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# Asymptotic Methods for some third order Nonlinear Differential Equations

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# ASYMPTOTIC METHODS FOR SOME THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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# ABSTRACT

In this thesis, we investigate the oscillations of third order nonlinear systems by the asymptotic method. The asymptotic method of Krylov-Bogoliubov-Mitropolskii (KBM) is a popular technique for obtaining analytic solution of a second order nonlinear oscillatory system.

First a third order nonlinear differential system modeling nonoscillatory process and characterized by critical damping is considered and a new perturbation technique is developed, based on the work of Krylov-Bogoliubov-Mitropolskii, to find the solution of the system. Then a method is presented unifying both third order damped and overdamped systems. This method is a generalization of Bogoliubov's asymptotic method and covers all the cases when the roots of the corresponding linear equation are real, real and complex, and real and purely imaginary. Later a third order forced nonlinear differential system modeling oscillatory process is considered and a new perturbation technique is developed to find the solution of the system. The methods are illustrated by several examples.

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# Introduction

Most physical and many biological laws and relations appear mathematically in the form of differential equations, ordinary or partial, linear or nonlinear, autonomous or nonautonomous. It may be noted that, the mathematical techniques available for the treatment of linear equations are well developed. In contrast, techniques for the analyses of the nonlinear systems are less well known and difficult to apply. Exact solutions, that are known, are relatively few, and a large part of the progress in the knowledge of nonlinear systems comes from approximate and graphical solutions. Although most of the differential equations involving physical problems are nonlinear, we can impose some restrictions and linearize the system and then solve them. In the case of small amplitude of oscillations many useful systems are linear, but they become nonlinear when amplitude is increased.

In this thesis, we shall discuss problems on oscillations that can be described by a dynamical systems of third order nonlinear ordinary differential equations. Dynamical systems usually contain a certain number of parameters. Our works will involve only one small positive parameter representing perturbation in the dynamical system.

The purpose of this thesis is to study the effect of small perturbation on a third order nonlinear oscillatory system by the asymptotic method of Krylov-Bogoliubov-Mitripolskii, a widely used technique for obtaining analytic solution of nonlinear differential equation with small nonlinearities. These results involving third order nonlinear oscillatory systems may be used in physics and in various branches of engineering, especially mechanical and electrical engineering.

# Chapter 1

## THE SURVEY AND THE PROPOSAL

In this chapter, we discuss the second and third order nonlinear oscillatory systems. The important results of these works have been summarized in brief.

### 1.1 The Second Order Nonlinear Systems

In 1926 Van dar Pol [1] discussed a technique to investigate the periodic solutions of the nonlinear differential equation

$$\frac{d^2x}{dt^2}+\omega_o^2x=\varepsilon(1-x^2)\frac{dx}{dt}+\varepsilon k\lambda\cos\lambda t \tag{1.1}$$

of the form

$$x=a_1(t)\cos\lambda t+a_2(t)\sin\lambda t \tag{1.2}$$

where $a_1(t)$ and $a_2(t)$ are slowly varying functions of time, such that

$$\frac{da_i}{dt}=O(\epsilon) \quad \text{and} \quad \frac{d^2a_i}{dt^2}=O(\epsilon^2), \text{ where } i = 1, 2.$$

Later in 1947 Krylov-Bogoliubov [2] developed a technique, similar to Van der Pol's technique, obtaining also a periodic solution of the weakly nonlinear second order differential equation,

$$\frac{d^2x}{dt^2}+\omega_o^2x=\varepsilon f(x,\frac{dx}{dt}) \tag{1.3}$$

of the form

$$x=a\cos\varphi \tag{1.4}$$

where

$$\varphi=\omega_o t+\vartheta,$$

$$\frac{da}{dt}=-\frac{\varepsilon}{\omega_o}\sin\varphi f(a\cos\varphi,-a\omega_o\sin\varphi) \tag{1.5a}$$

and

$$\frac{d\vartheta}{dt}=-\frac{\varepsilon}{a\omega_o}\cos\varphi f(a\cos\varphi,-a\omega_o\sin\varphi) \tag{1.5b}$$

To improve the first approximate solution, they also developed a technique to determine the solution of equation (1.2) to any approximation.

In 1956 this method has been extended by Popov [3] for nonlinear damped oscillatory systems and in 1961 it has been amplified and justified by Bogoliubov and Mitropolskii [4]. It was also extended to nonstationary vibrations by Mitropolskii [5] in 1965. They developed an asymptotic expansion of the form

$$x = a\cos\psi + \sum_{n=1}^{N} \varepsilon^n u_n(a,\psi) + O(\varepsilon^{N+1}) \qquad (1.6)$$

where each $u_n$ is a periodic function of $\psi$ with a period of $2\pi$, and $a$ and $\psi$ vary with time according to

$$\frac{da}{dt} = \sum_{n=1}^{N} \varepsilon^n A_n(a) + O(\varepsilon^{N+1}) \qquad (1.7a)$$

and

$$\frac{d\psi}{dt} = \omega_o + \sum_{n=1}^{N} \varepsilon^n \psi_n(a) + O(\varepsilon^{N+1}) \qquad (1.7b)$$

where the functions $u_n$, $A_n$ and $\psi_n$ are chosen such that (1.6) through (1.7b) satisfy the differential equation (1.3).

In 1969, I. S. N. Murty, B. L. Deekshatulu and G. Krisna [6] developed a method to obtain an approximate solution of a second order overdamped nonlinear system governed by the differential equation

$$\frac{d^2x}{dt^2} + k_1\frac{dx}{dt} + k_{11}x = \mu f(x) \qquad (1.8)$$

where $\mu$ is a small parameter with $k_1 = (\lambda_1 + \lambda_2)$, $k_{11} = \lambda_1\lambda_2$.

They found an approximate solution of (1.8) in the form,

$$x(t) = a(t)e^{-\lambda_1 t} + b(t)e^{-\lambda_2 t} + \mu u_1(a,b.t) + \mu^2 u_2(a,b,t) + \ldots \qquad (1.9)$$

They also used another technique to find the solution of (1.8) in the form,

$$x(t)=a(t)+b(t)+\mu u_1(a)+\mu v_1(b)+\mu^2 u_2(a)+\mu^2 v_2(b)+... \quad (1.9a)$$

and they concluded that the results of the two forms are close together when $\mu$ is small.

In 1971, I. S. N. Murty [7] developed a unified method for solving second order nonlinear systems. The method is a generalization of Bogoliubov's asymptotic method and covers all three cases when the characteristic roots of the corresponding linear equation are both real negative (unequal), complex conjugate (with negative real parts), or purely imaginary. He considered the differential equation

$$\frac{d^2x}{dt^2}+2k_1\frac{dx}{dt}+k_{11}x=\mu f(x) \quad (1.10)$$

He assumed a solution according to the asymptotic method in the form

$$x(t)=\frac{a}{2}e^{\psi}+\frac{a}{2}e^{-\psi}+\mu u_1(a,\psi)+\mu^2 u_2(a,\psi)+... \quad (1.11a)$$

or

$$x(t)=\frac{a}{2}e^{\psi}-\frac{a}{2}e^{-\psi}+\mu u_1(a,\psi)+\mu^2 u_2(a,\psi)+... \quad (1.11b)$$

where a and $\psi$ are defined by

$$\frac{da}{dt}=-k_1+\mu A_1(a)+\mu^2 A_2(a)+\ldots \qquad (1.12a)$$

and

$$\frac{d\psi}{dt}=k_2+\mu B_1(a)+\mu^2 B_2(a)+\ldots \qquad (1.12b)$$

with $2k_2=(\lambda_1-\lambda_2)$ and $u_1$, $u_2$,.. are functions of a and $\psi$.

If the roots of the characteristic equation of the corresponding linear equation of (1.10) are real, then $\psi$ is real and (1.11a) and (1.11b) become

$$x=a\cosh\psi+\mu u_1(a,\psi)+\mu^2 u_2(a,\psi)+\ldots \qquad (1.13a)$$

and

$$x=a\sinh\psi+\mu u_1(a,\psi)+\mu^2 u_2(a,\psi)+\ldots \qquad (1.13b)$$

But when the roots are complex conjugate, $\psi$ is purely imaginary and then $\psi$ is replaced by $i\psi$ in (1.12a) and (1.12b); thus the corresponding equations of (1.13a) and (1.13b) are

$$x=a\cos\psi+\mu u_1(a,\psi)+\mu^2 u_2(a,\psi)+\ldots \qquad (1.14a)$$

and

$$x=a\sin\psi+\mu u_1(a,\psi)+\mu^2 u_2(a,\psi)+\ldots \qquad (1.14b)$$

respectively.

In 1986, Sattar [8] has examined the critical damping of second order nonlinear system

$$\frac{d^2x}{dt^2}+2k_1\frac{dx}{dt}+k_2x=\varepsilon f(x) \tag{1.15}$$

on the basis of KBM method, where $\varepsilon$ is a small positive parameter and f(x) is a nonlinear function.

He obtained a solution of (1.15) in the form

$$x(t,\varepsilon)=a(1+\psi)+\varepsilon u(a,\psi)+\varepsilon^2... \tag{1.16}$$

where **a** and $\psi$ are function of t, defined by the differential equations

$$\frac{da}{dt}=-k_1a+\varepsilon A_1(a)+\varepsilon^2+... \tag{1.17a}$$

and

$$\frac{d\psi}{dt}=\omega+\varepsilon B_1(a)+\varepsilon^2... \tag{1.17b}$$

## 1.2 The Third Order Nonlinear Systems

In 1962, Osinski [9] studied damped oscillations modeled by third order nonlinear ordinary and partial differential equations under a special assumption; later Lardner and Bojadziev [10] have investigated a third order partial differential equation removing this restriction. Mulholand [11] also studied the third order nonlinear oscillations.

In 1982, Bojadziev [12] studied the damped oscillations modeled by a 3-dimensional weakly nonlinear autonomous differential system

$$\frac{dx}{dt}=Ax+\varepsilon f(x) \qquad (1.18)$$

where $\varepsilon$ is a small positive parameter $x=(x_1,x_2,x_3)$ is a vector, $f(x)=(f_1(x),f_2(x),f$ is a real vector function in a domain G, and f(Ø)=Ø. A is a real 3x3 constant matrix which has one real nonpositive eigen value $-\xi\,(\xi\geq 0)$ and two complex eigenvalues $-\zeta\pm i\omega$ with a nonpositive real part $-\zeta\,(\zeta\geq 0)$. The strong linear damping in the system is represented by the real parts of the eigen values $-\xi$ and $-\zeta$.

He obtained a solution of (1.18), in the form

$$x(t,\varepsilon)=\varphi a+b[\phi e^{i\psi}+\phi^{*}e^{-i\psi}]+\varepsilon u(a,b,\psi)+\varepsilon^2\ldots \qquad (1.19)$$

where the unknown vector functions $u=(u_1,u_2,u_3)$ is $2\pi$ period in $\psi$.

The scalar variables a, b and $\psi$ are functions of t satisfying the differential equations

$$\frac{da}{dt}=-\xi a+\varepsilon A(a,b)+\varepsilon^2\ldots$$

$$\frac{db}{dt}=-\zeta b+\varepsilon B(a,b)+\varepsilon^2\ldots \qquad (1.20)$$

$$\frac{d\psi}{dt}=\omega+\varepsilon C(a,b)+\varepsilon^2\ldots$$

In 1993, Sattar [13] has studied third order overdamped nonlinear systems. He considered the third order autonomous nonlinear ordinary differential equation

$$\dddot{x}+k_1\ddot{x}+k_2\dot{x}+k_3x=\varepsilon f(x) \qquad (1.21)$$

where dots denote differentiation with respect to t, and f(x) is the given nonlinear function. The constants $k_1$, $k_2$ and $k_3$ are specified by $k_1=\lambda_1+\lambda_2+\lambda_3$, $k_2=\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1$ and $k_3=\lambda_1\lambda_2\lambda_3$ and he found an asymptotic solution of (1.21) in the form

$$x(t,\varepsilon)=a+b(e^{\psi}+e^{-\psi})+\varepsilon u(a,b,\psi)+\varepsilon^2 v(a,b,\psi)+\cdots \qquad (1.22)$$

where **u**, **v** are functions of a, **b** and $\psi$, while a, **b** and $\psi$ are functions of time **t** defined by the differential equations

$$\begin{aligned}\dot{a}&=-k_4a+\varepsilon A(a,b)+\varepsilon^2M(a,b)+\cdots\\ \dot{b}&=-k_5b+\varepsilon B(a,b)+\varepsilon^2N(a,b)+\cdots \qquad (1.23)\\ \dot{\psi}&=-k_6+\varepsilon C(a,b)+\varepsilon^2D(a,b)+\cdots\end{aligned}$$

with $k_4=\lambda_1$, $k_5=\frac{1}{2}(\lambda_2+\lambda_3)$, $k_6=\frac{1}{2}(\lambda_2-\lambda_3)$.

The solutions are obtained as a power series in $\varepsilon$; the series itself is not convergent, but for a fixed number of terms the approximate solution tends to the exact solution as $\varepsilon$ tends to zero.

## 1.3 The proposal

We propose a perturbed system of a third order nonlinear ordinary differential equation

$$\frac{d^3x}{dt^3}+k_1\frac{d^2x}{dt^2}+k_2\frac{dx}{dt}+k_3x=\varepsilon f(x)+E_0\cos\omega t \qquad (1.24)$$

where $\varepsilon$ is a small positive parameter, **f(x)** is a nonlinear function and $E_o$ is a constant. If $E_o = 0$, the system is called autonomous.

In **chapter 2**, the critical damping of third order nonlinear systems are investigated. **Chapter 3** contains a unified method for solving third order nonlinear damped and overdamped systems, and **in chapter 4**, forced oscillation of a third order nonlinear system is investigated.

# Chapter 2

## The Third Order Critically Damped Oscillations

### 2.1 Introduction:

In vibration, frictional and other damping forces act to decrease the amplitude of the oscillation. An interesting case occurs when damping is such that any decrease in it produces oscillations. Such a motion is called critically damped.

Critical damping for a third-order nonlinear differential equation occurs when the discriminent of the characteristic equation of the corresponding linear equation vanishes and therefore, at least two of the characteristic roots are equal. First, we derive an asymptotic solution of a third-order nonlinear differential equation where all roots of the characteristic equation of its corresponding linear equation are equal and then we consider the case where two of the roots are equal.

The KBM (Krylov-Bogoliubov-Mitroploskii) method [2,4] is one of the widely used techniques for obtaining analytic solutions of the systems with small nonlinearites. The method which was developed originally for finding periodic solutions of nonlinear equation, has been extended by Popov [3] for second order nonlinear damped oscillatory systems, and later by Murty, Deekshatulu and Krishna [6] for second order overdamped nonlinear systems. Murty [7] has also presented a unified KBM method for solving second order damped and overdamped nonlinear systems. Bojadziev and Edwards [14], on the basis of the extended KBM method, have studied damped and overdamped second order systems with slowly varying parameters. Sattar [8] has examined a second order critically damped nonlinear system.

Damped nonlinear oscillations of a third order ordinary differential equation have been investigated by Bojadziev [12] and Mulholland [11]. Sattar [13] has studied a third order overdamped nonlinear system.

In this chapter we develop a new asymptotic method to find an approximate solution of a third order critically damped weakly nonlinear autonomous ordinary differential equation. Two examples are solved to illustrate the method.

## 2.2 The Method for Three Equal Roots of The Charactrestic Equation of The Corresponding Linear Equation

Consider a third order nonlinear autonomous ordinary differential equation

$$\frac{d^3x}{dt^3}+c_1\frac{d^2x}{dt^2}+c_2\frac{dx}{dt}+c_3x=\varepsilon x^3 \qquad (2.1)$$

where $\varepsilon$ is a small parameter which represent perturbation in the system and f(x) is the given nonlinear function. The constants c1, c2 and c3 are such that $c_1=3k$, $c_2=3k^2$ and $c_3=k^3$ where -k, -k and -k are the real negative repeated roots of the characteristic equation of (2.1) for $\varepsilon=0$. The critically damping force in the system is represented by these real negative repeated eigenvalues. For $\varepsilon=0$, the solution of (2.1) is

$$x(t)=(a+bt+ct^2)\,e^{-kt} \qquad (2.2)$$

where a, b and c arbitrary constants to be determined from the initial conditions $[\ddot{x}(0), \dot{x}(0), x(0)]$.

When $\varepsilon\neq 0$, we propose an asymptotic solution of (2.1) in the form

$$x(t,\varepsilon)=e^{\xi}(1+\vartheta+\varphi+\vartheta\varphi)+\varepsilon u(\xi,\vartheta,\varphi)+\varepsilon^2 v(\xi,\vartheta,\varphi)+\ldots \qquad (2.3)$$

where u, v,.. are functions of $\xi$, $\eta$ and $\varphi$ while $\xi$, $\eta$ and $\varphi$ are functions of t defined by the differential equations

$$\frac{d\xi}{dt}=-k+\varepsilon e^{-\xi}A_1(\xi)+\varepsilon^2 e^{-\xi}A_2(\xi)+\ldots$$

$$\frac{d\vartheta}{dt}=1+\varepsilon B_1(\xi)+\varepsilon^2 B_2(\xi)+\ldots \qquad (2.4)$$

$$\frac{d\varphi}{dt}=1+\varepsilon C_1(\xi)+\varepsilon^2 C_2(\xi)+\ldots$$

where $A_1$, $A_2$, .., $B_1$, $B_2$,.. and $C_1$, $C_2$,.. are functions of $\xi$.

Confining attention to the first few m terms in the series expansion of equations (2.3) and (2.4), we evaluate the functions **u**, **v**,.., $A_1$, $A_2$,.., $B_1$, $B_2$,.. and $C_1$, $C_2$,.. such that $\xi(t)$, $\vartheta(t)$ and $\varphi(t)$ appearing in (2.3) and (2.4) satisfy the given differential equation (2.1) with an accuracy of $\varepsilon^{m+1}$. In order to determine these functions, we impose an additional condition that the functions **u**, **v**,.. do not contain the fundamental terms involving $e^{\xi}$, $e^{\xi}\vartheta$, $e^{\xi}\varphi$ and $e^{\xi}\vartheta\varphi$, since these are already included in the first four terms of the series in (2.3).

Differentiating (2.3) three times with respect to t, using relations (2.4), substituting (2.3) and the values of derivatives $\dddot{x}, \ddot{x}, \dot{x}$ in the original equation (2.1), and comparing the coefficient of $\varepsilon$, we obtain

$$[k^2(\frac{d}{d\xi}-1)^2-6k(\frac{d}{d\xi}-1)+6]A_1+(k^2\frac{d^2}{d\xi^2}-3k\frac{d}{d\xi})(B_1+C_1)$$

$$+[k^2(\frac{d}{d\xi}-1)^2A_1-3k(\frac{d}{d\xi}-1)A_1+k^2\frac{d^2C_1}{d\xi^2}]\vartheta+[k^2(\frac{d}{d\xi}-1)^2A_1$$

$$-3k(\frac{d}{d\xi}-1)A_1+k^2\frac{d^2B_1}{d\xi^2}]\varphi+k^2(\frac{d}{d\xi}-1)^2A_1\vartheta\varphi$$

$$+(-k\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\vartheta}+\frac{\partial}{\partial\varphi}+k)^3u=f(x) \qquad (2.5)$$

We assume that the right hand side of (2.5) can be expanded in the Taylor series in powers of $\vartheta$ and $\varphi$ with

$$f(x)=f(\xi,\vartheta,\varphi)=\Sigma_{r=o}^{\infty}[g_r(\xi)\vartheta^r+h_{r+1}(\xi)\vartheta^{r+1}+j_r(\xi)\varphi^r+k_{r+1}(\xi)\varphi^{r+1}]$$

$$+\Sigma_{s,s'=o}^{\infty}l_{s,s'}(\xi)\vartheta^{s+1}\varphi^{s'+1} \qquad (2.6)$$

where r is an even integer and s, s' are integers and $g_o$, $h_o$, $j_o$, $k_o$, $l_{oo}$; $g_r$, $h_{r+1}$, $j_r$, $k_{r+1}$ and $l_{ss}(r,s,s>1)$ are the coefficients of the fundamental and higher argument terms in $\vartheta$ and $\varphi$.

Substituting (2.6) in (2.5) and equating the coefficient of $\vartheta$, $\varphi$ and higher argument terms of $\vartheta$, $\varphi$, we obtain

$$[k^2(\frac{d}{d\xi}-1)^2-6(\frac{d}{d\xi}-1)+6]A_1+(k^2\frac{d^2}{d\xi^2}-3k\frac{d}{d\xi})(B_1+C_1)$$

$$=g_o(\xi)+j_o(\xi) \qquad (2.7)$$

$$[k^2(\frac{d}{d\xi}-1)^2-3k(\frac{d}{d\xi}-1)]A_1+k^2\frac{d^2C_1}{d\xi^2}=h_1(\xi) \qquad (2.8)$$

$$[k^2(\frac{d}{d\xi}-1)^2-3k(\frac{d}{d\xi}-1)]A_1+k^2\frac{d^2B_1}{d\xi^2}=k_1(\xi) \qquad (2.9)$$

$$k^2\left(\frac{d}{d\xi}-1\right)^2 A_1\vartheta\varphi+\left(-k\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\vartheta}+\frac{\partial}{\partial\varphi}+k\right)^3 u=\Sigma_{r=2}^{\infty}[g_r(\xi)\vartheta^r$$

$$+h_{r+1}(\xi)\vartheta^{r+1}+j_r(\xi)\varphi^r+k_{r+1}(\xi)\varphi^{r+1}]+\Sigma_{s,s'=0}^{\infty}l_{s,s'}(\xi)\vartheta^{s+1}\varphi^{s'+1} \quad (2.10)$$

Equations (2.7), (2.8) and (2.9) are three nonhomogeneous linear partial differential equations with constant coefficients and particular solutions of these three simultaneous equations gives the functions $A_1$, $B_1$ and $C_1$. Substitution of these values of $A_1$, $B_1$ and $C_1$ in (2.4) yields the first approximate solution of (2.1).

## Determination of the first order correction term $u(\xi,\eta,\varphi)$

In order to obtain to the second approximation (or first improved approximation), we now determine the first order correction term $u(\xi,\eta,\varphi)$. Equation (2.10) is a third order nonhomogeneous linear partial differential equation with constant coefficients and we can easily obtain a particular solution of this equation.

### Example

Consider the third order differential equation

$$\frac{d^3x}{dt^3}+6\frac{d^2x}{dt^2}+12\frac{dx}{dt}+8x=\varepsilon x^3 \qquad (2.11)$$

For $\varepsilon=0$, the roots of the characteristic equation of (2.11) are -2, -2 and -2, i.e., k=2. From (2.6) and

$$f(x)=x^3=e^{3\xi}[1+3(\vartheta+\varphi)+3(\vartheta^2+3\vartheta\varphi+\varphi^2)+(\vartheta^3+9\vartheta^2\varphi+9\vartheta\varphi^2+\varphi^3)$$

$$+3(\vartheta^3\varphi+3\vartheta^2\varphi^2+\vartheta\varphi^3)+3(\vartheta^3\varphi^2+\vartheta^2\varphi^3)+\vartheta^3\varphi^3)]$$

we get $g_o+j_o=e^{3\xi}$, $h_1=k_1=3e^{3\xi}$ and

$$\Sigma_{r=2}^{\infty}[g_r\vartheta^r+h_{r+1}\vartheta^{r+1}+j_r\varphi^r+k_{r+1}\varphi^{r+1}]+\Sigma_{s,s'=0}^{\infty}l_{s,s'}\vartheta^{s+1}\varphi^{s'+1}=$$

$$e^{3\xi}[3(\vartheta^2+3\vartheta\varphi+\varphi^2)+(\vartheta^3+9\vartheta^2\varphi+9\vartheta\varphi^2+\varphi^3)+3(\vartheta^3\varphi+3\vartheta^2\varphi^2+\vartheta\varphi^3) \qquad (2.12)$$

$$+3(\vartheta^3\varphi^2+\vartheta^2\varphi^3)+\vartheta^3\varphi^3]$$

Use of (2.12) in (2.7)-(2.10) yields

$$[4(\frac{d}{d\xi}-1)^2-12(\frac{d}{d\xi}-1)+6]A_1+e^{\xi}(4\frac{d^2}{d\xi^2}-6\frac{d}{d\xi})(B_1+C_1)=e^{3\xi} \qquad (2.13)$$

$$[4(\frac{d}{d\xi}-1)^2-6(\frac{d}{d\xi}-1)]A_1+4e^{\xi}\frac{d^2C_1}{d\xi^2}=3e^{3\xi} \qquad (2.14)$$

$$[4(\frac{d}{d\xi}-1)^2-6(\frac{d}{d\xi}-1)]A_1+4e^{\xi}\frac{d^{B_1}}{d\xi^2}=3e^{3\xi} \qquad (2.15)$$

$$4(\frac{d}{d\xi}-1)^2A_1\vartheta\varphi+(-2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\vartheta}+\frac{\partial}{\partial\varphi}+2)^3u=e^{3\xi}[3(\vartheta^2+3\vartheta\varphi+\varphi^2)+(\vartheta^3+9\vartheta^2\varphi$$

$$+9\vartheta\varphi^2+\varphi^3)+3(\vartheta^3\varphi+3\vartheta^2\varphi^2+\vartheta\varphi^3)+3(\vartheta^3\varphi^2+\vartheta^2\varphi^3)+\vartheta^3\varphi^3] \qquad (2.16)$$

Substituting $A_1=le^{3\xi}$, $B_1=me^{2\xi}$ and $C_1=ne^{2\xi}$ in (2.13), (2.14) and (2.15) and equating the coefficients of $e^{3\xi}$ on both sides of these three equations, we get three linear algebraic equations and solving these equations then obtain $l=\frac{1}{8}$ and $m=n=\frac{5}{32}$.

Therefore $A_1=\frac{1}{8}e^{3\xi}$ and $B_1=C_1=\frac{5}{32}e^{2\xi}$ (2.17)

Substituting (2.17) in equations (2.4) and integrating, we obtain

$$e^{\xi}=\frac{e^{\xi_o}e^{-2t}}{[1+\frac{e^{2\xi_o}}{16}(e^{-4t}-1)\varepsilon]^{\frac{1}{2}}}$$

$$\vartheta=\vartheta_o+t+\frac{5}{64}e^{2\xi_o}(1-e^{-4t})\varepsilon \qquad (2.18)$$

$$\varphi=\varphi_o+t+\frac{5}{64}e^{2\xi_o}(1-e^{-4t})\varepsilon$$

where $e^{\xi_0}=e^{\xi}(0)$, $\vartheta_o=\vartheta(0)$ and $\varphi_o=\varphi(0)$.

Hence the first approximate solution of (2.11) is

$$x(t,\varepsilon)=e^{\xi}(1+\vartheta+\varphi+\vartheta\varphi)$$

where $e^{\xi}$, $\vartheta$ and $\varphi$ are given by equation (2.18).

The particular solution of (2.16) is

$$u(\xi,\vartheta,\varphi)=-\frac{e^{3\xi}}{64}[\frac{4434}{64}+\frac{11373}{128}(\vartheta+\varphi)+\frac{1311}{32}(\vartheta^2+\varphi^2)+\frac{1247}{32}\vartheta\varphi+\frac{103}{16}(\vartheta^3+9\vartheta^2\varphi+9\vartheta\varphi^2+\varphi^3)+\frac{39}{4}(\vartheta^3\varphi+3\vartheta^2\varphi^2+\vartheta\varphi^3)+\frac{21}{4}(\vartheta^3\varphi^2+\vartheta^2\varphi^3)+\vartheta^3\varphi^3] \quad (2.19)$$

Therefore the second approximate solution of (2.11) is

$$x(t,\varepsilon)=e^{\xi}(1+\vartheta+\varphi+\vartheta\varphi)+\varepsilon u$$

where $e^{\xi}$, $\vartheta$, $\varphi$ and u are given by equations (2.18) and (2.19) respectively.

## 2.3 The Method for Two Equal Roots of The Charactrestic Equation of The Corresponding Linear Equation

Again we consider equation (2.1) and the constants $c_1$, $c_2$ and $c_3$ are chosen in such a way that

$$c_1=\sigma+2\kappa, \quad c_2=\kappa(2\sigma+\kappa) \quad \text{and} \quad c_3=\sigma\kappa^2 \tag{2.20}$$

where $-\sigma$, $-\kappa$ and $-\kappa$ are real negative roots of the characteristic equation of (2.1) for $\varepsilon=0$. In this case damping force in the system is represented by these real negative eigen values where two of them are equal. For $\varepsilon=0$, the solution of (2.1) is

$$x(t,0)=ae^{-\sigma t}+(b+ct)e^{-\kappa t} \tag{2.21}$$

where a, b and c are arbitrary constants to be determined from the set of initial conditions $[\ddot{x}(0), \dot{x}(0), x(0)]$.

When $\varepsilon\neq0$, we propose an asymptotic solution of (2.1) in the form

$$x(t,\varepsilon)=e^{\xi}+e^{\eta}(1+\varphi)+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2 v(\xi,\eta,\varphi)+\ldots \tag{2.22}$$

where u, v,.. are functions of $\xi$, $\eta$ and $\varphi$ while $\xi$, $\eta$ and $\varphi$ are functions of t, defined by the differential equations

$$\frac{d\xi}{dt}=-\sigma+\varepsilon e^{-\xi}A_1(\xi,\eta)+\varepsilon^2 e^{-\xi}A_2(\xi,\eta)+\ldots$$

$$\frac{d\eta}{dt}=-\kappa+\varepsilon e^{-\eta}B_1(\xi,\eta)+\varepsilon^2 e^{-\eta}B_2(\xi,\eta)+\ldots \tag{2.23}$$

$$\frac{d\varphi}{dt}=1+\varepsilon C_1(\xi,\eta)+\varepsilon^2 C_2(\xi,\eta)+\ldots$$

where $A_1$, $A_2$,.., $B_1$, $B_2$,.. and $C_1$, $C_2$,.. are functions of $\xi$ and $\eta$.

Confining attention to the first few m terms in the series expansion of equations (2.22) and (2.23), we evaluate the functions u, v,.., $A_1$, $A_2$,.., $B_1$, $B_2$,.. and $C_1$, $C_2$,.. such that $\xi(t)$, $\eta(t)$ and $\varphi(t)$ appearing in (2.22) and (2.23) satisfy the given differential equation (2.1) with an accuracy of $\varepsilon^{m+1}$. In order to determine these functions, we impose an additional condition that the functions u, v,.. do not contain the fundamental terms involving $e^{\xi}$, $e^{\eta}$ and $e^{\eta}\varphi$ since these are already included in the first three terms of the series in (2.22).

Differentiating (2.22) three times with respect to t, using relations (2.23), substituting (2.22) and the derivatives $\dddot{x}$, $\ddot{x}$, $\dot{x}$ in the original equation (2.1) and comparing the coefficient of $\varepsilon$, we obtain

$$(\Omega-\kappa)^2 A_1+[(\Omega-\sigma)(\Omega-\kappa)-3\Omega+(2\sigma+\kappa)]B_1+e^{\eta}[\Omega^2+(\kappa-\sigma)\Omega]C_1(\Omega-\sigma)(\Omega-\kappa)B_1\varphi$$

$$+[(-\Omega+\frac{\partial}{\partial\varphi})^3+c_1(-\Omega+\frac{\partial}{\partial\varphi})^2+c_2(-\Omega+\frac{\partial}{\partial\varphi})+c_3]u=f(x) \qquad (2.24)$$

where $\Omega=\sigma\frac{\partial}{\partial\xi}+\kappa\frac{\partial}{\partial\eta}$.

We assume that the right hand side of (2.24) can expanded in the Taylor series in powers of $\varphi$ with

$$f(x)=\Sigma_{j=0}^{\infty}S_j(\xi,\eta)\varphi^j \qquad (2.25)$$

where j is an integer and $S_o$, $S_1$ and $S_j$ $(j \geq 2)$ are the coefficients of the fundamental and higher argument terms in $\varphi$.

Substituting (2.25) in (2.24) and equating the coefficients of $\varphi^o$, $\varphi$ and the higher argument terms of $\varphi$, we obtain

$$(\Omega-\kappa)^2 A_1 + [(\Omega-\sigma)(\Omega-\kappa) - 3\Omega + 2\sigma + \kappa] B_1 + e^{\eta} [\Omega^2 + (\kappa-\sigma)] C_1$$

$$= S_o(\xi,\eta) \qquad (2.26)$$

$$(\Omega-\sigma)(\Omega-\kappa) B_1 = S_1(\xi,\eta) \qquad (2.27)$$

$$[(-\Omega+\frac{\partial}{\partial\varphi})^3 + c_1(-\Omega+\frac{\partial}{\partial\varphi})^2 + c_2(-\Omega+\frac{\partial}{\partial\varphi}) + c_3] u$$

$$= \Sigma_{j=2}^{\infty} S_j(\xi,\eta)\varphi^j \qquad (2.28)$$

With help of (2.27), (2.26) can be written as

$$(\Omega-\kappa)^2 A_1 + e^{\eta}[\Omega^2 + (\kappa-\sigma)\Omega] C_1 = S_o(\xi,\eta) - S_1(\xi,\eta)$$

$$+ [3\Omega - (2\sigma+\kappa)] B_1 \qquad (2.29)$$

Now equation (2.27) is a nonhomogeneous linear partial differential equation with constant coefficients and its particular solution gives the function $B_1$. If we substitute this value of $B_1$ in the right hand side of (2.29), it reduces to the form $g(\xi,\eta)e^{\xi} + h(\xi,\eta)e^{\eta}$.

Thus equation (2.29) becomes

$$(\Omega-\kappa)^2 A_1+e^{\eta}[\Omega^2+(\kappa-\sigma)\Omega]C_1=g(\xi,\eta)e^{\xi}+h(\xi,\eta)e^{\eta} \qquad (2.30)$$

Equating the coefficients of $e^{\xi}$ and $e^{\eta}$ on both sides of equation (2.30), we obtain

$$(\Omega-\kappa)^2 A_1=g(\xi,\eta)e^{\xi} \qquad (2.31)$$

and

$$[\Omega^2+(\kappa-\sigma)\Omega]C_1=h(\xi,\eta) \qquad (2.32)$$

Equations (2.31) and (2.32) both are nonhomogeneous partial differential equations with constant coefficients and their particular solutions gives the functions $A_1$ and $C_1$. Substituting these values of $A_1$, $B_1$ and $C_1$ in (2.23) and then integrating (numerically) the first order approximate solution of equation (2.1) is obtained.

**Determination of the first order correction term $u(\xi,\eta,\varphi)$**

Equation (2.28) is a third order nonhomogeneous linear partial differential equation with constant coefficients and its particular solution gives the first order correction term $u(\xi,\eta,\varphi)$ and thus the first improved approximation of the original nonlinear equation (2.1) is determined.

Example

Consider the third order nonlinear equation

$$\frac{d^3x}{dt^3}+4\frac{d^2x}{dt^2}+5\frac{dx}{dt}+2x=\varepsilon x^3 \qquad (2.33)$$

When $\varepsilon=0$, the characteristic roots of equation (2.33) are -2, -1 and -1; i.e., $\sigma=2$ and $\kappa=1$, then we have

$$f=e^{3\xi}+3e^{2\xi+\eta}+3e^{\xi+2\eta}+e^{3\eta}+3(e^{2\xi+\eta}+2e^{\xi+2\eta}+e^{3\eta})\varphi+3(e^{\xi+2\eta}+e^{3\eta})\varphi^2+e^{3\eta}\varphi^3,$$

$$S_o=e^{3\xi}+3e^{2\xi+\eta}+3e^{\xi+2\eta}+e^{3\eta}, \qquad S_1=3(e^{2\xi+\eta}+2e^{\xi+2\eta}+e^{3\eta}) \qquad (2.34)$$

and $\qquad \Sigma_{j=2}^{\infty}S_j(\xi,\eta)\varphi^j=3(e^{\xi+2\eta}+e^{3\eta})\varphi^2+e^{3\eta}\varphi^3$

Therefore equations (2.27) and (2.28) become

$$[(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-2)(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1)]B_1=3(e^{2\xi+\eta}+2e^{\xi+2\eta}+e^{3\eta}) \qquad (2.35)$$

$$[(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1)^3+4(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1)^2+5(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1)+2]u$$

$$=3(e^{\xi+2\eta}+e^{3\eta})\varphi^2+e^{3\eta}\varphi^3 \qquad (2.36)$$

Solving (2.35) we obtain,

$$B_1=\frac{1}{4}e^{2\xi+\eta}+e^{\xi+2\eta}+\frac{3}{2}e^{3\eta} \tag{2.37}$$

Substituting this value of $B_1$ and the values of $S_o$ and $S_1$ from (2.34) in the right hand side of (2.29), we obtain,

$$\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)^2 A_1+e^{\eta}\left[\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)^2-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right]C_1=e^{\xi}(e^{2\xi}+4e^{2\eta})$$

$$+e^{\eta}\left(\frac{5}{2}e^{2\xi}+4e^{2\eta}\right) \tag{2.38}$$

where $g=e^{2\xi}+4e^{2\eta}$ and $h=\frac{5}{2}e^{2\xi}+4e^{2\eta}$.

Therefore equations (2.31) and (2.32) become

$$\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1\right)^2 A_1=e^{3\xi}+4e^{\xi+2\eta} \tag{2.39}$$

and

$$\left[\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)^2-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}\right]C_1=\frac{5}{2}e^{2\xi}+4e^{2\eta} \tag{2.40}$$

The particular solution of equations (2.39) and (2.40) are

$A_1=\frac{1}{25}e^{3\xi}+\frac{4}{9}e^{\xi+2\eta}$ and $C_1=\frac{5}{24}e^{2\xi}+2e^{2\eta}$.

Now putting the values of $A_1$, $B_1$ and $C_1$ in equations (2.23), we obtain

$$\frac{d\xi}{dt}=-2+\varepsilon\left(\frac{1}{25}e^{2\xi}+\frac{4}{9}e^{2\eta}\right)$$

$$\frac{d\eta}{dt}=-1+\varepsilon\left(\frac{1}{4}e^{2\xi}+e^{\xi+\eta}+\frac{3}{2}e^{2\eta}\right) \qquad (2.41)$$

$$\frac{d\varphi}{dt}=1+\left(\frac{5}{24}e^{2\xi}+2e^{2\eta}\right)$$

By using a **Computer** the above system has been integrated numerically by means of a fourth-order Runge-Kutta procedure with initial conditions $\xi(0)=\xi_o$, $\eta(0)=\eta_o$ and $\varphi(0)=\varphi_o$ for a small value of $\varepsilon$. If $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ for $\varepsilon=.1$, the approximate solution of this system is

| t | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 0.0 | -0.500000 | -0.100000 | 0.000000 |
| 0.1 | -0.696524 | -0.183129 | 0.115728 |
| 0.2 | -0.893610 | -0.269446 | 0.228898 |
| 0.3 | -1.091179 | -0.358362 | 0.339879 |
| 0.4 | -1.289156 | -0.449391 | 0.449006 |
| 0.5 | -1.487480 | -0.542135 | 0.556569 |
| 0.6 | -1.686092 | -0.636266 | 0.662823 |
| 0.7 | -1.884947 | -0.731519 | 0.767983 |
| 0.8 | -2.084003 | -0.827679 | 0.872236 |
| 0.9 | -2.283226 | -0.924571 | 0.975736 |
| 1.0 | -2.482588 | -1.022055 | 1.078614 |
| 1.1 | -2.682063 | -1.120016 | 1.180978 |
| 1.2 | -2.881632 | -1.218364 | 1.282919 |
| 1.3 | -3.081278 | -1.317024 | 1.384512 |
| 1.4 | -3.280988 | -1.415936 | 1.485818 |
| 1.5 | -3.480750 | -1.515053 | 1.586889 |
| 1.6 | -3.680555 | -1.614335 | 1.687767 |
| 1.7 | -3.880395 | -1.713752 | 1.788487 |
| 1.8 | -4.080264 | -1.813277 | 1.889076 |
| 1.9 | -4.280157 | -1.912891 | 1.989559 |

**Table 1.**

| t | x $(=e^{\xi}+e^{\eta}(1+\varphi))$ | x $(\varepsilon=0)$ |
|---|---|---|
| 0.0 | 1.511368 | 1.511368 |
| 0.1 | 1.427338 | 1.427643 |
| 0.2 | 1.347811 | 1.349037 |
| 0.3 | 1.272155 | 1.274816 |
| 0.4 | 1.199993 | 1.204429 |
| 0.5 | 1.131095 | 1.137470 |
| 0.6 | 1.065316 | 1.073638 |
| 0.7 | 1.002551 | 1.012711 |
| 0.8 | 0.942715 | 0.954522 |
| 0.9 | 0.885732 | 0.898945 |
| 1.0 | 0.831526 | 0.845882 |
| 1.1 | 0.780019 | 0.795252 |
| 1.2 | 0.731133 | 0.746986 |
| 1.3 | 0.684786 | 0.701023 |
| 1.4 | 0.640895 | 0.657304 |
| 1.5 | 0.599374 | 0.615771 |
| 1.6 | 0.560136 | 0.576365 |
| 1.7 | 0.523096 | 0.539025 |
| 1.8 | 0.488165 | 0.503690 |
| 1.9 | 0.455258 | 0.470294 |

**Table 2.**

Equation (2.36) is a third order linear nonhomogeneous differential equation with constant coefficients and its particular solution is

$$u=-\frac{e^{\xi+2\eta}}{36}(11+14\varphi+6\varphi^2)+\frac{e^{3\eta}}{8}(24-57\varphi-18\varphi^2-2\varphi^3)$$

Now using **Table 1**, we may compute **u** and obviously we shall get the first improved approximation of the form

$$x=e^{\xi}+e^{\eta}(1+\varphi)+\varepsilon u.$$

**Conclusions :**

An asymptotic method is developed to find the solution of a third order critically damped autonomous nonlinear differential equation. The solution is obtained as a power series in a small parameter $\varepsilon$. The series itself is not convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as $\varepsilon$ approaches zero.

# Chapter 3

# A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving Third Order Nonlinear Systems

## 3.1 Introduction:

Damped nonlinear oscillations of third order ordinary differential equations have been investigated by Bojadziev [12] and Mulholand [11]. Sattar [13] has studied a third order overdamped nonlinear system.

In this section, a method unifying both the damped and overdamped cases is presented. The method is a generalization of Bogoliubov's asymptotic method and covers all three cases when the roots of the corresponding linear equation are real, real and complex, and real and purely imaginary. Thus, the present method is independent of whether the corresponding linear equation of the system has three negative real roots, one negative real root and other two complex roots with negative real parts or one negative real root and two purely imaginary roots. Suitable examples are considered to show that by proper substitution for the roots in the general solution, the results lead to the various individual cases.

## 3.2 The Asymptotic Method

Consider a third order nonlinear ordinary differential equation

$$\frac{d^3x}{dt^3}+k_1\frac{d^2x}{dt^2}+k_2\frac{dx}{dt}+k_3x=\varepsilon f(x) \qquad (3.1)$$

where $\varepsilon$ is a small parameter which represents perturbation in the system and f(x) is the given nonlinear function. The constants are such that

$$k_1=\lambda+2\mu, \quad k_2=2\lambda\mu+\mu^2-\omega^2, \quad k_3=\lambda(\mu^2-\omega^2) \qquad (3.2)$$

where $-\lambda$, $-\mu+\omega$, $-\mu-\omega$ are the roots of the characteristic equation of (3.1) for $\varepsilon=0$. Therefore when $\varepsilon=0$, the solution of (3.1) becomes

$$x(t,0)=Ae^{-\lambda t}+Be^{-(\lambda-\omega)t}+Ce^{-(\lambda+\omega)t} \qquad (3.3)$$

where **A**, B and C are arbitrary constants to be determined from the given initial conditions $[\ddot{x}(0), \dot{x}(0), x(0)]$.

When $\varepsilon\neq0$, we propose an asymptotic solution of the equation (3.1) in the form,

$$x(t,\varepsilon)=e^{\xi}+e^{\eta}\cosh\varphi+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2v(\xi,\eta,\varphi)+\ldots \qquad (3.4)$$

where **u**, **v**,.. are functions of $\xi$, $\eta$ and $\varphi$ while $\xi$, $\eta$ and $\varphi$ are functions of **t** defined by the differential equations,

$$\frac{d\xi}{dt}=-\lambda+\varepsilon e^{-\xi}A_1(\xi,\eta)+\varepsilon^2\ldots$$

$$\frac{d\eta}{dt}=-\mu+\varepsilon e^{-\eta}B_1(\xi,\eta)+\varepsilon^2\ldots \qquad (3.5)$$

$$\frac{d\varphi}{dt}=\omega+\varepsilon C_1(\xi,\eta)+\varepsilon^2\ldots$$

where $A_1$, $A_2$,.. $B_1$, $B_2$,.. and $C_1$, $C_2$,.. are functions of $\xi$ and $\eta$.

Confining attention to the first few m terms in the series expansion of (3.4) and (3.5), we evaluate the functions **u**, **v**,.. $A_1$, $A_2$,.. $B_1$, $B_2$,.. and $C_1$, $C_2$,.. such that $\xi(t)$, $\eta(t)$ and $\varphi(t)$ appearing in (3.4) and (3.5) satisfy the given differential equation (3.1) with an accuracy of $\varepsilon^{m+1}$. In order to determine these functions, we impose an additional condition that the functions **u**, **v**,.. do not contain the fundamental terms $e^{\xi}$, $e^{\eta}\cosh\varphi$, since these are already included in the first two terms in the right hand side of equation (3.4).

Differentiating (3.4) three times with respect to t, using relations (3.5), substituting (3.4) and the derivatives $\dddot{x}$, $\ddot{x}$, $\dot{x}$ in the original equation (3.1), and comparing the coefficient of $\varepsilon$, we obtain

$$[(\Omega-\mu)^2-\omega^2]A_1+\cosh\varphi[(\Omega-\lambda)(\Omega-\mu)+2\omega^2]B_1+\omega e^{\eta}\cosh\varphi[-3\Omega+2(\lambda-\mu)]C_1$$

$$+\omega\sinh\varphi[-3\Omega+2\lambda+\mu]B_1+e^{\eta}\sinh\varphi[\Omega^2-(\lambda-\mu)\Omega+2\omega^2]C_1+[(\Omega+\omega\frac{\partial}{\partial\varphi})^3$$

$$+k_1(\Omega+\omega\frac{\partial}{\partial\varphi})^2+k_2(\Omega+\omega\frac{\partial}{\partial\varphi})+k_3]u=f(x) \qquad (3.6)$$

where $\Omega=-(\lambda\frac{\partial}{\partial\xi}+\mu\frac{\partial}{\partial\eta})$.

We now assume that the right hand side of (3.6) can be expanded in Tailor series in the from

$$f(x)=f(\xi,\eta,\varphi)=\Sigma_{r=o}^{\infty}g_r(\xi,\eta)\cosh\varphi \qquad (3.7)$$

with $g_r(\xi,\eta)=\Sigma_{j,k=1}^{\infty}h_{jk}e^{j\xi+k\eta}$.

where **r**, **j** and **k** are integers, and $g_r$ and thus $h_{jk}$ are the coefficients of the fundamental and higher argument terms of **coshrφ**.

Substituting (3.7) in (3.6) and comparing the constant term, the coefficients of **coshφ**, **sinhφ** and the coefficients of **coshrφ** (r>1), we obtain

$$[(\Omega-\mu)^2-\omega^2]A_1=g_o(\xi,\eta) \qquad (3.8)$$

$$[(\Omega-\lambda)(\Omega-\mu)+2\omega^2]B_1+\omega e^{\eta}[-3\Omega+2(\lambda-\mu)]C_1=g_1(\xi,\eta) \qquad (3.9)$$

$$\omega[-3\Omega+2\lambda+\mu]B_1+e^{\eta}[\Omega^2-(\lambda-\mu)\Omega+2\omega^2]C_1=0 \qquad (3.10)$$

$$[(\Omega+\omega\frac{\partial}{\partial\varphi})^3+K_1(\Omega+\omega\frac{\partial}{\partial\varphi})^2+K_2(\Omega+\omega\frac{\partial}{\partial\varphi})+K_3]\,u$$

$$=\Sigma_{r=2}^{\infty}\, g_r(\xi,\eta)\, cosh r\varphi \qquad (3.11)$$

Equation (3.8) is a second order linear partial differential equation with constant coefficients and equation (3.9) and (3.10) are simultaneous partial differential equations. Obviously a particular solution of (3.8) gives the function $A_1$. Substituting

$$B_1=\Sigma_{j,k=1}^{\infty} c_{jk} e^{j\xi+k\eta} \quad \text{and} \quad C_1=\Sigma_{j,k=1}^{\infty} d_{jk} e^{j\xi+(k-1)\eta} \qquad (3.12)$$

in (3.9) and (3.10) and equating the coefficient of $e^{j\xi+k\eta}$ from both sides we get a system of algebraic equations, whose solution give the values of $c_{jk}$ and $d_{jk}$. Thus $B_1$ and $C_1$ are obtained. Substituting these values of $A_1$, $B_1$ and $C_1$ in equation (3.5) and integrating (numerically) the first order approximate solution of the given equation (3.1) is obtained.

## Determination of the first order correction term $u(\xi,\eta,\varphi)$

Equation (3.11) is a third order nonhomogeneous linear partial differential equation with constant coefficients and its particular solution gives the first order correction term $u(\xi,\eta,\varphi)$ and thus the first improved approximation of the original nonlinear equation (3.1) is determined.

**Remarks:** One can proceed in a similar manner to determine the second and higher order approximations. However, we shall restrict ourselves here to the first approximation only.

For equation (3.1), it is also possible to assume a solution of the form

$$x(t,\varepsilon)=e^{\xi}+e^{\eta}\sinh\varphi+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2 v(\xi,\eta,\varphi)+\dots \qquad (3.13)$$

instead of the form (3.4) and determine the solution by a procedure similar to the above. The choice of the solution is dependent on the given initial conditions $[\dddot{x}(0), \dot{x}(0), x(0)]$.

When the roots of the characteristic equation of the corresponding linear equation of (3.1) are real, $\varphi$ being a real quantity, the second term on the right hand sides of equations (3.4) and (3.13) can be expressed in exponential form to give, respectively

$$x(t,\varepsilon)=e^{\xi}+\frac{1}{2}e^{\eta}(e^{\varphi}+e^{-\varphi})+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2 v(\xi,\eta,\varphi)+\dots \qquad (3.14a)$$

and

$$x(t,\varepsilon)=e^{\xi}+\frac{1}{2}e^{\eta}(e^{\varphi}-e^{-\varphi})+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2 v(\xi,\eta,\varphi)+\dots \qquad (3.14b)$$

But when the two roots of the corresponding linear equation are complex, one has to replace $\varphi$ by $i\varphi$ in equations (3.4) and (3.13). Thus, by using the identities $\cosh\varphi=\cos i\varphi$ and $\sinh i\varphi=i\sin\varphi$, the solutions can be written in the forms

$$x(t,\varepsilon)=e^{\xi}+e^{\eta}\cos\varphi+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2 v(\xi,\eta,\varphi)+\ldots \qquad (3.15a)$$

and

$$x(x,t)=e^{\xi}+e^{\eta}\sin\varphi+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2 v(\xi,\eta,\varphi)+\ldots \qquad (3.15b)$$

respectively.

## 3.3 Examples

**Case 1.** Three real roots.

Consider a third order nonlinear system governed by

$$\frac{d^3x}{dt^3}+8\frac{d^2x}{dt^2}+19\frac{dx}{dt}+12x=\varepsilon x^3 \qquad (3.16)$$

When $\varepsilon=0$, the characteristic roots of (3.16) are -4, -3 and -1.

Let us take $\lambda=4$, $\mu=2$ and $\omega=1$; then we have

$$f=e^{3\xi}+\frac{3}{2}e^{2\xi+\eta}+3\left(e^{2\xi+\eta}+\frac{1}{4}e^{3\eta}\right)\cosh\varphi+\frac{3}{2}e^{\xi+2\eta}\cosh 2\varphi+\frac{1}{4}e^{3\eta}\cosh 3\varphi,$$

$$g_o=e^{3\xi}+\frac{3}{2}e^{\xi+2\eta} \qquad \text{and} \qquad g_1=3e^{2\xi+\eta}+\frac{3}{4}e^{3\eta}.$$

Therefore equations (3.8), (3.9) and (3.10) become

$$[(4\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\eta}-2)^2-1]A_1=e^{3\xi}+\frac{3}{2}e^{\xi+2\eta} \qquad (3.17)$$

$$[(4\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\eta}-4)(4\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\eta}-2)+2]B_1+e^{\eta}[-3(4\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\eta})+4]C_1$$

$$=3e^{2\xi+\eta}+\frac{3}{4}e^{3\eta} \qquad (3.18)$$

$$[-3(4\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\eta})+10]B_1+e^{\eta}[(4\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\eta})^2$$

$$-2(4\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\eta})+2]C_1=0 \qquad (3.19)$$

Equation (3.17) is a second order nonhomogeneous linear partial differential equation with constant coefficients. A particular solution of this equation is

$$A_1=\frac{1}{99}e^{3\xi}+\frac{3}{70}e^{\xi+2\eta}$$

Equations (3.18) and (3.19) are second order simultaneous partial differential equations with constant coefficients. Now putting

$B_1=m_1e^{2\xi+\eta}+m_2e^{3\eta}$ and $C_1=n_1e^{2\xi}+n_2e^{2\eta}$

in equations (3.18) and (3.19) and then equating the coefficients of the exponents, we get the algebraic equations

$$m_1+4n_1=3/4$$
$$4m_1+n_1=0$$
$$-3m_2+4n_2=3/8$$
$$4m_2+3n_2=0$$

Solving these equations, we have

$m_1 = 1/14$, $m_2 = 5/24$, $n_1 = 1/35$ and $n_2 = 1/6$

and therefore

$$B_1 = \frac{1}{14} e^{2\xi+\eta} + \frac{5}{24} e^{3\eta} \quad \text{and} \quad C_1 = \frac{1}{35} e^{2\xi} + \frac{1}{6} e^{2\eta}.$$

Putting the values of $A_1$, $B_1$ and $C_1$ in (3.5) , we obtain

$$\frac{d\xi}{dt} = -4 + \varepsilon \left(\frac{1}{99} e^{2\xi} + \frac{3}{70} e^{2\eta}\right)$$

$$\frac{d\eta}{dt} = -2 + \varepsilon \left(\frac{1}{14} e^{2\xi} + \frac{5}{24} e^{2\eta}\right) \qquad (3.20)$$

$$\frac{d\varphi}{dt} = 1 + \varepsilon \left(\frac{1}{35} e^{2\xi} + \frac{1}{6} e^{2\eta}\right)$$

By using a **Computer** the above system has been integrated numerically by means of a fourth-order Runge-Kutta procedure with initial conditions $\xi(0)=\xi_0$, $\eta(0)=\eta_0$ and $\varphi(0)=\varphi_0$ for a small value of $\varepsilon$. If $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ for $\varepsilon=.1$ the approximate solution of this system is

| t | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 0.0 | -0.500000 | -0.100000 | 0.000000 |
| 0.1 | -0.899685 | -0.298411 | 0.101199 |
| 0.2 | -1.299479 | -0.497383 | 0.201988 |
| 0.3 | -1.699343 | -0.696711 | 0.302511 |
| 0.4 | -2.099252 | -0.896263 | 0.402859 |
| 0.5 | -2.499193 | -1.095975 | 0.503091 |
| 0.6 | -2.899153 | -1.295782 | 0.603246 |

| $t$ | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 0.7 | -3.299126 | -1.495650 | 0.703349 |
| 0.8 | -3.699108 | -1.695563 | 0.803419 |
| 0.9 | -4.099096 | -1.895505 | 0.903465 |
| 1.0 | -4.499088 | -2.095466 | 1.003496 |
| 1.1 | -4.899083 | -2.295400 | 1.103517 |
| 1.2 | -5.299079 | -2.495422 | 1.203531 |
| 1.3 | -5.699077 | -2.695410 | 1.303540 |
| 1.4 | -6.099075 | -2.895402 | 1.403546 |
| 1.5 | -6.499074 | -3.095397 | 1.503551 |
| 1.6 | -6.899074 | -3.295393 | 1.603553 |
| 1.7 | -7.299073 | -3.495391 | 1.703555 |
| 1.8 | -7.699073 | -3.695390 | 1.803556 |
| 1.9 | -8.099072 | -3.895388 | 1.903557 |

**Table 3.**

Thus the first approximate solution (numerical) of (3.16) with initial conditions $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ is

| $t$ | $x\ (=e\xi+e\eta\cosh\varphi)$ | $x\ (\varepsilon=0)$ |
|---|---|---|
| 0.0 | 1.511368 | 1.511368 |
| 0.1 | 1.152497 | 1.153409 |
| 0.2 | 0.893241 | 0.895991 |
| 0.3 | 0.703996 | 0.708696 |
| 0.4 | 0.564203 | 0.570602 |
| 0.5 | 0.459551 | 0.467267 |
| 0.6 | 0.380080 | 0.388690 |
| 0.7 | 0.318773 | 0.327918 |
| 0.8 | 0.270718 | 0.280088 |
| 0.9 | 0.232434 | 0.241787 |
| 1.0 | 0.201445 | 0.210596 |

| t | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 0.7 | -3.299126 | -1.495650 | 0.703349 |
| 0.8 | -3.699108 | -1.695563 | 0.803419 |
| 0.9 | -4.099096 | -1.895505 | 0.903465 |
| 1.0 | -4.499088 | -2.095466 | 1.003496 |
| 1.1 | -4.899083 | -2.295400 | 1.103517 |
| 1.2 | -5.299079 | -2.495422 | 1.203531 |
| 1.3 | -5.699077 | -2.695410 | 1.303540 |
| 1.4 | -6.099075 | -2.895402 | 1.403546 |
| 1.5 | -6.499074 | -3.095397 | 1.503551 |
| 1.6 | -6.899074 | -3.295393 | 1.603553 |
| 1.7 | -7.299073 | -3.495391 | 1.703555 |
| 1.8 | -7.699073 | -3.695390 | 1.803556 |
| 1.9 | -8.099072 | -3.895388 | 1.903557 |

Table 3.

Thus the first approximate solution (numerical) of (3.16) with initial conditions $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ is

| t | $x$ $(=e^{\xi}+e^{\eta}\cos\varphi)$ | x (for $\varepsilon=0$) |
|---|---|---|
| 0.0 | 1.511368 | 1.511368 |
| 0.1 | 1.152497 | 1.153409 |
| 0.2 | 0.893241 | 0.895991 |
| 0.3 | 0.703996 | 0.708696 |
| 0.4 | 0.564203 | 0.570602 |
| 0.5 | 0.459551 | 0.467267 |
| 0.6 | 0.380080 | 0.388690 |
| 0.7 | 0.318773 | 0.327918 |
| 0.8 | 0.270718 | 0.280088 |
| 0.9 | 0.232434 | 0.241787 |
| 1.0 | 0.201445 | 0.210596 |

| t | x $(=e^{\xi}+e^{\eta}\cos\varphi)$ | x (for $\varepsilon=0$) |
|---|---|---|
| 1.1 | 0.175976 | 0.184794 |
| 1.2 | 0.154746 | 0.163142 |
| 1.3 | 0.136821 | 0.144739 |
| 1.4 | 0.121514 | 0.128924 |
| 1.5 | 0.108311 | 0.115203 |
| 1.6 | 0.096826 | 0.103205 |
| 1.7 | 0.086764 | 0.092644 |
| 1.8 | 0.077896 | 0.083298 |
| 1.9 | 0.066449 | 0.074991 |

Table 4.

In this case equation (3.11) reduces to

$$[(-4\frac{\partial}{\partial\xi}-2\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\varphi})^3+8(-4\frac{\partial}{\partial}\xi-2\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\varphi})^2+19(-4\frac{\partial}{\partial\xi}-2\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\varphi})+12]u$$

$$=\frac{3}{2}e^{\xi+2\eta}\cosh 2\varphi+\frac{1}{4}e^{3\eta}\cosh 3\varphi \qquad (3.21)$$

This is a third order linear nonhomogeneous partial differential equation with constant coefficients, and its solution is

$$u=-\frac{e^{\xi+2\eta}}{1260}(34\cosh 2\varphi+29\sinh 2\varphi)+\frac{e^{3\eta}}{1920}[(6\varphi-1)\cosh 3\varphi+(6\varphi+1)\sinh 3\varphi]$$

Now using **Table 1**, we may compute **u** and obviously we shall get the second approximation of the form

$$x=e^{\xi}+e^{\eta}\cosh\varphi+\varepsilon u$$

**Case 2.** One real and two complex roots.

Consider the third order nonlinear differential equation

$$\frac{d^3x}{dt^3}+4\frac{d^2x}{dt^2}+9\frac{dx}{dt}+10x=\varepsilon x^3 \qquad (3.22)$$

When $\varepsilon=0$, the characteristic roots are -2, -1+i2 and -1-i2. Hence the corresponding equations of (3.17), (3.18), (3.19) and (3.20) in **Case 1**, are

$$\left[\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1\right)^2+4\right]A_1=e^{3\xi}+\frac{3}{2}e^{\xi+2\eta} \qquad (3.23)$$

$$\left[\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-2\right)\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1\right)-8\right]B_1-2e^{\eta}\left[-3\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)+2\right]C_1$$

$$=3e^{2\xi+\eta}+\frac{3}{4}e^{3\eta} \qquad (3.24)$$

$$2\left[-3\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)+5\right]B_1+e^{\eta}\left[\left(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)^2-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}-8\right]C_1=0 \qquad (3.25)$$

$$\left[\left(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+2\frac{\partial}{\partial\varphi}\right)^3+4\left(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+2\frac{\partial}{\partial\varphi}\right)^2+9\left(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+2\frac{\partial}{\partial\varphi}\right)+10\right]u$$

$$=\frac{3}{2}e^{\xi+2\eta}\cos 2\varphi+\frac{1}{4}e^{3\eta}\cos 3\varphi \qquad (3.26)$$

Equation (3.23) is a second order nonhomogeneous linear partial differential equation with constant cofficients. A particular solution of (3.23) is

$$A_1=\frac{1}{29}e^{3\xi}+\frac{3}{26}e^{\xi+2\eta}$$

Now putting

$B_1=m_1e^{2\xi+\eta}+m_2e^{3\eta}$ and $C_1=n_1e^{2\xi}+n_2e^{2\eta}$ in the equations (3.24) and (3.25) and then equating the cofficients of the exponents, we get the algebraic equations

$$m_1+5n_1 = 3/4$$
$$5m_1-n_1 = 0$$
$$-3m_2+4n_2 = 3/8$$
$$4m_2+3n_2 = 0$$

Solving these equations we have

$m_1=3/104$, $m_2=-9/200$, $n_1=15/104$ and $n_2=3/50$.

and therefore

$$B_1=\frac{3}{104}e^{2\xi+\eta}-\frac{9}{200}e^{3\eta}$$

and

$$C_1=\frac{15}{104}e^{2\xi}+\frac{3}{50}e^{2\eta}$$

Again putting the values of $A_1$, $B_1$ and $C_1$ in (3.5) we obtain

$$\frac{d\xi}{dt}=-2+\varepsilon\left(\frac{1}{29}e^{2\xi}+\frac{3}{26}e^{2\eta}\right)$$

$$\frac{d\eta}{dt}=-1+\varepsilon\left(\frac{3}{104}e^{2\xi}-\frac{9}{200}e^{2\eta}\right) \qquad (3.27)$$

$$\frac{d\varphi}{dt}=2+\varepsilon\left(\frac{15}{104}e^{2\xi}+\frac{3}{50}e^{2\eta}\right)$$

Numerical integration of the above system for $\varepsilon=.1$ with initial conditions $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ yields

| t | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 0.0 | -0.500000 | -0.100000 | 0.000000 |
| 0.1 | -0.699039 | -0.200246 | 0.200883 |
| 0.2 | -0.898269 | -0.300461 | 0.401541 |
| 0.3 | -1.097648 | -0.400645 | 0.602036 |
| 0.4 | -1.297147 | -0.500801 | 0.802413 |
| 0.5 | -1.496742 | -0.600933 | 1.002701 |
| 0.6 | -1.696413 | -0.701044 | 1.202924 |
| 0.7 | -1.896146 | -0.801136 | 1.403098 |
| 0.8 | -2.095929 | -0.901213 | 1.603235 |
| 0.9 | -2.295753 | -1.001277 | 1.803342 |
| 1.0 | -2.495609 | -1.101330 | 2.003428 |
| 1.1 | -2.695491 | -1.201373 | 2.203496 |
| 1.2 | -2.895395 | -1.301409 | 2.403550 |
| 1.3 | -3.095317 | -1.401438 | 2.603594 |
| 1.4 | -3.295253 | -1.501462 | 2.803630 |
| 1.5 | -3.495200 | -1.601482 | 3.003659 |
| 1.6 | -3.695158 | -1.701499 | 3.203682 |
| 1.7 | -3.895123 | -1.801512 | 3.403701 |
| 1.8 | -4.095094 | -1.901523 | 3.603716 |
| 1.9 | -4.295070 | -2.001532 | 3.803729 |

**Table 5.**

Thus the first approximate solution (numerical) of (3.22) for $\varepsilon=.1$ with initial conditions $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ is

| t | x ($=e\xi+e\eta\cos\varphi$) | x (for $\varepsilon=0$) |
|---|---|---|
| 0.0 | 1.511368 | 1.511368 |
| 0.1 | 1.299132 | 1.299098 |
| 0.2 | 1.088853 | 1.088633 |
| 0.3 | 0.885766 | 0.885165 |
| 0.4 | 0.694496 | 0.693349 |

| t | x $(=e^{\xi}+e^{\eta}\cos\varphi)$ | x (for $\varepsilon=0$) |
|---|---|---|
| 0.5 | 0.518858 | 0.517063 |
| 0.6 | 0.361741 | 0.359262 |
| 0.7 | 0.225060 | 0.221929 |
| 0.8 | 0.109786 | 0.106087 |
| 0.9 | 0.016014 | 0.011871 |
| 1.0 | -0.056928 | -0.061371 |
| 1.1 | -0.110350 | -0.114937 |
| 1.2 | -0.146055 | -0.150635 |
| 1.3 | -0.166197 | -0.170632 |
| 1.4 | -0.173142 | -0.177311 |
| 1.5 | -0.169340 | -0.173148 |
| 1.6 | -0.157215 | -0.160540 |
| 1.7 | -0.139071 | -0.141968 |
| 1.8 | -0.117022 | -0.119420 |
| 1.9 | -0.092937 | -0.094837 |

Table 6.

Equation (3.26) is a third order linear nonhomogeneous partial differential equation with constant coefficients and its solution is

$$u=\frac{e^{\xi+2\eta}}{2452}(6\sin 2\varphi+17\cos 2\varphi)+\frac{e^{3\eta}}{50320}(43\cos 3\varphi-36\sin 3\varphi)$$

Using Table 5, u may be computed and thus the second approximation is of the form

$$x=e^{\xi}+e^{\eta}\cos\varphi+\varepsilon u$$

**Case 3**. One real and two imaginary roots.

Consider the third order nonlinear differential equation

$$\frac{d^3x}{dt^3}+\frac{d^2x}{dt^2}+4\frac{dx}{dt}+4x=\varepsilon x^3 \qquad (3.28)$$

when $\varepsilon=0$ the characteristic roots are -1,+i2,-i2. Here the corresponding equations of (3.23), (3.24), (3.25) and (3.26) in **Case 2** are

$$\left(\frac{\partial^2}{\partial\xi^2}+4\right)A_1=e^{3\xi}+\frac{3}{2}e^{\xi+2\eta} \qquad (3.29)$$

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial}{\partial\xi}-8\right)B_1-2e^{\eta}\left(-3\frac{\partial}{\partial\xi}+2\right)C_1=3e^{2\xi+\eta}+\frac{3}{4}e^{3\eta} \qquad (3.30)$$

$$-2\left(-3\frac{\partial}{\partial\xi}+2\right)B_1-e^{\eta}\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial}{\partial\eta}-8\right)C_1=0 \qquad (3.31)$$

$$\left[\left(-\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\varphi}\right)^3+\left(-\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\varphi}\right)^2+4\left(-\frac{\partial}{\partial\xi}+2\frac{\partial}{\partial\varphi}\right)+4\right]u$$

$$=\frac{3}{2}e^{\xi+2\eta}\cos 2\varphi+\frac{1}{4}e^{3\eta}\cos 3\varphi \qquad (3.32)$$

Solving (3.29), (3.30) and (3.31) we obtain

$$A_1=\frac{1}{13}e^{3\xi}+\frac{3}{10}e^{\xi+2\eta}$$

$$B_1=-\frac{9}{50}e^{2\xi+\eta}-\frac{3}{40}e^{3\eta}$$

$$C_1=\frac{6}{25}e^{2\xi}-\frac{3}{80}e^{2\eta}$$

Putting these values of $A_1$, $B_1$ and $C_1$ in (3.5), we obtain

$$\frac{d\xi}{dt}=-1+\varepsilon\left(\frac{1}{13}e^{2\xi}+\frac{3}{10}e^{2\eta}\right)$$

$$\frac{d\eta}{dt}=-\varepsilon\left(\frac{9}{50}e^{2\xi}+\frac{3}{40}e^{2\eta}\right) \qquad (3.33)$$

$$\frac{d\varphi}{dt}=2+\varepsilon\left(\frac{6}{25}e^{2\xi}-\frac{3}{80}e^{2\eta}\right)$$

Numerical integration of the above system for $\varepsilon=.1$ with initial conditions $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ yields

| t | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 0.0 | -0.500000 | -0.100000 | 0.000000 |
| 0.1 | -0.597290 | -0.101215 | 0.200496 |
| 0.2 | -0.694631 | -0.102322 | 0.400850 |
| 0.3 | -0.792014 | -0.103340 | 0.601089 |
| 0.4 | -0.889433 | -0.104285 | 0.801231 |
| 0.5 | -0.986882 | -0.105170 | 1.001295 |
| 0.6 | -1.084356 | -0.106004 | 1.201295 |
| 0.7 | -1.181851 | -0.106797 | 1.401241 |
| 0.8 | -1.279364 | -0.107557 | 1.601143 |
| 0.9 | -1.376893 | -0.108287 | 1.801010 |
| 1.0 | -1.474434 | -0.108995 | 2.000847 |
| 1.1 | -1.571987 | -0.109684 | 2.200660 |
| 1.2 | -1.669549 | -0.110356 | 2.400453 |
| 1.3 | -1.767120 | -0.111015 | 2.600230 |
| 1.4 | -1.864699 | -0.111663 | 2.799994 |
| 1.5 | -1.962284 | -0.112302 | 2.999746 |
| 1.6 | -2.059875 | -0.112933 | 3.199490 |

| t | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 1.7 | -2.157472 | -Ø.113557 | 3.399226 |
| 1.8 | -2.255Ø74 | -Ø.114176 | 3.598957 |
| 1.9 | -2.35268Ø | -Ø.114791 | 3.798682 |

Table 7.

Thus the first approximate solution of (3.28) with initial conditions $\xi(\emptyset)=-.5$, $\eta(\emptyset)=-.1$ and $\varphi(\emptyset)=\emptyset$ is

| t | $x$ $(=e^{\xi}+e^{\eta}\cos\varphi)$ | x (for $\varepsilon=\emptyset$) |
|---|---|---|
| Ø.Ø | 1.511368 | 1.511368 |
| Ø.1 | 1.435936 | 1.435896 |
| Ø.2 | 1.33Ø437 | 1.33Ø141 |
| Ø.3 | 1.196681 | 1.195753 |
| Ø.4 | 1.Ø378Ø3 | 1.Ø35768 |
| Ø.5 | Ø.85812Ø | Ø.854463 |
| Ø.6 | Ø.662946 | Ø.657167 |
| Ø.7 | Ø.458362 | Ø.45ØØ24 |
| Ø.8 | Ø.25Ø966 | Ø.239736 |
| Ø.9 | Ø.Ø47595 | Ø.Ø3328Ø |
| 1.Ø | -Ø.144956 | -Ø.162389 |
| 1.1 | -Ø.32Ø212 | -Ø.34Ø622 |
| 1.2 | -Ø.472289 | -Ø.495357 |
| 1.3 | -Ø.596133 | -Ø.621369 |
| 1.4 | -Ø.687728 | -Ø.714489 |
| 1.5 | -Ø.744261 | -Ø.771776 |
| 1.6 | -Ø.764244 | -Ø.791647 |
| 1.7 | -Ø.747575 | -Ø.773941 |
| 1.8 | -Ø.695544 | -Ø.719937 |
| 1.9 | -Ø.61Ø794 | -Ø.6323Ø7 |

Table 8.

Solving (3.32), we obtain

$$u=\frac{3}{4384}e^{\xi+2\eta}(8cos2\varphi-11sin2\varphi)+\frac{1}{4736}e^{3\eta}(cos3\varphi-6sin3\varphi)$$

Using **Table 7**, **u** may be computed and thus we may obtain the second approximation of the form

$$x=e^{\xi}+e^{\eta}\cos\varphi+\varepsilon u$$

**Conclusion:**

A unified theory for the Krylov-Bogoliubov method is presented for obtaining the transient response of a third order weak nonlinear system. When the characteristic roots of the corresponding linear equation are three negative real or one real negative and two complex with negative real part or one negative real and two purely imaginary. Thus, there is no longer any need to treat the three cases separately.

# Chapter 4

# Forced Vibration of Third Order Nonlinear Systems

## 4.1 Introduction

In this chapter, a method is presented to obtain a solution of a third order nonlinear nonautonomous differential equation with small nonlinearties and a periodic forcing term.

The motion of the system

$$\frac{d^3x}{dt^3}+k_1\frac{d^2x}{dt^2}+k_2\frac{dx}{dt}+k_3x=\varepsilon f(x)+F_0\cos\omega t \qquad (4.1)$$

becomes periodic after some transient motions have died out, where $\varepsilon$ is a small parameter which represents perturbation in the system and $f(x)$ is the given nonlinear function. The period of the resulting oscillations is found to have a fundamental frequency of $\omega/2\pi$ and may therefore be represented by a Fourier series in multiple of $\omega$.

The amplitude of the steady state may be calculated by the method of iteration, which is essentially a process of successive approximation. An assumed solution is substituted in to the differential equation, which is integrated to obtain a solution of improved accuracy. The procedure may be repeated any number of times to achieve the desired accuracy.

## 4.2 Forced Vibrations for the Symmetric Restoring Force

When the restoring force is symmetric, we may take $f(x)=x^3$ in (4.1) and as a first approximation let us assume

$$x_1=a\cos\omega t+b\sin\omega t \tag{4.2}$$

where a and b are to be determined. If we substitute this expression for x in equation (4.1) and make use of the trigonometric identities

$$\cos^3\omega t=\frac{1}{4}\cos3\omega t+\frac{3}{4}\cos\omega t$$

$$\sin^3\omega t=\frac{3}{4}\sin\omega t-\frac{1}{4}\sin3\omega t \tag{4.3}$$

$$\cos2\omega t\sin\omega t=\frac{1}{2}(\sin3\omega t-\sin\omega t)$$

$$\cos2\omega t\cos\omega t=\frac{1}{2}(\cos\omega t+\cos3\omega t)$$

we obtain the equation

$$[(k_3-\omega^2k_1)a+\omega(k_2-\omega^2)]\cos\omega t+[\omega(\omega^2-k_2)a+(k_3-\omega^2k_1)b]\sin\omega t$$

$$=\varepsilon[\frac{3}{4}(a^3+ab^2)\cos\omega t+\frac{3}{4}(a^2b+b^3)\sin\omega t+\frac{1}{4}(a^3-3ab^2)\cos3\omega t$$

$$+\frac{1}{4}(3a^2b-b^3)\sin3\omega t]+F_0\cos\omega t \tag{4.4}$$

If the fundamental vibration is to satisfy (3.4), we must have

$$(k_3-\omega^2 k_1)\,a+\omega\,(k_2-\omega^2)\,b=\frac{3}{4}\,(a^3+ab^2)\,\varepsilon+F_o \qquad (4.5a)$$

and

$$\omega\,(\omega^2-k_2)\,a+(k_3-\omega^2 k_1)\,b=\frac{3}{4}\,(a^2b+b^3)\,\varepsilon \qquad (4.5b)$$

Equations (4.5a) and (4.5b) are two algebraic equations and using *Newton-Raphson's* method we may compute **a** and **b**.

## The Higher Approximation

To get a better results we may substitute

$$x_2=a\cos\omega t+b\sin\omega t+a_1\cos 3\omega t+b_1\sin 3\omega t \qquad (4.6)$$

in (4.1). Neglecting the higher order of $a_1$ and $b_1$ and the product terms of $\omega$ with $a_1$ and $b_1$ and equating the coefficients of $\cos 3\omega t$ and $\sin 3\omega t$ from both sides, we get

$$(k_3-9\omega^2 k_1)\,a_1+3\omega\,(k_2-9\omega^2)\,b_1=\frac{\varepsilon}{4}\,(a^3-3ab^2) \qquad (4.7a)$$

and

$$3\omega\,(9\omega^2-k_2)\,a_1+(k_3-9k_1\omega^2)\,b_1=\frac{\varepsilon}{4}\,(3a^2b-b^3) \qquad (4.7b)$$

Equation(4.7a) and (4.7b) are two linear algebraic equations. and solving them we easily get $a_1$ and $b_1$.

## Transient Behavior

For the corresponding autonomous equation of (4.1), i.e., when $F_o=0$, we have studied the transient behavior. We have found an asymptotic solution of (4.1), when $F_o=0$, in the form

$$x_t=e^{\xi}+e^{\eta}\cos\varphi+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2 v(\xi,\eta,\varphi)+\ldots \tag{4.8}$$

where **u**, **v**,.. are functions of **t** defined by the differential equations

$$\frac{d\xi}{dt}=-\lambda+\varepsilon e^{-\xi}A_1(\xi,\eta)+\varepsilon^2 e^{-\xi}A_2(\xi,\eta)+\ldots$$

$$\frac{d\eta}{dt}=-\mu+\varepsilon e^{-\eta}B_1(\xi,\eta)+\varepsilon^2 e^{-\eta}B_2(\xi,\eta)+\ldots \tag{4.9}$$

$$\frac{d\varphi}{dt}=\omega_o+\varepsilon C_1(\xi,\eta)+\varepsilon^2 C_2(\xi,\eta)+\ldots$$

When $F_o\neq 0$, we propose an asymptotic solution of (4.1), in the form,

$$x=x_s+x_t+\varepsilon y(t)+\varepsilon^2 z(t)+\ldots \tag{4.10}$$

where $x_s$ is a steady state solution which has been defined by (4.2),

$$x_t=e^{\xi}+e^{\eta}\cos\varphi+\varepsilon u(\xi,\eta,\varphi)+\varepsilon^2 v(\xi,\eta,\varphi)+\ldots \tag{4.11}$$

and **y**, **z**,.. are functions of **t**.

Differentiating equations (4.10) and (4.11) three times with respect to **t**, using the relations (4.9) and substituting (4.10), (4.11) and the values of derivatives $\dddot{x}, \ddot{x}, \dot{x}$ in (4.1), we obtain

$$\frac{d^3x_s}{dt^3}+k_1\frac{d^2x_s}{dt^2}+k_2\frac{dx_s}{dt}+k_3x_s+\varepsilon[((\Omega-\mu)^2+\omega_o^2)A_1+\cos\varphi((\Omega-\lambda)(\Omega-\mu)-2\omega_o^2)B_1$$

$$-\omega_o e^{\eta}\cos\varphi(-3\Omega+2(\lambda-\mu))C_1-\omega_o\sin\varphi(-3\Omega+2\lambda+\mu)B_1-e^{\eta}\sin\varphi(\Omega^2-(\lambda-\mu)\Omega$$

$$-2\omega_o^2)C_1+((\Omega+\omega\frac{\partial}{\partial\varphi})^3+k_1(\Omega+\omega\frac{\partial}{\partial\varphi})^2+k_2(\Omega+\omega\frac{\partial}{\partial\varphi})+k_3)u+\frac{d^3y}{dt^3}+k_1\frac{d^2y}{dt^2}+k_2\frac{dy}{dt}$$

$$+k_3y]+\varepsilon^2\ldots=F_o\cos\psi+\varepsilon[x_s^3+3x_s^2(e^{\xi}+e^{\eta}\cos\varphi)+3x_s(e^{\xi}+e^{\eta}\cos\varphi)^2$$

$$+(e^{\xi}+e^{\eta}\cos\varphi)^3]+\varepsilon^2\ldots \qquad (4.12)$$

where $\Omega\equiv-\lambda\frac{\partial}{\partial\xi}-\mu\frac{\partial}{\partial\eta}$. Since $x_s$ satisfy the differential equation

$$\frac{d^3x_s}{dt^3}+k_1\frac{d^2x_s}{dt^2}+k_2\frac{dx_s}{dt}+k_3x_s=F_o\cos\psi+\varepsilon x_s^3 \qquad (4.13)$$

we may omit these terms from (4.12) and then comparing the coefficient of $\varepsilon$ on both sides, we obtain,

$$[(\Omega-\mu)^2+\omega_o^2]A_1+\cos\varphi[(\Omega-\lambda)(\Omega-\mu)-2\omega_o^2]B_1-\omega_o e^{\eta}\cos\varphi[-3\Omega+2(\lambda-\mu)]C_1$$

$$-\omega_o\sin\varphi(-3\Omega+2\lambda+\mu)B_1-e^{\eta}\sin\varphi(\Omega^2-(\lambda-\mu)\Omega-e^{\eta}\sin\varphi[\Omega^2-(\lambda-\mu)\Omega-2\omega_o^2]C_1$$

$$+[(\Omega+\omega\frac{\partial}{\partial\varphi})^3+k_1(\Omega+\omega\frac{\partial}{\partial\varphi})^2+k_2(\Omega+\omega\frac{\partial}{\partial\varphi})+k_3]u+\frac{d^3y}{dt^3}+k_1\frac{d^2y}{dt^2}+k_2\frac{dy}{dt}+k_3y$$

$$=F_o\cos\psi+\varepsilon[x_s^3+3x_s^2(e^{\xi}+e^{\eta}\cos\varphi)+3x_s(e^{\xi}+e^{\eta}\cos\varphi)^2$$

$$+(e^{\xi}+e^{\eta}\cos\varphi)^3] \qquad (4.14)$$

Substituting $x_s=a\cos\psi+b\sin\psi$ in (4.14) and then comparing the coefficients of $\cos\varphi$, $\sin\varphi$, $\cos r\varphi$ and $\sin r\varphi$ $(r\geq 2)$ (not product with $\cos\psi$ and $\sin\psi$) , we obtain

$$[(\Omega-\mu)^2+\omega_o^2]A_1=e^{3\xi}+\frac{3}{2}e^{\xi+2\eta}+\frac{3}{2}(a^2+b^2)e^{\xi} \qquad (4.15)$$

$$[(\Omega-\lambda)(\Omega-\mu)-2\omega_o^2]B_1-\omega_o e^{\eta}[-3\Omega+2(\lambda-\mu)]C_1=3e^{2\xi+\eta}$$

$$+\frac{3}{4}e^{3\eta}+\frac{3}{2}(a^2+b^2)e^{\eta} \qquad (4.16)$$

$$-\omega_o[-3\Omega+2\lambda+\mu]B_1-e^{\eta}[\Omega^2-(\lambda-\mu)\Omega-2\omega_o^2]C_1=0 \qquad (4.17)$$

$$[(\Omega+\omega\frac{\partial}{\partial\varphi})^3+k_1(\Omega+\omega\frac{\partial}{\partial\varphi})^2+k_2(\Omega+\omega\frac{\partial}{\partial\varphi})+k_3]u=\frac{3}{2}e^{\xi+2\eta}\cos 2\varphi$$

$$+\frac{1}{4}e^{3\eta}\cos 3\varphi \qquad (4.18)$$

and

$$\frac{d^3y}{dt^3}+k_1\frac{d^2y}{dt^2}+k_2\frac{dy}{dt}+k_3y=3(e^{2\xi}+\frac{1}{2}e^{2\eta})(a\cos\psi+b\sin\psi)+e^{\xi}[\frac{3}{2}(a^2-b^2)$$

$$\cos 2\psi+3ab\sin 2\psi]+6e^{\xi+\eta}(a\cos\psi+b\sin\psi)+\frac{3}{2}e^{2\eta}(a\cos 2\varphi\cos\psi$$

$$+b\cos 2\varphi\sin\psi)+3e^{\eta}(\frac{a^2-b^2}{2}\cos\varphi\cos 2\psi+ab\cos\varphi\sin 2\psi) \qquad (4.19)$$

Equation (4.15) is a second order nonhomogeneous partial differential equation with constant coefficients and equations

(4.16) and (4.17) are simultaneous partial differential equations with constant coefficients. Obviously, a particular solution of (4.15) gives the function $A_1$.

Substituting $B_1 = c_1 e^{2\xi+\eta} + c_2 e^{3\eta} + c_3 e^{\eta}$ and $C_1 = d_1 e^{2\xi} + d_2 e^{2\eta} + d_3$ in (4.16) and (4.17) and equating the coefficients of $e^{2\xi+\eta}$, $e^{2\eta}$ and $e^{\eta}$ we get a system of algebraic equations, whose solution gives the values of $c_1$, $c_2$, $c_3$, $d_1$, $d_2$ and $d_3$. Thus $B_1$ and $C_1$ are obtained. Substituting these values of $A_1$, $B_1$ and $C_1$ in (4.9) and integrating (numerically) $\xi$, $\eta$ and $\varphi$ and then $x_t$ are obtained.

## Determination of $u(\xi,\eta,\varphi)$

Equation (4.18) is a third order nonhomogeneous linear partial differential equation with constant coefficients and its particular solution gives the first order correction term $u(\xi,\eta,\varphi)$ which determines the transient behavior of (4.1).

## Determination of y

Equation (4.19) is a third order ordinary nonhomogeneous differential equation. If we substitute

$$y=e^{2\xi}(l_1\cos\psi+l_2\sin\psi)+e^{2\eta}(l_3\cos\psi+l_4\sin\psi)+e^{\xi}(l_5\sin2\psi+l_6\cos2\psi)$$

$$+e^{\xi}(m_1\cos\varphi\cos\psi+m_2\cos\varphi\sin\psi+m_3\sin\varphi\cos\psi+m_4\sin\varphi\sin\psi)+e^{2\eta}(n_1\cos\varphi$$

$$\cos2\psi+n_2\cos\varphi\sin2\psi+n_3\sin\varphi\sin2\psi+n_4\sin\varphi\sin2\psi) \qquad (4.20)$$

in (4.19) and then equate the coefficients of $e^{2\xi}\cos\psi$, $e^{2\xi}\sin\psi,\ldots,$ $e^{2\eta}\sin\varphi\sin\psi$, we get a set of algebraic equations and solving them $l_1$, $l_2,\ldots,n_4$ are obtained.

### Example

Consider the equation

$$\frac{d^3x}{dt^3}+4\frac{d^2x}{dt^2}+9\frac{dx}{dt}+10x=.1x^3+2\cos t \qquad (4.21)$$

In this case, the corresponding equations of (4.5a) and (4.5b) are

$$6a+8b=\frac{3}{40}(a^3+ab^2)+2$$

and

$$-8a+6b=\frac{3}{40}(a^2b+b^3)$$

and the approximate solution of this system is

a=0.119983177 and b=0.160057627.

Therefore, the approximate value of $x_{91}$ is

| t | $x_{91}$ |
|---|---|
| Ø.Ø | Ø.119983 |
| Ø.1 | Ø.135363 |
| Ø.2 | Ø.14939Ø |
| Ø.3 | Ø.161925 |
| Ø.4 | Ø.172841 |
| Ø.5 | Ø.182Ø31 |
| Ø.6 | Ø.1894Ø2 |
| Ø.7 | Ø.19488Ø |
| Ø.8 | Ø.198411 |
| Ø.9 | Ø.19996Ø |
| 1.Ø | Ø.199511 |
| 1.1 | Ø.197Ø68 |
| 1.2 | Ø.192657 |
| 1.3 | Ø.18632Ø |
| 1.4 | Ø.178122 |
| 1.5 | Ø.168144 |
| 1.6 | Ø.156486 |
| 1.7 | Ø.143264 |
| 1.8 | Ø.128611 |
| 1.9 | Ø.112673 |

**Table 9.**

Putting the values of **a** and **b**, and the values of $k_1$, $k_2$, $k_3$ and $\omega$ in equations (4.7a) and (4.7b) and solving them, we obtain

$$a_1=7.205\times10^{-6} \quad \text{and} \quad b_1=-2.703\times10^{-6}$$

Equations (4.15), (4.16) and (4.17) then become

$$[(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1)^2+4]A_1=e^{3\xi}+\frac{3}{2}e^{\xi+2\eta}+0.06002161e^{\xi} \qquad (4.21)$$

$$[(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-2)(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1)-8]B_1-2e^{\eta}[-3(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta})+2]C_1$$

$$=3e^{2\xi+\eta}+\frac{3}{4}e^{3\eta}+0.06002161e^{\eta} \qquad (4.22)$$

and

$$2[-3(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta})+5]B_1+e^{\eta}[(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta})^2-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}-8]C_1=0 \qquad (4.23)$$

Solving (4.21), (4.22) and (4.23), we obtain,

$$A_1=\frac{1}{29}e^{3\xi}+\frac{3}{26}e^{\xi+2\eta}+0.012004322e^{\xi}$$

$$B_1=\frac{3}{104}e^{2\xi+\eta}-\frac{9}{200}e^{3\eta}-0.006002161e^{\eta}$$

and

$$C_1=\frac{15}{104}e^{2\xi}+\frac{3}{50}e^{2\eta}-0.00300108$$

Thus equations (4.9) become

$$\frac{d\xi}{dt}=-2+.1\left(\frac{1}{29}e^{2\xi}+\frac{3}{26}e^{2\eta}+0.012004322\right)$$

$$\frac{d\eta}{dt}=-1+.1\left(\frac{3}{104}e^{2\xi}-\frac{9}{200}e^{2\eta}-0.006002161\right) \qquad (4.24)$$

$$\frac{d\varphi}{dt}=2+.1\left(\frac{15}{104}e^{2\xi}+\frac{3}{50}e^{2\eta}-0.00300108\right)$$

By using a **Computer** the above system has been integrated numerically by means of a fourth-order Runge-Kutta procedure with initial conditions $\xi(0)=\xi_o$, $\eta(0)=\eta_o$ and $\varphi(0)=\varphi_o$. If $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$, the approximate solution of this system is

| t | ξ | η | φ |
|---|---|---|---|
| 0.0 | -0.500000 | -0.100000 | 0.000000 |
| 0.1 | -0.698919 | -0.200306 | 0.200853 |
| 0.2 | -0.898029 | -0.300581 | 0.401481 |
| 0.3 | -1.097288 | -0.400825 | 0.601947 |
| 0.4 | -1.296668 | -0.501041 | 0.802293 |
| 0.5 | -1.496143 | -0.601233 | 1.002551 |
| 0.6 | -1.695694 | -0.701404 | 1.202744 |
| 0.7 | -1.895307 | -0.801556 | 1.402888 |
| 0.8 | -2.094970 | -0.901693 | 1.602994 |
| 0.9 | -2.294674 | -1.001816 | 1.803072 |
| 1.0 | -2.494410 | -1.101929 | 2.003128 |
| 1.1 | -2.694172 | -1.202032 | 2.203166 |
| 1.2 | -2.893957 | -1.302128 | 2.403190 |
| 1.3 | -3.093758 | -1.402218 | 2.603204 |
| 1.4 | -3.293574 | -1.502302 | 2.803209 |
| 1.5 | -3.493402 | -1.602382 | 3.003208 |

| t | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 1.6 | -3.693240 | -1.702458 | 3.203201 |
| 1.7 | -3.893085 | -1.802531 | 3.403190 |
| 1.8 | -4.092936 | -1.902603 | 3.603175 |
| 1.9 | -4.292793 | -2.002672 | 3.803158 |

Table 10.

Thus, when $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$, the first approximate value of $x_t$ is

| t | $x_t$ |
|---|---|
| 0.0 | 1.511368 |
| 0.1 | 1.299148 |
| 0.2 | 1.088887 |
| 0.3 | 0.885821 |
| 0.4 | 0.694578 |
| 0.5 | 0.518974 |
| 0.6 | 0.361892 |
| 0.7 | 0.225247 |
| 0.8 | 0.110008 |
| 0.9 | 0.016265 |
| 1.0 | -0.056655 |
| 1.1 | -0.110063 |
| 1.2 | -0.145764 |
| 1.3 | -0.165912 |
| 1.4 | -0.172872 |

| t | $x_t$ |
|---|---|
| 1.5 | -Ø.169Ø93 |
| 1.6 | -Ø.156998 |
| 1.7 | -Ø.138889 |
| 1.8 | -Ø.116878 |
| 1.9 | -Ø.Ø92832 |

**Table 11.**

Using **Table 1** and **Table 3**, we obtain the first approximate solution of (4.21) with initial conditions $\xi(Ø)=-.5$, $\eta(Ø)=-.1$ and $\phi(Ø)=Ø$,

| t | x |
|---|---|
| Ø.Ø | 1.631351 |
| Ø.1 | 1.434511 |
| Ø.2 | 1.238277 |
| Ø.3 | 1.Ø47746 |
| Ø.4 | Ø.867419 |
| Ø.5 | Ø.7Ø1ØØ5 |
| Ø.6 | Ø.551294 |
| Ø.7 | Ø.42Ø127 |
| Ø.8 | Ø.3Ø8419 |
| Ø.9 | Ø.216225 |
| 1.Ø | Ø.142856 |
| 1.1 | Ø.Ø87ØØ5 |
| 1.2 | Ø.Ø46893 |
| 1.3 | Ø.Ø2Ø4Ø8 |
| 1.4 | Ø.ØØ525Ø |
| 1.5 | -Ø.ØØØ949 |

| t | x |
|---|---|
| 1.6 | -0.000512 |
| 1.7 | 0.004375 |
| 1.8 | 0.011733 |
| 1.9 | 0.019841 |

Table 12.

Now equations (4.18) and (4.19) for example (4.21) become

$$[(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\varphi})^3+4(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+2\frac{\partial}{\partial\varphi})^2+9(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+2\frac{\partial}{\partial\varphi})+10]u$$

$$=\frac{3}{2}e^{\xi+2\eta}\cos 2\varphi+\frac{1}{4}e^{3\eta}\cos 3\varphi \qquad (25)$$

and

$$\frac{d^3y}{dt^3}+4\frac{d^2y}{dt^2}+9\frac{dy}{dt}+103y=3(e^{2\xi}+\frac{1}{2}e^{2\eta})(a\cos\psi+b\sin\psi)+e^{\xi}[\frac{3}{2}(a^2-b^2)$$

$$\cos 2\psi+3ab\sin 2\psi]+6e^{\xi+\eta}(a\cos\psi+b\sin\psi)+\frac{3}{2}e^{2\eta}(a\cos 2\varphi\cos\psi$$

$$+b\cos 2\varphi\sin\psi)+3e^{\eta}(\frac{a^2-b^2}{2}\cos\varphi\cos 2\psi+ab\cos\varphi\sin 2\psi) \qquad (4.26)$$

The particular solutions of (4.25) and (4.26) are

$$u=\frac{e^{\xi+2\eta}}{2452}(6\sin 2\varphi+17\cos 2\varphi)+\frac{e^{3\eta}}{50320}(43\cos 3\varphi-36\sin 3\varphi) \qquad (4.27)$$

and

$$y=e^{2\xi}(0.119983177\cos\psi+0.160057627\sin\psi)+e^{2\eta}(-0.060003597\cos\psi$$

$$-0.030019811\sin\psi)+e^{\xi}(-0.003674928\cos 2\psi+0.00628285\sin 2\psi)$$

$$+e^{\xi+\eta}(0.038528021\cos\varphi\cos\psi-0.027051323\cos\varphi\sin\psi$$

$$-0.000573131\sin\varphi\cos\psi+0.026118949\sin\varphi\sin\psi)$$

$$+e^{2\eta}(0.000136265\cos 2\varphi\cos\psi+0.000534775\cos 2\varphi\sin\psi$$

$$-0.000573131\sin 2\varphi\cos\psi-0.000074981\sin 2\varphi\sin\psi)$$

$$+e^{\eta}(-0.000499375\cos\varphi\cos 2\psi+0.00240847\cos\varphi\sin 2\psi$$

$$-0.002392584\sin\varphi\cos 2\psi-0.000903434\sin\varphi\sin 2\psi) \quad (4.28)$$

## 4.3 Forced Vibrations for the Antisymmetric Restoring Force

When the restoring force is antisymmetric, we may use $f(x)=x^2$ in (4.1) and assume

$$x_1=a_0+a\cos\omega t+b\sin\omega t \quad (4.29)$$

as the first approximate solution, where $a_0$, $a$ and $b$ are to be determined. If we substitute this expression for $x$ in the equation (4.1) and make use of the trigonometric identities,

$$\cos^2\omega t=\frac{1}{2}(1+\cos 2\omega t),$$

$$2\cos\omega t\sin\omega t=\sin 2\omega t, \qquad (4.30)$$

and

$$\sin^2\omega t=\frac{1}{2}(1-\cos 2\omega t)$$

We obtain the equation,

$$[(k_3-\omega^2k_1)a+\omega(k_2-\omega^2)b]\cos\omega t+[\omega(\omega^2-k_2)a+(k_3-\omega^2k_1)b]\sin\omega t+k_3a_0=$$

$$\varepsilon[a_0^2+\frac{1}{2}(a^2+b^2)+2a_0(a\cos\omega t+b\sin\omega t)+\frac{1}{2}(a^2-b^2)\cos 2\omega t+ab\sin 2\omega t]$$

$$+F_0\cos\omega t \qquad (4.31)$$

If the constant term and the fundamental vibrations is to satisfy (4.31), we must have

$$k_3a_0=[a^2+\frac{1}{2}(a^2+b^2)]\varepsilon \qquad (4.32)$$

$$(k_3-\omega^2k_1)a+\omega(k_2-\omega^2)b=2a_0a\varepsilon+F_0 \qquad (4.33)$$

and

$$\omega(\omega^2-k_2)a+(k_3-\omega^2k_1)b=2a_0b\varepsilon \qquad (4.34)$$

Equations (4.32), (4.33) and (4.34) are three nonlinear algebraic equations and by *Newton-Raphson* method we may compute the approximate values of $a_0$, a and b.

## The Higher Approximation

To get a better result, we may substitute

$$x_2=a_0+a\cos\omega t+b\sin\omega t+a_1\cos 2\omega t+b_1\sin 2\omega t \qquad (4.35)$$

in (4.1). Neglecting the higher order of $a_1$ and $b_1$ and the terms of $\omega$ with $a_1$, $b_1$ and equating the coefficients of $\cos\omega t$ and $\sin\omega t$ from both sides we get,

$$(k_3-4\omega^2 k_1)\,a_1+2\omega\,(k_2-4\omega^2)\,b_1=\frac{1}{2}\,(a^2-b^2)\,\varepsilon \qquad (4.36a)$$

and

$$2\omega\,(4\omega^2-k_2)\,a_1+(k_3-4\omega^2 k_1)\,b_1=ab\varepsilon \qquad (4.36b)$$

Equations (4.36a) and (4.36b) are two linear algebraic equations and solving them we get $a_1$ and $b_1$.

## Transient behavior

In this case, we also propose an asymptotic solution of the form (10) and the corresponding equations of (4.15), (4.16), (4.17), (4.18) and (4.19) in case 1, are

$$[\,(\Omega-\mu)^2+\omega_0{}^2]\,A_1=e^{2\xi}+\frac{1}{2}\,e^{2\eta}+2a_0e^{\xi} \qquad (4.37)$$

$$[\,(\Omega-\lambda)\,(\Omega-\mu)-2\omega_0{}^2]\,B_1-\omega_0e^{\eta}\,[-3\Omega+2\,(\lambda-\mu)\,]\,C_1=2e^{\xi+\eta}+2a_0e^{\eta} \qquad (4.38)$$

$$-\omega_0\,[-3\Omega+(2\lambda+\mu)\,]\,B_1-e^{\eta}\,[\Omega^2-(\lambda-\mu)\,\Omega-2\omega_0{}^2]\,C_1=0 \qquad (4.39)$$

$$[(\Omega+\omega\frac{\partial}{\partial\varphi})^3+k_1(\Omega+\frac{\partial}{\partial\varphi})^2+k_2(\Omega+\omega\frac{\partial}{\partial\varphi})+k_3]u=\frac{1}{2}e^{2\eta}\cos 2\varphi \quad (4.40)$$

and

$$\frac{d^3y}{dt^3}+k_1\frac{d^2y}{dt^2}+k_3\frac{dy}{dt}+k_3=2e^{\xi}(a\cos\psi+b\sin\psi)$$

$$+2(a\cos\varphi\cos\psi+b\cos\varphi\sin\psi) \quad (4.41)$$

In a similar way, discussed in case 1, we can solve the equations (4.37)-(4.41).

## Example

Consider the equation

$$\frac{d^3x}{dt^3}+4\frac{d^2x}{dt^2}+9\frac{dx}{dt}+10x=0.1x^2+2\cos t \quad (4.42)$$

In this case, the corresponding equations of (4.32), (4.33) and (4.34) are

$$10a=\frac{1}{20}(2a_o{}^2+a^2+b^2)$$

$$6a+8b=\frac{1}{5}a_oa+2$$

and

$$-8a+6b=\frac{1}{5}a_ob$$

The approximate solution of this system is

$$a_0=0.0002, \quad a=0.119999776, \quad b=0.160000764.$$

Therefore the approximate values of $x_8$ are

| t | $x_8$ |
|---|---|
| 0.0 | 0.120200 |
| 0.1 | 0.135574 |
| 0.2 | 0.149595 |
| 0.3 | 0.162124 |
| 0.4 | 0.173034 |
| 0.5 | 0.182218 |
| 0.6 | 0.189583 |
| 0.7 | 0.195056 |
| 0.8 | 0.198582 |
| 0.9 | 0.200126 |
| 1.0 | 0.199672 |
| 1.1 | 0.197225 |
| 1.2 | 0.192810 |
| 1.3 | 0.186470 |
| 1.4 | 0.178269 |
| 1.5 | 0.168288 |
| 1.6 | 0.156629 |
| 1.7 | 0.143406 |
| 1.8 | 0.128752 |
| 1.9 | 0.112814 |

Table 13.

Putting these values of $a_o$, $a$, $b$, $k_1$, $k_2$, $k_3$ and $\omega$ in equations (4.36a) and (4.36b) and then solving them, we obtain

$$a_1=-0.00011647, \quad b_1=-0.000125883.$$

Equations (4.37), (4.38) and (4.39) become

$$[(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1)+4]A_1=e^{2\xi}+\frac{1}{2}e^{2\eta}+0.0004\,e^{\xi} \qquad (4.43)$$

$$[2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-2)(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}-1)-8]B_1-2e^{\eta}[-3(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta})+2]C_1$$

$$=2e^{\xi+\eta}+0.0004\,e^{\eta} \qquad (4.44)$$

and

$$2[-3(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta})+5]B_1+e^{\eta}[(2\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta})^2-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}-8]C_1=0 \qquad (4.45)$$

Solving equations (4.43), (4.44) and (4.45), we obtain

$$A_1=\frac{1}{13}e^{2\xi}+\frac{1}{10}e^{2\eta}+0.00008e^{\xi}$$

$$B_1=-\frac{3}{25}e^{\xi+\eta}-0.00004e^{\eta}$$

and

$$C_1=\frac{4}{25}-0.00002$$

Thus in this case equations (4.9) become

$$\frac{d\xi}{dt}=-2+0.1\left(\frac{1}{13}e^{\xi}+\frac{1}{10}e^{-\xi+2\eta}+0.00008\right)$$

$$\frac{d\eta}{dt}=-1+0.1\left(-\frac{3}{25}e^{\xi}-0.00004\right) \qquad (4.46)$$

$$\frac{d\varphi}{dt}=2+0.1\left(\frac{4}{25}e^{\xi}-0.00002\right)$$

Numerical integration of the above system for $\varepsilon=.1$ with initial conditions $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ yields

| t | $\xi$ | $\eta$ | $\varphi$ |
|---|---|---|---|
| 0.0 | -0.500000 | -0.100000 | 0.000000 |
| 0.1 | -0.698228 | -0.200660 | 0.200880 |
| 0.2 | -0.896537 | -0.301201 | 0.401602 |
| 0.3 | -1.094911 | -0.401645 | 0.602194 |
| 0.4 | -1.293340 | -0.502008 | 0.802679 |
| 0.5 | -1.491813 | -0.602307 | 1.003077 |
| 0.6 | -1.690324 | -0.702551 | 1.203403 |
| 0.7 | -1.888866 | -0.802751 | 1.403671 |
| 0.8 | -2.087433 | -0.902916 | 1.603890 |
| 0.9 | -2.286022 | -1.003050 | 1.804070 |
| 1.0 | -2.484628 | -1.103161 | 2.004217 |
| 1.1 | -2.683249 | -1.203251 | 2.204338 |
| 1.2 | -2.881883 | -1.303325 | 2.404437 |
| 1.3 | -3.080527 | -1.403385 | 2.604518 |
| 1.4 | -3.279180 | -1.503435 | 2.804585 |
| 1.5 | -3.477841 | -1.603475 | 3.004639 |
| 1.6 | -3.676509 | -1.703509 | 3.204680 |
| 1.7 | -3.875182 | -1.803536 | 3.404720 |

| t | ξ | η | φ |
|---|---|---|---|
| 1.8 | -4.073860 | -1.903558 | 3.604750 |
| 1.9 | -4.272543 | -2.003576 | 3.804774 |

Table 14.

Thus when $\xi(0)=-.5$, $\eta(0)=-.1$ and $\varphi(0)=0$ the first approximate value of $x_t$ is

| t | $x_t$ |
|---|---|
| 0.0 | 1.511368 |
| 0.1 | 1.299204 |
| 0.2 | 1.089038 |
| 0.3 | 0.886069 |
| 0.4 | 0.694914 |
| 0.5 | 0.519386 |
| 0.6 | 0.362371 |
| 0.7 | 0.225783 |
| 0.8 | 0.110592 |
| 0.9 | 0.016889 |
| 1.0 | -0.056001 |
| 1.1 | -0.109388 |
| 1.2 | -0.145079 |
| 1.3 | -0.165227 |
| 1.4 | -0.172197 |
| 1.5 | -0.168438 |
| 1.6 | -0.156370 |

| $t$ | $x_t$ |
| --- | --- |
| 1.7 | -Ø.138296 |
| 1.8 | -Ø.116324 |
| 1.9 | -Ø.Ø92322 |

Table 15.

Using **Table 5** and **Table 7**, we obtain the first approximate solution of (4.21) with initial conditions $\xi(Ø)=-.5$, $\eta(Ø)=-.1$ and $\phi(Ø)=Ø$,

| t | x |
| --- | --- |
| Ø.Ø | 1.631568 |
| Ø.1 | 1.434778 |
| Ø.2 | 1.238633 |
| Ø.3 | 1.Ø48193 |
| Ø.4 | Ø.867948 |
| Ø.5 | Ø.7Ø16Ø4 |
| Ø.6 | Ø.551954 |
| Ø.7 | Ø.42Ø839 |
| Ø.8 | Ø.3Ø9174 |
| Ø.9 | Ø.217Ø15 |
| 1.Ø | Ø.143671 |
| 1.1 | Ø.Ø87837 |
| 1.2 | Ø.Ø47731 |
| 1.3 | Ø.Ø21243 |
| 1.4 | Ø.ØØ6Ø72 |
| 1.5 | -Ø.ØØØ15Ø |

| t | x |
|---|---|
| 1.6 | -0.000259 |
| 1.7 | 0.005110 |
| 1.8 | 0.012428 |
| 1.9 | 0.020492 |

Table 16.

Now equations (4.40) and (4.41) for example (4.42) become

$$\left[\left(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+2\frac{\partial}{\partial\varphi}\right)^3+4\left(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+2\frac{\partial}{\partial\varphi}\right)^2+9\left(-2\frac{\partial}{\partial\xi}-\frac{\partial}{\partial\eta}+2\frac{\partial}{\partial\varphi}\right)+10\right]u$$

$$=\frac{1}{2}e^{2\eta}\cos 2\varphi \tag{4.46}$$

and

$$\frac{d^3y}{dt^3}+4\frac{d^2y}{dt^2}+9\frac{dy}{dt}+10y=e^{\xi}(2a\cos\psi+2b\sin\psi)$$

$$+e^{\eta}(2a\cos\varphi\cos\psi+2b\cos\varphi\sin\psi) \tag{4.47}$$

The particular solutions of these two equations are

$$u=\frac{1}{1480}e^{2\eta}(8\cos 2\varphi-11\sin 2\varphi) \tag{4.48}$$

and

$$y=e^{\xi}(-0.04000035\cos\psi+0.44799991\sin\psi)+e^{\eta}(0.00179556\cos\varphi\cos\psi$$

$$-0.000124439\cos\varphi\sin\psi-0.000568894\sin\varphi\cos\psi$$

$$+0.001315556\sin\varphi\sin\psi) \tag{4.49}$$

Conclusions :

A method is developed to find the forced vibrations for the symmetric and antisymmetric restoring forces which are modeled by a third order nonlinear nonautonomous differential equation with a small nonlinearities and a periodic forcing term. The amplitude of the steady state is calculated by the method of iteration and the period of the resulting oscillations in this case is found to have a fundamental frequency of $\frac{\omega}{2\pi}$.

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