Department of Mathematics PhD Thesis

2016

Bekenstein-Hawking Entropy by Energy Quantization from Different Black Holes

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University of Rajshahi

Bekenstein-Hawking Entropy by Energy Quantization from Different Black Holes

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

FROM

THE UNIVERSITY OF RAJSHAHI

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RAJSHAHI-6205, BANGLADESH

OCTOBER 2016

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PhD Dissertation

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Session: 2014-2015

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CERTIFICATE OF THE SUPERVISOR

This is to certify that the PhD thesis titled "**Bekenstein-Hawking Entropy by Energy Quantization from Different Black Holes**" has been prepared by Md. Jakir Hossain, Assistant Professor, Department of Mathematics, University of Rajshahi under my direct supervision and guidance for submission to the Department of Mathematics, University of Rajshahi, Bangladesh in fulfillment of the requirements for the degree of **Doctor of Philosophy**. The work is an original one and it is the result of the work carried out by Md. Jakir Hossain during the period of his study. I have gone through the draft and final version of the dissertation and found it satisfactory for submission. I am fully convinced that the results embodied in the thesis are new and this thesis has not been previously submitted to any other university or institute for any diploma or degree or fellowship.

To the best of my knowledge Md. Jakir Hossain bears a good moral character and is mentally and physically fit to get the degree. I wish him a bright future and every success in life.

(**Dr. Md. Atiqur Rahman**)

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CERTIFICATE FROM THE CO-SUPERVISOR

This is to certify that the dissertation titled “**Bekenstein-Hawking Entropy by Energy Quantization from Different Black Holes**” has been prepared by Md. Jakir Hossain, Assistant Professor, Department of Mathematics, University of Rajshahi as a fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** in Mathematics, University of Rajshahi, Bangladesh. It is an original work and it embodies the result of his own study and investigation conducted during the period he worked as a PhD research scholar.

I have gone through the draft and final version of the dissertation thoroughly and found it satisfactory for submission to the Department of Mathematics, University of Rajshahi, Bangladesh as a partial fulfillment for the award of the degree of **Doctor of Philosophy**. To the best of my knowledge, this dissertation has not been submitted previously to any other university or institute for any diploma or degree or fellowship.

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DECLARATION

I declare that the contents in my PhD thesis titled "**Bekenstein-Hawking Entropy by Energy Quantization from Different Black Holes**" which is an original work and the result of my own investigation. Neither the whole nor any part of the dissertation was submitted earlier to any university or institute for any award of degree or diploma. The works of other researchers have been duly acknowledged at the relevant places of the dissertation.

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Dedicated

to

My Parents

And

My Loving Daughters

ACKNOWLEDGEMENT

It is my pleasure to present a bouquet of thanks to all those persons who have helped me, in one way or the other, in completion of the present study. My first and foremost gratitude is to my learned supervisor Dr. Md. Atiqur Rahman, Associate Professor, Department of Applied Mathematics, University of Rajshahi, for his scholastic guidance and supervision. Without his whole hearted involvement, valuable guidance and constant encouragement, this research work would not be in the present shape.

I am extremely grateful to Dr. Md. Ilias Hossain, Co- supervisor, Assistant Professor, Department of Mathematics for showing his keen interest in my work and encouraging me throughout the period of my research work. I also thank my wife Rudaba Afrin Zitu for supporting me in each and every step of this journey.

I also wish to give my special thanks to all the learned faculty members of the Department of Mathematics and Applied Mathematics, University Rajshahi, for their co-operation and valuable suggestions at the time of need. I am also very grateful to the office staff of the Department of Mathematics, University of Rajshahi, for helping and supporting me at different stages of my work.

ACKNOWLEDGEMENT

I shall be failing in my duty if I do not express my profound feeling of my gratitude to my mother and my family members, especially my daughters Zareefa Tasnia and Zafreen Zunaira, they were very much deprived of my attention and love throughout the period of this study. They were constant sources of inspiration and encouragement to me.

Above all, I submit to the Almighty Allah Who kept me physically and mentally fit to pursue this research work.

Needless to say, I am alone responsible for the deficiencies that might have been left in this research work.

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ABSTRACT

We investigate the Bekenstein-Hawking entropy from different types of non-rotating black holes in de Sitter and Anti-de Sitter spaces by using the energy quantization mechanism in analogy with Bohr's atomic model. We quantize the energy of the particle from the quantization of angular momentum. We also investigate the change of entropy between two nearby states as well as the thermal emission rate. In the limiting case all the results coincide with one another for the black holes in de Sitter and Anti- de Sitter spaces.

The thesis is organized as follows:

In **chapter 1** we provide a brief discussion about our work of studying Bekenstein-Hawking entropy from black hole spacetime.

In **chapter 2 to 7** by using the quantization method we investigate the Lagrangian and canonical momenta of test particle, Radial motion and Effective Potential, Quantization of Circular Orbit, Energy quantization and Hawking Radiation for **Schwarzschild, Schwarzschild-de Sitter (SdS)**, **Schwarzschild Anti-de Sitter (SAdS)**, **Reissner-Nordström (RN)**, **Reissner-Nordström-de Sitter (RNdS)**, **Reissner-Nordström Anti-de Sitter (RNAdS)**, black holes. Our new process is universally robust and the entropy framework given in this work indeed support the new perspective on quantum properties of gravity beyond classical physics, however, suggests a new idea to unify gravity with quantum theory and in the limiting case, the results are in line with that obtained by Sakalli et al. and He et al.'s method of the black hole.

CONTENTS

Notations, Conventions and Acronyms

- Greek indices μ, ν, ...at tensors cycle the numbers 0 to 3 and Latin indices i, j, ... cycle only spatial coordinates from 0 to 3. The temporal index is denoted by t and number 0.

- M → Mass of black hole
- m → Mass of the particle
- r_c → Compton radius
- V_{eff} → Effective potential
- τ → Proper time
- £ → Lagrangian
- WKB → Wentzel–Kramers–Brillouin
- CFT → Conformal Field Theory
- dS → de Sitter
- AdS → Anti-de Sitter
- ADM → Arnowitt, Deser, and Misner
- AdS → Anti-de Sitter
- SdS → Schwarzschild-de Sitter
- SAdS → Schwarzschild Anti-de Sitter
- RN → Reissner-Nordström
- RNdS → Reissner-Nordström-de Sitter
- RNAdS → Reissner-Nordström Anti-de Sitter

List of Publications

1. **M. Jakir Hossain, M. Atiqur Rahman, and M. Ilias Hossain**, "Bekenstein–Hawking entropy by energy quantization from Reissner-Nordström black hole", *International Journal of Modern Physics D*, **V**ol. **25, No.2**, P.1-14(2016), **(World scientific, Impact factor: 1.963(2015)).**

2. **M. Atiqur Rahman, M. Jakir Hossain and M. Ilias Hossain**, "Bekenstein–Hawking Entropy by energy quantization from Schwarzschild-de Sitter black hole", *Journal of Astroparticle Physics*, **V**ol. **71**, P.71-75(2015), **(Elsevier, Impact factor: 3.425 (2015)).**

In Preparation

1. **M. Jakir Hossain, M. Atiqur Rahman, and M. Ilias Hossain**, "Bekenstein–Hawking entropy by energy quantization from Reissner-Nordström-de Sitter black hole".

2. **M. Jakir Hossain, M. Atiqur Rahman, and M. Ilias Hossain**, "Bekenstein–Hawking entropy by energy quantization from Schwarzschild black hole".

Chapter 1

Introduction

Black hole is one of the most extreme prediction of Einstein's general theory of relativity. Black holes challenge common sense to such a level that Einstein himself was unenthusiastic to accept their reality. In spite of initial resistance, the existence of black holes is broadly accepted these days by the physics community. Obviously, the astronomical proof of black holes is convincing. Now the most enthralling object in astrophysics, cosmology and high-energy physics is black hole and it is an important component of our universe that has been confirmed to exist. The term "black hole" was first introduced by John Wheeler in his public lecture "Our Universe: the Known and Unknown" at New York Hilton hotel on 29 December, 1967 [1]. Inquiry into the properties of black holes has attracted the notice of more and more astronomers and physicists for about four decades. Physicists are grappling the theory of black holes, while astronomers are searching for real-life examples of black holes in the universe. Both the physicists and astronomers want to discover the mystery of black holes by detecting their fingerprints [2]. It is now believed that there exist many super-massive black holes in the universe including our Milky Way galaxy having masses from a million to a billion solar masses at the centers of many galaxies. Even much smaller black holes have been formed and evaporated away since very beginning [3]. The behavior of black hole in classical theory is quite simple depends only on few parameters due to the well known 'No Hair Theorem' [4].

The unification of the two fundamental theories: general relativity and quantum mechanics is one of the unsolved problem in modern theoretical physics. The first theory concerns the gravitation dynamics on large cosmological scale in a fully classical ambit, the second one concerns, mainly, the atomic or subatomic quantum phenomena and the fundamental interactions [5, 6, 7]. The wide spread concurrence among physicists that general relativity and quantum mechanics are mismatched each other derives by the complexity of matching the two models. Even though this is a superficial deduction, it can be of help if put in the right context. In reality, the incongruity between the two approaches originates from another big problem of the modern physics: to couple the quantum mechanics (QM) [5] with the classical one in which the general relativity is embedded.

There are two most important physical effects related to the quantum gravity, one is the Hawking radiation and the another is the black hole entropy. In classical theory, the black hole horizon is surrounded by membrane and does not emitted radiation. The spacetime having a black hole in it, first, has a singularity, and second, has a horizon preventing an external observer from seing it. The singularity in general relativity is radically different from field theory singularities because it is a property not of some field but of the spacetime itself. The most convincing viewpoint is that arising of singularities demonstrate the incompleteness of classical gravitational theory and need to replace it by quantum theory. So, if one believes in general relativity as a right theory of gravitation then he can assert that the singularity problem will be solved only after discovering of the quantum gravity theory.

In 1974, Stephen Hawking first proved that a black hole was not totally black but can emit radiation (Hawking radiation) with temperature (Hawking temperature) [8]. His theory attracted a lot of attentions by theoretical physicists. Many research works have appeared to discuss deeply on the quantum radiation of black holes in different methods [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. Till now, one can derive Hawking radiation from black holes in many ways such as Euclidean quantum gravity [36, 37], string theory [38, 39], a tunnelling picture [9, 11, 15, 18, 22, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49], gravitational anomaly [50, 51, 52, 53] and Hamilton-Jacobi [54, 55, 56, 57, 58, 59, 60] methods.

In 1973, Bekenstein [61] first considered the black hole horizon area as an adiabatic invariant quantity known as black hole entropy which is proportional to the area spectrum [62, 63, 64]. A wonderful fact of black hole radiation [65, 66] have discovered by Hawking in 1975 and several works have been done to calculate this quantum effect [67]. In accordance with the Ehrenfest principle, any classical adiabatic invariant stands for a quantum entity with discrete spectrum, Bekenstein observed that the (non-extremal) black hole horizon area behaves as a classical adiabatic invariant and therefore conjectured that it should exemplify a discrete eigenvalue spectrum with quantum transitions. In this scenario, temperature of black holes is proportional to surface gravity while area of black hole horizon plays the role of its entropy. The entropic framework given in Ref. [68, 68] indeed support new idea on quantum properties of gravity beyond classical physics. The quantization of gravity presented in this work, however, should not be interpreted as only a support of quantization of black hole as an entropic force [69, 70, 71], it also suggests a new way to unify gravity with quantum theory and therefore will lead to new understanding and perspectives on gravity of a black hole. This issue is still remain unclear, for a review see [72].

In the background of a classical black hole, the quantum-mechanical treatment of matter has indicated that the black hole radiates as a black body with a certain temperature. Since the black hole radiation involves a mixture of gravity and quantum mechanics, the study of black holes seems to lead us into the territory of quantum gravity, which is the missing link in a complete picture of the fundamental forces in nature.

Hod [73] opined that the analysis of Bekenstein was only similar to the famous semi classical determination of a lower bound on the ground state energy of the hydrogen atom, and that as an alternative one should consider a wave analysis of black hole perturbations. Hod [73] first related the real parts of quasi-normal modes with the quantization of the black-hole area by using Bohr's correspondence principle and found that the area spectrum of a black hole can be ascertained by analyzing the asymptotic behavior of the highly damped quasi-normal mode frequencies.

The area spectrum was found by using the real part of the quasi-normal mode frequency of the Schwarzschild black hole. Through combining the proposal of Bekenstein, Kunstatter [74] again conformed the result in 2002 that the black hole horizon area is adiabatic invariant with the proposal of Hod that the quasi-normal mode frequency is responsible for the area spectrum. A lot of works related to the area spectrum and entropy spectrum of black holes spacetime have been examined so far based on Hod's ideas, Maggiore's suggestion, and Kunstatter's method [75, 76, 77, 78, 79, 80, 81]. In Einstein theory of gravity, the area spectrum and the entropy spectrum are equidistant in nature. It is also evident in modified gravity theory that the area spectrum is not equidistant, but the entropy spectrum is equidistant [82]. When huge energy density is intensified in a small region in space, it naturally collapses to form a black hole. Therefore, in the high energy regime, an elementary particle metamorphoses into a black hole though a sharp disagreement between black holes and elementary particles. For instance, the density of states in gravity, which is deduced by the Bekenstein-Hawking entropy formula, is different from the density of states in any renormalizable quantum field theory [83].

One proposed technique in passing to classical mechanics from quantum mechanics is the decoherence of quantum mechanics induced by fluctuations [84, 85, 86]. Decoherence was first introduced in 1970 by H. Dieter Zeh and has been a subject of active research since the 1980s [87]. Decoherence occurs when a system interacts with its environment in a thermodynamically irreversible way. Respect to this point, the general relativity with the finite transmission of interaction united to the initial condition of the universe e.g., finite size of its initial volume [88] can lead to a background of never ending vacuum fluctuations and hence to produce the quantum decoherence, breaking the quantum mechanics on large scale. Conversely, the quantum properties of vacuum on very small scale break the scale invariance of the classical approach (i.e., the large scale phenomena cannot reproduce themselves on a whatsoever small scale). If we apply this quantum property of vacuum [89] to the Planck-scale black hole, it instantly follows that its formation can be hindered by quantum effect leading to a limiting low value for the mass of a black hole. This fact is a positive physical phenomenon forbidding the formation of very dangerous microscopic black hole (with a mass comparable to elementary particles) leading in this way to a safe stable classical laboratory-scale world. To put in a different way, at the Planck scale, any quantum field theory decreases to a conformal field theory in the same spacetime dimension whereas the Bekenstein-Hawking entropy formula for black holes implies that gravity does to a conformal field theory in one less spacetime dimension [90]. This observation has ended in success of the AdS/CFT correspondence [91].

From beginning, quantization of black holes is an important issues for the researchers who deal with quantum gravity theory of physics [5, 92]. For the quantization of black holes or its gravity there has been no satisfactory solution yet. Since the radius is the only one parameter for the black holes having no charge and angular momentum, for an observer it is not possible to observe what happen inside it from outside. One should consider all the information are reserved on the surface which is known as event horizon of black hole. To solve this problem researchers have pursued the problem in two separate theories, string theory and loop quantum gravity. With the beginning of string theory [93, 94], as a candidate for quantum gravity and loop quantum gravity [95], a new window was opened to the problem of black hole radiation. For this reason, the nature of black hole radiation is such that quantum gravity effects cannot be neglected [96, 97]. According to all of the above incredible works, it is believed that a black hole is a quantum mechanical system and thus resembling any other quantum system. Its physical states can be illustrated by a wave function. We can utilize it as a starting point for testing different constructions of quantum gravity [98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108] because of its fundamental theoretical role in quantum general relativity.

Accordingly, more in-depth approaches have been emerged to solve the problem of quantum gravity such as string theory [93, 94], loop quantum gravity [109] and the theory of casual fermion system [110]. Highly speaking, the quantum gravity purposes only to describe the quantum behavior of the gravitational field and does not mean the unification of the fundamental interactions into a single mathematical structure. However, the propagation of the theory to the fundamental forces would be a direct consequence once the quantum and the classical general relativity were made compatible.

Since there are no definite approaches to calculate Hawking radiation. In this research, we have calculated Bekenstein-Hawking entropy from quantized energy of the test particle orbiting around different stable circular orbit like Bohr's atomic model. The study of the quantization of the motion of particles as well as electric charge around different spacetimes background have attracted considerable interest in the literature [111]. In 1931, it has been shown by Dirac that the existence of magnetic monopoles lead to quantization of electric charge [112]. Similar to the Dirac theory Zee [113] has also presented a new gravitational analog of Dirac quantization condition in 1985 which is known as the theory of gravitoelectromagnetism(GEM) [114, 115]. In this GEM analogy, the electric charge and the electric field of Maxwell electromagnetic theory play the role of the mass of the test particle and the gravitational acceleration, respectively. Also, the source of magnetic field is considered as the matter current density in accordance with the Biot-Savart law and is called the GSM magnetic field which is a divergenceless quantity everywhere. Again in GEM analogy, Zee [113] has considered the existence of a gravitipole following Dirac. Due to the hypothetical nature of the gravitipole one can splitting the upper bound of energy without the quantization effects on energy level splittings in atoms and molecules [113]. Also, by expanding the action of the test particle one can quantize the mass of the test particle.

In the present work, we have used the method of Simanek [116] to obtain the Hawking purely thermal emission rate as well as the Bekenstein-Hawking entropy of different nonrotating black hole spacetimes. We have quantized the energy from the angular momentum of a test particle orbiting in different energy states around black holes like Bohr's atomic model. It is noted for the spherically symmetric black hole that the canonical formulation can be developed to study quantization by proposing a foliation in the radial parameter because it is only a function of the Lagrangian coordinates. The structure of this thesis is as follows:

The second chapter presents the quantization of energy from Schwarzschild black hole. The new line element by taking the approximation of the event horizon is derived which is the totally new idea of the present research. We have shown that the different energy labels of black hole can be performed in the same way as that for the electron signal inside the atom like Bohr's quantum theory.

In the third chapter, we have investigated the Bekenstein-Hawking entropy of Schwarzschild-de Sitter (SdS) black hole [117]. Using energy quantization method and considering the Bohr's atomic model using effective potential due to Lagrangian and canonical momenta, we have shown that the change of entropy as well as purely thermal emission rate is dependent on quantum number and approach to zero for large quantum number for these black hole[117].

In the fourth chapter, for test orbiting particles, energy quantization method has been used to explore the Bekenstein-Hawking entropy of Schwarzschild Anti-de Sitter (SAdS) black hole. Using the similar view of chapter three we have shown the change of entropy for two nearby circular orbit around these black hole.

In the fifth chapter, we have investigated the Bekenstein-Hawking entropy of Reissner-Nordström (RN) black hole [118] by including charge parameter. We have developed the effective potential for radial motion of RN black hole. In view of the same assumption of SdS black hole [117], we have shown that the purely thermal emission rate is dependent on quantum number and approach to zero for large quantum number for the RN black hole.

In the sixth chapter, we have generalized our findings in chapter 3 (SdS black hole [117]) with charge parameter and introduce the Bekenstein-Hawking entropy of Reissner-Nordström-de Sitter (RNdS) black hole [118] by the energy quantization method. Like RN black hole here we have also shown that the change of entropy as well as purely thermal emission rate is dependent on quantum number and approach to zero for large quantum number for RNdS black hole and recovered the new result for Bekenstein-Hawking entropy of RNdS black hole.

In the seventh chapter, for test orbiting particles, energy quantization method has been used to explore the Bekenstein-Hawking entropy from Reissner-Nordström Anti-de Sitter (RNAdS) black hole. Applying the similar view of chapter six we have shown the purely thermal emission rate is depend on quantum number and approach to zero for very large quantum number for these black hole.

Finally, in chapter eight we have given a brief description of the results of our prime work from chapter two to chapter seven with a concluding note and direction for further study.

Chapter 2

Bekenstein-Hawking Entropy by Energy Quantization from Schwarzschild Black Hole

2.1 Introduction

Schwarzschild black hole is the simplest black hole in asymptotically de Sitter space. It is static and spherically symmetric black hole having only parameter M called the mass of the black hole and has no charge or angular momentum. The study of Schwarzschild black hole is interesting for various reasons [119, 120]. So many authors articulated the idea that the interior of a black hole can be considered as an anisotropic cosmological model [121]. Very recently, a good deal of work has been dedicated to the canonical formulation of the spherically symmetric gravity and to its quantization [122, 123, 124]. We have employed the method of Simanek [116] to investigate the entropy as well as Hawking radiation of Schwarzschild black hole.

We arrange this chapter as follows: In section 2.2, we describe the Lagrangian and canonical momenta of test particle for a near horizon approximation of Schwarzschild black hole spacetime. In section 2.3, we calculate the effective potential for radial motion using the Lagrangian and canonical momentum. In section 2.4, we discuss Hawking temperature and quantization of energy of Schwarzschild black hole. Finally, in section 2.5, we present our remarks.

2.2 Lagrangian and Canonical Momenta of Test Particle

In this section, we consider a test particle of constant mass m orbiting around a Schwarzschild black hole. The spacetime describing Schwarzschild black hole can be written as [125]

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu && (2.1)\\ &= -\left(1-\frac{2M}{r}\right)c^2dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2), && (2.2)\end{aligned}$$

where $r_s = 2M$ is the Schwarzschild radius for a black hole of mass M . Let us consider that M is much larger than the Planck mass and ensuring that r_s is much larger than the Compton radius, $r_c = \hbar/mc$. In this situation the quantum fluctuations of the black hole can be disregarded [126].
The Lagrangian of the test particle can be expressed in terms of the metric components $g_{\mu\nu}$ as follows

$$\pounds = -\frac{m}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\frac{m}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right). \qquad (2.3)$$

Using Eq. (2.1) we have from Eq. (2.3)

$$\pounds = -\frac{m}{2}\left(\frac{ds}{d\tau}\right)^2. \qquad (2.4)$$

Since we have $ds^2 = -c^2 d\tau^2$, therefore Eq. (2.4) gives

$$£ = \frac{m}{2}c^2. \tag{2.5}$$

Using Eqs. (2.5) and (2.4) into Eq. (2.3) we have

$$\begin{aligned} -\frac{m}{2}\left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) &= \frac{m}{2}c^2 \\ \text{or} \left(g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) &= -c^2. \end{aligned} \tag{2.6}$$

Multiplying both sides by m^2 and setting momenta $p^\mu = m\frac{dx^\mu}{d\tau}$, we get from Eq. (2.6)

$$g_{\mu\nu}p^\mu p^\nu + m^2c^2 = 0. \tag{2.7}$$

The corresponding conjugate equation can be given as

$$g^{\mu\nu}p_\mu p_\nu + m^2c^2 = 0. \tag{2.8}$$

For the purpose of radial motion of a geodesic, the above equation can be written in the following form as

$$g^{00}p_0^2 + g^{rr}p_r^2 + g^{\phi\phi}p_\phi^2 + g^{\theta\theta}p_\theta^2 + m^2c^2 = 0. \tag{2.9}$$

The Euler-Lagrange equation [127] can be written as the form

$$\frac{\partial}{\partial\tau}\left(\frac{\partial £}{\partial \dot{x}^\mu}\right) - \frac{\partial £}{\partial x^\mu} = 0, \tag{2.10}$$

where μ represents four components t, r, θ and ϕ. The t and ϕ components of Eq. (2.10) can be written respectively as

$$\frac{\partial}{\partial\tau}\left(\frac{\partial £}{\partial \dot{t}}\right) - \frac{\partial £}{\partial t} = 0, \tag{2.11}$$

and

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \pounds}{\partial \dot{\phi}}\right) - \frac{\partial \pounds}{\partial \phi} = 0. \tag{2.12}$$

Since the Lagrangian $\pounds$ is independent of t and ϕ, so that $\frac{\partial \pounds}{\partial t} = 0$ and $\frac{\partial \pounds}{\partial \phi} = 0$, but depends on $\dot{t}$ and $\dot{\phi}$ so that Eq. (2.3) with the help of Eq. (2.2) gives

$$\frac{\partial \pounds}{\partial \dot{t}} = -mc^2\left(1 - \frac{2M}{r}\right)\dot{t}, \tag{2.13}$$

and

$$\frac{\partial \pounds}{\partial \dot{\phi}} = -mr^2 \sin^2\theta \dot{\phi}. \tag{2.14}$$

Thus, the t component of the equation of motion, Eq. (2.11) can be written with the help of Eq. (2.13) as

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \pounds}{\partial \dot{t}}\right) = 0$$

$$\text{or} \qquad \frac{\partial}{\partial \tau}\left(-mc^2\left(1 - \frac{2M}{r}\right)\dot{t}\right) = 0. \tag{2.15}$$

Integrating Eq. (2.15) we get

$$-mc^2\left(1 - \frac{2M}{r}\right)\dot{t} = constant. \tag{2.16}$$

This constant can be taken as corresponding to particle energy E so that

$$mc^2\left(1 - \frac{2M}{r}\right)\dot{t} = E. \tag{2.17}$$

Similarly, the ϕ component, Eq. (2.12) can be written with the help of Eq. (2.14) as

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \pounds}{\partial \dot{\phi}}\right) = 0$$

$$\text{or} \qquad \frac{\partial}{\partial \tau}\left(-mr^2 \sin^2\theta \dot{\phi}\right) = 0. \tag{2.18}$$

Integrating Eq. (2.18) we get

$$-mr^2\sin^2\theta\dot{\phi} = constant. \tag{2.19}$$

This constant can be taken as the angular momentum L of the test particle so that

$$mr^2\sin^2\theta\dot{\phi} = L. \tag{2.20}$$

Therefore, the canonical momenta $p_\alpha = \frac{\partial \pounds}{\partial \dot{x}^\alpha}$ can be written with the help of Eqs. (2.2), (2.3), (2.17) and (2.20) of the following form

$$\begin{aligned} p_0 &= \frac{\partial \pounds}{\partial \dot{t}} = mc^2 g_{00}\dot{t} = mc^2\left(1 - \frac{2M}{r}\right)\dot{t} = E, \\ p_r &= \frac{\partial \pounds}{\partial \dot{r}} = -mg_{rr}\dot{r} = -m\left(1 - \frac{2M}{r}\right)^{-1}\dot{r}, \\ p_\theta &= \frac{\partial \pounds}{\partial \dot{\theta}} = -mg_{\theta\theta}\dot{\theta} = -mr^2\dot{\theta}, \\ p_\phi &= \frac{\partial \pounds}{\partial \dot{\phi}} = mg_{\phi\phi}\dot{\phi} = mr^2\sin^2\theta\dot{\phi} = L. \end{aligned} \tag{2.21}$$

2.3 Effective Potential for Radial Motion due to Schwarzschild Black Hole

For Schwarzschild black hole spacetime, Eq. (2.9) with the help of Eq. (2.21) can be written to the following new form

$$-\frac{E^2}{c^2}\left(1 - \frac{2M}{r}\right)^{-1} + m^2\left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + \frac{L^2}{r^2\sin^2\theta} + m^2r^2\dot{\theta}^2 + m^2c^2 = 0, \tag{2.22}$$

where the first four terms represents the magnitude of the energy-momentum four-vector ($p_0 = E/c, \mathbf{p}$).

We consider the particle moving along the equatorial plane $\theta = \frac{\pi}{2}$. Therefore, we have $\sin^2\theta = 1$ and $\dot{\theta} = 0$. Applying these conditions into Eq. (2.22), we get

$$-\frac{E^2}{c^2}\left(1-\frac{2M}{r}\right)^{-1} + m^2\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + \frac{L^2}{r^2} + m^2c^2 = 0. \quad (2.23)$$

According to Ref. [116], we introduce the energy and momentum of the test particle per unit rest mass as follows

$$\tilde{E} = \frac{E}{m}; \quad \tilde{L} = \frac{L}{m}. \quad (2.24)$$

Setting above relations into Eq. (2.23), we get

$$m^2c^2\left[\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{\dot{r}^2}{c^2} - \frac{\tilde{E}^2}{c^4}\right) + \frac{\tilde{L}^2}{c^2r^2} + 1\right] = 0, \quad (2.25)$$

which gives after few steps

$$\frac{\tilde{E}^2}{2c^2} = \frac{\dot{r}^2}{2} + \frac{1}{2}\left(\frac{\tilde{L}^2}{r^2} + c^2\right)\left(1-\frac{2M}{r}\right). \quad (2.26)$$

Equation (2.26) is energy equation for unit mass. The first and second terms of the right hand side are kinetic energy and potential energy respectively. Therefore, the effective potential V_{eff} for the radial motion can be defined from the above equation as

$$V_{eff} = \frac{1}{2}\left(\frac{\tilde{L}^2}{r^2} + c^2\right)\left(1-\frac{2M}{r}\right). \quad (2.27)$$

For the purpose of maximum potential we have $\frac{\partial V_{eff}}{\partial r} = 0$. Therefore, Eq. (2.27) respectively gives

$$\frac{\partial V_{eff}}{\partial r} = \frac{1}{2}\left[\frac{2M}{r^2}\left(\frac{\tilde{L}^2}{r^2} + c^2\right) + \left(1-\frac{2M}{r}\right)\left(\frac{-2\tilde{L}^2}{r^3}\right)\right] = 0$$

$$\text{or} \qquad \frac{M\tilde{L}^2}{r^4} + \frac{c^2 M}{r^2} - \frac{\tilde{L}^2}{r^3} + \frac{2\tilde{L}^2 M}{r^4} = 0$$

$$\text{or} \qquad \frac{3M\tilde{L}^2}{r^4} + \frac{c^2 M}{r^2} - \frac{\tilde{L}^2}{r^3} = 0$$

$$\text{or} \qquad \frac{c^2 M}{r^4}\left(r^2 - \frac{\tilde{L}^2}{c^2 M}r + \frac{3\tilde{L}^2}{c^2}\right) = 0. \tag{2.28}$$

The Eq. (2.28) is a quadratic equation of r. It has two roots describe the position of the circular orbit of the particle as follows

$$R_{\pm} = \frac{\tilde{L}^2}{2Mc^2} \pm \left[\left(\frac{\tilde{L}^2}{2Mc^2}\right)^2 - \frac{3\tilde{L}^2}{c^2}\right]^{\frac{1}{2}}. \tag{2.29}$$

Simplifying Eq. (2.29) we get

$$R_{\pm} = \frac{\tilde{L}^2}{2c^2 M}\left(1 \pm \left(1 - \frac{12c^2 M^2}{\tilde{L}^2}\right)^{\frac{1}{2}}\right). \tag{2.30}$$

We observe that the roots $R_{\pm}$ of the above Eq. (2.30) is real only when $\tilde{L}^2 \geq 12c^2M^2$ and for the smallest stable orbit the square root on the right hand side of Eq. (2.30) dissolves. Therefore, we must have

$$\tilde{L}^2 = 12c^2 M^2. \tag{2.31}$$

For large and largest stable circular orbits the conditions $\tilde{L}^2 \geq 12c^2M^2$ and $\tilde{L}^2 >> 12c^2M^2$ will be hold respectively.

2.4 Hawking Temperature and Energy Quantization

As discussed before, for the orbiting test particle the quantization of energy is closely related to the quantization of the angular momentum. Thus multiply the linear momentum $m\nu$ and the orbital radius r, we get the angular momentum in a circular orbit. Thus the Bohr's quantum circumstance is given by

$$m\nu r = n\hbar, \tag{2.32}$$

where $\hbar = h/2\pi$ and the integer value n represents the various stationary states which is called a quantum number.
Thus the quantum condition Eq. (2.32) is a special case of the condition [128]

$$\oint pdq = nh, (n = 0, 1, 2, 3, \cdots), \tag{2.33}$$

called Bohr-Sommerfeld quantum rule. Here q represent the generalized coordinate and p, the analogous canonical conjugate momentum. For periodic motion, the condition is applicable and the integration is to be performed over full period of the generalized co-ordinate. Eq. (2.32) corresponds to taking q to be angular position ϕ fluctuating from 0 to 2π for one period and the conjugate p is the angular momentum in the circular orbit, which is independent of ϕ, being a constant of motion. This can be shown as follows

$$\begin{aligned} &\oint pd\phi = nh, \\ \text{or} \quad &\oint m\nu r = nh, \\ \text{or} \quad &m\nu r = n\hbar. \end{aligned} \tag{2.34}$$

Bohr's two presumes composed with the generalized quantum law, Eq. (2.33) postulated by Sommerfeld [128] in 1916 constitute which is nowadays known as the old quantum theory or Bohr-Sommerfeld theory.

In the circumstance of hydrogen atom, Sommerfeld extended the Bohr model of hydrogenic atom to include the elliptical orbits in 1916.
The quantum condition Eq. (2.33) has actually two parts [128]

$$J_\phi = \oint p_\phi d\phi = nh, (n = 0, 1, 2, 3, \cdots), \tag{2.35}$$

and

$$J_r = \oint p_r dr = n_r h. (n_r = 0, 1, 2, 3, \cdots). \tag{2.36}$$

Thus angular momentum can be quantized as a periodic function of time following Wilson [129] and Sommerfeld [130] idea and which is involved in quantizing angular momentum of the orbiting test particle. In order to quantize angular momentum J_ϕ with the help of Eq. (2.35) and the canonical momentum L conjugate to the angular variable of the form

$$J_\phi = \int_0^{2\pi} L d\phi = nh. \tag{2.37}$$

While L is a constant of motion, Eq. (2.37) yields the quantization condition for the angular momentum

$$L = m\tilde{L} = n\hbar, \quad \text{so that} \quad \tilde{L}_0 = n_0 \hbar / m. \tag{2.38}$$

With the help of compton radius $r_c = \hbar/mc$ and Eq. (2.38), the Eq. (2.31) can be written as

$$\begin{aligned} & \frac{n_0^2 \hbar^2}{m^2} = 12 c^2 M^2 \\ \Rightarrow \quad & \frac{n_0^2 \hbar^2}{m^2 c^2} = 12 M^2 \\ \therefore \quad & n_0^2 r_c^2 = 12 M^2. \end{aligned} \tag{2.39}$$

By using Eq. (2.39) into Eq. (2.30), we get the radius of the different stable circular orbit of the particle corresponds to n_0 of the form

$$R_+ = \frac{n_0^2 r_c^2}{2M}\left(1 + \left[1 - \frac{12M^2}{r_c^2 n_0^2}\right]^{\frac{1}{2}}\right). \tag{2.40}$$

Inserting Eq. (2.39) into Eq. (2.40) we get the approximate radius of the first circular orbit R_0 of the form

$$R_0 \approx \frac{n_0^2 r_c^2}{2M}. \tag{2.41}$$

The position of the next higher circular orbit R_1 can be obtained from Eq. (2.40) by replacing n_0 with $n_1 = n_0 + 1$ of the form

$$R_1 = \frac{(n_0+1)^2 r_c^2}{2M}\left(1 + \left[1 - \frac{12M^2}{(n_0+1)^2 r_c^2}\right]^{\frac{1}{2}}\right). \tag{2.42}$$

Let us consider $2M >> r_c$ so that $n_0 >> 1$ and therefore we can write

$$(n_0+1)^2 = n_0^2 + 2n_0 + 1 = n_0^2\left[1 + \frac{2}{n_0} + \frac{1}{n_0^2}\right] \approx n_0^2\left[1 + \frac{2}{n_0}\right]. \tag{2.43}$$

Using Eq. (2.43) into Eq. (2.42) we get

$$R_1 = \frac{n_0^2(1+\frac{2}{n_0}) r_c^2}{2M}\left(1 + \left[1 - \frac{12M^2}{n_0^2(1+\frac{2}{n_0}) r_c^2}\right]^{\frac{1}{2}}\right). \tag{2.44}$$

By using the Eq. (2.39) into the right side of the above equation can be approximated of the form

$$\begin{aligned} 1 + \left[1 - \frac{12M^2}{n_0^2(1+\frac{2}{n_0}) r_c^2}\right]^{\frac{1}{2}} &\approx 1 + \left[1 - \frac{1}{(1+2/n_0)}\right]^{1/2} \\ &= 1 + \left[1 - (1+2/n_0)^{-1}\right]^{1/2} \\ &= 1 + \left[1 - \left(1 - \frac{2}{n_0} + \cdots\right)\right]^{1/2} \\ &= 1 + \sqrt{\frac{2}{n_0}}. \end{aligned} \tag{2.45}$$

Simplifying Eq. (2.44) with the help of Eqs. (2.41) and (2.45) as

$$R_1 = R_0 \left(1 + \frac{2}{n_0}\right)\left(1 + \sqrt{\frac{2}{n_0}}\right). \tag{2.46}$$

Thus the next higher stage n_2^2 defined as

$$\begin{aligned} n_2^2 = (n_0+2)^2 = n_0^2 + 4n_0 + 4 \\ = n_0^2\left[1 + \frac{4}{n_0} + \frac{4}{n_0^2}\right] \approx n_0^2\left[1 + \frac{4}{n_0}\right]. \end{aligned} \tag{2.47}$$

Progressing in this way, the radius of the next higher circular orbit R_2 of the test particle with the help of Eq. (2.47) can be obtained from Eq. (2.40) in the form of

$$\begin{aligned} R_2 &= \frac{n_0^2(1+\frac{4}{n_0})r_c^2}{2M}\left(1 + \left[1 - \frac{12M^2}{n_0^2(1+\frac{4}{n_0})r_c^2}\right]^{\frac{1}{2}}\right), \\ &= R_0\left(1 + \frac{4}{n_0}\right)\left(1 + \sqrt{\frac{4}{n_0}}\right) = R_1 \frac{\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right)}{\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right)}. \end{aligned} \tag{2.48}$$

Proceeding in the similar way the $(n+1)$th radius of the stable circular orbits of the particle can be written as

$$R_{n+1} = R_n \frac{\left(1+\frac{2n+2}{n_0}\right)\left(1+\sqrt{\frac{2n+2}{n_0}}\right)}{\left(1+\frac{2n}{n_0}\right)\left(1+\sqrt{\frac{2n}{n_0}}\right)}. \tag{2.49}$$

When $n_0 \to \infty$, we have $R_{n+1} = R_n$. Therefore, for large quantum number we can say that the two nearby states coincide. The true energy $E = m\tilde{E}$ is obtained from Eq. (2.26), yielding

$$\tilde{E}^2 = c^2\left(\frac{\tilde{L}^2}{r^2} + c^2\right)\left(1 - \frac{2M}{r}\right). \tag{2.50}$$

From Eq. (2.28) we obtain for $r = R$

$$
\begin{aligned}
& R^2 - \frac{\tilde{L}^2}{c^2 M} R + \frac{3\tilde{L}^2}{c^2} = 0 \\
\text{or} \qquad & 1 - \frac{\tilde{L}^2}{c^2 M R} + \frac{3\tilde{L}^2}{c^2 R^2} = 0 \\
\text{or} \qquad & \frac{\tilde{L}^2}{c^2 R^2} \left(\frac{R}{M} - 3 \right) = 1 \\
& \therefore \frac{\tilde{L}^2}{R^2} = \frac{c^2}{\left(\frac{R}{M} - 3\right)} \qquad (2.51)
\end{aligned}
$$

Substituting Eq. (2.51) into Eq. (2.50), we get the true energy as a function of R as

$$
\begin{aligned}
\frac{E^2}{m^2} &= \tilde{E}^2 = c^2 \left(1 - \frac{2M}{R}\right) \left(\frac{c^2}{(\frac{R}{M} - 3)} + c^2 \right) \\
\text{or} \qquad E^2 &= m^2 c^4 \left(1 - \frac{2M}{R}\right) \left(\frac{1}{\left(\frac{R-3M}{M}\right)} + 1 \right) \\
&= m^2 c^4 \left(1 - \frac{2M}{R}\right) \left(\frac{M + R - 3M}{R - 3M} \right) \\
&= m^2 c^4 \left(1 - \frac{2M}{R}\right) \left(\frac{R - 2M}{R - 3M} \right) \\
&= m^2 c^4 \left(1 - \frac{2M}{R}\right) \frac{\left(1 - \frac{2M}{R}\right)}{\left(1 - \frac{3M}{R}\right)} \\
\therefore \quad E &= mc^2 \frac{\left(1 - \frac{2M}{R}\right)}{\left(1 - \frac{3M}{R}\right)^{1/2}}. \qquad (2.52)
\end{aligned}
$$

Now, we consider large orbits such that $R >> 2M$. From Eq. (2.30), we have

$$
R_+ = \frac{\tilde{L}^2}{2c^2 M} \left(1 + \left(1 - \frac{12 c^2 M^2}{\tilde{L}^2} \right)^{\frac{1}{2}} \right). \qquad (2.53)
$$

Using the condition of mechanical stability, $\tilde{L}^2 = MGR$, the second term in the parenthesis can be written as

$$\begin{aligned}\frac{12c^2M^2}{\tilde{L}^2} &= 3c^2\frac{2GM}{c^2}\frac{2M}{MGR} \\ &= \frac{12M}{R} << 1. \end{aligned} \tag{2.54}$$

Neglecting higher terms and therefore, Eq. (2.52) can be written as

$$\begin{aligned} E = mc^2\frac{\left(1-\frac{2M}{R}\right)}{\left(1-\frac{3M}{R}\right)^{1/2}} &= mc^2\left(1-\frac{2M}{R}\right)\left(1-\frac{3M}{R}\right)^{-1/2} \\ \therefore \quad E &\approx mc^2\left(1-\frac{2M}{R}\right)\left(1+\frac{3M}{2R}\right) \\ &\approx mc^2\left(1-\frac{2M}{R}+\frac{3M}{2R}\right) \\ \therefore \quad E &\approx mc^2\left(1-\frac{M}{2R}\right). \end{aligned} \tag{2.55}$$

Using Eq. (2.38) into Eq. (2.41) we have

$$2R \approx \frac{n^2r_c^2}{M} = \frac{n^2\hbar^2}{m^2c^2M} = \frac{\tilde{L}^2}{c^2M}. \tag{2.56}$$

Introducing the radius R from Eq. (2.56), Eq. (2.55) yields

$$E \approx mc^2\left(1-\frac{c^2M^2}{\tilde{L}^2}\right). \tag{2.57}$$

The quantized energy E_n follows from Eq. (2.57) by replacing $\tilde{L}$ by quantized angular momentum $\tilde{L}_n = \frac{n\hbar}{m}$ and using $\hbar = r_c mc$. Proceeding in this way, we obtain

$$\begin{aligned} E_n &\approx mc^2\left(1-\frac{c^2M^2}{\tilde{L}_n^2}\right) \\ &\approx mc^2\left(1-\frac{c^2M^2}{n^2c^2r_c^2}\right) \\ \therefore \quad E_n &\approx mc^2\left(1-\frac{M^2}{n^2r_c^2}\right). \end{aligned} \tag{2.58}$$

The $(n+1)th$ position of quantized energy E_{n+1} defined as

$$\therefore \quad E_{n+1} \approx mc^2\left[1-\frac{M^2}{r_c^2(n+1)^2}\right]. \tag{2.59}$$

The dependence of E_n on the quantum number n is reminiscent of Bohr's quantization of the total energy of an atomic electron as expected for a Kepler problem. Using this result, we obtain the energy splitting

$$\therefore \quad \delta E_n = E_{n+1}-E_n \approx mc^2\frac{M^2}{r_c^2}\left[\frac{1}{n^2}-\frac{1}{(n+1)^2}\right]. \tag{2.60}$$

For Schwarzschild black hole, the Eq. (2.60) can be written as

$$\delta E \approx \frac{mc^2r_S^2}{4r_c^2}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right). \tag{2.61}$$

For large values of n, the first parenthesis of Eq. (2.61) can be replaced by $2/n^3$ so that

$$\delta E \approx \frac{c^4m^3r_S^2}{2\hbar^2n^3}. \tag{2.62}$$

When $n \to \infty$ Eq. (2.62) gives $\delta E \to 0$. Which indicate that for a larger circular orbit the change of quantized energy between two nearby states approaches to zero.

According to the first law of thermodynamics and taken temperature we get $dE = TdS$. Thus, the Bekenstein-Hawking Entropy for Schwarzschild black hole with the help of Eq. (2.61) in the form of

$$\delta S = \frac{\delta E}{T_H} = \frac{mc^2 r_S^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right], \tag{2.63}$$

where T_H is the Hawking temperature of Schwarzschild black hole.

According to the WKB approximation, the emission rate for an out going particle with positive energy E coming from inside to outside which related to the imaginary part of the particles action as

$$\Gamma \sim exp(-2Im(I)), \tag{2.64}$$

where $Im(I) = -\frac{1}{2}[S(M-E) - S(M)] = \frac{1}{2}\delta S$. For the Schwarzschild black hole the thermal emission rate and the change of entropy satisfies

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2 r_S^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]\right). \tag{2.65}$$

Therefore the above equation can be written in another form as

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\right). \tag{2.66}$$

The Hawking temperature for Schwarzschild black hole can be written as

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_S} = \frac{1}{8\pi M}, \tag{2.67}$$

where $\kappa = \frac{1}{2r_S}$ is the surface gravity of Schwarzschild black hole.

2.5 Concluding Remarks

Due to evaporation process Schwarzschild black hole loses its mass so that $M \to 0$. In this situation Eq. (2.67) gives $T_H \to \infty$ so that Eq. (2.66) yields entropy $S \to 0$ & thermal emission rate $\Gamma \to 0$. Those results agree with the results obtained by Sakalli et al. [131]. On the other hand, when $n \to \infty$, the Eq. (2.66) again gives entropy $S \to 0$ & thermal emission rate $\Gamma \to 0$. That is, for large circular orbit the change of entropy between two nearby states vanishes.

Our present work shows that the different energy labels of black hole can be performed in the same way as that for the electron signal inside the atom like Bohr's quantum theory which agrees with the result obtain by He et al.[132] for quantization of black holes.

Chapter 3

Bekenstein-Hawking Entropy by Energy Quantization from Schwarzschild-de Sitter Black Hole

3.1 Introduction

The Schwarzschild-de Sitter (SdS) black hole is the simplest black hole in de Sitter space. It is generalized of Schwarzschild black hole with a positive cosmological constant. The study of entropy and Hawking radiation of black holes in de Sitter space with a positive cosmological constant is important for two reasons. First, the recently observed accelerating expansion of our universe indicates the cosmological constant might be a positive one [133, 134, 135], and conjecture about de Sitter/CFT correspondence [136, 137]. All black holes in de Sitter space have two horizons- the black hole event horizon and the cosmological horizon. The de Sitter spacetime cannot be in thermodynamic equilibrium due to different temperatures for two different horizons. The black hole horizon and cosmological horizon are treated as two independent thermodynamic systems. We can study black hole horizon by separating the cosmological horizon by a thermally opaque membrane or box [138, 139, 140, 141, 142] and can be considered as two independent thermodynamic systems and have their respective first law of thermodynamics. For this reason the present study on SdS black hole is important and meaningful.

The chapter is described as follows: Section 3.2 is devoted to calculate the Lagrangian and canonical momenta of a test particle of the schwarzschild-de Sitter line element near the event horizon according to Simanek [116]. Effective potential for radial motion has been calculated in section 3.3. In section 3.4, we have quantized the circular orbit of the test particle moving around the black hole due to the fact that the azimuthal angle is a periodic function of time. The Hawking purely thermal emission rate has been derived through quantizing the energy of the SdS black hole gravity in section 3.5. Finally, we have finished with the concluding remarks in Section 3.6.

3.2 Lagrangian and Canonical Momenta of Test Particle

In this section we have used the canonical formulation [122, 123, 124, 143] to quantize the SdS black hole gravity. Cavagli'a et. al.[124, 143] have developed a canonical formulation deal with the spherically symmetric spacetime to quantize the Schwarzschild black hole by imposing a restriction in the radial parameter r due to the fact that the Lagrangian coordinates is only the functions of r. The solution of Einstein equations with a positive $\Lambda(= 3/\ell^2)$ term of Schwarzschild-de Sitter black hole can be configured as

[58]

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{r^2}{\ell^2}\right) c^2 dt^2 + \left(1 - \frac{2M}{r} - \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.1}$$

where M being the mass of the black hole, the coordinates are defined such that $-\infty \leq t \leq \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. Note that this spacetime is not asymptotically flat. At large r, the metric (3.1) tends to the dS space limit. The SdS black hole horizons are described by the real positive roots of $\frac{1}{\ell^2 r}(r - r_{sds})(r - r_c)(r_- - r) = 0$. From which we get [58]

$$r_{sds} = 2M\left(1 + \frac{4M^2}{\ell^2} + \cdots\right). \tag{3.2}$$

The terms in the bracket is greater than one and indicate that SdS black hole radius is larger than Schwarzschild Black hole ($r_s = 2M$). For simplicity we can rewrite the Eq. (3.1) in the following form

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 + \frac{r^3}{2M\ell^2}\right)\right) c^2 dt^2 + \left(1 - \frac{2M}{r}\left(1 + \frac{r^3}{2M\ell^2}\right)\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.3}$$

If we take the first approximation $r_0 = 2M$ then the above metric (3.3) can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right) c^2 dt^2 + \left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.4}$$

Now, we consider a test particle of mass m orbiting along the circular geodesics in the equatorial plane around SdS black hole. Then according to the Refs. [5, 144] the metric given in Eq. (3.4) can be written in the following form

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)c^2dt^2 + \left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)^{-1}dr^2 + r^2d\phi^2. \tag{3.5}$$

We assume that the black hole mass M is larger than the Planck mass so that the Compton radius, $r_c = \hbar/mc$ is the smallest than the SdS black hole r_{sds}. In this situation, the quantum fluctuations of the black hole is disregarded [126]. One can define the Lagrangian of the test particle in terms of the metric components g_{ij} as

$$\begin{aligned} \pounds &= -\frac{m}{2}g_{ij}\dot{x}^i\dot{x}^j \\ &= -\frac{m}{2}\left[\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)c^2\dot{t}^2\right] \\ &-\frac{m}{2}\left[\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + \mathrm{r}^2\dot{\theta}^2\right]. \end{aligned} \tag{3.6}$$

Since the SdS spacetime is static and spherically symmetric, there exist two constants of motion for the test particles, associated with two Killing vectors as

$$E = \frac{\partial \pounds}{\partial \dot{t}} = mc^2\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)\dot{t}, \tag{3.7}$$

$$L = \frac{\partial \pounds}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}, \tag{3.8}$$

where E and L denote corresponding specific energy and the specific angular momentum of the orbiting test particle respectively. With the help of the canonical momenta defined by $p_\alpha = \partial \pounds / \partial \dot{x}^\alpha$ the other two components can be written as

$$p_r = \frac{\partial \pounds}{\partial \dot{r}} = -m\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r}, \tag{3.9}$$

$$p_\theta = \frac{\partial \pounds}{\partial \dot{\theta}} = -mr^2\dot{\theta}. \tag{3.10}$$

3.3 Radial Motion and Effective Potential

The radial motion of a geodesic can be written as

$$g^{00}p_0^2 + g^{rr}p_r^2 + g^{\phi\phi}p_\phi^2 + g^{\theta\theta}p_\theta^2 + m^2c^2 = 0, \tag{3.11}$$

the four-vector ($p_0 = E/c, \mathbf{p}$) describe the magnitude of the energy-momentum. Inserting Eqs. (3.7) to (3.10) we have [116]

$$-\frac{E^2}{c^2}\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)^{-1} + m^2\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r}^2 + \frac{L^2}{r^2\sin^2\theta} + \frac{m^2}{r^2}\dot{\theta}^2 + m^2c^2 = 0. \tag{3.12}$$

When $\dot{\theta}^2 = 0$, and $\sin^2\theta = 1$ then the above Eq. (3.12) becomes

$$-\frac{E^2}{c^2}\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)^{-1} + m^2\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r}^2 + \frac{L^2}{r^2} + m^2c^2 = 0. \tag{3.13}$$

We introduce the energy and momentum of the test particle per unit rest mass as given in Ref. [116] in the form of

$$\tilde{E} = \frac{E}{m}, \quad \tilde{L} = \frac{L}{m} \tag{3.14}$$

and using this into Eq. (3.13) we get

$$m^2c^2\left[-\frac{\tilde{E}^2}{c^4}\left(1-\frac{2M}{r}\left(1+\frac{4M^2}{\ell^2}\right)\right)^{-1}\right]$$
$$+m^2c^2\left[\frac{1}{c^2}\left(1-\frac{2M}{r}\left(1+\frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r}^2+\frac{\tilde{L}^2}{c^2r^2}+1\right]=0. \quad (3.15)$$

The radial motion of the test particle can be obtained from the above equation in the form of

$$\frac{\dot{r}^2}{2}=\frac{\tilde{E}^2}{2c^2}-\frac{1}{2}\left(\frac{\tilde{L}^2}{r^2}+c^2\right)\left(1-\frac{2M}{r}\left(1+\frac{4M^2}{\ell^2}\right)\right). \quad (3.16)$$

For the time like particle orbit here the velocity is described by two parameters- the energy and the angular momentum. Note that here $\dot{r}$ is in different form and as such it is called the effective potential V_{eff}. Therefore, the effective potential for the radial motion can be written as

$$V_{eff}=\frac{1}{2}\left(\frac{\tilde{L}^2}{r^2}+c^2\right)\left(1-\frac{2M}{r}\left(1+\frac{4M^2}{\ell^2}\right)\right). \quad (3.17)$$

The radial acceleration of the test particle can be obtained by taking the derivative of Eq. (3.17) with respect to the proper time and by taking $\frac{\partial V_{eff}}{\partial r}=0$ the maximum potential can be obtained as

$$\frac{c^2}{r^4}M\left(1+\frac{4M^2}{\ell^2}\right)(r^2-\frac{\tilde{L}^2}{c^2M\left(1+\frac{4M^2}{\ell^2}\right)}r+\frac{3\tilde{L}^2}{c^2})=0. \quad (3.18)$$

Equation (3.18) is a quadratic equation and gives the two roots in the form of

$$R_{\pm}=\frac{\tilde{L}^2}{2c^2M\left(1+\frac{4M^2}{\ell^2}\right)}\pm\left[\left(\frac{\tilde{L}^2}{2c^2M\left(1+\frac{4M^2}{\ell^2}\right)}\right)^2-\frac{3\tilde{L}^2}{c^2}\right]^{\frac{1}{2}}, \quad (3.19)$$

where R is the radius of the circular orbit. We can write the Eq. (3.19) in an alternative form as

$$R_{\pm} = \frac{\tilde{L}^2}{2c^2M\left(1+\frac{4M^2}{\ell^2}\right)}\left(1 \pm \left(1 - \frac{12c^2M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{\tilde{L}^2}\right)^{\frac{1}{2}}\right). \quad (3.20)$$

It is clear from the Eq. (3.20) that $R_{\pm}$ is real only when $\tilde{L}^2 \geq 12c^2M^2\left(1+\frac{4M^2}{\ell^2}\right)^2$. For the smallest stable orbit, the square root on the right hand side of Eq. (3.20) vanishes, hence we have $\tilde{L}^2 = 12c^2M^2\left(1+\frac{4M^2}{\ell^2}\right)^2$. We also correspond to the conditions $\tilde{L}^2 \geq 12c^2M^2\left(1+\frac{4M^2}{\ell^2}\right)^2$ and $\tilde{L}^2 >> 12c^2M^2\left(1+\frac{4M^2}{\ell^2}\right)^2$ for large and largest stable circular orbits respectively.

3.4 Quantization of Circular Orbit

According to the Wilson [129] and Sommerfeld [130] opinion angular momentum can be quantized as a periodic function of time and help quantize energy because it is closely related to the quantize angular momentum of the orbiting test particle. For the periodic motion we can quantized angular momentum J_ϕ with the help of canonical momentum L conjugate to the angular variable in the form of

$$J_\phi = \int_0^{2\pi} L d\phi = nh. \quad (3.21)$$

Since L is a constant of motion, Eq. (3.14) gives the quantization condition for the angular momentum in the form of

$$L = m\tilde{L} = n\hbar, \quad \text{so that} \quad \tilde{L}_0 = n_0/m\hbar. \quad (3.22)$$

As mentioned in the previous section, we have for smallest circular orbit

$$\tilde{L}^2 = 12c^2M^2\left(1+\frac{4M^2}{\ell^2}\right)^2, \tag{3.23}$$

which can be written with the help of Eq. (3.22) in terms of n_0 as

$$n_0^2 = \frac{12M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{r_c^2}, \tag{3.24}$$

where r_c is the Compton radius of the test particle defined by $\frac{\hbar}{mc}$. The radius of the different stable circular orbits of the particle correspond to n_0 can be obtained by using Eq. (3.22) into Eq. (3.20) in the form of

$$R_\pm = n_0^2\frac{r_c^2}{2M\left(1+\frac{4M^2}{\ell^2}\right)}\left(1+\left[1-\frac{12M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{r_c^2n_0^2}\right]^{\frac{1}{2}}\right). \tag{3.25}$$

Using Eq. (3.24) into Eq. (3.25), the radius of the first circular orbit denoted by R_0 can be approximated as

$$R_0 \approx n_0^2\frac{r_c^2}{2M\left(1+\frac{4M^2}{\ell^2}\right)}, \tag{3.26}$$

in the limiting case that is when $\ell \to \infty$ this becomes $R_0 = n_0^2\frac{r_c^2}{r_s}$ and agree with the result given in Ref. [116], where $r_s = 2M$ is the Schwarzschild radius. The position of the next higher circular orbit R_1 can be obtained from Eq. (3.25) by replacing n_0 with $n_1 = n_0 + 1$ in the form of

$$R_1 = (n_0+1)^2\frac{r_c^2}{2M\left(1+\frac{4M^2}{\ell^2}\right)}\left(1+\left[1-\frac{12M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{r_c^2(n_0+1)^2}\right]^{\frac{1}{2}}\right). \tag{3.27}$$

For simplicity, we consider $2M\left(1+\frac{4M^2}{\ell^2}\right) >> r_c$ so that $n_0 >> 1$. We therefore can write $(n_0+1)^2 \approx n_0^2[1+\frac{2}{n_0}]$ and using this into Eq. (3.27) we get

$$R_1 = n_0^2(1+\frac{2}{n_0})\frac{r_c^2}{2M\left(1+\frac{4M^2}{\ell^2}\right)}\left(1+\left[1-\frac{12M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{r_c^2 n_0^2(1+\frac{2}{n_0})^2}\right]^{\frac{1}{2}}\right). \quad (3.28)$$

The term in the first bracket can be approximated with the help of Eq. (3.24) in the form of

$$1+\left[1-\frac{12M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{r_c^2 n_0^2(1+\frac{2}{n_0})^2}\right]^{\frac{1}{2}} \approx 1+\left[1-\frac{1}{(1+2/n_0)^2}\right]^{1/2}$$
$$= 1+\sqrt{\frac{2}{n_0}}. \quad (3.29)$$

Therefore, Eq. (3.28) reduced to

$$R_1 = R_0\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right). \quad (3.30)$$

In the similar fashion, the radius of the next higher circular orbit R_2 of the particle can be calculated from Eq. (3.25) in the form of

$$R_2 = R_0\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right) = R_1\frac{\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right)}{\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right)}. \quad (3.31)$$

Finally, in general form, the radius of the stable circular orbits of the particle can be written as

$$R_{n+1} = R_n\frac{\left(1+\frac{2n+2}{n_0}\right)\left(1+\sqrt{\frac{2n+2}{n_0}}\right)}{\left(1+\frac{2n}{n_0}\right)\left(1+\sqrt{\frac{2n}{n_0}}\right)}, \quad (3.32)$$

which means that gravity of the SdS black holes can be quantized in discrete states with radius R_{n+1}. When $n \to \infty$ we have $R_{n+1} = R_n$. Therefore, we may conclude that for large quantum number two nearby states coincide.

3.5 Energy Quantization and Hawking Radiation

In this section, we have quantized the energy of the orbiting test particle with the help of angular momentum because the quantization of energy is closely related to the quantization of the angular momentum. Equation (3.16) gives the energy for zero velocity at $r = R$

$$\tilde{E}^2 = c^2 \left(\frac{\tilde{L}^2}{R^2} + c^2 \right) \left(1 - \frac{2M\left(1 + \frac{4M^2}{\ell^2}\right)}{R} \right). \tag{3.33}$$

But we have from Eq. (3.18) at $r = R$

$$\frac{\tilde{L}^2}{R^2} = \frac{c^2}{\left(\frac{2R}{2M\left(1+\frac{4M^2}{\ell^2}\right)} - 3 \right)}. \tag{3.34}$$

Therefore, Eq. (3.33) can be rewritten as

$$\begin{aligned} \frac{E^2}{m^2} = \tilde{E}^2 &= c^2 \left(1 - \frac{2M\left(1 + \frac{4M^2}{\ell^2}\right)}{R} \right) \left(\frac{c^2}{\left(\frac{R}{M\left(1+\frac{4M^2}{\ell^2}\right)} - 3 \right)} + c^2 \right), \\ &= m^2 c^4 \left(1 - \frac{2M\left(1 + \frac{4M^2}{\ell^2}\right)}{R} \right) \frac{\left(1 - \frac{2M\left(1+\frac{4M^2}{\ell^2}\right)}{R} \right)}{\left(1 - \frac{3M\left(1+\frac{4M^2}{\ell^2}\right)}{R} \right)}, \end{aligned} \tag{3.35}$$

and after some calculations gives

$$E = mc^2 \frac{\left(1 - \frac{2M\left(1+\frac{4M^2}{\ell^2}\right)}{R}\right)}{\left(1 - \frac{3M\left(1+\frac{4M^2}{\ell^2}\right)}{R}\right)^{1/2}}. \tag{3.36}$$

To simplify the above Eq. (3.36), we recall the Eq. (3.20) and setting the mechanical stability condition $\tilde{L}^2 = MGR$. The second term in the parenthesis of the Eq. (3.20) can be written as

$$\begin{aligned} \frac{12c^2M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{\tilde{L}^2} &= 3c^2\frac{2GM}{c^2}\frac{2M\left(1+\frac{4M^2}{\ell^2}\right)}{MGR} \\ &= \frac{12M\left(1+\frac{4M^2}{\ell^2}\right)}{R}, \end{aligned} \tag{3.37}$$

which gives with the help of Eq. (3.23) for the circular orbit corresponding to $n_0 >> 1$

$$\frac{12M\left(1+\frac{4M^2}{\ell^2}\right)}{R} << 1. \tag{3.38}$$

Therefore, Eq. (3.36) can be approximated to

$$E \approx mc^2\left(1 - \frac{M\left(1+\frac{4M^2}{\ell^2}\right)}{2R}\right). \tag{3.39}$$

Substituting Eq. (3.22) into Eq. (3.39) we have

$$E \approx mc^2\left(1 - \frac{c^2M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{\tilde{L}^2}\right). \tag{3.40}$$

The quantized energy E_n for n energy label can be obtained as

$$E_n \approx mc^2 \left(1 - \frac{4c^2M^2\left(1+\frac{M^2}{\ell^2}\right)^2}{\tilde{L}_n^2}\right), \tag{3.41}$$

which with the help of Eqs. (3.23) and (3.24) becomes

$$E_n \approx mc^2 \left(1 - \frac{M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{n^2 r_c^2}\right). \tag{3.42}$$

The energy corresponding to $(n+1)th$ label can be written from Eq. (3.42) as

$$E_{n+1} \approx mc^2 \left[1 - \frac{M^2\left(1+\frac{4M^2}{\ell^2}\right)^2}{r_c^2(n+1)^2}\right]. \tag{3.43}$$

Thus, from the quantized energy formula we get the energy difference between two nearby states in the form of

$$\begin{aligned}\delta E &= E_{n+1} - E_n \\ &\approx \frac{mc^2}{r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times M^2\left(1+\frac{4M^2}{\ell^2}\right)^2. \end{aligned} \tag{3.44}$$

If we neglect the terms corresponding to 4th and the higher power of (M/ℓ) of the SdS black hole radius given in Eq. (3.2) then we have

$$r_{sds} \approx 2M\left(1+\frac{4M^2}{\ell^2}\right). \tag{3.45}$$

Therefore, the Eq. (3.44) can be written for SdS black hole as

$$\delta E = \frac{mc^2 r_{sds}^2}{4r_c^2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right). \tag{3.46}$$

For large values of n, the bracket can be replaced by $2/n^3$ so that

$$\delta E \approx \frac{c^4 m^3 r_{sds}^2}{2\hbar^2 n^3}. \tag{3.47}$$

It is clear that δE decreases with the increase of n. When $n \to \infty$ we have $\delta E \to 0$. Therefore, for a larger circular orbit the change of energy between two nearby states approaches to zero.

In classical thermodynamics one can define the change in the entropy for a given temperature as $dS(E) = dE/T(E)$. Therefore, the Bekenstein-Hawking Entropy of SdS black hole can be written with the help of Eq. (3.46) in the form of

$$\begin{aligned} \delta S &= \frac{\delta E}{T_H} \\ &= \frac{mc^2 r_{SdS}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \\ &= \frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 + \frac{4M^2}{\ell^2}\right)^2, \end{aligned} \tag{3.48}$$

where T_H is the Hawking temperature of SdS black hole. According to the WKB approximation, the emission rate for an outgoing particle with positive energy E coming from inside $R_{in}(M)$ to outside $R_{out}(M-E)$ of a circular orbit can be related to the imaginary part of the particles action $Im(I)$ as

$$\Gamma \sim exp(-2Im(I)), \tag{3.49}$$

where $Im(I) = -\frac{1}{2}[S(M-E) - S(M)] = \frac{1}{2}\delta S$. Therefore, for the SdS black hole the thermal emission rate and the entropy change satisfies

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2 r_{SdS}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]\right). \tag{3.50}$$

Inserting the value of r_{SdS} into the Eq. (3.50) we have

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 + \frac{4M^2}{\ell^2}\right)^2\right). \tag{3.51}$$

The Hawking temperature for SdS black hole is defined as

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M\left(1 + \frac{4M^2}{\ell^2}\right)}, \tag{3.52}$$

where $\kappa = \frac{1}{4M\left(1+\frac{4M^2}{\ell^2}\right)}$ is the surface gravity of SdS black hole. When the cosmological parameter $\ell \to \infty$, the Eq. (3.52) reduces to

$$T_H = \frac{1}{8\pi M}. \tag{3.53}$$

Which agree with the results of Schwarzschild black hole.

3.6 Concluding Remarks

When the evaporation process take place SdS black hole continuously loses its mass M so that the tunneling rate and entropy changes. At a stage its mass goes to near zero and the evaporation process will further stop. In this situation, Eq. (3.52) gives $T_H \to \infty$ so that Eq. (3.51) represents entropy $S \to 0$ & thermal emission rate $\Gamma \to 0$. Those results agree with the results obtained by Sakalli et al.[131]. In the other side, when n is very very large, Eq. (3.51) also gives the entropy $S \to 0$ & thermal emission rate $\Gamma \to 0$. Thus we may conclude that, for large circular orbit the change of entropy between two nearby states approaches to zero. Our obtaining results also agree with the results of He et al.[132] for quantization of black holes. When the cosmological constant $\ell \to \infty$ the result is similar to chapter 2 for Schwarzschild black hole.

Chapter 4

Bekenstein-Hawking Entropy by Energy Quantization from Schwarzschild Anti-de Sitter Black Hole

4.1 Introduction

The black hole solutions in Anti-de Sitter (AdS) spaces come from the Einstein's field equation with a negative cosmological constant. The difference consisting Anti-de Sitter black holes and de Sitter black holes is due to minimum temperature that occur when their sizes are in the order of the characteristic radius of the AdS space. The red-shifted temperature measured at infinity of larger AdS black holes are greater and can be in stable equilibrium with thermal radiation at a certain temperature.

Recently, thermodynamic criticality and phase transition of AdS black holes have been studied extensively. Since the Hawking-Page phase transition for Schwarzschild-AdS black hole was proposed, various phase structures for different black holes in AdS space have been found. Moreover, much attention has been paid to AdS/CFT correspondence similar in spirit to dS/CFT correspondence. So our study on the Schwarzschild Anti-de Sitter black holes is reasonable and meaningful.

The plan of this chapter is as follows: In Section 4.2 we investigate the Lagrangian and canonical momenta for SAdS black hole. We calculate the effective potential for radial motion of a Schwarzschild Anti-de Sitter solution in Section 4.3. We use these results in Section 4.4 to study the quantization of energy and Hawking temperature. In Section 4.5, we investigate the microcanonical ensemble.

4.2 Investigation of Lagrangian and Canonical Momenta for SAdS Black Hole

The Schwarzschild Anti-de Sitter black hole with mass M and a negative cosmological constant $\Lambda(= -3/\ell^2)$ can be configured as [58]

$$\begin{aligned} ds^2 = &-\left(1-\frac{2M}{r}+\frac{r^2}{\ell^2}\right)c^2dt^2+\left(1-\frac{2M}{r}+\frac{r^2}{\ell^2}\right)^{-1}dr^2 \\ &+r^2(d\theta^2+\sin^2\theta d\phi^2). \end{aligned} \tag{4.1}$$

The coordinates are defined such that $-\infty \leq t \leq \infty$, $r \geq 0$, $0 \leq \theta \leq \pi$, and $0 \leq \phi \leq 2\pi$. The black hole horizon for SAdS Black hole can be found from the cubic equation $r^3 + \ell^2 r - 2M\ell^2 = 0$ as [58]

$$r_{SAdS} = 2M\left(1-\frac{4M^2}{\ell^2}+\cdots\right). \tag{4.2}$$

Since $\left(1-\frac{4M^2}{\ell^2}+\cdots\right) < 1$, the radius of SAdS black hole is smaller than Schwarzschild black hole ($r_s = 2M$). For the simplicity we can modify the

Eq. (4.1) of the following form

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 - \frac{r^3}{2M\ell^2}\right)\right)c^2dt^2 + \left(1 - \frac{2M}{r}\left(1 - \frac{r^3}{2M\ell^2}\right)\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (4.3)$$

Let us consider a test particle of mass m orbiting along the circular geodesics in the equatorial plane around SAdS black hole. Then according to the Refs. [5, 144] and using $r_0 = 2M$ into the above metric (4.3) we have

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2}{\ell^2}\right)\right)c^2dt^2 + \left(1 - \frac{2M}{r}\left(1 - \frac{4M^2}{\ell^2}\right)\right)^{-1}dr^2 + r^2d\phi^2. \quad (4.4)$$

If we take the black hole mass is larger than the Planck mass so that the Compton radius, $r_c = \hbar/mc$ is the smallest than the radius of SAdS black hole r_{SAdS}. For this reason the quantum fluctuations of the black hole is disregarded [126]. The Lagrangian of the test particle in terms of the metric components g_{ij} is defined as

$$\pounds = -\frac{m}{2}g_{ij}\dot{x}^i\dot{x}^j = -\frac{m}{2}\left[\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2}{\ell^2}\right)\right)c^2\dot{t}^2\right] -\frac{m}{2}\left[\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + \mathrm{r}^2\dot{\theta}^2\right]. \quad (4.5)$$

While the SAdS spacetime is static and spherically symmetric, there exist two constants of motion for the test particles, associated with two Killing vectors in terms of energy E and angular momentum L and the other two components can be written with the help of the canonical momenta $p_\alpha = \frac{\partial \pounds}{\partial \dot{x}^\alpha}$ defined as

$$E = \frac{\partial \pounds}{\partial \dot{t}} = mc^2 \left(1 - \frac{2M}{r}\left(1 - \frac{4M^2}{\ell^2}\right)\right)\dot{t},$$
$$L = \frac{\partial \pounds}{\partial \dot{\phi}} = mr^2 \sin^2\theta\dot{\phi}. \tag{4.6}$$

$$p_r = \frac{\partial \pounds}{\partial \dot{r}} = -m\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r},$$
$$p_\theta = \frac{\partial \pounds}{\partial \dot{\theta}} = -mr^2\dot{\theta}. \tag{4.7}$$

4.3 Quantization of Circular Orbit due to Radial Motion and Effective Potential

The radial motion of a geodesic can be written as

$$g^{\alpha\alpha}p_\alpha^2 + m^2c^2 = 0, \tag{4.8}$$

where $\alpha = 0, r, \theta, \phi$, and $(p_0 = E/c, \mathbf{p})$ expresses the magnitude of the energy-momentum. Inserting Eqs. (4.6) to (4.7) and setting $\dot{\theta} = 0$, $\sin^2\theta = 1$ into the above Eq. (4.8) we have

$$-\frac{E^2}{c^2}\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2}{\ell^2}\right)\right)^{-1}$$
$$+m^2\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r}^2 + \frac{L^2}{r^2} + m^2c^2 = 0. \tag{4.9}$$

In Ref. [116], we get the energy and angular momentum of the test particle in the form of

$$\tilde{E} = \frac{E}{m}, \quad \tilde{L} = \frac{L}{m}. \tag{4.10}$$

Using the above relation into Eq. (4.9) we get

$$m^2c^2\left[-\frac{\tilde{E}^2}{c^4}\left(1-\frac{2M}{r}\left(1-\frac{4M^2}{\ell^2}\right)\right)^{-1}\right]$$
$$+m^2c^2\left[\frac{1}{c^2}\left(1-\frac{2M}{r}\left(1-\frac{4M^2}{\ell^2}\right)\right)^{-1}\dot{r}^2+\frac{\tilde{L}^2}{c^2r^2}+1\right]=0. \quad (4.11)$$

For the purpose of radial motion of the test particle the above Eq. (4.11) can be written as

$$\frac{\dot{r}^2}{2}=\frac{\tilde{E}^2}{2c^2}-\frac{1}{2}\left(\frac{\tilde{L}^2}{r^2}+c^2\right)\left(1-\frac{2M}{r}\left(1-\frac{4M^2}{\ell^2}\right)\right). \quad (4.12)$$

We can describe the velocity for the the time like particle by the two parameters energy E and angular momentum L. Thus the effective potential V_{eff} for the radial motion can be written as

$$V_{eff}=\frac{1}{2}\left(\frac{\tilde{L}^2}{r^2}+c^2\right)\left(1-\frac{2M}{r}\left(1-\frac{4M^2}{\ell^2}\right)\right). \quad (4.13)$$

For the purpose of maximum potential, taking the first derivative of Eq. (4.13) with respect to the proper time and then equating to zero, we obtain

$$\frac{c^2}{r^4}M\left(1-\frac{4M^2}{\ell^2}\right)(r^2-\frac{\tilde{L}^2}{c^2M\left(1-\frac{4M^2}{\ell^2}\right)}r+\frac{3\tilde{L}^2}{c^2})=0. \quad (4.14)$$

Solving equation (4.14) we get the two roots for $r=R$ in the form as

$$R_\pm=\frac{\tilde{L}^2}{2c^2M\left(1-\frac{4M^2}{\ell^2}\right)}\left(1\pm\left(1-\frac{12c^2M^2\left(1-\frac{4M^2}{\ell^2}\right)^2}{\tilde{L}^2}\right)^{\frac{1}{2}}\right), \quad (4.15)$$

where R is the radius of the circular orbit.

From the Eq. (4.15), we observe that $R_\pm$ is real only when

$\tilde{L}^2 \geq 12c^2M^2\left(1-\frac{4M^2}{\ell^2}\right)^2$ and for the smallest stable orbit, the square root on the right hand side of Eq. (4.15) vanishes. Therefore, we must have

$$\tilde{L}^2 = 12c^2M^2\left(1-\frac{4M^2}{\ell^2}\right)^2. \tag{4.16}$$

According to the conditions $\tilde{L}^2 \geq 12c^2M^2\left(1-\frac{4M^2}{\ell^2}\right)^2$ and $\tilde{L}^2 >> 12c^2M^2\left(1-\frac{4M^2}{\ell^2}\right)^2$ holds for large and the largest stable circular orbits respectively. Angular momentum can be quantized as a periodic function of time and help to quantize energy which idea developed by Wilson and Sommerfeld [129, 130] and those are closely related to the quantize angular momentum of the orbiting test particle. In order to quantize angular momentum J_ϕ with the help of canonical momentum L conjugate to the angular variable in the form of

$$J_\phi = \int_0^{2\pi} L d\phi = nh. \tag{4.17}$$

Thus L is a constant of motion, Eq. (4.10) gives the quantization condition for the angular momentum in the form of

$$L = m\tilde{L} = n\hbar, \quad \text{so that} \quad \tilde{L}_0 = n_0\hbar/m. \tag{4.18}$$

With the help of Compton radius and using Eq. (4.18) into Eq. (4.16), we get

$$n_0^2 r_c^2 = 12M^2\left(1-\frac{4M^2}{\ell^2}\right)^2. \tag{4.19}$$

The radius of the different stable circular orbit of the particle corresponds to n_0 can be obtained by using Eq. (4.18) into Eq. (4.15) in the form of

$$R_+ = n_0^2 \frac{r_c^2}{2M\left(1-\frac{4M^2}{\ell^2}\right)} \left(1 + \left[1 - \frac{12M^2\left(1-\frac{4M^2}{\ell^2}\right)^2}{r_c^2 n_0^2}\right]^{\frac{1}{2}}\right). \tag{4.20}$$

Inserting Eq. (4.19) into Eq. (4.20), we get the approximate radius of the initial circular orbit R_0 as

$$R_0 \approx n_0^2 \frac{r_c^2}{2M\left(1-\frac{4M^2}{\ell^2}\right)}. \tag{4.21}$$

In the limiting case, $R_0 = n_0^2 \frac{r_c^2}{r_s}$ and which agree with the result given in Ref. [116], where $r_s = 2M$ is the Schwarzschild radius. By replacing $n_1 = n_0 + 1$ in Eq. (4.20), we get the next higher circular orbit R_1 in the form of

$$R_1 = (n_0+1)^2 \frac{r_c^2}{2M\left(1-\frac{4M^2}{\ell^2}\right)} \left(1 + \left[1 - \frac{12M^2\left(1-\frac{4M^2}{\ell^2}\right)^2}{r_c^2 (n_0+1)^2}\right]^{\frac{1}{2}}\right). \tag{4.22}$$

For simplicity, let us consider $2M\left(1-\frac{4M^2}{\ell^2}\right) >> r_c$ so that $n_0 >> 1$. Therefore, we can write Eq. (4.22) in the following form

$$R_1 = n_0^2 (1+\frac{2}{n_0}) \frac{r_c^2}{2M\left(1-\frac{4M^2}{\ell^2}\right)} \left(1 + \left[1 - \frac{12M^2\left(1-\frac{4M^2}{\ell^2}\right)^2}{r_c^2 n_0^2 (1+\frac{2}{n_0})^2}\right]^{\frac{1}{2}}\right). \tag{4.23}$$

The parenthesis in the right side of the above equation can be approximated with the help of Eq. (4.19) in the form of

$$1 + \left[1 - \frac{12M^2\left(1-\frac{4M^2}{\ell^2}\right)^2}{r_c^2 n_0^2 (1+\frac{2}{n_0})^2}\right]^{\frac{1}{2}} \approx 1 + \sqrt{\frac{2}{n_0}}. \tag{4.24}$$

Therefore, Eq. (4.23) reduced to

$$R_1 = R_0\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right). \tag{4.25}$$

In the similar manner, the radius of the next higher circular orbit R_2 of the test particle can be evaluated from Eq. (4.20) as

$$R_2 = R_0\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right) = R_1\frac{\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right)}{\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right)}. \tag{4.26}$$

Proceeding in the similar way, the radius of the stable circular orbits of the particle can be written as

$$R_{n+1} = R_n\frac{\left(1+\frac{2n+2}{n_0}\right)\left(1+\sqrt{\frac{2n+2}{n_0}}\right)}{\left(1+\frac{2n}{n_0}\right)\left(1+\sqrt{\frac{2n}{n_0}}\right)}. \tag{4.27}$$

When $n_0 \to \infty$, we observe that $R_{n+1} = R_n$. Therefore, we may conclude that for large quantum number two nearby states coincide.

4.4 Quantized Energy and Hawking Temperature

Our intend is to quantize the energy of the orbiting test particle with the help of angular momentum. Thus the equation (4.12) gives the energy for zero velocity at $r = R$ as

$$\tilde{E}^2 = c^2\left(\frac{\tilde{L}^2}{R^2}+c^2\right)\left(1-\frac{2M\left(1-\frac{4M^2}{\ell^2}\right)}{R}\right). \tag{4.28}$$

But we have from Eq. (4.14) at $r = R$

$$\frac{\tilde{L}^2}{R^2} = \frac{c^2}{\left(\frac{2R}{2M\left(1-\frac{4M^2}{\ell^2}\right)}-3\right)}. \tag{4.29}$$

Using Eqs. (4.10) and (4.29) into Eq. (4.28), we have

$$E^2 = m^2c^4 \left(1 - \frac{2M\left(1 - \frac{4M^2}{\ell^2}\right)}{R}\right)^2 \left(1 - \frac{3M\left(1 - \frac{4M^2}{\ell^2}\right)}{R}\right)^{-1}. \quad (4.30)$$

Therefore, the above equation can be written as

$$E = mc^2 \left(1 - \frac{2M\left(1 - \frac{4M^2}{\ell^2}\right)}{R}\right) \left(1 - \frac{3M\left(1 - \frac{4M^2}{\ell^2}\right)}{R}\right)^{-1/2}. \quad (4.31)$$

Using the mechanical stability condition $\tilde{L}^2 = MGR$ into Eq. (4.15), the second term in the parenthesis of the Eq. (4.15) can be written as

$$\frac{12c^2M^2\left(1 - \frac{4M^2}{\ell^2}\right)^2}{\tilde{L}^2} = \frac{12M\left(1 - \frac{4M^2}{\ell^2}\right)}{R}, \quad (4.32)$$

which gives with the help of Eq. (4.16) for the circular orbit corresponding to $n_0 >> 1$

$$\frac{12M\left(1 - \frac{4M^2}{\ell^2}\right)}{R} << 1. \quad (4.33)$$

Therefore, Eq. (4.31) can be approximated to

$$E \approx mc^2 \left(1 - \frac{M\left(1 - \frac{4M^2}{\ell^2}\right)}{2R}\right). \quad (4.34)$$

Applying Eq. (4.16) into Eq.(4.21) can be written in the following form

$$2R \approx \frac{n^2 r_c^2}{M\left(1 - \frac{4M^2}{\ell^2}\right)} = \frac{n^2\hbar^2}{m^2c^2M\left(1 - \frac{4M^2}{\ell^2}\right)} = \frac{\tilde{L}^2}{c^2M\left(1 - \frac{4M^2}{\ell^2}\right)}. \quad (4.35)$$

Substituting Eq. (4.35) into Eq. (4.34) we have

$$E \approx mc^2 \left(1 - \frac{c^2M^2\left(1 - \frac{4M^2}{\ell^2}\right)^2}{\tilde{L}^2}\right). \quad (4.36)$$

The nth label quantized energy E_n can be obtained form

$$E_n \approx mc^2 \left(1 - \frac{c^2 M^2 \left(1 - \frac{M^2}{\ell^2}\right)^2}{\tilde{L}_n^2} \right). \tag{4.37}$$

Using $\hbar = r_c mc$ and Eq. (4.18) into above equation we obtain

$$E_n \approx mc^2 \left(1 - \frac{M^2 \left(1 - \frac{4M^2}{\ell^2}\right)^2}{n^2 r_c^2} \right). \tag{4.38}$$

The corresponding $(n+1)$th label energy can be written from the above Eq. (4.38) as

$$E_{n+1} \approx mc^2 \left[1 - \frac{M^2 \left(1 - \frac{4M^2}{\ell^2}\right)^2}{r_c^2 (n+1)^2} \right]. \tag{4.39}$$

Therefore, the quantized energy difference between two nearby states is

$$\begin{aligned} \delta E &= E_{n+1} - E_n \\ &\approx \frac{mc^2}{r_c^2} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right] M^2 \left(1 - \frac{4M^2}{\ell^2}\right)^2. \end{aligned} \tag{4.40}$$

Neglecting the 4th and the higher powers of (M/ℓ) of the SAdS black hole radius, which gives Eq. (4.2) then we have

$$r_{SAdS} \approx 2M \left(1 - \frac{4M^2}{\ell^2}\right). \tag{4.41}$$

We observe that when $\ell \to \infty$, then $r_{SAdS} = r_S$. Therefore, the Eq. (4.40) can be written for SAdS black hole

$$\delta E = \frac{mc^2 r_{SAdS}^2}{4 r_c^2} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right). \tag{4.42}$$

For large values of n, the bracket can be replaced by $2/n^3$ so that

$$\delta E \approx \frac{c^4 m^3 r_{SAdS}^2}{2\hbar^2 n^3}. \tag{4.43}$$

We observe that when n increases then δE decreases. As $n \to \infty$ we have shown that the change of energy between two nearby states becomes zero. Therefore, the Bekenstein-Hawking Entropy of SAdS black hole can be written with the help of Eq. (4.42) and in classical mechanics from the first law of thermodynamics as

$$\begin{aligned} \delta S &= \frac{\delta E}{T_H} \\ &= \frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times r_{SAdS}^2 \\ &= \frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 - \frac{4M^2}{\ell^2}\right)^2, \end{aligned} \tag{4.44}$$

where T_H is Hawking temperature and δS is the change of entropy between two nearby states. Thus, for larger circular orbit the change of entropy between two nearby states approaches to zero and possesses low energy continuum behavior. For this large circular orbit, the temperature approaches to zero.

Using WKB approximation, the imaginary part of the particles action of the emission rate for an out going particle with positive energy E coming from inside to outside as

$$\Gamma \sim exp(-2Im(I)), \tag{4.45}$$

where $Im(I) = -\frac{1}{2}[S(M-E) - S(M)] = \frac{1}{2}\delta S$. Therefore, for the SAdS black hole the thermal emission rate and the entropy change satisfies

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2 r_{SAdS}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]\right). \tag{4.46}$$

Using Eq. (4.41) into the above equation, we have

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 - \frac{4M^2}{\ell^2}\right)^2\right). \tag{4.47}$$

The Hawking temperature of SAdS black hole is given by

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M\left(1 - \frac{4M^2}{\ell^2}\right)}, \tag{4.48}$$

where $\kappa = \frac{1}{4M\left(1-\frac{4M^2}{\ell^2}\right)}$ is the surface gravity of SAdS black hole. When $\ell \to \infty$, the tunneling rate, entropy and the Hawking temperature agree similar to Schwarzschild black hole.

4.5 Concluding Remarks

For the evaporation process the SAdS black hole loses its mass i.e. $M \to 0$. For this circumstance Eq. (4.48) yields $T_H \to \infty$ so that the Eq. (4.47) gives $S \to 0$ & $\Gamma \to 0$. Those results agree with the results of Sakalli et al.[131] and for quantization of black holes in He et al.[132]. Again for large n, the equation (4.47) gives $S \to 0$ & $\Gamma \to 0$. That is, for large circular orbit the change of entropy between two nearby states vanishes. If we replace $-\ell^2$ into ℓ^2, then the results coincide with the previous results of SdS black hole [117] in chapter 3. Also we observe that if $\ell \to \infty$ then the results of SAdS black hole reduces to Schwarzschild black hole in chapter 2.

Chapter 5

Bekenstein-Hawking Entropy by Energy Quantization from Reissner-Nordström Black Hole

5.1 Introduction

The Reissner-Nordström (RN) black hole is generalized of Schwarzschild black hole with charge parameter Q in asymptotically de Sitter space. However, if in the RN setup the gravitating mass M and the electric charge Q obey the superextremality condition, $Q > M$, the space-time geometry corresponds not to a black hole but to a naked singularity. With $Q < M$, it describe two horizons- inner and outer horizons. The inner horizon is called Cauchy horizon. The extremal RN space is also special in admitting supersymmetry in the contest of N = 2 supergravity [145, 146, 147, 148, 149]. Since the RN singularity is electrically charged, it may be neutralized by spontaneous pair creation provided that charges are sufficiently high. Thus aspects of the RN solution must be of interest in a broader contest.

The organization of this chapter goes as follows: In section 5.2 we review RN line element near the event horizon approximation. Section

5.3 we develop the effective potential for radial motion. Section 5.4 and section 5.5 are respectively devoted to quantize of circular orbit around the black hole and quantization gravitational energy. Section 5.6 ends up with concluding remarks.

5.2 Reissner-Nordström Line Element

The line element of Reissner-Nordström black hole with mass M and charge Q as [150]

$$ds^2 = -f(r)c^2dt^2 + g((r))^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.1}$$

where $f(r)$ and $g(r)$ are the functions of the parameters M, Q and r. These are represented in the form of

$$f(r) = g(r) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right).$$

For the static spherical charged black hole $-\infty \leq t \leq \infty, r \geq 0, 0 \leq \theta \leq 2\pi$. We get the event horizon [150] of black hole in the form of

$$r_+ = r_{RN} = M\left(1 + \sqrt{1 - \frac{Q^2}{M^2}}\right) = 2M\left(1 - \frac{Q^2}{4M^2} - \frac{Q^4}{16M^4} + \cdots\right). \tag{5.2}$$

Now simplifying the Eq. (5.1), we get the following form

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 - \frac{Q^2}{2Mr}\right)\right)c^2dt^2 + \left(1 - \frac{2M}{r}\left(1 - \frac{Q^2}{2Mr}\right)\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.3}$$

Taking as a first approximation $r_0 = 2M\left(1 - \frac{Q^2}{4M^2}\right)$, the above metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)c^2dt^2 + \left(1 - \frac{2M}{r}\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.4}$$

Now, let us consider a test particle of mass m orbiting along the circular geodesics in the equatorial plane around RN black hole. Then according to the Refs. [5, 144] the metric given in Eq. (5.4) written as the following form

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)c^2dt^2 + \left(1 - \frac{2M}{r}\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)^{-1}dr^2 + r^2d\theta^2. \tag{5.5}$$

5.3 Radial Motion and Effective Potential due to Lagrangian and Canonical Momenta

In this section to quantize the RN black hole gravity we have used the canonical formulation [122, 123, 124, 143]. We have assumed the black hole mass M is larger than the Planck mass so that the Compton radius, $r_c = \hbar/mc$ is very small and therefore the change of quantum effect of the black hole is disregarded [126]. We use the Lagrangian of the test particle in terms of the metric components g_{ij} as

$$
\begin{aligned}
\pounds &= -\frac{m}{2} g_{ij}\dot{x}^i\dot{x}^j \\
&= -\frac{m}{2}\left[\left(1-\frac{2M}{r}\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)c^2\dot{t}^2\right] \\
&-\frac{m}{2}\left[\left(1-\frac{2M}{r}\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)^{-1}\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + \mathrm{r}^2\dot{\theta}^2\right].
\end{aligned}
\tag{5.6}
$$

For static spherical charged black hole, here we have two Killing vectors in terms of energy E and angular momentum L as

$$
\begin{aligned}
E &= \frac{\partial \pounds}{\partial \dot{t}} = mc^2\left(1-\frac{2M}{r}\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)\dot{t}, \\
L &= \frac{\partial \pounds}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}.
\end{aligned}
\tag{5.7}
$$

The radial motion of a geodesic and the canonical momenta defined by $p_\alpha = \frac{\partial \pounds}{\partial \dot{x}^\alpha}$ and the other two components obtain in the following form

$$
\begin{aligned}
p_r &= \frac{\partial \pounds}{\partial \dot{r}} = -m\left(1-\frac{2M}{r}\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)^{-1}\dot{r}, \\
p_\theta &= \frac{\partial \pounds}{\partial \dot{\theta}} = -mr^2\dot{\theta}.
\end{aligned}
\tag{5.8}
$$

$$
g^{00}p_0^2 + g^{rr}p_r^2 + g^{\phi\phi}p_\phi^2 + g^{\theta\theta}p_\theta^2 + m^2c^2 = 0, \tag{5.9}
$$

where the four-vectors ($p_0 = E/c, \mathbf{p}$) express the magnitude of the energy-momentum. Inserting Eqs. (5.7) and (5.8) into Eq. (5.9) and using $\dot{\theta} = 0$ and $\sin^2\theta = 1$, we obtain in Ref. [116]

$$\left(1-\frac{2M}{r}\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right)^{-1}\left(m^2\dot{r}^2-\frac{E^2}{c^2}\right)$$
$$+\frac{L^2}{r^2}+m^2c^2=0. \tag{5.10}$$

Again in Ref. [116], the energy and momentum of the test particle per unit rest mass in the form of

$$\tilde{E}=\frac{E}{m},\tilde{L}=\frac{L}{m} \tag{5.11}$$

To search the radial motion of the test particle, inserting Eq. (5.11) into Eq. (5.10) we have

$$\frac{\dot{r}^2}{2}=\frac{\tilde{E}^2}{2c^2}-\frac{1}{2}\left(\frac{\tilde{L}^2}{r^2}+c^2\right)\left(1-\frac{2M}{r}\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right). \tag{5.12}$$

The energy and the angular momentum are two parameters by which we can describe the velocity for the time like particle orbit. Thus we have the effective potential V_{eff} for the radial motion in following form

$$V_{eff}=\frac{1}{2}\left(\frac{\tilde{L}^2}{r^2}+c^2\right)\left(1-\frac{2M}{r}\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\right). \tag{5.13}$$

Differentiate Eq. (5.13) with respect to the proper time and we get the radial acceleration of the test particle. To get the maximum potential, we set $\frac{\partial V_{eff}}{\partial r}=0$ and which implies

$$\frac{c^2}{r^4}M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)\left(r^2-\frac{\tilde{L}^2}{c^2M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}r+\frac{3\tilde{L}^2}{c^2}\right)=0. \tag{5.14}$$

The above equation gives the two roots of the following form

$$R_{\pm} = \frac{\tilde{L}^2}{2c^2M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)} \pm \left[\left(\frac{\tilde{L}^2}{2c^2M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}\right)^2 - \frac{3\tilde{L}^2}{c^2}\right]^{\frac{1}{2}}. \tag{5.15}$$

Here R represents the radius of the circular orbit and the another form of the Eq. (5.15) is

$$R_{\pm} = \frac{\tilde{L}^2}{2c^2M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)} \left(1 \pm \left(1 - \frac{12c^2M^2\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{\tilde{L}^2}\right)^{\frac{1}{2}}\right). \tag{5.16}$$

We observe that $\tilde{L}^2 \geq 12c^2M^2\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2$ holds when $R_{\pm}$ is real and $\tilde{L}^2 = 12c^2M^2\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2$ holds for the smallest stable orbit while the square root on the right hand side of Eq. (5.16) vanishes. Also for large and the largest stable circular orbits we have the conditions $\tilde{L}^2 \geq 12c^2M^2\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2$ and $\tilde{L}^2 >> 12c^2M^2\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2$ respectively.

5.4 Quantization of the Orbiting Test Particle

According to Wilson [129] and Sommerfeld idea [130] the angular momentum can be quantized as a periodic function and which is intricated to the quantize angular momentum of the orbiting test particle. With the help of canonical momentum L the quantize angular momentum J_ϕ can be written as

$$J_\phi = \int_0^{2\pi} L d\phi = nh. \tag{5.17}$$

Here L is a constant of motion, then the Eq. (5.11) provides the quantization condition for the angular momentum in the following form

$$L = m\tilde{L} = n\hbar, \quad \text{so that} \quad \tilde{L}_0 = n_0\hbar/m. \tag{5.18}$$

As discussed before for the smallest circular orbits we have

$$\tilde{L}^2 = 12c^2M^2\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2. \tag{5.19}$$

The above Eq. (5.19) can be written with the help of Eq. (5.18) as

$$n_0^2 = \frac{12M^2\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{r_c^2}, \tag{5.20}$$

where $r_c = \frac{\hbar}{mc}$ the Compton radius. Using Eq. (5.18) into Eq. (5.16), we can obtain the radius of the different stable circular orbits of the particle correspond to n_0 is

$$R_+ = n_0^2\frac{r_c^2}{2M\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)} \times\left(1 + \left[1 - \frac{12M^2\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{r_c^2n_0^2}\right]^{\frac{1}{2}}\right). \tag{5.21}$$

Applying Eq. (5.20) into Eq. (5.21), the first approximation of the circular orbit radius R_0 can be written as

$$R_0 \approx n_0^2 \frac{r_c^2}{2M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}, \tag{5.22}$$

when charge becomes zero then $R_0 = n_0^2 \frac{r_c^2}{r_s}$ which agrees with the result given in Ref. [116]. Now our prime task for quantization, the position of the next higher circular orbit R_1 can be obtained from Eq. (5.21) by replacing n_0 with $n_1 = n_0 + 1$ as follows

$$R_1 = (n_0+1)^2 \frac{r_c^2}{2M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)} \times \left(1 + \left[1 - \frac{12M^2\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{r_c^2(n_0+1)^2}\right]^{\frac{1}{2}}\right). \tag{5.23}$$

Let us consider $2M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right) >> r_c$ so that $n_0 >> 1$. Using $(n_0+1)^2 \approx n_0^2[1+\frac{2}{n_0}]$ into the Eq. (5.23) we get

$$R_1 = n_0^2\left(1+\frac{2}{n_0}\right) \frac{r_c^2}{2M\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)} \times \left(1 + \left[1 - \frac{12M^2\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{r_c^2 n_0^2(1+\frac{2}{n_0})}\right]^{\frac{1}{2}}\right). \tag{5.24}$$

With the help of Eq. (5.20), the term in the first bracket can be approximated in the form of

$$1+\left[1-\frac{12M^2\left(1-\frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{r_c^2n_0^2(1+\frac{2}{n_0})}\right]^{\frac{1}{2}} \approx 1+\sqrt{\frac{2}{n_0}}. \quad (5.25)$$

Therefore, Eq. (5.24) reduces to the form

$$R_1 = R_0\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right). \quad (5.26)$$

Similarly, we can get the radius of the next higher circular orbit R_2 of the particle can be intended from Eq. (5.21) in the form of

$$R_2 = R_0\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right) = R_1\frac{\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right)}{\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right)}. \quad (5.27)$$

Finally, we get the radii of the stable circular orbits of the particle can be written in general form as

$$R_{n+1} = R_n\frac{\left(1+\frac{2n+2}{n_0}\right)\left(1+\sqrt{\frac{2n+2}{n_0}}\right)}{\left(1+\frac{2n}{n_0}\right)\left(1+\sqrt{\frac{2n}{n_0}}\right)}. \quad (5.28)$$

Thus the gravity of the RN black holes can be quantized in discrete states with radius R_{n+1} and when n tends to infinity then $R_{n+1} = R_n$. Hence our conclusion is that for large quantum number two nearby states coincide.

5.5 Quantization of Gravitational Energy

We know that the quantization of energy is strongly connected to the quantization of the angular momentum and therefore, we will quantize the energy of the orbiting test particle with the help of angular momentum. Equation (5.12) gives the energy for zero velocity at $r = R$

$$\tilde{E}^2 = c^2 \left(\frac{\tilde{L}^2}{R^2} + c^2 \right) \left(1 - \frac{2M\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}{R} \right). \tag{5.29}$$

But we have from Eq. (5.14) at $r = R$

$$\frac{\tilde{L}^2}{R^2} = c^2 \Bigg/ \left(\frac{2R}{2M\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)} - 3 \right). \tag{5.30}$$

Using Eq. (5.30) into Eq. (5.29) and therefore we get

$$E = mc^2 \frac{\left(1 - \frac{2M\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}{R} \right)}{\left(1 - \frac{3M\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}{R} \right)^{1/2}}. \tag{5.31}$$

We introduce the mechanical stability condition $\tilde{L}^2 = MGR$ into Eq. (5.16) and we have the second term in the parenthesis of the Eq. (5.16) as

$$\frac{12c^2M^2\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{\tilde{L}^2} = \frac{12M\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}{R}.$$

With the help of Eq. (5.19) for the circular orbits corresponding to $n_0 >> 1$ we have $\frac{12M\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}{R} << 1$. Therefore, Eq. (5.31) can be approximated to

$$E \approx mc^2 \left(1 - \frac{M\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)}{2R} \right). \tag{5.32}$$

By using Eq. (5.18) into Eq. (5.22) and using this in Eq. (5.32) we can get the approximation value of E as

$$E \approx mc^2 \left(1 - \frac{c^2M^2\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{\tilde{L}^2} \right). \tag{5.33}$$

For n energy label the quantized energy E_n can be written as

$$E_n \approx mc^2 \left(1 - \frac{c^2M^2\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{\tilde{L}_n^2} \right). \tag{5.34}$$

With the help of Eq. (5.18) and using $\hbar = r_c mc$ the above equation becomes

$$E_n \approx mc^2 \left(1 - \frac{M^2\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{n^2 r_c^2} \right). \tag{5.35}$$

From above equation the corresponding $n+1$ energy label can be written as

$$E_{n+1} \approx mc^2 \left[1 - \frac{M^2\left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2}{r_c^2(n+1)^2} \right]. \tag{5.36}$$

Therefore, the energy difference between two nearby states is as follows

$$\begin{aligned}\delta E &= E_{n+1} - E_n \\ &\approx \frac{mc^2}{r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] M^2 \left(1 - \frac{Q^2}{4M^2(1-\frac{Q^2}{4M^2})}\right)^2 . \end{aligned} \quad (5.37)$$

Neglecting the 3rd and higher terms of (Q/M) of the RN black hole radius given in Eq. (5.2) then we get $r_{RN} \approx 2M\left(1 - \frac{Q^2}{4M^2}\right)$ and therefore, the above equation for RN black hole stands as

$$\delta E \approx \frac{mc^2 r_{RN}^2}{4r_c^2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right). \quad (5.38)$$

If $n >> 1$, we can put $2/n^3$ in the place of the parenthesis in above equation as the following form

$$\delta E \approx \frac{c^4 m^3 r_{RN}^2}{2\hbar^2 n^3}, \quad (5.39)$$

which ensures that the change of energy between two nearby states approaches to zero for large values of n. Thus, the Bekenstein-Hawking Entropy of RN black hole with the help of Eq. (5.39) and applying the first law of thermodynamics as

$$\begin{aligned}\delta S &= \frac{\delta E}{T_H} \\ &= \frac{mc^2 r_{RN}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \\ &= \frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 - \frac{Q^2}{4M^2}\right)^2 . \end{aligned} \quad (5.40)$$

Here T_H is the Hawking temperature of RN black hole. Thus, the change of entropy between two nearby states approaches to zero and possesses low energy continuum behavior which means that between two nearby states by transition a black hole can absorb a particle with small energy or between far-away states a big one can be absorbed. The imaginary part of the particles action of the emission rate for an out going particle with positive energy E coming from inside to outside by WKB approximation is

$$\Gamma \sim exp(-2Im(I)), \tag{5.41}$$

where $Im(I) = -\frac{1}{2}[S(M-E) - S(M)] = \frac{1}{2}\delta S$. Therefore, for the RN black hole the thermal emission rate and the entropy change satisfies

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2 r_{RN}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]\right). \tag{5.42}$$

Inserting the RN black hole radius into the Eq. (5.42) becomes as

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 - \frac{Q^2}{4M^2}\right)^2\right). \tag{5.43}$$

The Hawking temperature can be rewritten in terms of surface gravity κ as

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{4\pi r_{RN}} = \frac{1}{8\pi M\left(1 - \frac{Q^2}{4M^2}\right)}. \tag{5.44}$$

If we set $Q = 0$, we get

$$T_H = \frac{1}{8\pi M}, \tag{5.45}$$

which agree with Schwarzschild black hole.

5.6 Concluding Remarks

During the evaporation process RN black hole loses its mass. At a stage its mass $M \to 0$ so that Eq. (5.44) gives Hawking temperature $T_H \to \infty$ and therefore, Eq. (5.43) yields entropy $S \to 0$ & thermal emission rate $\Gamma \to 0$. Those results agree with the results of Sakalli et al.[131] and He et al.[132] for quantization of black holes. On the other hand, if we put $n \to \infty$ into Eq. (5.43) we get $S \to 0$, $\Gamma \to 0$. Which means that for large circular orbit the change of entropy between two nearby states equal to zero. When $Q \to 0$ then the results of RN black hole in chapter 5 coincide with the results of Schwarzschild black hole in chapter 2.

Chapter 6

Bekenstein-Hawking Entropy by Energy Quantization from Reissner-Nordström-de Sitter Black Hole

6.1 Introduction

The Reissner-Nordström-de Sitter spacetime has some specific features such that the Cauchy horizon and the timelike singularity exist. Generally, all de Sitter black holes have the black hole event horizon and the cosmological horizon which can be treated as a thermodynamic system. The different temperatures for the two horizons make the whole spacetime cannot be in thermodynamic equilibrium except for the Nariai or lukewarm case. The SdS black hole is always thermodynamically stable and RNdS black hole may undergo phase transition at some points.

The rest of this chapter is designed as follows: Section 6.2 deals with the simplification of RNdS line element and we express the energy and angular momentum in terms of RNdS metric. In Section 6.3, we have calculated the effective possibility of radial motion using the Lagrangian and canonical momentum of a test particle of the Reissner-Nordström-de Sitter black hole according to our previous work [117]. Here we have also quantized the circular orbit of the test particle moving around the black hole while in section 6.4, the Hawking purely thermal emission rate has been established by quantizing the energy of the RNdS black hole gravity. Finally in section 6.5, we present our concluding remarks.

6.2 Lagrangian and Canonical Momenta due to RNdS Spacetime

In this section to quantize the RNdS black hole gravity, we have the line element of Reissner-Nordström-de Sitter black hole with a positive cosmological constant Λ ($\Lambda = 3/\ell^2$) in refs. [151, 152] as

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{r^2}{\ell^2} + \frac{Q^2}{r^2}\right) c^2 dt^2 + \left(1 - \frac{2M}{r} - \frac{r^2}{\ell^2} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6.1}$$

where M being the mass, ℓ is the cosmological radius, Q the total charge with respect to the static de Sitter space are defined such that $-\infty \leq t \leq \infty, r \geq 0, 0 \leq \theta \leq 2\pi$. At large r, the metric (6.1) tends to the dS space limit. It is seen that the explicit dS case is obtained by setting $M = 0$ while the explicit Reissner-Nordström case is obtained by taking the limit $\ell \to \infty$. For the simplicity we can rewrite the metric given in Eq. (6.1) of the following form

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 + \frac{r^3}{2M\ell^2} - \frac{Q^2}{2Mr}\right)\right)c^2dt^2$$
$$+\left(1 - \frac{2M}{r}\left(1 + \frac{r^3}{2M\ell^2} - \frac{Q^2}{2Mr}\right)\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.2}$$

The black hole parameters M, Q, and ℓ are related to the roots of $r^4 - \ell^2r^2 + 2M\ell^2r - \ell^2Q^2 = 0$. Solving this equation, the position of the black hole horizon [152] can be written as

$$r_h = r_{RNdS} = \frac{1}{\alpha}\left(1 + \frac{4M^2}{\ell^2\alpha^2} + \cdots\right)\left(M + \sqrt{M^2 - Q^2\alpha}\right), \tag{6.3}$$

with $\alpha = \sqrt{1 + \frac{4Q^2}{\ell^2}}$. Since $\ell^2 >> Q^2$, therefore $\alpha \approx 1$ and therefore Eq. (6.3) reduced to

$$r_h = r_{RNdS} \approx \left(1 + \frac{4M^2}{\ell^2} + \cdots\right)M\left(1 + \sqrt{1 - \frac{Q^2}{M^2}}\right). \tag{6.4}$$

Since $\left(1 + \frac{4M^2}{\ell^2} + \cdots\right) > 1$, The radius of RNdS black hole is greater then RN black hole. For a negligible charge $Q \approx 0$ and the first approximated position of horizon r_0 tends to SdS black hole and which is grater than RN black hole as

$$r_0 \approx 2M\left(1 + \frac{4M^2}{\ell^2} + \cdots\right) = 2M\left(1 + \frac{4M^2}{\ell^2}\right). \tag{6.5}$$

Using Eq. (6.5), the metric equation given in (6.2) can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2(1+\frac{4M^2}{\ell^2})^3}{\ell^2} - \frac{Q^2}{4M^2(1+\frac{4M^2}{\ell^2})}\right)\right)c^2dt^2$$
$$+\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2(1+\frac{4M^2}{\ell^2})^3}{\ell^2} - \frac{Q^2}{4M^2(1+\frac{4M^2}{\ell^2})}\right)\right)^{-1}dr^2$$
$$+r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.6}$$

If we let $\xi = 1 + \frac{4M^2}{\ell^2}$, then the above Eq. (6.6) gives to

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)\right)c^2dt^2$$
$$+\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)\right)^{-1}dr^2$$
$$+r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{6.7}$$

Suppose a test particle of mass m orbiting along the circular geodesics in the equatorial plane $\theta = \pi/2$ around RNdS black hole. Thus the above metric as described in [5, 128] can be written as

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)\right)c^2dt^2$$
$$+\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)\right)^{-1}dr^2 + r^2d\theta^2. \tag{6.8}$$

Considering the black hole mass M is larger than the Planck mass so that the Compton radius, $r_c = \hbar/mc$ is very small than the radius of the RNdS black hole r_h. In this situation, the quantum fluctuations of the black hole is disregarded [126]. The Lagrangian of the test particle in terms of the metric components g_{ij} is defined as

$$\begin{aligned}\pounds &= -\frac{m}{2}g_{ij}\dot{x}^i\dot{x}^j \\ &= -\frac{m}{2}\left[\left(1-\frac{2M}{r}\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)\right)c^2\dot{t}^2\right] \\ &-\frac{m}{2}\left[\left(1-\frac{2M}{r}\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)\right)^{-1}\dot{r}^2+r^2(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)\right].\end{aligned} \tag{6.9}$$

Since the RNdS spacetime is spherically charged, there exist two constants of motion for the test particles, associated with two Killing vectors in terms of energy E and angular momentum L as

$$E = \frac{\partial \pounds}{\partial \dot{t}} = mc^2\left(1-\frac{2M}{r}\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)\right)\dot{t}, \tag{6.10}$$

$$L = \frac{\partial \pounds}{\partial \dot{\phi}} = mr^2\sin^2\theta\dot{\phi}. \tag{6.11}$$

From the canonical momenta equations $p_\alpha = \frac{\partial \pounds}{\partial \dot{x}^\alpha}$, the other two components take the following form

$$p_r = \frac{\partial \pounds}{\partial \dot{r}} = -m\left(1-\frac{2M}{r}\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)\right)^{-1}\dot{r}, \tag{6.12}$$

$$p_\theta = \frac{\partial \pounds}{\partial \dot{\theta}} = -mr^2\dot{\theta}. \tag{6.13}$$

6.3 Quantization of Circular Orbit due to Radial Motion and Effective Potential

The radial motion of a geodesic corresponding to Eq. (6.9) can be written as

$$g^{\alpha\alpha}p_\alpha^2 + m^2c^2 = 0, \tag{6.14}$$

where $\alpha = 0, r, \theta, \phi$, and $(p_0 = E/c, \mathbf{p})$ expresses the magnitude of the energy-momentum. Using Eqs. (6.10) to (6.13) and Eq. (6.8) with $\dot{\theta} = 0$ and $\sin^2\theta = 1$ into Eq. (6.14), we have

$$\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)\right)^{-1}\left(-\frac{E^2}{c^2} + m^2\dot{r}^2\right) + \frac{L^2}{r^2} + m^2c^2 = 0. \tag{6.15}$$

We introduce the energy and angular momentum in Refs. [116, 117] of the test particle per unit rest mass $\tilde{E} = \frac{E}{m}$, $\tilde{L} = \frac{L}{m}$ respectively and setting these into Eq. (6.15) we get

$$\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)\right)^{-1}\left(-\frac{\tilde{E}^2}{c^4} + \frac{\dot{r}^2}{c^2}\right) + \frac{\tilde{L}^2}{c^2r^2} + 1 = 0. \tag{6.16}$$

Simplifying the above equation for the radial motion of the test particle as follows

$$\frac{\dot{r}^2}{2} = \frac{\tilde{E}^2}{2c^2} - \frac{1}{2}\left(\frac{\tilde{L}^2}{r^2} + c^2\right)\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)\right). \tag{6.17}$$

We can describe the velocity for the the time like particle by the parameters energy and angular momentum. Thus, we have the effective potential V_{eff} for the radial motion from Eq. (6.17) in the form of

$$V_{eff} = \frac{1}{2}\left(\frac{\tilde{L}^2}{r^2} + c^2\right)\left(1 - \frac{2M}{r}\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)\right). \tag{6.18}$$

To find the maximum potential, taking the derivative of Eq. (6.18) with respect to the proper time and then setting to zero, we obtain

$$\frac{c^2}{r^4}M\left(1-\frac{2M}{r}\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)\right)$$
$$\times\left(r^2-\frac{\tilde{L}^2}{c^2M\left(1-\frac{2M}{r}\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)\right)}r+\frac{3\tilde{L}^2}{c^2}\right)=0. \quad (6.19)$$

Now our prime task in this section is to find the radius of the circular orbit namely R. For this purpose, solving the above equation for r the radius of the circular orbit can be written in the following form

$$R_\pm=\frac{\tilde{L}^2}{2c^2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)}$$
$$\pm\left[\left(\frac{\tilde{L}^2}{2c^2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)}\right)^2-\frac{3\tilde{L}^2}{c^2}\right]^{\frac{1}{2}}. \quad (6.20)$$

Equivalently we can write the Eq. (6.20) as

$$R_\pm=\frac{\tilde{L}^2}{2c^2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)}$$
$$\times\left(1\pm\left(1-\frac{12c^2M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2}{\tilde{L}^2}\right)^{\frac{1}{2}}\right). \quad (6.21)$$

From the above equation we see that $R_\pm$ is real only when $\tilde{L}^2\geq 12c^2M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2$ and the square root on the right hand side of Eq. (6.21) vanishes for the smallest stable orbit. Therefore, we must have

$$\tilde{L}^2 = 12c^2M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2. \tag{6.22}$$

Also the conditions $\tilde{L}^2 \geq 12c^2M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2$ and $\tilde{L}^2 >> 12c^2M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2$ hold for large and largest stable circular orbits respectively. Angular momentum can be quantized as a periodic function of time following Wilson and Sommerfeld idea [129, 130] and which is involved in quantizing angular momentum of the orbiting test particle. In order to quantize angular momentum J_ϕ with the help of canonical momentum L conjugate to the angular variable in the form of

$$J_\phi = \int_0^{2\pi} L d\phi = nh. \tag{6.23}$$

As L is a constant of motion, thus $\tilde{L} = \frac{L}{m}$ gives the quantization condition for the angular momentum in the form of

$$L = m\tilde{L} = n\hbar, \quad \text{so that} \quad \tilde{L}_0 = n_0\hbar/m. \tag{6.24}$$

Now inserting equation Eq. (6.24) with the compton radius $r_c = \hbar/mc$ of the test particle in Eq. (6.22) we obtain

$$n_0^2 = \frac{12M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2}{r_c^2}. \tag{6.25}$$

The radius of the different stable circular orbit of the particle corresponds to n_0 can be obtained by using Eq. (6.25) into Eq. (6.21) in the form of

$$R_+ = \frac{n_0^2 r_c^2}{2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)}\left(1+\left[1-\frac{12M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2}{r_c^2 n_0^2}\right]^{\frac{1}{2}}\right). \tag{6.26}$$

We get the approximate radius of the first circular orbit namely R_0 by inserting Eq. (6.25) into Eq. (6.26) as

$$R_0 \approx \frac{n_0^2 r_c^2}{2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)}. \tag{6.27}$$

In the limiting case when $\ell \to \infty$ with vanishing charge this reduced to $R_0 = n_0^2 \frac{r_c^2}{r_s}$ which agrees with the result given in Ref. [116], where $r_s = 2M$ is the Schwarzschild radius. The position of the next higher circular orbit R_1 can be obtained from Eq. (6.26) by replacing n_0 with $n_1 = n_0 + 1$ in the form of

$$R_1 = \frac{(n_0+1)^2 r_c^2}{2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)}\left(1+\left[1-\frac{12M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2}{(n_0+1)^2 r_c^2}\right]^{\frac{1}{2}}\right). \tag{6.28}$$

Now $n_0 >> 1$ while $2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right) >> r_c$ and therefore we can write $(n_0+1)^2 \approx n_0^2[1+\frac{2}{n_0}]$ and using this into Eq. (6.28) we obtain

$$R_1 = \frac{n_0^2(1+\frac{2}{n_0})r_c^2}{2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)} \times\left(1+\left[1-\frac{12M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2}{n_0^2(1+\frac{2}{n_0})r_c^2}\right]^{\frac{1}{2}}\right). \tag{6.29}$$

The parenthesis in the right side of the above equation can be approximated with the help of Eq. (6.25) in the form of

$$1+\left[1-\frac{12M^2\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)^2}{n_0^2(1+\frac{2}{n_0})r_c^2}\right]^{\frac{1}{2}} \approx 1+\sqrt{\frac{2}{n_0}}. \tag{6.30}$$

Thus the Eq. (6.29) is reduced to the simplest form as

$$R_1 = R_0\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right). \tag{6.31}$$

If we advance in this way, the radius of the next higher circular orbit R_2 of the particle can be calculated from Eq. (6.26) in the form of

$$R_2 = R_0\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right) = R_1\frac{\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right)}{\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right)}. \tag{6.32}$$

Proceeding in the similar manner the $(n+1)$th radius of the stable circular orbits of the particle can be written as

$$R_{n+1} = R_n\frac{\left(1+\frac{2n+2}{n_0}\right)\left(1+\sqrt{\frac{2n+2}{n_0}}\right)}{\left(1+\frac{2n}{n_0}\right)\left(1+\sqrt{\frac{2n}{n_0}}\right)}. \tag{6.33}$$

Here it is clear that when n_0 goes to infinity we have $R_{n+1} = R_n$. Thus, we can say that for large quantum number two nearby states coincide for RNdS black hole.

6.4 Entropy in terms of Quantized Energy and Hawking Temperature

In this section, our aim is to quantize the energy of the orbiting test particle with the help of angular momentum. We have the energy for zero velocity from Eq. (6.17) by setting $r = R$ of the following form

$$\tilde{E}^2 = c^2\left(\frac{\tilde{L}^2}{R^2}+c^2\right)\left(1-\frac{2M\left(1+\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\xi}\right)}{R}\right). \tag{6.34}$$

But Eq. (6.19) gives for $r = R$ as

$$\frac{\tilde{L}^2}{R^2} = c^2 \Big/ \left(\frac{2R}{2M(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi})} - 3 \right). \tag{6.35}$$

By the symmetric simplification as in [118], using Eq. (6.35) into Eq. (6.34), we get

$$E = mc^2 \left(1 - \frac{2M\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)}{R} \right) \times \left(1 - \frac{3M\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)}{R} \right)^{-1/2}. \tag{6.36}$$

Using the the mechanical stability condition $\tilde{L}^2 = MGR$ in Eq. (6.21), the second term in the parenthesis can be written as

$$\begin{aligned} \frac{12c^2M^2\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2}{\tilde{L}^2} &= 3c^2 \frac{2GM}{c^2} \frac{2M\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)}{MGR} \\ &= \frac{12M\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)}{R}, \end{aligned} \tag{6.37}$$

which gives with the help of Eq. (6.22) for the circular orbits corresponding to $n_0 >> 1$

$$\frac{12M\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)}{R} << 1. \tag{6.38}$$

Therefore, Eq. (6.36) can be approximated after some steps as

$$E \approx mc^2 \left(1 - \frac{M\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)}{2R} \right). \tag{6.39}$$

Applying Eq. (6.24) into Eq.(6.27) and substituting this into Eq. (6.39) we obtain

$$E \approx mc^2 \left(1 - \frac{c^2 M^2 \left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2}{\tilde{L}^2} \right). \tag{6.40}$$

The nth quantized energy label E_n can be defined as

$$E_n \approx mc^2 \left(1 - \frac{c^2 M^2 \left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2}{\tilde{L}_n^2} \right). \tag{6.41}$$

Using $\hbar = r_c mc$ and Eq. (6.24) we obtain from Eq. (6.41) as

$$E_n \approx mc^2 \left(1 - \frac{M^2 \left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2}{n^2 r_c^2} \right). \tag{6.42}$$

The corresponding $(n+1)th$ energy label can be written from Eq. (6.42) as

$$E_{n+1} \approx mc^2 \left[1 - \frac{M^2 \left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2}{r_c^2 (n+1)^2} \right]. \tag{6.43}$$

Thus, the energy difference between two nearby states in the form of

$$\begin{aligned} \delta E &= E_{n+1} - E_n \\ &\approx \frac{mc^2}{r_c^2} \left[\frac{1}{n^2} - \frac{1}{(n+1)^2} \right] M^2 \left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2 . \end{aligned} \tag{6.44}$$

If we neglect the terms corresponding to 4th and higher power of (M/ℓ) of the RNdS black hole radius given in Eq. (6.3) then we have

$$r_{RNdS} \approx 2M \left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right). \tag{6.45}$$

We observe that when $\ell \to \infty$ $r_{RNdS} \approx r_{RN}$ that has been developed in [118]. Therefore, the Eq. (6.44) can be written for RNdS black hole as

$$\delta E \approx \frac{mc^2 r_{RNdS}^2}{4r_c^2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right). \tag{6.46}$$

For large values of n, the bracket can be replaced by $2/n^3$ so that

$$\begin{aligned}\delta E &\approx \frac{c^4 m^3 r_{RNdS}^2}{2\hbar^2 n^3} \\ &= \frac{c^4 m^3}{2\hbar^2 n^3} \times 4M^2\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2. \end{aligned} \tag{6.47}$$

When $n \to \infty$ we have $\delta E \to 0$, which indicates for a large circular orbit the change of energy between two nearby states approaches to zero.

Thus, the Bekenstein-Hawking Entropy for RNdS black hole with the help of Eq. (6.46) and from chapter 2 in Eq. (2.63), we get

$$\begin{aligned}\delta S &= \frac{\delta E}{T_H} \\ &= \frac{mc^2 r_{RNdS}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \\ &= \frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2, \end{aligned} \tag{6.48}$$

where T_H is the Hawking temperature of RNdS black hole. Using WKB approximation, the imaginary part of the particles action of the emission rate for an out going particle with positive energy E coming from inside to outside as

$$\Gamma \sim exp(-2Im(I)), \tag{6.49}$$

where $Im(I) = -\frac{1}{2}[S(M-E) - S(M)] = \frac{1}{2}\delta S$. Therefore, for the RNdS black hole the thermal emission rate and the entropy change satisfies

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2 r_{RNdS}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]\right). \tag{6.50}$$

Inserting Eq. (6.45) into Eq. (6.50), then the above equation can be written in another form as

$$\begin{aligned}\Gamma &\sim \exp(\delta S) = \exp\left(\frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times r_{RNdS}^2\right). \\ &= \exp\left(\frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)^2\right).\end{aligned} \tag{6.51}$$

The Hawking temperature of RNdS black hole can be given as

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M\left(1 + \frac{4M^2\xi^3}{\ell^2} - \frac{Q^2}{4M^2\xi}\right)}. \tag{6.52}$$

When $\ell \to \infty$, then we get $\xi = 1$ and the above equation (6.52) reduces to

$$T_H = \frac{1}{8\pi M\left(1 - \frac{Q^2}{4M^2}\right)}, \tag{6.53}$$

which agree with RN black hole.
When $Q = 0$, Eq. (6.52) gives

$$T_H = \frac{1}{8\pi M\left(1 + \frac{4M^2\left(1+\frac{4M^2}{\ell^2}\right)^3}{\ell^2}\right)} \approx \frac{1}{8\pi M\left(1 + \frac{4M^2}{\ell^2}\right)}, \tag{6.54}$$

which agree with SdS black hole result. When $\ell \to \infty$, Eq. (6.54) gives $T_H = \frac{1}{8\pi M}$, which is the same as Schwarzschild black hole.

6.5 Concluding Remarks

When Hawking radiation take place in spacetime, RNdS black hole loses its mass and charge so that $M \to 0$ and $Q \to 0$. In this case Eq. (6.52) gives $T_H \to \infty$. Therefore, from Eq. (6.51) we can say that entropy and thermal emission rate both vanishes. Those results agree with the results obtained by Sakalli et al.[131] and He et al.[132]. Again, if we set $n \to \infty$ Eq. (6.51) gives $S \to 0$ & $\Gamma \to 0$. Thus we may conclude that, for large circular orbit the change of entropy between two nearby states vanishes. The RNdS black hole reduced to Schwarzschild black hole in chapter 2 when $\ell \to \infty$ and $Q \to 0$. When the charge $Q \to 0$, the results coincide with the results of SdS black hole [117] in chapter 3. By considering $Q \to 0$ and replacing ℓ^2 in the place of $-\ell^2$, we have the same results in chapter 6 and SAdS black hole in chapter 4. Again, when $\ell \to \infty$ then the results of RNdS black hole is same as the results of RN black hole [118] in chapter 5.

Chapter 7

Bekenstein-Hawking Entropy by Energy Quantization from Reissner-Nordström Anti-de Sitter Black Hole

7.1 Introduction

During the last four decades researchers have been curious to investigate Hawking radiation in Anti-de Sitter space because of AdS/CFT correspondence [153, 154]. In accordance with the AdS/CFT correspondence [153, 154], a big static black hole in asymptotically AdS spacetime corresponds to a (approximately) thermal state in the CFT. Thus, the time scale for the decay of the black hole perturbation that is given by the imaginary part of its action, corresponds to the timescale to attain thermal equilibrium in the powerfully combined CFT [155]. Besides, the most recent development in string/M-theory stimulates the study of black holes in Anti-de Sitter spaces to great extent. Hence, our study on the Reissner-Nordström Anti-de Sitter black holes is reasonable, contemporary and useful.

In this chapter, we investigate the quantization of Reissner-Nordström Anti-de Sitter(RNAdS) black hole using the same method that has been used in chapter two with a charged parameter and a negative cosmological constant. The rest of this chapter is planned as follows: Firstly in section 7.2 we have discussed Reissner-Nordström Anti-de Sitter(RNAdS) Spacetime and secondly in section 7.3 we have calculated the Lagrangian and canonical momentum of a test particle of the RNAdS metric according to our previous work [117]. In Section 7.4, we have designed the effective possibility of radial motion using the Lagrangian and canonical momentum. Here we have also quantized the angular momentum of the test particle moving around the black hole while in section 7.5 and in section 7.6 we have established by quantizing the energy of the RNAdS black hole gravity. Finally in section 7.7, we present our concluding remarks.

7.2 Reissner-Nordström Anti-de Sitter Spacetime

With the negative cosmological constant Λ ($\Lambda = -3/\ell^2$), the metric of the Reissner-Nordström Anti-de Sitter black hole is given by [55, 156]

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{r^2}{\ell^2} + \frac{Q^2}{r^2}\right) c^2dt^2 + \left(1 - \frac{2M}{r} + \frac{r^2}{\ell^2} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (7.1)$$

where M, ℓ represents the mass, cosmological radius and Q is the total charge with respect to the static de Sitter space are defined such that $-\infty \leq t \leq \infty, r \geq 0, 0 \leq \theta \leq 2\pi$. The metric (7.1) tends to the AdS space limit for large r. We observe that the explicit AdS case is obtained by setting $M = 0$ while the explicit Reissner-Nordström case is obtained by taking the limit $\ell \to \infty$. For simplicity we can rewrite the metric given in Eq. (7.1) in the following form

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 - \frac{r^3}{2M\ell^2} - \frac{Q^2}{2Mr}\right)\right)c^2dt^2$$
$$+\left(1 - \frac{2M}{r}\left(1 - \frac{r^3}{2M\ell^2} - \frac{Q^2}{2Mr}\right)\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7.2}$$

The characteristic polynomial of RNAdS spacetime can be written as $r^4 + \ell^2 r^2 - 2M\ell^2 r + \ell^2 Q^2 = 0$. Solving this and the position of the black hole horizon [156] can be obtained as

$$r_h = r_{RNAdS} = \frac{1}{\alpha}\left(1 - \frac{4M^2}{\ell^2\alpha^2} - \cdots\right)\left(M + \sqrt{M^2 - Q^2\alpha}\right),$$

with $\alpha = \sqrt{1 - \frac{4Q^2}{\ell^2}}$. Since $\ell^2 >> Q^2$, therefore $\alpha \approx 1$ and therefore the above equation becomes as

$$r_h = r_{RNAdS} \approx \left(1 - \frac{4M^2}{\ell^2} + \cdots\right)M\left(1 + \sqrt{1 - \frac{Q^2}{M^2}}\right). \tag{7.3}$$

Since $\left(1 - \frac{4M^2}{\ell^2} + \cdots\right) < 1$, the radius of RNAdS black hole is smaller than RN black hole. For a negligible charge $Q \approx 0$ and the Eq. (7.3) give the first approximated position of horizon r_0 tends to SAdS black hole as

$$r_0 \approx 2M\left(1 - \frac{4M^2}{\ell^2} + \cdots\right) = 2M\left(1 - \frac{4M^2}{\ell^2}\right). \tag{7.4}$$

By using Eq. (7.4) into Eq. (7.2), we get

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)c^2dt^2 + \left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7.5}$$

where $\chi = 1 - \frac{4M^2}{\ell^2}$. Now let us consider a test particle of mass m orbiting along the circular geodesics in the equatorial plane $\theta = \pi/2$ around RNAdS black hole. Thus, the above Eq. (7.5) as described in [5, 144] can become in the following form

$$ds^2 = -\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)c^2dt^2 + \left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)^{-1}dr^2 + r^2d\theta^2. \tag{7.6}$$

7.3 Lagrangian and Canonical Momenta

Here we assume that the black hole mass is larger than the Planck mass so that the Compton radius $r_c = \hbar/mc << r_h$, where r_h is the radius of the RNAdS black hole. In this circumstances, the quantum fluctuations of the black hole is disregarded [126]. In terms of the metric components g_{ij}, the Lagrangian of the test particle can be written as

$$\begin{aligned}\pounds &= -\frac{m}{2}g_{ij}\dot{x}^i\dot{x}^j \\ &= -\frac{m}{2}\left[\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)c^2\dot{t}^2\right] \\ &-\frac{m}{2}\left[\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right].\end{aligned} \tag{7.7}$$

For static and spherically charged RNAdS black hole, there exist two constants of motion for the test particles, associated with two Killing vectors in terms of energy E, angular momentum L and the canonical momenta components p_r, p_θ can be written in the following form

$$\begin{aligned} E &= \frac{\partial \pounds}{\partial \dot{t}} = mc^2 \left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)\dot{t}, \\ L &= \frac{\partial \pounds}{\partial \dot{\phi}} = mr^2 \sin^2\theta\dot{\phi}. \end{aligned} \tag{7.8}$$

$$\begin{aligned} p_r &= \frac{\partial \pounds}{\partial \dot{r}} = -m \left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)^{-1}\dot{r}, \\ p_\theta &= \frac{\partial \pounds}{\partial \dot{\theta}} = -mr^2\dot{\theta}. \end{aligned} \tag{7.9}$$

7.4 Radial Motion and Effective Potential for RNAdS Line Element

The radial motion of a geodesic can be written as

$$g^{00}p_0^2 + g^{rr}p_r^2 + g^{\phi\phi}p_\phi^2 + g^{\theta\theta}p_\theta^2 + m^2c^2 = 0, \tag{7.10}$$

the four vectors ($p_0 = E/c, \mathbf{p}$) can be expressed in the magnitude of the energy-momentum. Inserting Eqs. (7.8), (7.9) and (7.6) into Eq. (7.10), we have

$$\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)^{-1}\left(-\frac{E^2}{c^2} + m^2\dot{r}^2\right) + \frac{L^2}{r^2\sin^2\theta} + \frac{m\dot{\theta}^2}{r^2} + m^2c^2 = 0. \tag{7.11}$$

Setting $\dot{\theta}^2 = 0$ and $\sin^2\theta = 1$, the above equation becomes as

$$\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)^{-1}\left(-\frac{E^2}{c^2} + m^2\dot{r}^2\right) + \frac{L^2}{r^2} + m^2c^2 = 0. \tag{7.12}$$

In Refs. [116, 117], the energy and momentum of the test particle per unit rest mass $\tilde{E} = \frac{E}{m}$ and $\tilde{L} = \frac{L}{m}$ are respectively and using these are into Eq. (7.12), we have

$$\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right)^{-1}\left(-\frac{\tilde{E}^2}{c^4} + \frac{\dot{r}^2}{c^2}\right) + \frac{\tilde{L}^2}{c^2r^2} + 1 = 0. \tag{7.13}$$

Simplifying the above equation, we get the radial motion of the test particle in the following form

$$\frac{\dot{r}^2}{2} = \frac{\tilde{E}^2}{2c^2} - \frac{1}{2}\left(\frac{\tilde{L}^2}{r^2} + c^2\right)\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right). \tag{7.14}$$

For the time like particle orbit the velocity is expressed in terms of two parameters energy and angular momentum. Thus, we have the effective potential V_{eff} for the radial motion from Eq. (7.14) of the following form

$$V_{eff} = \frac{1}{2}\left(\frac{\tilde{L}^2}{r^2} + c^2\right)\left(1 - \frac{2M}{r}\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)\right). \tag{7.15}$$

The radial acceleration of the test particle can be obtained by taking the derivative of Eq. (7.15) with respect to the proper time and then equating to zero, we obtain

$$\frac{c^2}{r^4}M\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)$$
$$\times\left(r^2-\frac{\tilde{L}^2}{c^2M\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)}r+\frac{3\tilde{L}^2}{c^2}\right)=0. \tag{7.16}$$

Now our prime task in this section is to find the radius of the circular orbit namely R. For this purpose, we can write the above equation in the following form

$$\left(R^2-\frac{\tilde{L}^2}{c^2M\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)}R+\frac{3\tilde{L}^2}{c^2}\right)=0. \tag{7.17}$$

Solve the above equation for the radius of the circular orbit R, we get the two roots in the form as

$$R_\pm=\frac{\tilde{L}^2}{2c^2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}$$
$$\times\left(1\pm\left(1-\frac{12c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{\tilde{L}^2}\right)^{\frac{1}{2}}\right). \tag{7.18}$$

We observe that the roots $R_\pm$ of the above equation is real only when $\tilde{L}^2\geq 12c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2$ and for the smallest stable orbit the square root on the right hand side of Eq. (7.18) dissolves. Therefore, we must have

$$\tilde{L}^2=12c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2. \tag{7.19}$$

For large and the largest stable circular orbits the conditions $\tilde{L}^2 \geq 12c^2M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2$ and $\tilde{L}^2 >> 12c^2M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2$ are holds respectively.

7.5 Quantization of Angular Momentum of the Test Particle

According to Wilson [129] and Sommerfeld [130] idea, angular momentum can be quantized as a periodic function of time and with the help of quantize energy for the reason that it is closely related to the quantize angular momentum of the orbiting test particle. In order to quantize angular momentum J_ϕ with the help of canonical momentum L conjugate to the angular variable in the form as

$$J_\phi = \int_0^{2\pi} L d\phi = nh. \tag{7.20}$$

Since L is a constant of motion, thus $\tilde{L} = \frac{L}{m}$ gives the quantization condition for the angular momentum in the following form

$$L = m\tilde{L} = n\hbar, \quad \text{so that} \quad \tilde{L}_0 = n_0\hbar/m. \tag{7.21}$$

With the help of Compton radius $r_c = \hbar/mc$ and Eq. (7.21), the Eq. (7.19) can be written as

$$n_0^2 r_c^2 = 12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2. \tag{7.22}$$

By using Eq. (7.22) into Eq. (7.18), we get the radius of the different stable circular orbits of the particle corresponds to n_0 in the following form

$$R_+ = \frac{n_0^2 r_c^2}{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)} \times \left(1 + \left[1 - \frac{12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{r_c^2 n_0^2}\right]^{\frac{1}{2}}\right). \tag{7.23}$$

Inserting Eq. (7.22) into Eq. (7.23) we get the approximate radius of the first circular orbit R_0 in the form of

$$R_0 \approx \frac{n_0^2 r_c^2}{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}. \tag{7.24}$$

In the limiting case when $Q \to 0$ and $\ell \to \infty$ the above equation reduces to $R_0 = n_0^2 \frac{r_c^2}{r_s}$ which agree with the result given in Ref. [116], where $r_s = 2M$ represents the Schwarzschild radius. The position of the next higher circular orbit R_1 can be obtained from Eq. (7.23) by replacing n_0 with $n_1 = n_0 + 1$ in the form of

$$R_1 = \frac{(n_0+1)^2 r_c^2}{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)} \times \left(1 + \left[1 - \frac{12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{(n_0+1)^2 r_c^2}\right]^{\frac{1}{2}}\right). \tag{7.25}$$

Now $n_0 >> 1$ while $2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right) >> r_c$ and therefore we can write

$$(n_0+1)^2 \approx n_0^2\left[1 + \frac{2}{n_0}\right]. \tag{7.26}$$

Using Eqs. (7.26) and (7.22) into Eq. (7.25) we get

$$R_1 = R_0 \left(1 + \frac{2}{n_0}\right)\left(1 + \sqrt{\frac{2}{n_0}}\right). \tag{7.27}$$

Thus the next higher stage n_2^2 defined as

$$n_2^2 \approx n_0^2 \left[1 + \frac{4}{n_0}\right]. \tag{7.28}$$

Progressing in this way, the radius of the next higher circular orbit R_2 of the test particle with the help of Eq. (7.28) can be obtained from Eq. (7.23) in the form of

$$\begin{aligned} R_2 &= \frac{n_0^2(1+\frac{4}{n_0})r_c^2}{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)} \left(1 + \left[1 - \frac{12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{n_0^2(1+\frac{4}{n_0})r_c^2}\right]^{\frac{1}{2}}\right) \\ &= R_0\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right) = R_1 \frac{\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right)}{\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right)}. \end{aligned} \tag{7.29}$$

In the similar manner, the $(n+1)$th radius of the stable circular orbits of the particle can be written as

$$R_{n+1} = R_n \frac{\left(1+\frac{2n+2}{n_0}\right)\left(1+\sqrt{\frac{2n+2}{n_0}}\right)}{\left(1+\frac{2n}{n_0}\right)\left(1+\sqrt{\frac{2n}{n_0}}\right)}. \tag{7.30}$$

Here we have $R_{n+1} = R_n$, when $n_0 \to \infty$. For large quantum number we can say that the two nearby states coincide for RNAdS black hole.

7.6 Energy Quantization and Hawking Temperature

With the help of angular momentum we want to quantize the energy of the orbiting test particle in this section. For zero velocity at $r = R$ the

Eq. (7.14) gives as

$$\tilde{E}^2 = c^2\left(\frac{\tilde{L}^2}{R^2} + c^2\right)\left(1 - \frac{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}{R}\right). \tag{7.31}$$

Using Eq. (7.16) for $r = R$ into the Eq. (7.31) we get

$$E = mc^2\left(1 - \frac{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}{R}\right) \times \left(1 - \frac{3M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}{R}\right)^{-1/2}. \tag{7.32}$$

With the help of mechanical stability condition $\tilde{L}^2 = MGR$ in Eq. (7.18) and neglecting higher terms of Eq. (7.32) and therefore, Eq. (7.32) can be written as

$$E \approx mc^2\left(1 - \frac{M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}{2R}\right). \tag{7.33}$$

Using Eq. (7.21) and Eq.(7.24) into the above equation we have

$$E \approx mc^2\left(1 - \frac{c^2M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{\tilde{L}^2}\right). \tag{7.34}$$

The quantized energy E_n for nth energy label can be written as

$$E_n \approx mc^2\left(1 - \frac{c^2M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{\tilde{L}_n^2}\right). \tag{7.35}$$

The above equation can be written with the help of Eq. (7.21) and $\hbar = r_c mc$ as

$$E_n \approx mc^2 \left(1 - \frac{M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{n^2 r_c^2}\right). \tag{7.36}$$

From the Eq. (7.36), we obtain the corresponding $(n+1)$th label energy in the form of

$$E_{n+1} \approx mc^2 \left[1 - \frac{M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{r_c^2 (n+1)^2}\right]. \tag{7.37}$$

Thus, the quantized energy difference between two nearby states in the form of

$$\begin{aligned} \delta E &= E_{n+1} - E_n \\ &\approx \frac{mc^2}{r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times M^2 \left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2. \end{aligned} \tag{7.38}$$

Neglecting the 4th and higher powers of (M/ℓ) of the RNAdS black hole radius given in Eq. (7.3) then we have

$$r_{RNAdS} \approx 2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right). \tag{7.39}$$

When $\ell \to \infty$, we observe that the radius of the RNAdS black hole approaches to the radius of the RNdS black hole and which has been developed in [118]. For RNAdS black hole, the Eq. (7.38) can be written as

$$\delta E \approx \frac{mc^2 r_{RNAdS}^2}{4r_c^2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right). \tag{7.40}$$

For large values of n, the first parenthesis of Eq. (7.40) can be replaced by $2/n^3$ so that

$$\begin{aligned}\delta E &\approx \frac{c^4 m^3 r_{RNAdS}^2}{2\hbar^2 n^3} \\ &= \frac{c^4 m^3}{2\hbar^2 n^3} \times 4M^2 \left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2 .\end{aligned} \tag{7.41}$$

We observe that $\delta E \to 0$ as $n \to \infty$ and which indicates for a larger circular orbit the change of quantized energy between two nearby states approaches to zero.

Therefore, the Bekenstein-Hawking Entropy for RNAdS black hole with the help of Eq. (7.40) can be written as

$$\begin{aligned}\delta S &= \frac{\delta E}{T_H} \\ &= \frac{mc^2 r_{RNAdS}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \\ &= \frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2 \left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2 ,\end{aligned} \tag{7.42}$$

where T_H is the Hawking temperature of RNAdS black hole. According to the WKB approximation, the emission rate for an out going particle with positive energy E coming from inside to outside which is related to the imaginary part of the particles action as

$$\Gamma \sim exp(-2Im(I)), \tag{7.43}$$

where $Im(I) = -\frac{1}{2}[S(M-E) - S(M)] = \frac{1}{2}\delta S$. For the RNAdS black hole the thermal emission rate and the change of entropy satisfies

$$\Gamma \sim \exp(\delta S) = \exp\left(\frac{mc^2 r_{RNAdS}^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right]\right). \tag{7.44}$$

Putting the value of r_{RNAdS} into the above equation, we get

$$\begin{aligned}\Gamma &\sim \exp(\delta S) = \exp\left(\frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times r_{RNAdS}^2\right) \\ &= \exp\left(\frac{mc^2}{4T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times 4M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2\right).\end{aligned} \tag{7.45}$$

The Hawking temperature of RNAdS black hole is defined by

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}. \tag{7.46}$$

When $\ell \to \infty$ we get $\chi = 1$ so that

$$T_H = \frac{1}{8\pi M\left(1 - \frac{Q^2}{4M^2}\right)}, \tag{7.47}$$

which is same as RN black hole. When $Q = 0$, Eq. (7.46) reduced to

$$T_H = \frac{1}{8\pi M\left(1 - \frac{4M^2\left(1-\frac{4M^2}{\ell^2}\right)^3}{\ell^2}\right)} \approx \frac{1}{8\pi M\left(1 - \frac{4M^2}{\ell^2}\right)}, \tag{7.48}$$

which is similar as SAdS black hole. In the absence of cosmological parameter Eq. (7.48) agree with the result of Schwarzschild black hole as $T_H = \frac{1}{8\pi M}$.

7.7 Concluding Remarks

We observe that when evaporation processes take place in spacetime, RNAdS black hole loses its mass and charge so that $M \to 0$ and $Q \to 0$.

Therefore, Eq. (7.46) gives $T_H \to \infty$ so that the entropy and the thermal emission rate vanish and agree with the results obtained by Sakalli et al.[131] and like Bohr's quantum theory for quantization of black hole in He et al.[132]. On the other hand, for a large black hole i.e., $n \to \infty$ the Eq. (7.45) gives the change of entropy between two nearby states and the thermal emission rate vanishes for RNAdS black hole. Our result for RNAdS black hole reduced to Schwarzschild black hole in chapter 2 when $\ell \to \infty$ and $Q \to 0$. If we put $-\ell^2$ in the place of ℓ^2 and $Q \to 0$ and then the results of RNAdS black hole coincide of the results of SdS black hole [117] in chapter 3. When the charge $Q \to 0$, the results coincide with the results of SAdS black hole in chapter 4. Again when $\ell \to \infty$ then the results RNAdS black hole is same as the results of RN black hole [118] in chapter 5. If we replace ℓ^2 by $-\ell^2$ then the results coincide with the results of RNdS black hole in chapter 6.

Chapter 8

Conclusion

In this thesis, we have studied Hawking radiation of different non-rotating black holes in asymptotically-de Sitter and Anti-de Sitter spaces. We have considered the test particle orbiting different circular orbits around a black hole in analogy with Bohr's atomic model and have successfully applied for non-rotating black holes mentioned above. We have developed quantization formula of different energy label according to the Wilson [129] and Sommerfeld [130] openion from conservation of energy and angular momentum of the particle. The quantized formula of the energy labels for each black holes takes the following form

$$R_{n+1} = R_n \frac{\left(1 + \frac{2n+2}{n_0}\right)\left(1 + \sqrt{\frac{2n+2}{n_0}}\right)}{\left(1 + \frac{2n}{n_0}\right)\left(1 + \sqrt{\frac{2n}{n_0}}\right)}.$$

This means that all the non-rotating black holes have same discrete energy states. The position of these energy labels are independent of the charge of the black hole and cosmological constant either positive or negative which is very interesting. If the gravitational mass is large, the first label index, $n_0 \to \infty$ yields $R_{n+1} = R_n$. Therefore, we may conclude that for large quantum number all the energy states coincide. But the difference of energy between two nearby estates depends on both cosmological constant and charge of the black holes.

We have developed a new formula for the Bekenstein-Hawking entropy using classical thermodynamics equation $dS(E) = dE/T(E)$ for a constant temperature T_H called Hawking temperature as

$$\delta S = \frac{mc^2}{4\pi T_H r_c^2}\left[\frac{1}{n^2} - \frac{1}{(n+1)^2}\right] \times (\text{radius of a black hole})^2.$$

Where the Hawking temperature is defined as $T_H = \frac{\kappa}{2\pi}$, κ being the surface gravity of black hole. For a large black hole $n \to \infty$, which means that the change of entropy becomes zero. Also, the temperature approaches to zero for this situation. If the black hole continuously losses it mass due to radiation and becomes a tinny one its radius will approach to zero but $T_H \to \infty$. In this situation, the entropy δS becomes zero and the radiation process will be stopped. This means that a mini black hole can only be emitted or absorbed particles with finite energy and definite quantum numbers like the atomic spectrum of quantum mechanism.

Since the tunneling rate is related to the change of entropy, the tunneling rate becomes zero when the change of entropy between two energy states becomes zero. The present research shows that the different energy labels of black hole in the nature can be performed in the same way as that for the electron occupy different energy labels outside the atom like quantum theory which is interesting same as [132]. The entropy framework given in this work indeed support the new perspectives on quantum properties of gravity given in Refs. [68, 69] beyond classical physics, however, suggests a new idea to unify gravity with quantum theory.

The results we have found in the chapter 7 for the RNAdS black hole, reduce to the RN black hole [118] for $\ell \to \infty$ (chapter 5), the Schwarzschild black hole for $\ell \to \infty$ and $Q = 0$ (chapter 2), the Schwarzschild Anti-de Sitter black hole for $M = Q = 0$ (chapter 4), the Schwarzschild-de Sitter black hole [117] for $M = Q = 0$ and replacing ℓ^2 by $-\ell^2$ (chapter 3), and finally for the Reissner-Nordström-de Sitter spacetime with ℓ^2 instead of $-\ell^2$ (chapter 6). Moreover the results of the RNdS black hole reduce to the results for the lukewarm black hole when $Q = M$ [157]. So our present research work is thus well motivated.

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