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A Study on Standard n-Ideals of a Lattice

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University of Rajshahi

THE UNIVERSITY OF RAJSHAHI

BANGLADESH

A STUDY ON STANDARD n-IDEALS OF A LATTICE

A thesis

Presented for the degree of Master of philosophy

BY

ABU SADAT SYEED AHMED

B.Sc. Hons. (R.U); M.Sc.(Pure Math.)(Rajshahi University)

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In the

Department of Mathematics

University of Rajshahi,

Rajshahi, Bangladesh.

June, 2014.

Dedicated

To

My Beloved Parents and Brother

Declaration

This thesis does not incorporate without acknowledgement of any materials previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain any materials previously published or written by another person expect where due reference is made in the text.

Professor Md. Abdul Latif
Supervisor

সুপারভাইজার,
এম.ফিল ও পিএইচ.ডি
গণিত বিভাগ
রাজশাহী বিশ্ববিদ্যালয়

STATEMENT OF ORIGINALITY

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Abu Sadat Syeed Ahmed.

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The Author

SUMMARY

This thesis studies the nature of standard n-ideals of a lattice. The idea of n-ideals in a lattice was first introduced by Cornish and Noor. For a fixed element n of a lattice L, a convex sublattice containing n is called an n-ideal. If L has a ‘0’, then replacing n by 0, an n-ideal becomes an ideal. Moreover if L has 1, an n-ideal becomes a filter by replacing n by 1. Thus, the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. So any result involving n-ideals will give a generalization of the results on ideals and filters with 0 and 1 respectively in a lattice. In this thesis we give a series of results on n-ideals of a lattice which certainly extend and generalize many works in lattice theory.

Chapter-1, discusses n-ideals, finitely generated n-ideals and other results on n-ideals of a lattice which are basic to this thesis. We have shown that, a lattice L is modular (distributive) if and only if $I_n(L)$, the lattice of n-ideals is modular (distributive).

In chapter-2, we have discussed lattices and elements with special properties. Here we have proved the coincidence of standard and neutral elements in a wide class of lattices including modular lattices, weakly modular lattices as well as relatively complemented lattices. In modular lattices and relatively complemented lattices the proves of the results are trivial but in weakly modular lattices this prove is not so simple. In this chapter, we have proved the following results:

(i) In a weakly modular lattices L, an element d is distributive if and only if it is neutral.
(ii) Let a,b,c be neutral elements of a lattice L, an $a<b<c$ if d is relative complement of b in the interval [a,c], then it is also neutral and uniquely determined.

(iii) The lattice of all n-ideals of a weakly modular lattice is not necessarily weakly modular.

(iv) Given the n-ideal I of the lattice L and a covering system $\bar{I}$ of I and the lattice polynomials f_a, g_a ($a \in A$). If every element of $\bar{I}$ is of the type $f_a = g_a$ $(a \in A)$, then I as an element of $I_n(L)$ is of the type $f_a = g_a$ $(a \in A)$.

In chapter-3, we have given some definitions of standard elements and standard n-ideals. We have proved the fundamental characterization theorems of standard elements and standard n-ideals. Also we have deduced some important properties of standard elements and standard n-ideals. Then we have given some notions and notations of standard n-ideals which is more general than that of neutral n-ideals. We have given some basic concept of congruence relation of lattices. Here we have given The First General Isomorphism Theorem and The Second General Isomorphism Theorem.

In chapter-4, we discuss on standard n-ideal of a lattice. Standard elements and ideals have been studied by many authors including Grätzer. From an open problem given by him, Fried and Schmidt have extended the idea to standard (convex) sublattices. In the light of their work we have developed the notion of standard n-ideals and showed that an n-ideal is standard if and only if it is a standard sublattice. We have also given a characterization of a standard n-ideal S interms of the congruence $\Theta(S)$. Then we have proved the following results:-

(i) For a neutral element n, the principal n-ideal $<a>_n$ of a lattice L is a standard n-ideal if and only if $a \vee n$ is standard and $a \wedge n$ is dual standard.

(ii) Let I be an arbitrary n-ideal and S be a standard n-ideal of a lattice L, where n is neutral. If $I \vee S$ and $I \cap S$ are principal n-ideals, then I itself is a principal n-ideal.

(iii) Let n be neutral element of a lattice L. Let S and T be two standard n-ideals of L. Then

(i) $\Theta(S \cap T) = \Theta(S) \cap \Theta(T)$

(ii) $\Theta(S \vee T) = \Theta(S) \vee \Theta(T)$

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CHAPTER-1

"Basic concept of n-ideals of a lattice"

Introduction: The idea of n-ideals in a lattice was first introduced by Cornish and Noor in several papers [3],[14],[15]. Let L be a lattice and $n \in L$ is a fixed element, a convex sublattice containing n is called an n-ideal. If L has a "0", then replacing n by "0" an n-ideal becomes an ideal. Moreover if L has 1, an n-ideal becomes a filter by replacing n by 1. Thus, the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. So any result involving n-ideals will give a generalization of the results on ideals and filters with 0 and 1 respectively in a lattice.

The set of all n-ideals of L is denoted by $I_n(L)$ which is an algebraic lattice under set-inclusion. Moreover, $\{n\}$ and L are respectively the smallest and largest elements of $I_n(L)$ while the set-theoretic intersection is the infimum.

For any two n-ideals I and J of L, we have,

$$I \wedge J = \{x : x = m(i, n, j) \text{ for some } i \in I,\ j \in J\},$$

$$\text{where } m(x,y,z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$$

and $I \vee J = \{x : i_1 \wedge j_1 \leqq x < i_2 \vee j_2,$

for some $i_1, i_2 \in I$ and $j_1, j_2 \in J\}$.

The n-ideal generated by $a_1, a_2, a_3 \ldots \ldots \ldots a_m$ is denoted by $< a_1, a_2, a_3 \ldots \ldots \ldots a_m > n$.

Clearly $< a_1, a_2, a_3 \ldots \ldots \ldots a_m >_n = < a_1 >_n \vee \ldots \vee < a_m >_n$.

The n-ideal generated by a finite number of elements is called a finitely generated n-ideal. The set of all finitely generated n-ideals is denoted by $F_n(L)$. Of course $F_n(L)$ is a lattice. The n-ideal generated by a single element is called a principal n-ideal. The set of all principal n-ideals of L is denoted by $P_n(L)$. We have

$$< a >_n \quad \{x \in L : a \wedge n < x < a \vee n\}$$

The median operation

$m(x,y,z) = (x\wedge y)\vee(y\wedge z)\vee(z\wedge x)$ is very well known in lattice theory. This has been used by several authors including Birkhoff and Kiss [1] for bounded distributive lattices, Jakubik and Kalibiar [12] for distributive lattices and Sholander [18] for median algebra.

An n-ideal P of a lattice L is called prime if $m(x,n,y)\in P$; $x,y\in L$ implies either $x\in P$ or $y\in P$.

Standard and neutral elements in a lattice were studied extensively in [11] and [9, chapter-3]. An element s of a lattice L is called standard if for all

$x,y\in L$, $x\wedge(y\vee s) = (x\wedge y)\vee(x\wedge s)$. An element $n\in L$ is called neutral if it is standard and for all $x,y\in L$, $n\wedge(x\vee y)=(n\wedge x)\vee(n\wedge y)$. Of course 0 and 1 of a lattice are always neutral. An element $n\in L$ is called central if it is neutral and complemented in each interval containing n. A lattice L with 0 is called sectionally complemented for all $x\in L$. A distributive lattice with

0, which is sectionally complemented is called a generalized boolean lattice. For the background material we refer the reader to the texts of G. Grätzer [8], Birkhoff [02] and Rutherford [17].

In section 1, we have given some fundamental results on finitely generated n-ideals. We have shown that for a neutral element n of a lattice L, $P_n(L)$ is a lattice if and only if n is central. We have also shown that for a neutral element n, a lattice L is modular (distributive) if and only if $I_n(L)$ is modular (distributive). We proved that, in a distributive lattice L, if both supremum and infimum of two n-ideals are principal, then each of them is principal.

In section 2, we have studied the prime n-ideals of a lattice. Here we have generalized the separation property for distributive lattices given by M. H. Stone [8, Th. 15, p-74] in terms of prime n-ideals. Then we showed that in a distributive lattice, every n-ideal is the intersection of prime n-ideals containing it.

1. Finitely generated n-ideals.

1.1.1 We start this section with the following proposition which gives some descriptions of $F_n(L)$.

1.1.2 Proposition: Let L be a lattice and $n \in L$. For $a_1, a_2, \ldots\ldots, a_m \in L$,

(i) $<a_1, a_2, a_3 \ldots \ldots \ldots a_m>n \subseteq \{y \in L: (a_1] \wedge (a_2] \wedge \ldots\ldots \wedge (a_m] \wedge (n] \subseteq (y] \subseteq (a_1] \vee (a_2] \ldots\ldots \vee (a_m] \vee (n]\}$

(ii) $<a_1, a_2, a_3 \ldots \ldots \ldots a_m>n = \{y \in L: a_1 \wedge a_2 \wedge a_3 \wedge \ldots \ldots \wedge a_m \wedge n < y < a_1 \vee a_2 \vee \ldots \ldots \vee a_m \vee n\}$.

(iii) $<a_1, a_2, a_3 \ldots \ldots \ldots a_m>n = \{y \in L: a_1 \wedge a_2 \wedge \ldots \ldots \wedge a_m \wedge n \leq y = (y \wedge a_1) \vee (y \wedge a_2) \vee \ldots \ldots \vee (y \wedge a_m) \vee (y \wedge n)$,

when L is distributive.

(iv) For any $a \in L$,

$$<a>n = \{y \in L: a \underline{\wedge n \leq} y = (y \wedge a) \vee (y \wedge n)\}$$

$$=\{y\epsilon L: y=(y\wedge a)\vee(y\wedge n)\vee(a\wedge n)\}$$

whenever n is standard.

(v) Each finitely generated n-ideal is two generated.

Indeed $<a_1, a_2, a_3\ldots\ \ldots\ \ldots\ a_m>n$ $<a_1\wedge a_2\wedge\ldots\ \ldots\wedge a_m\wedge n, a_1\vee\ldots\ \ldots\vee a_m\vee n>n$.

(vi) $F_n(L)$ is a lattice and its members are simply the intervals [a,b] such that $a<n<b$ and for each intervals

$$[a,b]\vee[a_1,b_1]=[a\wedge a_1,b\vee b_1]$$

and $$[a,b]\wedge[a_1,b_1]=[a\vee a_1,b\wedge b_1].$$

Proof: (i) Right hand side is clearly an n-ideal containing $a_1, a_2, a_3\ldots\ \ldots\ \ldots\ a_m$.

(ii) This clearly follows from (i) and by the convexity of n-ideals.

(iii) When L is distributive, then by (ii) $y<a_1\vee a_2\vee\ldots\ \ldots\vee a_m\vee n$ implice that

$y = y\wedge[\, a_1 \vee a_2 \vee \ldots \ \ldots \vee a_m \vee n] = (y\wedge a_1)\vee(y\wedge a_2)\vee \ldots$

$\ldots\vee(y\wedge a_m)\vee(y\wedge n)$, and (iii) follows.

(iv) By (ii) $<a>_n = \{y\epsilon\ L : a\wedge n<y<a\vee n\}$.

Then $y=y\wedge(a\vee n)=(y\wedge a)\vee(y\wedge n)$, when n is standard. This proves (iv)

(v) This clearly follows from (ii)

(vi) First part is readily verifiable. For the second part, consider the intervals $[a,b]$ and $[a_1,b_1]$ where $a\leq n\leq b$, and $a_1<n<b_1$.

Then using (ii)we have,$[a,b]\vee[a_1,b_1]=<a,a_1,b,b_1>n$

$=[a\wedge a_1\wedge b\wedge b_1\wedge n, a\vee a_1\vee b\vee b_1\vee n]$

$= [a\wedge a_1,\ b\vee b_1]$, while

$[a,b]\wedge[a_1,b_1] = [a\vee a_1, b\wedge b_1]$ is trivial.

In general, the set of principal n-ideals $P_n(L)$ is not necessarily a lattice. The case is different when n is a central element. The following theorem also gives a characterization of central element of a lattice L.

1.1.3 Theorem: Let n be a neutral element of a lattice L. Then $P_n(L)$ is a lattice if and only if n is central.

Proof: Suppose n is central. Let $<a>n, <b>n \epsilon P_n(L)$. Then using neutrality of n and proposition-1.1.2(vi),

$$<a>n \wedge <b>n \quad [a\wedge n, a\vee n]\wedge[b\wedge n, b\vee n]$$

$$=[(a\vee b)\wedge n, (a\wedge b)\vee n]$$

And $\quad <a>n \vee <b>n = [a\wedge b\wedge n, a\vee b\vee n]$.

Since n is central, there exist c and d such that

$$c\wedge n=(a\vee b)\wedge n,\ c\vee n=(a\wedge b)\vee n$$

and $\quad d\wedge n = a\wedge b\wedge n, \quad d\vee n = a\vee b\vee n$.

Which implices that $<a>n \wedge <b>n = <c>n$ and $<a>n \vee <b>n = <d>n$ and so $P_n(L)$ is a lattice.

Conversely, suppose that $P_n(L)$ is a lattice and $a \le n \le b$. Then $[a,b] = <a>n \vee <b>n$ .Since $P_n(L)$ is a lattice, $<a>n \vee <b>n = <c>n$ for some $c \epsilon L$.

This implies that c is the relative complement of n in [a,b]. Therefore n is central.■

Now, we like to discuss $F_n(L)$ when it is sectionally complemented.

1.1.4 Theorem: Let L be a lattice. Then $F_n(L)$ is sectionally complemented if and only if for each $a,b \in L$, with $a \leq n \leq b$, the intervals [a,n] and [n,b] are complemented.

Proof: Suppose $F_n(L)$ is sectionally complemented. Consider $a \leq c \leq n$ and $n \leq d \leq b$. Then $\langle n \rangle \subseteq [c,d] \subseteq [a,b]$. Since $F_n(L)$ is sectionally complemented, so there exists $[c', d']$ such that $[c,d] \wedge [c', d'] = \langle n \rangle$ and $[c,d] \vee [c', d'] = [a,b]$. This implies $c \vee c' = n$, $c \wedge c' = a$ and $d \wedge d' = n$, $d \vee d' = b$. That is c' is the relative complement of c in [a,n] and d' is the relative complement of d in [n,b]. Hence [a,n] and [n, b] are complemented for all $a,b \in L$ with $a \leq n \leq b$.

Conversely, suppose that [a,n] and [n, b] are complemented for all $a,b \in L$ with $a \leq n \leq b$. Consider

$<n> \subseteq [c,d] \subseteq [a,b]$. Then $a \leq c \leq n \leq d \leq b$. since $[a,n]$ and $[n, b]$ are complemented so there exist c' and d' such that $c \vee c' = n$, $c \wedge c' = a$ and $d \wedge d' = n$, $d \vee d' = b$. Thus $[c,d] \wedge [c',d'] = [c \vee c', d \wedge d'] = [n,n] = <n>$ and $[c,d] \vee [c',d'] = [c \wedge c', d \vee d'] = [a,b]$, which implies that $[c,d]$ has a relative complement $[c', d']$. Hence $F_n(L)$ is sectionally complemented.■

We have the following corollaries:

1.1.5 Corollary: For a distributive lattice L, $F_n(L)$ is generalized boolean if only if $[a,n]$ and $[n,b]$ are complemented for each $a,b \in L$ with $\underline{a<n<b}$.

1.1.6 Corollary: For a distributive lattice L, $F_n(L)$ is generalized boolean if only if both $(n]^d$ and $[n)$ are generalized boolean where $(n]^d$ denotes the dual of the lattice $(n]$

In lattice theory, it is well known that a lattice L is modular (distributive) if and only if the lattice of ideals I(L) is modular (distributive). Our following theorems are nice generalizations of this results in terms of n-ideals when n is a neutral element. The following Lemma is needed for the next theorem, which is due to Grätzer [10].

1.1.7 Lemma: An element n of a lattice L is neutral if and only if $m(x,n,y)=(x\wedge y)\vee(x\wedge n)\vee(y\wedge n)$

$$=(x\vee y)\wedge(x\vee n)\wedge(y\vee n).$$

1.1.8 Theorem: Let L be a lattice with neutral element n. Then L is modular if and only if $I_n(L)$ is modular.

Proof: First assume that L is modular. Let $I,J,K\in I_n(L)$ with $\underline{K\subset I}$. Obviously,

$(I\wedge J)\underline{\vee K\subseteq} I\wedge(J\vee K)$.

To prove the reverse inequality, let $x\in I\wedge(J\vee k)$. Then $x\in I$ and $x\in j\vee k$. Then $j_1\wedge k_1<x<j_2\vee k_2$ for some $j_1,j_2\in J$, $k_1,k_2\in K$. Since $\underline{I\supset K}$ so $x\wedge k_1\in I$ and $x\vee k_2\in I$.

Then by Lemma 1.1.7

$m(x\wedge k_1,n,j_1)\wedge k_1=k_1\wedge[((x\wedge k_1)\vee n)\wedge(n\vee j_1)\wedge((x\wedge k_1)\vee j_1)]$

$=[(x\wedge k_1)\vee n]\wedge(n\vee j_1)\wedge[(x\wedge k_1)\vee(k_1\wedge j_1)]$,

as L is modular.

$\leq x$, as $j_1\wedge k_1\leq x$

On the other hand

$m(x\vee k_2,n,j_2)\vee k_2=$

$\{[(x\vee k_2)\wedge n]\vee(n\wedge j_2)\vee[(x\vee k_2)\wedge j_2]\}\vee k_2$,

$=[(x\vee k_2)\wedge n]\vee(n\wedge j_2)\vee[(x\vee k_2)\wedge (k_2\vee j_2)]$,

as L is modular.

$>x$ as $j_2\vee k_2>x$

So we have

$m(x\wedge k_1,n,j_1)\wedge k_1<x<m(x\vee k_2\ ,n,j_2)\vee k_2$

Hence $x\epsilon(I\wedge J)\vee k$.

Therefore

$I\wedge(J\vee K)=(I\wedge J)\vee K$ with $\underline{k\subseteq I}$ and so $I_n(L)$ is modular.

Conversely, suppose that $I_n(L)$ is modular.

Then for any $a,b,c \in L$ with $c \leq a$, consider the n-ideals $<a\vee n>_n$, $<b\vee n>_n$ and $<c\vee n>_n$. Then of course $<c\vee n>_n \subset <a\vee n>_n$. Since $I_n(L)$ is modular, So

$$<a\vee n>_n \wedge [<b\vee n>_n \vee <c\vee n>_n]$$

$$=[<a\vee n>_n \wedge <b\vee n>_n] \vee <c\vee n>_n.$$

Then by proposition 1.1.2 (vi) and by neutrality of n, it is easy to show that

$$[a\wedge(b\vee c)]\vee n = [(a\wedge b)\vee c]\vee n \qquad (A)$$

Again, consider the n-ideals $<a\wedge n>_n$, $<b\wedge n>_n$ and $<c\wedge n>_n$, $\underline{c\leq a}$ implies $<a\wedge n>_n \subset <c\wedge n>_n$. Then using modularity of $I_n(L)$, we have

$$<a\wedge n>_n \vee (<b\wedge n>_n \wedge <c\wedge n>_n)$$

$$=(<a\wedge n>_n \vee <b\wedge n>_n) \wedge <c\wedge n>_n.$$

Then using proposition 1.1.2 (vi) again and the neutrality of n, it is easy to see that

$$[a\wedge(b\vee c)]\wedge n \quad [(a\wedge b)\vee c]\wedge n \;\ldots\;\ldots\; (B)$$

From (A) & (B) we have $a\wedge(b\vee c)=(a\wedge b)\vee c$, with $c \leq a$, as n is neutral. Therefore L is modular.■

From the proof of above theorem, it can be easily seen that the following corollary holds which is an improvement of the theorem.

1.1.9 Corollary: For a neutral element n of a lattice L, the following conditions are equivalent:-

(i) L is modular,

(ii) $I_n(L)$ is modular ,

(iii) $F_n(L)$ is modular.

We have the following theorem;

1.1.10 Theorem: Let L be a lattice with neutral element n. Then L is distributive if and only if $I_n(L)$ is distributive.

Proof: First assume that L is distributive. Let I, J, K ϵ $I_n(L)$. Then obviously,

$(I\wedge J)\vee(I\wedge K)\subseteq I\wedge(J\vee K)$. To prove the reverse inequality, let $x \in I\wedge(J\vee K)$ which implies $x \in I$ and $x\in J\vee K$. Then $j_1\wedge k_1\leq x\leq j_2\vee k_2$ for some $j_1,j_2 \in J$, $k_1,k_2 \in K$. Since L is distributive,

$$m(x,n,j_1)\wedge m(x,n,k_1)= [(x\wedge n)\vee(x\wedge j_1)\vee(n\wedge j_1)] \wedge [(x\wedge n)\vee(x\wedge k_1)\vee(n\wedge k_1)]$$

$$= (x\wedge n)\vee(n\wedge j_1\wedge k_1)\vee(x\wedge j_1\wedge k_1)$$

$$\leq x\vee(j_1\wedge k_1)=x$$

Also, $m(x,n,j_2)\vee m(x,n,k_2) = [(x\wedge n)\vee(x\wedge j_2)\vee(n\wedge j_2)] \vee [(x\wedge n)\vee(x\wedge k_2)\vee(n\wedge k_2)]$

$$= (n\wedge(x\vee j_2\vee k_2)) \vee (x\wedge(j_2\vee k_2)),$$

$$=[n\wedge(j_2\vee k_2)] \vee x \geq x$$

Then we have

$$m(x,n,j_1) \wedge m(x,n,k_1) \leq x \leq m(x,n,j_2) \vee m(x,n,k_2)$$

and so $x \in (I\wedge J) \vee (I\wedge K)$.

Therefore $I\wedge(J\vee K) = (I\wedge J) \vee (I\wedge K)$, and so $I_n(L)$ is distributive.

The converse follows form the proof of above theorem.■

Following corollary immediately follows from the above proof which is also an improvement of the above theorem.

1.1.11 Corollary: Let L be a lattice with a neutral element n. Then the following conditions are equivalent:

(i) L is distributive,

(ii) $I_n(L)$ is distributive,

(iii) $F_n(L)$ is distributive.

We conclude this section with a nice generalization of [8: Lemma-5, P-71]. To prove this we need the following lemma:

1.1.12 Lemma: Let L be a distributive lattice. Then, any finitely generated n-ideal which is contained in a principal n-ideal is principal.

Proof: Let $[b,c]$ be a finitely generated n-ideal such that $b<n<c$. Let $<a>_n$ be a principal n-ideal such that $[b,c]\subseteq<a>_n = [a\wedge n, a\vee n]$.

Then $a\wedge n<b<n<c<a\vee n$. Suppose $t \quad (a\wedge c)\vee b$.

Then

$$t\wedge n \quad [(a\wedge c)\vee b] \wedge n =(n\wedge a\wedge c) \vee (n\wedge b),$$

as L is distributive.

$$=b\wedge n=b$$

and $\quad t\vee n =[(a\wedge c) \vee b] \vee n= (a\wedge c)\vee n$

$=(a\vee n)\wedge(c\vee n)$, as L is distributive.

$=c\vee n=c$

Hence $[b,c] =[t\wedge n, t\vee n]=<t>_n$.

Therefore, $[b,c]$ is a principal n-ideal.■

1.1.13 Theorem: Let I and J be n-ideals of a distributive lattice L. If $I \vee J$ and $I \wedge J$ are principal n-ideals, then I and J are also principal.

Proof: Let,b $I \vee J = \langle a \rangle_n$ and $I \wedge J = \langle b \rangle_n$. Then for all $i \epsilon I$, $j \epsilon J$, $I, j \leq a \vee n$ and $I, j \geq a \wedge n$.

So there exist $i_1, i_2 \epsilon I$ and $j_1, j_2 \epsilon J$ such that $a \wedge n = i_1 \wedge j_1$ and $a \vee n = i_2 \vee j_2$.

Consider the n-ideal $[b \wedge i_1 \wedge n, b \vee i_2 \vee n]$. Since
$[b \wedge i_1 \wedge n, b \vee i_2 \vee n] \subset I \subseteq \langle a \rangle_n$,
$[b \wedge i_1 \wedge n, b \vee i_2 \vee n] = \langle t \rangle_n$, by lemma 1.1.12for some $t \epsilon L$. Then

$$\langle a \rangle_n = J \vee I \supset J \vee [b \wedge i_1 \wedge n, b \vee i_2 \vee n]$$

$$\supset [j_1 \wedge n, j_2 \vee n] \vee [b \wedge i_1 \wedge n, b \vee i_2 \vee n]$$

$$[j_1 \wedge n \wedge b \wedge i_1, j_2 \vee n \vee b \vee i_2]$$

$$\supset [a \wedge n, a \vee n] \quad \langle a \rangle_n.$$

This implies that

$$I \vee J = J \vee [b \wedge i_1 \wedge n, b \vee i_2 \vee n] = J \vee \langle t \rangle_n$$

Further,

$$\langle b \rangle_n = J \wedge I \supset J \wedge [b \wedge i_1 \wedge n, b \vee i_2 \vee n]$$

$$\supset J\wedge [b\wedge n,\ b\vee n] = \langle b\rangle_n$$

Which implies that

$$J\wedge I = J\wedge[b\wedge i_1\wedge n,\ b\vee i_2\vee n]$$

$$= J\langle t\rangle_n.$$

Since L is distributive, $I_n(L)$ is also distributive by lemma 1.1.12 and using this distributivity we obtain that $I = \langle t\rangle_n$. Similarly we can show that J is also principal.■

2. Prime n-ideals.

1.2.1 Recall that an n-ideal P of a L is prime if $m(x,n,y)\in P$, $x,y\in L$ implies either $x\in P$ or $y\in P$. The set of all prime n-ideals of L is denoted by P(L). In M.H. Stone [8, Th.15, p-74], we have the following separation property.

1.2.2 Theorem: Let L be a distributive lattice, let I be an ideal, let D be a dual ideal of L, and let

$I \cap D = \Phi$. Then there exists a prime ideal P of L such that $P \supset I$ and $P \cap D = \Phi$.

From the proof of above theorem given in [8], it can easily seen that the following result also holds which is certainly an improvement of above.

1.2.3 Theorem: Let L be a distributive lattice, let I be an ideal, let D be a convex sublattice of L, and let $I \cap D = \Phi$. Then there exists a prime ideal P of L such that $P \supset I$ and $P \cap D = \Phi$.

Our next result gives a separation property for distributive lattices interms of prime n-ideals which is of course an extension of the above results.

1.2.4 Theorem: In a distributive lattice L, suppose I is an n-ideal and D is a convex sublattice of L with $I \cap D = \Phi$. Then there exists a prime n-ideal P of L such that $P \supset I$ and $P \cap D = \Phi$.

Proof: Let x be the set of all n-ideals of L that contains I and that are disjoint from D. Since $I \epsilon x$, x is non-empty. Let C be a chain in x and let $T = \cup\{x | x \epsilon C\}$. If $a, b \epsilon T$, then $a \epsilon X$, $b \epsilon Y$ for some $X, Y \epsilon C$. Since C is a chain, either $\underline{X \subset Y}$ or $\underline{Y \subset X}$. Suppose $\underline{X \subset Y}$. Then $a, b \epsilon Y$ and so $a \wedge b$, $a \vee b \epsilon Y \subset T$, as Y is an n-ideal. Thus, T is a sublattice.

If $a, b \epsilon T$ and $a \underline{\leq r \leq} b$, $r \epsilon L$, then $a, b \epsilon Y$ for some $Y \epsilon C$, and so $r \epsilon Y \subset T$ as Y is convex. Moreover $n \epsilon T$. Therefore T is an n-ideal. Obviously $T \supset I$ and $T \cap D = \Phi$, which verifies that T is the maximum element of C. Hence by Zorn's lemma, x has a maximal element, say P. We claim that P is a prime n-ideal.

Indeed, if P is not prime, then there exist $a, b \epsilon L$ such that $a, b \notin P$ but $m(a,n,b) \epsilon P$. Then by the maximality of P, $(P \vee \langle a \rangle_n) \cap D \neq \Phi$. Then there exist $x, y \epsilon D$ such that $p_1 \wedge a \wedge n < x < p_2 \vee a \vee n$ and $p_3 \wedge b \wedge n < y < p_4 \vee b \vee n$ for some $p_1, p_2, p_3, p_4 \epsilon P$. Since

$m(a,n,b)=(a\wedge n)\vee(b\wedge n)\vee(a\wedge b)\in P$, taking infimum with $p_1\wedge p_3\wedge n$, we have $(p_1\wedge p_3\wedge a\wedge n)\vee(p_1\wedge p_3\wedge b\wedge n)\in P$.

Choosing $r=((p_1\wedge p_3\wedge a\wedge n)\vee(p_1\wedge p_3\wedge b\wedge n)$, we have $r<x\vee y$ with $r\in P$. Since $x<r\vee x<x\vee y$, $y<r\vee y<x\vee y$ and D is a convex sublattice, so $r\vee x, r\vee y\in D$. Therefore $(r\vee x)\wedge(r\vee y)\in D$.

Again, $r\vee x<p_2\vee a\vee n<p_2\vee p_4\vee a\vee n$ and $r\vee y<p_4\vee b\vee n<p_2\vee p_4\vee b\vee n$ implies $(r\vee x)\wedge(r\vee y)\leq(p_2\vee p_4\vee a\vee n)\wedge(p_2\vee p_4\vee b\vee n)=s$(say).

Since $m(a,n,b)=(a\vee n)\wedge(b\vee n)\wedge(a\vee b)\in P$, taking supremum with $p_2\vee p_4\vee n$, we have $s\in P$. Also, $r<(r\vee x)\wedge(r\vee y)<s$. Thus, again by convexity of P, $(r\vee x)\wedge(r\vee y)\in P$. This implies $P\cap D\neq\Phi$, which leads to a contradiction. Therefore, P is a prime n-ideal.■

We conclude this section with the following corollaries.■

1.2.5 Corollary: Let I be an n-ideal of a distributive lattice L and let $a \notin I$, $a \in L$. Then there exists a prime n-ideal P of L such that $P \supseteq I$ and $a \notin P$.

1.2.6 Corollary: Every n-ideal I of a distributive lattice L is the intersection of all prime n-ideals containing it.

Proof: Let $I_1 = \cap\{P: P \supset I, P \text{ is a prime n-ideal of } L\}$. If $I \neq I_1$, then there is an $a \in I_1 - I$. Then by above corollary, there is a prime n-ideal P with $P \supset I$, $a \notin P$. But $a \notin P \supset I_1$ gives a contradiction.■

CHAPTER-2

Lattices and elements with special properties:

Let, L denote the non-modular lattice of five elements, generated by the elements p,q,r that is $p>q$, $p \vee r \vee n = q \vee r \vee n = L$, $p \wedge r \wedge n = n$. Where $\vee$ will denote the modular, non-distributive lattice of five elements with the generators p,q,r that is $p \vee q = q \vee r \vee n = r \vee p \vee n = L$, $p \wedge q = q \wedge r \wedge n = r \wedge p \wedge n = n$.

An element d of the lattice L is called distributive if $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$ for all $x,y \in L$ also we have that d is distributive if and only if $x \equiv y(\Theta\langle d\rangle_n)$ implies $x \vee y = [(x \wedge y) \vee d \vee n] \wedge (x \vee y)$.An element n of L is said to be neutral if the sublattice {n,x,y} is distributive , where x and y are arbitraty elements of L. we have the following theorem:

Theorem 2.1 The elements $x,y,z \in L$ generate a distributive sublattice of L if and only if for all permutations a,b,c of x,y,z the following equalities hold:

(6) $a\vee(b\wedge c) \quad (a\vee b)\wedge(a\vee c)$

(7) $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$

(8) … … $(a\wedge b)\vee(b\wedge c)\vee(c\wedge a) = (a\vee b)\wedge(b\vee c)\wedge(c\vee a)$.

Theorem 2.2 An element n of L is neutral if and only if

(i) … $n\vee(x\wedge y) - (n\vee x)\wedge(n\vee y)$ for all $x,y \in L$

(i′) $n\wedge(x\vee y) = (n\wedge x)\vee(n\wedge y)$ for all $x,y \in L$

(ii) … … $n\wedge x=n\wedge y$ and $n\vee x=n\vee y$ $(x,y\in L)$

Imply $x=y$ i.e. the relative complements of n are unique.

Theorem 2.3 An element n of a modular lattice L is neutral if and only if condition (i) (or equivalently, condition i′) is satisfied.

An ideal I of L is called distributive element of I(L). I is neutral if it is a neutral element of I(L). The lattice L is weakly modular (see GRATZER and SCCHMIDT [11]) if from $a,b \to c,d$ ($a,b,c,d \in L; c \neq d$) it follows the existence of $a_1, b_1 \in L$ satisfying $a \wedge b \leqq a_1 < b_1 \leqq a \vee b$ and $c,d \to a_1, b_1$.

We have the following lemma:

LEMMA 2.4 (GRATZER and SCCHMIDT [11]) Let the lattice L be

A) modular, or

B) relatively complemented,

Then L is weakly modular.■

A lattice L with n is called section complemented if all of its intervals of type [n,a] and [a,n] are complemented as lattices. In general, the lattice L is section complemented if any element of L is contained in a suitable principal dual n-ideal which is section complemented as a lattice.

The following assertion is trivial:

LEMMA 2.5 Any relatively complemented lattice is section complemented. Finally, we mention the

V-distributive law:

$$x \wedge \vee y_\alpha = \vee (x \wedge y_\alpha).$$

A complete lattice L is called V-distributive if this law unrestrictedly holds in L.

In this chapter, our aim is to prove the coincidence of distributive and standard and neutral elements in weakly modular lattices. This result is the same in modular lattice. There the proof was trivial, in consequence of the application of Theorem 2.4. But in weakly modular lattices the proof is not so simple.

Theorem 2.6 In a weakly modular lattice L, an element d is distributive if and only if it is neutral.

Proof: It follows easily from the fact that d is distributive if and only if $x \quad y(\Theta\langle d\rangle_n)$ is equivalent to $[(x\wedge y)\vee d\vee n]\wedge(x\vee y)= x\vee y$. It follows that the kernel of the homomorphism induced by the congruence relation $\Theta\langle d\rangle_n$ is $\langle d\rangle_n$. Further, if x,y $>$ d and $x\equiv y(\Theta\langle d\rangle_n)$ then x=y, because $x\vee y = [(x\wedge y)\vee(d\vee n)]\wedge(x\vee y)=x\wedge y$. From these facts we will use only the following:

(⋆) If $a<b<d<c<e$ and d is a distributive element then $a,b\to c,e$ implies c=e.

Indeed, under the stated conditions $a,b\to c,e$ implies $c\equiv e(\Theta\langle d\rangle_n)$ and so c=e.

Now let d be a distributive element of the weakly modular lattice L. First we prove that d is standard, that is we prove for any $x,y\in L$,
$x\wedge(d\vee n\vee y)=(x\wedge d\wedge n)\vee(x\wedge y)$ … … (A)

Suppose (A) does not hold.

Then $x\wedge(d\vee n\vee y)>(x\wedge d\wedge n)\vee(x\wedge y)$. Let $x\wedge(d\vee n\vee y)=a$ and $(x\wedge d\wedge n)\vee(x\wedge y)=b$, then we have $a>b$.

We prove that

$$\text{(B)} \quad \ldots \qquad d, d\wedge n\wedge x\rightarrow a, b$$

namely, $d, d\wedge n\wedge x\rightarrow(d\vee n\vee x)\wedge(d\vee n\vee y), b\rightarrow a, b$.

Indeed, because of $d\wedge n\wedge x<b$ we have to prove for the validity of $d, d\wedge n\wedge x\rightarrow(d\vee n\vee x)\wedge(d\vee n\vee y), b$ only $d\vee n\vee b=(d\vee n\vee x)\wedge(d\vee n\vee y)$.

But $d\vee n\vee b = (d\vee n)\vee(x\wedge d\wedge n)\vee(x\wedge y)$

$$=(d\vee n)\vee(x\wedge y)=(d\vee n\vee x)\wedge(d\vee n\vee y),$$

for d is distributive. Now using the inequalities $a<(d\vee n\vee x)\wedge(d\vee n\vee y)$ and $a>b$, we see that $b=b\wedge a$ and $a=(d\vee n\vee x)\wedge(d\vee n\vee y)\wedge a$ are trivial. Thus

$(d\vee n\vee x)\wedge(d\vee n\vee y), b\rightarrow c, b$ and (B) is proved.

Next we verify that

$$\text{(C)} \ldots \quad \ldots. \qquad d, d\vee n\vee y\rightarrow a, b,$$

namely $\quad d, d\vee n\vee y \rightarrow d\wedge n\wedge x, a \rightarrow a, b.$

To prove the first part of this statement, we have to show only $a\wedge d\wedge n = d\wedge n\wedge x$, but $a\wedge d\wedge n = (d\wedge n\wedge x)\wedge(d\vee n\vee y) = d\wedge n\wedge x$. The second part of the assertion is clear.

Let us use the condition $a>b$ and the weak modularity of L from these it follows the existence of elements u,r for which

(D) $\quad a,b \rightarrow u,r, \; d<r<u<d\vee n\vee y.$

From (B) and (D) it follows $d, d\wedge n\wedge x \rightarrow u,r$, in contradiction to $(\star)$. Thus we have got a contradiction from $a>b$, so $a=b$,

i.e d is standard. Now we have to prove that d is standard, then it is neutral.

If this statement is not true, then we conclude the existence of elements x,y of L such that

$$(d\wedge n)\wedge(x\vee y) > (d\wedge n\wedge x)\vee(d\wedge n\wedge y),$$

i.e the condition (i′) of Theorem 2.2 does not hold. Putting $s_1=(d\wedge n)\wedge(x\vee y)$ and $s_2=(d\wedge n\wedge x)\vee(d\wedge n\wedge y)$ let us suppose $s_1>s_2$. First we prove that $s_1\vee x>s_2\vee x$ and $s_1\vee y>s_2\vee y$.

Suppose that one of these does not hold, for instance, $s_1\vee x>s_2\vee x$; then from $s_1>s_2$ we have $s_1\vee x>s_2\vee x$. We will see that it follows $d\wedge x, x\rightarrow s_1, s_2$,

namely $d\wedge n\wedge x, x\rightarrow s_2\vee(d\wedge n\wedge x), s_2\vee x\rightarrow s_1, s_2$.

To prove this it is enough to show that $s_1\wedge[s_2\vee(d\wedge n\wedge x)]=s_2$ and $s_1\wedge(s_2\vee x)=s_1$. Indeed, $s_1\wedge[s_2\vee(d\wedge n\wedge x)]=s_1\wedge s_2=s_2$

and $s_1\wedge(s_2\vee x)=s_1\wedge(s_1\vee x)=s_1$ (we have used $s_1\vee x=s_2\vee x$ in this step). Again from $s_1>s_2$ and from the weak modularity it follows the existence of elements u,v with $d\wedge n\wedge x<u<r<x$ and $s_1,s_2\rightarrow u,r$. But $s_1,s_2 \underline{\leq d}$ and so $s_1\equiv s_2$ (Θ_d).consequently $u\equiv v(\Theta_d)$. Therefore we have $v=u\cup d_1$ with a suitable $d_1<d$. Then $v=u\vee d_1<u\vee(d\wedge n\wedge x)=u$, for we get from $v=u\vee d_1$ that $d_1\underline{\leq v\leq x}$ and hence $d_1<d\wedge n\wedge x$. The

inequality we have just proved is in contradiction to the hyp0thesis $r>u$. Thus we have proved that $s_1 \vee x > s_2 \vee x$, and in a similar way one can prove $s_1 \vee y > s_2 \vee y$.

Now, using $s_1 \vee x > s_2 \vee x$ and $s_1 \vee y = s_2 \vee y$, we prove that

$$(d \wedge n) \wedge (s_2 \vee x),\ s_2 \vee x \rightarrow s_1 \wedge (s_2 \vee y), s_1,$$

namely, $(d \wedge n) \wedge (s_2 \vee x)$,

$s_2 \vee x \rightarrow d \wedge x, x \rightarrow s_2 \vee y, s_2 \vee (x \vee y) \rightarrow (s_2 \vee y) \wedge s_1, s_1$.

From these $(d \wedge n) \wedge (s_2 \vee x)$, $s_2 \vee x \rightarrow d \wedge n \wedge x, x$ is clear. To verify $d \wedge n \wedge x, x \rightarrow s_2 \vee y, s_2 \vee (x \vee y)$ we use the inequality $d \wedge n \wedge x < (d \wedge n \wedge x) \vee (d \wedge n \wedge y) = s_1 < s_2 \vee y$. nd so $(d \wedge n \wedge x) \vee (s_2 \vee y) = s_2 \vee y$, further $x \vee (s_2 \vee y) = s_2 \vee (x \vee y)$. To prove $s_2 \vee y, s_2 \vee (x \vee y) \rightarrow (s_2 \vee y) \wedge s_1, s_1$ we have only to observe the inequality

$s_1 = (d \wedge n) \wedge (x \vee y) < s_2 \vee (x \vee y) = x \vee y$, and then

$[s_2 \vee (x \vee y)] \wedge s_1 = s_1$.

Before applying weak modularity we have to show that $s_1 \neq s_1 \wedge (s_2 \vee y)$. Indeed, in case $s_1 = s_1 \wedge (s_2 \vee y)$ it follows $s_1 < s_2 \vee y$, and then $s_1 \vee y = s_2 \vee y$, which is a contradiction to $s_1 \vee y > s_2 \vee y$. From this we see that $(d \wedge n) \wedge (s_2 \vee x) = s_2 \vee x$ is also impossible, for $(d \wedge n) \wedge (s_2 \vee x)$, $s_2 \vee x$ $s_1 \wedge (s_2 \vee y), s_1$, and so $(d \wedge n) \wedge (s_2 \vee x) = s_2 \vee x$ implies $s_1 \wedge (s_2 \vee y) = s_1$. Now, using the weak modularity and $(d \wedge n) \wedge (s_2 \vee x)$,

$s_2 \vee x \rightarrow s_1 \wedge (s_2 \vee y), s_1$, it follows the existence of u,v such that $(d \wedge n) \wedge (s_2 \vee x) < u < v < s_2 \vee x$ and $s_1 \wedge (s_2 \vee y), s_1 \rightarrow u, v$. It follows now $u \equiv v(\Theta_d)$ in a similar way as in the first step of the proof, thus $v = u \vee d'$ $(d' < d)$.But from $v < s_2 \vee x$ we have $d' < (d \wedge n) \wedge (s_2 \vee x)$ for

$d \geqq s_1 > s_2$

Consequently, $v = u \vee d' < u \vee [(d \vee n) \wedge (s_2 \vee x)] = u$,

a contradiction to $v > u$.

Thus we have verified the validity of the conditions of Theorem 2.2, thus d is neutral.■

We have the following corollaries:

Corollary 2.7 In a weakly modular lattice every standard element is neutral.

Corollary 2.8 If $I_n(L)$ is weakly modular, then any standard n- ideal of L is neutral.

Corollary 2.9 In a relatively complemented lattice L any standard element is neutral.

Corollary 2.10 In a modular lattice any standard element and n-ideal is neutral.

Corollary 2.9 and 2.10 are immediate consequences of Lemma 2.3.

Unfortunately, we cannot establish Theorem 2.6 for distributive n-ideals, not even the more important Corollary 2.7 for standard n-ideals. A detailed discussion of the proof shows that the idea of the proof essentially uses the distributive n-ideals. But we cannot get the results for n-ideals by a simple application of Theorem 2.6 to $I_n(L)$.

We shall now deal separately with (standard, i.e.) neutral elements of a special class of weakly modular lattices. We intend to show that in relatively complemented lattices the set of all neutral elements is again a relatively complemented lattice. We have the following result.

LEMMA 2.11 Let a,b,c be neutral elements of a lattice L and $a<b<c$. If a relative complement d of b in the interval [a,c] exists , then it is also neutral and uniquely determined.

Proof: We know that $L=\langle b\rangle_n \times \langle b\rangle_n$ under the correspondence $x \to (x\wedge b\wedge n, x\vee b\vee n)$. Under this $d\to(a,c)$therefore d is neutral (for both component of d are neutral) in L and consequently it is neutral in L. The uniqueness assertion is trivial.

We have the following corollaries:

Corollary 2.12 Any complement of a neutral element is neutral.

Corollary 2.13 The neutral elements (if any) of a relatively complemented lattice form a relatively complemented distributive sublattice.

We note that from Corollary 2.12 we do not get Lemma 2.11, only that d is neutral in [a,c].

Lemma 2.11 is not true for standard elements. As an example take the lattice L where n,p,L are standard, while (the unique) relative complement of p in [n,L] is r which is not standard.

THEOREM 2.14: *The lattice of all n-ideals of a weakly modular lattice is not necessarily weakly modular.*

Proof: We have to construct a weakly modular lattice K such that $I_n(K)$ is not weakly modular. Consider the chain of non-negative integers and take the direct product of this chain by the chain of two elements. The elements of this lattice are of the form (m,0) and (m,1),where 0 and 1 are the zero and unit elements of the two elements and n is an arbitrary non-negative integer. Further, we define the elements x_m $(m=1,2,3,\ldots\ldots.)$ satisfying the following relations:

$$x_m \vee (m-1,1)=x_m \vee (m,0)=(m,1),$$

$$x_m \wedge (m-1,1)=x_m \wedge (m,0)=(m-1,0).$$

Thus we have got a lattice L. Finally, we define three further elements x,y,1 subject to

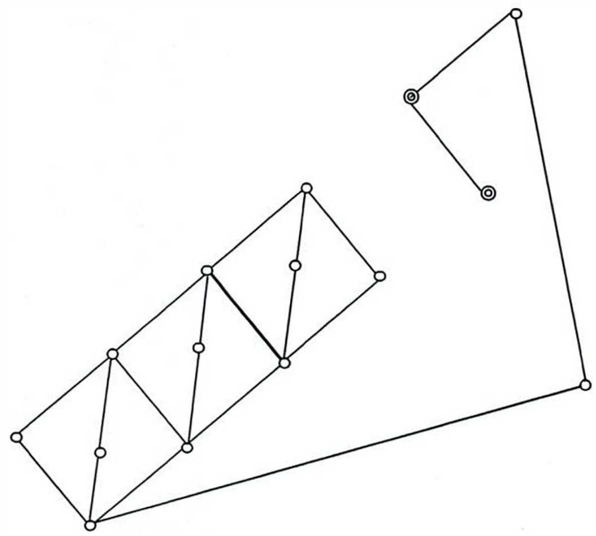
$$x \vee y=x \vee z=y \vee z=1,$$

$$x \wedge y=x \wedge z=y \wedge z=(0,0) \qquad (z\neq 0, z\in L).$$

Denote the partially ordered set of all these elements by K. The elements of K are denoted by this symbol in this given figure.

It is easy to see that K is a lattice. Also, we have K is weakly modular. All but two n-ideals of K are principal n-ideals, these exceptional ones are denoted by ⊙ in the diagram, thus the diagram of K, completed by these two elements, gives the diagram of $I_n(K)$. Now, it is easy to see that K is not weakly modular. Indeed, under the congruence relation generated by the congruence of the two new elements, no two different elements of K are congruent. While from the congruence of any two different elements of K it follows the congruence of the two new elements, we have considered K to be imbedded in $I_n(K)$. The existence of the lattice K proves the Theorem.

So far we could assure the weak modularity only of the lattice of all n-ideals of a modular lattice. Naturally, the same is true for every weakly modular lattice in which the ascending chain condition holds, because in this case the lattice of all n-ideals is identical with (more precisely isomorphic to) the original lattice. The following question arises is it possible that the lattice of all n-ideals of a relatively complemented lattice is weakly modular if in the lattice the ascending chain condition does not hold? Is it possible



FIGER 6

that the n-ideal lattice of the same is relatively complemented? The interest of this latter question is that in modular lattices the answer is always negative. Despite this, the following assertion is true:

There exists a relatively complemented lattice L, not satisfying the ascending chain condition, such that $I_n(L)$ is relatively complemented . This lattice may be chosen to be semi-modular.

To construct L, consider an infinite set H. We say that the partition p of H, which divides the set H into the disjoint subsets H_α, is finite, if all but a finite number of the H_α consist of one element, and every H_α consists of a finite number of elements. We denote by *FP(H)* the set of all finite and by P(H) the set of all partitions of H.

It is clear that the join and meet of any two finite partitions are finite again, and if a partition is smaller than a finite partition, then it is also finite. It follows that FP(H) is an n-ideal of the lattice P(H). Now, it is easy to prove that just the finite partitions are the elements of the lattice P(H) which are inaccessible from below. Indeed, if p is a finite partition, then the interval $[\omega,p]$ of the lattice P(H) is finite, therefore p is inaccessible from below. Now suppose p is not finite, and let $\{H_\alpha\}$ be the corresponding partition of H (the H_α are pairwise disjoint). Either infinitely many H_α are containing more than one element, or at least one H_α contains an infinity of elements. In the first case, assume that $H_1, H_2, \ldots \ldots$ contain more one element. We define the partition p, to be the same as p on the set $H \setminus \vee H_j$ ($j=i+1, \ldots \ldots$infinity) while on the $\vee H_j$ ($j=i+1, \ldots \ldots$infinity) let all the classes of p, consist of one element.

Obviously, $p_1 < p_2 < \ldots$ and $\vee p_i$ p, so p is accessible from below. It is also clear that every partition is the complete join of finite partitions and finally, it is well known that P(H) is meet continuous. It follows that P(H) is isomorphic to the lattice of all n-ideals of FP(H).

Now we will prove that FP(H) satisfies the requirements. We have to prove yet that in FP(H) the ascending chain condition does not hold, that FP(H) and P(H) are relatively complemented , and finally that FP(H) is semi-modular. The first of these assertions is trivial, since H is infinite. The second and the third assertions have been proved for P(H), but these properties are preserved under taking an n-ideal of the lattice, therefore these hold in FP(H).

We could assure the weak modularity of the n-ideal lattice of a modular lattice, for the modularity of a lattice may be defined by an equality. We now show that if the weak modularity of a lattice is a consequence of the fulfillment of a system of equalities, then the n-ideal lattice is also weakly modular. First we prove a general theorem which will serve for other purposes as well.

To formulate the theorem we need two notions. We call a subset $\bar{I}$ of the n-ideal I a covering system of I if $I=\{x;\exists y\epsilon\ \bar{I},\ x<y\}$. Thus, for instance, $\bar{I}=I$ is always a covering system and if $I=\langle a\rangle_n$ then $\langle a\rangle_n$ is a covering system. If I is generated by
the set $\{x_\alpha\}$, then the finite join of the x_α form a covering system.

Let $f_\alpha\ (y,x_1,x_2,\dots\ \dots.\ x_n)$ and $g_\alpha\ (y,\ x_1,x_2,\dots\ \dots.\ x_n\ )$ be lattice polynomials, where n depends a and a_1 runs over an arbitrary set of indices A.(It is not a restriction that f_a $(y,x_1,x_2,\ldots\ \ldots.\ x_n)$ and g_α $(y,\ x_1,x_2,\ldots\ \ldots.\ x_n)$ depend on the same number of variables. Indeed, if $g_a= g_\alpha$ $(y,\ x_1,x_2,\ldots\ \ldots.\ x_r\ )$, $r<n$, then define g_α $(y,\ x_1,x_2,\ldots\ \ldots.\ x_n\ )= g_\alpha$ $(y,\ x_1,x_2,\ldots\ \ldots.\ x_r)\vee(x_1\wedge x_2\wedge\ldots\ \ldots\wedge x_r\wedge\ldots\ \ldots\wedge x_n\wedge y)$. Independently of the values of the $x_1,x_2,\ldots\ \ldots.\ x_n$, the equality g_α $(y,\ x_1,x_2,\ldots\ \ldots.\ x_n)= g'_\alpha$ $(y,\ x_1,x_2,\ldots\ \ldots.\ x_n)$ always holds.) We say that the element s is of the type $f_a = g_a(a\epsilon A)$, if for all $a_1,a_2,\ldots\ \ldots,a_n$ ϵL and $a\epsilon A$ we have $f_a(s,\ a_1,a_2,\ldots\ \ldots,a_n)= g_a(s,\ a_1,a_2,\ldots\ \ldots,a_n)$. It is clear that the standard elements are of the type $f_a = g_a$ with the polynomials $f_1(y,x_1,x_2)=x_1\wedge(y\vee x_2)$ and $g_1(y,x_1,x_2)=(x_1\wedge y)\vee(x_1\wedge x_2)$ and $A=\{1\}$. Similarly, the neutral elements are also of the type $f_a = g_a$; we get a system of five polynomials from the Corollary of Theorem 2.1.

We conclude this section with the following result.

Theorem 2.15 *Given the n-ideal I of the lattice L and a covering system $\bar{I}$ of I and the lattice polynomials*

f_a, g_a $(a\epsilon A)$. If every element of $\bar{I}$ is of the type $f_a = g_a$ $(a\epsilon A)$, then I as an element of $I_n(L)$ is of the type $f_a = g_a$ $(a\epsilon A)$.

Proof: It is enough to prove the theorem for one pair of polynomials $f_a = g_a$. For if the theorem failed to be true, then there would be a pair of polynomials f = g such that I does not satisfy the corresponding equality.

Consider the polynomials f and g, and construct the following satisfy sets of L:

$$F=\{t;t\leqq f(a,j_1,\dots \dots,j_n),\ a\epsilon\bar{I},j_1\epsilon J_1,\dots \dots,j_n\epsilon J_n\},$$

$$G=\{t;t\leqq g(a,j_1,\dots \dots,j_n),\ a\epsilon\bar{I},j_1\epsilon J_1,\dots \dots,j_n\epsilon J_n\}$$

where $j_1,\dots \dots,j_n$ are fixed n-ideals of L. We prove that F is an n-ideal. It is enough to prove that t_1, $t_2\epsilon F$ implies $t_1 \vee t_2 \epsilon F$. Indeed, if $t_1,t_2\epsilon F$, then there exist $a_i\epsilon\bar{I}$ and $j_{1,i}\epsilon J_1,\dots \dots,j_{n,i}\epsilon J_n$ $(i=1,2)$ with $t_i\leqq f(a_I,\ j_{1,i}\dots \dots,j_{n,i})$.

Now choose an element a of $\bar{I}$ for which $a_1 \vee a_2 \leqq a$. Then $f(a,j_{1,1}\vee j_{1,2},\ \dots \dots,j_{n,1}\vee j_{n,2})$ is an element of F, and since the lattice polynomials are isotone functions of their variables, $t_1 \vee t_2 \leqq f(a, j_{1,1} \vee j_{1,2}, \ldots \ldots, j_{n,1} \vee j_{n,2})$ is clear, and so $t_1 \vee t_2 \in F$. Similarly, we can prove that G is also an ideal. $t \in F$, then

$t \leqq f(a, j_1, \ldots \ldots, j_n)$, but $f(a, j_1, \ldots \ldots, j_n) = g(a, j_1, \ldots \ldots, j_n)$, for a is an element of the type f=g, and so $t \leqq g(a, j_1, \ldots \ldots, j_n)$, that is , $t \in G$.

We get $F \subseteq G$ and similarly $G \subseteq F$, that is, F=G. Owing to Lemma I, $F = f(I, j_1, \ldots \ldots, j_n)$ is clear. $G = g(I, j_1, \ldots \ldots, j_n)$, holds as well. Summing up, we got that $f(I, j_1, \ldots \ldots, j_n) = g(I, j_1, \ldots \ldots, j_n)$.■

Now we turn our attention to corollaries of this theorem. We say that the lattice L is of the type $f_a = g_a$ if every element of L is of the same type, i.e. if the equalities $f_a = g_a$ $(a \in A)$ identically hold. We have a corollary.

COROLLARY 2.16 Let f_a, g_a $(a \in A)$ be lattice polynomials and suppose L is of the type $f_a = g_a$

$(a \in A)$.Then this system of equalities holds in $I_n(L)$ too. Also it follows immediately from Theorem 2.15 taking $\bar{I}=I$ for all n-ideals $I \in I_n(L)$.■

CHAPTER-3

STANDARD ELEMENT AND n-IDEALS.

3.1. Some notions and notations

The partial ordering relation will be denoted by $<$, in case of set lattice (that is lattices the elements of which are certain subsets of a given set) by $\subset$. In lattices the meet and the join will be designated by $\cap$ and $\cup$. And the complete meet and complete join by $\wedge$ and $\vee$. The least and greatest element of a partially ordered set (or of a lattice) we denote by 0 and 1. If a covers b (i.e. $a>b$, but $a>x>b$ for no x), then we write $a\succ b$.

If $a(x)$ is a property defined on the set H, then we define $\{x;\ a(x)\}$ as the set of all $x\in H$ for which $a(x)$ is true. Hence in partially ordered sets $<a>_n=\{x: x\wedge a<x<x\vee a\}$ is the principal n-ideal generated by a, while $\{x;a<x<b\}$ is the interval $[a,b]$ provided that $a<b$. If b covers a, then the interval $[a,b]$ is a prime

interval. The dual principal n-ideal is denoted by $<a>_n{}^d$.

If any two elements a,b of L, satisfying a<b, may be connected by a finite maximal chains of the lattice L are finite and bounded, then L is called of finite length. If all intervals of the lattice L are of finite length, then L is of locally finite length. If L has a "n` and is of locally finite length, furthermore for all $a \in$L, in [n,a] any two maximal chains are of the same length, then we say that in L' the Jordan-Dedekind chain condition is satisfied. In this case the length of any maximal chain of the interval [n,a] will be denoted by L(a), and d(x) is called the dimension function.

Let P and Q be partially ordered sets. The ordinal sum of P and Q is defined as the partially ordered set, which is the set union of P and Q, and the partial ordering remains unaltered in P and Q, while $x<y$ holds for all $x \in P$ and $y \in Q$; this partially

ordered set will be denoted by P⊕Q. The set of all n-ideals of a lattice L, partially ordered under set inclusion, form a lattice, which will be denoted by $I_n(L)$.

LEMMA 3.2 $I_n(L)$ is a conditionally complete lattice. The meet of a set of n-ideals (if it exists) is the set-theoretical meet. The join of the n-ideals I_α ($\alpha \in A$) is the set of all x such that

$i_{\alpha 1} \wedge \ldots \wedge i_{\alpha n} < x < i_{\alpha 1} \vee \ldots \vee i_{\alpha n}$ ($i_{\alpha j} \in I_{\alpha j}$) for some elements α_j of A.■

If A is a general algebra and **Θ** is a congruence relation of A, then the congruence classes of A modulo **Θ** form a general algebra A(**Θ**). This is a homomorphic image of A. According to [20], we have the following two general isomorphism theorems.

3.3 THE FIRST GENERAL ISOMORPHISM THEOREM

Let A be a general algebra and A' a subalgebra of A, further let Θ be an equivalence relation of A such that every equivalence class of A may be represented by an element of A' . Let Θ' denote the equivalence relation of A' induced by Θ. If Θ is a congruence relation, then so is Θ' and

$$A(\Theta) \sim A'(\Theta').$$

The natural isomorphism makes a congruence class of A correspond to the contained congruence class of A'.

3.4 THE SECOND GENERAL ISOMORPHISM THEOREM

Let A' be a homomorphic image of the general algebra A, let Θ be an equivalence relation of A, and denote Θ' the equivalence relation of A' under which the equivalence classes are the homomorphic images of those of A modulo Θ, and suppose that no two different equivalence classes of A modulo Θ have the same homomorphic image. Then Θ is a congruence relation if and only if Θ' is one and in this case

$$A(\Theta) \cong A'(\Theta').$$

The natural isomorphism makes an equivalence class of A correspond to its homomorphic image.

3.5 Congruence relations in lattices

Let Θ be a congruence relation of the lattice L and denote by L/Θ be homomorphic image of L induced by the congruence relation Θ that is the lattice of all congruence classes. If L/Θ has a [n], then the complete inverse image of the [n] is an n-ideal of L, called the kernel of the homomorphism $L \to L/\Theta$.

A simple criterion for a binary relation η to be a congruence relation is formulated in the following Lemma.

LEMMA 3.6 (GRATZER and SCHMIDT [21]) Let η be a binary relation defined on the lattice L. η is a congruence relation if and only if the following conditions hold for all $x,y,z \in L$:

(a) $x \equiv x(\eta)$;

(b) $x \vee y \equiv x \wedge y(\eta)$ if and only if $x \equiv y(\eta)$;

(c) $x > y > z$, $x \equiv y(\eta)$, $y{\_}z(\eta)$ imply $x \equiv z(\eta)$;

(d) $x > y$ and $x{\_}y(\eta)$, then $x \vee z \equiv y \vee z(\eta)$ and $x \wedge z \equiv y \wedge z(\eta)$.

The congruence relations of L will be denoted by $\Theta, \Phi, \ldots$. The set of all congruence relations of L, partially ordered by $\Theta < \Phi$ if and only if $x{\_}y(\Theta)$ implies $x \equiv y(\Phi)$, will be denoted by C(L).

LEMMA 3.7 (BIRKHOFF [22] and KRISGNAN [23]) C(L) is a complete lattice $x{\_}y(\wedge \Theta_{\alpha})(\alpha \in A)$ if and only if $x \equiv y(\Theta_{\alpha})$ for all $\alpha \in A$; $x \equiv y(\vee \Theta_{\alpha})(\alpha \in A)$ if and only if there exists a sequence of elements in L,ι

$x \vee y = \underline{z_0 \geq z_1 \geq} \ldots > z_n = x \wedge y$ such that

$z_i \equiv z_{i-1}(\Theta_{a_i})(i=1,2,\ldots\ldots,n)$ for suitable $a_1,\ldots\ldots,a_n \in A$.■

3.8 The least and greatest elements of the lattice C(L) will be denoted by $\boldsymbol{\omega}$ and ι respectively.

Let H be a subset of L, $\Theta[H]$ denote the least congruence relation under which any pair of elements of H is congruent. This we call the congruence relation induced by H. If H has just two elements, $H=\{a,b\}$, then $\Theta[H]$ will be written as Θ_{ab}. The congruence relation Θ_{ab} is called minimal. First we describe the following minimal congruence relation Θ_{ab}. To do this, we have to make some preparations. Given two pairs of elements a,b and c,d of L, suppose that either $c\wedge d \geqq a\wedge b$

And $(c\wedge d)\vee(a\vee b)=c\vee d$, or $c\vee d \leqq a\vee b$ and $(c\vee d)\wedge(a\wedge b)=c\wedge d$.

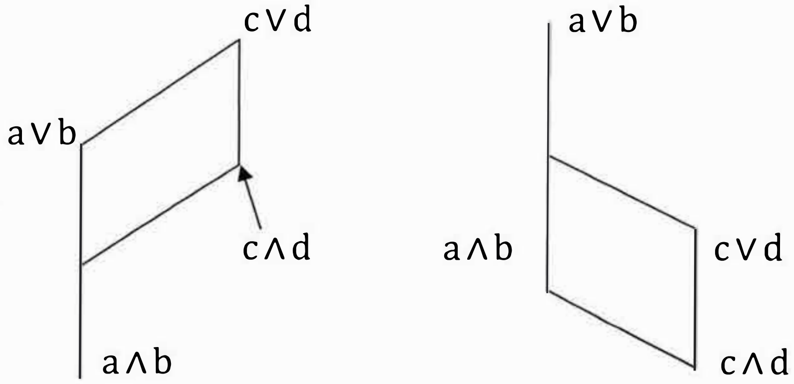


Fig.1

Then we say that a,b is weakly projective in one step to c,d and write $a,b \rightarrow c,d$. The situation is given in Fig.1. In other words $a,b \rightarrow c,d$ if and only if the intervals $[(a\vee b)\wedge c\wedge d, a\vee b],[c\wedge d, c\vee d]$ or $[a\wedge b,(a\wedge b)\vee c\vee d],[c\wedge d,c\vee d]$ are transposes. If there exist two finite sequences of elements $a=x_0,x_1,\ldots\ldots,x_n=c$ and $b=y_0,\ldots\ldots,y_n=d$ in L such that

(1) ……………… $a,b=x_0,y_0\rightarrow x_1,y_1\rightarrow \ldots\ldots\ldots\rightarrow x_n,y_n=c,d$. then we say that a,b is weakly projective to c,d, in notation: $a,b\rightarrow c,d$, or if we are also interested in the number n, then we write $a,b\rightarrow c,d$.

If $a,b\rightarrow c,d$ and $c,d\rightarrow a,b$, then a,b and c,d are transposes, and we write $a,b\rightarrow c,d$. If the sequence

(1) may be chosen in such a way that the neighbouring members are transpose, then a,b and c,d are called projective and we write $a,b \rightarrow c,d$.

The importance of this notion is shown by the fact that $a,b \rightarrow c,d$ and $a \equiv b(\boldsymbol{\Theta})$ imply $c \equiv d(\boldsymbol{\Theta})$ (applying this to $\boldsymbol{\Theta} = \boldsymbol{\omega}$, we get that $a=b$ implies $c=d$, a fact which will be used several times).

Now we are able to describe $\boldsymbol{\Theta}_{ab}$:

According to [24], we have the following describtion: Let a,b,c,d be elements of the lattice L. $c \equiv d(\boldsymbol{\Theta}_{ab})$ holds if and only if there exist $y_i \in L$ with

(2)……………… $c \vee d = y_0 > y_1 > \ldots > y_k = c \wedge d$ and $a,b \rightarrow y_{i-1}, y_i$ $(i=1,2,\ldots\ldots,k)$.It is easy to describe $\boldsymbol{\Theta}[H]$, using Lemma 3.7 and above. We have the following trivial identity:

(3)………… $\boldsymbol{\Theta}[H] = \bigvee \boldsymbol{\Theta}_{ab}(a,b \in H)$. The symbol $\boldsymbol{\Theta}[H]$ will be used mostly in case H is an n-ideal. Then one can prove the following important identity.

(4)…………… $\Theta[\vee I_a]=\vee\Theta[I_a]$ $(I_a \in I(L))$.

The following definition is more importance in this chapter. Let L be a lattice and I an ideal of L. By the factor lattice L/I of the lattice L modulo the ideal I is meant the homomorphic image of L induced by $\Theta(I)$, I.e. $L/I \cong L(\Theta[I])$.

Finally, we mention the definition of permutability: the congruence relations Θ and Φ are called permutable if $a\equiv x(\Theta)$ and $x^{-}b(\Phi)$ imply the existence of a, y such that $a\equiv y(\Phi)$ and $y\equiv b(\Theta)$.

We recall the definition of standard elements:

The element s of the lattice L is standard if the equality

(A) … $x\wedge(s\vee y) = (x\wedge s)\vee(x\wedge y)$ holds for all $x,y\in L$.

First of all, let us see some examples for standard elements, in the lattice L. p is a standard element. At the same time it is clear that p is not neutral.

(Furthermore, in the same lattice $<r>_n$ is a homomorphism kernel but r is not standard.)

Obviously, any element of a distributive lattice is standard. Furthermore, in any lattice the elements n and L (if exist) are standard element. The simplest from for defining standard elements is the equality (A) however; it is not the most important property of a standard element. Some important characterizations of standard elements are given in the following theorem.

We conclude this chapter with the following results.

Theorem3.9: (The fundamental characterization theorem of standard elements) the following conditions upon an element s of the lattice L are equivalent:

(α) s is a standard element;

the equality $u=(u \wedge s) \vee (u \wedge t)$ holds whenever $u < s \vee t$ $(u,t \in L)$;

(γ) the relation Θ_s ,defined by " $x \equiv y(\Theta_s)$ if and only if $(x \wedge y) \vee s_1 = x \vee y$ for some $s_1 \leqq s$ is a congruence relation ;

(δ) for all $x, y \in L$

(i) $s \vee (x \wedge y) = (s \vee x) \wedge (s \vee y)$

(ii) $s \wedge x = s \wedge y$ and $s \vee x = s \vee y$ imply $x = y$.

Proof: We have proved the equivalence of the four conditions cyclically

(α) implies (β). Indeed if (α) holds and $u < s \vee t$, then $u = u \wedge (s \vee t)$ Owing to (A) we get $u = (u \wedge s) \vee (u \wedge t)$, which was to be proved.

(β) implies (γ). Using condition (β) and Lemma 3.6 we will prove that Θ_s is a congruence relation.

(a) $x \equiv x(\Theta_s)$. Indeed for any $x \in L$, the equality $(x \wedge x) \vee (x \wedge s) = x$ trivially holds, so if we put $s_1 = x \wedge s$, we get the assertion.

(b) $x \wedge y \equiv x \vee y$ (Θ_s). This is trivial from the definition of Θ_s.

(c) $x \geqq y \geqq z$, $x \equiv y$ (Θ_s) and $y \equiv z(\Theta_s)$. By hypothesis $x = y \vee s_1$ and $y = z \vee s_2$ for suitable elements $s_1, s_2 \leqq s$ Consequently $x = y \vee s_1 = (z \vee s_2) \vee s_1 = z \vee (s_1 \vee s_2)$ for $s_1 \vee s_2 \leqq s$, that means $x \equiv z(\Theta_s)$.

(d) In case $x \geqq y$ and $x \equiv y$ (Θ_s) holds, $x \vee z \equiv y \vee z$ (Θ_s) and $x \wedge z \equiv y \wedge z$ (Θ_s) . In fact, by assumption $x = y \vee s_1$ ($s_1 \leqq s$), and hence we get $x \vee z = (y \vee z) \vee s_1$, that is $x \vee z \equiv y \vee z$ (Θ_s) . To prove the second assertion we start from the relations $x = y \vee s_1$ and $x \wedge z \leqq y \vee s_1 \leqq y \vee s$. Applying condition (β) to $u = x \wedge z$, $t = y$ and using $x \wedge y = y$, we get

$$x \wedge z = (x \wedge z \wedge s) \vee (x \wedge z \wedge y) = (y \wedge z) \vee s_2, \text{ where } s_2$$

$= x \wedge z \wedge s \leqq s$, which means $x \wedge z \equiv y \wedge z$ (Θ_s)

(γ) implies (δ). First we prove that (γ) implies (i). According to the definition of Θ_s, the congruences $x \equiv s \vee x$ (Θ_s) and $y \equiv s \vee y$ (Θ_s) hold for arbitrary $x, y \in L$. We get $x \wedge y \equiv (s \vee x) \wedge (s \vee y)$ (Θ_s).By monotonicity.

$x \wedge y < (s \vee x) \wedge (s \vee y)$, hence again by the definition of Θ_s. it follows that $(s \vee x) \wedge (s \vee y) = (x \wedge y) \vee s_1$ with suitable $s_1 < s$. Joining with s and keeping the inequalities $s_1 < s$ and $s < (s \vee x) \wedge (s \vee y)$ in view, we derive $s \vee (x \wedge y) = (s \vee x) \wedge (s \vee y)$, which is nothing else than (i).

Secondly, we prove that (γ) implies (ii). Let the elements x and y be chosen as in (ii). We know that $s \vee y \equiv y$ (Θ_s), so meeting with x and using

$x \vee s = y \vee s$ we get $x = (x \vee s) \wedge = (y \vee s) \wedge x \equiv y \wedge x$ (Θ_s), consequently, using (γ), $(x \wedge y) \vee s_1 = x$ with suitable $s_1 \leqq s$. From the last equality $s_1 \leqq x$, accordingly $s_1 < s \wedge x = s \wedge y < y$ (in the meantime we have used the sup-position $s \wedge x = s \wedge y$ of (ii)), thus $x = (x \wedge y) \vee s_1 < (x \wedge y) \vee y = y$. We may conclude similarly that $y < x$, and thus $x = y$, which was to be proved.

(δ) implies (α). Let x and y be arbitrary elements of L and define $a = x \wedge (s \vee y)$ and $b = (x \wedge s) \vee (x \wedge y)$. By (ii), it suffices to prove that $s \wedge a = s \wedge b$ and $s \vee a = s \vee b$.

To prove the equality we start from $s \wedge a$:

$$s \wedge a = s \wedge [x \wedge (s \vee y)] = x \wedge [s \wedge (s \vee y)] = x \wedge s.$$

It follows from the monotonicity that $x \wedge s < b$ $(x \wedge s) \vee (x \wedge y) < [x \wedge (s \vee y)] \vee [x \wedge (s \vee y)] = a$. Meeting with s, we get $s \wedge x < s \wedge b < s \wedge a$. But we have already proved that $s \wedge x = s \wedge a$, and so $s \wedge a = s \wedge b$. To prove $s \vee a = s \vee b$ we start from $s \vee a$ and use (i) several times:

$$s \vee a = s \vee [x \wedge (s \vee y)] = (s \vee x) \wedge [s \vee (s \vee y)] =$$

$$(s \vee x) \wedge (s \vee y) = s \vee (x \wedge y) = s \vee (x \wedge s) \vee (x \wedge y) = s \vee b,$$

and so Theorem 3.9 is completely proved.■

We have the following lemma:

LEMMA 3.10 An element s of L is standard if and only if the following two conditions are satisfied:

(i*) the correspondence $x \rightarrow x \vee s$ is an endomorphism of L ;

(ii*) if $x > y$, $s \vee x = s \vee y$ and $s \wedge x \quad s \wedge y$, then $x = y$.

It is easy to see that (i) is equivalent to (i*). Indeed, for any fixed s, the correspondence $x \to x \vee s$ is a join-endomorphism. That it is meet-endomorphism as well is guaranted just by (i). In the proof of Theorem 3.9, at the step "(δ) implies (α)" we have used (ii) only for x=a and y=b, and in this case $y \leqq x$ holds. Consequently, in the proof we have only used (ii*), and so one can replace (ii) by (ii*). From condition (γ) of Theorem 3.9 we derive easily the following lemma:

LEMMA 3.11 Let s be a standard element of the lattice L. Then $<s>_n$ is a homomorphism kernel, namely $\boldsymbol{\Theta}[<s>_n] = \boldsymbol{\Theta}_s$. Conversely, if $x \equiv y$ $\boldsymbol{\Theta}[<s>_n]$ hold when and only when $(x \wedge y) \vee s_1 = x \vee y$ with a suitable $s_1 \leqq s$, then s is a standard element.

Proof: The congruence relation $\boldsymbol{\Theta}_s$ obviously satisfies $\boldsymbol{\Theta} \to \boldsymbol{\Theta}[<s>_n]$. Consequently $<s>_n$ is in the kernel of the homomorphism induced by $\boldsymbol{\Theta}_s$. We have to prove that $<s>_n$ is just the kernel. Otherwise there exists an $x>s$ with $x \equiv s$ (Θ_s). By definition, it follows $x=s \vee s_1$ ($s_1<s$) which is obviously a contradiction. Conversely, if $\Theta[<s>_n]= \Theta_s$, then Θ_s is a congruence relation, since $\Theta[<s>_n]$ is one and then from condition (γ) of Theorem 3.9 it follows that s is a standard element.

We have formulated Lemma 3.11 separately despite the fact that it is an almost trivial variant of condition (γ) of Theorem 3.9 because it points out that property of the standard elements which we think to be the most important one. It may be reformulated as follows: if (s] is a principal ideal of L, then $x \equiv y$ $\Theta[<s>_n]$ if and only if there exist a sequence of elements $x \vee y = z_0 > z_1 > z_2 > \dots \dots \geqq z_m = x \wedge y$ of L, an $s_1 < s$, and a sequence of integers n_1, n_2, … … n_m such that $s_1, s \rightarrow z_{i-1}, z_i$ $(i=1,2,3\dots\dots,m)$. Now the definition of standardness is as follows:

s is standard if and only if $n_i=1$ may be chosen for all i. It follows then we may suppose $m=1$ as well.■

CHAPTER – 4

Standard n-ideals

Introduction: Standard elements and ideals in a lattice were introduced by Grätzer and Schmidt [11]. Some additional work has done by Janowitz [13] while Fried and Schmidt [07] have extended the idea of standard ideals to convex sublatttices.

According to Grätzer and Schmidt [11], if a is an element of a lattice L, then

(i) a is called distributive if $a \vee (x \wedge y)$

$$= (a \vee x) \wedge (a \vee y), \text{ for all } x, y \in L.$$

(ii) a is called standard if $x \wedge (a \vee y)$

$$= (x \wedge a) \vee (x \wedge y) \text{ for all } x, y \in L.$$

(iii) a is called neutral if for all $x, y \in L$,

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y),$$

i.e. a is standard

and (b) $a \wedge (x \vee y) \quad (a \wedge x) \vee (a \wedge y)$.

Grätzer [10] has shown that an element n in a lattice L is neutral if and only if

$$(n \wedge x) \vee (n \wedge y) \vee (x \wedge y)$$

$$= (n \vee x) \wedge (n \vee y) \wedge (x \vee y),$$

for all $x, y \in L$.

An ideal S of lattice L is called standard if it is a standard element of the lattice of ideals I(L).

Fried and Schmidt [7] have extended the idea of standard ideals to convex sublattices. Moreover, Nieminen(convex) sublattices. On the other hand, in a more recent paper Dixit and paliwal [5], [6] have established some results on standard, neutral and distributive (convex) sublattices. But their technique is quite different from those of the above authors. We denote the set of all convex sublattices of L by Csub (L). According to [7] and [9], we define two operations $\wedge$ and $\dot{\vee}$ (these notations have been used by Nieminen in [9] on Csub (L)) by

$$A \wedge B \qquad <\{a \wedge b : a \in A,\ \ b \in B\}>$$

And $$A \dot{\vee} B \qquad <\{a \dot{\vee} b : a \in A,\ \ b \in B\}>$$

For all A, B $\in$ Csub(L) where <H> denotes the convex sublattice generated by a subset H of L.

If A and B are both ideals then $A \dot{\vee} B$ and $A \wedge B$ are exactly the join and meet of A and B in the ideal Lattice.

However, in general case neither $A \subset A \dot{\vee} B$ and $A \wedge B \subseteq A$ are valid. For example if $A = \{a\}$ and $B = \{b\}$, then both inequalities imply $A = B$.

According to [11], a convex sublattics of a lattice L is called a standard convex sublattice (or simply a "standard sublattice") if

$$I \wedge <S, K> = <I \wedge S, I \wedge K>$$

And $I \dot{\vee} \langle S, K\rangle = \langle I \dot{\vee} S, I \dot{\vee} K\rangle$ hold for any pair $\{I, K\}$ of Csub (L) whenever either $S \cap K$ nor $I \cap \langle S, K\rangle$ are empty, where $\cap$ denotes the set theoretical intersection.

We call an n-ideal of a lattice L, a standard n-ideal if it is a standard element of the lattice of n-ideals $I_n(L)$.

In this chapter, we have given a characterization of standard n-ideals using the concept of standard sublattice when n is a neutral element. For a neutral element n of a lattice L, we prove the following:

(i) an n-ideal is standard if and only if it is a standard sublattice.

(ii) the intersection of a standard n-ideal and n-ideal I of a lattice L is a standard n-ideal in I.

(iii) the principal n-ideal $\langle a\rangle_n$ of a lattice L is a standard n-ideal if and only if $a \vee n$ is standard and $a \wedge n$ is dual standard.

(iv) For an arbitrary n-ideal I and a standard n-ideal S of a lattice L, if $I \vee S$ and $I \wedge S$ are principal n-ideals, then I itself is a principal n-ideal.

"Standard n-ideals"

According to Fried and Schmidt [7, Th.-1], we have a fundamental characterization theorem for standard convex sublattices:

4.1 Theorem: The following conditions are equivalent for each convex sublattice S of a lattice L:

(α) S is a standard sublattice,

(β) Let K be any convex sublattice of L such that $K \cap S \neq \Phi$. Then to each $x \in \langle S, K \rangle$, there exist $s_1, s_2 \in S$, $a_1, a_2 \in K$ such that

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \wedge s_2) \vee (x \wedge a_2)$$

(β') For any convex sublattice K of L and for each s_2, $s_1' \in S$, there are elements s_1, $s_2' \in S$, a_1, $a_2 \in K$ such that

$$x = (x \wedge s_1) \vee (x \wedge (a_1 \vee s_2))$$

$$= (x \wedge s_2') \wedge (x \wedge (a_2 \wedge s_1')),$$

(γ) The relation $\Theta[S]$ on L defined by

$x \equiv y\ (\Theta\ [S])$ if and only if

$$x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$$

and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$ with suitable $t, s \in S$ is a congruence relation.

Following result which is due to [7] shows that the concept of standard sublattices and standard ideals coincides in case of ideals.

4.2 Proposition: [7, Pro.2] An ideal S of a lattice L is Standard if and only if it is a standard sublattice. Recall that an n-ideal I of a lattice L is called a standard n-ideal if it is a standard element of $I_n(L)$, the lattice of n-ideals.

The following theorem gives an extension of proposition 4.2 above.

4.3 Theorem: For a neutral element n of a lattice L, an n-ideal is standard if and only if it is a standard sublattice.

Proof: First assume that an n-ideal S of a lattice L is a standard sublattice. That is, for all convex sublattice I & K of L with

$$S \cap K \neq \Phi \text{ and } I \cap <S, K> \neq \Phi,$$

We have $I \wedge <S, K> = <I \wedge S, I \wedge K>$ and

$$I \dot{\vee} <S, K> = <I \dot{\vee} S, I \dot{\vee} K>.$$

We are to show that S is a standard n-ideal in $I_n(L)$.

That is for all n-ideal I, K $\in I_n(L)$,

$$I \cap (S \vee K) = (I \cap S) \vee (I \cap K).$$

Clearly, $(I \cap S) \vee (I \cap K) \subseteq I \cap (S \vee K)$.

So let $x \in I \cap (S \vee K)$. Then $x \in I$ and $x \in S \vee K$ so by theorem 4.1 we have

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \vee s_2) \wedge (x \vee a_2),$$

for some $s_1, s_2 \in S$ and $a_1, a_2 \in K$.

Now $x = (x \wedge s_1) \vee (x \wedge a_1)$

$$\leqq [(x \wedge s_1) \vee (x \wedge n) \vee (s_1 \wedge n] \vee$$

$$[(x \wedge a_1) \vee (x \wedge n) \vee (a_1 \wedge n)]$$

$$= m(x, n, s_1) \vee m(x, n, a_1),$$

that is $x \leqq m(x, n, s_1) \vee m(x, n, a_1)$

again $x = (x \vee s_2) \wedge (x \vee a_2)$

$$\geqq [(x \vee s_2) \wedge (x \vee n) \wedge (s_2 \vee n)] \wedge$$

$$[(x \vee a_2) \wedge (x \vee n) \wedge (a_2 \vee n)]$$

$$= m^d(x, n, s_2) \wedge m^d(x, n, a_2)$$

$$= m(x, n, s_2) \wedge m(x, n, a_2) \quad \text{as } n \text{ is neutral.}$$

Hence $m(x, n, s_2) \wedge m(x, n, a_2)$

$\leq x \leq m(x, n, s_1) \vee m(x, n, a_1)$

Which implies $x \in (I \cap S) \vee (I \cap K)$.

Thus, $I \cap (S \vee K) \quad (I \cap S) \vee (I \cap K)$ and so S is a standard n-ideal.

Conversely, Suppose that n-ideal S of a Lattice L is standard. Consider any convex sublattice K of L such that $S \cap K \neq \Phi$. Since S is an n-ideal, clearly

$<S, K> = <S, <K>_n>$. Let $x \in <x>_n \cap (S, <K>_n)$

$= (<x>_n \cap S) \vee (<x>_n \cap <K>_n)$, as S is a standard n-ideal. This implies

$$<x>_n = (<x>_n \cap S) \vee (<x>_n \cap <K>_n) \quad \ldots\ldots\ldots \; (1)$$

Since $x \vee n$ is the largest element of $<x>_n$,

So we have $x \vee n = m(x \vee n, n, s_1) \vee m(x \vee n, n, t)$

for some $s \in S$. $t \in <K>_n$.

$$((x \vee n) \wedge s_1) \vee ((x \vee n) \wedge t) \vee n$$

$= (x \wedge s_1) \vee ((x \wedge t) \vee n)$, as n is neutral.

Now, $t \in \langle K\rangle_n$ implies $t \leq t_1 \vee n$ for some $t_1 \in K$.

Then $x \vee n \leq (x \wedge s_1) \vee (x \wedge (t_1 \vee n)) \vee n$

$(x \wedge s_1) \vee (x \wedge t_1) \vee n$

$\leq (x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n \leq x \vee n$

which implies that

$$x \vee n = (x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n$$

Then $x = x \wedge (x \vee n)$

$x \wedge [(x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n]$

$[x \wedge \{(x \wedge (s_1 \vee n)) \vee (x \wedge t_1)\}] \vee (x \wedge n)$,

as n is neutral.

$=(x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee (x \wedge n)$

$=(x \wedge (s_1 \vee n)) \vee (x \wedge t_1)$,

where $s_1 \vee n \in S$, $t_1 \in K$.

Since $x \wedge n$ is the smallest element of $\langle x\rangle_n$, using the relation (1) a dual proof of above shows that

$x=(x\vee(s_2\wedge n))\wedge(x\vee t_2)$ for some $s_2\in S$, $t_2\in K$. Hence from Th.4.1.(β) we obtain that S is a standard sublattice.

Now, we give characterizations for standard n-ideals when n is a neutral element. We prefer to call it the "Fundamental characterization Theorem" for standard n-ideals.

4.4 Theorem: If n is a neutral element of a lattice L. Then the following conditions are equivalent:

(a) S is a standard n-ideal;

(b) For any n-ideal K,

$$S\vee K = \{x: x=(x\wedge s_1)\vee(x\wedge k_1)$$

$$=(x\wedge s_1')\vee(x\wedge k_1')\vee(x\wedge n)$$

and $$x=(x\vee s_2)\wedge(x\vee k_2)$$

$$= (x\vee s_2')\wedge(x\vee k_2')\wedge(x\vee n)$$

For some $s_1, s_2, s_1', s_2' \in S$; $k_1, k_2, k_1', k_2' \in K$ }.

(c) The relation $\Theta(S)$ on L defined by $x\equiv y$ $\Theta(S)$ if and only if $x\wedge y=((x\wedge y)\vee t)\wedge(x\vee y)$ and $x\vee y=((x\vee y)\wedge s)\vee(x\wedge y)$, for some $t,s\in S$, is a congruence relation.

Proof: (a) → (b). Suppose S is a standard n-ideal and K be any n-ideal. Let $x\in S\vee K$. Since K is also a convex sublattice of L, we have from the proof of theorem 4.1.3, $x=(x\wedge(s_1\vee n))\vee(x\wedge t_1)$

$=(x\vee(s_2\wedge n))\wedge(x\vee t_2)$ for some $s_1,s_2\in S$; $t_1,t_2\in K$. Since n is neutral, from above we also have

$$x=(x\wedge s_1)\vee(x\wedge t_1)\vee(x\wedge n)$$

$$=(x\vee s_2)\wedge(x\vee t_2)\wedge(x\vee n).$$

Thus (b) holds. (b) → (c). Let (b) holds. Let $\Theta(S)$ be defined as $x\equiv y$ $\Theta(S)$ if and only if

$x\wedge y=((x\wedge y)\vee t)\wedge(x\vee y)$ and $x\vee y=((x\vee y)\wedge s)\vee(x\wedge y)$.

For $x>y$,

$y=(y\vee t)\wedge x$ and $x=(x\wedge s)\vee y$, for some $t,s\in S$, with $s>t$.

Obviously, $\Theta(S)$ is reflexive and symmetric.

Moreover $x\equiv y\ \Theta(S)$ if and only if $x\wedge y\equiv x\vee y\ \Theta(S)$ Now suppose $x>y>z$ with $x\_y\ \Theta(S)$ and $y\equiv z\ \Theta(S)$.

Then $x=(x\wedge s_1)\vee y$, $y=(y\vee t_1)\wedge x$ and

$y=(y\wedge s_2)\vee z$, $z=(z\vee t_2)\wedge y$ for some $s_1,s_2,t_1,t_2 \in S$.

Then $x=(x\wedge s_1)\vee y = (x\wedge s_1)\vee(y\wedge s_2)\vee z$

$<(x\wedge s_1)\vee(x\wedge s_2)\vee z$

$<(x\wedge(s_1\vee s_2))\vee z<x$,

Which implies $x=(x\wedge(s_1\vee s_2))\vee z$.

Similarly, we can show that $z=(z\vee(t_1\wedge t_2))\wedge x$.

This shows that $x \equiv z\ \Theta(S)$.

For the substitution property, suppose $x>y$ and $x\equiv y\ \Theta(S)$. Then $x=(x\wedge s)\vee y$ and $y=(y\vee t)\wedge x$, for some $s,t\in S$. From these relations it is easy to find $s,t\in S$ with $\underline{t<s}$ satisfying the relations. Then for every $z\in L$, $y\wedge z<x\wedge z$

and $y\wedge z < t\vee(y\wedge z)$.

Therefore $y\wedge z < (t\vee(y\wedge z))\wedge(x\wedge z)$

$$< (t\vee y)\wedge(x\wedge z)$$

$$= ((t\vee y)\wedge x)\wedge z$$

$$- y\wedge z.$$

This implies $y\wedge z = (t\vee(y\wedge z))\wedge(x\wedge z)$.

Let K be the n-ideal $<t\wedge y\wedge z, y>_n$.

Since $s, t\wedge y\wedge z \in S\vee K$, so by the convexity of

$S\vee K$, $t\wedge y\wedge z < t\wedge y < t\wedge x < s\wedge x < s$ as $t<s$.

This implies that $s\wedge x \in S\vee K$

Hence $x = (s\wedge x)\vee y \in S\vee K$.

Also, by the convexity of $S\vee K$, $t\wedge y\wedge z<y\wedge z<x\wedge z<x$

implies $y\wedge z$, $x\wedge z \in S\vee K$. Then by (b)

we have

$$x\wedge z = (x\wedge z\wedge s_1)\vee(x\wedge z\wedge k_1)\vee(x\wedge z\wedge n)$$

for some $s_1\in S$, $k_1\in K$.

$$(x\wedge z\wedge s_1)\vee(x\wedge z\wedge(y\vee n))\vee(x\wedge z\wedge n),$$

as $y \vee n$ is the largest element of K.

$=(x \wedge z \wedge s_1) \vee (y \wedge z) \vee (x \wedge z \wedge n)$, as n is neutral.

$= ((x \wedge z) \wedge (s_1 \vee n)) \vee (y \wedge z)$,

where $s_1 \vee n \in S$. Therefore $x \wedge z \equiv y \wedge z\ \Theta(S)$ dually we can prove $x \vee z \equiv y \vee z \Theta(S)$. Therefore using [15, Lemma 8.p-74], $\Theta(S)$ is a congruence relation.

Hence (c) holds.

Finally, we shall show that (c) → (a).

Let (c) holds. For any n-ideals I,K of L, obviously

$(I \cap S) \vee (I \cap K) \subseteq I \cap (S \vee K)$. To prove the reverse

Inequality, suppose $x \in I \cap (S \vee K)$.

Then $x \in I$ and $x \in S \vee K$. Since $x \in S \vee K$, it is easy to find the elements $s_1, s_2 \in S$, $k_1, k_2 \in K$ with $s_1 < n < s_2$ and $\underline{k_1 < n < k_2}$ such that $s_1 \wedge k_1 < x < s_2 \vee k_2$.

Now, $s_1 \equiv s_2\ \Theta(s)$ implies $s_2 \vee k_2 \equiv s_1 \vee k_2 = k_2\ \Theta(S)$.

Since $x < s_2 \vee k_2$,

we have $x = x\wedge(s_2 \vee k_2)$

$\equiv x\wedge k_2\ \Theta(S)$. Then by (c)

$x = (x\wedge s)\vee(x\wedge k_2)$ for some $s \in S$.

$\leq m(x,n,s)\vee m(x,n,k_2)$.

Also $s_1 \equiv s_2\ \Theta(S)$ implies $s_1\wedge k_1 \equiv s_2\wedge k_1 = k_1\ \Theta(s)$.

So, $x = x\vee(s_1\wedge k_1) \equiv x\vee k_1\ \Theta(s)$.

Applying (c) again we have

$x = (x\vee t)\wedge(x\vee k_1)$ for some $t\in S$.

$\geq m^d(x,n,t)\wedge m^d(x,n,k_1)$

$= m(x,n,t) \wedge m(x,n,k_1)$, as n is neutral.

Hence $x\in(I\cap S)\vee(I\cap K)$.

This implies $I\cap(S\vee K) = (I\cap S)\vee(I\cap K)$.

Therefore (a) holds.

4.5 Corollary: Suppose n is a neutral element of a lattice L. Then for a standard n-ideal S of L, $\Theta(S)$ is the smallest congruence relation of L containing S as a class.

Proof: Clearly any two elements of S are related by $\Theta(S)$.

Now suppose $x \equiv y\ \Theta(S)$ with $x > y$.

Then by theorem 4.4, we have $y=(y \vee t) \wedge x$ and

$x= (x \wedge s) \vee y$ for some $s,t \in$. Suppose $y \in S$.

Then $y \leq x = (x \wedge s) \vee y < y \vee s$. Then, by the convexity of S,

$x \in S$. On the other hand, if $x \in S$, then

$x > y=(y \vee t) \wedge x > t \wedge x$ implies $y \in S$.

Hence $\Theta(S)$ contains S as a class.

Let Φ be a congruence relation containing S as a class. We have $x \equiv y\ \Theta(S)$ with $x > y$,

$x=(x \wedge s) \vee y$ and $y=(y \vee t) \wedge x$ for some $s,t \in S$.

Now, $x = (x \wedge s) \vee y \equiv (x \wedge n) \vee y \Phi$

$(x \vee y) \wedge (n \vee y)$, as n is neutral.

$= x \wedge (n \vee y) - x \wedge (y \vee t) \Phi = y\Phi$.

This implies $\Theta(S) \subset \Phi$. Hence $\Theta(S)$ is the smallest congruence containing S as a class.

4.6 Corollary: If n is a neutral element and S and T are two standard n-ideals of a lattice L, then $S \cap T$ is a standard n-ideal.

Proof: Clearly $S \cap T$ is an n-ideal. Suppose

$x \equiv y \ (\Theta(S) \cap \Theta(T))$ with $x > y$. Since $x \equiv y\ \Theta(S)$, so

we have $x = (x \wedge s_1) \vee y$ and $y = (y \vee s_2) \wedge x$, for some $s_1, s_2 \in S$. Here we can consider $s_2 < n < s_1$.
Now $x \equiv y\ \Theta(T)$ implies $x \wedge s_1 - y \wedge s_1\ \Theta(T)$, and so there exists $t_1 \in T$, $t_1 > n$ such that

$x \wedge s_1 = ((x \wedge s_1) \wedge t_1) \vee (y \wedge s_1)$.

Then $x = (x \wedge s_1) \vee y = [((x \wedge s_1) \wedge t_1) \vee (y \wedge s_1)] \vee y$

$(x \wedge s_1 \wedge t_1) \vee y = (x \wedge (s_1 \wedge t_1)) \vee y$.

Again $x \equiv y\ \Theta(T)$ implies $x \vee s_2 \equiv y \vee s_2\ \Theta(T)$. Then we can find $t_2 \in T$ with $t_2 \leqq n$ such that

$$y \vee s_2 = ((y \vee s_2) \vee t_2) \wedge (x \vee s_2).\ \text{Then}$$

$$\begin{aligned} y = (y \vee s_2) \wedge x &= [((y \vee s_2) \vee t_2) \wedge (x \vee s_2)] \wedge x \\ &= (y \vee s_2 \vee t_2) \wedge (x \vee s_2) \wedge x \\ &= (y \vee (s_2 \vee t_2)) \wedge x. \end{aligned}$$

Now, $n \leqq s_1 \wedge t_1 \leqq s_1$ and $n \leqq s_1 \wedge t_1 \leqq t_1$ implies $s_1 \wedge t_1 \in S \cap T$. Also $s_2 \leqq s_2 \vee t_2 \leqq n$ and $t_2 \leqq s_2 \vee t_2 \leqq n$ implies $s_2 \vee t_2 \in S \cap T$. Hence $x \equiv y\ \Theta(S \cap T)$. Therefore

$$\Theta(S \cap T) = (\Theta(S) \cap \Theta(T)).$$

Hence by Theorem 4.4 $S \cap T$ is also a standard n-ideal.

4.7 Corollary: Let n be a neutral element of a lattice L and S be a standard n-ideal. Then $x \equiv y\ \Theta(S)$ if and only if

$$\langle x \rangle_n \vee S = \langle y \rangle_n \vee S.$$

Proof: Let $x \quad y\ \Theta(S)$. Then for $x > y$, we have

$x = (x \wedge s_1) \vee y$ and $y = (y \vee s_2) \wedge x$ for some s_1 , $s_2 \in S$.

This implies $x \vee s_1 = y \vee s_1$, $x \wedge s_2 = y \wedge s_2$

Now, $y < x < x \vee s_1 = y \vee s_1$, which implies $x \in \langle y \rangle_n \vee S$. On the other hand, $x \wedge s_2 = y \wedge s_2 < y < x$ implies $y \in \langle x \rangle_n \vee S$

Hence $\langle x \rangle_n \vee S = \langle y \rangle_n \vee S$. Conversely

suppose that $\langle x \rangle_n \vee S = \langle y \rangle_n \vee S$.

As $x \in \langle y \rangle_n \vee S \quad \langle y \rangle_n \vee S$, so

By Theorem 4.4, $x \quad (x \wedge y_1) \vee (x \wedge s)$,

for some $y_1 \in \langle y \rangle_n$, $s \in S$.

$$(x \wedge (y \vee n)) \vee (x \wedge s)$$

$$(x \wedge y) \vee (x \vee n)) \vee (x \wedge s)$$

$$= y \vee [x \wedge (n \vee s), \text{ as } n \text{ is neutral.}$$

Also, $y \in \langle y \rangle_n \vee S = \langle x \rangle_n \vee S$. Then applying Th. 4.4 again we have $y = (y \vee x_1) \wedge (y \vee s')$, For some $x \in \langle x \rangle_n$, $s' \in S$.

Then $y = (y \vee (x \wedge n)) \wedge (y \vee s')$

$= (y \vee x) \wedge (y \vee n)) \wedge (y \vee s')$

$= x \wedge [y \vee (n \wedge s')]$, as n is neutral.

Since $n \vee s$, $n \wedge s' \in S$, so we have

$$x \equiv y\ \Theta(S).$$

We know from [18] that the intersection of a standard ideal with an arbitrary ideal I of a lattice L is standard in I.

Following lemma is a generalization of this result.

4.8 Lemma : The intersection of a standard n-ideal and an n-ideal I of a lattice L is a standard n-ideal in I, where n is a neutral element.

Proof: Let S be a standard n-ideal of L. We are to show that $S \cap I$ is a standard n-ideal in I. Consider an n-ideal K of I, which is also an n-ideal of L. Now, let $x \in (S \cap I) \vee K \subseteq S \vee K$. Since S is standard, so we have by theorem 4.4, $x = (x \wedge s) \vee (x \wedge k)$, for some $s \in S$, $k \in K$. By the monotionity, we can choose both $s > n$, $k > n$.

put s' $(x \vee n) \wedge s$. Then $s' < s$

and $n = (x \vee n) \wedge n < (x \vee n) \wedge s = s' < x \vee n$.

Since $x \vee n \in I$, so by convexity of S and I,

$s' \in S \cap I$. Also $x \wedge s' = x \wedge s$. Thus

$x = (x \wedge s') \vee (x \wedge k)$, for some $s' \in S \cap I$, $k \in K$.

Also, by duality we get $x = (x \vee s'') \wedge (x \vee k')$

for some $s'' \in S \cap I$, $k' \in K$.

Hence by theorem 4.4,

We have $S \cap I$ is standard in I.■

4.9 Lemma: Let n be a neutral element of a lattice L and Φ is a homomorphism of L onto a lattice L' such $\Phi(n) = n'$, $n' \in L'$. Then for any standard n-ideal I for L, $\Phi(I)$ is a standard n'-ideal of L'.

Proof: Clearly $\Phi(I)$ is a sublattice of L'. Let $p<t<q$, where $p, q \in \Phi(I)$, $t \in L'$. Then $p = \Phi(x)$ and $q = \Phi(y)$ for some $x,y \in I$. Since Φ is onto, $t = \Phi(r)$

for some $r \in L$.

Then $\Phi(r) = \Phi(r) \wedge \Phi(y) \quad \Phi(r\wedge y)$

And $\Phi(r) = \Phi(r) \vee \Phi(x)$

$\Phi(x) \vee \Phi(r\wedge y)$

$= \Phi(x \vee(r \wedge y))$

Now, $x<x\vee(r\wedge y)<x\vee y$ and so by convexity we have

$x \vee(r \wedge y) \in I$. Thus $t = \Phi(x \vee(r \wedge y)) \in \Phi(I)$.

Hence $\Phi(I)$ is a convex sublattice of L'.

Moreover $\Phi(n)=n'$ implies $\Phi(I)$ is an n'-ideal of L'.

For standardness, we shall prove (b) of theorem 4.4 for $\Phi(I)$. Let k' be any n'-ideal of L'. Then $k' = \Phi(k)$ for some n-ideal K of L.

Let $y \in \Phi(I) \vee \Phi(K) \subseteq \Phi(I \vee K)$.

Then $y = \Phi(x)$ for some $x \in I \vee K$. Since I is a standard n-ideal of L, using (b) of Theorem 4.4

we have $x = (x \wedge i_1) \vee (x \wedge k_1) \vee (x \wedge n)$,

for some $i_1 \in I$, $k_1 \in K$

$= (x \vee i_2) \wedge (x \vee k_2) \wedge (x \vee n)$,

For some $i_2 \in I$, $k_2 \in K$.

Then $y = \Phi(x)$

$= \Phi(x \wedge i_1) \vee \Phi(x \wedge k_1) \vee \Phi(x \wedge n)$

$= [\Phi(x) \wedge \Phi(i_1)] \vee [\Phi(x) \wedge \Phi(k_1)] \vee [\Phi(x) \wedge \Phi(n)]$

$= [y \wedge \Phi(i_1)] \vee [y \wedge \Phi(k_1)] \vee [y \wedge n']$.

Also, $y = \Phi(x)$

$= [y \vee \Phi(i_2)] \wedge [y \vee \Phi(k_2)] \wedge [y \vee n']$.

Then using (b) of theorem 4.4 again, $\Phi(I)$ is a standard n'-ideal of L'.

From Grätzer and Schmidt [18], we know that ideal (s] is standard if and only if s is standard in L. One may ask the question heather this is true for principal n-ideal when n is a neutral element. In fact this not even true when L is a complemented lattice. Figure 4.1 and Figure 4.2 Exhibits the complemented lattice L, where n is neutral. There $\langle a\rangle_n$ is standard in $I_n(L)$ but a is not standard in L. Moreover b is standard in L but $\langle b\rangle_n$ is not standard.

4.10 Lemma: For a neutral element n, the principal n-ideal $\langle a\rangle_n$ of a lattice L is a standard n-ideal if and only if $a\vee n$ is standard and $a\wedge n$ is dual standard.

Proof: First suppose that $a\vee n$ is standard and $a\wedge n$ is dual standard. We are to show that $\langle a\rangle_n$ is a standard n-ideal. Let us define a relation

$\Theta(\langle a\rangle_n)$ on L by $x \equiv y\ \Theta(\langle a\rangle_n)$ if and only if

$$x\wedge y = ((x\wedge y)\vee t)\wedge(x\vee y)$$

and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$ for some $t, s \in \langle a \rangle_n$. For $x > y$, we have

$x = (x \wedge s) \vee y$ and $y = (y \vee t) \wedge x$. Clearly $\Theta(\langle a \rangle_n)$ is reflexive and symmetric.

Also $x \equiv y\Theta(\langle a \rangle_n)$ if and only if $x \wedge y \equiv x \vee y$ $\Theta(\langle a \rangle_n)$. Now, let $x > y > z$ and $x \equiv y$ $\Theta(\langle a \rangle_n)$ and $y \equiv z$ $\Theta(\langle a \rangle_n)$. Then

$X = (x \wedge s) \vee y$, $y = (y \vee t) \wedge x$ and $y = (y \wedge p) \vee z$,

$z - (z \vee q) \wedge y$, for some $s, t, p, q \in \langle a \rangle_n$.

Now $x - (x \wedge s) \vee y$

$= (x \wedge s) \vee (y \wedge p) \vee z$

$< (x \wedge s) \vee (x \wedge p) \vee z$

$< [x \wedge (s \vee p)] \vee z < x$,

which implies $x = (x \wedge (s \vee p)) \vee z$.

Also, $z = (z \vee q) \wedge y$

$= (z \vee q) \wedge (y \vee t) \wedge x$

$$\geq (z \vee q) \wedge (z \vee t) \wedge x$$

$$\geq (z \vee (q \wedge t)) \wedge x \geq z,$$

which implies $z = (z \vee (q \wedge t)) \wedge x$.

Hence $x \equiv z\ \Theta(<a>_n)$.

To prove the substitution property,

let $x \equiv y \Theta(<a>_n)$, $x \geq y$ and $r \in L$. Then $x \quad (x \wedge s) \vee y$

and $y = (y \vee t) \wedge x$ for some $s, t \in <a>_n$.

Since $s, t \in <a>_n$, $a \wedge n \leq s, t \leq a \vee n$. Set $s = a \vee n$,

$t = a \wedge n$.

Then we have

$$x = (x \wedge s) \vee y = y \vee [x \wedge (a \vee n)]$$

$$= x \wedge (y \vee a \vee n), \text{ as } a \vee n \text{ is standard.}$$

Therefore, $x \wedge r = x \wedge r \wedge (y \vee a \vee n)$

$$- (x \wedge r \wedge y) \vee [(x \wedge r) \wedge (a \vee n)]$$

$$[(x \wedge r) \wedge (a \vee n)] \vee (y \wedge r).$$

On the other hand, $y = (y \vee t) \wedge x$

$$= (y \vee (a \wedge n)) \wedge x$$

and so $\quad y \wedge r = [(y \vee (a \wedge n)) \wedge x] \wedge r$

$$= (y \vee (a \wedge n)) \wedge (x \wedge r)$$

$$> [(y \wedge r) \vee (a \wedge n)] \wedge (x \wedge r)$$

$$> y \wedge r.$$

Thus, $y \wedge r = [(y \wedge r) \vee (a \wedge n)] \wedge (x \wedge r)$.

Therefore, $x \wedge r \equiv y \wedge r \ \Theta(<a>_n)$.

Again, $y = (y \vee t) \wedge x \ = x \wedge (y \vee (a \wedge n))$

$= y \vee (x \wedge (a \wedge n))$, as $a \wedge n$ is dual standard.

Therefore, $y \vee r = y \vee r \vee (x \wedge (a \wedge n))$

$- (y \vee r \vee x) \wedge ((y \vee r) \vee (a \wedge n))$,

$$= (x \vee r) \wedge [(y \vee r) \vee (a \wedge n)].$$

On the other hand $x = (x \wedge s) \vee y$

$$= (x \wedge (a \vee n)) \vee y$$

and so, $x\vee r = (x\wedge(a\vee n))\vee y\vee r$

$$< [(x\vee r)\wedge(a\vee n)]\vee(y\vee r)$$

$$< x\vee r.$$

Thus $x\vee r = [(x\vee r)\wedge(a\vee n)]\vee(y\vee r)$

Therefore $x\vee r \equiv y\vee r\ \Theta(<a>_n)$. Hence $\Theta(<a>_n)$ is a congruence relation. Thus by theorem 4.4, $<a>_n$ is a standard n-ideal.

Conversely, suppose that $<a>_n$ is a standard n-ideal. We shall show that $a\vee n$ is standard and $a\wedge n$ is dual standard. Since $<a>_n$ is standard so for any principal n-ideals $<x>_n$, $<y>_n$ we have $<x>_n \cap (<a>_n \vee <y>_n) = (<x>_n \cap <a>_n) \vee (<x>_n \cap <y>_n)$.

Then by some routine calculations , we get

$[(x\wedge n)\vee\{(a\wedge n)\wedge(y\wedge n)\} ,(x\vee n)\wedge\{(a\vee n)\vee(y\vee n)\}]$

$=[\{(x\wedge n)\vee(a\wedge n)\}\wedge\{(x\wedge n)\vee(y\wedge n)\},\{(x\vee n)\wedge(a\vee n)\}\vee\{(x\vee n)\wedge(y\vee n)\}]$ … … (1)

This implies, $(x\vee n)\wedge\{(a\vee n)\vee(y\vee n)\}$

$$=\{(x\vee n)\wedge(a\vee n)\}\vee\{(x\vee n)\wedge(y\vee n)\}.$$

Since n is neutral, so

$$\text{L.H.S} = (x\vee n)\wedge\{(a\vee n)\vee(y\vee n)\}$$

$$- (x\vee n)\wedge(a\vee n\vee y)$$

$$- [x\wedge(a\vee n\vee y)]\vee n,$$

and

$$\text{R.H.S} = [(x\vee n)\wedge(a\vee n)]\wedge[(x\vee n)\wedge(y\vee n)]$$

$$= n\vee(x\wedge(a\vee n))\vee(x\wedge y)\vee n,$$

$$= (x\wedge y)\vee(x\wedge(a\vee n))\vee n.$$

Let $A = x\wedge(y\vee(a\vee n))$

and $B = (x\wedge y)\vee(x\wedge(a\vee n))$.

Now, $A\wedge n = x\wedge(y\vee(a\vee n))\wedge n = x\wedge n$

and $B\wedge n = [(x\wedge y)\vee(x\wedge(a\vee n))]\wedge n = x\wedge n$.

So by neutrality of n, A=B. That is,

$$x\wedge(y\vee(a\vee n)) = (x\wedge y)\vee(x\wedge(a\vee n)).$$

This implies $a \vee n$ is standard.

Also, from (1) we get

$(x \wedge n) \vee \{(a \wedge n) \wedge (y \wedge n)\}$

$\{(x \wedge n) \vee (a \wedge n)\} \wedge \{(x \wedge n) \vee (y \wedge n)\}$.

Then, from (1) we get

$(x \wedge n) \vee \{(a \wedge n) \wedge (y \wedge n)\} =$

$\{(x \wedge n) \vee (a \wedge n)\} \wedge \{(x \wedge n) \vee (y \wedge n)\}$.

Then applying the similar technique we can show that

$$x \vee ((a \wedge n) \wedge y) - (x \vee (a \wedge n)) \wedge (x \vee y).$$

This implies $a \wedge n$ is dual standard.

In a distributive lattice, it is well known that if the infimum and supremum of two ideals are principal, the infimum and supremum of two ideals are principal, then both of them are principal. In [18, lemma 8.], Grätzer and Schmidt have generalized that result for standard ideals. They showed that in an arbitrary lattice L, if I is an arbitrary ideal and S is standard ideal of L, and if $I \vee S$ and $I \wedge S$ are principal, then I itself is a principal ideal. The following theorem is a generalization of their result. To prove this we need the following Lemma:

4.11 Lemma: Let n be a neutral element of a lattice L. Then any finitely generated n-ideal which is contained in a principal n-ideal is principal.

Proof: Let $[b,c]$ be a finitely generated n-ideal such that $b < n < c$. Let $\langle a \rangle_n$ be a principal n-ideals which contains $[b,c]$. Then $a \wedge n \leqq b \leqq n \leqq c \leqq a \vee n$. Suppose $t = (a \vee b) \wedge c$. Since n is neutral, we have

$$n \wedge t = n \wedge [(a \vee b) \wedge c] = n \wedge (a \vee b)$$

$$= (n \wedge a) \vee (n \wedge b) = n \wedge b = b,$$

and $\quad n \vee t \quad n \vee [(a \vee b) \wedge c]$

$$(n \vee a \vee b) \wedge (n \vee c)$$

$= (n \vee a) \wedge c \quad c.$

Hence $[b,c] = [n \wedge t,\ n \vee t] = \langle t \rangle_n$.

Therefore $[b,c]$ is a principal n-ideal.

4.12 Theorem: Let I be an arbitrary n-ideal and S be a standard n-ideal of a lattice L, where n is neutral. If $I \vee S$ and $I \cap S$ are principal n-ideals, then I itself is a principal n-ideal.

Proof: Let $I \vee S = \langle a \rangle_n = [a \wedge n,\ a \vee n]$ and $I \cap S = \langle b \rangle_n = [b \wedge n,\ b \vee n]$. Since S is a standard n-ideal, then by theorem 4.4,

$a \vee n = [(a \vee n) \wedge s] \vee ((a \vee n) \wedge x) \quad$ for some $s \in S$, $x \in I$

$s \vee x.$

Again, $a \wedge n \in S \vee I$ So by theorem 4.4, again there exist $s_1 \in S$ and $x_1 \in I$ such that

$a \wedge n = ((a \wedge n) \vee s_1) \wedge ((a \wedge n) \vee x_1) = s_1 \wedge x_1.$

Now, consider the n-ideal $[b \wedge x_1 \wedge n,\ b \vee x \vee n]$. Obviously, $[b \wedge x_1 \wedge n,\ b \vee x \vee n] \subseteq I \subseteq \langle a \rangle_n$. So by above lemma, $[b\wedge x_1\wedge n,\ b\vee x\vee n]$ is a principal n-ideal say $<t>_n$ for some $t\in L$.

Then $<a>_n \quad I\vee S \supset S \vee [b\wedge x_1\wedge n,\ b\vee x\vee n]$

$$\supseteq [s_1\wedge n,\ s\vee n] \vee [b\wedge x_1\wedge n,\ b\vee x\vee n]$$

$$[s_1\wedge n\wedge b\wedge x_1\wedge n,\ s\vee n\vee b\vee x\vee n]$$

$$= [a\wedge n,\ a\vee n] = <a>_n.$$

This implies $S\vee I = S \vee [b\wedge x_1\wedge n,\ b\vee x\vee n]$

$$= S\vee <t>_n \ \ldots\ \ldots\ \ldots\ (A)$$

Further, $<b>_n \quad S\cap I \supset S\cap[b\wedge x_1\wedge n,\ b\vee x\vee n]$

$$\supset S\cap[b\wedge n,\ b\vee n] = <b>_n, \text{ as}$$

$b\wedge x_1\wedge n < b\wedge n < b\vee n < b\vee x\vee n$. This implies

$S\cap I = S\cap[b\wedge x_1\wedge n,\ b\vee x\vee n] = S\cap <t>_n \ \ldots\ ..\ ..\ \ (B)$

Since S is standard so we have from (A) & (B),

$I = <t>_n$. Therefore I is a principal n-ideal.

In this section we shall deduce some important properties of standard elements and n-ideals from the fundamental characterization theorem. If S is a standard n-ideal, then we call the congruence relation $\Theta(S)$, generated by S, a standard n-congruence relation. If $S= \langle s\rangle_n$, then $\Theta(S) = \Theta(\langle s\rangle_n)$ and so $\Theta(\langle s\rangle_n)$ is a standard n-congruence relation which we call principal standard n-congruence . Firstly, we prove some results on the connection between standard n-ideals and standard n-congruence relations.

4.13 Theorem: Let n be neutral element of a lattice L. Let S and T be two standard n-ideals of L. Then

$$\text{(i)} \quad \Theta(S\cap T) = \Theta(S) \cap \Theta(T)$$

$$\text{(ii)} \quad \Theta(S\vee T) = \Theta(S) \vee \Theta(T).$$

Proof: (i) This has already been proved in corollary 4.6,

(ii) Clearly, $\Theta(S) \vee \Theta(T) \subseteq \Theta(S\vee T)$. To prove the reverse inequality,

let $x\equiv y$ $\Theta(S\vee T)$ with $x\geqq y$.

Then $y = (y \vee p) \wedge x$ and $x = (x \wedge q) \vee y$,

for some p , $q \in S \vee T$.

Then by theorem 4.4,

$P = (p \wedge s_1) \vee (p \wedge t_1)$ and $p = (p \vee s_2) \wedge (p \vee t_2)$,

$q = (q \wedge s_3) \vee (q \wedge t_3)$ and $q = (q \vee s_4) \wedge (q \vee t_4)$

for some s_1 , s_2 , s_3 , $s_4 \in S$ and t_1 , t_2 , t_3 , $t_4 \in T$.

Now, $P = (p \wedge s_1) \vee (p \wedge t_1)$

$\equiv (p \wedge n) \vee (p \wedge t_1) \;\; \Theta(S)$

$(p \wedge n) \vee (p \wedge n) \;\; \Theta(T)$

$= p \wedge n.$

Thus, $p \equiv p \wedge n \; (\Theta(S) \vee \Theta(T))$

Again, $p = (p \vee s_2) \wedge (p \vee t_2)$

$\equiv (p \vee n) \wedge (p \vee t_2) \;\; \Theta(S)$

$\equiv (p \vee n) \wedge (p \vee n) \;\; \Theta(T)$

$= p \vee n.$

Thus, $p \equiv p \vee n \; (\Theta(S) \vee \Theta(T))$. This implies

$$p \wedge n \equiv p \vee n \ (\Theta(S) \vee \Theta(T))$$

and so $p \equiv n \ (\Theta(S) \vee \Theta(T))$.

Similarly, we have $q \equiv n \ (\Theta(S) \vee \Theta(T))$.

Now, $y = (y \vee p) \wedge x$

$\equiv (y \vee n) \wedge x \ \ (\Theta(S) \vee \Theta(T))$

$= (y \wedge x) \vee (n \wedge x)$, as n is neutral.

$= y \vee (x \wedge n)$

$\equiv y \vee (x \wedge q) \ (\Theta(S) \vee \Theta(T))$

$= x$.

This implies $x \equiv y \ (\Theta(S) \vee \Theta(T))$.

Therefore, $\Theta(S \vee T) = (\Theta(S) \vee \Theta(T))$,

which proves (ii).

4.14 Lemma: Let s be a standard element of alattice L and 'a' be an arbitrary element of L. Then $m(a,n,s)$ is standard in $\langle a \rangle_n$, where n is neutral in L.

Proof: Let $p,q \in \langle a \rangle_n$. Then $a\wedge n < p,q < a\vee n$.

Also $p = p\wedge(a\vee n) = (p\wedge a)\vee(p\wedge n)$, and

$q = q\wedge(a\vee n) = (q\wedge a)\vee(q\wedge n)$, as n is neurtal.

Let $r = m(a,n,s)$.

Now, $p\wedge(q\vee r) =$

$p\wedge[\{(q\wedge a)\vee(q\wedge n)\}\vee\{(a\wedge n)\vee(a\wedge s)\vee(n\wedge s)\}]$

$= p\wedge[\{(q\wedge a)\vee(q\wedge n)\}\vee\{(a\wedge s)\vee(n\wedge s)\}]$, as $q\wedge a > a\wedge n$.

$= p\wedge[\{q\wedge(a\vee n)\}\vee\{s\wedge(a\vee n)\}]$

$= p\wedge(a\vee n)\wedge(q\vee s)$,

as s is standard.

$= p\wedge(q\vee s)$, as $p < a\vee n$,

$= (p\wedge q)\vee(p\wedge s)$,

as s is standard.

$= (p\wedge q)\vee(p\wedge s)\vee(a\wedge n)$ … … (A)

Also, $p\wedge r = p\wedge m(a,n,s)$

$= p\wedge[(a\wedge n)\vee(a\wedge s)\vee(n\wedge s)]$

$- \ [p\wedge\{(a\wedge n)\vee(a\wedge s)\}]\vee(p\wedge n\wedge s),$

as $n\wedge s$ is standard.

$- \ [p\wedge\{a\wedge(n\vee s)\}]\vee\ (p\wedge n\wedge s),$

as s is standard.

$- \ (p\wedge a\wedge n)\vee(p\wedge a\wedge s)\vee(p\wedge n\wedge s)$

$- \ (p\wedge a\wedge n)\vee[(p\wedge s)\wedge(a\vee n)],$

as n is neutral.

$= (a\wedge n)\vee(p\wedge s).$

Hence from (A), $p\wedge(q\vee r) = (p\wedge q)\vee(p\wedge r)$ and

So $r = m(a,n,s)$ is standard in $\langle a\rangle_n$.■

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