-\frac{4m^2}{\ell^2\alpha^2}+\cdot\cdot\right)^2(m+\sqrt{m^2-q^2\alpha})^2}{\frac{1}{\alpha}\left(1-\frac{4m^2}{\ell^2\alpha^2}+\cdot\cdot\right)(m+\sqrt{m^2-q^2\alpha})-m+A}\omega, \qquad (6.18)$$

where $A=\frac{2}{\ell^2\alpha^3}\left(1-\frac{4m^2}{\ell^2\alpha^2}+\cdot\cdot\right)^3(m+\sqrt{m^2-q^2\alpha})^3$.

$$\mathrm{Im}I = \frac{\pi.\frac{1}{\alpha^2}(m+\sqrt{m^2-q^2\alpha})^2}{\frac{1}{\alpha}\left[\left(1+\frac{4m^2}{\ell^2\alpha^2}+\cdot\cdot\right)(m+\sqrt{m^2-q^2\alpha})-m\alpha\left(1+\frac{8m^2}{\ell^2\alpha^2}+\cdot\cdot\right)+B\right]}\omega,$$

where $B=\frac{2}{\ell^2\alpha^2}\left(1-\frac{4m^2}{\ell^2\alpha^2}+\cdot\cdot\right)(m+\sqrt{m^2-q^2\alpha})^3$.

Now for the simplicity, neglecting $m^3$ and its higher order terms, we then get

$$\mathrm{Im}I = \pi.\frac{1}{\alpha}.\frac{(m+\sqrt{m^2-q^2\alpha})^2}{(m+\sqrt{m^2-q^2\alpha})-m\alpha}\omega. \qquad (6.19)$$

In presence of cosmological constant, RNAdS spacetime is dynamic, we fix the ADM(Amowitt-Deser-Misner) mass and angular momentum of the total spacetime and allow the RNAdS black hole to fluctuate. When a particle with energy $\omega$ tunnels out, the mass of the RNAdS black hole changed into $m-\omega$. Since the angular velocity of the particle at the horizon is zero ($\Omega_+=0$), the angular momentum is equal to zero. Taking the self-gravitational interaction into account, the imaginary part of the true action can be calculated from Eq. (6.19) in the following integral form

$$\mathrm{Im}I = \pi\frac{1}{\alpha}.\int_0^\omega\frac{(m+\sqrt{m^2-q^2\alpha})^2}{(m+\sqrt{m^2-q^2\alpha})-m\alpha}d\omega' \qquad (6.20)$$

$$\mathrm{Im}I = \pi\frac{1}{\alpha}.\int_0^\omega\frac{(m+\sqrt{m^2-q^2\alpha})^2}{\sqrt{m^2-q^2\alpha}+(1-\alpha)m}d\omega'. \qquad (6.21)$$

For the maximum value of the above integral, neglecting $(1-\alpha)m$. Thus euation (6.21) becomes

$$\text{Im}I = \pi\frac{1}{\alpha}.\int_0^{\omega}\frac{\left(m+\sqrt{m^2-q^2\alpha}\right)^2}{\sqrt{m^2-q^2\alpha}}d\omega'. \quad (6.22)$$

Replacing $m$ by $m-\omega$ we have

$$\text{Im}I = -\pi\frac{1}{\alpha}.\int_m^{(m-\omega)}\frac{\left(m-\omega'+\sqrt{(m-\omega')^2-q^2\alpha}\right)^2}{\sqrt{(m-\omega')^2-q^2\alpha}}d(m-\omega') \quad (6.23)$$

$$\begin{aligned}\text{Im}I &= -\pi\frac{1}{\alpha}.\int_m^{(m-\omega)}\frac{2(m-\omega')^2+2(m-\omega')\sqrt{(m-\omega')^2-q^2\alpha}-q^2\alpha}{\sqrt{(m-\omega')^2-q^2\alpha}}\times \\ & \quad d(m-\omega'). \end{aligned} \quad (6.24)$$

Finishing the integral we get

$$\text{Im}I = -\pi\frac{1}{\alpha}.[(m-\omega)\sqrt{(m-\omega)^2-q^2\alpha}+(m-\omega)^2-m\sqrt{m^2-q^2\alpha}-m^2]. \quad (6.25)$$

Therefore, the non-thermal tunneling rate is given by

$$\begin{aligned}\Gamma\sim\exp(-2\text{Im}I) &= \exp\{\pi.\frac{1}{\alpha}[2(m-\omega)^2 \\ &\quad +2(m-\omega)\sqrt{(m-\omega)^2-q^2\alpha}-2m\sqrt{m^2-q^2\alpha}-2m^2]\} \\ &= \exp[\pi(r_f^2-r_i^2)] \\ &= \exp(\Delta S_{BH}). \end{aligned} \quad (6.26)$$

Here, $r_i=\frac{1}{\sqrt{\alpha}}[m+\sqrt{m^2-q^2\alpha}]$ and $r_f=\frac{1}{\sqrt{\alpha}}[(m-\omega)+\sqrt{(m-\omega)^2-q^2\alpha}]$ are the locations of the RNAdS event horizon before and after the particle emission, and $\Delta S_{BH}=S_{BH}(m-\omega)-S_{BH}(m)$ is the change of Bekenstein-Hawking entropy [51].

## 6.4 Purely Thermal Radiation

The radiation spectrum described by Eq. (6.26) is not pure thermal although gives a correction to the Hawking radiation of RNAdS black hole. The purely thermal spectrum can be derived from Eq. (6.26) by expanding the tunneling rate in power of $\omega$ up to second order as discussed by Liu et al. [65] of the form

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{-\omega\frac{\partial S_{BH}(m)}{\partial m} + \frac{\omega^2}{2}\frac{\partial^2 S_{BH}(m)}{\partial m^2}\right\}. \qquad (6.27)$$

From Eq.(6.26)

$$S_{BH}(m-\omega) = \frac{\pi}{\alpha}\{(m-\omega) + \sqrt{(m-\omega)^2 - q^2\alpha}\}^2. \qquad (6.28)$$

Using (6.28) in (6.27), we get

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp[\pi(-\omega\beta + \frac{\omega^2}{2}\gamma)], \qquad (6.29)$$

where $\beta = \frac{2}{\alpha}\left[2m + \sqrt{m^2 - q^2\alpha} + \frac{m^2}{(m^2-q^2\alpha)^{\frac{3}{2}}}\right]$ and
$\gamma = \frac{2}{\alpha}\left[2 + \frac{3m}{\sqrt{m^2-q^2\alpha}} - \frac{m^3}{(m^2-q^2\alpha)^{\frac{3}{2}}}\right]$.
If we put $-\ell^2$ in the place of $\ell^2$, the Hawking non thermal spectrum and pure thermal spectrum agree with these of RNdS black hole.

## 6.5 Concluding Remarks

Hawking radiation as massive particle tunneling method from RNAdS black hole [131] have been presented here. By taking into account the self-gravitational interaction of particle, the background spacetime as dynamical and the energy as conservation, we have found the non-thermal and purely thermal tunneling probabilities of RNAdS black hole. The non-thermal tunneling probability of particle emission is proportional to the phase space factor depending on the initial and final entropy of the system (the change of the Bekenstein-Hawking entropy), which implies that the emission spectrum actually deviates from perfect thermally but is in agreement with an underlying unitary theory. The similar results have been shown under the same assumption for massive particles tunneling across the event horizon of SAdS [130] and Taub-NUT [58, 126, 182] black holes. Our motivation also indeed support the results obtained by massless or massless charged particles tunneling from different spacetimes such as charged black hole with a global monopole [99, 128], Kerr-NUT black hole [65] and Kerr and Kerr-Newman black holes [60, 103] as well as other cases [48, 172]. We therefore come to the conclusion that the actual radiation spectrum of RNAdS black hole is not precisely thermal, which provides an interesting correction to Hawking pure thermal spectrum. In the limiting case, i.e., when $\ell \to \infty$, $\alpha \to 1$ our results for non-thermal and purely thermal radiations reduced to

$$\Gamma \sim \exp(-2\mathrm{Im}I) = \exp\{\pi[\{(m-\omega) + \sqrt{(m-\omega)^2 - q^2}\}^2 - \{m + \sqrt{m^2 - q^2}\}^2]\}, \tag{6.30}$$

and

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\Big[ - 2\pi\omega\{(2m + \sqrt{m^2-q^2} + \frac{m^2}{\sqrt{m^2-q^2}}) - \frac{\omega}{2}(2 + \frac{3m}{\sqrt{m^2-q^2}} - \frac{m^3}{(m^2-q^2)^{\frac{3}{2}}})\}\Big]. \tag{6.31}$$

These are the non-thermal and purely thermal tunneling rates of Reissner-Nordström black hole, where $r_i = m + \sqrt{m^2 - q^2}$ and $r_f = (m - \omega) + \sqrt{(m - \omega)^2 - q^2}$ are the positions of the event horizon of Reissner-Nordström black hole before and after the particles emission. Obviously, both the results are fully consistent with that obtained by Parikh and Wilczek [51] and is also reduced to our previous result of SAdS [130] black hole when $q = 0$.

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# Chapter 7

## Hawking Non-thermal and Purely thermal radiations of Kerr-de Sitter Black Hole by Hamilton-Jacobi method

### 7.1 Introduction

A prominent role in black hole physics, the discovery by Hawking in 1974 that black hole can emit particles [24, 25] from the event horizon. At once time, the picture of the primary state of composing matter leads to a paradoxical claim of black hole information loss. This is so-called "the paradox of black hole information loss". Taking quantum process into account, the black holes mass becomes small and smaller and eventually completely evaporated [27] that is the situation is changed. In the tunneling mechanism, virtual particles and anti-particles create due to vacuum fluctuation and a positive energy particle tunnels out of the horizon and materializes as a real particle, while the negative energy particle tunnels into the horizon and is absorbed and moved to infinite distance whenever the positive energy particle is left outside the horizon and forms the Hawking radiation. A semiclassical tunneling method innovated by Kraus and Wilczek [53, 54] and then reinterpreted by Parikh and Wilczek [51] and have shown that the actual emission spectrum deviates from the purely thermal. People have growing interest and therefore several works have been done for further development of the tunneling approach, but all of them are only focused on Hawking radiation of various black hole spacetime such as those in de Sitter [67, 82, 98, 99, 183], anti-de Sitter [38, 75, 77, 184] spacetimes, charged black holes [100, 183], rotating black holes [79, 103] and many other cases in references [55, 56, 57, 58, 60, 61, 62, 63, 64, 68, 69, 71, 72, 74, 76, 78, 80, 81]. All of these works are limited to massless particle and gives a correction to the emission rate arising from loss of mass of the black hole corresponding to the energy carried by radiated quantum when the energy conservation and the self gravitation interaction are considered . To calculate the action, most of the researchers introduced Painlevé or dragging coordinates that are well regular at the horizon of the black hole and found out the motion equation of the particle, and then calculated Hamilton equation to get it. We have also used dragging coordinates and express the event horizon of Kerr-de sitter (KdS) black hole in terms of black hole parameters in an infinite series and finally show that the tunneling rate is related to the change of Bekenstein-Hawking entropy.

Based on semiclassical tunneling picture, Angheben et al. [90] proposed 'Hamilton-Jacobi method' and in fact this method is an extension of the complex path analysis proposed by Padmanabhan et al. [85, 86, 87, 88, 89]. This method involves calculating the imaginary part of the action from relativistic Hamilton-Jacobi equation in which the derive radiation spectrum was only a leading term due to the fact that the self-gravitation interaction and energy conservation of emitted particle were ignored. Parikh and Wilczek [51] have shown that the true radiation spectrum is not strictly thermal but satisfies the underlying unitary theory when self-gravitation interaction and energy conservation are considered and the background geometry of a radiating black hole can be altered (unfixed) with the loss of radiated energy.

Kerner and Mann promoted quantum tunneling methods for various black hole spacetime [106, 107, 108, 109] and considering this process Chen, Zu and Yang [126] reformed Hamilton-Jacobi method for massive particle tunneling and investigate Hawking radiation of Kerr-NUT black hole [65], the charged black hole with a global monopole [99, 128] and also applied to higher dimensional black holes [110, 111, 112], black holes in string theory [113], black strings [114, 115, 116], accelerating and rotaing black holes [118, 119, 120] and many other black holes in references [66, 73]. Following their work, several researches have been carried out as charged particle tunneling [79, 101, 102, 103] and all the results supported Parikh and Wilczek's [51] opinion and gave a correction to the Hawking pure thermal spectrum.

In recent times, we have developed Hamilton-Jacobi method and investigated the hawking purely thermal and non-thermal radiations of the SdS [129] black hole where the position of the black hole horizon is taken in a series of black hole's parameters so that the spacetime metric becomes dynamical and self-gravitation interaction are taken into account and the changed of background geometry can be treated as the loss of radiated energy of the black hole. In this chapter, we have been applied the same method to investigate the Hawking radiation of Kerr-de Sitter (KdS) black hole [161]. The method of Chen et al. [125, 126] is used to describe Hawking-Radiation from the action of radiation particles. Since our prime concern of this work is to calculate the imaginary part of action from Hamilton-Jacobi equation avoid by exploring the equation of motion of the radiation particle in Painlevé coordinate system and calculating the Hamilton equation. The equation of motion of massive particles are different from massless particle though the radiation particles do not vary. After considering the self-gravitational interaction and the unfixed background spacetime, the derived radiation spectrum deviates from the purely thermal one and the tunneling rate is related to the change of Bekenstein-Hawking entropy.

It is noticed that the cosmological constant plays an important role in our research because the accelerating expansion of our universe indicates the cosmological constant might be a positive one [132, 153, 154], and the conjecture about de Sitter/conformal field theory (CFT) correspondence [166, 167] has been suggested that there is a dual relation between quantum gravity on a dS space and Euclidean conformal field theory (CFT) on a boundary of dS space [157, 159]. The outgoing particles tunnel from black hole horizon and incoming particles tunnel from cosmological horizon and formed Hawking radiation and the incoming particles can fall into the horizon along classically permitted trajectories for black hole horizon, but outgoing particles can fall classically out of the horizon for cosmological horizon. Thus our study of black hole on Kerr-de Sitter black hole [161] is of great consequence and significant.

We arrange this chapter as follows. The later section describes the KdS black hole spacetime with the position of event horizon. The new line element of KdS black hole near the event horizon is derived in section 7.3 and the unfixed background spacetime and the self-gravitational interaction are taken into account, we review the non-thermal radiation of KdS black hole from massive particle tunneling method. In section 7.4 we discuss the pure thermal radiation. Finally, section 7.5 includes our remarks.

## 7.2 Kerr-de Sitter black hole

The line element, describing Kerr-de Sitter black hole solution with a positive cosmological constant $\Lambda(= 3/\ell^2)$, rotating black hole in four-dimensional spacetime with asymptotic-de Sitter behavior in the Boyer-Lindguist coordinates [185] is given by

$$\begin{aligned} ds^2 &= -\frac{f(r) - f(\theta)a^2\sin^2\theta}{\rho^2}dt^2 + \frac{f(\theta)(r^2+a^2)^2 - f(r)a^2\sin^2\theta}{\rho^2\Sigma^2}\sin^2\theta d\phi^2 \\ &\quad + \frac{\rho^2}{f(r)}dr^2 + \frac{\rho^2}{f(\theta)}d\theta^2 - \frac{2a[(r^2+a^2)f(\theta) - f(r)]\sin^2\theta}{\rho^2\Sigma}dtd\phi, \quad (7.1) \end{aligned}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad f(\theta) = 1 + \frac{a^2\cos^2\theta}{\ell^2}, \quad \Sigma = 1 + \frac{a^2}{\ell^2},$$
$$f(r) = (r^2+a^2)(1 - \frac{r^2}{\ell^2}) - 2Mr = (1 - \frac{a^2}{\ell^2})r^2 - 2Mr + a^2 - \frac{r^4}{\ell^2}. \quad (7.2)$$

Here $\ell$ is the cosmological radius, $M$ and $a$ are the mass of the black hole and angular momentum per unit mass. The specific angular momentum

$a = (J\Sigma^2)/M$ is kept as a constant through this chapter. The de Sitter space are defined such that $-\infty \leq t \leq \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. The metric (7.1) describes an interesting rotating AdS black hole called the Kerr-Anti-de Sitter (KAdS) black hole if we replace $\ell^2$ by$-\ell^2$.

The only single positive real root is obtained by $r^4 - (\ell^2 - a^2) + 2M\ell^2 r - \ell^2 a^2 = 0$ and which is located at the black hole (event) horizon $r_h$ such that

$$r_h = \frac{\ell\beta}{\sqrt{3}}.\sin\Big[\frac{1}{3}\sin^{-1}\frac{3M\sqrt{3}}{\ell\Sigma\beta}\Big] \times\Big(1 + \sqrt{1 - \frac{a^2\ell}{\sqrt{3}M\beta}.\frac{2}{1+\delta}\text{cosec}\Big[\frac{1}{3}\sin^{-1}\frac{3M\sqrt{3}}{\ell\Sigma\beta}\Big]}\Big), \tag{7.3}$$

where

$$\delta = \sqrt{1 - \frac{4a^2\beta^2}{3M^2}\sin^2\Big[\frac{1}{3}\sin^{-1}\frac{3\sqrt{3}M}{\ell\Sigma\beta}\Big]}, \quad \Sigma = 1 + \frac{a^2}{\ell^2}, \quad \beta = \sqrt{1 - \frac{a^2}{\ell^2}}. \tag{7.4}$$

Expanding $r_h$ in terms of $\ell$, $M$ and $a$ with $a^2(1 + \frac{a^2}{\ell^2}) < M^2$ and setting $\delta = 1$, we obtain

$$r_h = \frac{M}{\Sigma}\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdots\right)\left(1 + \sqrt{1 - \frac{a^2\Sigma}{M^2}}\right), \tag{7.5}$$

which can be written as

$$r_h = \frac{1}{\Sigma}\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdots\right)\left(M + \sqrt{M^2 - a^2\Sigma}\right). \tag{7.6}$$

It is clear that the event horizon of the Kerr-de Sitter black hole is greater than the Kerr event horizon $r_{Ke} = M + \sqrt{M^2 - a^2}$. Again it also shows that Kerr [81] event horizons for $\ell \to \infty(\Sigma \to 1)$ and Schwarzschild-de Sitter [129] event horizons for $a = 0$.

## 7.3 The Hamilton-Jacobi Method

Two new methods have been employed to calculate the imaginary part of the action, one the null geodesic method developed by Parikh and Wilczek [51, 82, 83] and the other method is called Hamilton-Jacobi method [85, 86, 87, 88, 89]. The difference of later method from Parikh's is mainly that such method concentrates on introducing the proper spatial distance and upon calculating the relaivistic Hamilton-Jacobi equation. For calculating the imaginary part of the action for the process of s-wave emission across the horizon, which in turn is related, using the WKB approximation [168], satisfies $\Gamma \sim \exp(-2\mathrm{Im}I)$, where $I$ is the action of the outgoing particle and $\Gamma$ is the emission rate.

In the Hamilton-Jacobi method we avoid the exploration of the equation of motion in the Painlevé coordinates systems. In order to calculate the imaginary part of the action from the relativistic Hamilton-Jacobi equation, the action $I$ of the outgoing particle from the black hole horizon satisfies the relativistic Hamilton-Jacobi equation

$$g^{ab}\left(\frac{\partial I}{\partial x^a}\right)\left(\frac{\partial I}{\partial x^b}\right)+m^2=0, \tag{7.7}$$

in which $m$ and $g^{ab}$ are the mass of the particle and the inverse metric tensors respectively.

We now define $\dot{\phi}=\frac{d\phi}{dt}=-\frac{g_{14}}{g_{44}}$ on the line element (7.1) and hence the Kerr-de Sitter black hole can be written as

$$ds^2=-\frac{f(r)f(\theta)\rho^2}{f(\theta)(r^2+a^2)^2-f(r)a^2\sin^2\theta}dt^2+\frac{\rho^2}{f(r)}dr^2+\frac{\rho^2}{f(\theta)}d\theta^2. \tag{7.8}$$

The position of the event horizon is same as given in Eq. (7.6). The action can be derived from the line element (7.1) and (7.8) respectively. For the convenience, we select the line element (7.8) and make a treatment to it.

Near the event horizon, the line element (7.8) takes on form as

$$ds^2 = -\frac{f_{,r}(r_h)(r-r_h)\rho^2(r_h)}{(r_h^2+a^2)^2}dt^2 + \frac{\rho^2(r_h)}{f_{,r}(r_h)(r-r_h)}dr^2 + \frac{\rho^2(r_h)}{f(\theta)}d\theta^2. \quad (7.9)$$

In which $\rho^2(r_h) = r_h^2 + a^2\cos^2\theta$ and $f_{,r}(r_h) = \frac{df}{dr}\Big|_{r=r_h} = \frac{2}{\Sigma^2}(\beta^2 r_h - M - 2\frac{r_h^3}{\ell^2})$. The non-null inverse metric tensors for the metric (7.9) are namely

$$\bar{g}^{11} = -\frac{(r_h^2+a^2)^2}{f_{,r}(r_h)(r-r_h)\rho^2(r_h)}, g^{22} = \frac{f_{,r}(r_h)(r-r_h)}{\rho^2(r_h)}, g^{33} = \frac{f(\theta)}{\rho^2(r_h)}. \quad (7.10)$$

We can write Eq. (7.7) with the help of Eq. (7.10) as

$$-\frac{(r_h^2+a^2)^2}{\rho^2(r_h)f_{,r}(r_h)(r-r_h)}\left(\frac{\partial I}{\partial t}\right)^2 + \frac{f_{,r}(r_h)(r-r_h)}{\rho^2(r_h)}\left(\frac{\partial I}{\partial r}\right)^2 + \frac{f(\theta)}{\rho^2(r_h)}\left(\frac{\partial I}{\partial \theta}\right)^2 + m^2 = 0. \quad (7.11)$$

To find the solution of the action $I$ for $I(t,r,\theta,\phi)$ in a easy way, we consider the properties of black hole spacetime, the separation of variables can be taken as follows

$$I = -\omega t + R(r) + H(\theta) + j\phi, \quad (7.12)$$

where $\omega$ is the energy of the particle, $R(r)$ and $H(\theta)$ are the generalized momentums, and $j$ is the angular momentum with respect to $\phi$-axis.

So we have $\frac{\partial I}{\partial t} = -\omega + j\Omega_h$, $\frac{\partial I}{\partial r} = \frac{\partial R(r)}{\partial r}$, $\frac{\partial I}{\partial \theta} = \frac{\partial H}{\partial \theta}$, where $\Omega_h = \frac{d\phi}{dt}\Big|_{r=r_h} = \frac{a\Sigma}{r_h^2+a^2}$ is the angular velocity at the event horizon and $j = (Ma)/\Sigma^2$.

Therefore, inserting above values into Eq.(7.11) and solving $R(r)$ yields an expression of

$$R(r) = \pm\frac{r_h^2+a^2}{f_{,r}(r_h)}\int\frac{dr}{(r-r_h)}\times$$

$$\sqrt{(\omega - j\Omega_h)^2 - \frac{\rho^2(r_h) f_{,r}(r_h)(r-r_h)}{(r_h^2+a^2)^2}\left[\frac{f(\theta)}{\rho^2(r_h)}.(\frac{\partial H}{\partial \theta})^2 + m^2\right]}.$$

Finishing the above integral we get

$$R(r) = \pm \frac{\pi i (r_h^2 + a^2)}{f_{,r}(r_h)}(\omega - j\Omega_h) + \sigma, \tag{7.13}$$

where $\pm$ sign comes from the square root and $\sigma$ is the constant of complex integration. The imaginary part of the action arising due to pole at the event horizon can be obtained from the complex constant $\sigma$ and therefore, we can write the probabilities of ingoing and outgoing particles whenever crossing $r_h$ as follows:

$$\begin{aligned} P_{in} &= \exp(-2\text{Im}I) = \exp[-2(\text{Im}R_- + \text{Im}\sigma)], \\ P_{out} &= \exp(-2\text{Im}I) = \exp[-2(\text{Im}R_+ + \text{Im}\sigma)]. \end{aligned} \tag{7.14}$$

In the classical point of view [169], when there is no reflection for the ingoing waves, the incoming probability "$P_{in}$" be unity that is everything is absorbed by the black hole for any ingoing particles passing its horizon. In this case, we take $\text{Im}\sigma = \frac{\pi(r_h^2+a^2)}{f_{,r}(r_h)}(\omega - j\Omega_h)$, which implies that the imaginary part of the action $I$ for a massive tunneling particle can only come out $R_+$. Therefore, we obtain the imaginary part of action $I$ corresponding to the outgoing particle of the form, namely

$$\begin{aligned} \text{Im}I &= \text{Im}R_+ + \text{Im}\sigma \\ &= \frac{2\pi(r_h^2+a^2)}{f_{,r}(r_h)}(\omega - j\Omega_h) \\ &= \frac{\Sigma^2\pi(r_h^2+a^2)}{\beta^2 r_h - M - 2\frac{r_h^3}{\ell^2}}(\omega - j\Omega_h). \end{aligned} \tag{7.15}$$

We now focus on a classical treatment of the associated radiation and adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out while the negative one is absorbed by the black hole resulting in a decrease in the mass. We consider the emitted particle as an ellipsoid shell of energy $\omega$ and fix the Arnowitt-Deser-Misner(ADM) mass and angular momentum of the total spacetime since in presence of cosmological constant KdS spacetime is dynamic and allow the KdS black hole to fluctuate. When a particle with energy $\omega$ and angular momentum $j$ tunnels out, the mass and angular momentum of the KdS black hole changed into $M - \omega$ and $J - j$. Assuming the self-gravitational interaction into account, the imaginary part of the true action can be calculated from Eq. (7.15) in the following integral form

$$
\begin{aligned}
\mathrm{Im} I &= \pi\Sigma^2 . \int_{(0,0)}^{(\omega,j)} \frac{(r_h^2 + a^2)}{\beta^2 r_h - M - 2\frac{r_h^3}{\ell^2}} (d\omega' - \Omega_h dj') \\
&= -\pi\Sigma^2 . \int_{(M,J)}^{(M-\omega,J-j)} \frac{(r_h^2 + a^2)}{\beta^2 r_h - (M-\omega') - 2\frac{r_h^3}{\ell^2}} \times \\
&\quad [d(M-\omega') - \frac{a\Sigma}{r_h^2 + a^2} d(J - j')] \\
&= -\pi\Sigma^2 . \int_{M}^{(M-\omega)} \frac{r_h^2}{\beta^2 r_h - (M-\omega') - 2\frac{r_h^3}{\ell^2}} d(M-\omega') \\
&\quad -\pi\Sigma^2 . \int_{M}^{(M-\omega)} \frac{a^2}{\beta^2 r_h - (M-\omega') - 2\frac{r_h^3}{\ell^2}} d(M-\omega') \\
&\quad +\pi\Sigma^3 . \int_{J}^{(J-j)} \frac{a}{\beta^2 r_h - (M-\omega') - 2\frac{r_h^3}{\ell^2}} d(J - j'),
\end{aligned}
$$

. (7.16)

where $J - j' = (M - \omega')a/\Sigma^2$ and $M$ will be replaced by $M - \omega$ in $r_h$.

Using Eq. (7.6) we evaluate the following

$$\begin{aligned} \text{Im}I &= \frac{\Sigma^2\pi(r_h^2 + a^2)}{\beta^2 r_h - M - 2\frac{r_h^3}{\ell^2}}(\omega - j\Omega_h) \\ &= \frac{\pi\Sigma^2(r_h^2 + a^2)}{\beta^2 r_h - M - 2\frac{r_h^3}{\ell^2}}\omega - \frac{\pi a\Sigma^3}{\beta^2 r_h - M - 2\frac{r_h^3}{\ell^2}}j. \end{aligned} \tag{7.17}$$

$$\begin{aligned} \text{Im}I = & \frac{\pi\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)^2 (M + \sqrt{M^2 - a^2\Sigma})^2}{\frac{\beta^2}{\Sigma}\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Sigma}) - M - A}\omega \\ & + \frac{\Sigma^2\pi a^2}{\frac{\beta^2}{\Sigma}\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Sigma}) - M - A}\omega \\ & - \frac{\Sigma^3\pi a}{\frac{\beta^2}{\Sigma}\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Sigma}) - M - A}j, \end{aligned}$$

where $A = \frac{2}{\ell^2\Sigma^3}\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)^3 (M + \sqrt{M^2 - a^2\Sigma})^3$.

$$\begin{aligned} \text{Im}I \;=\; & \frac{\pi(M + \sqrt{M^2 - a^2\Sigma})^2}{\frac{\beta^2}{\Sigma}\left[\left(1 - \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Sigma}) - \frac{M\Sigma}{\beta^2}\left(1 - \frac{8M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right) - B\right]}\omega \\ & + \frac{\Sigma^2\pi a^2}{\frac{\beta^2}{\Sigma}\left[\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Sigma}) - \frac{M\Sigma}{\beta^2} - \frac{\Sigma A}{\beta^2}\right]}\omega \\ & - \frac{\Sigma^3\pi a}{\frac{\beta^2}{\Sigma}\left[\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Sigma}) - \frac{M\Sigma}{\beta^2} - \frac{\Sigma A}{\beta^2}\right]}j, \end{aligned}$$

where $B = \frac{2}{\ell^2\beta^2\Sigma^2}\left(1 + \frac{4M^2}{\ell^2\Sigma\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Sigma})^3$.

Now for the simplicity, neglecting $M^3$ and its higher order terms, we then get

$$\text{Im}I \;=\; \frac{\pi\Sigma}{\beta^2}.\frac{(M + \sqrt{M^2 - a^2\Sigma})^2}{(M + \sqrt{M^2 - a^2\Sigma}) - \frac{M\Sigma}{\beta^2}}\omega + \frac{\Sigma^3\pi a^2}{\beta^2\left[M + \sqrt{M^2 - a^2\Sigma} - \frac{M\Sigma}{\beta^2}\right]}\omega$$

$$-\frac{\Sigma^4\pi a}{\beta^2\left[M+\sqrt{M^2-a^2\Sigma}-\frac{M\Sigma}{\beta^2}\right]}j. \tag{7.18}$$

To obtain the maximum value, neglecting $(1-\frac{\Sigma}{\beta^2})M$. Equation (7.18) becomes

$$\begin{aligned}\mathrm{Im}I &= \frac{\pi\Sigma}{\beta^2}.\frac{(M+\sqrt{M^2-a^2\Sigma})^2}{\sqrt{M^2-a^2\Sigma}}\omega+\frac{\Sigma^3\pi a^2}{\beta^2\left[\sqrt{M^2-a^2\Sigma}\right]}\omega \\ &\quad -\frac{\Sigma^4\pi a}{\beta^2\left[\sqrt{M^2-a^2\Sigma}\right]}j. \end{aligned}\tag{7.19}$$

Now if we replace $M$ by $M-\omega$ and $j$ by $J-j$ in the integral form of Eq. (7.19), then from Eq. (7.16), we obtain [second and third integral vanish]

$$\begin{aligned}\mathrm{Im}I &= -\frac{\pi\Sigma}{\beta^2}.\int_M^{(M-\omega)}\frac{(M-\omega+\sqrt{(M-\omega)^2-a^2\Sigma})^2}{\sqrt{(M-\omega)^2-a^2\Sigma}}d(M-\omega') \\ &= -\frac{\pi\Sigma}{\beta^2}.\int_M^{(M-\omega)}\frac{2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-a^2\Sigma}}{\sqrt{(M-\omega)^2-a^2\Sigma}}d(M-\omega') \\ &\quad +\frac{\pi\Sigma}{\beta^2}.\int_M^{(M-\omega)}\frac{a^2\Sigma}{\sqrt{(M-\omega)^2-a^2\Sigma}}d(M-\omega'). \end{aligned}\tag{7.20}$$

Doing the $\omega'$ integral finally yields

$$\begin{aligned}\mathrm{Im}I &= -\frac{\pi\Sigma}{\beta^2}\{(M-\omega)\sqrt{(M-\omega)^2-a^2\Sigma} \\ &\quad +(M-\omega)^2-M\sqrt{M^2-a^2\Sigma}-M^2\} \\ &= -\frac{\pi\Sigma}{2\beta^2}\{2(M-\omega)\sqrt{(M-\omega)^2-a^2\Sigma} \\ &\quad +2(M-\omega)^2-2M\sqrt{M^2-a^2\Sigma}-2M^2\} \\ &= -\frac{1}{2}\exp[\pi(r_f^2-r_i^2)] \\ &= -\frac{1}{2}\exp(\Delta S_{BH}), \end{aligned}\tag{7.21}$$

where the Bekenstein-Hawking entropy of the black hole is $S_{BH}(M) = \pi r_i^2$ and $S_{BH}(M-\omega) = \pi r_f^2$, and $\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$ is the difference of Bekenstein-Hawking entropy. Setting $r_i = \frac{\sqrt{\Sigma}}{\beta}[M + \sqrt{M^2 - a^2\Sigma}]$ and $r_f = \frac{\sqrt{\Sigma}}{\beta}[(M-\omega) + \sqrt{(M-\omega)^2 - a^2\Sigma}]$, which are the locations of the KdS event horizon before and after the particles emission respectively.

Utilizing WKB approximation [168], the relationship between the tunneling rate and the imaginary part of the action of the radiative particle for the KdS black hole is given by

$$\Gamma \sim \exp(-2\text{Im}I) = \exp(\Delta S_{BH}). \tag{7.22}$$

## 7.4 Purely Thermal Radiation

From Eq. (7.22) we observe that the tunneling rate at the event horizon is related to the Bekenstein-Hawking entropy, and is consistent with an underlying unitary theory. Again the radiation spectrum is not pure thermal although gives a correction to the Hawking radiation of KdS black hole. In the form of a thermal spectrum, using the WKB approximation the tunneling rate is also related to the energy and the Hawking temperature of the radiative particle as $\Gamma \sim \exp(-\frac{\Delta\omega}{T})$. If $\Delta\omega < 0$ is the energy of the emitted particle then due to energy conservation, the energy of the outgoing shell must be $-\Delta\omega$, then above expression becomes $\Gamma \sim \exp(\frac{\Delta\omega}{T})$. By the first law of thermodynamics, it can be written as $\Gamma \sim \exp(\Delta S)$, which is related to the change of Bekenstein-Hawking entropy as follows

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\{S_{BH}(M-\omega) - S_{BH}(M)\}. \tag{7.23}$$

We establish Eq.(7.23) as developed by Hossain et al. [131] in power of $\omega$ upto second order using Taylor's theorem of the form

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{-\omega \frac{\partial S_{BH}(M)}{\partial M} + \frac{\omega^2}{2}\frac{\partial^2 S_{BH}(M)}{\partial M^2}\right\}. \qquad (7.24)$$

From Eq.(7.21), we can write

$$S_{BH}(M-\omega) = \frac{\pi\Sigma}{\beta^2}[(M-\omega) + \sqrt{(M-\omega)^2 - a^2\Sigma}]^2. \qquad (7.25)$$

At $\omega = 0$,

$$\frac{\partial S_{BH}(M)}{\partial M} = \frac{2\Sigma}{\beta^2}\left[2M + \sqrt{M^2 - a^2\Sigma} + \frac{M^2}{\sqrt{M^2 - a^2\Sigma}}\right], \qquad (7.26)$$

$$\frac{\partial^2 S_{BH}(M)}{\partial M^2} = \frac{2\Sigma}{\beta^2}\left[2 + \frac{3M}{\sqrt{M^2 - a^2\Sigma}} - \frac{M^3}{(M^2 - a^2\Sigma)^{\frac{3}{2}}}\right]. \qquad (7.27)$$

Therefore, the tunneling rate in power of $\omega$ upto second order, the purely thermal spectrum can be revealed from Eq. (7.24) as follows:

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp[\pi(-\omega\gamma + \frac{\omega^2}{2}\lambda)], \qquad (7.28)$$

where $\gamma = \frac{2\Sigma}{\beta^2}\left[2M + \sqrt{M^2 - a^2\Sigma} + \frac{M^2}{\sqrt{M^2 - a^2\Sigma}}\right]$ and
$\lambda = \frac{2\Sigma}{\beta^2}\left[2 + \frac{3M}{\sqrt{M^2 - a^2\Sigma}} - \frac{M^3}{(M^2 - a^2\Sigma)^{\frac{3}{2}}}\right]$.

The radiation spectrum given by (7.28) is more accurate and provides an interesting correction to Hawking pure thermal spectrum.

## 7.5 Concluding Remarks

In this chapter, we have discussed the purely thermal and non-thermal Hawking radiations as massive particle tunneling process from KdS black hole by taking into account the self-gravitational interaction, the background spacetime is dynamical and the energy as conservation by employing standard Hamilton-Jacobi method. We have explored that the tunneling rate at the event horizon of KdS black hole is related to the change of Bekenstein-Hawking entropy. In the limiting case $\Sigma = 1, \beta = 1$ the KdS black hole reduces to the Kerr black hole [81]. The positions of the event horizon of Kerr black hole before and after the emission of the particles with energy $\omega$ are $r_i = M + \sqrt{M^2 - a^2}$ and $r_f = (M - \omega) + \sqrt{(M - \omega)^2 - a^2}$. From Eq. (7.22), the tunneling rate of Kerr black hole can be written as

$$\begin{aligned} \Gamma \sim \exp(-2\text{Im}I) &= \exp\Big\{\pi\Big[\{(M-\omega) + \sqrt{(M-\omega)^2 - a^2}\}^2 \\ &\quad -\{M + \sqrt{M^2 - a^2}\}^2\Big]\Big\} \\ &= \exp[\pi(r_f^2 - r_i^2)] \\ &= \exp(\Delta S_{BH}). \end{aligned} \tag{7.29}$$

Again Eq.(7.22) reduced to our previous result of SdS [129] black hole when $a = 0$ and also which is fully consistent with that obtained by Parikh and Wilczek [51, 82, 83] from spherically symmetric black holes.

In addition, our discussion made here can be directly to the anti-de Sitter case by changing the sign of the cosmological constant to a negative one, which have been discussed in later chapter.

# Chapter 8

# Hawking Non-thermal and Purely thermal radiations of Kerr-anti-de Sitter Black Hole by Hamilton-Jacobi method

## 8.1 Introduction

About four decades ago, an extraordinary invention made by Stephen William Hawking that black holes radiate thermally. By the black hole thermodynamics, the thermal radiation with the Hawking temperature determined by the surface gravity at the event horizon [24, 25] is taken as entropy [16, 17, 21] and surface gravity is the acceleration measured at the spatial infinity that a stationary particle should undergo to withstand the gravity at the event horizon. The two important case, one is the information lost and the other one is the technical problem arisen during the study of Hawking thermal radiation. The loss of information was not a serious problem in the classical theory, since the information could be thought of as preserved inside the black hole but just not very accessible. However, taking the quantum effect into consideration, the situation is changed. On the thermal radiation of black holes [38, 51, 52, 53], the emission of Hawking radiation [24, 25], black holes could lose energy, shrink, and eventually becomes smaller and smaller until disappears completely. It seems that pure quantum states (the original matter that forms the black hole) can evolve into mixed states (the thermal spectrum at infinity) and such an evolution violates the fundamental principles of quantum theory, as these prescribe a unitary time evolution of basis states. Derivations in string theory support the idea that Hawking radiation can be described within a manifestly unitary theory, it remains a mystery how information is returned. Moreover, when Hawking first proved the existence of black hole radiation, he described it as tunneling triggered by vacuum fluctuations near the horizon [24, 25] but actual derivation of Hawking radiation did not proceed in this way at all. This method also gives a leading correction to the emission rate arising from loss of mass of the black hole corresponding to the energy carried by the radiated quantum. Carrying this method, the Hawking radiation from AdS black holes have investigated by Hemming and Keski-Vakkuri [38] and Medved has studied those from a de Sitter cosmological horizon [67]. All these spherically symmetric investigations are successful.

Many researchers developed two universal methods to correctly recover Hawking radiation of black holes. First one is the gravitational anomaly method [41] in which the Hawking radiation can be determined by anomaly canceled conditions and regularity requirement at the event horizon. Following, this method is widely used to calculate the Hawking radiation for different black holes [42, 43, 44, 45, 46, 47, 48, 49, 50]. The other is the semi-classical tunneling method initiated by Kraus and Wilczek [52, 53, 54]. Actually their development is not advanced since that the method applied, is limited to discuss the tunneling rate of the uncharged massless particles only [38, 55, 57, 58, 59, 60, 61, 62, 64, 65, 67, 70, 71, 72, 73, 74, 75, 76, 78, 79, 80, 81]. For black holes with a charge, the emitted outgoing particles can be charged also, not only should the energy conservation but also the charge conservation be considered [34, 54]. This tunneling picture can be described in another way, that is, a particle/anti-particle pair is created just outside the horizon, the negative energy particle tunnels into the horizon because the negative energy orbit exists only inside the horizon, the positive energy "partner" is left outside and emerges at infinity.

Parikh's [83, 84] and Parikh-Wilczek' s [51] original calculation only considered the tunneling process of a massless and uncharged particle, recently it is shown that such tunneling method can be easily extended to study the massive [79] and charged particle's tunneling process [63]. Zhang and Zhao was first proposed by Hawking radiation from massive uncharged particle tunneling [98] and charged particle tunneling [100] from black hole and in 2005, Zhang and Zhao et al. extended their work to the Hawking radiation of massive and charged particles and made a great deal of success [99, 101, 102, 103, 104], which has effective significance on the further cognition and research on black holes and also in the same year, a different method was introduced by Angheben et al. [90]. It is called Hamilton-Jacobi method. In fact, the method of Angheben et al. [90] is an extension of the complex path analysis proposed by Padmanabhan et al. [85, 86, 87, 88, 89]. Using the null-geodesic and Hamilton-Jacobi methods, for analyzing the temperature of Taub-NUT black holes , Kerner and Mann [109] developed quantum tunneling methods and Hamilton-Jacobi method is rolled up for calculating the relativistic Hamilton-Jacobi equation. Parikh and Wilczek applied the semi-classical quantum tunneling model to research on the Hawking radiation of the static Schwarzschild and Reissner-Nordström black holes [51, 82, 83] and them opinion the true radiation spectrum is not strictly thermal but satisfies the underlying unitary theory when self-gravitation interaction and energy conservation are considered. It is clear that the background geometry of a radiating black hole should be altered (unfixed) with the loss of energy. Chen, Zu and Yang reformed Hamilton-Jacobi method for massive particle tunneling and investigate the Hawking radiation of the Taub-NUT black hole [126] and using this method Hawking radiation of Kerr-NUT black hole [65] and the charged black hole with a global monopole [99, 128] developed using Painlevé coordinate system. In this chapter, this method is applied to investigate the Hawking radiation of Kerr-anti-de Sitter (KAdS) black hole and to calculate the imaginary part of action from Hamilton-Jacobi equation avoid by exploring the equation of motion of the radiation particle in Painlevé coordinate system and calculating the Hamilton equation. In this method tunneling rate is related to the change of Bekenstein-Hawking entropy [51, 82].

Recently, Rahman et al. [129, 130] developed the Hawking radiation of Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter black holes by Hamilton-Jacobi method when the self-gravitational interaction and the unfixed background spacetime are taken into account. In this method, the imaginary part of the action come from the relativistic Hamilton-Jacobi equation and the actual Hawking radiation spectrum deviates from the purely thermal one, satisfies the underlying unitary theory and gives a leading correction to the radiation spectrum. The imaginary part of the action for the process of s-wave emission across the horizon, which in turn is related to the change of Bekenstein-Hawking entropy and using WKB approximation we get

$$\Gamma \sim \exp(-2\mathrm{Im}I), \tag{8.1}$$

where $\Gamma$ is the emission rate, $I$ is the action for an outgoing positive energy particle.

Properties of black holes in anti-de Sitter (AdS) spaces especially those of thermodynamics [109] investigated thoroughly in recent years within the context of the AdS/CFT correspondence [166, 167] and a large static black hole in asymptotically AdS spacetime corresponds to an (approximately) thermal state in the CFT. So the time scale for the decay of the black hole perturbation, which is given by the imaginary part of its action, corresponds to the timescale to reach thermal equilibrium in the strongly coupled CFT [158]. The accelerating expansion of our universe indicates the cosmological constant might be a positive one [132, 153, 154] and the recent development in string /M-theory greatly stimulate the study of black holes in anti-de Sitter spaces and hence our study on the Kerr-anti-de Sitter black holes is plausible and meaningful.

In order to carry-over this chapter we describe the KAdS black hole spacetime and near the event horizon the new line element of KAdS black hole is derived in the later section. In section 8.3, the unfixed background spacetime and the self-gravitational interaction are taken into account, we review the Hawking non-thermal radiation of KAdS black hole from massive particle tunneling method. In section 8.4, we derived the Hawking purely thermal radiation from non-thermal rate. Finally, in section 8.5, we present our remarks.

## 8.2 Kerr-anti-de Sitter black hole

Kerr-anti-de Sitter black hole, which is the exact solution of the Einstein field equations with a negative cosmological constant describes rotating black hole in four-dimensional spacetime with asymptotic-anti-de Sitter behavior in the Boyer-Lindguist coordinates [185] with cosmological radius $\ell$, mass $M$ and the angular momentum per unit mass $a$ has the form

$$\begin{aligned} ds^2 &= -\frac{1}{\rho^2}(\Delta_r - \Delta_\theta a^2 \sin^2\theta)dt^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ &+\frac{1}{\rho^2\Xi^2}[\Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta]\sin^2\theta d\phi^2 \\ &-\frac{2a}{\rho^2\Xi}[\Delta_\theta(r^2+a^2) - \Delta_r]\sin^2\theta dt d\phi, \end{aligned} \tag{8.2}$$

where

$$\begin{gathered} \rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta_\theta = 1 - \frac{a^2\cos^2\theta}{\ell^2}, \quad \Xi = 1 - \frac{a^2}{\ell^2}, \\ \Delta_r = (r^2+a^2)(1+\frac{r^2}{\ell^2}) - 2Mr = (1+\frac{a^2}{\ell^2})r^2 - 2Mr + a^2 + \frac{r^4}{\ell^2}. \end{gathered} \tag{8.3}$$

The coordinates are defined such that $-\infty \leq t \leq \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. The only positive real root is obtained from the zeroes of $r^4 + (\ell^2 + a^2) - 2M\ell^2 r + \ell^2 a^2 = 0$ and which is located at the black hole (event) horizon $r_+$ such that

$$r_+ = \frac{\ell\alpha}{\sqrt{3}}.\sinh\Big[\frac{1}{3}\sinh^{-1}\frac{3M\sqrt{3}}{\ell\Xi\alpha}\Big] \times\Bigg(1+\sqrt{1-\frac{a^2\ell}{\sqrt{3}M\alpha}.\frac{2}{1+\delta}\text{cosech}\Big[\frac{1}{3}\sinh^{-1}\frac{3M\sqrt{3}}{\ell\Xi\alpha}\Big]}\Bigg), \quad (8.4)$$

where

$$\delta = \sqrt{1-\frac{4a^2\alpha^2}{3M^2}\sinh^2\Big[\frac{1}{3}\sinh^{-1}\frac{3\sqrt{3}M}{\ell\Xi\alpha}\Big]}, \qquad \alpha=\sqrt{1+\frac{a^2}{\ell^2}}. \quad (8.5)$$

Expanding $r_+$ in terms of Kerr-anti-de Sitter black hole parameters with $a^2(1-\frac{a^2}{\ell^2}) < M^2$ and setting $\delta = 1$, we obtain

$$\begin{aligned} r_+ &= \frac{M}{\Xi}\left(1-\frac{4M^2}{\ell^2\Xi\alpha^2}+\cdots\right)\left(1+\sqrt{1-\frac{a^2\Xi}{M^2}}\right) \\ &= \frac{1}{\Xi}\left(1-\frac{4M^2}{\ell^2\Xi\alpha^2}+\cdots\right)\left(M+\sqrt{M^2-a^2\Xi}\right). \end{aligned} \quad (8.6)$$

Therefore, we can write $r_+ = (M+\sqrt{M^2-a^2\Xi})\beta$, with $\beta < 1$. Obviously, the event horizon of the Kerr-anti-de-Sitter black hole is less than the Kerr event horizon $r_{Ke} = M + \sqrt{M^2-a^2}$. It also gives Kerr black hole [81] for $\ell \to \infty(\Xi \to 1)$ and Schwarzschild-anti-de Sitter black hole [130] for $a = 0$. To study the Hawking radiation of Kerr-anti-de Sitter black hole effectively, we choose dragging coordinate transformation as follows

$$\frac{d\phi}{dt} = -\frac{g_{03}}{g_{33}} = \frac{a\Xi[\Delta_\theta(r^2+a^2)-\Delta_r]}{\Delta_\theta(r^2+a^2)^2-\Delta_r a^2\sin^2\theta}. \quad (8.7)$$

Now applying (8.7) on the line element (8.2), then the new line element of the Kerr-anti-de Sitter black hole takes on form as

$$ds^2 = -\frac{\Delta_r\Delta_\theta\rho^2}{\Delta_\theta(r^2+a^2)^2-\Delta_r a^2\sin^2\theta}dt^2+\frac{\rho^2}{\Delta_r}dr^2+\frac{\rho^2}{\Delta_\theta}d\theta^2. \quad (8.8)$$

The position of the event horizon is same as given in Eq. (8.6). We select the line element (8.8) and make a treatment to it and therefore, near the event horizon, the line element (8.8) can be written as

$$ds^2 = -\frac{\Delta_{r,r}(r_+)(r-r_+)\rho^2(r_+)}{(r_+^2+a^2)^2}dt^2 + \frac{\rho^2(r_+)}{\Delta_{r,r}(r_+)(r-r_+)}dr^2 + \frac{\rho^2(r_+)}{\Delta_\theta}d\theta^2, \quad (8.9)$$

where

$$\rho^2(r_+) = r_+^2 + a^2\cos^2\theta,$$
$$\Delta_{r,r}(r_+) = \frac{d\Delta_r}{dr}\Big|_{r=r_+} = \frac{2}{\Xi^2}(\beta^2 r_+ - M - 2\frac{r_+^3}{\ell^2}). \quad (8.10)$$

## 8.3 The HJ Method for Non-thermal Radiation

To calculate the imaginary part of the action from the relativistic Hamilton-Jacobi equation, we use the method of Chen et al. [125, 126] by giving up the exploration of the equation of motion in the Painlevé coordinates systems. In order to calculate the imaginary part of the action from the relativistic Hamilton-Jacobi equation, the action $I$ of the outgoing particle from the black hole horizon satisfies the relativistic Hamilton-Jacobi equation

$$g^{ab}(\partial_a I)(\partial_b I) + m^2 = 0, \quad (8.11)$$

in which $m$ and $g^{ab}$ are the mass of the particle and the inverse metric tensors derived from the line element (8.9). The non-null inverse metric tensors for the metric (8.9) are

$$g^{00} = -\frac{(r_+^2+a^2)^2}{\Delta_{r,r}(r_+)(r-r_+)\rho^2(r_+)}, \qquad g^{11} = \frac{\Delta_{r,r}(r_+)(r-r_+)}{\rho^2(r_+)},$$

$$g^{22} = \frac{\Delta_\theta}{\rho^2(r_+)} \tag{8.12}$$

Using (8.12) in Eq. (8.11), we have

$$g^{00}(\partial_t I)^2 + g^{11}(\partial_r I)^2 + g^{22}(\partial_\theta I)^2 + m^2 = 0. \tag{8.13}$$

The action $I$ for $I(t, r, \theta, \phi)$ is too difficult to solve. To find the solution in a convenient way, the separation of variables can be taken as follows

$$I = -\omega t + R(r) + H(\theta) + j\phi, \tag{8.14}$$

where $\omega$ is the energy of the particle, $j$ is the angular momentum with respect to the angular $\phi$, $R(r)$ and $H(\theta)$ are the generalized momentums. From Eq.(8.13) and Eq. (8.14), we get

$$\begin{aligned} R(r) &= \pm\frac{r_+^2 + a^2}{\Delta_{r,r}(r_+)}\int \frac{dr}{(r - r_+)} \times \\ &\sqrt{(\omega - j\Omega_+)^2 - \frac{\rho^2(r_+)\Delta_{r,r}(r_+)(r - r_+)}{(r_+^2 + a^2)^2}\left[\frac{\Delta_\theta}{\rho^2(r_+)}(\partial_\theta H)^2 + m^2\right]} \\ &= \pm\frac{\pi i(r_+^2 + a^2)}{\Delta_{r,r}(r_+)}(\omega - j\Omega_+) + \zeta, \end{aligned} \tag{8.15}$$

where $\Omega_+ = \frac{d\phi}{dt}\Big|_{r=r_+} = \frac{a\Xi}{r_+^2 + a^2}$ express the angular velocity of the particle at the event horizon and $\pm$ sign comes from the square root and $\zeta$ is the constant of integration. Inserting Eq. (8.15) into Eq. (8.14), the imaginary part of two different actions corresponding to the outgoing and incoming particles can be written as

$$\mathrm{Im}I_\pm = \pm\frac{\pi(r_+^2 + a^2)}{\Delta_{r,r}(r_+)}(\omega - j\Omega_+) + \mathrm{Im}(\zeta). \tag{8.16}$$

With classical limit as mentioned in refs. [169], we make sure the incoming probability to be unity when there is no reflection i.e., everything is absorbed by the horizon. In this situation the appropriate value of $\zeta$ instead of zero or infinity can be taken as $\zeta = \frac{\pi i(r_+^2+a^2)}{\Delta_{r,r}(r_+)}(\omega - j\Omega_+) + \mathrm{Re}(\zeta)$. Therefore, $\mathrm{Im}I_- = 0$ and $I_+$ give the imaginary part of action $I$ corresponding to the outgoing particle of the form

$$\begin{aligned}
\mathrm{Im}I &= \frac{2.\pi(r_+^2+a^2)}{\Delta_{r,r}(r_+)}(\omega - j\Omega_+) \\
&= \frac{\pi\Xi^2(r_+^2+a^2)}{\alpha^2 r_+ - M + 2\frac{r_+^3}{\ell^2}}(\omega - j\Omega_+) \\
&= \frac{\pi\Xi^2(r_+^2+a^2)}{\alpha^2 r_+ - M + 2\frac{r_+^3}{\ell^2}}\left(\omega - \frac{ja\Xi}{r_+^2+a^2}\right) \\
&= \frac{\pi\Xi^2 r_+^2}{\alpha^2 r_+ - M + 2\frac{r_+^3}{\ell^2}}\omega + \frac{\pi\Xi^2 a^2}{\alpha^2 r_+ - M + 2\frac{r_+^3}{\ell^2}}\omega - \frac{\pi\Xi^3 a}{\alpha^2 r_+ - M + 2\frac{r_+^3}{\ell^2}}j.
\end{aligned} \tag{8.17}$$

Using Eq. (8.6) into Eq. (8.17), we get the imaginary part of the action as

$$\begin{aligned}
\mathrm{Im}I &= \frac{\pi\left(1 - \frac{4M^2}{\ell^2\Xi\alpha^2} + \cdot\cdot\right)^2 (M + \sqrt{M^2 - a^2\Xi})^2}{\frac{\alpha^2}{\Xi}\left(1 - \frac{4M^2}{\ell^2\Xi\alpha^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Xi}) - M + A}\omega \\
&\quad + \frac{\pi\Xi^2 a^2}{\frac{\alpha^2}{\Xi}\left(1 - \frac{4M^2}{\ell^2\Xi\alpha^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Xi}) - M + A}\omega \\
&\quad - \frac{\pi a\Xi^3}{\frac{\alpha^2}{\Xi}\left(1 - \frac{4M^2}{\ell^2\Xi\alpha^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Xi}) - M + A}j,
\end{aligned} \tag{8.18}$$

where $A = \frac{2}{\ell^2\Xi^3}\left(1 - \frac{4M^2}{\ell^2\Xi\alpha^2} + \cdot\cdot\right)^3 (M + \sqrt{M^2 - a^2\Xi})^3$.

$$\begin{aligned}
&= \frac{\pi(M + \sqrt{M^2 - a^2\Xi})^2}{\frac{\alpha^2}{\Xi}\left[\left(1 + \frac{4M^2}{\ell^2\Xi\alpha^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Xi}) - \frac{M\Xi}{\alpha^2}\left(1 + \frac{8M^2}{\ell^2\Xi\alpha^2} + \cdot\cdot\right) + B\right]}\omega \\
&\quad + \frac{\pi\Xi^2 a^2}{\frac{\alpha^2}{\Xi}\left[\left(1 - \frac{4M^2}{\ell^2\Xi\alpha^2} + \cdot\cdot\right)(M + \sqrt{M^2 - a^2\Xi}) - \frac{M\Xi}{\alpha^2} + \frac{\Xi A}{\alpha^2}\right]}\omega
\end{aligned}$$

$$-\frac{\pi a\Xi^3}{\frac{\alpha^2}{\Xi}\left[\left(1-\frac{4M^2}{\ell^2\Xi\alpha^2}+\cdot\cdot\right)(M+\sqrt{M^2-a^2\Xi})-\frac{M\Xi}{\alpha^2}+\frac{\Xi A}{\alpha^2}\right]}j, \tag{8.19}$$

where$B=\frac{2}{\ell^2\alpha^2\Xi^2}\left(1-\frac{4M^2}{\ell^2\Xi\alpha^2}+\cdot\cdot\right)(M+\sqrt{M^2-a^2\Xi})^3$.

Now for the simplicity, neglecting $M^3$ and its higher order terms, we then get

$$\begin{aligned}\mathrm{Im}I &= \frac{\pi\Xi}{\alpha^2}.\frac{(M+\sqrt{M^2-a^2\Xi})^2}{(M+\sqrt{M^2-a^2\Xi})-\frac{M\Xi}{\alpha^2}}\omega+\frac{\pi a^2\Xi^3}{\alpha^2\left[M+\sqrt{M^2-a^2\Xi}-\frac{M\Xi}{\alpha^2}\right]}\omega.\\ &\quad -\frac{\pi a\Xi^4}{\alpha^2\left[M+\sqrt{M^2-a^2\Xi}-\frac{M\Xi}{\alpha^2}\right]}j.\end{aligned} \tag{8.20}$$

Suppose the Arnowitt-Deser-Misner (ADM) mass of the total spacetime to be fixed and in presence of cosmological constant KAdS spacetime is dynamic, and hence allow KAdS black hole to fluctuate. When a particle with energy $\omega$ and angular momentum $j$ tunnels out, the mass and angular momentum of the KAdS black hole should be replaced by $M-\omega$ and $J-j$ respectively. Taking the self-gravitational interaction into account, the imaginary part of the true action can be calculated from Eq. (8.20) in the following integral form

$$\begin{aligned}\mathrm{Im}I &= \frac{\pi\Xi}{\alpha^2}.\int_0^\omega\frac{(M+\sqrt{M^2-a^2\Xi})^2}{M+\sqrt{M^2-a^2\Xi}-\frac{M\Xi}{\alpha^2}}d\omega'\\ &\quad+\frac{\pi\Xi^3}{\alpha^2}.\int_0^\omega\frac{a^2}{M+\sqrt{M^2-a^2\Xi}-\frac{M\Xi}{\alpha^2}}d\omega'\\ &\quad-\frac{\pi\Xi^4}{\alpha^2}.\int_0^j\frac{a}{M+\sqrt{M^2-a^2\Xi}-\frac{M\Xi}{\alpha^2}}dj'\\ &= \frac{\pi\Xi}{\alpha^2}.\int_0^\omega\frac{(M+\sqrt{M^2-a^2\Xi})^2}{\sqrt{M^2-a^2\Xi}+(M-\frac{M\Xi}{\alpha^2})}d\omega'\\ &\quad+\frac{\pi\Xi^3}{\alpha^2}.\int_0^\omega\frac{a^2}{\sqrt{M^2-a^2\Xi}+(M-\frac{M\Xi}{\alpha^2})}d\omega'\end{aligned}$$

$$-\frac{\pi\Xi^4}{\alpha^2}.\int_0^j \frac{a}{\sqrt{M^2-a^2\Xi}+(M-\frac{M\Xi}{\alpha^2})}dj'. \tag{8.21}$$

For the maximum value of integration, neglecting $(1-\frac{\Xi}{\alpha^2})M$. Equation (8.21) becomes

$$\begin{aligned}\mathrm{Im}I &= \frac{\pi\Xi}{\alpha^2}.\int_0^\omega \frac{(M+\sqrt{M^2-a^2\Xi})^2}{\sqrt{M^2-a^2\Xi}}d\omega' + \frac{\pi\Xi^3}{\alpha^2}.\int_0^\omega \frac{a^2}{\sqrt{M^2-a^2\Xi}}d\omega' \\ &\quad -\frac{\pi\Xi^4}{\alpha^2}.\int_0^j \frac{a}{\sqrt{M^2-a^2\Xi}}dj'. \end{aligned}\tag{8.22}$$

Now replacing $M$ by $M-\omega$ and $j$ by $J-j$, we have

$$\begin{aligned}\mathrm{Im}I &= -\frac{\pi\Xi}{\alpha^2}.\int_M^{(M-\omega)} \frac{(M-\omega+\sqrt{(M-\omega)^2-a^2\Xi})^2}{\sqrt{(M-\omega)^2-a^2\Xi}}d(M-\omega') \\ &\quad -\frac{\pi\Xi^3}{\alpha^2}.\int_M^{(M-\omega)} \frac{a^2}{\sqrt{(M-\omega)^2-a^2\Xi}}d(M-\omega') \\ &\quad +\frac{\pi\Xi^4}{\alpha^2}.\int_J^{(J-j)} \frac{a}{\sqrt{(M-\omega)^2-a^2\Xi}}d(J-j'), \end{aligned}\tag{8.23}$$

where $J-j'=(M-\omega')a/\Xi^2$. Therefore Eq. (8.23) becomes

$$\begin{aligned}\mathrm{Im}I &= -\frac{\pi\Xi}{\alpha^2}.\int_M^{(M-\omega)} \frac{(M-\omega+\sqrt{(M-\omega)^2-a^2\Xi})^2}{\sqrt{(M-\omega)^2-a^2\Xi}}d(M-\omega') \\ &= -\frac{\pi\Xi}{\alpha^2}.\int_M^{(M-\omega)} \frac{2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-a^2\Xi}}{\sqrt{(M-\omega)^2-a^2\Sigma}}d(M-\omega') \\ &\quad +\frac{\pi\Xi}{\alpha^2}.\int_M^{(M-\omega)} \frac{a^2\Xi}{\sqrt{(M-\omega)^2-a^2\Xi}}d(M-\omega'). \end{aligned}\tag{8.24}$$

Finishing the $\omega'$ integral finally yields

$$\begin{aligned}\mathrm{Im}I &= -\frac{\pi\Xi}{\alpha^2}.\{(M-\omega)\sqrt{(M-\omega)^2-a^2\Xi} \\ &\quad +(M-\omega)^2-M\sqrt{M^2-a^2\Xi}-M^2\}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\Big\{\frac{\pi\Xi}{\alpha^2}[2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-a^2\Xi} \\
&\quad -2M\sqrt{M^2-a^2\Xi}-2M^2]\Big\} \\
&= -\frac{1}{2}\exp[\pi(r_f^2-r_i^2)] \\
&= -\frac{1}{2}\Delta S_{BH}, \qquad (8.25)
\end{aligned}
$$

where $\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$ is the change of Bekenstein-Hawking entropies of the Kerr-anti-de Sitter black hole before and after the emission of the particles by setting $r_i = \frac{\sqrt{\Xi}}{\alpha}[M+\sqrt{M^2-a^2\Xi}]$ and $r_f = \frac{\sqrt{\Xi}}{\alpha}[(M-\omega)+\sqrt{(M-\omega)^2-a^2\Xi}]$ respectively.

Therefore, using Eq. (8.1) the tunneling probability for the KAdS black hole can be obtained as

$$
\Gamma \sim \exp(-2\mathrm{Im}I) = \exp(\Delta S_{BH}). \qquad (8.26)
$$

The result shows the actual radiation spectrum deviates from the purely thermal one and gives a correction to the Hawking radiation of the black hole.

## 8.4 Purely Thermal Radiation

It is obvious from Eq. (8.26) the radiation spectrum is not pure thermal although gives a correction to the Hawking radiation of KAdS black hole. The purely thermal spectrum can be derived from Eq. (8.26) by expanding as discussed by Hossain et al. [131] of the form

$$
\Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{-\omega\frac{\partial S_{BH}(M)}{\partial M}+\frac{\omega^2}{2}\frac{\partial^2 S_{BH}(M)}{\partial M^2}\right\}. \qquad (8.27)
$$

From Eq.(8.25)

$$S_{BH}(M-\omega)=\frac{\pi\Xi}{\alpha^2}\Big[(M-\omega)+\sqrt{(M-\omega)^2-a^2\Xi}\Big]^2. \qquad (8.28)$$

Using (8.28) in (8.27) and finishing the calculation, the purely thermal emission rate reduces to the form

$$\begin{aligned}&\Gamma\sim\exp(\Delta S_{BH})\\&=\exp\Big[\frac{-2\pi\Xi\omega}{\alpha^2}\{(2M+\sqrt{M^2-a^2\Xi}+\frac{M^2}{\sqrt{M^2-a^2\Xi}})\\&-\frac{\omega}{2}(2+\frac{3M}{\sqrt{M^2-a^2\Xi}}-\frac{M^3}{(M^2-a^2\Xi)^{\frac{3}{2}}})\}\Big]. \qquad (8.29)\end{aligned}$$

In the limiting case i.e. when $\Lambda=0$, then $\Xi=1,\alpha=1$ and in this case, non-thermal and purely thermal tunneling rates for the KAdS black hole reduces to

$$\begin{aligned}\Gamma\sim\exp(-2\text{Im}I) &= \exp\Big\{\pi\Big[\{(M-\omega)+\sqrt{(M-\omega)^2-a^2}\}^2\\&-\{M+\sqrt{M^2-a^2}\}^2\Big]\Big\}, \qquad (8.30)\end{aligned}$$

and

$$\begin{aligned}&\Gamma\sim\exp(\Delta S_{BH})\\&=\exp\Big[-2\pi\omega\{(2M+\sqrt{M^2-a^2}+\frac{M^2}{\sqrt{M^2-a^2}})\\&-\frac{\omega}{2}(2+\frac{3M}{\sqrt{M^2-a^2}}-\frac{M^3}{(M^2-a^2)^{\frac{3}{2}}})\}\Big]. \qquad (8.31)\end{aligned}$$

Which is consistent to the results for the Kerr black hole [81] and where $r_i=M+\sqrt{M^2-a^2}$ and $r_f=(M-\omega)+\sqrt{(M-\omega)^2-a^2}$. These are the positions of the event horizon of Kerr black hole before and after the emission of the particles respectively.

For another special case, only when $a = 0$, the line element (8.2) is reduced to the SAdS black hole [130]. So the tunneling probabilities for the SAdS black hole can be written as

$$\begin{aligned} \Gamma \sim \exp(-2\mathrm{Im}I) &= \exp\{\pi[4(M-\omega)^2\left(1-\frac{8(M-\omega)^2}{\ell^2}\right) \\ &\quad -4M^2\left(1-\frac{4M^2}{\ell^2}\right)]\} \end{aligned} \tag{8.32}$$

and

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{-8\pi\omega\left[\left(M-\frac{16M^3}{\ell^2}\right)-\frac{\omega}{2}\left(1-\frac{48M^2}{\ell^2}\right)\right]\right\}, \tag{8.33}$$

where $r_i = 2M\left(1-\frac{4M^2}{\ell^2}\right)$ and $r_f = 2(M-\omega)\left(1-\frac{4(M-\omega)^2}{\ell^2}\right)$ are the locations of the SAdS [130] event horizon before and after the particles emission.

Furthermore only when $a = 0$ and $\Lambda = 0$, the line element (8.2) is reduced to the Schwarzschild black hole [51] and therefore the non-thermal and purely thermal tunneling rates can be written as

$$\begin{aligned} \Gamma \sim \exp(-2\mathrm{Im}I) &= \exp\left\{\pi\left[4(M-\omega)^2-4M^2\right]\right\} \\ &= \exp[\pi(r_f^2-r_i^2)] \end{aligned} \tag{8.34}$$

and

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\left[-8\pi\omega\left(M-\frac{\omega}{2}\right)\right], \tag{8.35}$$

which are full accordant with Parikh and Wilczek's results [51, 82, 83] and where $r_i = 2M$ and $r_f = 2(M-\omega)$ are the locations of the Schwarzschild black hole event horizon before and after the particles emission.

## 8.5 Concluding Remarks

We have been presented Hawking radiation as massive particle tunneling method from KAdS black hole in this chapter. Considering the self-gravitational interaction, the background spacetime as dynamical and the energy as conservation, we have explored the Hawking non-thermal and purely thermal tunneling probabilities of KAdS black hole at the event horizon. It is noted that the similar result have been shown taking the same proposition for massive particle tunneling at the event horizon of SAdS black hole [129], RNAdS black hole [131] and Taub-NUT black holes [126] and also agree by massless or massless charged particle tunneling from various spacetime like as charged black hole with a global monopole [99, 128], kerr-NUT black hole [65], Kerr and Kerr-Newman black holes [60, 103] etc. Thus the actual radiation spectrum of KAdS black hole is not precisely thermal and the tunneling probability is related to the change of Bekenstein-Hawking entropy but satisfies an underlying unitary theory and also gives a correction to Hawking radiation.

# Chapter 9

# Hawking Non-thermal and Purely thermal radiations of Kerr-Newman-de Sitter Black Hole by Hamilton-Jacobi method

## 9.1 Introduction

Many researchers have attempt to provide various methods to correctly find out the Hawking radiations of different black holes because Hawking proved that black holes have emission of thermal radiation [24]. To describe Hawking radiation as a quantum tunneling process, a new window first opened by Kraus and Wilczek [52, 53, 54] where a particle moves in a dynamical geometry and then formulated by many researchers [51, 79, 85, 86, 88, 89, 90]. Recently, several works on rotating black holes [55, 65, 80, 81, 102, 103, 118, 119, 120, 125, 127, 186, 187, 188] have been done by using Painleavé or dragging or tortoise or Eddington-Finkelstein coordinate transformations but most of them are focus on studying Hawking radiation of massless/scalar particles tunneling from different rotating black holes. Here we have used the dragging coordinate transformation to obtain the same results from the Kerr-Newman-de Sitter (KNdS) black hole using massive particle tunneling process by expressing the event horizon of KNdS black hole in terms of black hole parameters in an infinite series and is very interesting point in this research.

This chapter is devoted to investigate the Hawking non-thermal and purely thermal tunneling rates of the Kerr-Newman black hole in the de Sitter space. To obtain the correct tunneling rates, we use the method which regards the action of the emitted particles satisfies the relativistic Hamilton-Jacobi equation and solving it contains the imaginary part of the action [90, 186, 187]. It is noticed that the analysis of massive particles tunneling from the Kerr-Newman-de Sitter black hole parallels to the case that we have made for Kerr-de Sitter black hole. Here the energy as well as charge conservation are taken into account.

This chapter is arranged as follows: In section 9.2 we describe the Kerr-Newman-de Sitter black hole spacetime with the position of event horizon and derive the new line element of KNdS black hole near the event horizon. In section 9.3 we describe the Hamilton-Jacobi method for the KNdS spacetime. Again, we consider the spacetime background as dynamical and self-gravitational interaction of the emitted particles, the non-thermal tunneling rate of KNdS black hole from massive particle tunneling process have been reviewed in section 9.4. In section 9.5 we develop the Hawking purely thermal rate from non-thermal rate. Finally, in section 9.6 we give our remarks.

## 9.2 Kerr-Newman-de Sitter black hole

The most general black hole [188] solution can be expressed in the Boyer-Lindguist coordinate [185] as an exact solution of the Einstein field equations with a positive cosmological constant $\Lambda(=3/\ell^2)$ describes charged rotating black hole in asymptotically de Sitter space with cosmological radius $\ell$, mass $M$, charge $q$ and angular momentum per unit mass $a$ of the form

$$\begin{aligned} ds^2 \;=\; & -\frac{f(r)-f(\theta)a^2\sin^2\theta}{\rho^2}dt^2+\frac{f(\theta)(r^2+a^2)^2-f(r)a^2\sin^2\theta}{\rho^2\Sigma^2}\sin^2\theta d\phi^2 \\ & +\frac{\rho^2}{f(r)}dr^2+\frac{\rho^2}{f(\theta)}d\theta^2-\frac{2a[(r^2+a^2)f(\theta)-f(r)]\sin^2\theta}{\rho^2\Sigma}dtd\phi, \quad (9.1)\end{aligned}$$

where

$$\rho^2=r^2+a^2\cos^2\theta, \quad f(\theta)=1+\frac{a^2\cos^2\theta}{\ell^2},$$
$$\Sigma=1+\frac{a^2}{\ell^2}, \quad f(r)=(1-\frac{a^2}{\ell^2})r^2-2Mr+a^2-\frac{r^4}{\ell^2}+q^2. \quad (9.2)$$

The de Sitter space are defined such that $-\infty\leq t\leq\infty$, $r\geq0$, $0\leq\theta\leq\pi$, and $0\leq\phi\leq2\pi$. The metric (9.1) describes an interesting charged rotating AdS black hole called the Kerr-Newman-Anti-de Sitter (KNAdS) black hole if we replace $\ell^2$ by $-\ell^2$. There are apparent singularities in the metric at the values of $r$ for which

$$f(r)=(1-\frac{a^2}{\ell^2})r^2-2Mr+a^2-\frac{r^4}{\ell^2}+q^2=0 \quad (9.3)$$

The function $f(r)=0$ with $\ell^2>a^2$ has four distinct roots: $r_h, r_-, r_c$, and $r_{--}$. The real root $r_h$ corresponds to the radius of the black hole's outer event horizon, while the other real root $r_-$ represents the radius of the inner cauchy horizon. Here we indicate $r_c$ as the cosmological horizon and $r_{--}$ the negative root of $f(r)$ another cosmological horizon. Equation (9.3) can be written as

$$r^4 - (\ell^2 - a^2)r^2 + 2M\ell^2 r - \ell^2(a^2 + q^2) = 0. \tag{9.4}$$

Solving the above equation, the black hole event horizon and cosmological horizon can be written respectively of the form

$$\begin{aligned} r_h &= \frac{\ell\beta}{\sqrt{3}}.\sin\Big[\frac{1}{3}\sin^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big] \\ &\times\Big(1 + \sqrt{1 - \frac{(a^2 + q^2)\ell}{\sqrt{3}M\beta}.\frac{2}{1+\delta}\mathrm{cosec}\Big[\frac{1}{3}\sin^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big]}\Big), \end{aligned} \tag{9.5}$$

and

$$\begin{aligned} r_c &= \frac{\ell\beta}{\sqrt{3}}.\sin\Big[\frac{1}{3}\sin^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big] \\ &\times\Big(\sqrt{1 + \frac{1+\delta}{2}.\frac{3M\sqrt{\ell}}{\sqrt{3}\beta^2}\mathrm{cosec}^3\Big[\frac{1}{3}\sin^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big]} - 1\Big), \end{aligned} \tag{9.6}$$

where

$$\delta = \sqrt{1 - \frac{4(a^2 + q^2)\beta^2}{3M^2}\sin^2\Big[\frac{1}{3}\sin^{-1}\frac{3\sqrt{3}M}{\ell\alpha\beta}\Big]}, \tag{9.7}$$

$$\alpha = \sqrt{\{1 + \frac{a^2}{\ell^2}\}^2 + \frac{4q^2}{\ell^2}}, \quad \beta = \sqrt{1 - \frac{a^2}{\ell^2}} \tag{9.8}$$

and $r_{--} = -(r_h + r_- + r_c)$ is the another cosmological horizon. With $\delta \approx 1$ the black hole event horizon can be approximated as

$$r_h \approx \frac{\ell\beta}{\sqrt{3}}.\sin\Big[\frac{1}{3}\sin^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big].\Big(1 + \sqrt{1 - \frac{(a^2 + q^2)\alpha}{M^2}}\Big). \tag{9.9}$$

Expanding $r_h$ in terms of $\ell$, $M$, $q$ and $a$ with $(a^2+q^2)\alpha < M^2$ , we obtain

$$r_h = \frac{M}{\alpha}\left(1+\frac{4M^2}{\ell^2\beta^2\alpha}+\cdots\right)\left(1+\sqrt{1-\frac{(a^2+q^2)\alpha}{M^2}}\right), \tag{9.10}$$

which can be written as

$$r_h = \frac{1}{\alpha}\left(1+\frac{4M^2}{\ell^2\beta^2\alpha}+\cdots\right)\left(M+\sqrt{M^2-(a^2+q^2)\alpha}\right). \tag{9.11}$$

Obviously, the event horizon of the Kerr-Newman-de Sitter black hole is greater than the Kerr-Newman event horizon $r_{Ke} = M+\sqrt{M^2-(a^2+q^2)}$. It is interesting to note that it reduced to the Kerr-Newman black hole [125] for $\ell \to \infty$, Kerr-de Sitter black hole for $q = 0$, Kerr black hole [81] for $\ell \to \infty, q = 0$ and Schwarzschild-de Sitter black hole [129] for $a = 0$ and $q = 0$. We perform the following effective transformation to obtain the Hawking radiation of the KNdS black hole.

$$\frac{d\phi}{dt} = \frac{a\Sigma[f(\theta)(r^2+a^2)-f(r)]}{f(\theta)(r^2+a^2)^2-f(r)a^2\sin^2\theta}. \tag{9.12}$$

Using (9.12) in the line element (9.1), then the new line element of the Kerr-Newman-de Sitter black hole becomes

$$ds^2 = -\frac{f(r)f(\theta)\rho^2}{f(\theta)(r^2+a^2)^2-f(r)a^2\sin^2\theta}dt^2+\frac{\rho^2}{f(r)}dr^2+\frac{\rho^2}{f(\theta)}d\theta^2. \tag{9.13}$$

The position of the event horizon is same as given in Eq. (9.11). The action can be derived from the line element (9.1) and (9.13) respectively. Near the event horizon, the line element (9.13) can be rewritten as

$$ds^2 = -\frac{f_{,r}(r_h)(r-r_h)\rho^2(r_h)}{(r_h^2+a^2)^2}dt^2+\frac{\rho^2(r_h)}{f_{,r}(r_h)(r-r_h)}dr^2+\frac{\rho^2(r_h)}{f(\theta)}d\theta^2, \tag{9.14}$$

when

$$\rho^2(r_h) = r_h^2+a^2\cos^2\theta$$
$$f_{,r}(r_h) = \frac{df}{dr}\Big|_{r=r_h} = \frac{2}{\Sigma^2}(\beta^2 r_h - M - 2\frac{r_h^3}{\ell^2}). \tag{9.15}$$

## 9.3 The HJ Method for KNdS Spacetime

In this section, we have applied the standard Hamilton-Jacobi method [85, 86, 87, 88, 89] developed by Angheben et al.[90, 186] which is an extension of complex path analysis proposed by Padmanabhan et al. [85, 86, 87, 88, 89] to calculate the imaginary part of the true action for the process of s-wave emission across the horizon. Using the WKB approximation [168], the emission rate satisfies the following relation

$$\Gamma \sim \exp(-2\text{Im}I), \tag{9.16}$$

where $I$ is the action of the outgoing particle.

In the Hamilton-Jacobi method we avoid the exploration of the equation of motion in the Painlevé coordinates systems. The classical action of the radiation particle tunnels across the event horizon satisfies the relativistic Hamilton-Jacobi equation

$$g^{ij}\left(\frac{\partial I}{\partial x^i}\right)\left(\frac{\partial I}{\partial x^j}\right)+u^2=0, \tag{9.17}$$

where $u$ is the mass of the radiating particle and $g^{ij}$ are the inverse metric tensors derived from the metric (9.14), namely

$$g^{\bar{1}1}=-\frac{(r_h^2+a^2)^2}{\rho^2(r_h)f_{,r}(r_h)(r-r_h)},g^{\bar{2}2}=\frac{f_{,r}(r_h)(r-r_h)}{\rho^2(r_h)},$$
$$g^{\bar{3}3}=-\frac{f(\theta)}{\rho^2(r_h)}, \tag{9.18}$$

and others are null. Substituting them in the Eq. (9.17), we get

$$g^{\bar{1}1}\left(\frac{\partial I}{\partial t}\right)^2+g^{\bar{2}2}\left(\frac{\partial I}{\partial r}\right)^2+g^{\bar{3}3}\left(\frac{\partial I}{\partial \theta}\right)^2+u^2=0. \tag{9.19}$$

For the HJ equation, to find the solution we use the separation of variables method for the action $I(t, r, \theta, \phi)$ as follows

$$I = -\omega t + R(r) + H(\theta) + j\phi, \tag{9.20}$$

where $\omega$ is the energy of the emitted particle, $j$ is the angular momentum with respect to angular $\phi$, $R(r)$ and $H(\theta)$ are the generalized momentums. The angular velocity of the particle at the event horizon is

$$\Omega_h = \frac{d\phi}{dt}\Big|_{r=r_h} = \frac{a\Sigma}{r_h^2 + a^2}. \tag{9.21}$$

Using Eq. (9.20) in Eq. (9.19), we obtain an expression as follows

$$\begin{aligned} \frac{dR(r)}{dr} &= \sqrt{-\bar{g}^{11}\bar{g}_{22}} \times \left\{(\omega - j\Omega_h)^2 + \bar{g}_{11}\left[\bar{g}^{33}\left(\frac{dH(\theta)}{d\theta}\right)^2 + u^2\right]\right\} \\ R(r) &= \pm\frac{r_h^2 + a^2}{f_{,r}(r_h)}\int \frac{dr}{r - r_h} \times \\ & \sqrt{(\omega - j\Omega_h)^2 - \frac{\rho^2(r_h) f_{,r}(r_h)(r - r_h)}{(r_h^2 + a^2)^2}\left[\bar{g}^{33}\left(\frac{dH(\theta)}{d\theta}\right)^2 + u^2\right]} \\ & . \end{aligned} \tag{9.22}$$

## 9.4 Non-thermal Tunneling Rate

For the convenience of research, let's the emitted particle as an ellipsoid shell of energy $\omega$ to tunnel across the event horizon. The quadratic form of Eq. (9.19) is the reason of $\pm$ signatures that summarized in the above equation. Solution of Eq. (9.22) with "+" signature corresponds to outgoing particles and the other solution i.e., the solution with "-"signature refers to the ingoing particles. The solution given by Eq.(9.22) is singular at $r = r_h$ which corresponds to the event horizon. Finishing the above integral by using the Cauchy's integral formula, we obtain

$$R(r) = \pm\frac{2\pi i(r_h^2 + a^2)}{f_{,r}(r_h)}(\omega - j\Omega_h). \qquad (9.23)$$

Substituting the above result in Eq. (9.20), the imaginary part of the action $I$ corresponding to the outgoing particle is obtained by $\pi$ times the residue of the integrand

$$\begin{aligned} \mathrm{Im}I &= \frac{2\pi(r_h^2 + a^2)}{f_{,r}(r_h)}(\omega - j\Omega_h) \\ &= \frac{\pi\Sigma^2(r_h^2 + a^2)}{\beta^2 r_h - M - 2\frac{r_h^3}{\ell^2}}(\omega - j\Omega_h). \end{aligned} \qquad (9.24)$$

Using Eqs. (9.11) and (9.21) into Eq. (9.24), we get the imaginary part of the action as

$$\begin{aligned} \mathrm{Im}I = &\frac{\frac{\pi\Sigma^2}{\alpha^2}\left(1 + \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)^2 (M + \sqrt{M^2 - (a^2 + q^2)\alpha})^2}{\frac{\beta^2}{\alpha}\left(1 + \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - M - A}\omega \\ &+\frac{\pi a^2\Sigma^2}{\frac{\beta^2}{\alpha}\left(1 + \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - M - A}\omega \\ &-\frac{\pi a\Sigma^3}{\frac{\beta^2}{\alpha}\left(1 + \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - M - A}j, \end{aligned}$$

where $A = \frac{2}{\ell^2\alpha^3}\left(1 + \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)^3 (M + \sqrt{M^2 - (a^2 + q^2)\alpha})^3$.

$$\begin{aligned} \mathrm{Im}I \; = \; &\frac{\pi\Sigma^2(M + \sqrt{M^2 - (a^2 + q^2)\alpha})^2}{\beta^2\alpha\left[\left(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - B\right]}\omega \\ &+\frac{\pi a^2\Sigma^2}{\frac{\beta^2}{\alpha}\left[\left(1 + \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - \frac{M\alpha}{\beta^2} - \frac{\alpha A}{\beta^2}\right]}\omega \end{aligned}$$

$$-\frac{\pi a\Sigma^3}{\frac{\beta^2}{\alpha}\left[\left(1+\frac{4M^2}{\ell^2\alpha\beta^2}+\cdot\cdot\right)(M+\sqrt{M^2-(a^2+q^2)\alpha})-\frac{M\alpha}{\beta^2}-\frac{\alpha A}{\beta^2}\right]}j,$$

where $B=\frac{M\alpha}{\beta^2}(1-\frac{8M^2}{\ell^2\alpha\beta^2}+\cdot\cdot)+\frac{2}{\ell^2\beta^2\alpha^2}(1+\frac{4M^2}{\ell^2\alpha\beta^2}+\cdot\cdot)(M+\sqrt{M^2-(a^2+q^2)\alpha})^3$.

To get the maximum value of the integration, neglecting above second order terms of black hole parameter 'mass' from the denominator, we then get

$$\begin{aligned}\mathrm{Im}I &= \frac{\pi\Sigma^2}{\beta^2\alpha}.\frac{(M+\sqrt{M^2-(a^2+q^2)\alpha})^2}{(M+\sqrt{M^2-(a^2+q^2)\alpha})-\frac{M\alpha}{\beta^2}}\omega \\ &\quad+\frac{\Sigma^2\pi a^2\alpha}{\beta^2\left[M+\sqrt{M^2-(a^2+q^2)\alpha}-\frac{M\alpha}{\beta^2}\right]}\omega \\ &\quad-\frac{\Sigma^3\pi a\alpha}{\beta^2\left[M+\sqrt{M^2-(a^2+q^2)\alpha}-\frac{M\alpha}{\beta^2}\right]}j \qquad (9.25)\end{aligned}$$

Let us now focus on a semiclassical treatment of the associated radiation and adopt the picture of a pair of virtual particles spontaneously created just inside the horizon. The positive energy virtual particle can tunnel out -no classical escape route exists - where it materializes a real particle while the negative energy particle is absorbed by the black hole, resulting in a decrease in the mass and angular momentum of the black hole. If the particle's self-gravitational interaction is taken into account, equations (9.1) to (9.25) should be changed. Fixing the ADM mass, charge and angular momentum of the total spacetime and allow mass and angular momentum of the black hole to vary. Then we should replace $M$ by $M-\omega$ and $j$ by $J-j$, and therefore the imaginary part of the true action can be calculated from Eq. (9.25) in the following integral

$$
\begin{aligned}
\text{Im}I &= \frac{\pi\Sigma^2}{\beta^2\alpha}.\int_0^{\omega}\frac{(M+\sqrt{M^2-(a^2+q^2)\alpha})^2}{\sqrt{M^2-(a^2+q^2)\alpha}+(M-\frac{M\alpha}{\beta^2})}d\omega' \\
&+\frac{\pi\Sigma^2a^2\alpha}{\beta^2}.\int_0^{\omega}\frac{1}{\sqrt{M^2-(a^2+q^2)\alpha}+(M-\frac{M\alpha}{\beta^2})}d\omega' \\
&-\frac{\pi a\Sigma^3\alpha}{\beta^2}.\int_0^{j}\frac{1}{\sqrt{M^2-(a^2+q^2)\alpha}+(M-\frac{M\alpha}{\beta^2})}dj'. \qquad (9.26)
\end{aligned}
$$

For the maximum value of integration, neglecting $(1-\frac{\alpha}{\beta^2})M$. Equation (9.26) becomes

$$
\begin{aligned}
\text{Im}I &= \frac{\pi\Sigma^2}{\beta^2\alpha}.\int_0^{\omega}\frac{(M+\sqrt{M^2-(a^2+q^2)\alpha})^2}{\sqrt{M^2-(a^2+q^2)\alpha}}d\omega' \\
&+\frac{\pi\Sigma^2a^2\alpha}{\beta^2}.\int_0^{\omega}\frac{1}{\sqrt{M^2-(a^2+q^2)\alpha}}d\omega' \\
&-\frac{\pi a\Sigma^3\alpha}{\beta^2}.\int_0^{j}\frac{1}{\sqrt{M^2-(a^2+q^2)\alpha}}dj'. \qquad (9.27)
\end{aligned}
$$

Replacing $M$ by $M-\omega$ and $j$ by $J-j$, we have

$$
\begin{aligned}
\text{Im}I &= -\frac{\pi\Sigma^2}{\beta^2\alpha}.\int_M^{(M-\omega)}\frac{(M-\omega+\sqrt{(M-\omega)^2-(a^2+q^2)\alpha})^2}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}d(M-\omega') \\
&-\frac{\pi\Sigma^2a^2\alpha}{\beta^2}.\int_M^{(M-\omega)}\frac{1}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}d(M-\omega') \\
&+\frac{\pi a\Sigma^3\alpha}{\beta^2}.\int_J^{(J-j)}\frac{1}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}d(J-j'), \qquad (9.28)
\end{aligned}
$$

where

$$J-j'=\frac{(M-\omega')a}{\Sigma^2}. \qquad (9.29)$$

Using Eq. (9.29) into Eq. (9.28), and finishing the integral, the imaginary part of the action can be obtained.

$$\begin{aligned}\mathrm{Im}I &= -\frac{\pi\Sigma^2}{\beta^2\alpha}.\int_M^{(M-\omega)}\frac{2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}\\ &\quad\times d(M-\omega')+\frac{\pi\Sigma^2}{\beta^2\alpha}.\int_M^{(M-\omega)}\frac{(a^2+q^2)\alpha}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}d(M-\omega')\\ &.\end{aligned} \tag{9.30}$$

Doing the $\omega'$ integral, finally we get

$$\begin{aligned}\mathrm{Im}I &= -\frac{\pi\Sigma^2}{\beta^2\alpha}\{(M-\omega)\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}\\ &\quad+(M-\omega)^2-M\sqrt{M^2-(a^2+q^2)\alpha}-M^2\}\\ &= -\frac{\pi\Sigma^2}{2\beta^2\alpha}\{2(M-\omega)\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}\\ &\quad+2(M-\omega)^2-2M\sqrt{M^2-(a^2+q^2)\alpha}-2M^2\}\\ &= -\frac{\pi\Sigma^2}{2\beta^2\alpha}\{(M-\omega)+\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}\}^2\\ &\quad-(M+\sqrt{M^2-(a^2+q^2)\alpha})^2\\ &= -\frac{1}{2}\exp[\pi(r_f^2-r_i^2)]\\ &= -\frac{1}{2}\exp(\Delta S_{BH}).\end{aligned} \tag{9.31}$$

Here $r_i = \frac{\Sigma}{\beta\sqrt{\alpha}}[(M+\sqrt{(M^2-(a^2+q^2)\alpha}]$ and $r_f = \frac{\Sigma}{\beta\sqrt{\alpha}}[(M-\omega) + \sqrt{(M-\omega)^2-(a^2+q^2)\alpha}]$ are the locations of the KNdS event horizon before and after the particle emission respectively, and $\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$ is the difference of Bekenstein-Hawking entropy.

Utilizing Eq.(9.16), the relationship between the tunneling rate and the imaginary part of the action of the radiative particle for the KNdS black hole is given by

$$\Gamma \sim \exp(-2\mathrm{Im}I) = \exp(\Delta S_{BH}). \tag{9.32}$$

## 9.5 Purely Thermal Radiation

The non-thermal emission rate described by Eq. (9.32) is related to the change of Bekenstein-Hawking entropy, and is consistent with an underlying unitary theory and the radiation spectrum is not pure thermal although gives a correction to the Hawking radiation of KNdS black hole. The pure thermal radiation spectrum can be derived from Eq.(9.32) by expanding the tunneling rate in power of $\omega$ upto second order as follows

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{-\omega\frac{\partial S_{BH}(M)}{\partial M} + \frac{\omega^2}{2}\frac{\partial^2 S_{BH}(M)}{\partial M^2}\right\}. \quad (9.33)$$

From Eq.(9.31), we can write

$$S_{BH}(M-\omega) = \frac{\pi\Sigma^2}{\beta^2\alpha}[(M-\omega) + \sqrt{(M-\omega)^2 - (a^2+q^2)\alpha}]^2. \quad (9.34)$$

At $\omega = 0$,

$$\frac{\partial S_{BH}(M)}{\partial M} = \frac{2\Sigma^2}{\beta^2\alpha}\left[2M + \sqrt{M^2-(a^2+q^2)\alpha} + \frac{M^2}{\sqrt{M^2-(a^2+q^2)\alpha}}\right] \quad (9.35)$$

and

$$\frac{\partial^2 S_{BH}(M)}{\partial M^2} = \frac{2\Sigma^2}{\beta^2\alpha}\left[2 + \frac{3M}{\sqrt{M^2-(a^2+q^2)\alpha}} - \frac{M^3}{(M^2-(a^2+q^2)\alpha)^{\frac{3}{2}}}\right] \quad (9.36)$$

With the help of Eqs. (9.35) and (9.36), the pure thermal emission rate is of the form

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp[\pi(-\omega\eta + \frac{\omega^2}{2}\lambda)], \quad (9.37)$$

where $\eta = \frac{2\Sigma^2}{\beta^2\alpha}\left[2M + \sqrt{M^2-(a^2+q^2)\alpha} + \frac{M^2}{\sqrt{M^2-(a^2+q^2)\alpha}}\right]$ and $\lambda = \frac{2\Sigma^2}{\beta^2\alpha}\left[2 + \frac{3M}{\sqrt{M^2-(a^2+q^2)\alpha}} - \frac{M^3}{(M^2-(a^2+q^2)\alpha)^{\frac{3}{2}}}\right]$.

## 9.6 Concluding Remarks

We have developed the non-thermal and purely thermal tunneling rates using massive particles tunneling process from KNdS black hole [161] by taking into account the self-gravitational interaction, the background spacetime as dynamical and the energy as conservation. We have explored that the tunneling rate at the event horizon of KNdS black hole is related to the change of Bekenstein-Hawking entropy. The results are in accordance with Parikh and Wilczek's opinion [51, 82, 83] from spherically symmetric black holes. We also conclude that the actual radiation spectrum of KNdS black hole is not precisely thermal, which provides an interesting correction to the Hawking pure thermal spectrum.

We now like to point out that some of the previous results existed in this chapter which can be enclosed as special cases. In particular, when cosmological constant vanishes, then $\Sigma = \beta = \alpha = 1$ and hence the pure thermal spectrum can be reduced for the Kerr-Newman black hole [125]. The position of the event horizon before and after the emission of the particles with energy $\omega$ are $r_i = M + \sqrt{(M^2 - (a^2+q^2)}$ and $r_f = (M-\omega) + \sqrt{(M-\omega)^2 - (a^2+q^2)}$ respectively. From Eq.(9.32), the non-thermal tunneling rate for the Kerr-Newman black hole can be written as

$$\begin{aligned}\Gamma \sim \exp(-2\text{Im}I) &= \exp[\pi\{(M-\omega) + \sqrt{(M-\omega)^2 - (a^2+q^2)}\}^2 \\ &\quad + \{M + \sqrt{M^2 - (a^2+q^2)}\}^2] \\ &= \exp[\pi(r_f^2 - r_i^2)] \\ &= \exp(\Delta S_{BH}), \end{aligned} \tag{9.38}$$

and the purely thermal rate of the Kerr-Newman black hole can be written as

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp[-8\pi\omega(\eta - \frac{\omega}{2}\lambda)], \tag{9.39}$$

where $\eta = \frac{1}{4}\left[2M + \sqrt{M^2 - (a^2+q^2)} + \frac{M^2}{\sqrt{M^2-(a^2+q^2)}}\right]$ and
$\lambda = \frac{1}{4}\left[2 + \frac{3M}{\sqrt{M^2-(a^2+q^2)}} - \frac{M^3}{(M^2-(a^2+q^2))^{\frac{3}{2}}}\right]$.

It is interesting that for $q = 0$ , it reduces to the result of Kerr-de Sitter black hole (chapter 7), and for $q = 0, a = 0$, it becomes to the result of SdS black hole [129]. Finally, if one sets $\ell \to \infty$, $a = 0$ and $q = 0$ gives the result for the Schwarzschild black hole [51].

In addition, our discussion made here can be directly to the anti-de Sitter case by changing the sign of the cosmological constant to a negative one, which have been discussed in later chapter.

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# Chapter 10

# Hawking Non-thermal and Purely thermal radiations of Kerr-Newman-anti-de Sitter Black Hole by Hamilton-Jacobi method

## 10.1 Introduction

Recently, a semiclassical tunneling process applied to find the Hawking radiation of the static Schwarzschild and Reissner-Nordström black holes by Parikh and Wilczek [51, 82, 83] and their result shows that the radiation spectrum is not pure thermal but satisfies the unitary principle and support the result of information conservation. In their process, the tunneling potential barrier is produced by the self-gravitation interaction and the position of the horizons before and after the particles emission. Following this method, several researchers studied the Hawking radiation of various spacetime [38, 56, 67, 68, 69, 70, 75, 76, 86, 90] by using Painleavé or dragging or tortoise or Eddington-Finkelstein coordinate transformations and these radiations are limited to uncharged massless particle only.

In this chapter, we use the Parikh and Wilczek's opinion [51, 82, 83] and employing standard Hamilton-Jacobi method to investigate the Hawking non-thermal and purely thermal tunneling rates of the Kerr-Newman-anti-de Sitter (KNAdS) black hole for massive particle. The Kerr-Newman-anti-de Sitter (KNAdS) black hole which is the KAdS black hole generalized with a charge parameter, described by the metric

$$\begin{aligned} ds^2 &= -\frac{\Delta_r - \Delta_\theta a^2 \sin^2\theta}{\rho^2} dt^2 + \frac{\Delta_\theta (r^2+a^2)^2 - \Delta_r a^2 \sin^2\theta}{\rho^2 \Xi^2} \sin^2\theta d\phi^2 \\ &+ \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{2a[(r^2+a^2)\Delta_\theta - \Delta_r] \sin^2\theta}{\rho^2 \Xi} dt d\phi, \end{aligned} \tag{10.1}$$

where

$$\begin{aligned} \rho^2 = r^2 + a^2\cos^2\theta, \quad \Delta_\theta = 1 - \frac{a^2\cos^2\theta}{\ell^2}, \\ \Xi = 1 - \frac{a^2}{\ell^2}, \quad \Delta_r = (1+\frac{a^2}{\ell^2})r^2 - 2Mr + a^2 + \frac{r^4}{\ell^2} + q^2. \end{aligned} \tag{10.2}$$

Here the parameters $M$, $a$, $\ell$ and $q$ are the associated with the mass, angular momentum, cosmological radius, and charge parameters of the spacetime respectively in the background of the rotating anti de Sitter space. The spacetime causal structure depend strongly on the singularities of the metric given by the zeros of $\Delta_r$ as follows

$$\Delta_r = (1+\frac{a^2}{\ell^2})r^2 - 2Mr + a^2 + \frac{r^4}{\ell^2} + q^2 = 0. \tag{10.3}$$

Depending on the black hole parameters, the function $\Delta_r = 0$ with $\ell^2 > a^2$ has four distinct roots. For the KNAdS black hole case we are interested to find the real root of $\Delta_r = 0$, namely the real root $r_+$ corresponds to the radius of the black hole's outer event horizon, while the other real root $r_-$ represents the radius of the inner cauchy horizon and $r_c$ as the cosmological horizon. Equation (10.3) can be written as

$$r^4 + (\ell^2 + a^2)r^2 - 2M\ell^2 r + \ell^2(a^2 + q^2) = 0. \tag{10.4}$$

Solving the above equation, the position of the black hole horizons are given by

$$\begin{aligned} r_\pm &= \frac{\ell\beta}{\sqrt{3}}.\sinh\Big[\frac{1}{3}\sinh^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big] \\ &\times\Big(1 \pm \sqrt{1 - \frac{(a^2+q^2)\ell}{\sqrt{3}M\beta}.\frac{2}{1+\delta}\text{cosech}\Big[\frac{1}{3}\sinh^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big]}\Big), \end{aligned} \tag{10.5}$$

and

$$\begin{aligned} r_c &= \frac{\ell\beta}{\sqrt{3}}.\sinh\Big[\frac{1}{3}\sinh^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big] \\ &\times\Big(\sqrt{1 + \frac{1+\delta}{2}.\frac{3M\sqrt{\ell}}{\sqrt{3}\beta^2}\text{cosech}^3\Big[\frac{1}{3}\sinh^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big]} - 1\Big), \end{aligned} \tag{10.6}$$

where

$$\delta = \sqrt{1 - \frac{4(a^2+q^2)\beta^2}{3M^2}\sinh^2\Big[\frac{1}{3}\sinh^{-1}\frac{3\sqrt{3}M}{\ell\alpha\beta}\Big]},$$

$$\alpha = \sqrt{\{1 - \frac{a^2}{\ell^2}\}^2 - \frac{4q^2}{\ell^2}}, \quad \beta = \sqrt{1 + \frac{a^2}{\ell^2}} \tag{10.7}$$

and $r_{--} = -(r_+ + r_- + r_c)$ is the another cosmological horizon. With $\delta \approx 1$ the black hole horizons can be approximated as

$$r_\pm \approx \frac{\ell\beta}{\sqrt{3}}.\sinh\Big[\frac{1}{3}\sinh^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big].\Big(1 \pm \sqrt{1 - \frac{(a^2+q^2)\alpha}{M^2}}\Big). \tag{10.8}$$

Taking only the positive sign i.e.the event horizon of KNAdS black hole as follows

$$r_+ = \frac{\ell\beta}{\sqrt{3}}.\sinh\Big[\frac{1}{3}\sinh^{-1}\frac{3M\sqrt{3}}{\ell\alpha\beta}\Big].\Big(1+\sqrt{1-\frac{(a^2+q^2)\alpha}{M^2}}\Big). \quad (10.9)$$

Expanding $r_+$ in terms of black hole parameters with negative cosmological constant under the condition $(a^2+q^2)\alpha < M^2$ , we obtain

$$r_+ = \frac{M}{\alpha}\left(1-\frac{4M^2}{\ell^2\beta^2\alpha}+\cdots\right)\left(1+\sqrt{1-\frac{(a^2+q^2)\alpha}{M^2}}\right), \quad (10.10)$$

which can be written as

$$r_+ = \frac{1}{\alpha}\left(1-\frac{4M^2}{\ell^2\beta^2\alpha}+\cdots\right)\left(M+\sqrt{M^2-(a^2+q^2)\alpha}\right). \quad (10.11)$$

Now if we set $\mu = \frac{1}{\alpha}\left(1-\frac{4M^2}{\ell^2\beta^2\alpha}+\cdots\right)$, then $r_+ = \left(M+\sqrt{M^2-(a^2+q^2)\alpha}\right)\mu$ with $\mu < 1$ and hence the event horizon of KNAdS black hole is less than the Kerr-Newman [125] event horizon $r_{KN} = M+\sqrt{M^2-(a^2+q^2)}$. As the event horizon of KNAdS black hole coincides with the outer infinite red-shift surface, we apply the geometrical optical limit and the "s-wave" approximation. Using the semiclassical WKB method [168], the tunneling probability is found to be related to the imaginary part of the action as the following form

$$\Gamma \sim \exp(-2\mathrm{Im}I), \quad (10.12)$$

where $I$ is the action of the radiating particle.

## 10.2 The HJ Method for KNAdS Spacetime

The Hamilton-Jacobi method was applied extensively to the non-thermal radiation in 1990s and attracted people's attention [88, 90, 109]. In 2005, applying semiclassical tunneling method, Angheben, Nadalini, Vanzo and Zerbini [90] developed Hamilton-Jacobi method [85, 86, 87, 88, 89] ignoring the self-gravitational effect of the emitted scalar particles. Here we now consider the method of Chen et al. [125, 126] to calculate the imaginary part of the action from the relativistic Hamilton-Jacobi equation. The action of the radiating particle $I$ satisfies the relativistic Hamilton-Jacobi equation

$$g^{ij}(\partial_i I)(\partial_j I) + m^2 = 0, \tag{10.13}$$

where $m$ and $g^{ij}$ are the mass of the particle and the inverse metric tensors respectively.

In this method, we avoid the exploration of the equation of motion in the Painlevé coordinates systems for calculate the imaginary part of the action $I$. For the convenience of our research to study the Hawking radiation, adopting the transformation $\frac{d\phi}{dt} = -\frac{g_{14}}{g_{44}}$ on the line element (10.1), we obtain the new line element of the Kerr-Newman-anti-de Siter black hole as

$$ds^2 = -\frac{\Delta_r \Delta_\theta \rho^2}{\Delta_\theta (r^2 + a^2)^2 - \Delta_r a^2 \sin^2\theta} dt^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2. \tag{10.14}$$

The position of black hole horizon of the metric given by Eq.(10.14) is same as given in Eq. (10.11). Therefore, the line element near the event horizon rewritten as

$$ds^2 = -\frac{\Delta_{r,r}(r_+)(r - r_+)\rho^2(r_+)}{(r_+^2 + a^2)^2} dt^2 + \frac{\rho^2(r_+)}{\Delta_{r,r}(r_+)(r - r_+)} dr^2 + \frac{\rho^2(r_+)}{\Delta_\theta} d\theta^2, \tag{10.15}$$

where $\rho^2(r_+)$ and $\Delta_{r,r}$ are defined as follows

$$\rho^2(r_+) = r_+^2 + a^2 \cos^2\theta$$

$$\Delta_{r,r}(r_+) = \frac{d\Delta_r}{dr}\bigg|_{r=r_+} = \frac{2}{\Xi^2}(\beta^2 r_+ - M + 2\frac{r_+^3}{\ell^2}). \tag{10.16}$$

Calculating the non-null inverse metric tensors from the metric (10.15) and employing these in Eq. (10.13) as follows

$$-\frac{(r_+^2+a^2)^2}{\rho^2(r_+)\Delta_{r,r}(r_+)(r-r_+)}\left(\partial_t I\right)^2 + \frac{\Delta_{r,r}(r_+)(r-r_+)}{\rho^2(r_+)}\left(\partial_r I\right)^2 + \frac{\Delta_\theta}{\rho^2(r_+)}\left(\partial_\theta I\right)^2 + m^2 = 0. \tag{10.17}$$

To solve action $I(t, r, \theta, \phi)$, we consider the properties of the black hole spacetime and carry out the separation of variables as

$$I = -\omega t + R(r) + H(\theta) + j\phi, \tag{10.18}$$

where $\omega$ is the energy of the emitted particle, $R(r)$ and $H(\theta)$ are the generalized momentums, and $j$ s the angular momentum of the particle with respect to $\phi$-axis. Inserting Eq. (10.18) into Eq. (10.17) to seek a solution of the following form

$$R(r) = \pm\frac{r_+^2+a^2}{\Delta_{r,r}(r_+)}\int\frac{dr}{(r-r_+)}\times \sqrt{(\omega - j\Omega_+)^2 - \frac{\rho^2(r_+)\Delta_{r,r}(r_+)(r-r_+)}{(r_+^2+a^2)^2}\left[\frac{\Delta_\theta}{\rho^2(r_+)}\left(\partial_\theta H\right)^2 + m^2\right]}, \tag{10.19}$$

where the angular velocity of the particle at the event horizon is

$$\Omega_+ = \frac{d\phi}{dt}\bigg|_{r=r_+} = \frac{a\Xi}{r_+^2+a^2} \tag{10.20}$$

We treat the emitted particle as an ellipsoid shell of energy $\omega$ to tunnel across the event horizon. Finishing the above integral by using the

Cauchy's integral formula, we obtain

$$R(r) = \pm\frac{2\pi i(r_+^2 + a^2)}{\Delta_{r,r}(r_+)}(\omega - j\Omega_+), \tag{10.21}$$

where $\pm$ sign comes from the square root. Therefore, the imaginary part of the action $I$ corresponding to the outgoing particle is obtained by $\pi$ times the residue of the integrand

$$\begin{aligned} \text{Im}I &= \frac{2\pi(r_+^2 + a^2)}{\Delta_{r,r}(r_+)}(\omega - j\Omega_+) \\ &= \frac{\Xi^2\pi(r_+^2 + a^2)}{\beta^2 r_h - M + 2\frac{r_+^3}{\ell^2}}(\omega - j\Omega_+). \end{aligned} \tag{10.22}$$

Using Eqs. (10.11) and (10.20) into Eq. (10.22), we get the imaginary part of the true action of the radiation particle as

$$\begin{aligned} \text{Im}I = & \frac{\frac{\pi\Xi^2}{\alpha^2}\left(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)^2 (M + \sqrt{M^2 - (a^2 + q^2)\alpha})^2}{\frac{\beta^2}{\alpha}\left(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - M + A}\omega \\ & + \frac{\Xi^2\pi a^2}{\frac{\beta^2}{\alpha}\left(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - M + A}\omega \\ & - \frac{\Xi^3\pi a}{\frac{\beta^2}{\alpha}\left(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - M + A}j, \end{aligned}$$

where $A = \frac{2}{\ell^2\alpha^3}\left(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)^3 (M + \sqrt{M^2 - (a^2 + q^2)\alpha})^3$.

$$\begin{aligned} \text{Im}I = & \frac{\pi\Xi^2(M + \sqrt{M^2 - (a^2 + q^2)\alpha})^2}{\beta^2\alpha\left[\left(1 + \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) + B\right]}\omega \\ & + \frac{\Xi^2\pi a^2}{\frac{\beta^2}{\alpha}\left[\left(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - \frac{M\alpha}{\beta^2} + \frac{\alpha A}{\beta^2}\right]}\omega \\ & - \frac{\Xi^3\pi a}{\frac{\beta^2}{\alpha}\left[\left(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot\right)(M + \sqrt{M^2 - (a^2 + q^2)\alpha}) - \frac{M\alpha}{\beta^2} + \frac{\alpha A}{\beta^2}\right]}j, \end{aligned}$$

where $B = -\frac{M\alpha}{\beta^2}(1 + \frac{8M^2}{\ell^2\alpha\beta^2} + \cdot\cdot) + \frac{2}{\ell^2\beta^2\alpha^2}(1 - \frac{4M^2}{\ell^2\alpha\beta^2} + \cdot\cdot)C$ and $C = (M + \sqrt{M^2 - (a^2+q^2)\alpha})^3$.

To get the maximum value of the integration, neglecting higher order terms above and equal $M^3$ in the denominator, we then get

$$\begin{aligned}\mathrm{Im}I &= \frac{\pi\Xi^2}{\beta^2\alpha}.\frac{(M + \sqrt{M^2 - (a^2+q^2)\alpha})^2}{(M + \sqrt{M^2 - (a^2+q^2)\alpha}) - \frac{M\alpha}{\beta^2}}\omega \\ &+ \frac{\Xi^2\pi a^2\alpha}{\beta^2\left[M + \sqrt{M^2 - (a^2+q^2)\alpha} - \frac{M\alpha}{\beta^2}\right]}\omega \\ &- \frac{\Xi^3\pi a\alpha}{\beta^2\left[M + \sqrt{M^2 - (a^2+q^2)\alpha} - \frac{M\alpha}{\beta^2}\right]}j. \end{aligned} \tag{10.23}$$

## 10.3 Non-thermal Tunneling Rate

Since the emitted particle can be treated as a shell of energy $\omega$, Eqs. (10.22) and (10.23) should be modified when the particle's self-gravitational interaction is incorporated. Taking into account the energy conservation as well as angular momentum, the mass parameter and the angular momentum in these equations will be replaced with $M \to M - \omega$ and $j \to J - j$ when the particle with energy $\omega$ and angular momentum $j$ tunnels out of the event horizon. We fix the ADM mass, charge and angular momentum of the total spacetime and in presence of comological constant KNAdS spacetime is dynamic and allow mass and angular momentum of the black hole to fluctuate. Then the imaginary part of the true action can be calculated from Eq. (10.23) in the following integral

$$\mathrm{Im}I = \frac{\pi\Xi^2}{\beta^2\alpha}.\int_0^{\omega} \frac{(M + \sqrt{M^2 - (a^2+q^2)\alpha})^2}{\sqrt{M^2 - (a^2+q^2)\alpha} + (M - \frac{M\alpha}{\beta^2})} d\omega'$$

$$+\frac{\pi\Xi^2a^2\alpha}{\beta^2}.\int_0^{\omega}\frac{1}{\sqrt{M^2-(a^2+q^2)\alpha}+(M-\frac{M\alpha}{\beta^2})}d\omega'$$
$$-\frac{\pi\Xi^3a\alpha}{\beta^2}.\int_0^{j}\frac{1}{\sqrt{M^2-(a^2+q^2)\alpha}+(M-\frac{M\alpha}{\beta^2})}dj'. \quad (10.24)$$

For the maximum value of integration, neglecting $(1-\frac{\alpha}{\beta^2})M$. Equation (10.24) becomes

$$\mathrm{Im}I=\frac{\pi\Xi^2}{\beta^2\alpha}.\int_0^{\omega}\frac{(M+\sqrt{M^2-(a^2+q^2)\alpha})^2}{\sqrt{M^2-(a^2+q^2)\alpha}}d\omega'$$
$$+\frac{\pi\Xi^2a^2\alpha}{\beta^2}.\int_0^{\omega}\frac{1}{\sqrt{M^2-(a^2+q^2)\alpha}}d\omega'$$
$$-\frac{\pi\Xi^3a\alpha}{\beta^2}.\int_0^{j}\frac{1}{\sqrt{M^2-(a^2+q^2)\alpha}}dj'. \quad (10.25)$$

Replacing $M$ and $j$ by $M-\omega$ and $J-j$ respectively, we obtain

$$\mathrm{Im}I = -\frac{\pi\Xi^2}{\beta^2\alpha}.\int_M^{(M-\omega)}\frac{(M-\omega+\sqrt{(M-\omega)^2-(a^2+q^2)\alpha})^2}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}d(M-\omega')$$
$$-\frac{\pi\Xi^2a^2\alpha}{\beta^2}.\int_M^{(M-\omega)}\frac{1}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}d(M-\omega')$$
$$+\frac{\pi\Xi^3a\alpha}{\beta^2}.\int_J^{(J-j)}\frac{1}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}d(J-j'), \quad (10.26)$$

where $J-j'=\frac{(M-\omega')a}{\Xi^2}$ and so there is

$$\mathrm{Im}I = -\frac{\pi\Xi^2}{\beta^2\alpha}.\int_M^{(M-\omega)}\frac{2(M-\omega)^2+2(M-\omega)\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}$$
$$\times d(M-\omega')+\frac{\pi\Xi^2}{\beta^2\alpha}.\int_M^{(M-\omega)}\frac{(a^2+q^2)\alpha}{\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}}d(M-\omega'). \quad (10.27)$$

Finishing the $\omega'$ integral, we obtain

$$\mathrm{Im}I = -\frac{\pi\Xi^2}{\beta^2\alpha}\{(M-\omega)\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}$$

$$
\begin{aligned}
& +(M-\omega)^2 - M\sqrt{M^2-(a^2+q^2)\alpha} - M^2\} \\
= & -\frac{\pi\Xi^2}{2\beta^2\alpha}\{2(M-\omega)\sqrt{(M-\omega)^2-(a^2+q^2)\alpha} \\
& +2(M-\omega)^2 - 2M\sqrt{M^2-(a^2+q^2)\alpha} - 2M^2\} \\
= & -\frac{\pi\Xi^2}{2\beta^2\alpha}\{(M-\omega)+\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}\}^2 \\
& -(M+\sqrt{M^2-(a^2+q^2)\alpha})^2. \qquad (10.28)
\end{aligned}
$$

Therefore, the non-thermal tunneling rate for the KNAdS black hole is given by

$$
\begin{aligned}
\Gamma \sim \exp(-2\mathrm{Im}I) &= \exp[\frac{\pi\Xi^2}{\beta^2\alpha}\{(M-\omega)+\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}\}^2 \\
& -(M+\sqrt{M^2-(a^2+q^2)\alpha})^2] \\
&= \exp[\pi(r_f^2-r_i^2)] \\
&= \exp(\Delta S_{BH}). \qquad (10.29)
\end{aligned}
$$

Here we find that $\Delta S_{BH} = S_{BH}(M-\omega) - S_{BH}(M)$ is the change of Bekenstein-Hawking entropy of the KNAdS black hole before and after the massive particles emission by taking into account $r_i = \frac{\Xi}{\beta\sqrt{\alpha}}[(M+\sqrt{(M^2-(a^2+q^2)\alpha}]$ and $r_f = \frac{\Xi}{\beta\sqrt{\alpha}}[(M-\omega)+\sqrt{(M-\omega)^2-(a^2+q^2)\alpha}]$.

## 10.4 Purely Thermal Radiation

The radiation spectrum given by Eq.(10.29) is not pure thermal, which gives a correction to the Hawking radiation of the KNAdS black hole and is consistent with an underlying unitary theory. We now expand (10.29) in power of $\omega$ upto second order as discussed by Hossain et al. [131] of the form

$$\Gamma \sim \exp(\Delta S_{BH}) = \exp\left\{-\omega\frac{\partial S_{BH}(M)}{\partial M} + \frac{\omega^2}{2}\frac{\partial^2 S_{BH}(M)}{\partial M^2}\right\}. \quad (10.30)$$

From Eq.(10.29), we can write

$$S_{BH}(M-\omega) = \frac{\pi\Xi^2}{\beta^2\alpha}[(M-\omega) + \sqrt{(M-\omega)^2 - (a^2+q^2)\alpha}]^2. \quad (10.31)$$

Using Eq.(10.31) in Eq.(10.30), we obtain

$$\begin{aligned}\Gamma &\sim \exp(\Delta S_{BH}) \\ &= \exp\Big[\frac{-2\pi\Xi^2\omega}{\alpha\beta^2}\{(2M + \sqrt{M^2-(a^2+q^2)\alpha} + \frac{M^2}{\sqrt{M^2-(a^2+q^2)\alpha}}) \\ &-\frac{\omega}{2}(2 + \frac{3M}{\sqrt{M^2-(a^2+q^2)\alpha}} - \frac{M^3}{(M^2-(a^2+q^2)\alpha)^{\frac{3}{2}}})\}\Big]. \quad (10.32)\end{aligned}$$

If we put $-\ell^2$ in the place of $\ell^2$, the Hawking non-thermal spectrum and pure thermal spectrum agree with that of KNdS black hole.

## 10.5 Concluding Remarks

In a nutshell, we have investigated the Hawking non-thermal and purely thermal radiations of massive particles as a semiclassical tunneling process from the KNAdS black hole event horizon by taking into account the self-gravitation effect of the emitted particles, the unfixed background spacetime. The results of our work show that the radiant spectrum is not a pure thermal one and the tunneling rate is related to the change of Bekenstein-Hawking entropy and is consistent with an underlying unitary theory. The results we have obtained in this chapter provides further evidence to support the Parikh and Wilczek's opinion [51, 82, 83] from spherically symmetric black holes.

The study of this chapter gives the result for the Kerr-Newman black hole [125] when $\ell \to \infty$. For $q = 0$, the study provides the result of chapter 8 for the Kerr-anti-de Sitter black hole, while for $\ell \to \infty, q = 0$, the result reduces for the Kerr black hole [81]. The result of chapter 3 for the Schwarzschild-anti-de Sitter black hole [130] is obtained if one sets $a = 0$ and $q = 0$. Moreover, the choice $\ell \to \infty$, $a = 0$ and $q = 0$ gives the result for the Schwarzschild black hole [51].

# Chapter 11

# Conclusion

In this thesis, we have studied Hawking radiations of nonrotating black holes (SdS, SAdS, RNdS, RNAdS) and rotating black holes (KdS, KAdS, KNdS, KNAdS) in de Sitter and anti-de Sitter spaces. We have used the massive particle tunneling mechanism and discuss the Hawking non-thermal and purely thermal emission rates employing standard Hamilton-Jacobi method. For nonrotating black holes no coordinates transformation have been used. By considering the spacetime background to be dynamical, incorporate the self-gravitation effect of the emitted particles and the conservation laws of energy and angular momentum, we have shown that the non-thermal and purely thermal tunneling rates of all nonrotating black holes in de/Anti-de Sitter space are related to the change of Bekenstein-Hawking entropy and the derived emission spectrum deviates from the pure thermal spectrum. Such result satisfies an underlying unitary theory and also gives a correction to the Hawking radiation of the nonrotating black holes in de Sitter and anti-de Sitter spaces. For rotating black holes only dragging coordinates transformation have been used to obtain a near horizon metric with rotation. The results we have obtained for all rotating black holes by taking the self-gravitation effect into consideration agree with the results obtained for nonrotating black holes. It is noted that for all the black holes the spacetime metric has been made dynamical by taking the position of all black holes as a series of infinite terms in presence of cosmological constant.

We find some virtues in the investigation of Hawking radiation by Hamilton-Jacobi method.

Firstly, we do not introduce the Painleavé coordinate transformation which is appropriate for massless particle tunneling method though we have used the dragging coordinate transformation for massive particle tunneling from rotating black holes. For simplicity, we have considered a new form of spacetime metric.

Secondly, since the derivation of action only depends on the Hamilton-Jacobi equation, we avoid differentiating massive particles because the massive particles need to differentiate for time-like character.

Thirdly, we need not to solve the Hamilton canonical equations because Hamilton-Jacobi method can be used to obtain the actions of radiation particles from any type of black holes either stationary or non-stationary.

Therefore, the massive particle tunneling method can successfully be applied to a wide range of spacetimes. We have extended the method to various types of de Sitter and anti-de Sitter black holes at the event horizon. Actually this is the extension of the classical framework [58, 99, 103, 128, 181, 189] for spherically symmetric black hole to deal with Hawking radiation of massive particle tunneling through the event horizon. The results we have obtained from chapter 3 to chapter 10 of this thesis also indeed in accordance with the results obtained by massless or massless charged particle tunneling from different spacetime such as charged black hole with a global monopole [99, 128], Kerr NUT black hole [65] and Kerr and Ker-Newman [103] black holes.

We have derived the the Hawking non-thermal and the purely thermal tunneling probabilities for the SdS black hole [129] (in chapter 3) and SAdS black hole [130] (in chapter 4). In particular, results obtained in Ref. [51, 82, 83] can be recovered from the SdS black hole [130] and SAdS black hole [130]. For example, if the cosmological radius becomes infinite, in this case $\Lambda = 0$ and therefore the results we have obtained for the the Schwarzschild-de Sitter black hole [129] and the Schwarzschild-anti-de Sitter black hole [130] reduce to the results for the Schwarzschild black hole.

The Hawking non-thermal and purely thermal emission rates developed in chapter 5 and 6 for the RNdS and RNAdS [131] black holes reduce to the result for the RN black hole [99] for $\ell \to \infty$, the SdS [129] (chapter 3) and SAdS [130] (chapter 4) black holes respectively for $q = 0$ , and finally the Schwarzschild black hole for $q = 0$ and $\Lambda = 0$ and which is fully consistent with the result obtained by Parikh and Wilczek's [51, 82, 83].

The recovered tunneling rates in chapter 7 and 8 for the KdS and KAdS black holes reduce to the result for the Kerr black hole [81] for $\ell \to \infty$, the SdS [129] (chapter 3) and SAdS [130] (chapter 4) black holes respectively for $a = 0$ , and finally the Schwarzschild black hole for $a = 0$ and $\Lambda = 0$ and which is full accordant with Parikh and Wilczek's [51, 82, 83] result.

The results we have obtained in chapter 9 and 10 for the KNdS and

KNAdS black holes reduce to the result for the Kerr-Newman black hole [125] when $\ell \to \infty$, the KdS (chapter 7) and KAdS (chapter 8) black holes respectively for $q = 0$, the Kerr black hole [81] for $\ell \to \infty$ and $q = 0$, the SdS [129] (chapter 3) and SAdS [130] (chapter 4) black holes respectively for $a = 0$ and $q = 0$, and finally support the Parikh and Wilczek's [51, 82, 83] opinion for $\ell \to \infty$, $a = 0$ and $q = 0$.

We conclude that non-thermal and purely thermal emission rates can be expressed as standard form for each black hole. Both the emission rates are related to the exponent of the difference in the Bekenstein-Hawking entropy, $\Delta S$, before and after emission [51, 82, 83]. It is noted that due to hard calculation very little work have been investigated either for massless/charged particle or massive particle tunneling from black hole in de Sitter/anti-de Sitter spaces. Therefore, massive particle tunneling from black hole with cosmological constant in general relativity, both in astrophysics as well as in cosmology, seems to be a field open to investigation.

So our present work of this thesis is thus well motivated.

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July, 1999 To June, 2000 (Exam. held in May 2002), University of Rajshahi, Department of Mathematics, Master of Science (M.Sc), Major Subject: Applied Mathematics.

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