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On Finitely Generated n-Ideals of a Nearlattice

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It is certified that the thesis entitled **"On Finitely Generated n-Ideals of a Nearlattice"** submitted by **Md. Mizanur Rahman** in fulfilment of the requirments for the degree of **Doctor of Philosophy** in Mathematics, University of Rajshahi, Rajshahi, Bangladesh has been completed under our supervision. We believe that this research work is an original one and it has not been submitted elsewhere for any degree.

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STATEMENT OF ORIGINALITY

This thesis entitled **"On Finitely Generated n-Ideals of a Nearlattice"** does not incorporate without acknowledgement with any material previously submitted for a degree or diploma in any University and to the best of my knowledge and belief, does not contain material previously published or written by another person except where due reference is made in the text.

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ABSTRACT

The thesis studies extensively the nature of finitely generated n-ideals of a nearlattice. Many authors including [1], [34] and [35] have studied about n-ideals of a lattice. A convex sublattice containing a fixed element n of a lattice L, is called an n-*ideal*. If L has '0', then replacing n by '0', an n-ideal becomes an ideal.Similarly, if L has 1, an n-ideal becomes a filter replacing n by 1. Thus the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. Many authors including [27], [40], [47] and [56] done some work on n-ideals of a nearlattice. A nearlattice is a meet semilattice with the property that any two elements possessing a common upper bound, have a supremum. For two n-ideals I and J of a nearlattice S, [27] has given a neat description of $I \vee J$, while the set theoretic intersection is the infinimum. So, $I_n(S)$, the set of all n-ideals of a nearlattice S, is a lattice. An n-ideal generated by a finite number of elements $a_1, a_2, ..., a_r$ is called a *finitely generated* n-*ideal* and denoted by $<a_1, a_2, ...,a_r>_n$, while the n-ideal generated by a single element x is called a *principal* n-*ideal*, denoted by $<x>_n$. $F_n(S)$ and $P_n(S)$ denotes the set of all finitely generated n-ideals and the set of all principal n-ideals respectively. When $n \in S$ is medial and standard, then $P_n(S)$ is meet semilattice. Moreover, when n is sesquimedial and neutral, then $P_n(S)$ is again a nearlattice. In general $F_n(S)$ is a join semilattice. But when S is distributive, then $F_n(S)$ is also a distributive lattice. In this thesis we prove several results on finitely generated n-ideals of a distributive nearlattice S when n is an upper element. These results certainly extend and generalize many results on disjunctive, generalized Boolean, generalized Stone and relatively Stone nearlattices. Since for a distributive nearlattice S with $n \in S$, $I_n(S)$ is a distributive algebraic lattice, so it is pseudocomplemented. Since we can talk

about the pseudocomplementation of $F_n(S)$ only when $0, 1 \in S$ (that is, S is bounded lattice), so for a general $F_n(S)$ we consider only the sectional and relative pseudocomplementation.

In chapter 1, we describe some basic properties of nearlattices which are essential for the rest of the thesis. Here we discuss the ideals and congruences of a nearlattice. Then we discuss on n-ideals of a nearlattice. Recently [25] and [46] have established some results on n-ideals. Here we extend their work and establish some important results.

Chapter 2 discusses the prime n-ideals of a nearlattice and establishes several properties of prime n-ideals. We include a proof of the generalization of Stone's seperation theorem. We make a correction to a certain inaccuracy in the statement of the result $P_n(S) \cong (n]^d \times [n)$ if and only if n is a central element, due to [42]. Then we show that $F_n(S)$ is sectionally complemented if and only if $P_n(S)$ is so. Using this result we show that $F_n(S)$ is generalized Boolean if and only if the prime n-ideals of S are unordered by set inclusion. Then we discuss on the congruences of S containing an n-ideal I as a class. We show that when S is distributive, $I_n(S) \cong I(F_n(S))$ if $F_n(S)$ is generalized Boolean. Moreover, there is an isomorphism between $C(F_n(S))$ and C(S) when n is an upper element. Finally, we include a result on the permutability of the congruences $\Theta(I)$ and $\Theta(J)$ for n-ideals I and J of a distributive nearlattice S when n is upper.

In chapter 3, we study the n-kernels of skeletal congruences on a distributive nearlattice. Previously, skeletal congruences have been studied by Cornish [8]. Then Latif in [34] studied the n-kernels of skeletal congruences on a distributive lattice. This chapter generalizes several results of their work. Here we give a description on $\Theta(J)^*$ for an n-ideal J of a distributive nearlattice S. For a nearlattice S, we define the skeleton

$$SC(S) = \{\Theta \in C(S): \Theta = \Phi^* \text{ for some } \Phi \in C(S)\}$$
$$= \{\Theta \in C(S) : \Theta = \Phi^{**}\}$$

We define $J^+ = \{x \in S: m(x, n, j) = n \text{ for all } j \in J\}$ when n is a medial element of S. Obviously J^+ is is an n-ideal and $J \cap J^+ = \{n\}$. We also define $Ker_n\Theta = \{x \in S: x \equiv n\Theta\}$ and $K_nSC(S) = \{Ker_n\Theta : \Theta \in SC(S)\}$.

This chapter establishes the following fundamental results:

(i) J^+ is the n-kernel of $\Theta(J)^*$, when n is an upper element of a distributive nearlattice S.

(ii) $F_n(S)$ is disjunctive if and only if $P_n(S)$ is disjunctive, when n is an upper element of S.

(iii) $F_n(S)$ is disjunctive if and only if $\langle a\rangle_n = \langle a\rangle_n^{++}$, when n is upper.

(iv) $F_n(S)$ is disjunctive if and only if $\Theta(J^*) = (\Theta(J))^*$ for dense n-ideal J.

(v) $F_n(S)$ is generalized Boolean if and only if for any n-ideal J, $\Theta(J^+) = \Theta(J)^*$.

(vi) $F_n(S)$ is generalized Boolean if and only if the map $\Theta \longrightarrow Ker_n\Theta$ is a lattice isomorphism of SC(S) onto $K_nSC(S)$, whose inverse is the map $J \rightarrow \Theta(J)$, when J is an n-ideal of a distributive nearlattice S with an upper element n.

Chapter 4 discusses on minimal prime n-ideals of a nearlattice. We give some characterizations of minimal prime n-ideals which are essential for the further development of this chapter. Here we provide a number of results which are

generalizations of the results on generalized Stone nearlattices. We also give several characterizations of those $F_n(S)$ which are generalized Stone nearlattices, in terms of n-ideals. We prove that when $F_n(S)$ is an sectionally pseudocomplemented distributive nearlattice, then $F_n(S)$ is generalized Stone if and only if for any two minimal prime ideals P and Q, $P \vee Q = S$. We give some characterizations of those $F_n(S)$ which are relatively Stone in terms of n-ideals and relative n-annihilators. These results are certainly generalizations of several results on relatively Stone nearlattices. Also, we show that for a nearlattice S, when $F_n(S)$ is a relatively pseudocomplemented distributive nearlattice and n is an upper element, then $F_n(S)$ is relatively Stone if and only if two incomparable prime n-ideals P and Q are comaximal. That is $P \vee Q = S$.

Pseudocomplemented distributive nearlattices satisfying Lee's identities form equational subclasses denoted by B_m, $-1 \leq m < \omega$. Cornish [9] and Mandelker [38] have studied distributive lattices analouges to B_1-lattices. Moreover, Cornish[10], Beazer [4], Davey [14] and ayub [1] have each independently obtained several characterizations of sectionally B_m-lattices and relative B_m – lattices. In chapter 5 we generalize their results by studying principal n-ideals which are sectionally in B_m and relatively in B_m. We show that if for a central element n, $P_n(S)$ is sectionally pseudocomplemented and distributive, then $P_n(S)$ is sectionally in B_m if and only if for any $x_0, x_1, x_2, \ldots, x_m \in S$,
$\langle x_0\rangle_n^+ \vee \langle x_1\rangle_n^+ \vee \ldots \vee \langle x_m\rangle_n^+ = S$, which is also equivalent to the condition that for any m+1 distinct minimal prime n-ideals $P_0, \ldots, P_m$ of S, $P_0 \vee P_1 \vee \ldots \vee P_m = S$. At the end, we also show that if $P_n(S)$ is relatively pseudocomplemented, then $P_n(S)$ is relatively in B_m if and only if for any m+1 pairwise incomparable prime n-ideals $P_0, \ldots, P_m$, $P_0 \vee P_1 \vee \ldots \vee P_m = S$.

Chapter 1

CHAPTER 1

NEARLATTICES AND n-IDEALS OF NEARLATTICES

1.1. Preliminaries

In this section it is intended only to outline and fix the notation for some of the concepts of nearlattices which are basic to this thesis. We also described some results on arbitrary nearlattices which we have developed independently. For the background material in lattice theory we refer the reader to the texts of G. Birkhoff [5], G. Gratzer [16], [18], D.E. Rutherford [58], V.K. Khanna [31] and Szasz [63].

By a *nearlattice* S we will always mean a (lower) semilattice which has the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman, in their paper [11], referred this property as the upper bound property, and a semilattice of this nature as a semilattice with the upper bound property. These types of semilattices have been studied extensively by [11], [12], [27], [39], [41], [46] and [51]. They have noticed that the behaviour of such a semilattice is closer to that of a lattice than an ordinary semilattice. So they preferred to use the term "nearlattice" in place of semilattice with the upper bound property.

Of course, a nearlattice with a largest element is a lattice. Since any semilattice satisfying the descending chain condition has the upper bound property, all finite semilattices are nearlattices.

Now we give an example of a meet semilattice which is not a nearlattice.

Example: In $\mathbf{R}^2$ consider the set

$$S = \{(0,0)\} \cup \{(2,0)\} \cup \{(0,2)\} \cup \{(2,y) : y > 2\}$$

Shown by the following figure 1.1.

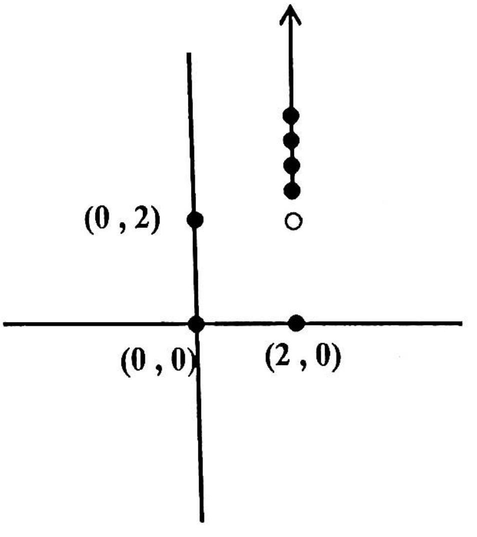


Figure 1.1

Now, define the partial ordering ≤ on S by $(x, y) \le (x_1, y_1)$ if and only if $x \le x_1$ and $y \le y_1$. Observe that (S, ≤) is a meet semilattice . Both (2, 0) and (0, 2) have common upper bounds. In fact $\{(2, y) : y > 2\}$ are common upper bounds of them. But the supremum of (2, 0) and (0, 2) does not exist. Therefore, (S,≤) is not a nearlattice.

The upper bound property first appeared in Gratzer and Lakser [19], while Rozen [57] has shown that it is the result of placing certain associativity conditions on the partial join operation.

Evans in his paper [15] referred nearlattices as *conditional lattices.* By a conditional lattice he means a (meet) semilatice S with the condition that for each $x \in S$, $\{y \in S: y \le x\}$ is a lattice and it can be easily seen that this condition is equivalent to the upper bound property of S. Also Nieminen refers to nearlattices as *"partial lattices"* in his paper [39].

The least element of a nearlattice is denoted by 0. If $x_1, x_2, x_3, \ldots, x_n$ are the elements of a nearlattice then by $x_1 \vee x_2 \vee \ldots\ldots \vee x_n$, we mean that the supremum of $x_1, x_2, x_3, \ldots, x_n$ exists and $x_1 \vee x_2 \vee \ldots\ldots \vee x_n$ is the symbol denoting this supremum.

A non-mpty subset K of a nearlattice S is called a *subnearlattice* of S if for any a, b $\in$ K, both $a \wedge b$ and $a \vee b$ (whenever it exists in S) belong to K ($\wedge$ and $\vee$ are taken in S) and the $\wedge$ and $\vee$ of K are the restrictions of the $\wedge$ and $\vee$ of S to K. Moreover, a subnearlattice K of a nearlattice S is called a sublattice of S if $a \vee b \in K \quad \forall a, b \in K$.

A nearlattice S is called *modular* if for any $a, b, c \in S$ with $c \leq a$, $a \wedge (b \vee c) = (a \wedge b) \vee c$ whenever $b \vee c$ exists.

By [47] , a nearlattice S is *modular* if and only if for all $t, x, y \in S$ with $z \leq x$, $x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

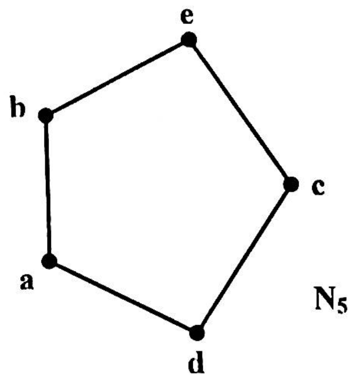
A nearlattice S is called *distributive* if for any $x_1, x_2, x_3, \ldots\ldots, x_n$ $x \wedge (x_1 \vee x_2 \vee \ldots\ldots \vee x_n) = (x \wedge x_1) \vee (x \wedge x_2) \vee \ldots\ldots \vee (x \wedge x_n)$ whenever $x_1 \vee x_2 \vee \ldots\ldots \vee x_n$ exists.

Notice that the right hand expression always exists by the upper bound property of S.

By [47], a nearlattice S is *distributive* if and only if for all $t, x, y, z \in S$, $t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

Lemma 1.1.1. *A nearlattice S is distributive (modular) if and only if* $(x] = \{ y \in S : y \leq x \}$ *is a distributive (modular) lattice for each* $x \in S$. ●

Consider the following lattices:



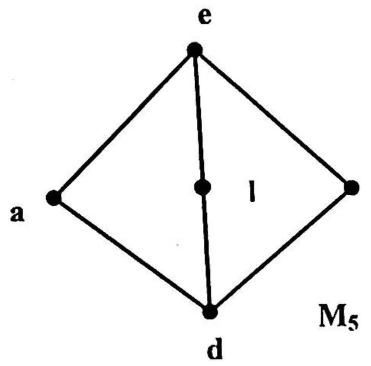


Figure 1.2 **Figure 1.3**

Hickman in [11] has given the following extension of a very fundamental result of lattice theory.

Theorem 1.1.2. *A nearlattice S is distributive if and only if S does not contain a sublattice isomorphic to N_5 or M_5.* ●

Following result is also an extension of a fundamental result of lattice theory which is due to [11].

Theorem 1.1.3. *A nearlattice S is modular if and only if S does not contain a sublattice isomorphic to N_5.* ●

In this context it should be mentioned that many lattice theorists e.g. R. Balbes [2a], J.C. Varlet [65], R.C. Hickman [24], [25] and K.P.Shum [61] have worked with a class of semilattices S which has the property that for each $x, a_1, a_2, a_3, \ldots\ldots, a_r \in S$ if $a_1 \vee a_2 \vee a_3 \vee \ldots\ldots \vee a_r$ exsits then $(x \wedge a_1) \vee (x \wedge a_2) \vee (x \wedge a_3)$

$\vee \ldots\ldots\ldots \vee (x \wedge a_r)$ exists and equals $x \wedge (a_1 \vee a_2 \vee a_3 \vee \ldots\ldots\ldots \vee a_r)$. [2a] called them as *Prime semi-lattices* while [24] referred them as *weakly distributive semilattices.*

Hickman in [25] has defined a ternary operation j by $j(x, y, z) = (x \wedge y) \vee (y \wedge z)$ on a nearlattice S (which exists by the upper bound property of S). In fact, he has shown that (also see Lyndon [37, Theorem 4], the resulting algebras of the type (S; j) form a variety, which is referred to as the variety of join algebras and following are its defining identities.

(i) $j(x, x, x) = x,$

(ii) $j(x, y, x) = j(y, x, y)$

(iii) $j(j(x, y, x), z, j(x, y, x)) = j(x, j(y, z, y), x)$

(iv) $j(x, y, z) = j(z, y, x).$

(v) $j(j(x, y, z), j(x, y, x), j(x, y, z)) = j(x, y, x).$

(vi) $j(j(x, y, x), y, z)) = j(x, y, z)$

(vii) $j(x, y, j(x, z, x) = j(x, y, x)$

(viii) $j(j(x, y, j(w, y, z)), j(x, y, z), j(x, y, j(x, y, z))) = j(x, y, z).$

We call a nearlattice S a *medial nearlattice* if for all $x, y, z \in S$,
$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists .
For a (meet) semilattice S, if $m(x, y, z)$ exists for all $x, y, z \in S$, then it is not hard to see that S has the upper bound property and hence is a nearlattice. Distributive medial nearlattices were first studied by Sholander in [59] and [60] and recently by Evans in [15]. Sholander preferred to call these as median semilattices. There he showed that every medial nearlattice S can be characterized by means of an algebra (S; m) of type < 3 >, known as median algebra, satisfying the following two identities:

(i) m (a, a, b) = a

(ii) m(m(a, b, c), m(a, b, d), e) = m(m(c, d, e), a, b)

A nearlattice S is said to have the *three property* if for any a, b, c $\in$ S, $a \vee b \vee c$ exists whenever $a \vee b$, $b \vee c$ and $c \vee a$ exist.

Nearlattices with the three property were discussed by Evans in [15], where he referred it as *strong conditional lattice.*

Following result shows that the Evan's conditional lattices are precisely the medial nearlattices. The equivalence of (i) and (iii) of the following lemma is trivial, while the proof of (i) $\Leftrightarrow$ (ii) is inductive.

Lemma 1.1.4. (Evans [15]). *For a nearlattice S the following conditions are equivalent.*

(i) S *has the three property.*

(ii) *Every pair of a finite number* n ($\geq$ 3) *of elements of* S *possess a supremum ensures the existence of the supremum of all the n elements* .

(iii) S *is medial.* ●

A family $\mathcal{A}$ of subsets of a set A is called a *closure system* on A if

(i) A $\in \mathcal{A}$ and

(ii) $\mathcal{A}$ is closed under arbitrary intersection.

Suppose $\mathcal{B}$ is a subfamily of $\mathcal{A}$. $\mathcal{B}$ is called a *directed system* if for any X, Y $\in \mathcal{B}$ there exists Z in $\mathcal{B}$ such that X, Y $\subseteq$ Z

If $\bigcup\{x: x\in \mathcal{B}\} \in \mathcal{A}$ for every directed system $\mathcal{B}$ contained in the closure system $\mathcal{A}$, then $\mathcal{A}$ is called *algebraic.* When ordered by set inclusion an algebraic system forms an algebraic lattice.

A nonempty subset H of a nearlattice S is called *hereditary* if for any $x \in S$ and $y \in H$, $x \leq y$ implies $x \in H$. The set H(S) of all hereditary subsets of S is a complete distributive lattice when partially ordered by set-inclusion, where the meet and join in H(S) are given by set theoretic intersection and union respectively.

1.2. Ideals of Nearlattices

A nonempty subset I of a nearlattice S is called an *ideal* if it is hereditary and closed under existent finite suprema. In other words, any subnearlattice with hereditary property is an *ideal.*

We denote the set of all ideals of S by I(S). If S has a smallest element 0, then I(S) is an algebraic closure system on S and is consequently an algebraic lattice.

However, if S does not possess smallest element then we can only assert that $I(S) \cup \{\Phi\}$ is an algebraic closure system. For any subset K of a near lattice S, (K] denotes the ideal generated by K.

Infinimum of two ideals of a nearlattice is their set theoretic intersection. In a general nearlattice the formula for the supremum of two ideals is not very easy. We start this section with the following lemma which gives the formula for the supremum of two ideals. It is in fact exercise 22 of Gratzer [16, p-54] for the partial lattices.

Lemma 1.2.1. *Let* I *and* J *be ideal of a nearlattice* S. *Let* $A_0 = I \cup J$,

$A_n = \{x \in S: x \leq y \vee z,\ y \vee z$ *exists and* $y, z \in A_{n-1}\}$, *for* $n = 1, 2, 3, \ldots$ *and*

$K = \bigcup_{n=0}^{\infty} A_n$. *Then* $K = I \vee J$.

Proof : Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots\ldots\ldots \subseteq A_n \subseteq \ldots\ldots$, K is an ideal containing I and J. Suppose H is any ideal containing I and J. Of course $A_0 \subseteq H$. We proceed by induction. Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$ and let $x \in A_n$. Then $x \leq y \vee z$ with y, $z \in A_{n-1}$. Since $A_{n-1} \subseteq H$ and H is an ideal, so $y \vee z \in H$ and $x \in H$. That is $A_n \subseteq H$ for every n. Thus $K = I \vee J$. ●

Following result can also be proved in a similar way and it is in fact due to [28]. This will be needed for further development of the thesis.

Lemma 1.2.2. *Let* K *be a non-empty subset of a nearlattice* S.

Then $(K] = \bigcup_{m=0}^{\infty} A_m$, *where* $A_0 = \{t \in S : t = (k_1 \wedge t) \vee (k_2 \wedge t)$ *for some* $k_1, k_2 \in K\}$ *and*

$A_m = \{ t \in S : t = (a_1 \wedge t) \vee (a_2 \wedge t)$ *for some* $a_1, a_2 \in A_{m-1} \}$. ●

Cornish and Hickman in [11, Theorem 1.1] have established the following result.

Theorem 1.2.3. *The following conditions on a nearlattice* S *are equivalent.*

(i) S *is distributive.*

(ii) *For any* $H \in H(S)$,

$$(H] = \{h_1 \vee h_2 \vee \ldots\ldots\ldots \vee h_n : h_1, h_2, \ldots. h_n \in H\}.$$

(iii) *For any* $I, J \in I(S)$,

$$I \vee J = a_1 \vee a_2 \vee \ldots\ldots\ldots\ldots \vee a_n : \ a_1, a_2, \ldots\ldots\ldots\ldots a_n \in I \cup J\}.$$

(iv) I(S) *is a distributive lattice.*

(v) *The map* H ⟶ (H] *is a lattice homomorphism of* H(S) *onto* I(S) (*which preserves arbitrary suprema*). ●

From above Theorem it is easy to see that S is distributive if and only if for any $I, J \in I(S)$,

$$I \vee J = \{ i \vee j : i \in I , j \in J \}$$

Let $I_f(S)$ denote the set of all finitely generated ideals of a nearlattice S. $I_f(S)$ is obviously an upper subsemilattice of I(S). Also for any $x_1, x_2, \ldots\ldots, x_m \in S$,

$(x_1, x_2, \ldots\ldots, x_m]$ is clearly the supremum of $(x_1] \vee (x_2] \vee \ldots\ldots \vee (x_m]$.

When S is distributive,

$(x_1, x_2, \ldots\ldots, x_m] \cap (y_1, y_2, \ldots\ldots, y_n]$

$= ((x_1] \vee (X_2] \vee \ldots\ldots \vee (x_m]) \cap ((y_1] \vee (y_2] \vee \ldots\ldots \vee (y_n])$

$= \bigvee_{i,j} (x_i \wedge y_j]$ for any $x_1, x_2, \ldots\ldots x_m, y_1, y_2, \ldots\ldots, y_n \in S$ (by Th. 1.2.3) and so $I_f(S)$ is a distributive sublattice of I(S) , c.f. Cornish and Hickman [11].

A nearlattice S is said to be *finitely smooth* if the intersection of two finitely generated ideals is itself finitely generated. For example,

(i) distributive nearlattices.

(ii) finite nearlattices,

(iii) lattices are finitely smooth.

In [25], Hickman exhibited a nearlattice which is not finitely smooth.

By theorem 1.2.3, a nearlattice S is distributive if and only if I(S) is distributive. But for modular nearlattices, the case is different.

By [51], we know that S is modular if I(S) is so. But the following example shows that its converse may not be true.

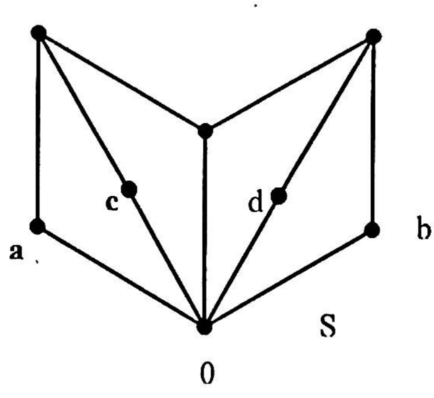


Figure 1.4

Notice that in S, (r] is modular for each $r \in S$. But in I(S), { (0], (a], (a, d], (b, c], S} is a pentagonal sublattice.

A *filter* F of a nearlattice S is a non-empty subset of S such that if $f_1, f_2 \in F$ and $x \in S$ with $f_1 \leq x$, then both $f_1 \wedge f_2$ and x are in F.

A filter G is called a *prime filter* if $G \neq S$ and at least one of $x_1, x_2, \ldots\ldots, x_n$ is in G whenever $x_1 \vee x_2 \vee \ldots\ldots\ldots \vee x_n$ exists in G.

An ideal P of a nearlattice S is called a *prime ideal* if $P \neq S$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$. It is not hard to see that a filter F of a nearlattice S is prime if and only if S - F is a prime ideal.

The set of filters of a nearlattice is an upper semilattice; yet it is not a lattice in general, as there is no guarantee that the intersection of two filters is non-empty. The join of two filters is given by $F_1 \vee F_2 = \{ s \in S: s \geq f_1 \wedge f_2$ for some $f_1 \in F_1, f_2 \in F_2\}$. The smallest filter containing a subset H is denoted by [H). Moreover, the description of the join of filters shows that for all $a, b \in S$,

$$[a) \vee [b) = [a \wedge b).$$

A subnearlattice K of a nearlattice S is called a *convex sub nearlattice* if $a \leq c \leq b$ with $a, b \in K$, $c \in S$ implies $c \in K$.

Now we study some properties of convex subnearlattices of a nearlattice.

Theorm 1.2.4. *In a nearlattice* S, *suppose* K *is a convex subnearlattice. Then*

$[K) = \{x \in S : x \geq k$ *for some* $k \in K\}$. ●

We omit details as it is very trivial.

In a lattice L, it is well known that for a convex sublattice C of L,

$$C = (C] \cap [C).$$

Following figure 1.5 shows that for a convex subnearlattice C in a general nearlattice , this may not be true.

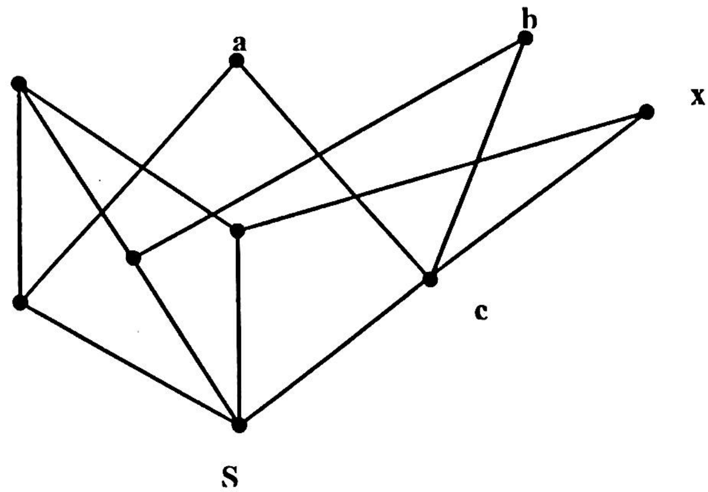


Figure 1.5

Here C = {a, b, c} is a convex subnearlattice of S. Observe that (C] = S and [C) = {a, b, c, x}, hence $(C] \cap [C) = [C) \neq C$

But this result holds when the nearlattice S is distributive. To prove this we need the following lemma.

Lemma 1.2.5. *Suppose* C *is a convex subnearlattice of a distributive near- lattice* S. *Then* $(C] = \{x \in S : x = (x\wedge c_1) \vee \ldots \vee (x\wedge c_n)$ *for some* $c_1,\ldots,c_n \in C\}$.

Proof : Let x , y ∈ R.H.S such that $x\vee y$ exists.

Then $x = (x\wedge p_1) \vee (x\wedge p_2) \vee \ldots\ldots\ldots \vee (x\wedge p_m)$ and $y = (y\wedge q_1) \vee (y\wedge q_2) \vee \ldots\ldots\ldots \vee (y\wedge q_n)$ for some $p_1, p_2, \ldots\ldots\ldots, p_m, q_1,\ldots,q_n \in C$.

Thus $x\vee y = (x\wedge p_1) \vee \ldots\ldots\ldots \vee (x\wedge p_m) \vee (y\wedge q_1) \vee \ldots\ldots\ldots \vee (y\wedge q_n)$

$\leq ((x\vee y) \wedge p_1)\vee\ldots\vee((x\vee y)\wedge p_m)\vee((x\vee y) \wedge q_1) \vee\ldots\vee ((x\vee y) \wedge q_n)$

$\leq x \vee y$ implies

$$x \vee y = ((x \vee y) \wedge p_1) \vee \dots \vee ((x \vee y) \wedge p_m) \vee ((x \vee y) \wedge q_1) \vee \dots \vee ((x \vee y) \wedge q_n).$$

Therefore, $x \vee y \in$ R.H.S.

If $x \in$ R.H.S. and $t \in S$ with $t \leq x$, then

$x = (x \wedge p_1) \vee \dots \vee (x \wedge p_m)$ for some $p_1, \dots, p_m \in C$.

Thus $t = t \wedge x = t \wedge [(x \wedge p_1) \vee \dots \vee (x \wedge p_m)]$

$= (t \wedge p_1) \vee \dots \vee (t \wedge p_m)$ as S is distributive which implies $t \in$ R.H.S.

and so, R.H.S. is an ideal. For any $c \in C$, $c = c \wedge c$ implies $c \in$ R.H.S.

Hence, R.H.S. is an ideal containing C.

Finally, suppose that I is any ideal containing C. Then for any $x \in$ R.H.S. implies $x = (x \wedge p_1) \vee \dots \vee (x \wedge p_m)$ for some $p_1, \dots, p_m \in C$. Then $x \wedge p_1, \dots, x \wedge p_m \in I$ and hence $x \in I$. Therefore R.H.S $= (C]$. ●

Theorem 1.2.6. *For a convex subnearlattice* C *of a distributive nearlattice* S, $(C] \cap [C) = C$.

Proof: Obviously, $C \subseteq (C] \cap [C)$.

For the reverse inclusion let $x \in (C] \cap [C)$. Then $x \in [C)$ implies $x \geq c$ for some $c \in C$.

By above lemma, $x \in (C]$ implies $x = (x \wedge c_1) \vee \dots \vee (x \wedge c_n)$ for some $c_1, c_2, \dots, c_n \in C$. Then $c \wedge c_i \leq x \wedge c_i \leq c_i$ and so by convexity of C, $x \wedge c_i \in C$ for each $i = 1, 2, 3, \dots, n$. Hence $x \in C$.

Therefore, $(C] \cap [C) = C$. ●

For a convex sublattice C of a lattice it is well known that $x \in (C]$ implies $x \leq c$ for some $c \in C$. Again $x \in [C)$ implies $x \geq c_1$ for some $c_1 \in C$ and so by convexity of C, $C = (C] \cap [C)$.

But $x \in (C]$ implies $x \leq c$ for some $c \in C$ is not true in a general nearlattice.

Figure 1.6 shows that this is not true even in a distributive nearlattice, although $C = (C] \cap [C)$ holds there by theorem 1.2.6

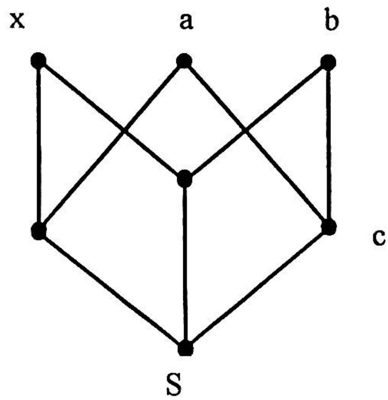


Figure 1.6

Here clearly S is distributive. Let $C = \{a, b, c\}$.

Here $(C] = S$. Thus $x \in (C]$ but $x \nleq c$ for any $c \in C$.

Because of this fact it is very difficult to work with the convex subnearlattices of a nearlattice. But the things become much easier in case of a medial nearlattice.

Recall that a nearlattice S is *medial* if for all $x, y, z \in S$,

$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists in S.

Though the following result is due to [50], we prefer to include its proof for the convenience of the reader.

Theorem 1.2.7. *If* C *is a convex subnearlattice in a medial nearlattice* S, *then* $x \in (C]$ *implies for some* $c \in C$. *Hence* $C = (C] \cap [C)$.

Proof: By lemma 1.2.2, $(C] = \bigcup_{m=0}^{\infty} A_m$ where A_m's are defined as in the lemma .

If $x \in A_0$, then $x = (x \wedge c_1) \vee (x \wedge c_2)$ for some $c_1, c_2 \in C$. Observe that

$c_1 \wedge c_2 \leq (x \wedge c_1) \vee (c_1 \wedge c_2) \leq c_1$ implies $(x \wedge c_1) \vee (c_1 \wedge c_2) \in C$, by convexity.

Similarly, $(x \wedge c_2) \vee (c_1 \wedge c_2) \in C$.

Thus $m(c_1, x, c_2) = (x \wedge c_1) \vee (x \wedge c_2) \vee (c_1 \wedge c_2) \in C$, and so $x \leq c$ where $c = (x \wedge c_1) \vee (x \wedge c_2) \vee (c_1 \wedge c_2)$.

Now we use the method of induction. Suppose $x \in A_{m-1}$ implies $x \leq c$ for $c \in C$. Let $y \in A_m$. Then $y = (y \wedge a_1) \vee (y \wedge a_2)$ for some $a_1, a_2 \in A_{m-1}$.
Now $a_1, a_2 \in A_{m-1}$ implies $a_1 \leq p$ and $a_2 \leq q$ for some $p, q \in C$.
Thus $y \leq (y \wedge p) \vee (y \wedge q) \leq y$ and so $y = (y \wedge p) \vee (y \wedge q)$. This implies $y \in A_0$, and so $y \leq c$ for some $c \in C$.

This completes the proof. ●

From the above proof we have the following corollary.

Corollary 1.2.8. *For a convex subnearlattice* C *of a medial nearlattice* S, $(C] = A_m$ *for each* m = 0, 1, 2,....*where* A_m *are defined as in Lemma* 1.2.2 .
In other words, $(C] = \{t \in S : t = (t \wedge c_1) \vee (t \wedge c_2)$ *for some* $c_1, c_2 \in C\}$. ●

1.3. Congruences

Suppose S is a nearlattice. An equivalence relation Θ on S is called a ***congruence*** relation if $x_i \equiv y_i(\Theta)$ for i = 1, 2 $(x_i, y_i \in S)$ implies

(i) $x_1 \wedge x_2 \equiv y_1 \wedge y_2 (\Theta)$ and

(ii) $x_1 \vee x_2 \equiv y_1 \vee y_2 (\Theta)$ provided $x_1 \vee x_2$ and $y_1 \vee y_2$ exist.

It is easy to show that for an equivalence relation Θ on S, the above conditions are equivalent to the conditions that for $x, y \in S$, if $x \equiv y(\Theta)$, then

(1) $x \wedge t \equiv y \wedge t (\Theta)$ for all $t \in S$ and

(2) $x \vee t \equiv y \vee t (\Theta)$ for all $t \in S$, provided both $x \vee t$ and $y \vee t$ exist.

The set of all congruences C(S) on S is an algebraic closure system on S × S and hence, when ordered by set inclusion, is an algebraic lattice. It is due to Cornish and Hickman [11] that for an ideal I of a distributive nearlattice S, the relation $\Theta(I)$, defined by $x \equiv y\ (\Theta(I))$ if and only if $(x] \vee I = (y] \vee I$, is the smallest congruence having I as a class.

Also, the equivalence relation R(I) defined by $x \equiv y$ R(I)) if and only if for any $s \in S$, $x \wedge s \in I$ if and only if $y \wedge s \in I$, is the largest congruence having I as a class.

Suppose S is a distributive nearlattice and $x \in S$, we will use Θ_x for $\Theta((x])$. Moreover Ψ_x denotes the congruence, defined by $a \equiv b\ (\Psi_x)$ if and only if $a \wedge x = b \wedge x$. In a distributive nearlattice S, for any $a, b \in S$, $\Theta(a, b)$ denotes the smallest congruence identifying a and b. Due to Cornish and Hickman [11] if $a \le b$, then $\Theta(a, b) = \Psi a \cap \Theta_b$. Also, in a distributive nearlattice S, they observed that if S has a smallest element 0, then clearly $\Theta_x = \Theta(0, x)$ for any $x \in S$. Moreover it is easy to see that:

(i) $\Theta_a \vee \Psi a = \iota$, the largest congruence of S.

(ii) $\Theta_a \cap \Psi a = \omega$, the smallest congruence of S.

and $\Theta(a, b)' = \Theta_a \vee \Psi_b$ where a ≤ b and ' ' ' denotes the complement.

Suppose S is an arbitrary nearlattice and E(S) denotes its lattice of equivalence relations. For $\Phi_1, \Phi_2 \in E(S)$, $\Phi_1 \vee \Phi_2$ denotes their supremum, $x \equiv y\ (\Phi_1 \vee \Phi_2)$ if and only if there exists $x = z_0, z_1, \ldots\ldots, z_n = y$ such that $z_{i-1} \equiv z_i\ (\Phi_1$ or $\Phi_2)$ for i = 1, 2, …, n.

Gratzer and Lakser in [19], stated the following result without proof and a proof, different than given below, appears in Cornish and Hickman [11] and [28]; also see Hickman [24].

Theorem1.3.1. *For any nearlattice* S, C(S) *is a distributive (complete) sub lattice of* E(S).

Proof: Suppose $\Theta, \Phi \in C(S)$. Define Ψ to be the supremum of Θ and Φ in the lattice of equivalence relations E(S) on S.

Let $x \equiv y(\Psi)$. Then there exists $x = z_0, z_1, \ldots, z_n = y$ such that $z_{i-1} \equiv z_i\ (\Theta \text{ or } \Phi)$. Thus for any $t \in S$, $z_{i-1} \wedge t \equiv z_i \wedge t\ (\Theta \text{ or } \Phi)$ as $\Theta, \Phi \in C(S)$. Hence $x \wedge t \equiv y \wedge t\ (\Psi)$ and consequently Ψ is a semilattice congruence.

Then, in particular $x \wedge y \equiv x\ (\Psi)$ and $x \wedge y \equiv y\ (\Psi)$, with $x \leq y$, and choose any $t \in S$, such that both $x \vee t$ and $y \vee t$ exist. Then there exits $z_0, z_1, z_2, \ldots, z_n$ such that $x = z_0$, $z_n = y$ and $z_{i-1} \equiv z_i\ (\Theta \text{ or } \Phi)$.

Put $\omega_i = z_i \wedge y$ for all i = 0 , 1,..,n. Then $x = \omega_0$, $y = \omega_n$, $\omega_{i-1} \equiv \omega_i\ (\Theta \text{ or } \Phi)$. Hence by the upper bound property , $\omega_i \vee t$ exists for all i = 0,1,2,…….,n (as ω_i, $t \leq y \vee t$) and $\omega_{i-1} \vee t \equiv \omega_i \vee t\ (\Theta \text{ or } \Phi)$ for all i = 1,2,……,n (as $\Theta, \Phi \in C(S)$) , i.e. $x \vee t \equiv y \vee t\ (\Psi)$. Then by [12, Lemma 1.2.3], Ψ is a congruence on S. Therefore, C(S) is a sublattice of the lattice E(S).

To show the distributivity of C(S), let $x \equiv y\ (\Theta \cap (\Theta_1 \vee \Theta_2))$.

Then $x \wedge y \equiv y\ (\Theta)$ and $(\Theta_1 \vee \Theta_2)$. Also, $x \wedge y \equiv x(\Theta)$ and $(\Theta_1 \vee \Theta_2)$.

Since $x \wedge y \equiv y(\Theta_1 \vee \Theta_2)$, there exists $t_0, t_2, \ldots, t_n$ such that (as we have seen in the proof of the first part), $x \wedge y = t_0$, $t_n = y$, $t_{i-1} \equiv t_i$ $(\Theta_1 \text{or } \Theta_2)$ and $x \wedge y = t_0 \leq t_i \leq y$ for each $i = 0, 1, 2, \ldots, n$.

Hence $t_{i-1} \equiv t_i$ (Θ) for all $i = 1, 2, \ldots, n$ and so $t_{i-1} \equiv t_i$ $(\Theta \cap \Theta_1)$ or $(\Theta \cap \Theta_2)$.

Thus $x \wedge y \equiv y((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$. By symmetry, $x \wedge y \equiv x((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$ and the proof completes by transivity of the congruences. ●

1.4. n-ideals of a nearlattice

Noor, Latif and Ayub Ali have studied extensively the n-ideals of a lattice in different contexts. But the idea of n-ideals was first appeared by Cornish and Noor in the paper [13]. The n-ideals have also been used in proving some results in [40] and [45].

A convex sublattice containing a fixed element n of a lattice L, is called an n-ideal. If L has '0' then replacing n by '0' an n-ideal becomes an ideal. Similarly if L has 1, an n-ideal becomes a filter by replacing n by 1. Thus the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. So any result involving n-ideals of a lattice L will give a generalization work of the results on ideals and filters of L.

A convex subnearlattice containing a fixed element n of a nearlattice S, is called an *n-ideal* of S. Recently [27] and [46] have done some work on n-ideals of a nearlattice. In this thesis we are interested to try to extend their work and to establish several results of [1] and [34] in nearlattices.

The set $I_n(L)$ of all n-ideals of a lattice L is an algebraic lattice under set inclusion. Moreover $\{n\}$ and L are respectively the smallest and the largest elements of $I_n(L)$. In $I_n(L)$ the set theoretic intersection is the infimum. Moreover, for two n-ideals I and J of a lattice L,

$I \vee J = \{x \in L: i_1 \wedge j_1 \leq x \leq i_2 \vee j_2 \text{ for some } i_1, i_2 \in I \text{ and } j_1, j_2 \in J\}$.

In a nearlattice S the following result gives a description of supremum of two n-ideals which is due to [27].

Lemma 1.4.1. *Let* I *and* J *be* n*-ideals of a nearlattice* S. *Suppose* $A_0 = I \cup J$.

$A_m = \{x \in S : i \wedge j \leq x \leq i_1 \vee j_1$, *where* $i_1 \vee j_1$ *exists and* $I, i_1, j, j_1 \in A_{m-1}\}$ *for* $m = 1, 2, \ldots$. *Then* $I \vee J = \bigcup_{m=0}^{\infty} A_m$. ●

An n-ideal generated by a finite number of elements $a_1, \ldots, a_m$ is called a *finitely generated* n*-ideal,* denoted by $<a_1, \ldots, a_m>_n$. The set of finitely generated n-ideals is denoted by $F_n(S)$.

Clearly $< a_1, a_2, \ldots, a_m >_n = < a_1 >_n \vee < a_2 >_n \vee \ldots \vee < a_m >_n$.

An n-ideal generated by single element a is called a *principal* n*-ideal,* denoted by

$< a >_n$. The set of principal n-ideals is denoted by $P_n(S)$.

Standard and neutral elements (ideals) in lattices have been studied by several authors including [17], [18] and [23] .Then [12] has studied them very extensively in nearlattices. By [12] an element s of a nearlattice S is called *standard* if for all

$t, x, y \in S,\quad t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$

The element s is called *neutral* if

(i) s is standard and

(ii) for all $x, y, z \in S$,

$$s \wedge [(x \wedge y) \vee (x \wedge z)] = (s \wedge x \wedge y) \vee (s \wedge x \wedge z).$$

In a distributive nearlattice every element is neutral and hence standard. For detailed literature on standard and neutral elements in a nearlattice, we again refer the reader to see [12].

Following result is due to [13] which gives a description of finitely generated n-ideals of a nearlattice .

Proposition 1.4.2. *Let* S *be a nearlattice and* $x \in S$. *For* $a_1, a_2, \ldots, a_m \in S$,

(i) $< a_1, a_2, \ldots, a_m >_n \subseteq \{y \in S : (a_1] \cap (a_2] \cap \ldots \cap (a_m] \cap (n]$

$\subseteq (y] \subseteq (a_1] \vee \ldots \vee (a_m] \vee (n]\}$.

(ii) $< a_1, a_2, \ldots, a_m >_n = \{y \in S : a_1 \wedge \ldots \wedge a_m \wedge n \leq y$

$= (y \wedge a_1) \vee (y \wedge a_2) \vee \ldots \vee (y \wedge a_m) \vee (y \wedge n)\}$,

provided S *is distributive.*

(iii) *For any* $a \in S$, $< a >_n = \{y \in S: a \wedge n \leq y = (y \wedge a) \vee (y \wedge n)\}$

$= \{y \in S: y = (y \wedge a) \vee (y \wedge n) \vee (a \wedge n)\}$,

whenever n *is standard in* S.

(iv) *When S is a lattic, each finitely generated* n*-ideal is two generated. Indeed,*

$< a_1, \ldots, a_m >_n = < a_1 \wedge \ldots \wedge a_m \wedge n, \ a_1 \vee \ldots \vee a_m \vee n >_n$

(v) *When* S *is a lattice ,* $F_n(S)$ *is a lattice and its members are simply the intervals* $[a, b]$ *such that* $a \leq n \leq b$ *and*

$[a, b] \vee [a_1, b_1] = [a \wedge a_1, b \vee b_1]$ *and*

$[a, b] \cap [a_1, b_1] = [a \vee a_1, b \wedge b_1]$. ●

An element n of a nearlattice S is called a *medial element* if

$m(x, n, y) = (x\wedge n) \vee (x \wedge y) \vee (y \wedge n)$ exists for all $x, y \in S$. Of course, in a medial nearlattice every element is medial.

An element $n\in S$ is called *sesqui-medial* if for all $x, y, z\in S$, $([x\wedge n) \vee (y\wedge n)] \wedge [(y \wedge n) \vee (z \wedge n)]) \vee (x\wedge y) \vee (y \wedge z)$ exists in S. It is easy to see that every sesqui-medial element is medial and in a medial nearlattice every element is sesqui-medial. For detailed literature on these elements, see [13] and [40].

An element $n\in S$ is called an *upper element* if $x \vee n$ exists for all $x\in S$. Every upper element is of course a sesqui-medial element. An element $n\in S$ is called a *central element* if it is upper and complemented in each interval containing it. A nice description of a central element has been given by Cornish and Noor in [12].

Now we prove the following result.

Theorem 1.4.3. *If* n *is a medial element of a nearlattice* S, *then for* n-*ideals* I *and* J *of* S,

$$I \cap J = \{m(i, n, j) : i \in I,\ j\in J\}.$$

Proof: If $x \in I\cap J$, then $x = m(x, n, x)$ implies $I \cap J \subseteq$ R.H.S. Conversely, for any $i\in I$ and $j\in J$, $i \wedge n \leq (i \wedge j) \vee (i\wedge n) \leq i$ implies $(i\wedge j) \vee (i \wedge n) \in I$ by convexity.

Also $i\wedge n \leq (i\wedge n) \vee (j \wedge n) \leq n$ implies $(i \wedge n) \vee (j\wedge n) \in I$.

Therefore, $m(i, n, j) = ((i\wedge j) \vee (i\wedge n)) \vee ((i\wedge n) \vee (j \wedge n)) \in I$ as I is an n-ideal. Similarly $m(i, n, j) \in J$ and so $m(i, n, j)\in I \cap J$.

This completes the proof. ●

Following results are due to [40].

Theorem 1.4.4. *If* n *is standard and medial element of a nearlattice* S, *then* P_n (S) *is a meet semilattice. In fact, for all* a, b $\in$ S,

$< a >_n \cap < b >_n = < m(a, n, b) >_n.$

Moreover, when n *is neutral and sesquimedial, Then* P_n(S) *is also a near- lattice. Thus when n is upper and neutral, then* P_n(S) *is a nearlattice.* ●

Corollary1.4.5.*If n is neutral and sesquimedial in a nearlattice* S, *then any finitely generated* n-*ideal which is contained in a principal* n-*ideal is principal.* ●

It should be noted that the set of finitely generated n-ideals F_n(S) is merely a join semilattice for a general nearlattice. As the intersection of two finitely generated n-ideals of a nearlattice is not necessarily finitely generated, F_n(S) is not a lattice for a general nearlattice. But if S is distributive and n is medial , then F_n(S) is a lattice. In fact , we have the following result due to [27].

Theorem1.4.6. *Let* S *be a nearlattice with a neutral and medial element* n. *Then the following conditions are equivalent.*

(i) S *is distributive*

(ii) I_n(S) *is a distributive lattice.*

(iii) F_n(S) *is a distributive lattice.* ●

We conclude this chapter with the following result which will be needed in proving several results of this thesis.

Theorem 1.4.7. *Let* S *be a distributive nearlattice with an upper element* n *and let* I , J *be two* n-*ideals of* S. *Then for any* $x \in I \vee J$, $x \vee n = i_1 \vee j_1$ *and* $x \wedge n = i_2 \wedge j_2$ *for some* $i_1, i_2 \in I$, $j_1, j_2 \in J$ *with* $i_1, j_1 \geq n$ *and* $i_2, j_2 \leq n$.

Proof: Let $x \in I \vee J$. Then $x \in (I] \vee (J]$. Then by theorem1.2.3, $x = i' \vee j'$ for some $i' \in (I]$ and $j' \in (J]$. So by lemma 1.2.5, $i' = (i' \wedge c_1) \vee \ldots\ldots \vee (i' \wedge c_r)$ and $j' = (j' \wedge d_1) \vee \ldots\ldots \vee (j' \wedge d_s)$ for some $c_1, c_2, \ldots\ldots, c_r \in I$ and $d_1, d_2, \ldots\ldots, d_s \in J$. Now $n \leq (i' \wedge c_p) \vee n \leq c_p \vee n$ implies by convexity that $(i' \wedge c_p) \vee n \in I$, $p = 1, 2, \ldots\ldots\ldots, r$. Therefore, $i' \vee n \in I$. Similarly $j' \vee n \in J$. $x \vee n = (i' \vee n) \vee (j' \vee n) = i_1 \vee j_1$ where $i_1 = i' \vee n \in I$, $j_1 = j' \vee n \in J$.

By the dual proof of above similarly, we can show that $x \wedge n = i_2 \wedge j_2$ for some $i_2 \in I$, $j_2 \in J$. ●

Chapter 2

CHAPTER 2

FINITELY GENERATED n-IDEALS WHICH ARE GENERALIZED BOOLEAN

Introduction

In case of lattices prime n-ideals have been studied extensively by [43]. In this chapter we studied the prime n-ideals, principal n-ideals, finitely generated n-ideals, semi-Boolean algebras and congruences corresponding to n-ideals in a distributive nearlattice. We have given several characterizations of prime n-ideals for nearlattices which helped us in proving many results of the thesis. For a medial element n of a nearlattice S, an n-ideal P of S is called *prime* if $P \neq S$ and $m(x, n, y) \in P$ $(x, y \in S)$ implies either $x \in P$ or $y \in P$.

In section 4 of chapter 1, we have discussed the principal n-ideals, $< a >_n$ generated by $a \in S$. The set of principal n-ideals of a nearlattice S is denoted by $P_n(S)$. When n is standard ,

$$< a >_n = \{y \in S:\ a \wedge n \leq y = (y \wedge a) \vee (y \wedge n)\}$$

$$= \{y \in S:\ y = (y \wedge a) \vee (y \wedge n) \vee (a \wedge n) = m(y, n, a)\}$$

Recall that an element $n \in S$ is an *upper element* if $x \vee n$ exists for all $x \in S$. If $n \in S$ is an upper element, then clearly $< a >_n = [a \wedge n,\ a \vee n]$. Also, recall that an element $n \in S$ is *central* if

(i) n is neutral and upper.

and (ii) it has a complement in each interval containing it.

A nice description of semi-Boolean algebras have been given by Cornish and Hickman in [11]. According to [11], a semilattice S is a *semi-Boolean algebra* if and only if the following conditions are satisfied.

(i) S has the upper bound property.

(ii) S is distributive.

(iii) S has a 0 and for any $x \in S$, $(x]^* = \{y \in S: y \wedge x = 0\}$ is an ideal and

$(x] \vee (x]^* = S$.

A nearlattice S is called *relatively complemented* if each interval [x, y] in S is complemented. That is for $x \le t \le y$ there exists s in [x, y] such that $t \wedge s = x$ and $t \vee s = y$.

A nearlattice S is called *sectionally complemented* if [0, x] is complemented for each $x \in S$. Of course, every relatively complemented nearlattices S with 0 is sectionally complemented. Thus a nearlattice S with 0 is semi-Boolean if and only if it is sectionally complemented and distributive.

In section 1 of this chapter, we have discussed the prime n-ideals of a nearlattice. We have established several properties of prime n-ideals. Finally we have generalized the separation property for n-ideals in a distributive nearlattice.

In section 2, we have shown that for a distributive nearlatice S, $F_n(S)$ is generalized Boolean if and only if $P_n(S)$ is semi-Boolean when n is an upper element. Moreover, $F_n(S)$ is generalized Boolean if and only if [a, n] and [n , b] are complemented for all $a < n < b$. We have also shown that $F_n(S)$ is generalized Boolean if and only if the set of all prime n-ideals of S are unordered by set inclusion.

In section 3, we studied the smallest congruences generated by n-ideals of a distributive nearlattice. Here we have shown that for an upper element n of a distributive nearlattice S, each $\theta \in C(F_n(S))$, define a relation $\rho(\theta)$ on a distributive nearlattice S given by $x \equiv y\ \rho(\theta)$ if and only if $< x >_n \equiv < y >_n(\theta)$ and so $\rho(\theta)$ is a congruence relation on S. Moreover, we have also shown that $\theta_i \in C(F_n(S))$, $i \in \mathcal{A}$, where $\mathcal{A}$ is an index set,

(i) $\rho(\cap \theta_i) = \cap \rho(\theta_i)$

(ii) $\rho(\vee \theta_i) = \vee \rho(\theta_i)$.

At the end, we have shown the permutability of congruences $\theta(I)$ and $\theta(J)$, for n-ideals I and J of a distributive medial nearlattice S. We have proved that above congruences permute for all I and J if and only if S is a lattice and n is complemented in each interval containing it.

2.1. Prime n-ideals of a nearlattice

Recall that an element n of a nearlattice S is *medial* if m(x , n , y) exists for all x , y $\in$ S. Also recall that for a medial element n , an n-ideal P of a near- lattice S is *prime* if $P \neq S$ and $m(x , n , y) \in P$ $(x , y \in S)$ implies either $x \in P$ or $y \in P$. The set of all prime n-ideals of S is denoted by P(S). Now we will study these prime n-ideals more elaborately. Following results will give a clear idea about prime n-ideals.

Theorem 2.1.1. *If* n *is medial element and* P *is a prime* n-*ideal of a nearlattice* S, *then* P *contains either* (n] *or* [n), *but not both.*

Proof: Suppose P is prime n-ideal and $P \not\supseteq (n]$. Then there exists r < n such that $r \notin P$. Now let $s \in [n)$. Then $m(r, n, s) = (r \wedge n) \vee (n \wedge s) \vee (r \wedge s) = r \vee n \vee r = n \in P$. That is, $m(r , n , s) \in P$. Since P is prime, this implies $s \in P$ and so $P \supseteq [n)$. Similarly, if $P \supseteq [n)$, then we can show that $P \supseteq (n]$. Finally, if P contains both (n] and [n) , then by convexity of P, P = S which is impossible. ●

The following results are due to [42] which give a clear idea on prime n-ideals.

Theorem 2.1.2. *Let* n *be a neutral and medial element of a nearlattice* S. *Then every prime* n-*ideal* P *of* S *is either an ideal filter. If it is an ideal, then it is also prime ideal. If it is a filter then it is a prime filter.* ●

Lemma 2.1.3. *For a medial element* n, *any prime ideal* P *containing n of a nearlattice* S *is a prime n-ideal.* ●

Lemma 2.1.4. *Let* n *be a neutral and medial element of a nearlattice* S. *Then any prime filter Q containing* n *is a prime* n-*ideal.* ●

Following result has been proved by [42] when n is a medial nearlattice. We improve the result for a general nearlattice with n as a medial element.

Lemma 2.1.5. *In a distributive nearlattice* S *if* I *is an n-ideal and* D *is a convex subnearlattice with* $I \cap D = \Phi$. *Then either* $(I] \cap D = \Phi$ *or* $[I) \cap D = \Phi$.

Proof: Suppose, $(I] \cap D \neq \Phi$ and $[I) \cap D \neq \Phi$. Let $x \in (I] \cap D$. This implies $x \in (I]$ and $x \in D$. Since S is distributive, so by lemma 1.2.5,

$x = (x \wedge i_1) \vee (x \wedge i_2) \vee \ldots\ldots \vee (x \wedge i_r)$ for some $i_1, i_2, \ldots, i_r \in I$.

Again let $y \in [I) \cap D$, this implies $y \in D$ and $y \geq i'$ for some $i' \in I$.

Now, $x \wedge y \leq y \wedge [(x \wedge i_1) \vee (x \wedge i_2) \vee \ldots\ldots \vee (x \wedge i_r)]$

$= (y \wedge x \wedge i) \vee \ldots\ldots\ldots \vee (y \wedge x \wedge i_r)$

$\leq (y \wedge i_1) \vee (y \wedge i_2) \vee \ldots\ldots \vee (y \wedge i_r) \leq y$.

So by convexity, $(y \wedge i_1) \vee (y \wedge i_2) \vee \ldots\ldots \vee (y \wedge i_r) \in D$.

Again, $i' \wedge i_1 \leq y \wedge i_1 \leq i_1$ implies by convexity that $y \wedge i_1 \in I$.

Similarly $y \wedge i_2 \in I, \ldots, y \wedge i_r \in I$. Hence, $(y \wedge i_1) \vee (y \wedge i_2) \vee \ldots\ldots \vee (y \wedge i_r) \in I$.

Thus, $I \cap D \neq \Phi$ which is a contradiction.

Therefore, if $I \cap D = \Phi$, then either $(I] \cap D = \Phi$ or $[I) \cap D = \Phi$. ●

Separation property is a well known result in lattice theory. Following result is an extension of that result for nearlattices which is due to [51].

Theorem 2.1.6. *Let* I *be an ideal and* D *be a convex subnearlattice of a distributive nearlattice* S *with* $I \cap D = \Phi$. *Then there exists a prime ideal* $P \supseteq I$ *such that* $P \cap D = \Phi$. ●

Now we generalize this result for n-ideal in a distributive nearlattice. This also generalizes [34, Theorem 1.2.4.] .

Theorem 2.1.7. *Let* S *be a distributive nearlattice and* n *be a medial element of* S. *Let* I *be an* n-*ideal and* D *be a convex subnearlattice with* $I \cap D = \Phi$. *Then there exists a prime* n-*ideal* P *of* S *such that* $P \supseteq I$ *and* $P \cap D = \Phi$.

Proof: Since $I \cap D = \Phi$, so by lemma 2.1.5, either $(I] \cap D = \Phi$ or $[I) \cap D = \Phi$. If $(I] \cap D = \Phi$, then by theorem 2.1.6, there exists a prime ideal $P \supseteq (I]$ such that $P \cap D = \Phi$. Since $n \in P$, so by lemma 2.1.3, P is a prime n-ideal. On the other hand if $[I) \cap D = \Phi$, then by dual result of theorem 2.1.6, there exists a prime filter $Q \supseteq [I)$ such that $Q \cap D = \Phi$. Thus $n \in Q$. Since S is distributive, so n is a neutral element of S and so by lemma 2.1.4, Q is also a prime n-ideal. This completes the proof. ●

Following corollary trivially follows from above result.

Corollary 2.1.8. *Let* n *be medial element and* I *be an* n-*ideal of a distributive nearlattice* S *and let* $a \in S$ *such that* $a \notin I$. *Then there exists a prime* n-*ideal* P *of* S *such that* $P \supseteq I$ *and* $a \notin P$. ●

Following result is also an extension of a result in [51] for n-ideals. This also generalizes [34, Corollary 1.2.6.] and [1 , Corollary 1.2.10. for n-ideals in a lattice].

Lemma 2.1.9. *Let* n *be a medial of a distributive nearlattice* S. *Then every n-ideal I of* S *is the intersection of all prime* n-*ideals containing it.*

Proof: Let $I_1 = \cap \{p : P \supseteq I, \ P$ is a prime n-ideal of S $\}$.

If $I \neq I_1$, then there is an element $a \in I_1 - I$. Then by above corollary, there is a prime n-ideal P with $P \supseteq I$, $a \notin P$. But $a \notin P \supseteq I$, gives a contradiction.

This completes the proof. ●

2.2. Principal and Finitely generated n-ideals

In section 4 of chapter 1, we have defined the principal n-ideal $< a >_n$, generated $a \in S$. The set of principal n-ideals of a nearlattice S is denoted by $P_n(S)$. By proposition 1.4.2, if n is standard element of a nearlattice S, then for any $a \in S$,

$$< a >_n = \{y \in S: a \wedge n \leq y = (y \wedge a) \vee (y \wedge n)\}.$$

By 1.4.4, we know that when n is standard and medial, then $P_n(S)$ is a meet semilattice and $< a >_n \cap < b >_n = < m(a, n, b) >_n$ for all $a, b \in S$.

Also by 1.4.4, when n is neutral and sesquimedial, then $P_n(S)$ is in fact a nearlattice. Thus when n is upper and neutral, $P_n(S)$ is a nearlattice. Moreover, [42] have improved that result by the following theorem.

Theorem 2.2.1. *If* S *is a medial nearlattice and* n *is a neutral element of* S, *then* $P_n(S)$ *is also a medial nearlattice.* ●

In this thesis, the central elements play a very important role. [42 , Th.2.2.2] have proved that for an element $n \in S$, n is central element if and only if n is upper and $P_n(S) \cong (n]^d \times [n)$. But we find certain inaccuracy in their result. Observe that in five-element lattice $\{ 0, a, b, n, 1 : a \wedge b = a \wedge n = b \wedge n = 0 ; a \vee n = a \vee b = b \vee n = 1\}$, $P_n(S)$ is a four-element Boolean lattice and $P_n(S) \cong (n]^d \times [n)$. Thus n is complemented in each interval containing it. But n is not central as it is not neutral. Thus we can restate that result as follows:

Theorem 2.2.2. *For a neutral element n of a nearlattice* S, n *is central if and only if n is upper and* n *is upper and* $P_n(S) \cong (n]^d \times [n)$. ●

Recall that a nearlattice S with 0 is *sectionally complemented*, if the interval [0, x] is complemented for each $x \in S$.

Of course, every relatively complemented nearlattice S with 0 is sectionally complemented.

A nearlattice S with 0 is called *semi-Boolean* if it is distributive and the interval [0, x] is complemented for each $x \in S$.

Following results are easy consequences of the above theorem.

Corollary 2.2.3. *Let* S *be a nearlattice and* $n \in S$ *be a central element. Then* $P_n(S)$ *is sectionally complemented if and only if the intervals* [a , n] *and* [n ,b] *are complemented for each* a , b $\in$ S, ($a \le n \le b$). ●

We know by 1.4.6, that if n is medial in distributive nearlattice S, then $I_n(S)$ is also distributive and hence $P_n(S)$ (if it s a nearlattice) is also distributive.

Corollary 2.2.4. *If* n *is central element of a distributive nearlattice* S, *then* $P_n(S)$ *is semi-Boolean if and only if the intervals* [a, n] *and* [n, b] *are complemented for each* a, b $\in$ S ($a \le n \le b$). ●

Following results are due to [51] . These will be needed for further development of the thesis.

Lemma 2.2.5. *If* S_1 *is a subnearlattice of a distributive nearlattice* S *and* P_1 *is a prime ideal (filter) in* S_1, then *there exists a prime ideal* P *in* S *such that* $P_1 = P \cap S_1$. ●

In lattice theory it is well known that a distributive lattice L with 0 and 1 is Boolean if and only if its set of prime ideals is unordered by set inclusion. Following result due to [51] have generalized this result for distributive nearlattices with 0.

Theorem 2.2.6. *If* S *is a distributive nearlattice with* 0, *then* S *is semi-Boolean if and only if its set of prime ideals (filters) is unordered by set inclusion.* ●

Now we would like to generalize this result for n-ideals. To do this we need the following results.

Theorem 2.2.7. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is generalized Boolean.*

(ii) $P_n(S)$ *is semi-Boolean.*

Proof: (i) $\Rightarrow$(ii) is obvious by corollary 1.4.5. Conversely, let (ii) holds.

Let $\{n\} \subseteq \langle x_1, x_2, \ldots, x_p \rangle_n \subseteq \langle y_1, y_2, \ldots, y_s \rangle_n$.

That is , $\{n\} \subseteq \langle x_1, x_2, \ldots\ldots, x_p \rangle_n \cap \langle y_1 \rangle_n \subseteq \langle y_1 \rangle_n$, which implies

$\{n\} \subseteq (\langle x_1 \rangle_n \vee \langle x_2 \rangle_n \vee \ldots\ldots\ldots \vee \langle x_p \rangle_n) \cap \langle y_1 \rangle_n \subseteq \langle y_1 \rangle_n$ and so

$\{n\} \subseteq [(\langle x_1 \rangle_n \cap \langle y_1 \rangle_n) \vee \ldots\ldots\ldots \vee (\langle x_p \rangle_n \cap \langle y_1 \rangle_n)] \subseteq \langle y_1 \rangle_n$.

Thus, $\{n\} \subseteq \langle m(x_1, n, y_1) \rangle_n \vee \ldots\ldots\ldots\ldots \vee \langle m(x_p, n, y_1) \rangle_n \subseteq \langle y_1 \rangle_n$. By corollary 1.4.5, $\langle m(x_1, n, y_1) \rangle_n \vee \ldots\ldots\ldots \vee \langle m(x_p, n, y_1) \rangle_n$ is a principal n-ideal.

Now let , $\langle t_1 \rangle_n = \langle m(x_1, n, y_1) \rangle_n \vee \ldots\ldots\ldots \vee \langle m(x_p, n, y_1) \rangle_n$ and let $\langle r_1 \rangle_n$ be a complement of $\langle t_1 \rangle_n$ such that $\langle r_1 \rangle_n \vee \langle t_1 \rangle_n = \langle y_1 \rangle_n$ and $\langle r_1 \rangle_n \cap \langle t_1 \rangle_n = \{n\}$.

So, we can get $\langle t_i \rangle_n$; $i = 1,2,\ldots,s$ and the complement , $\langle r_i \rangle_n$; $i = 1,2,\ldots,s$ of

$< t_i >_n$ such that $< r_1, r_2, \ldots, r_s >_n \vee < x_1, x_2, \ldots, x_p >_n = < y_1, y_2, \ldots, y_s >_n$.

and $< r_1, r_2, \ldots, r_s >_n \cap < x_1, x_2, \ldots, x_p >_n$

$= (< r_1 >_n \cap < x_1, x_2, \ldots, x_p >_n) \vee \ldots \vee (< r_s >_n \cap < x_1, x_2, \ldots, x_p >_n)$

$= \{n\} \vee \{n\} \vee \ldots\ldots \vee \{n\}$

$= \{n\}$

Therefore, $F_n(S)$ is generalized Boolean. ●

Lemma 2.2.8. *Let* n *be an upper element of a distributive nearlattice* S. *Then the following conditions are equivalent.*

(i) $P_n(S)$ *is semi Boolean*

(ii) $[a, n]$ *and* $[n, b]$ *are complemented for all* $a < n < b$.

Proof : (i) $\Rightarrow$ (ii). Suppose $P_n(S)$ is semi-Boolean and let $a \le y \le n$.

Therefore, $\{n\} \subseteq < y >_n \subseteq < a >_n$ which implies

$\{n\} \subseteq [y, n] \subseteq [a, n]$.

Let $< t >_n$ be the relative complement of $< y >_n$ in $[\{n\}, < a >_n]$. Then $t \le n$.

Also, $< t >_n \cap < y >_n = \{n\}$ and $< t >_n \vee < y >_n = < a >_n$

Now $< t >_n \cap < y >_n = \{n\}$ implies

$[t, n] \wedge [y, n] = \{n\}$ and so

$[t \vee y, n] = \{n\}$ implies $t \vee y = n$.

Also, $< t >_n \vee < y >_n = < a >_n$ implies

$[t, n] \vee [y, n] = [a, n]$ and so

$[t \wedge y, n] = [a, n]$. Thus $t \wedge y = a$.

Hence, $[a, n]$ is complemented. Similarly we can prove dually that $[n, b]$ is also complemented.

(ii) $\Rightarrow$ (i). Suppose $[a, n]$ and $[n, b]$ are complemented for all $a < n < b$. Consider $\{n\} \subseteq < p >_n \subseteq < q >_n$. Then $q \wedge n \leq p \wedge n \leq n \leq p \vee n \leq q \vee n$. Since $[n, q \vee n]$ is complemented, so there exists $s \in [n, q \vee n]$, such that $(p \vee n) \wedge s = n$ and $p \vee n \vee s = q \vee n$. Again as $[q \wedge n\ n]$ is complemented, so there exists $r \in [q \wedge n, n]$ such that $r \wedge p \wedge n = q \vee n$ and $r \vee (p \wedge n) = n$.

Then $\quad [r, s] \cap < p >_n = \{n\}$.

$[r, s] \vee < p >_n = < q >_n$.

That is $[r, s]$ is the relative complement of $< p >_n$ in $[\{n\}, < q >_n]$. But by corollary 1.4.5, we know that any finitely generated n-ideal contained in a principal n-ideal is principal. Hence $[r, s] \in P_n(S)$ and so $P_n(S)$ is semi- Boolean. ●

Now we extend the result [34, Th. 1.2.9.] for nearlattices.

Theorem 2.2.9. *Let* S *be a distributive nearlattice and* $n \in S$ *be an upper element. Then the following conditions are equivalent.*

(i) $F_n(S)$ *is generalized Boolean.*

(ii) *The set of prime* n-ideals P(S) *of* S *is unordered by set inclusion.*

Proof : (i) $\Leftrightarrow$ (ii). Suppose $F_n(S)$ is generalized Boolean. Then by theorem 2.2.7

and 2.2.8, the interval [x, n] and [n, y] are complemented for each $x, y \in S$ with $x \le n \le y$. If P(S) is not unordered, suppose there are prime n-ideals P, Q with $P \subset Q$. Let $b \in Q - P$. Now as Q is prime, there exists $a \in S$ such that $a \notin Q$. Then either $a \wedge n \notin Q$ or $a \vee n \notin Q$ (here $a \vee n$ exists an n is upper). For otherwise $a \in Q$, by convexity.

Suppose $a \vee n \notin Q$, Since $[n , a \vee n]$ is complemented and $n \le (a \wedge b) \vee n \le a \vee n$, so there exists $t \in [n , a \vee n]$ such that $t \wedge [(a \wedge b) \vee n] = n$ and $t \vee [(a \wedge b) \vee n] = a \vee n$.

Since $t \wedge [(a \wedge b) \vee n] = m(t , n ,(a \wedge b) \vee n) = n$, thus $t \wedge [(a \wedge b) \vee n] = m(t , n , (a \wedge b) \vee n) \in P$. Since P is prime, so either $t \in P$ or $(a \wedge b) \vee n \in P$. Now $n \le (a \wedge b) \vee n \le b \vee n$ implies $(a \wedge b) \vee n \in Q$. If $t \in P$, then $t \in Q$ and so $a \vee n = t \vee [(a \wedge b) \vee n] \in Q$, which gives a contradiction.

If $(a \wedge b) \vee n \in P$, then $(a \wedge b) \vee n = m(a \vee n, n, b) \in P$ implies $b \in P$ which is again a contradiction. Therefore, $a \vee n \in Q$.

Now if $a \wedge n \notin Q$, then $a \wedge b \wedge n \notin Q$ as $n \in Q$ and Q is convex. Since $b \wedge n$ has relative complement in $[a \wedge b \wedge n, n]$. Proceeding as above, again we arrive at a contradiction. Thus $a \wedge n \in Q$. Since both $a \wedge n$ and $a \vee n$ belong Q, so by convexity $a \in Q$. This gives a contradiction. Therefore the set of prime n-ideals P(S) is unordered.

(ii)$\Rightarrow$ (i). Suppose that P(S) is unordered. Consider any interval [n , b] in S. Let P_1, Q_1 be two prime ideals of [n, b] . Then by lemma 2.2.5, there exist prime ideals P and Q of S such that $P_1 = P \cap [n , b]$ and $Q_1 = Q \cap [n , b]$.

Since P and Q contains n, so by lemma 2.1.3, they are prime n-ideals. Since P(S) is unordered, so P and Q are incomparable. This follows that P_1 and Q_1 are also incomparable.

If not, let $P_1 \subset Q_1$. Then for any $z \in P$, $(z \vee n) \wedge b \in [n, b]$ and $n \leq (z \vee n) \wedge b \leq z \vee n$ implies, $(z \vee n) \wedge b \in P_1 \subset Q_1$. Thus $(z \vee n) \wedge b \in Q$. But $b \notin Q$ as Q_1 is prime in [n, b]. Therefore $z \vee n \in Q$ as Q is a prime ideal of S and so $z \in Q$. Hence $P \subset Q$, which is a contradiction. Therefore by [16, Th. 22, p-46], [n, b] is complemented..

Again consider the interval [a, n]. Since the prime filters are the complements of prime ideals, so considering two prime filters of [a, n] and using the same argument as above we see that [a, n] is also complemented. Hence by lemma 2.2.8,, $P_n(S)$ is semi-Boolean and by theorem 2.2.7, $F_n(S)$ is generalized Boolean. ●

2.3. Congruences corresponding to n-ideals in a distributive nearlattice

In this section we discuss on the permutability of the congruences $\Theta(I)$ and $\Theta(J)$ in the distributive nearlattice S, where I and J are n-ideals of S. In a nearlattice S, two congruences Θ and Φ *permute* if for a, b, c $\in$S with $a \equiv b\ (\Theta)$ and $b \equiv c\ (\Phi)$ imply that there exists some $d \in S$ such that $a \equiv d(\Phi)$ and $d \equiv c(\Theta)$. For an n-ideal I in a distributive nearlattice, the congruence $\Theta(I)$ have been studied by [42]. The following result of [42] gives a description of the smallest congruence relation of a distributive nearlattice S containing an n-ideal as a class, where n is a fixed element of S.

Theorem 2.3.1. *For an* n-*ideal* C *of a distributive nearlattice* S, *the relation* $\Theta(C)$ *on* S *defined by* $x \equiv y\ \Theta(C)$ $(x,y \in S)$ *if and only if* $x \wedge c = y \wedge c$ *for some* $c \in C$ *and* $(x] \vee (C] = (y] \vee (C]$, *is the smallest congruence containing* C *as a class.* ●

[42, Th.2.3.4.] has also shown that a nearlattice S with a neutral element n is distributive if and only if each n-ideal C is a class of some congruence of S.

Following result is due to [42]

Lemma 2.3.2. *Let* n *be a medial element of a distributive nearlattice* S. *Then for any two* n-*ideals* I *and* J *of* S,

(i) $\Theta(I \cap J) = \Theta(I) \cap \Theta(J)$.

(ii) $\Theta(I \vee J) = \Theta(I) \vee \Theta(J)$. ●

Theorem 2.3.3. *Let* S *be a distributive nearlattice and* n *be an upper element of* S. *Then the map* $\rho : C(F_n(S)) \longrightarrow C(S)$ *is an isomorphism where for each* $\Theta \in C(F_n(S))$, $\rho(\Theta)$ *is defined by* $x \equiv y\ \rho(\Theta)$ *if and only if* $\langle x \rangle_n \equiv \langle y \rangle_n \Theta$.

Proof: By above lemma it is sufficient to prove that ρ is one-one and onto . Suppose $\rho(\Theta) = \rho(\Phi)$.Then $< a >_n \equiv < c >_n \ \Theta$ if and only if $a \equiv c \ \rho(\Theta) = \rho(\Phi)$ if and only if $< a >_n \equiv < c >_n(\Phi)$. Now let, $< a_1, a_2 \ldots a_r >_n \equiv < c_1, c_2, \ldots c_s >_n (\Theta)$.Then,

$$< a_1,a_2,\ldots, a_r >_n \cap < c_i >_n \equiv < c_1,c_2,\ldots\ldots c_s >_n \cap < c_i >_n \ \Theta ; i = 1,2, \ldots\ldots\ldots,s.$$

Thus, $< m(a_1, n, c_i), \ldots\ldots\ldots\ldots, m(a_r, n, c_i) >_n \equiv < c_i >_n (\Theta)$.

But by corollary 1.4.5, $< m(a_1, n, c_i), \ldots\ldots\ldots\ldots, m(a_r, n, c_i) >_n = < t_i >_n$ for some $t_i \in S$. Thus,$< t_i >_n \equiv < c_i >_n (\Theta)$. Hence $t_i \equiv c_i \ \rho(\Theta) = \rho(\Phi)$.This implies $< t_i >_n \equiv < c_i >_n(\Phi)$.

Thus, $< m(a_1, n, c_i), \ldots\ldots, m(a_r, n, c_i) >_n \equiv < c_i >_n(\Phi)$ for each $i = 1, 2, \ldots, s$.

Therefore, $< a_1,a_2,\ldots\ldots a_r >_n \cap < c_1,c_2,\ldots, c_s >_n \equiv < c_1,c_2,\ldots, c_s >_n (\Phi)$.

Similarly, $< a_1,a_2,\ldots, a_r >_n \cap < c_1,c_2,\ldots, c_s >_n \equiv < a_1,a_2,\ldots, a_r >_n (\Phi)$

and so $< a_1,a_2,\ldots, a_r >_n \equiv < c_1,c_2,\ldots, c_s >_n (\Phi)$. This implies $\Theta \subseteq \Phi$. Similarly, $\Phi \subseteq \Theta$, and so $\Theta = \Phi$. Therefore, ρ is one-one.

For ontoness, let $\Phi \in C(S)$. Define $\Theta \in C(F_n(S))$ by $\Theta = \vee\{\Theta(< a >_n, < b >_n) : a \equiv b \ \Phi\}$. If $x \equiv y(\Phi)$, Then $< x >_n \equiv < y >_n \Theta(< x >_n , < y >_n)$, and so $< x >_n \equiv < y >_n \Theta$ .This implies $x \equiv y \ \rho(\Theta)$ and so $\Phi \subseteq \rho(\Theta)$.

To prove the reverse inclusion , let $x \equiv y \ \rho(\Theta)(< a >_n, < b >_n) : a \equiv b \Phi)$.Then $< x >_n \equiv < y >_n \Theta(< a >_n \cap < b >_n , < a >_n \vee < b >_n)$. This implies $< x >_n \cap < a >_n \cap < b >_n = < y >_n \cap < a >_n \cap < b >_n$ and $< x >_n \vee < a >_n \vee < b >_n = < y >_n \vee < a >_n \vee < b >_n$. Then by some routine calculations we get,

$(x\wedge n) \vee (a\wedge n) \vee (b\wedge n) = (y\wedge n) \vee (a\wedge n) \vee (b\wedge n)$

$(x\vee n) \wedge (a\vee n) \wedge (b\vee n) = (y\vee n) \wedge (a\vee n) \wedge (b\vee n)$

And $x\wedge a\wedge b\wedge n = y\wedge a\wedge b\wedge n$

$(x] \vee (a] \vee (b] \vee (n] = (y] \vee (a] \vee (b] \vee (n]$

Now $x\wedge n = (x\wedge n) \wedge [(x\wedge n) \vee (a\wedge n)\vee(b\wedge n)]$

$= (x\wedge n) \wedge [(y\wedge n) \vee (a\wedge n)\vee(b\wedge n)]$

$\equiv (x\wedge n) \wedge [(y\wedge n) \vee (b\wedge n)]\ \Theta(a, b)$

$= (x\wedge y\wedge n) \vee (x\wedge b\wedge n)$ as S is distributive.

$\equiv (x\wedge y\wedge n) \vee (x\wedge a\wedge b\wedge n)\ \Theta(a, b)$

$= (x\wedge y\wedge n) \vee (y\wedge a\wedge b\wedge n)$

$= (y\wedge n) \wedge [(x\wedge n) \vee (a\wedge b\wedge n)]$

$\equiv (y\wedge n) \wedge [(x\wedge n) \vee (a\wedge n)\vee(b\wedge n)]\ \Theta(a, b)$

$= (y\wedge n) \wedge [(y\wedge n) \vee (a\wedge n)\vee(b\wedge n)] = y\wedge n.$

Hence $x\wedge n \equiv y\wedge n\ \Theta(a, b)$.

Again $x\vee n = (x\vee n) \vee [(x\vee n) \wedge (a\vee n) \wedge (b\vee n)]$

$= (x\vee n) \vee [(y\vee n) \wedge (a\vee n) \wedge (b\vee n)]$

$$\equiv (x \vee n) \vee [(y \vee n) \wedge (b \vee n)]\ \Theta(a, b)$$

Then , $(x \vee n] \equiv (x \vee n] \vee ((y \vee n] \wedge (b \vee n])\ \Theta(<a>_n, <b>_n)$

$$= ((x] \vee (y] \vee (n]) \wedge ((x] \vee (b] \vee (n])$$

$$\equiv ((x] \vee (y] \vee (n]) \wedge ((x] \vee (a] \vee (b] \vee (n])\ \Theta(<a>_n, <b>_n)$$

$$= ((x] \vee (y] \vee (n]) \wedge ((y] \vee (a] \vee (b] \vee (n])$$

$$\equiv ((x] \vee (y] \vee (n]) \wedge (((y] \vee (n]) \vee ((a] \vee (n]) \wedge ((b] \vee (n]))\ \Theta(<a>_n, <b>_n)$$

$$= ((y] \vee (n]) \vee [((x] \vee (n]) \wedge ((a] \vee (n]) \wedge ((b] \vee (n]))$$

$$= (y \vee n] \vee ((x \vee n) \wedge (a \vee n) \wedge (b \vee n)]$$

$$= (y \vee n] \vee ((y \vee n) \wedge (a \vee n) \wedge (b \vee n)]$$

$$= (y \vee n]$$

That is, $(x \vee n] \equiv (y \vee n]\ \Theta(<a>_n, <b>_n)$ and so $x \vee n \equiv y \vee n \Theta(a, b)$. Then by distributivity, $x \equiv y \Theta(a, b)$. Also $\Theta(a, b) \subseteq \Phi$. Thus $x \equiv y(\Phi)$.

Therefore, by above lemma $\rho(\Theta) \subseteq \Phi$. Hence $\rho(\Theta) = \Phi$ and so ρ is onto. ●

By [52], it is well known that for a distributive nearlattice S with 0, S is semi-Boolean if and only if I(S) $\cong$ C(S). [42] have generalized the result for n-ideals given by following result.

Theorem 2.3.4. *Let* n *be a central element of a nearlattice* S , *then* $I_n(S) \cong C(S)$ *if and only if* $P_n(S)$ *is sem-Boolean.* ●

Since the lattice of ideals of a distributive nearlattice S is isomorphic to the lattice of congruences if and only if S is semi-Boolean, so using 2.3.3 and above theorem, we obtain the following corollary.

Corollary 2.3.5. *For a central element* n *of a distributive nearlattice* S, $I_n(S) \cong I(F_n(S))$ *if* $F_n(S)$ *is generalized Boolean.* ●

Now, we describe an isomorphism between $C(F_n(S))$ and $C(S)$ in presence of distributivity. We prove this with the help of the following lemma.

Lemma 2.3.6. *Let* n *be an upper element of a distributive nearlattice* S. *For each* $\Theta \in C(F_n(S))$, *define a relation* $\rho(\Theta)$ *on* S *given by* $x \equiv y\ \rho(\Theta)$ *if and only if* $< x >_n \equiv < y >_n (\Theta)$. *Then* $\rho(\Theta)$ *is a congruence relation on* S.

Moreover, for $\Theta_i \in C(F_n(S))$, $i \in \mathcal{A}$ *where* $\mathcal{A}$ *is an index set,*

(i) $\rho(\cap\Theta_i) = \cap\rho(\Theta_i)$

and (ii) $\rho(\vee\Theta_i) = \vee\rho(\Theta_i)$.

Proof: Clearly $\rho(\Theta)$ is an equivalence relation. To prove the substitution property, suppose $x \equiv y\ \rho(\Theta)$ and $t \in S$. Let $x \vee t$, $y \vee t$ exist. Then $< x >_n \equiv < y >_n\ \Theta$ and so, $< x >_n \vee < t \vee n >_n \equiv < y >_n \vee < t \vee n >_n\ \Theta$. That is,

$[n, x \vee t \vee n] \equiv [n, y \vee t \vee n]\ \Theta$ . ...(i)

Again $< x >_n \cap < t \wedge n >_n \equiv < y >_n \cap < t \wedge n >_n (\Theta)$. That is

$[(x \wedge n) \vee (t \wedge n), n] \equiv [(y \wedge n) \vee (t \wedge n), n]\ (\Theta)$ ....(ii)

Taking supremum of (i) and (ii), we have

$[(x \wedge n) \vee (t \wedge n),\ x \vee t \vee n] \equiv [(y \wedge n) \vee (t \wedge n),\ y \vee t \vee n](\Theta)$

Thus, $[(x \vee t) \wedge n,\ x \vee t \vee n] \equiv [(y \vee t) \wedge n,\ y \vee t \vee n]\ (\Theta)$, as n is neutral.

That is, $< x \vee t >_n \equiv < y \vee t >_n(\Theta)$, and so $x \vee t \equiv y \vee t\ \rho(\Theta)$. Similarly a dual proof of above shows that $x \wedge t \equiv y \wedge t\ \rho(\Theta)$ and so $\rho(\Theta)$ is a congruence of S.

For the second part, the proof of (i) is trivial. For the proof of (ii), Since ρ is order preserving, obviously $\vee \rho(\Theta_i) \subseteq \rho(\vee \Theta_i)$.

To prove the reverse inequality, assume that $x \equiv y\ \rho(\vee \Theta_i)$. Then

$< x >_n \equiv < y >_n (\vee \Theta_i)$. Thus $< x >_n \cap < y >_n = < m(x, n, y) >_n \equiv < x >_n (\vee \Theta_i)$ so by corollary 1.4.5, there exists a sequence of principal n-ideals.

$< m(x, n, y) >_n = < z_0 >_n ,\ < z_1 >_n, \ldots, < z_r >_n = < y >_n,$

with $< z_{j-1} >_n \equiv < z_j >_n (\Theta_{i_k})$; $i_k \in \mathcal{A}$ $k, j = 1, 2, \ldots, r$.

This implies $z_{j-1} \equiv z_j\ \rho(\Theta_{i_k})$, which shows that $m(x, n, y) \equiv x\ (\vee \rho(\Theta_i))$.

Similarly, $m(x, n, y) \equiv y(\vee \rho(\Theta_i))$. Hence $x \equiv y\ (\vee \rho(\Theta_i))$.

So we have $\rho(\vee \Theta_i) \subseteq \vee \rho(\Theta_i)$.

Hence $\rho(\vee \Theta_i) = \vee \rho(\Theta_i)$. ●

We now discuss on the permutability of the congruences $\Theta(I)$ and $\Theta(J)$ in a distributive nearlattice S, where I and J are n-ideals of S.

It is well known in lattice theory that for any two ideals I and J of a distributive nearlattice S, $\Theta(I)$ and $\Theta(J)$ always permute. But this is not true in general for n-ideals. For example, consider the 3 – element chain

$L = \{0, n, 1\}$.

Let $I = \{0, n\}$ and $J = \{n, 1\}$. Here $0 \equiv n\ \Theta(I)$ and $n \equiv 1\ \Theta(J)$. But there exists no $x \in S$ such that $0 \equiv x\ \Theta(J)$ and $x \equiv 1\ \Theta(I)$.

In [34, Theorem 2.1.11.] Latif has proved a result on the permutability of the congruences $\Theta(I)$ and $\Theta(J)$ in a distributive lattice L, where I and J are n- ideals. We conclude this chapter with the following result which generalizes [34 ,Theorem 2.1.11]

Theorem 2.3.7. *Let* S *be a distributive medial nearlattice and* n *is an upper element of* S *.Then for* $I, J \in I_n(S)$, *the following conditions are equivalent* .

(i) $\Theta(I)$ *and* $\Theta(J)$ *permute* .

(ii) S *is a lattice and* n *is complemented in each interval containing it.*

(iii) $P_n(S)$ *is a lattic.*

Proof: (i) ⇒ (ii). Let $a, b \in S$. Consider $\Theta(n, a \vee n)$ and $\Theta(n, b \vee n)$. Now, $a \vee n \equiv (a \vee n) \wedge (b \vee n)\ \Theta(n, a \vee n)$ and $(a \vee n) \wedge (b \vee n) \equiv b \vee n\ \Theta(n, b \vee n)$.

Since $\Theta(n, a \vee n)$ and $\Theta(n, b \vee n)$ permute, so there exists $t \in S$ such that $a \vee n \equiv t\ \Theta(n, b \vee n)$ and $t \equiv b \vee n\ \Theta(n, a \vee n)$. These imply $(a \vee n] \vee (b \vee n] = (t] \vee (b \vee n]$ and $(t] \vee (a \vee n] = (b \vee n] \vee (a \vee n]$

Thus, $a \vee n = ((a \vee n) \wedge t) \vee ((a \vee n) \wedge (b \vee n))$

and $b \vee n = ((b \vee n) \wedge t) \vee ((b \vee n) \wedge (a \vee n))$.

Since S is a medial nearlattice, So $a \vee b \vee n = ((a \vee n) \wedge t) \vee ((b \vee n) \wedge t) \vee ((a \vee n) \wedge (b \vee n))$ exists. Therefore, $a \vee b$ exists in S, and so S is a lattice.

Now let $x \leq n \leq y$. Also, $x \equiv n \Theta(x, n)$ and $n \equiv y \Theta(n, y)$. Since $\Theta(x, n)$ and $\Theta(n, y)$ permute, so there exists $t \in S$ such that $x \equiv t \Theta(n, y)$ and $t \equiv y \Theta(x, n)$. These imply $x = t \wedge n$ and $y = t \vee n$. Thus, t is the relative complement of n in $[x, y]$, So (ii) is proved.

(ii) $\Rightarrow$ (iii). Let $< a >_n, < b >_n \in P_n(S)$.

As S is a lattice, so $< a >_n \vee < b >_n = [a \wedge n, a \vee n] \vee [b \wedge n, b \vee n] = [a \wedge b \wedge n, a \vee b \vee n]$.

Since n is complemented in each interval containing it, so there exists $c \in [a \wedge b \wedge n, a \vee b \vee n]$ such that $c \wedge n = a \wedge b \wedge n$ and $c \vee n = a \vee b \vee n$.

Therefore, $< a >_n \vee < b >_n = < c >_n$ and hence $P_n(S)$ is a lattice.

(iii) $\Rightarrow$ (ii). Suppose $P_n(S)$ is a lattice. Let $a, b \in S$.

Now $< a >_n \vee < b >_n = < t >_n$ for some $t \in S$.

That is $[a \wedge n, a \vee n] \vee [b \wedge n, b \vee n] = [t \wedge n, t \vee n]$.

This implies $a \vee n, b \vee n \leq t \vee n$. Then by the upper bound property, $a \vee b \vee n$ exists, and so $a \vee b$ exists. Therefore, S is a lattice.

Also for $x \leq n \leq y$, $[x, y] = < x >_n \vee < y >_n = < t >_n = [t \wedge n, t \vee n]$ for some $t \in S$. This implies $t \wedge n = x$ and $t \vee n = y$. Therefore n is complemented in each interval containing it.

Finally, (ii) $\Rightarrow$ (i) follows from the proof of (ii) $\Rightarrow$ (i) in [34, Theorem 2.1.11]. ●

Chapter 3

CHAPTER 3

THE n-KERNELS OF SKELETAL CONGRUENCES ON A DISTRIBUTIVE NEARLATTICE

Introduction

Throughout this chapter we will be concerned with a distributive nearlattice S with a fixed element n. Skeletal congruences on distributive lattices have been studied extensively by Cornish in [8].

For any $\Theta \in C(S)$, Θ^* denotes the *pseudocomplement* of Θ. By its very definition $\Theta \cap \Phi = \omega$ if and only if $\Phi \leq \Theta^*$, $\Phi \in C(S)$. Since C(S) is a distributive algebraic lattice, so Θ^* must exists.

The skeleton $SC(S) = \{\Theta \in C(S) : \Theta = \Theta^{**}\}$. The set I(S) of all ideals of a distributive nearlattice S with 0 is pseudocomplemented. The pseudo complement J^* of an ideal J is the *annihilator ideal* $J^* = \{x \in S : x \wedge j = 0 \text{ for all } j \in J\}$. we also denote $KSC(S) = \{\ker \Theta : \Theta \in SC(S)\}$. The *kernel* of congruence Θ is $\ker \Theta = \{x \in S : x \equiv 0(\Theta)\}$. Of course $\ker \Theta\,(J) = J$.

The set $I_n(S)$ of all n-ideals of a distributive nearlattice S is a distributive algebraic lattice. So $I_n(S)$ is pseudocomplemented . For a medial element n and for any n-ideal J of a distributive nearlattice S, we define $J^+ = \{x \in S: m(x, n, j) = n \text{ for all } j \in J\}$. Then J^+ is an n-ideal and $J \cap J^+ = \{n\}$. J^+ is called the *annihilator* n-*ideal* of J, which is the pseudocomplement of J in $I_n(S)$ .We define n-kernel of a congruence Θ by $\ker_n \Theta = \{x \in S: x \equiv n\,(\Theta)\}$, which is clearly an n-ideal. $\Theta \in C(S)$ is called *dense* if $\Theta^* = \omega$, while an n-ideal J is called *dense* if $J^+ = \{n\}$.

A non-empty subset T of a nearlattice S is called *join-dense* if each $z \in S$ is the join of its predecessors in T. Also, T is called *meet-dense* if each $z \in S$ is the meet of its successors in T.

For $a, b \in S$, $< a, b > = \{x \in S : x \wedge a \leq b\}$ is called the *annihilator of a relative* to b, or simply a *relative annihilator*. In presence of distributivity it is easy to see that each relative annihilator is an ideal.
In a lattice L, we define $< a, b >_d = \{x \in L: x \vee a \geq b\}$ is known as *relative dual annihilator*. For relative annihilator ideals of a distributive lattice we refer the reader to see [38]. Recently [2] have extended these results for nearlattices.

A distributive nearlattice S with 0 is called *disjunctive* (weakly comple-mented and sectionally semi-complemented are alternative terms) if for $0 \leq a < b$, there is an element $x \in S$ such that $x \wedge a = 0$ and $0 < x \leq b$. For details on these lattices see [8] , [5] , [65] and recently [56] have extended the results of [8] for nearlattices.

M. A. Latif has studied the skeletal congruences in a lattice L in [34] extensively and established several equivalent conditions for F_n (L) to be disjunctive and generalized Boolean in terms of skeletal congruences of L.

In this chapter, we have discussed the skeletal congruence extensively in case of nearlattice and studied those $F_n(S)$, which are disjunctive.
In section 1, we have studied the skeletal congrences Θ^* of a distributive nearlattice S, where * represents the pseudocomplement. Then we have given a descriptions of $\Theta(J)^*$, where Θ (J) is the smallest congruence of S containing n-ideal J as a class and showed that J^+ is the n-kernel of $\Theta(J)^*$. We have proved that for a convex subsemilattice J of a distributive nearlattice S is small if and only if it is

meet-dense in S.

In section 2,we have shown that for an upper element n , $F_n(S)$ is disjunctive if and only if the intervals [a, n] is dual disjunctive and [n, b] is disjunctive for all $a \leq n \leq b$, $a, b \in S$. We have also shown that $F_n(S)$ is disjunctive if and only if the n-kernel of each skeletal congruence is an annihilator n-ideal. Moreover, we have given some other equivalent conditions for $F_n(S)$ to be disjunctive and generalized Boolean. Finally it is proved that $F_n(S)$ is generalized Boolean if and only if the map $\Theta \rightarrow Ker_n\Theta$ is a lattice isomorphism of SC(S) onto $K_nSC(S)$ whose inverse is the map $J \rightarrow \Theta(J)$, where J is an n-ideal and n is an upper element of S.

3.1. Skeletal congruence on a distributive nearlattice

For any $\Theta \in C(S)$, Θ^* denotes the pseudocomplement of Θ. For a near- lattice S, we define the skeleton

$$SC(S) = \{ \Theta \in C(S) : \Theta = \Phi^* \text{ for some } \Phi \in C(S)\}.$$
$$= \{ \Theta \in C(S) : \Theta = \Theta^{**} \}.$$

Recall that for a, b$\in$S, $< a, b > = \{x \in S : x \wedge a \leq b\}$ is the *annihilator of* a *relative* to b, or simply a *relative annihilator*. In presence of distributivity, it is easy to show that each relative annihilator is an ideal. Also note that $< a , b > = < a , a \wedge b >$. In case of a lattice L, we define

$< a, b >_d = \{x\in : x \vee a \geq b\}$ which is known as a *dual annihilator of* a *relative* to b , or simply *relative dual annihilator*. It is very easy to see that $<a, b>_d$ in a dual ideal (filter) of L when L is distributive. Since in a general nearlattice, supremum of two elements may not exist, so it is not possible to define a dual relative annihilator ideal for any two elements a and b.

But if n is an upper element of S, then $x \vee n$ exists for all $x\in S$. Then for any $a\in (n]$, $a \vee x$ exists for all $x \in S$ by the upper bound property of S. Thus for any $a\in (n]$, we can talk about dual relative annihilator ideal of the form $< a , b >_d$ for any $b \in S$. That is, for any $a \leq n$ in S,
$<a, b >_d = \{x \in S : x \vee a \geq b \}$ is a relative dual annihilator and in presence of distributivity, it is a filter of S.

The following theorem gives a neat description of the pseudocomplenent Θ^* of $\Theta \in C(S)$, which is due to [56]. This could also be deduced from Paperts description in [55, Theorem 2], also c .f [55, Th. 3.1 and 3.2].

Theorem 3.1.1.. *For a distributive nearlattice* S *with* 0, *the following conditions hold.*

(i) *For* $a \le b$ $(a, b \in S)$, $x \equiv y(\Theta(a, b)')$ *if and only if* $(x \wedge b) \vee a = (y \wedge b) \vee a$, *where* $(\Theta(a, b)')$ *is the complement of* $\Theta(a, b)$.

(ii) *For any* $\Theta \in C(S)$, $x \equiv y(\Theta^*)$ $(x, y \in S)$ *if and only if for each* $a, b \in S$ *with* $a \le b$ *and* $a \equiv b\ \Theta$; $(x \wedge b) \vee a = (y \wedge b) \vee a$.

(iii) *For any* $\Theta \in C(S)$, $x \equiv y(\Theta^*)$ *if and only if* $\Theta(0, x) \cap \Theta = \Theta(0, y) \cap \Theta$. That is, $\Theta_x \cap \Theta = \Theta_y \cap \Theta$ *if and only if* $\Psi_x \cap \Theta = \Psi_y \cap \Theta$. ●

Following result is due to [2] which gives a nice characterization of Θ^* when n is an upper element.

Theorem 3.1.2. *Let* S *be a distributive nearlattice with an upper element* n. *Then for any* $\Theta \in C(S)$, $x \equiv y(\Theta^*)$ *if and only if* $\Theta(n, x) \cap \Theta = \Theta(n, y) \cap \Theta$. ●

The following result is due to [56] which is a generalization of a result of Cornish [8] for lattices.

Theorem 3.1.3. *For a distributive nearlattice* S *with* 0, *the following conditions hold.*

(i) *For any ideal* J, $x \equiv y\ (\Theta(J)^*$ $(x, y \in S)$ *if and only if* $(x] \cap J = (y] \cap J$ i.e. *if and only if* $x \wedge j = y \wedge j$ *for all* $j \in J$.

(ii) *For an ideal* J, *both* $\Theta(J)^*$ *and* $\Theta(J^*)$ *have* J^* *as their Kernel.*

(iii) *An ideal* J *is the Kernel of a skeletal congruence if and only if it is the intersection of relative annihilator ideals*.

(iv) *Each principal ideal is an intersection of relative annihilator ideals.* ●

Now we generalize the above results for n-ideals when n is an upper element of S.

Recall that for an n-ideal J of a distributive nearlattice S, if n is a medial element $J^+ = \{x \in S : m(x, n, j) = n \text{ for all } j \in J\}$. J^+ is known as the annihilator n-ideal of J and $J \cap J^+ = \{n\}$. Of course J^+ is an n-ideal. Also recall that the n-kernel of a congruence Θ is given by $Ker_n \Theta = \{x \in S : x \equiv n \Theta\}$, which is also an n-ideal.

Following result is a generalization of above result, which is also due to [2]. Here we prefer to include the proof for the convenience.

Theorem 3.1.4. *If* S *is a distributive nearlattice and* $n \in S$ *is an upper element, then the following conditions hold.*

(i) *For any* n-*ideal* J, $x \equiv y\ (\Theta(J)^*)$ $(x, y \in S)$ *if and only if* $< x >_n \cap J = < y >_n \cap J$ *i.e. if and only if* $m(x, n, j) = m(y, n, j)$ for *all* $j \in J$.

(ii) *For an* n-*ideal* J, *both* $\Theta(J)^*$ and $\Theta(J^+)$ *have* J^+ *as their* n-*kernel*.

(iii) *The* n-*kernels of the skeletal congruences are Precisely those n-ideals which are intersection of relative annihilator ideals and dual relative annihilator ideals whose end points are of the form* $x \vee n$ *and* $x \wedge n$ *respectively.*

(iv) *Each principal n-ideal in a distributive nearlattice is the intersection of relative annihilator ideals and dual relative annihilator ideals whose end points are of the form* $x \wedge n$ *and* $x \vee n$.

Proof: (i). For any two n-ideals I and J of S, we have $\Theta(I \cap J) = \Theta(I) \cap \Theta(J)$. Also, since n is upper so $\Theta(n, x) = \Theta(n \wedge x, n \vee x) = \Theta(< x >_n)$. Then by 3.1.2, $x \equiv y\ (\Theta(J)^*)$ if and only if $\Theta(n, x) \cap \Theta(J) = \Theta(n, y) \cap \Theta(J)$ if and only if $\Theta(< x >_n) \cap \Theta(J) = \Theta(< y >_n) \cap \Theta(J)$ if and only if $\Theta(< x >_n \cap J) = \Theta(< y >_n \cap J)$ if and only if $< x >_n \cap J = < y >_n \cap J$, by 2.3.6, if and only if $m(x, n, j) = m(y, n, j)$ for all $j \in J$ by theorem 1.4.3. Hence (i) holds.

(ii). If $x \in Ker_n(\Theta(J)^*)$, then $x \equiv n\ (\Theta(J)^*)$.Then by (i) above ,$<x>_n \cap J = <n>_n \cap J$ if and only if $m(x, n, j) = m(n, n, j) = n$ for all $j \in J$ and so $x \in J^+$, and thus (ii) holds.

(iii). Consider $a, b \in S$ with $a \leq b$, Since $\Theta(a, b)^* = \Theta(a, b)'$. So by theorem 3.1.1, $x \in Ker_n \Theta(a, b)^*$ if and only if $(x \wedge b) \vee a = (n \wedge b) \vee a$ (since $a \leq b$, if and only if $(x \wedge b) \vee a$ and $(n \wedge b) \vee a$ exist by the upper bound property of S.)

Now, we shall show that $(x \wedge b) \vee a = (n \wedge b) \vee a$ is equivalent to $x \in <b \vee n, a \vee n> \cap <a \wedge n, b \wedge n>_d$. Since $(x \wedge b) \vee a = (n \wedge b) \vee a$ implies $x \wedge b \leq a \vee n$, we have $x \wedge (b \vee n) = (x \wedge b) \vee (x \wedge n) \leq a \vee n$, and so $x \in <b \vee n, a \vee n>$

Again from $(x \wedge b) \vee a = (n \wedge b) \vee a$, we have $b \wedge n \leq (x \wedge b) \vee a$.

So, $b \wedge n \leq (x \wedge b \wedge n) \vee (a \wedge n) \leq x \vee (a \wedge n)$, which implies $x \in <a \wedge n, b \wedge n>_d$.

Hence $x \in <b \vee n, a \vee n> \cap <a \wedge n, b \wedge n>_d$.

Conversely, let $x \in <b \vee n, a \vee n> \cap <a \wedge n, b \wedge n>_d$.

Then, $x \in <b \vee n, a \vee n>$ and $x \in <a \wedge n, b \wedge n>_d$

Then $x \wedge (b \vee n) \leq a \vee n$ and $x \vee (a \wedge n) \geq b \wedge n$

Now $x \wedge (b \vee n) \leq a \vee n$ implies

$$x \wedge b = x \wedge b \wedge (b \vee n)$$
$$\leq (a \vee n) \wedge b$$
$$= (a \wedge b) \vee (b \wedge n)$$
$$= a \vee (b \wedge n) \text{ and so } (x \wedge b) \vee a \leq (b \wedge n) \vee a \quad \ldots\ldots\ldots\ldots(1)$$

On the other hand, $b \wedge n \leq x \vee (a \wedge n)$ implies

$$= b \wedge n \leq b \wedge (x \vee (a \wedge n))$$
$$= (x \wedge b) \vee (a \wedge b \wedge n)$$
$$= (x \wedge b) \vee (a \wedge n).$$

and so , $(n \wedge b) \vee a \leq (x \wedge b) \vee a$..............(2)

Form (1) and (2) we have, $(x \wedge b) \vee a = (n \wedge b) \vee a$.

Since for any $\Theta \in C(S)$,

$\Theta^{*} = \cap \{\Theta(a, b)^{*} : a \equiv b\ \Theta\}$, hence the result follows.

(iv). Since each principal n-ideal

$$< a >_n = Ker_n \Theta(< a >_n)$$

$= Ker_n \Theta(a \wedge n, a \vee n)$and since $\Theta(a \wedge n, a \vee n)$ is skeletal, so by(iii) the result follows. ●

A non-empty subset T of a nearlattice S is called *large* if $x \wedge t = y \wedge t$ for all $t \in T$, $x, y \in S$ implies $x = y$. Also recall that T is join-dense if each $z \in S$ is the join of its predecessors in T.

In a lattice L, a non-empty subset T is called *small* if $x \vee t = y \vee t$ for all $t \in T$, $x, y \in L$ implies $x = y$. In a distributive nearlattice S, we call a non-empty subset T *small* if for all $x, y \in S$, with $x \leq y$ and $y = x \vee (y \wedge t)$ for all $t \in T$ imply $x = y$.

Now we will show that these two definitions are equivalent in case of a lattice.

Suppose first definition holds for a subset T in a lattice L. Let $x \leq y$ with $y = x \vee (y \wedge t)$ for all $t \in T$.

Then $y \vee t = (x \vee (y \wedge t) \vee t) = x \vee t$.

Therefore, $x = y$ and so second definitition holds.

Conversely, suppose second definition is true. Let $x \vee t = y \vee t$ for all $t \in T$. Then $x \leq y$ and so $x \wedge y \leq y$. Thus, $y = y \wedge (y \vee t) = y \wedge (x \vee t) = (y \wedge x) \vee (y \wedge t) = x \vee (y \wedge t)$.

Hence by second definition, $x = y$, and so first definition holds. ●

Recall that T is meet-dense in S if each $z \in S$ is the meet of its successors in T. It can be easily shown that an ideal in a nearlattice is large if and only if it is join dense. It is clear from 3.1.3 that an ideal J of a distributive nearlattice S is join dense if and only if Θ (J) is dense in C(S), that is, $\Theta(J)^* = \omega$.

Following lemma is due to [56] and it will be needed for our next theorem.

Lemma 3.1.6. *A convex subnearlattice* J *of a distributive nearlattice* S *is large if and only if it is join-dense in* S. ●

Corollary 3.1.7. *An* n-*ideal of a distributive nearlattice* S *is large if and only if it is join-dense in* S. ●

Dually we can easily prove the following results.

Lemma 3.1.8. *A convex sub semi-lattice* J *of a distributive nearlattice* S *is small if and only if it is meet-dense in* S

Proof: Suppose J is meet-dense. Let $x \leq y$ and suppose $y = x\vee (y\wedge t)$ for all $t \in J$. Let t_r is a successor of x in J. Since $x \leq y$, and $x \leq t_r$. Therefore, $y\wedge t_r = (y\wedge t_r)\vee x = y$ implies t_r is a successor of y . So J is small.

Conversely, Suppose J is small. Let $x\in S$ and $\{t_i\}$ successors of x in J. we need to show $x = t_1\wedge t_2\wedge \ldots\ldots \wedge t_r$

Now, let r be the lower bound of $\{ t_i \}$. Then $x\vee r$ exists.

Let t be any element of J .

Then $t\wedge t_i \leq x\vee (t\wedge t_i) \leq t_i$. Then by convexity, $x\vee(t\wedge t_i) \in J$ and $x\vee (t\wedge t_i)$ is a successor of x in J .

Therefore, $x\vee (t\wedge t_i) = t_k$ for some k and and so, $x \vee (t\wedge t_i) \geq r$.

Thus, $r = r\wedge [x\vee (t\wedge t_i)]$

$= (r\wedge x) \vee (r\wedge t\wedge t_i)$

$= (r\wedge x) \vee (r\wedge t) \leq r$.

Hence $r = (r \wedge x) \vee (r \wedge t)$ for all $t \in J$.

Since J is small, this implies $r = r \wedge x$.

That is, $r \leq x$. Since r is the lower bound of $\{t_i\}$, so , $x = t_1 \wedge t_2 \wedge \ldots . \wedge t_r$

Therefore, J is meet-dense in S. ●

Corollary 3.1.9. *An* n-*ideal of a distributive nearlattice* S *is small if and only if it is meet-dense in* S. ●

Theorem 3.1.10. *Let* S *be a distributive nearlattice with an upper element* n.*Then for any* n-*ideal* J *of* S , Θ (J) *is dense in* C(S) *if and only if* J *is both meet and join dense.* ●

3.2. Disjunctive and SemiBoolean algebras

Recall that a distributive nearlattice S with 0 is disjunctive if $0 \leq a < b$ implies the existence of $x \in S$ such that $x \wedge a = 0$ and $0 < x \leq b$. We have already mentioned that the disjunctive (sectionally semi-complemented) lattices have been studied by many authors including [8] . Then [53] introduced the notion for nearlattices and generalized all the results of [8]. On the other hand, Latif in [34] generalized the results of [8] for n-ideals in lattices. In this section we will generalize the results of [53] in terms of n-ideals.

By [2, Th. 2.3.7.], we know that for any n-ideal J of a distributive medial nearlattice S, R(J) denotes the largest congruence having J as its kernel , where $x \equiv y$ R(J) if and only if for each $r \in L$, $m(x , n , r) \in J$ if and only if $m(y , n , r) \in J$.

The following result is due to [53] which gives a description of disjunctive nearlattice.

Theorem 3. 2.1. *For a distributive nearlattice* S *with* 0 , the following conditions are equivalent.

(i) S *is disjunctive.*

(ii) *For all* $a \in S$, $(a] = (a]^{**}$

(iii) $R((0]) = \omega$ ●

Theorem 3. 2. 2. *Let* S *be a distributive nearlattice with* 0 *and* n *be an upper element of* S *then the following conditions are equivalent.*

(i) $F_n(S)$ *is a disjunctive lattice.*

(ii) $P_n(S)$ *is a disjunctive nearlattice.*

(iii) *The interval* $[a, n]$ *is dual disjunctive and* $[n, b]$ *is disjunctive for all*

$a \le n \le b$, $a, b \in S$.

Proof: (i) $\Rightarrow$ (ii). Let $\{n\}\} \subseteq \langle a \rangle_n \subset \langle b \rangle_n$. Since (i) holds, there exists $\langle x_1, x_2, \ldots\ldots, x_r \rangle_n$ such that $\{n\} \subset \langle x_1, x_2, \ldots.., x_r \rangle_n \subset \langle b \rangle_n$ for which $\langle a \rangle_n \cap \langle x_1, x_2, \ldots.., x_r \rangle_n = \{n\}$. Now by corollary 1.4.5, $\langle x_1, x_2, \ldots.., x_r \rangle_n$ is a principal n-ideal. Therefore, (ii) holds.

(ii) $\Rightarrow$ (i). Let (ii) holds and $\{n\} \subseteq \langle x_1, x_2, \ldots.., x_r \rangle_n \subseteq \langle y_1, y_2, \ldots., y_s \rangle_n$. Then $\{n\} \subseteq \langle x_1, x_2, \ldots\ldots. x_r \rangle_n \cap \langle y_t \rangle_n \subset \langle y_t \rangle_n$ for $t = 1, 2, 3, \ldots\ldots s$.

That is, $\{n\} \subseteq \langle m(x_1, n, y_t) \rangle_n \vee \ldots\ldots \vee \langle m(x_r, n, y_t) \rangle_n \subset \langle y_t \rangle_n$. But by corollary 1.4.5, $\langle m(x_1, n, y_t) \rangle_n \vee \ldots\ldots\ldots \vee \langle m(x_r, n, y_t) \rangle_n$ is a principal n-ideal. Thus there exists $\langle c \rangle_n$ and $\{n\} \subset \langle c \rangle_n \subseteq \langle y_t \rangle_n$ such that $\langle x_1, x_2, \ldots, x_r \rangle_n \cap \langle y_t \rangle_n \cap \langle c \rangle_n = \{n\}$. This implies $\langle x_1, x_2, \ldots., x_r \rangle_n \cap \langle c \rangle_n = \{n\}$ where $\langle c \rangle_n \subseteq \langle y_1, y_2, \ldots\ldots., y_s \rangle_n$

Therefore, $F_n(S)$ is disjunctive.

(ii) $\Rightarrow$ (iii). Let (ii) holds and $a < b \le n$ for which $\{n\} \subseteq \langle b \rangle_n \subset \langle a \rangle_n$. Then by (ii), there exists $\langle t \rangle_n$ with $\{n\} \subset \langle t \rangle_n \subseteq \langle a \rangle_n$ such that $\langle b \rangle_n \cap \langle t \rangle_n = \{n\}$. This implies $n = [b, n] \cap [t, n] = [b \vee t, n]$ and so $b \vee t = n$. Moreover, $\{n\} \subset \langle t \rangle_n \subseteq \langle a \rangle_n$ implies $a \le t < n$. Therefore, $(n]$ is dual disjunctive.

Similarly, we can show that $[n)$ is disjunctive.

(iii) $\Rightarrow$ (ii). Let (iii) holds and $\{n\} \subseteq \langle a \rangle_n \subset \langle b \rangle_n$

So, $\{n\} \subseteq [a \wedge n, a \vee n] \subset [b \wedge n, b \vee n]$ implies either $b \wedge n < a \wedge n \le n$

or $n \le a \vee n < b \vee n$. If $b \wedge n < a \wedge n \le n$, then there exists t with $b \wedge n \le t < n$ such that $(a \wedge n) \vee t = n$

So, $< t >_n \cap < a >_n = [t, n] \cap [a \wedge n,\ a \vee n]$

$$= [t \vee (a \wedge n), n] = \{n\}. \text{ Where } \{n\} \subset < t >_n \subseteq < b >_n.$$

Again, if $n \leq a \vee n < b \vee n$, we can similarly show that there exists $< s >_n$ with $\{n\} \subset < s >_n \subseteq < b >_n$ such that $< s >_n \cap < a >_n = \{n\}$. Therefore, $P_n(S)$ is disjunctive. ●

Following result has been proved by [2] when S is a distributive medial nearlattice with an upper element. We generalize that result for an ordinary distributive nearlattice where n is merely an upper element.

Therom 3.2.3. *Suppose* S *is a distributive nearlattice with an upper element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is disjunctive.*

(ii) $P_n(S)$ *is disjunctive.*

(iii) *For each* $a \in S$, $< a >_n = < a >_n^{++}$.

(iv) $R(\{n\}) = \omega$.

Proof: (i) $\Leftrightarrow$ (ii) holds by Theorem 3. 2. 2.

(ii) $\Rightarrow$ (iii). Here n is upper. Suppose $P_n(S)$ is disjunctive and suppose that $< a >_n \neq < a >_n^{++}$ for some $a \in S$. Since $< a >_n \subseteq < a >_n^{++}$, so there exists $t \in < a >_n^{++}$ but $t \notin < a >_n = [a \wedge n, a \vee n]$, which implies either $a \wedge n \nleq t$ or, $t \nleq a \vee n$. Suppose $a \wedge n \nleq t$, then $t \wedge a \wedge n < a \wedge n$. Thus $[a \wedge n, n] \subset [t \wedge a \wedge n, n]$, and so $\{n\} \subseteq < a \wedge n >_n \subset < t \wedge a \wedge n >_n$. Since $P_n(S)$ is disjunctive, so there exists $< b >_n$

such that $\{n\} \subset \langle b \rangle_n \subseteq \langle t \wedge a \wedge n \rangle_n$ and $\langle a \wedge n \rangle_n \cap \langle b \rangle_n = \{n\}$. This implies $[(a \wedge n) \vee (b \wedge n)\ ,\ n] = \{n\}$, and so $(a \wedge n) \vee (b \wedge n) = n$.

Now $\langle a \rangle_n \cap \langle b \rangle_n = [(a \wedge n) \vee (b \wedge n)\ ,\ (a \vee n) \wedge (b \vee n)]$.

$$= [n\ ,\ (a \vee n) \wedge (b \vee n)]$$

$$= \{n\} \quad \text{as } b \le n. \text{ Hence } \langle b \rangle_n \subseteq \langle a \rangle_n^{+}.$$

Now, $\langle b \rangle_n = \langle b \rangle_n \cap \langle t \wedge a \wedge n \rangle_n$

$$= [(b \wedge n) \vee (t \wedge a \wedge n)\ ,\ n]$$

$$= [((t \wedge n) \vee (b \wedge n)) \wedge ((a \wedge n) \vee (b \wedge n))\ ,\ n]$$

$$= [((t \wedge n) \vee (b \wedge n)) \wedge n\ ,\ n]$$

$$= [(t \wedge n) \vee (b \wedge n)\ ,\ n]$$

$$= \langle t \wedge n \rangle_n \cap \langle b \rangle_n$$

$$= \{n\} \quad \text{as } t \wedge n \in \langle a \rangle_n^{++} \text{ and } \langle b \rangle_n \subseteq \langle a \rangle_n^{+}.$$

Thus $\langle b \rangle_n = \{n\}$, which is a contradiction. Therefore, $\langle a \rangle_n = \langle a \rangle_n^{++}$ for all $a \in S$, which is (iii).

Again suppose $t \nleq a \vee n$

Then $t \vee n \nleq a \vee n$. This implies $t \neq (t \wedge a) \vee (t \wedge n)$

That is $(t \wedge a) \vee (t \wedge n) < t$, and so $(t \wedge a) \vee n < t \vee n$

Thus $\{n\} \subseteq \langle (t \wedge a) \vee n \rangle_n \subset \langle t \vee n \rangle_n$.

Since $P_n(S)$ is disjunctive, so there exists $< c >_n$ such that $\{n\} \subset < c >_n \subseteq < t \vee n >_n$ and $< c >_n \cap < (t \wedge a) \vee n >_n = \{n\}$. This implies $[c \wedge n\ ,\ c \vee n] \cap [\ n\ ,\ (t \wedge a) \vee n] = \{n$, and so $[n,\ ((t \wedge a) \vee n) \wedge (c \vee n)] = \{n\}$

Thus, $((t \wedge a) \vee n) \wedge (c \vee n) = n$. That is $(t \wedge a \wedge c) \vee n = n$ and so, $t \wedge a \wedge c \leq n$.

Also, $(t \wedge a \wedge c) \vee n = n$ implies $[(t \wedge c) \vee n] \wedge [a \vee n] = n$.

Hence, $< (t \wedge c) \vee n >_n \subseteq < a >_n^{+}$.

Now, $< c >_n = < c >_n \cap < t \vee n >_n$.

$= [c \wedge n, c \vee n] \cap [n\ ,\ t \vee n]$

$= [n,\ (t \wedge c) \vee n]$

$= < t \vee n >_n \cap < (t \wedge c) \vee n >_n$

$= \{n\}$ as $< (t \wedge c) \vee n >_n \subseteq < a >_n^{+}$ and $t \vee n \in < a >_n^{++}$.

Thus $< c >_n = n$, which is a contradiction. Therefore, $< a >_n = < a >_n^{++}$ for all $a \in S$. Thus (iii) holds.

(iii)$\Rightarrow$ (ii). Suppose $< a >_n = < a >_n^{++}$ for all $a \in S$. Suppose $n \leq a < b$. Then $\{n\} \subseteq < a >_n \subset < b >_n$ and $< a >_n = < a >_n^{++}$, $< b >_n = < b \vee n >_n^{++}$ implies $< a >_n^{+} \supset < b >_n^{+}$, so there exists $r \in < a >_n^{+}$ such that $r \notin < b >_n^{+}$ this implies $m(r, n, a) = n$ and $m(r, n, x) \neq n$ for some $x \in < b >_n$. Then $n = m(r, n, a) = (r \vee n) \wedge a$ and as $x \geq n$, $m(r, n, x) = (r \vee n) \wedge x$. Then $\{n\} \subset < m(r, n, x) >_n \subseteq < b >_n$ and $n < (r \vee n) \wedge x \leq b$.

Moreover, $a \wedge (r \vee n) \wedge x = n \wedge x = n$. This implies $[n)$ is disjunctive. Similarly we can show that $(n]$ is dual disjunctive .

Hence, $(n]$ is dual disjunctive and $[n)$ is disjunctive . So by theorem 3.2.2, $P_n(S)$ is disjunctive, which implies (ii).

(ii) $\Rightarrow$ (iv). Suppose $P_n(S)$ is disjunctive. Let $x \equiv y$ $R(\{n\})$. If $x \neq y$, then either $x \wedge y < x$ or , $x \wedge y < y$. Suppose $x \wedge y < x$. Since S is distributive, so either $x \wedge y \wedge n < x \wedge n$ or, $(x \wedge y) \vee n < x \vee n$. If $x \wedge y \wedge n < x \wedge n$, then $< x >_n \subset < x >_n \vee < y >_n$ and so $< x >_n \cap < y >_n \subset < y >_n$.

If $(x \wedge y) \vee n < x \vee n$, then $< x >_n \cap < y >_n \subset < x >_n$ .Thus $x \neq y$ implies either $< x >_n \cap < y >_n \subset < x >_n$ or, $< x >_n \cap < y >_n \subset < y >_n$. Without loss of generality, suppose $< x >_n \cap < y >_n \subset < x >_n$. Since $P_n(S)$ is disjunctive, there exists $< t >_n$ such that $\{n\} \subset < t >_n \subseteq < x >_n$ and $< t >_n \cap < x >_n \cap < y >_n = \{n\}$ and so $< y >_n \cap < t >_n = \{n\}$. That is $m(y, n, t) = n$. Since $x \equiv y$ $R(\{n\})$, so $m(x, n, t) = n$ and so $< x >_n \cap < t >_n = \{n\}$. This implies $< t >_n = \{n\}$, which is a contradiction. Therefore $x = y$. Thus $R(\{n\}) = \omega$ which is (iv).

Finally, we show that (iv) $\Rightarrow$ (ii). Let $R(\{n\}) = \omega$.

Consider the interval $[n, b]$. If $[n, b]$ is not disjunctive, then there exists $x \in S$ with $n \leq x < b$ such that $x \wedge t > n$ for all t with $n < t \leq b$. Choose any $r \in S$. Then $m(x, n, r) = m(x, n, (r \wedge b) \vee n) = (x \wedge r) \vee n$. Also $m(b, n, r) = m(b, n, (r \wedge b) \vee n) = (b \wedge r) \vee n$. If $m(b, n, r) = n$, then $n \leq (x \wedge r) \vee n \leq (b \wedge r) \vee n = n$ implies $m(x, n, r) = n$.

Again $m(x, n, r) = n$ implies $n = m(x, n, (r \wedge b) \vee n)$

$$= n \vee (x \wedge [(r \wedge b) \vee n]).$$

This implies $x \wedge [(r \wedge b) \vee n] = n$ as $x \geq n$. Since $n \leq (r \wedge b) \vee n \leq b$, so by above condition$(r \wedge b) \vee n = n$. Thus $m(b, n, r) = m(b, n, (r \wedge b) \vee n) = m(b, n, n) = n$

Therefore, $m(x, n, r) = n$ if and only if $m(b, n, r) = n$ for any $r \in S$. This implies $x \equiv b\, R(\{n\})$ and so $x = b$, which is a contradiction to our assumption. Hence $[n, b]$ must be disjunctive. A dual proof of above shows that each interval $[a, n]$, $a \in S$ is a dual disjunctive. Therefore by theorem 3.2.2, $P_n(S)$ is disjunctive. ●

Following result is due to [53] which is a generalization of [10 , Th.2.1]

Theorem 3. 2. 4. *In a distributive nearlattice* S *with* 0 , *the following conditions are equivalent.*

(i) S *is disjunctive.*

(ii) *Each dense ideal* J (i. e., $J^{*} = (0]$) *is join -dense.*

(iii) *For each dense ideal* J, $\Theta(J^{*}) = \Theta(J)^{*}$.

(iv) *For each dense ideal* J, $\Theta(J^{**}) = \Theta(J)^{**}$. ●

We generalize the above result for n-ideals. This result is also an improvement of [2, Th. 3.2.4] . Here we have considered n as merely an upper element instead of central. We prefer to omit the proof as it can be proved by same technique as in the proof of [2, Th. 3.2.4].

Theorem: 3.2.5. *Let* S *be a distributive nearlattice and* $n \in S$ *be an upper element, then the following conditions are equivalent.*

(i) $F_n(S)$ *is disjunctive.*

(ii) $P_n(S)$ *is disjunctive.*

(iii) *Each dense* n*-ideal* J, *is both join and meet dense.*

(iv) *For each dense* n*-ideal* J, $\Theta(J^{+}) = \Theta(J)^{*}$.

(v) *For each dense* n*-ideal* J, $\Theta(J^{++}) = \Theta(J)^{**}$.●

The following theorem is a generalization of [34 , Th. 3.2.6] .

Theorem 3.2.6. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is disjunctive.*

(ii) *For each congruence* Φ, $\Phi^{*} = \Theta(\ker_n \Phi)^{*}$.

(iii) *For each n-ideal* J, $R(J)^{*} = \Theta(J)^{*}$.

(iv) *For each congruence* Φ, $\ker_n(\Phi^{*}) = (\ker_n \Phi)^{+}$.

(v) *For each congruence* Φ, $\ker_n(\Phi^{**}) = (\ker_n \Phi)^{++}$.

(vi) *Then n-kernel of each skeletal congruence is an annihilator* n*-ideal.*

Proof : (i) ⇒ (ii). Suppose $F_n(S)$ is disjunctive. Since $\Theta(\ker_n \Phi) \subseteq \Phi$, so we have $\Phi^{*} \subseteq \Theta(\ker_n \Phi)^{*}$. So it is sufficient to prove that $\Phi \cap \Theta(\ker_n \Phi)^{*} = \omega$. Suppose $x \leq y$ and $x \equiv y\ (\Phi \cap \Theta(\ker_n \Phi)^{*})$ implies $x \equiv y\ \Phi$ and $x \equiv y\ \Theta(\ker_n \Phi)^{*}$. If $x < y$, then either $x \wedge n < y \wedge n$ or $x \vee n < y \vee n$.

Suppose $x \vee n < y \vee n$. Since $F_n(S)$ is disjunctive, so by theorem 3.2.2, [n) is also a disjunctive. So there exists $n < a \leq y \vee n$ such that $a \wedge (x \vee n) = n$.

Now, $n = a \wedge (x \vee n) \equiv a \wedge (y \vee n) = a\ (\Phi)$ and so $a \in \ker_n \Phi$.

Since $x \equiv y\,\Theta(\ker_n \Phi)^*$, so $x \vee n \equiv y \vee n\ \Theta(\ker_n\ \Phi)^*$ and since $a \in \ker_n \Phi$, so by the theorem 3.1.4, $m(x \vee n, n, a) = m(y \vee n, n, a)$.That is, $((x \vee n) \wedge n) \vee (a \wedge (x \vee n)) \vee (n \wedge a) = ((y \vee n) \wedge n) \vee (a \wedge (y \vee n)) \vee (n \wedge a)$ and so, $n \vee (a \wedge (x \vee n)) = x \vee n$, this implies $n = a$, which is a contradiction. Therefore, $x = y$ and so $\Phi \cap \Theta(\ker_n \Phi)^* = \omega$.

Thus, $\Theta(\ker_n \Phi)^* \subseteq \Phi^*$. Hence, $\Phi^* = \Theta(\ker_n \Phi)^*$.

(ii) $\Rightarrow$ (iii) holds, since J is the kernel of $R(J)$ and $\Theta(J)$.

(iii) $\Rightarrow$ (i) . Suppose (iii) holds, since $\Theta(\{n\}) = \omega$ and since (iii) holds ,

so $R(\{n\})^* = \Theta(\{n\})^* = \iota$ implies $R(\{n\})^{**} = \omega$.

Then by 3.2.3, we have $F_n(S)$ is disjunctive.

Since by Theorem 3.1.4 (ii), $\Theta(J)^*$ and $\Theta(J^+)$ have J^+ as their n-kernels, so (ii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (vi) are obvious.

Finally , we need to prove (vi) $\Rightarrow$ (i). Suppose (vi) holds. Let $n \le a < c$. Then by theorem 3.1.4 (iii), $< c, a >$ is the n-kernel of a skeletal congruence. Since (vi) holds, so there is an annihilator n-ideal K such that $< c, a > = K = K^{**}$. As $a \wedge c \le a$ implies $< c, a > = K = K^{**}$. Also, since $a < c$, $a \notin < c, a > = K = K^{**}$. So there exists $e \in K^*$, such that $m(c, n, e) \neq n$. But $m(a, n, e) = n$ implies $(a \wedge e) \vee n = n$. That is, $a \wedge (e \vee n) = n$, and so $a \wedge (e \vee n) \wedge c) = n$. Also, $m(c, n, e) \neq n$ implies $(e \vee n) \wedge c > n$ and so $n < (e \vee n) \wedge c \le c$ with $a \wedge (e \vee n) \wedge c = n$. Therefore $[n)$ is disjunctive. A dual proof of this gives that $(n]$ is dual disjunctive and so by `theorem 3.2.2, $F_n(S)$ is disjunctive. ●

Recall that a nearlattice S with 0 is semi-Boolean if it is distributive and the interval [0, x] is complemented for each $x \in S$. By [51], we know that the lattice of all ideals of a nearlattice is isomorphic to the lattice of congruence if and only if S is semi-Boolean. Following theorem is due to [56].

Theorem 3.2.7. *Let* S *be a distributive nearlattice with* 0, *then the following conditions are equivalent.*

(i) S *is semi-Boolean.*

(ii) *For each congruence* Φ, $\Phi^* = \Theta(\ker \Phi^*)$

(iii) *For each ideal* J, $\Theta(J^*) = \Theta(J)^*$.

(iv) *For each ideal* J, $\Theta(J^{**}) = \Theta(J)^{**}$. ●

Now we have following generalization.

Theorem 3.2.8. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is generalized Boolean.*

(ii) *For each congruence* Φ, $\Phi^* = \Theta(\ker_n \Phi^*)$

(iii) *For each ideal* J, $\Theta(J^+) = \Theta(J)^*$.

(iv) *For each ideal* J, $\Theta(J^{++}) = \Theta(J)^{**}$.

Proof: (i) $\Rightarrow$(ii). Suppose $F_n(S)$ is generalized Boolean. Then by 2.2.7, $P_n(S)$ is semi-Boolean. Let Ψ be any congruence on S. Then by theorem 2.3.3, $\Psi = \Theta(\ker_n \Psi)$. Thus with $\Psi = \Phi^*$, we see that (i) implies (ii).

(ii) $\Rightarrow$(iii) follows from Theorem 3.1.4 and (iii) $\Rightarrow$(iv) is obvious.

(iv)$\Rightarrow$(i) . Suppose (iv) holds. Put $J = \langle a_1 \rangle_n \vee \langle a_1 \rangle_n^+$. Since $J^{++} = S$, (iv) implies $\Theta(\langle a_1 \rangle_n \vee \langle a_1 \rangle_n^+)^{**} = \iota$. It follows that $\Theta(\langle a \rangle_n^* \cap \Theta(\langle a \rangle_n^+)^* = \omega$ and so $\Theta(\langle a_1 \rangle_n^+)^* \subseteq \Theta(\langle a \rangle_n)^{**} = \Theta(\langle a \rangle_n)$.

Now, by 3.1.4, $\langle a \rangle_n^+ = \ker_n \Theta(\langle a \rangle_n)^*$.

Then, $\Theta(\langle a \rangle_n^+) \subseteq \Theta(\langle a \rangle_n)^*$ and so $\Theta(\langle a \rangle_n) = \Theta(\langle a \rangle_n)^{**} \subseteq \Theta(\langle a \rangle_n^+)^*$.

Therefore, $\Theta(\langle a \rangle_n) = \Theta(\langle a \rangle_n^+)^*$.

But $\langle a \rangle_n^+ = \langle a \rangle_n^{+++}$, so by (iv) $\Theta(\langle a \rangle_n)^* = \Theta(\langle a \rangle_n^+)^{**} = \Theta(\langle a \rangle_n^{+++})$ $= \Theta(\langle a \rangle_n^+)$.

Now, let $n \le a \le b$. then for all $j \in \langle a \rangle_n = [n, a]$, $m(a, n, j) = m(b, n, j) = j$

Thus by 3.1.4, $a \equiv b \Theta(\langle a \rangle_n)^* = \Theta(\langle a \rangle_n^+)$. Then $(a] \vee (\langle a \rangle_n^+] = (b] \vee (\langle a \rangle_n^+]$ implies by 1.2.5, that $b = (a \wedge b) \vee (b \wedge r_1) \vee \ldots\ldots \vee (b \wedge r_s)$ for some $r_1, r_2, \ldots, r_s \in \langle a_1 \rangle_n^+$. That is, $b = a \vee (b \wedge r_1) \vee \ldots \vee (b \wedge r_s)$.

Again, $r_i \in \langle a_1 \rangle_n^+$ implies $m(a, n, r_i) = (a \wedge n) \vee (a \wedge r_i) \vee (r_i \wedge n) = n$, and so $a \wedge r_i \le n$. Thus $a \wedge r = a \wedge r \wedge n = r \wedge n$.

Now, put $p_i = (b \wedge r_i) \vee n$ and $p = p_1 \vee p_2 \vee \ldots\ldots\ldots \vee p_s$.

Then $n \le p \le b$. Also $p \wedge a = (a \wedge b \wedge r_1) \vee \ldots\ldots\ldots\ldots\ldots \vee (a \wedge b \wedge r_s) \vee (a \wedge n) = n$ and $p \vee a = (b \wedge r_1) \vee \ldots\ldots\ldots\ldots\ldots \vee (b \wedge r_s) \vee a \vee n = b \vee n = b$.

Hence $[n, b]$ is complemented for each $b \in S$.

Similarly, a dual proof of above shows that $[c, n]$ is also complemented for each $c \le n$. Hence by theorem 2.2.7, $F_n(S)$ is generalized Boolean.●

For a nearlattice S, the skeletal $SC(S) = \{ \Theta \in C(S) : \Theta = \Phi^{*}$ for some $\Phi \in C(S)\} = \{ \Theta \in C(S) : \Theta = \Theta^{**}\}$ is a complete Boolean lattice. The meet of a set $\{\Theta_i\} \subseteq SC(S)$ is $\cap \Theta_i$; as in C(S), while the join is given by $\underline{\vee}\Theta_i = (\vee \Theta_i)^{**} = (\cap \Theta_i^{*})^{*}$ and the complement of $\Theta \in SC(S)$ is Θ^{*}.

The fact that SC(S) is complete follows from the fact that SC(S) is precisely the set of closed elements associated with the closure operation $\Theta \to \Theta^{**}$ on the complete lattice C(S) and SC(S) is Boolean because of Glivenko's theorem, c. f. Gratzer [16,Th.4 , P- 58].

The set $KSC(S) = \{\ker\Theta : \Theta \in SC(S)\}$ is closed under arbitrary set theoretic intersections and hence is a complete lattice.

Also, for any $n \in S$, $K_n SC(S) = \{ Ker_n\Theta : \Theta \in SC(S)\}$ is a complete lattice . We also denote $A(S) = \{ J : J \in I(S) ; J = J^{**}\}$, which is a Boolean lattice.

The following theorems are due to [53]. In fact Cornish proved these results for lattices in [8 , Theorem 2.4 and Theorem 2.5] , which are extensions of the classical Theorem of Hashimoto [16 , Theorem 8, p-9].

Theorem 3.2.9. *Let* S *be a distributive nearlattice with* 0. *Then the following conditions are equivalent.*

(i) S *is disjunctive.*

(ii) *The map* $\Theta \to \ker\Theta$ *of* SC(S) *onto* KSC(S) *is one to one.*

(ii) *The map* $\Theta \to \ker\Theta$ *of* SC(S) *onto* KS(S) *preserves finite joins.*

(iii) *The map* $\Theta \to \ker\Theta$ *is a lattice isomorphism of* SC(S) *onto* KSC(S),

whose inverse is the map $J \to \Theta(J)^{**}$. ●

Theorem 3. 2. 10. *Let* S *be a distributive nearlattice with* 0. *Then the nearlattice* S *is semi-Boolean if and only if the map* $\Theta \rightarrow \text{Ker}\ \Theta$ *is a lattice isomorphism of* SC(S) *onto* KSC(S), *whose inverse is the map* $J \rightarrow \Theta(J)$. ●

Following result has been proved by [2] for $P_n(S)$ when n is central element. We make a slight improvement by considering n as an upper element instead of central element. We omit the proof as it can be proved by using similar technique.

Theorem 3.2.11. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is disjunctive.*

(ii) *The map* $\Theta \rightarrow \text{Ker}_n\ \Theta$ *of* SC(S) *onto* K_n SC(S) *is one-to-one and so is a one-to-one correspondence.*

(iii) *The map* $\Theta \rightarrow \text{Ker}_n\Theta$ *of* SC(S) onto K_n SC(S) *preserves finite joins.*

(iv) *The map* $\Theta \rightarrow \text{Ker}_n\Theta$ *is a lattice isomorphism of* SC(S) *onto* K_nSC(S), *whose inverse is the map* $J \rightarrow \Theta(J)^{**}$ *for any* n- *ideal* J *in* S. ●

We conclude this chapter by the following result which is a generalization of [34,, Th. 3.2.12].

Theorem 3.2.12. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is generalized Boolean.*

(ii) $P_n(S)$ *is semi-Boolean.*

(iii) *The map* $\Theta \to \mathrm{Ker}_n \Theta$ *is a lattice isomorphism of* SC(S) *onto* $K_n SC(S)$, *whose inverse is the map* $J \to \Theta(J)$, J *is an n-ideal of* S.

Proof: By theorem 3.2.7, (i) $\Leftrightarrow$ (ii).

(ii)$\Rightarrow$(iii). Suppose (ii) holds. Then of course $P_n(S)$ is disjunctive and so by theorem 3.2.12, the inverse of $\Theta \to \mathrm{Ker}_n \Theta$ is $J \to \Theta(J)^{**}$.

Now by theorem 3.2.9, $\Theta(J)^{**} = \Theta(J^{++})$ for any $j \in K_n SC(S)$. So due theorem 3.2.5, $j = j^{++}$. Hence $J \to \Theta(J)$ is the inverse of $\Theta \to \mathrm{Ker}_n \Theta$.

Conversely, let $J \to \Theta(J)$ is the inverse of $\Theta \to \mathrm{Ker}_n \Theta$. Then by theorem 3.2.11, $P_n(S)$ is disjunctive and so by theorem 3.2.6, $\mathrm{Ker}_n(\Theta(J)^{**}) = [\mathrm{Ker}_n(\Theta(J))]^{++} = J^{++}$ for any n-ideal J of S. Then by theorem 3.1.4, we have $J^{++} \in K_n SC(S)$. Also we must have $\Theta(J^{++}) = \Theta(\mathrm{Ker}_n(\Theta(J))^{**}) = \Theta(J)^{**}$. Then by theorem 3.2.8, $P_n(S)$ is semi Boolean. ●

Chapter 4

CHAPTER 4

NEAR LATTICES WHOSE FINITELY GENERATED n-IDEALS FORM A GENERALIZED STONE LATTICE

Introduction

Many authors including [3] , [6], [7], [22], [29], [30], [32], [62] and [64] have studied about minimal prime ideals and Stone (generalized) lattices. Chen and Gratzer in [6] and [7] studied the construction and structures of Stone lattices. On the other hand, minimal prime ideals and generalized Stone nearlattices have been studied by [48] .

In this chapter, we introduce the concept of minimal prime n-ideals and generalize some of the results on minimal prime ideals. These results are used to generalize several important results on generalized Stone nearlattices in terms of n-ideals.

A prime n-ideal P is said to be a *minimal prime n-ideal* belonging to n-ideal I if ,

(i) $I \subseteq P$ and

(ii) there exists no prime ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

A prime n-ideal P of a nearlattice S is called a *minimal prime n-ideal* if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. Thus a minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

A distributive lattice L with 0 and 1 is called a *Stone lattice* if it is pseudo-complemented and for each $a \in L$, $a^* \vee a^{**} = 1$.

Also, we know that a distributive pseudocomplemented lattice is a Stone lattice if and only if for each $a , b \in L$, $(a \wedge b)^* = a^* \vee b^*$.

A nearlattice S with 1 is a lattice. So the idea of pseudocomplementation is not possible in case of a general nearlattice.

But for a nearlattice S with 0 , we can talk about sectional pseudocomplemention.

A nearlattice S with 0 is called *sectionally pseudocomplemented* if the interval [0, x] for each $x \in S$ is complemented. Of course, every finite distributive nearlattice is sectionally pseudocomplemented.

A nearlattice S is called *relatively pseudocomplemented* if the interval [a, b] for each $a, b \in S$ with $a < b$ is pseudocomplemented.

A distributive nearlattice S with 0 is called a *generalized Stone* nearlattice if $(x]^{*} \vee (x]^{**} = S$ for each $x \in S$.

It is proved in [48] that, a distributive nearlattice S with 0 is a generalized Stone nearlattice if and only if each interval [0, x], $0 < x \in S$ is a Stone lattice.

In chapter 3, we have already defined that for any n-ideal J of a nearlattice S, $J^{+} = \{ x \in S : m(x, n, j) = n \text{ for all } j \in J\}$

Observe that J^{+} is an n-ideal and $J \cap J^{+} = \{n\}$.

Though we can not talk about pseudocomplementation in a distributive near-lattice S with 0, I(S) the lattice of all ideals of S is pseudocomplemented as it is a distributive algebraic lattice. The n-ideals of S form an algebraic closure system on S and hence under set inclusion, they form an algebraic lattice, which we denote by $I_n(S)$. we have already mentioned that $I_n(S)$ is a distributive lattice if S is distributive. Thus $I_n(S)$ is pseudocomplemented.

From chapter 1, we know that for a distributive nearlattice S with a medial element n, $F_n(S)$ is a distributive lattice with the smallest element $\{n\}$.

Let $< a_1, a_2, .., a_r >_n \in F_n(S)$. By interval $[\{n\} , < a_1, a_2, .., a_r >_n]$ in $F_n(S)$, we mean the set of all finitely generated n-ideals contained in $< a_1, a_2, .., a_r >_n$. $F_n(S)$ is called *sectionally pseudocomplemented* if for each $< a_1, a_2, .., a_r >_n \in F_n(S)$, the interval $[\{n\}, < a_1, a_2, ..., a_r >_n]$ in $F_n(S)$ is pseudo -complemented. That is, each finitely generated n- ideal contained in $< a_1, a_2, ..., a_r >_n$ has a relative pseudocomplement in $[\{n\} , < a_1, a_2, .., a_r >_n]$ which is a member of $F_n(S)$. We shall denote the relative pseudocomplement of $< b_1, b_2, \ldots, b_s >_n$ in any interval by $< b_1, b_2, \ldots, b_s >_n^0$, while $< b_1, b_2, \ldots, b_s >_n^+$ denotes the pseudocomplement of $< b_1, b_2, \ldots, b_s >_n$ in $I_n(S)$.

If $F_n(S)$ is a distributive sectionally pseudocomplemented lattice, then $F_n(S)$ is a generalized Stone lattice if for each $< a_1, a_2, \ldots, a_r >_n \in F_n(S)$, the interval $[\{n\}, < a_1, a_2, \ldots, a_r >_n]$ in $F_n(S)$ is a Stone lattice.

For $b \le a \le n$, if $[b, n]$ is dual pseudocomplemented, then a^{0d} denotes the relative pseudocomplement of a in $[b, n]$. If $[n, d]$ is pseudo complemented, then for $c \in [n, d]$, c^0 denotes the relative pseudocomplement of c in $[n, d]$. Two prime n-ideals P and Q of a nearlattice S are called *co-maximal* if $P \vee Q = S$.

Many authors including Mandelker [38], Varlet [65], Latif [34] and [41] have been studied relative annihilators in lattices and semi-lattices. Also, Cornish [9] has used the annihilators in studying relatively normal lattices. In this chapter we introduce the notion of relative annihilators around a fixed element n of a nearlattice S which is used to generalize several results on relatively Stone nearlattices.

Recall from chapter 3 that for $a, b \in S$, $< a, b > = \{ x \in S : x \wedge a \le b \}$ denotes the relative annihilator. In presence of distributivity, each relative annihilator is an ideal. Also, $< a, b > = < a, a \wedge b >$. Consult [41] and [47] for details on this topic. Again for $a, b \in L$, where L is a lattice, $< a, b >_d = \{ x \in L : x \vee a \ge b \}$, denotes the relative dual annihilator. In presence of distributivity of L, $< a, b >_d$ is a dual ideal (filter).

In case of a nearlattice, it is not possible to define a dual relative annihilator ideal for any a and b. But if n is an upper element of S, then $x \vee n$ exists for all $x \in S$. Then for any $a \in (n]$, we can talk about dual relative annihilator ideal of $< a, b >_d$ for any $b \in S$. That is, for any $a \le n$ in S, $< a, b >_d = \{x \in S : x \vee a \ge b\}$.

For any $a, b \in S$ and upper element $n \in S$, we define

$< a, b >^n = \{ x \in S : m(a, n, x) \in < b >_n \}$

$= \{x \in S : b \wedge n \leq m(a, n, x) \leq b \vee n\}$ is called the *annihilator of* a *relative to* b *around the element* n or simply a *relative* n- *annihilator*. It is easy to see that for all $a, b \in S$, $< a, b >^n$ is always a convex subset containing n. In presence of distributivity it can be easily seen that $< a, b >^n$ is an n-ideal. If $0 \in S$, then putting n = 0, we have $< a, b >^n = < a, b >$.

For two n-ideals A and B of a nearlattice S,

$< A, B >$ denotes $\{x \in S : m(a, n, x) \in B$ for all $a \in A\}$, When n is medial element. In presence of distributivity, clearly $< A, B >$ is an ideal. Moreover, we can easily show that $< a, b >^n = << a >_n, < b >_n >$.

In section 1, we have studied minimal prime n-ideals of S. Here we have given some characterizations of minimal prime n-ideals. We have also shown that for an upper element n of a distributive nearlattice S, $F_n(S)$ is sectionally pseudocomplemented if and only if $(n]$ is sectionally dual pseudocomplemented and $[n)$ is sectionally pseudocomplemented.

In section 2, we have given several characterizations of those $F_n(S)$ which are generalized Stone nearlattices in terms of n-ideals. We proved that for an upper element n of a distributive nearlattice S, $F_n(S)$ is generalized Stone if and only if $(n]$ is dual generalized Stone and $[n)$ is generalized Stone. Then we have given several characterizations of those $F_n(S)$ which are generalized Stone.

In section 3, we have studied annihilator n-ideals $< a, b >^n$. We have give some characterizations of distributive and modular nearlattices in terms of relative annihilators. We prove that for an upper and neutral element n of a nearlattice S, $(n]^d$ and $[n)$ are modular nearlattices if and only if $P_n(S)$ is a modular nearlattice.

In section 4, we have given several characterizations of those $F_n(S)$ which are relatively Stone in terms of n-ideals and relative annihilators. We have shown that when n is an upper element of S, then $F_n(S)$ is relatively Stone if and only if any two incomparable prime n-ideals P and Q are co-maximal. That is $P \vee Q = S$.

4.1. Minimal prime n-ideals

Recall that a prime n-ideal P is a *minimal prime* n-*ideal belonging to* n-*ideal* I if ,

(i) I ⊆ P and

(ii) there exists no prime n -ideal Q such that Q ≠ P and I ⊆ Q ⊆ P.

Thus a prime n-ideal P of S is a *minimal prime* n-*ideal* if there exists no prime n-ideal Q such that Q ≠ P and Q ⊆ P. In other words, a minimal prime n-ideal is a minimal prime n-ideal belonging to {n}.

Recall that an element n of a nearlattice S is medial if m(x, n, y) exists for all x , y ∈ S. Since for the definition of prime n-ideal of S, the medial property of n is essential, so in talking about prime n-ideals of S, we will always assume n as a medial element.

We start this section with the following result due to [2] which is a generalization of a well known result on lattice theory.

Lemma 4.1.1. *Let* S *be a nearlattice with a medial element* n. *Then every prime n-ideal contains a minimal prime n-ideal.* ●

Theorem 4.1.2. *Let* S *be a distributive nearlattice with a sesquimedial element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is sectionally pseudocomplemented.*

(ii) $P_n(S)$ *is sectionally pseudocomplemented.*

Moreover, when n *is upper , then both* (i) *and* (ii) *are equivalent to*

(iii) (n] *is sectionally dual pseudocomplemented and* [n) *is sectionally pseudocomplemented.*

Proof: (i) ⇒ (ii) . Suppose (i) holds. Let $\{n\} \subseteq \langle a \rangle_n \subseteq \langle b \rangle_n$. Since (i) holds, so there exist $\langle t_1, t_2, \ldots, t_r \rangle_n$ in $F_n(S)$ with $\langle t_1, t_2, \ldots, t_r \rangle_n \subseteq \langle b \rangle_n$ which is the sectional pseudocomplement of $\langle a \rangle_n$ in $[\{n\}, \langle b \rangle_n]$. But by corollary 1.4.5, $\langle t_1, t_2, \ldots, t_r \rangle_n = \langle c \rangle_n$ for some c ∈ S. This implies $\langle c \rangle_n$ is the sectional pseudocomplement of $< a >_n$ in $[\{n\}, < b >_n]$. Therefore, $P_n(S)$ is sectionally pseudocomplemented.

(ii) ⇒ (i). Suppose (ii) holds. Let $\{n\} \subseteq < a_1,a_2,...,a_r >_n \subseteq < b_1, b_2, ...,b_s >_n$. Then $\{n\} \subseteq < a_1,a_2,.......,a_r >_n \cap < b_t >_n \subseteq < b_t >_n$ for some $t = 1,2,3,...,s$. Then by corollary 1.4.5, $< a_1,a_2,....,a_r >_n \cap < b_t >_n$ is principal and let $< p_t >_n = < a_1,a_2,.......,a_r >_n \cap < b_t >_n$. So, there exist $< c_t >_n$, with $\{n\} \subseteq < c_t >_n \subseteq < b_t >_n$ such that $< c_t >_n$ is the sectional pseudocomplement of $< p_t >_n$. So,

$< a_1,a_2,....,a_r >_n \cap < b_t >_n \cap < c_t >_n = \{n\}$.

That is, $< a_1,a_2,....,a_r >_n \cap < c_1,c_2,.......,c_s >_n = \{n\}$.

More over, $< c_1,c_2,.......,c_s >_n \subseteq < b_1,b_2,.......,b_s >_n$. Now let $< q_1,q_2,....,q_l >_n \subseteq < b_1,b_2,.......,b_s >_n$ and $< q_1,q_2,....,q_l >_n \cap < a_1,a_2,....,a_r >_n = \{n\}$.

Since $< q_1,q_2,....,q_l >_n \cap < b_1 >_n \subseteq < b_1 >_n$.

Then $< q_1,q_2,....,q_l >_n \cap < b_1 >_n \cap < a_1,a_2,.......,a_r >_n = \{n\}$ implies $< q_1,q_2,....,q_l >_n \cap < b_1 >_n \subseteq < c_1 >_n$ by corollary 1.4.5. Similarly, we can get,

$< q_1,q_2,....,q_l >_n \cap < b_1,b_2,....,b_s >_n \subseteq < c_1, c_2,.....,c_s >_n$

That is, $< q_1,q_2,....,q_l >_n \subseteq < c_1,c_2,....,c_s >_n$.

Hence $< c_1,c_2,....,c_s >_n$ is the sectional pseudocomplement of $<a_1,a_2,...,a_r >_n$ in $[\{n\} , < b_1,b_2,....,b_s >_n]$.

Therefore, $F_n(S)$ is sectionally pseudocomplemented.

(ii)⇒ (iii). Suppose (ii) holds. Let $a \leq b \leq n$. Then $\{n\} \subseteq < b_1 >_n \subseteq < a_1 >_n$. Then $\{n\} \subseteq [b , n] \subseteq [a , n]$. Since (ii) holds, so there exists $< t_1 >_n$ such that $< t_1 >_n$ is the sectional pseudocomplement of $< b >_n$ in $[\{n\} , < a >_n]$. Then $\{n\} \subseteq < t >_n \subseteq [a , n]$ implies $a \leq t \leq n$.

Now $[b, n] \cap [t, n] = \{n\}$. That is, $[b \vee t, n] = \{n\}$ which implies $b \vee t = n$.

Let $c \in S$ with $a \leq c \leq n$ such that $b \vee c = n$. This implies that $< b >_n \cap < c >_n = \{n\}$. Since $< t >_n$ is the sectional pseudocomplement of $< b >_n$, so $< c >_n \subseteq < t >_n$. That is, $[c, n] \subseteq [t, n]$ which implies $t \leq c$.

Therefore, $(n]$ is sectionally dual pseudocomplemented.

A dual proof of (ii)$\Rightarrow$(iii) shows that $[n)$ is sectionally pseudocomplemented.

Finally, (iii) $\Rightarrow$ (ii). Suppose (iii) holds. Let $\{n\} \subseteq < a >_n \subseteq < b >_n$.

Then $b \wedge n \leq a \wedge n \leq n \leq a \vee n \leq b \vee n$. By (iii), $[n)$ is sectionally pseudocomplemented. So, $n \leq a \vee n \leq b \vee n$. Then there exists t, $n \leq t \leq b \vee n$ such that t is the sectional pseudocomplement of $a \vee n$ in $[n, b \vee n]$.

Hence $t \wedge (a \vee n) = n$.

Again $b \wedge n \leq a \wedge n \leq n$. Since $(n]$ is sectionally dual pseudocomplemented, so there exists s with $b \wedge n \leq s \leq n$ such that s is a sectional dual pseudocomplement of $a \wedge n$ in $[b \wedge n, n]$. Then $s \vee (a \wedge n) = n$.

Now, $\{n\} \subseteq < s, t >_n \subseteq < b >_n$ implies $\{n\} \subseteq ([s, n] \vee [n, t]) \subseteq < b >_n$ and so $\{n\} \subseteq ([s \wedge n, t \vee n]) \subseteq < b >_n$. Thus, $\{n\} \subseteq [s, t] \subseteq < b >_n$ and so by corollary 1.4.5, $[s, t]$ is a principal n-ideal. We will show that this is the required sectional pseudocomplement of $< a >_n$ in $[\{n\}, < b >_n]$.

Now, $[s, t] \cap < a >_n$.

$$= [s, t] \cap [a \wedge n, a \vee n]$$

$$= [s \vee (a \wedge n), t \wedge (a \vee n)]$$

$$= \{n\}.$$

Let there exists $< p >_n$ in $[\{n\}, < b >_n]$ such that $< a >_n \cap < p >_n = \{n\}$.

This implies $[a \wedge n, a \vee n] \cap [p \wedge n, p \vee n] = \{n\}$.

That is, $[(a \wedge n) \vee (p \wedge n), (a \vee n) \wedge (p \vee n)] = \{n\}$.

This implies $(a \vee n) \wedge (p \vee n) = n$ and $(a \wedge n) \vee (p \wedge n) = n$.

and so $p \vee n \leq t$ and $p \wedge n \geq s$ and so $< p >_n \subseteq [s, t]$.

Therefore, $[s, t]$ is the sectionally pseudocomplemented of $< a >_n$ in $[\{n\}, < b >_n]$. Therefore, $[\{n\}, < b >_n]$ is pseudocomplemented in $P_n(S)$. Hence $P_n(S)$ is sectionally pseudocomplemented. ●

Now we give a characterization of minimal prime n-ideals of a distributive nearlattice S, when $F_n(S)$ is sectionally pseudocomplemented. To do this, we need the following lemmas.

Lemma 4.1.3. [2] *Let* S *be distributive nearlattice and* $n \in S$ *be a medial element. Then for any* $< a >_n \in P_n(S)$ *and for any* n-*ideal* I, $(I \cap < a >_n)^+ \cap < a >_n = I^+ \cap < a >_n$. ●

Lemma 4.1.4. *Suppose* $P_n(S)$ *is sectionally pseudocomplemented distributive nearlattice and* $< b >_n \subseteq < a >_n$ *in* $P_n(S)$ *then,*

(i) $< b >_n^0 = < b >_n^+ \cap < a >_n$ *and*

(ii) $< b >_n^{00} = < b >_n^{++} \cap < a >_n$.

Proof: (i) is trivial. For (ii), Using (i), we have $< b >_n^{00} = (< b >_n^0)^+ \cap < a >_n = (< b >_n^+ \cap < a >_n)^+ \cap < a >_n$. Thus by 4.1.2, $< b >_n^{00} = < b >_n^{++} \cap < a >_n$. ●

Theorem 4.1.5. *Let* n *be a sesquimedial element of a distributive nearlattice* S. *Suppose* $F_n(S)$ *is a sectionally pseudocomplemented distributive lattice and* P *is a prime* n-*ideal* of S. *The following conditions are equivalent.*

(i) P *is minimal*

(ii) $x \in P$ *implies* $< x >_n^+ \nsubseteq P$.

(iii) $x \in P$ *implies* $< x >_n^{++} \subseteq P$.

(iv) $P \cap D(< t >_n) = \Phi$ *for all* $t \in S - P$,

where $D(< t >_n) = \{x \in < t >_n : < x >_n^0 = \{n\}\}$.

Proof: (i) ⇒ (ii). Suppose P is minimal. If (ii) fails, then there exists $x \in P$ such that $< x >_n^+ \subseteq P$. Since P is prime n-ideal, so by theorem 2.1.2, P is a prime ideal or a prime filter. Suppose P is a prime ideal. Let $D = (S - P) \vee (x]$. We claim that $n \notin D$.

If $n \in D$, then $n = q \wedge x$ for some $q \in S - P$. Then $<q>_n \cap <x>_n = <(q \wedge x) \vee (q \wedge n) \vee (x \wedge n)>_n = \{n\}$ implies $<q>_n \subseteq <x>_n^+ \subseteq P$. Thus $q \in D$, which is a contradiction. Hence $n \notin D$. Then by theorem 2.1.7, there exists a prime n-ideal Q with $Q \cap D = \Phi$. Then $Q \subseteq P$ as $Q \cap (S - p) = \Phi$ and $Q \neq P$, since $x \notin Q$. But this contradicts the minimality of P. Hence $<x>_n^+ \subseteq P$. Similarly, we can prove that $<x>_n^+ \subseteq P$ if P is prime filter.

(ii)$\Rightarrow$ (iii). Suppose (ii) holds and $x \in P$. Then $<x>_n^+ \not\subseteq P$. Since $<x>_n^+ \cap <x>_n^{++} = \{n\} \subseteq P$ and P is prime, so, $<x>_n^{++} \subseteq P$.

(iii)$\Rightarrow$ (iv). Suppose (iii) holds and $t \in S - P$. Let $x \in P \cap D(<t>_n)$. Then $x \in P$, $x \in D(<t>_n)$. Thus, $<x>_n^0 = \{n\}$ and so $<x>_n^{00} = <t>_n$. By (iii), $x \in P$ implies $<x>_n^{++} \subseteq P$. Also by lemma 4.1.4, $<x>_n^{00} = <x>_n^{++} \cap <t>_n$.

Hence $<x>_n^{++} \cap <t>_n = <t>_n$ and so $<t>_n \subseteq <x>_n^{++} \subseteq P$. That is, $t \in P$, which is a contradiction. Therefore, $P \cap D(<t>_n) = \Phi$ for all $t \in S - P$.

(iv) $\Rightarrow$ (i). Suppose P is not minimal. Then there exists a prime n-ideal $Q \subset P$. Let $x \in P - Q$.

Since Q is prime, so $<x>_n \cap <x>_n^+ = \{n\} \subseteq Q$ implies that $<x>_n^+ \subseteq Q \subset P$. Chose any $t \in S - P$. Then $<t>_n \cap (<x>_n \vee <x>_n^+) \subseteq P$.

Now, $<t>_n \cap (<x>_n \vee <x>_n^+) = (<t>_n \cap <x>_n) \vee (<t>_n \cap <x>_n^+)$

$= <m(t, n, x)>_n \vee ((<t>_n \cap <x>_n)^+ \cap <t>_n)$ by lemma 4.1.3.

$= <m(t, n, x)>_n \vee (<m(t, n, x)>_n^+ \cap <t>_n)$

$= <m(t, n, x)>_n \vee <m(t, n, x)>_n^0$, [by lemma 4.1.4] , where $<m(t, n, x)>_n^0$ is the relative pseudocomplement of $<m(t, n, x)>_n$ in $[\{n\}, <t>_n]$. Since $F_n(S)$ is sectionally pseudocomplemented , so by theorem 4.1.2, $P_n(S)$ is also sectionally pseudocomplemented. Thus $<m(t, n, x)>_n^0$ is principal and so $<m(t, n, x)>_n \vee <m(t, n, x)>_n^0$ is finitely generated n-ideal contained in $<t>_n$. Therefore by corollary 1.4.5,

$< m(t, n, x) >_n \vee < m(t, n, x) >_n^0 = < r >_n$, for some $r \in < t >_n$,

Moreover, $< r >_n^0 = < m(t, n, x) >_n^0 \cap < m(t, n, x) >_n^{00} = \{n\}$.

Thus, $r \in P \cap D(< t >_n)$, which is a contradiction. Therefore P must be minimal.●

4.2. Some generalizations of the results on generalized Stone Nearlattices

For any $n \leq b \leq 1$, b^* denotes the relative pseudocomplement of b in[n , 1], while for $s \leq a \leq n$, a^{0d} denotes the relative dual pseudocomplement of a in [s, n].

Theorem 4.2.1. *Let* n *be an upper element and* $F_n(S)$ *be a sectionally pseudocomplemented distributive lattice .Then for*

$\{n\} \subseteq \langle a \rangle_n \subseteq \langle b \rangle_n$,

$\langle a \rangle_n^0 = [a\wedge n, a\vee n]^0 = [(a\wedge n)^{0d}, (a\vee n)^0]$.

Proof : Since $F_n(S)$ is sectionally pseudocomplemented , so by theorem 4.1.2 [a, n] is sectionally dual pseudocomplemented and [n, b] is sectionally pseudocomplemented . Here $b\wedge n \leq a\wedge n \leq n \leq a\vee n \leq b\vee n$. Since $(a\wedge n)^{0d}$ is the relative dual pseudocomplement of $(a\wedge n)$ in $[b\wedge n , n]$ and $(a\vee n)^0$ is the relative pseudocomplment of $a\vee n$, in $[n , b\vee n]$, so $[a\wedge n , a\vee n]\cap [(a\wedge n)^{0d} , (a\vee n)^0]$ $= [(a\wedge n) \vee (a\wedge n)^{0d}, (a\vee n) \wedge (a\vee n)^0] = [n, n] = \{n\}$.

Now let $t\in \langle a\rangle_n^0$. Then $[t\wedge n , t\vee n] \subseteq \langle a \rangle_n^0$. Thus $\{n\} = [t\wedge n, t\vee n]\cap [a\wedge n , a\vee n] = [(t\wedge n\vee(a\wedge n) , (t\vee n) \wedge (a\vee n)]$ and so $(t\wedge n)\vee(a\wedge n) = n = (t\vee n) \wedge (a\vee n)$. This implies $t\wedge n \geq (a\wedge n)^{0d}$ and $(t\vee n) \leq (a\vee n)^0$. Hence $[t\wedge n , t\vee n] \subseteq [(a\wedge n)^{0d}, (a\vee n)^0]$ and so $\langle a \rangle_n^0 \subseteq [(a\wedge n)^{0d}, (a\vee n)^0]$.

Therefore , $\langle a \rangle_n^0 = [a\wedge n , a\vee n]^0 = [(a\wedge n)^{0d} , (a\vee n)^0]$. ●

Suppose S is a distributive lattice with 0 and 1. For $n \leq b \leq 1$, let b^+ denotes the pseudocomplement of b in [n, 1] and for $0 \leq a \leq n$, suppose a^{+d} denotes the dual pseudocomplement of a in [0, n]. Thus we have the following result.

Corollary 4.2.2. *Let* L *be a distributive lattice with* 0 *and* 1 with $n \in L$ *and* $F_n(L)$ *is a pseudocomplemented distributive lattice. Then for any* $[a, b] \in F_n(L)$,

$[a, b]^+ = [a^{+d}, b^{+d}]$.

Proof: $[a, b]^+ = ([a, n] \vee [n, b])^+$

$= [a, n]^+ \cap [n, b]^+$

$= [a^{+d}, 1] \cap [n^{+d}, b^+]$ (by 4.2.1)

$= [a^{+d}, 1] \cap [0, b^+]$

$= [a^{+d}, b^+]$. ●

Recall that a distributive nearlattice S with 0 is a *generalized Stone near- lattice* if for each $x \in L$, $(x]^* \vee (x]^{**} = S$. By Katrinak [30] , S with 0 is a generalized Stone nearlattice if the interval $[0, x]$ is Stone for each $x \in S$. Generalized Stone lattices have been studied by many authors including [9] , [29] , [30] and [1]. Then [48] have extended their work for nearlattices. Following lemma is needed to prove one of the main result of the chapter.

Lemma 4.2.3. *Suppose* $F_n(S)$ *is a sectionally pseudocomplemented distributive nearlattice and* n *be a medial element of* S. *Let* $x, y \in S$ *with* $< x >_n \cap < y >_n = \{n\}$. *Then the following conditions are equivalent.*

(i) $< x >_n^+ \vee < y >_n^+ = S$.

(ii) *For any* $t \in S$, $< m(x, n, t) >_n^0 \vee < m(y, n, t) >_n^0 = < t >_n$ *where* $< m(x, n, t) >_n^0$ *denotes the relative pseudocomplement of* $< m(x, n, t) >_n$ *in* $[\{n\}, < t >_n]$.

Proof: (i) ⇒ (ii). Suppose (i) holds. Then for any $t \in S$,

$< m(x, n, t) >_n^0 \vee < m(y, n, t) >_n^0 = (< x >_n \cap < t >_n)^0 \vee (< y >_n \cap < t >_n)^0$

$= (< x >_n \cap < t >_n)^+ \cap < t >_n) \vee ((< y >_n \cap < t >_n)^+ \cap < t >_n)$. [by lemma 4.1.4.].

$= (< x >_n^+ \cap < y >_n^+) \vee (< y >_n^+ \cap < t >_n)$[by lemma 4.1.3]

$= (< x >_n^+ \vee < y >_n^+) \cap < t >_n$.

$= S \cap < t >_n \; = < t >_n.$

Hence (ii) holds.

(iii) $\Rightarrow$(i). Suppose (ii) holds and $t \in S$.

By (ii), $< m(x, n, t) >_n^0 \vee < m(y, n, t) >_n^0 = < t >_n$. Then using lemma 4.1.3 and 4.1.4 and the calculation of (i) $\Rightarrow$(ii) , we get $(< x >_n^+ \vee < y >_n^+) \cap < t >_n = < t >_n$. This implies $< t >_n \subseteq < x >_n^+ \vee < y >_n^+$ and so, $t \in < x >_n^+ \vee < y >_n^+$.

Therefore, $< x >_n^+ \vee < y >_n^+ = S$. ●

Following lemma is needed to prove our next theorem which is a key result of this chapter.

Lemma 4.2.4. *Suppose* $F_n(S)$ *and* $P_n(S)$ *are sectionally pseudocomple- mented. Let* $\{n\} \subseteq <a_1, a_2, \ldots, a_r>_n \subseteq <b_1, b_2, \ldots, b_s>_n$ and $< a_1, a_2, \ldots, a_r >_n^0$ *represents the relative pseudocomplement of* $< a_1, a_2, \ldots, a_r >_n$ *in* $[\, \{n\} , < b_1, b_2, \ldots, b_s >_n$. *If for each* $t = 1,2,3,\ldots,s$ $\; < c_t >_n$ *and* $< d_t >_n$ *represent the relative pseudocomplement and the double relative pseudocomplement of the principal* n*-ideal* $< a_1, a_2, \ldots, a_r >_n \cap < b_t >_n$ *in interval* $[\, \{n\} , < b_t >_n]$, *then*

(i) $<a_1, a_2, \ldots, a_r>_n^0 \cap < b_t >_n = < c_t >_n$

(ii) $< a_1, a_2, \ldots, a_r >_n^{00+} \cap < b_t >_n = < d_t >_n$, $t = 1,2,\ldots,s$.

Proof:(i). Since $< c_t >_n$ is the relative pseudocomplement of $<a_1, a_2, \ldots, a_r >_n \cap < b_t >_n$, so $<a_1, a_2, \ldots, a_r>_n \cap < b_t >_n \cap < c_t >_n = \{n\}$.

That is , $<a_1, a_2, \ldots, a_r>_n \cap < c_t >_n = \{n\}$. Moreover, $\{n\} \subseteq < c_t >_n \subseteq < b_t >_n \subseteq <b_1, b_2, \ldots, b_s >_n$. Thus, $< c_t >_n \subseteq <a_1, a_2, \ldots, a_r >_n^0$.Therefore, R.H.S.$\subseteq$ L..H.S.

For the reverse inclusion, $<a_1, a_2, \ldots, a_r >_n^0 \cap < b_t >_n \in F_n(S)$ as $F_n(S)$ is sectionally pseudocomplemented . Therefore by corollary 1.4.5,

$<a_1, a_2, \ldots, a_r >_n^0 \cap < b_t >_n \in P_n(S)$.

Now, $(<a_1, a_2, \ldots, a_r >_n \cap < b_t >_n) \cap (<a_1, a_2, \ldots, a_r >_n^0 \cap < b_t >_n)$

$= <a_1, a_2, \ldots, a_r >_n \cap <a_1, a_2, \ldots, a_r >_n^0 \cap < b_t >_n$.

$= \{n\}$.

This implies $<a_1, a_2, \ldots\ldots, a_r>_n^0 \cap <b_t>_n \subseteq <c_t>_n$.

Therefore, $<a_1, a_2, \ldots\ldots, a_r>_n^0 \cap <b_t>_n = <c_t>_n$.

(ii). $<c_t>_n \cap <d_t>_n = \{n\}$.

So, $<a_1, a_2, \ldots, a_r>_n^0 \cap <b_t>_n \cap <d_t>_n = \{n\}$.Then

Or, $<a_1, a_2, \ldots, a_r>_n^0 \cap <d_t>_n = \{n\}$, which implies

$<d_t>_n \subseteq <a_1, a_2, \ldots, a_r>_n^{00}$, and so $<d_t>_n \subseteq <a_1, a_2, \ldots, a_r>_n^{00} \cap <b_t>_n$.

Conversely, $<a_1, a_2, \ldots\ldots, a_r>_n^{00} \cap <a_1, a_2, \ldots\ldots, a_r>_n^0 \cap <b_t>_n = \{n\}$, implies $(<a_1, a_2, \ldots\ldots, a_r>_n^{00} \cap <b_t>_n) \cap <c_t>_n = \{n\}$ (by (i)).

Therefore, $<a_1, a_2, \ldots\ldots, a_r>_n^{00} \cap <b_t>_n \subseteq$ relative pseudocomplement of $<c_t>_n$ in $[\{n\}, <b_t>_n]$. Thus $<a_1, a_2, \ldots\ldots, a_r>_n^{00} \cap <b_t>_n \subseteq <d_t>_n$ and so (ii) holds. ●

Lemma 4.2.5. *Suppose* n *is an upper element of a distributive nearlattice* S. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is generalized Stone.*

(ii) $P_n(S)$ *is generalized Stone.*

(iii) $(n]$ *is dual generalized Stone and* $[n)$ *is generalized Stone.*

Proof: (i) ⇒ (ii). Suppose (i) holds. So $P_n(S)$ is sectionally pseudocomple -mented by 4.1.2. Let us consider $\{n\} \subseteq <a>_n \subseteq <b>_n$. Then by (i), obviously $<a>_n^0 \vee <a>_n^{00} = <b>_n$.This implies (ii) holds.

(ii)⇒ (i). Let $\{n\} \subseteq <a_1, a_2, \ldots, a_r>_n \subseteq <b_1, b_2, \ldots b_s>_n$.If for each $t = 1,2,3,\ldots,s$, $<c_t>_n$ is the relative pseudocomplement of $<a_1, a_2, \ldots\ldots, a_r>_n \cap <b_t>_n$ in $[\{n\}, <b_t>_n]$ and $<d_t>_n$ is double relative pseudocomplement of $<a_1, a_2, \ldots, a_r>_n \cap <b_t>_n$ in $[\{n\}, <b_t>_n]$, then

$$(<a_1, a_2, \ldots, a_r>_n^0 \vee <a_1, a_2, \ldots, a_r>_n^{00}) \cap <b_t>_n$$
$$= (<a_1, a_2, \ldots, a_r>_n^0 \cap <b_t>_n) \vee (<a_1, a_2, \ldots, a_r>_n^{00} \cap <b_t>_n)$$

$= <c_t>_n \vee <d_t>_n$ by lemma 4.2.4,

$= <b_t>_n$ as $P_n(S)$ is generalized Stone.

Therefore, $<a_1, a_2, \ldots, a_r>_n^0 \vee <a_1, a_2, \ldots, a_r>_n^{00}$

$= (<a_1, a_2, \ldots, a_r>_n^0 \vee <a_1, a_2, \ldots, a_r>_n^{00}) \cap <b_1, b_2, \ldots, b_s>_n$

$= <b_1>_n \vee <b_2>_n \vee \ldots \vee <b_s>_n$ by distributivity.

$= <b_1, b_2, \ldots, b_s>_n$.

Hence $[\{n\}, <b_1, b_2, \ldots, b_s>_n]$ is Stone and so $F_n(S)$ is generalized Stone.

(ii)⇒ (iii) . Let $P_n(S)$ be generalized Stone .Consider the interval $[n, b]$ and $n \leq a \leq b$. Thus, $\{n\} \subseteq <a>_n \subseteq <b>_n$. Since $P_n(S)$ is generalized Stone, so $<a>_n^0 \vee <a>_n^{00} = <b>_n$, where $<a>_n^0$ and $<a>_n^{00}$ are the relative and double relative pseudocomplements respectively of $<a>_n$ in $[\{n\}, <b>_n]$. Thus by theorem 4.2.1, $[n, b] = [n, a]^0 \vee [n, a]^{00} = [n, a^0] \vee [n, a^{00}] = [n, a^0 \vee a^{00}]$ where a^0 and a^{00} are the relative double relative pseudocomplement respectively of a in $[n, b]$, and so, $a^0 \vee a^{00} = b$. This implies $[n)$ is generalized Stone.

By dual proof of above we can prove that $(n]$ is dual generalized Stone.

(iii) ⇒ (ii). Suppose (iii) holds. Let $\{n] \subseteq <a>_n \subseteq <b>_n$. Then $b \wedge n \leq a \wedge n \leq n \leq a \vee n \leq b \vee n$. Since $[n)$ is generalized Stone, so there exists $(a \vee n)^0$ such that $(a \vee n)^0 \vee (a \vee n)^{00} = b \vee n$. Similarly, as $(n]$ is dual generalized Stone, so there exists $(a \wedge n)^{0d}$ such that $(a \wedge n)^{0d} \wedge (a \wedge n)^{00d} = b \wedge n$. Then by Theorem 4.2.1, $<a>_n^0 \vee <a>_n^{00} = [a \wedge n, a \vee n]^0 \vee [a \wedge n, a \vee n]^{00} = [(a \wedge n)^{0d}, (a \vee n)^0] \vee [(a \wedge n)^{00d}, (a \vee n)^{00}] = [(a \wedge n)^{0d} \wedge (a \wedge n)^{00d}, (a \vee n)^0 \vee (a \vee n)^{00}] = [b \wedge n, b \vee n] = <b>_n$. This implies $P_n(S)$ is generalized Stone. ●

Using above theorem we give the following nice characterization of those $F_n(S)$ which are generalized Stone. This characterization has been shown by [2] for $P_n(S)$

when n is a central element .So our result is an improvement of their result. Of course, the proof of most part of the theorem follows the same technique as the proof of [2, Th.4.2.6].

Theorem 4.2.6. *Let* n *be an upper element of* S *and* $F_n(S)$ *be a sectionally pseudocomplemented distributive nearlattice. Then the following conditions are equivalent.*

(i) $F_n(S)$ *is generalized Stone.*

(ii) $P_n(S)$ *is generalized Stone*

(iii) *For any* $x \in S$, $<x>_n^+ \vee <x>_n^{++} = S$.

(iv) *For all* $x, y \in S$, $(<x>_n \cap <y>_n)^+ = <x>_n^+ \vee <y>_n^+$

(v) *For all* $x, y \in S$, $<x>_n \cap <y>_n = \{n\}$ *implies that* $<x>_n^+ \vee <y>_n^+ = S$

Proof: (i) $\Leftrightarrow$ (ii) follows by theorem 4.2.4,

(ii)$\Rightarrow$ (iii). Suppose (ii) holds and $t \in S$, then for any $x \in S$, $m(x, n, t) \in <t>_n$ and so $<m(t, n, x)>_n \in [\{n\}, <t>_n]$. Since $P_n(S)$ is generalized Stone, so

$<m(t, n, x)>_n^0 \vee <m(t, n, x)>_n^{00} = <t>_n$.

Then by lemma 4.1.4,

$$<t>_n = (<m(t, n, x)>_n^+ \cap <t>_n) \vee <m(t, n, x)>_n^{++} \cap <t>_n)$$
$$= ((<x>_n \cap <t>_n)^+ \cap <t>_n) \vee ((<x>_n \cap <t>_n)^{++} \cap <t>_n)$$

Thus by lemma 4.1.3,

$$<t>_n = (<x>_n^+ \cap <t>_n) \vee (<x>_n^{++} \cap <t>_n).$$

Thus $<t>_n = (<x>_n^+ \vee <x>_n^{++}) \cap <t>_n$.

This implies $<t>_n \subseteq <x>_n^+ \vee <x>_n^{++}$ and so $t \in <x>_n^+ \vee <x>_n^{++}$.

Therefore, $<x>_n^+ \vee <x>_n^{++} = S$.

(iii)$\Rightarrow$ (iv). Suppose (iii) holds. For any $x, y \in S$,

$$(<x>_n \cap <y>_n) \cap (<x>_n^+ \vee <y>_n^+)$$

$= (<x>_n \cap <y>_n \cap <x>_n^+) \vee (<x>_n \cap <y>_n \cap <y>_n^+$

$= \{n\} \vee \{n\} = \{n\}.$

Now let $<x>_n \cap <y>_n \cap I = \{n\}$ for some n-ideal I.

Then $<y>_n \cap I \subseteq <x>_n^+$. Meeting $<x>_n^{++}$ with both sides, we have,

$<y>_n \cap I \cap <x>_n^{++} = \{n\}$. This implies $I \cap <x>_n^{++} \subseteq <y>_n^+$.

Hence $I = I \cap S = I \cap (<x>_n^+ \vee <x>_n^{++})$

$= (I \cap <x>_n^+) \vee (I \cap <x>_n^{++})$

$\subseteq (<x>_n^+ \vee <y>_n^+)$

Therefore, $<x>_n^+ \vee <y>_n^+ = (<x>_n \cap <y>_n)^+$.

(iv)$\Rightarrow$ (v). Let $<x>_n \cap <y>_n = \{n\}$ for some $x, y \in S$.

Then by (iv), $S = \{n\}^+ = (<x>_n \cap <y>_n)^+$

$= <x>_n^+ \vee <y>_n^+$. Thus (v) holds.

To complete the proof we shall show that (v) $\Rightarrow$ (i). Suppose (v) holds. Since $F_n(S)$ is sectionally pseudocomplemented, so by theorem 4.2.1, $(n]$ is sectionally dual pseudocomplemented and $[n)$ is sectionally pseudo -complemented.

Suppose $n \leq b \leq d$. Let b^0 be the relative pseudocomplement of b in [n, d]. Now $b^0 \wedge b^{00} = n$. Thus $<b^0>_n \cap <b^{00}>_n = [n, b^0 \wedge b^{00}] = [n, n] = \{n\}$.

Also, $<b^0>_n, <b^{00}>_n \subseteq <d>_n$. Then by equivalent conditions of (v) given in lemma 4.2.3, we have $<m(b^0, n, d)>_n^0 \vee <m(b^{00}, n, d)>_n^0 = <d>_n$. But $m(b^0, n, d) = b^0$ and $m(b^{00}, n, d) = b^{00}$ as $n \leq b^0$, $b^{00} \leq d$.

But by corollary 4.2.2,

$< b^0 >_n^0 = < b^{00} >_n$ and $< b^{00} >_n^0 = < b^{000} >_n = < b^0 >_n$.

Therefore, $< d >_n = < b^{00} >_n \vee < b^0 >_n = < b^0 \;\; b^{00} >_n$ which gives $b^0 \vee b^{00} = d$.

This implies [n, d] is a Stone lattice .That is [n) is generalized Stone .

A dual proof of above shows that (v) also implies that (n] is a dual generalized Stone lattice. Therefore by lemma 4.2.5, F_n (S) is generalized Stone . ●

Following corollary is an immediate consequence of above result.

Corollary 4.2.7. *Let* n *be any element of a distributive lattice* L *with* 0 *and* 1 *and let* $F_n(L)$ *be a pseudocomplemented distributive lattice. Then the following conditions are equivalent.*

(i) $F_n(L)$ *is Stone.*

(ii) *For all* $x \in L$, $< x >_n^+ \vee < x >_n^{++} = L$

(iii) *For all* $x, y \in L$, $(<x>_n \cap < y >_n) = < x >_n^+ \vee < y >_n^+$

(iv) *For all* $x, y \in L$, $< x >_n \cap < y >_n = \{n\}$ *implies that* $<x>_n^+ \vee < y >_n^+ = L$. ●

For a prime ideal P of a distributive nearlattice S with 0, we define $0(P) = \{ x \in S: x \wedge y = 0$ for some $y \in S - P\}$. Clearly, 0(P) is an ideal and $0(P) \subseteq P$. 0(P) is the intersection of all the minimal prime ideals of S which are contained in P.

For a prime n-ideal P of a distributive nearlattice S, we write, $n(P) = \{y \in S:$

$m(y, n, x) = n$ for some $x \in S - P\}$. Clearly, n(P) is an n-ideal and $n(P) \subseteq P$.

Following results are due to [2] which are generalizations of [56, Lemma 3.2.1] and [56, Proposition 3.2.2]

Lemma 4.2.8 *Let* S *be a distributive nearlattice with a medial element* n *and* P *be a prime* n-*ideal in* S *.Then each minimal prime n-ideal belonging to* n(P) *is contained in* P. ●

Proposition 4.2.9. *For a medial element* n , *if* P *is a prime* n-*ideal in a distributive near lattice* S , *then* n(P) is *the intersection of all minimal prime* n-*ideals contained in* P. ●

Following result on finitely generated n-ideals have been proved by [44] for lattices . Then [2] have generalized the result for $P_n(S)$ where n is a central element of nearlattices S. We have improved that result for $F_n(S)$ when n is merely an upper element. We prefer to omit the proof as it follows the same technique of proof of [2].

Theorem 4. 2.10. *Let* F_n (S) *be a sectionally pseudocomplemented distributive lattice and* n *be an upper element in* S. *Then the following conditions are equivalent.*

(i) *For any* $x \in S$, $< x >_n^+ \vee < x >_n^{++} = S$, *equivalently,* $F_n(S)$ *is generalized Stone.*

(ii) *For any two minimal prime* n-*ideals* P *and* Q, $P \vee Q = S$.

(iii) *Every prime* n-*ideal contains a unique minimal prime* n-*ideal.*

(iv) *For each prime* n-*ideal* P, n(P) *is a Prime* n-*ideal.* ●

Following corollary is an immediate consequence of above theorem.

Corollary 4.2.11. *Let* L *be a distributive lattice with* 0 *and* 1 *and* n *be an upper element of* L. $F_n(L)$ *is a pseudocomplemented distributive lattice, then the following conditions are equivalent.*

(i) $F_n(L)$ *is Stone.*

(ii) *For any two minimal prime* n-*ideals* P *and* Q, $P \vee Q = L$.

(iii) *Every Prime* n-*ideal contains a unique minimal prime* n-*ideal.*

(iv) *For each prime* n-*ideal* P, n(P) *is a prime* n-*ideal.* ●

4.3. Relative annihilators around an upper element of a nearlattice

We start with the following characterization of $<a, b>^n$.

Theorem: 4.3.1. *Let* S *be a nearlattice with an upper element* n . *Then for all* a, b $\in$ S, *the following conditions are equivalent.*

(i) $<a, b>^n$ *is an* n-*ideal.*

(ii) $< a\wedge n, b\wedge n >_d$ *is a filter and*

$< a\vee n, b\vee n >$ *is an ideal.*

Proof: (i) $\Rightarrow$(ii). Suppose (i) holds. Let x, y $\in < a\vee n , b\vee n >$ and $x\vee y$ exists. Then $x \wedge(a\vee n) \le b\vee n$. Thus $(x\wedge(a\vee n))\vee n \le b\vee n$, then by neutrality of n , $(x\vee n) \wedge(a\vee n) \le b\vee n$. Also

$$m (x\vee n, n, a) = (x \vee n \vee a) \wedge(x \vee n) \wedge(a \vee n)$$
$$= (x \vee n) \wedge(a \vee n) \le b \vee n.$$

This implies $x\vee n \in < a, b >^n$.

Similarly, $y\vee n \in<a , b >^n$. Since $< a , b >^n$ is an n-ideal , so $x \vee y \vee n \in < a , b >^n$.
This implies $m(x \vee y \vee n, n, a) \le b\vee n$. That is ,

$$(x\vee y\vee n) \wedge(a\vee n) \le b\vee n \text{ and so} (x \vee y) \wedge(a \vee n) \le b \vee n.$$

Therefore, $x \vee y \in< a\vee n , b\vee n >$.

Moreover , for $x \in< a\vee n , b\vee n >$ and $t \le x$ ($t \in S$), obviously $t \wedge(a \vee n) \le b\vee n$, so $t \in< a \vee n , b \vee n >$. Hence $< a\vee n, b\vee n >$ is an ideal.

A dual proof of the above shows that $< a \wedge n, b \wedge n >_d$ is a filter.

(ii)$\Rightarrow$(i). Suppose (ii) holds and x , $y\in<a , b >^n$. Then $m(x, n, a) \in < b >_n$. Then using the neutrality of n, $b\wedge n \le(x\wedge a) \vee(x\wedge n) \vee(a\wedge n) = (x \vee a) \wedge(x \vee n) \wedge(a\vee n) \le b\vee n$..

Similarly, $b\wedge n \le (y\wedge a) \vee (y\wedge n)\vee(a\wedge n) = (y \vee a)\wedge(y\vee n) \wedge(a\vee n) \le b\vee n$.

So, $b\wedge n \le [(x\wedge a) \vee(x\wedge n) \vee(a\wedge n)] \wedge n = (x\wedge n) \vee(a\wedge n)$.

This implies, $x\wedge n\in <a\wedge n , b\wedge n >_d$.

Similarly, $y \wedge n \in \langle a \wedge n\, ,\ b \wedge n \rangle_d$. Since $\langle a \wedge n\, ,\ b \wedge n \rangle_d$ is a filter, so we have $x \wedge y \wedge n \in \langle a \wedge n\, ,\ b \wedge n \rangle_d$. Thus, $(x \wedge y \wedge n) \vee (a \wedge n) \geq (b \wedge n)$.

But $m(x \wedge y \wedge n\, ,\ n\, ,\ a) = (x \wedge y \wedge n) \vee (a \wedge n) \geq b \wedge n$, so $x \wedge y \wedge n \in \langle a\, ,\ b \rangle^n$.

Again by the neutrality of n ,

$(x \vee n) \wedge (a \vee n) = [\,(x \vee a) \wedge (x \vee n) \wedge (a \vee n)\,] \vee n \leq b \vee n$.

Similarly, $(y \vee n) \wedge (a \vee n) \leq b \vee n$.

Thus $((x \wedge y) \vee n) \wedge (a \vee n) \leq b \vee n$.

But $((x \wedge y) \vee n) \wedge (a \vee n) = m((x \wedge y) \vee n\, ,\ n\, ,\ a)$, as n is neutral.

Therefore, $(x \wedge y) \vee n \in \langle a\, ,\ b \rangle^n$, and so by the convexity of $\langle a\, ,\ b \rangle^n$,

$x \wedge y \in \langle a\, ,\ b \rangle^n$.

A dual proof of above also shows that $x \vee y \in \langle a\, ,\ b \rangle^n$. Clearly $\langle a,\ b \rangle^n$ contains n. Therefore, $\langle a,\ b \rangle^n$ is an n - ideal. ●

Following result has been proved by [2] when n is a central element. This is also true when n is merely an upper element.

Proposition 4.3.2. *Let* S *be a nearlattice with an upper element* n . *For all* a, b $\in$ S, *the following conditions are hold.*

(i) $\langle a \vee n,\ b \vee n \rangle$ *is an ideal if and only if* $[\,n\,)$ *is a distributive subnearlattice of S.*

(ii) $\langle a \wedge n\, ,\ b \wedge n \rangle_d$ *is a filter if and only if* $(\,n\,]^d$ *is a distributive subnearlattice of* S. ●

By theorem 4.3.1 and the above result we have the following result.

Theorem 4.3.3. *Let* S *be a nearlattice and* $n \in S$ *be an upper element. Then for all* a , $b \in S$ $\langle a\, ,\ b \rangle^n$ *is an* n-*ideal if and only if* $(\,n\,]^d$ *and* $[\,n\,)$ *are distributive.* ●

Recall that a nearlattice S is distributive if for all $x, y, z \in S$, $x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)$ provided $y\vee z$ exists [47] has given an alternative definition of distributivity of S.

A nearlattice S is distributive if and only if for all $t, x, y, z\in S$,

$$t\wedge((x\wedge y)\vee(x\wedge z)) = (t\wedge x\wedge y)\vee(t\wedge x\wedge z).$$

Similarly by [47], a nearlattice S is modular if and only if for all $t, x, y, z\in S$ with $z\le x$, $x\wedge((t\wedge y)\vee(t\wedge z)) = (x\wedge t\wedge y)\vee(t\wedge z)$.

Recently [1] have generalized [38 , Th. 2] in terms of n-annihilators. [38,Th. 2] has also been extended by [47] in characterizing modular nearlattices. Here we extend the result for n-ideals in terms of relative n-annihilators.

Theorem 4.3.4. *Let* n *be an upper and neutral element of a nearlattice* S. *Then the following conditions are equivalent.*

(i) $(n]^d$ *and* $[n)$ *are modular nearlattices.*

(ii) $P_n(S)$ *is modular nearlattice.*

(iii) *For* $a, b\in S$ *with* $<b>_n \subseteq <a>_n$, $x\in<b>_n$ *and* $y\in<a, b>^n$ *imply* $x\wedge y, x\vee y\in<a, b>^n$ *if* $x\vee y$ *exists in* S.

Proof : (i) $\Rightarrow$(iii) . Suppose $[n)$ and $(n]$ are modular. Here $<b>_n \subseteq <a>_n$, so $a\wedge n \le b\wedge n \le n \le b\vee n \le a\vee n$. Since $x\in<b>_n$, so $b\wedge n \le n \le b\vee n$. Hence, $a\wedge n \le b\wedge n \le x\wedge n \le x\vee n \le b\vee n \le a\vee n$. Now $y\in<a, b>^n$ implies $m(y, n, a)\in<b>_n$. Then by neutrality of n, $(y\wedge a)\vee(y\wedge n)\vee(a\wedge n) \le b\vee n$ and so $((y\wedge a)\vee(y\wedge n)\vee(a\wedge n))\vee n = (y\vee n)\wedge(a\vee n) \le b\vee n$. Thus using the modularity of $[n)$ and the existence of $x\vee y$, $m(x\vee y\vee n, n, a) = (x\vee y\vee n)\wedge(a\vee n)$

$= [(a\vee n)\wedge(y\vee n)]\vee(x\vee n)$, as $x\vee n \le b\vee n \le a\vee n$.

This implies $m(x\vee y\vee n, n, a) \le b\vee n$ and so $x\vee y\vee n\in<a, b>^n$. Since n is neutral, so $a\wedge n \le b\wedge n \le x\wedge n$ implies that $b\wedge n \le (x\wedge n)\vee(y\wedge n)\vee(a\wedge n) = ((x\vee y)\wedge n)\vee(a\wedge n)$

$= m(x\vee y)\wedge n, n, a) \le b\vee n$.

Therefore, $(x \vee y) \wedge n \in \langle a, b \rangle^n$. Hence by the convexity of $\langle a, b \rangle^n$, $x \vee y \in \langle a, b \rangle^n$.

Again using the modularity of $(n]^d$, a dual proof of above shows that $x \wedge y \in \langle a, b \rangle^n$. Hence (iii) holds.

(iii)$\Rightarrow$ (i). Suppose (iii) holds. Let $x, y, z \in [n)$ with $x \le z$ and whenever $x \vee y$ exists.

Then $x \vee (y \wedge z) \le z$. This implies $\langle x \vee (y \wedge z) \rangle \subseteq \langle z \rangle_n$.

Now $x \le x \vee (y \wedge z)$ implies $x \in \langle x \vee (y \wedge z) \rangle_n$.

Again $y \wedge z \le x \vee (y \wedge z)$ implies $m(y, n, z) = y \wedge z \in \langle x \vee (y \wedge z) \rangle_n$.

Hence $y \in \langle z, x \vee (y \wedge z) \rangle^n$.

Thus by (ii), $x \vee y < z, x \vee (y \wedge z) >^n$. That is, $(x \vee y) \wedge n \le x \vee (y \wedge n)$ and so $(x \vee y) \wedge n = x \vee (y \wedge n)$. Therefore $[n)$ is modular.

Similarly, using the condition (iii), we can easily show that $(n]$ is also modular.

(i)$\Rightarrow$ (ii).

Let $(n]^d$ and $[n)$ be modular. Consider $\langle a \rangle_n, \langle b \rangle_n, \langle c \rangle_n \in P_n(S)$ with $\langle c \rangle_n \subseteq \langle a \rangle_n$.

Then $a \wedge n \le c \wedge n \le n \le c \vee n \le a \vee n$. Now for any $t \in S$,

$\langle a \rangle_n \cap [(\langle t \rangle_n \cap \langle b \rangle_n) \vee \langle t \rangle_n \cap \langle c \rangle_n]$

$= [a \wedge n, a \vee n] \cap ([t \wedge n, t \vee n] \cap (b \wedge n, b \vee n)) \vee ([t \wedge n, t \vee n] \cap [c \wedge n, c \vee n])))$.

By some continue calculation and using the fact that $[n)$ is modular, we find that the right hand part of above interval $= ((a \vee n) \wedge (t \vee n) \wedge (b \vee n)) \vee ((t \vee n) \wedge (c \vee n))$ which is same as the right part of the interval in $(\langle a \rangle_n \cap \langle t \rangle_n \cap \langle b \rangle_n) \vee (\langle t \rangle_n \cap \langle c \rangle_n)$. Similarly, using the modularity of $(n]^d$ and some routine calculation, we find that the left member of the interval in both $\langle a \rangle_n \cap [(\langle t \rangle_n \cap \langle b \rangle_n) \vee (\langle t \rangle_n \cap \langle c \rangle_n)]$ and $(\langle a \rangle_n \cap \langle t \rangle_n \cap \langle b \rangle_n) \vee (\langle t \rangle_n \cap \langle c \rangle_n)$ are also same. This implies,

$\langle a \rangle_n \cap [(\langle t \rangle_n \cap \langle b \rangle_n) \vee (\langle t \rangle_n \cap \langle c \rangle_n)]$

$= (\langle a \rangle_n \cap \langle t \rangle_n \cap \langle b \rangle_n) \vee (\langle t \rangle_n \cap \langle c \rangle_n)$ and so $P_n(S)$ is modular.

(ii)$\Rightarrow$(i). Suppose (ii) holds and a, b, c $\in$[n) with c $\leq$ a. suppose b$\vee$c exists. Here $< c >_n \subseteq < a >_n$. By (ii), $< a >_n \cap (< b >_n \vee < c >_n) = (< a >_n \cap < b >_n) \vee < c >_n$.
That is $[n, a] \cap ([n, b] \vee [n, c]) = ([n, a] \cap [n, b]) \vee [n, c]$.
That is $[n, a] \cap ([n, b\vee c] = [n, a\wedge b) \vee [n, c]$
That is $[n, a\wedge(b\vee c)]) = [n, (a\wedge b) \vee c]$ and this implies $a\wedge(b\vee c) = (a\wedge b) \vee c$ and so [n) is modular.

Similarly we can prove that $(n]^d$ is also modular.

Therefore, (i) holds.

Thus by (ii), $x\vee y \in < z, x\vee(y\wedge z) >^n$.

That is, $(x\vee y)\wedge z \leq x\vee(y\wedge z)$ and so $(x\vee y)\wedge z = x\vee(y\wedge z)$.

Therefore [n) is modular.

Similarly, using the condition (ii), we can easily show that (n] is also modular. ●

Following the proof of (i) $\Leftrightarrow$ (ii) in theorem 4.3.4, we can easily prove that $P_n(S)$ is distributive if and only if $(n]^d$ and [n) are distributive when n is an upper element. Therefore, by theorem 4.3.2, we have the following result.

Corollary 4.3.5. *Let* S *be a nearlattice and* n $\in$S *is an upper element. Then for all* a, b $\in$S, $<a, b >^n$ *is an* n-*ideal if and only if* $P_n(S)$ *is distributive.* ●

[34, Th. 1] gave characterizations of distributive lattices. Then [47] extended the result for nearlattices. [1] generalized the result for n-ideals in lattices. Following theorem generalizes all the above results for nearlattices using relative n-annihilators. This has been proved by [2] when n is central element. We omit the proof as it is exactly same as the proof of [2].

Theorem 4.3.6. *Let* n *be an upper and neutral element of a nearlattice* S. *Then the following conditions are equivalent.*

(i) $P_n(S)$ *is distributive.*

(ii) $< a \vee b, b \vee n >$ *is an ideal and* $< a \wedge n, b \wedge n >_d$ *is a filter whenever* $< a >_n \subseteq < b >_n$. ●

We conclude this section with the following characterization of minimal prime n-ideals belonging to an n-ideal. Since the proof of this is almost similar to theorem 4.1.5, we omit the proof.

Theorem 4.3.7. Let S *be a distributive nearlattice and* P *be a prime* n- *ideal of* S *belonging to an* n-*ideal* J. *Then the following conditions are equivalent.*

(i) P *is minimal prime* n -*ideal belonging to* J.

(ii) $x \in P$ *implies* $<< x >_n, J > \nsubseteq P$. ●

4.4. Some characterizations of those $F_n(S)$ which are relatively Stone lattices

$F_n(S)$ is relatively pseudocomplemented if the interval $[<a_1, a_2,\ldots,a_r>_n, < b_1,b_2,\ldots,b_s >_n]$ in $F_n(S)$ for each $<a_1,a_2,\ldots,a_r>_n, <b_1,b_2,\ldots,b_s>_n \in F_n(S)$, with $<a_1, a_2,\ldots,a_r>_n \subset < b_1,b_2,\ldots,b_s >_n$ is pseudocomplemented.

Moreover, $F_n(S)$ is a relatively Stone lattice if each interval $[<a_1, a_2,\ldots,a_r>_n, < b_1,b_2,\ldots,b_s >_n]$ with $<a_1, a_2,\ldots,a_r>_n \subset < b_1,b_2,\ldots,b_s >_n$ ($<a_1, a_2,\ldots,a_r>_n, < b_1,b_2,\ldots,b_s >_n \in F_n(S)$) is a Stone lattice.

Theorem 4.4.1. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions hold.*

(i) $<<x>_n \vee <y>_n, <x>_n> = <<y>_n, <x>_n>$;

(ii) $<<x>_n, J> = \vee_{y\in J} <<x>_n, <y>_n>$ *the supremum of* n*-ideals* $<<x>_n, <y>_n>$ *in the lattice of n-ideals of* S, *for any* $x \in S$ *and any* n*- ideal* J.

Proof: (i). Obviously L.H.S $\subseteq$ R.H.S.

To prove the reverse inclusion let $t \in$ R.H.S, then $t \in <<y>_n, <x>_n>$. This implies $m(y, n, t) \in <x>_n$. That is, $<m(y, n, t)>_n \subseteq <x>_n$ and so $(<y>_n \cap <t>_n) \vee (<x>_n \cap <t>_n) \subseteq <x>_n$. That is, $<t>_n \cap [<x>_n \vee <y>_n] \subseteq <x>_n$ which implies $t \in <<x>_n \vee <y>_n, <x>_n>$ thus $t \in$ L.H.S and so R.H.S. $\subseteq$ L.H.S. Hence L.H.S. = R.H.S.

(ii) Obviously R.H.S. $\subseteq$ L.H.S.

To prove the reverse inclusion, let $t \in$ L.H.S. Then $m(x, n, t) \in J$ that is $m(x, n, t) = j$ for some $j \in J$. This implies $t \in <<x>_n, <j>_n>$. Thus $t \in$ R.H.S. and so (ii) holds.●

Following lemma will be needed for the development of this chapter. This is in fact, the dual of [9, lemma 3.6] and very easy to prove. So we prefer to omit the proof.

Lemma 4.4.2. *Let* L *be a distributive lattice. Then the following conditions hold.*

(i) $< x \wedge y,\ x >_d = < y,\ x >_d$.

(ii) $< [x),\ F >_d = \vee_{y \in F} < x, y >_d$, *where* F *is filter of* L.

(iii) $\{< x, a >_d \vee < y, a >_d\} \cap [a, b] = \{< x, a >_d \cap [a, b]\} \vee \{< y, a >_d \cap [a, b]\}$, *where* $[a, b]$ *represents any interval in* L. ●

Lemma 4.4.3 and 4.4.4 are essential for the proof of our main result of this section. These results are due to [2].

Lemma 4.4.3. *Let* S *be a distributive nearlattice with an upper element n.* *Suppose* a, b, c ∈ S.

(i) *If* a, b, c ≥ n, *then* $<< m(a, n, b)>_n, c>_n > = << a>_n, < c>_n > \vee << b >_n, < c>_n >$ *is equivalent to* $< a \wedge b,\ c > = < a,\ c > \vee < b,\ c >$.

(ii) *If* a, b, c ≤ n, *then* $<< m(a, n, b)>_n, < c >_n > = << a >_n, < c >_n > \vee << b >_n, < c >_n >$ *is equivalent to* $< a \vee b,\ c > = < a,\ c > \vee < b,\ c >_d$. ●

Lemma 4.4.4. *Let* S *be a distributive nearlattice with an upper element n. Suppose* a, b, c ∈ S.

(i) If a, b, c ≥ n and $a \vee b$ exists, then $<< c >_n , < a >_n \vee < b >_n > = << c >_n , < a >_n > \vee << c >_n , < b >_n >$ is equivalent to $< c , a \vee b > = < c , a > \vee < c , b >$.

(ii) If a , b, c ≤ n , then $<< c >_n , < a >_n \vee < b >_n > = << c >_n , < a >_n > \vee << c >_n , < b >_n >$ is equivalent to $< c , a \vee b >_d = < c , a >_d \vee < c , b >_d$. ●

Following result on Stone lattices is well known due to [9] , [29]and [30]

Theorem 4.4.5. *Let* L *be a pseudocomplemented distributive lattice. Then the following conditions are equivalent.*

(i) L *is Stone*

(ii) *For each* $x, y \in L$, $(x \wedge y)^{*} = x^{*} \vee y^{*}$.

(iii) *If* $x \wedge y = 0$, $x, y \in L$, *thus* $x^{*} \vee y^{*} = 1$. ●

Similarly we can prove the following result which is dual to above theorem.

Theorem 4.4.6. *Let* L *be a dual pseudocomplemented distributive lattice. Then the following conditions are equivalent.*

(iii) L *is dual Stone*

(iv) *For each* $x, y \in L$, $(x \vee y)^{*d} = x^{*d} \wedge y^{*d}$.

(iii) *If* $x \vee y = 1$, $x, y \in L$, *then* $x^{*d} \wedge y^{*d} = 0$, *where* x^{*d} *denotes the dual pseudocomplement* of x. ●

Ayub in [1, Th. 3.2.7] has given a nice characterization of relatively dual Stone lattices in terms of dual relative annihilators, which is in fact the dual of [9, Th.3.7] . As we have mentioned earlier that in nearlattices the idea of dual relative annihilators is not always possible. But when n is an upper element in S then $x \vee n$ exists for all $x \in S$. Thus for any $a \in (n]$, $x \vee a$ exists for all $x \in S$. Hence we can define $< a, b >_d$ for all $a \in (n]$ and $b \in S$.

Following result is due to [2].

Theorem 4.4.7. *Let* n *be an upper element of a distributive nearlattice* S *such that* (n] *is relatively dual pseudocomplemented. Let* $a, b, c \in (n]$ *be arbitrary elements and* A, B *be arbitrary filters of* (n] . *Then the following conditions are equivalent.*

(i) (n] *is relative dual Stone.*

(ii) $< a, b >_d \vee < b, a > = (n]$

(iii) $< c, a \wedge b >_d = <c, a >_d \vee < c, b >_d.$

(iv) $[<C), A \vee B >_d = < [C), A >_d \vee < [C), B >_d$

(v) $< a \vee b, c >_d = < a, c >_d \vee < b, c >_d$. ●

Theorem 4.4.8 and 4.4.9 are needed for the further development of this section. Next two theorems are the main results of this section which give several characterizations of those $F_n(S)$ which are relatively Stone.

Theorem 4.4.8. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions are equivalent.*

(i) $F_n(S)$ *is relatively pseudocomplemented.*

(ii) $P_n(S)$ *is relatively pseudocomplemented.*

(iii) $(n]$ *is relatively dual pseudocomplemented and* $[n)$ *is relatively pseudocomplemented.*

Proof : (i) $\Rightarrow$(ii).Suppose (i) holds. Consider $< a >_n \subseteq < b >_n \subseteq < c >_n$. Since (i) holds, so there exists $< t_1, t_2, \ldots t_r >_n$ in $F_n(S)$ with $<t_1, t_2, \ldots t_r >_n \subseteq <c >_n$ which is the pseudocomplement of $< b >_n$ in $[< a >_n , < c >_n]$. But by Corollary 1.4.5, $< t_1, t_2, \ldots t_r >_n = < d >_n$ for some $d \in S$. This implies $<d >_n$ is the relative pseudocomplement of $< b >_n$ in $[< a >_n , < c >_n]$. Therefore, $P_n(S)$ is relatively pseudocomplemented.

(ii)$\Rightarrow$(i). Suppose (ii) holds.

Consider $< a_1, a_2, \ldots a_r >_n \subseteq < b_1, b_2, \ldots b_s >_n \subseteq < c_1, c_2, \ldots c_k >_n$ .Then $< a_1, a_2, \ldots a_r >_n \cap < c_t >_n \subseteq < b_1, b_2, \ldots b_s >_n \cap < c_t >_n \subseteq < c_t >_n$ for some $t = 1,2,\ldots,k$.

Then by Corollary 1.4.5, $< b_1, b_2, \ldots b_s >_n \cap < c_t >_n$ is principal and let $< p_t >_n = < b_1, b_2, \ldots, b_s >_n \cap < c_t >_n$. So there exists $< d_t >_n$, such that $< d_t >_n$ is the relative pseudocomplement of $< p_t >_n$ in $[< a_1, a_2, \ldots a_r >_n \cap < c_t >_n , < c_t >_n]$.

So, $< b_1, b_2, \ldots b_s >_n \cap < c_t >_n \cap < d_t >_n = < a_1, a_2, \ldots a_r >_n \cap < c_t >_n$.

Thus, $< b_1, b_2, \ldots b_s >_n \cap < d_t >_n = < a_1, a_2, \ldots a_r >_n \cap < c_t >_n$, for each t.

Hence, $< b_1, b_2, \ldots b_s >_n \cap < d_1, d_2, \ldots d_k >_n = < a_1, a_2, \ldots a_r >_n$

Moreover, $< d_1, d_2, \ldots d_k >_n \subseteq < c_1, c_2, \ldots c_k >_n$.

Now let, there exist $< q_1, q_2, \ldots, q_m >_n \subseteq < c_1, c_2, \ldots c_k >_n$ such that $< q_1, q_2, \ldots q_m >_n \cap < b_1, b_2, \ldots b_s >_n = < a_1, a_2, \ldots a_r >_n$.

Snce $< q_1, q_2, \ldots q_m >_n \cap < c_t >_n \subseteq < c_t >_n$, so $< q_1, q_2, \ldots q_m >_n \cap < c_t >_n \cap < b_1, b_2, \ldots b_s >_n = < a_1, a_2, \ldots a_r >_n \cap < c_t >_n$ which implies $< q_1, q_2, \ldots q_m >_n \cap < c_t >_n \subseteq < d_t >_n$ by Corollary 1.4.5, for each $t = 1, 2, \ldots, k$.

Thus, $< q_1, q_2, \ldots q_m >_n \cap < c_1, c_2, \ldots c_k >_n \subseteq < d_1, d_2, \ldots d_k >_n$.

That is $< q_1, q_2, \ldots q_m >_n \subseteq < d_1, d_2, \ldots d_k >_n$. Hence $< d_1, d_2, \ldots, d_k >_n$ is the relative pseudocomplement of $< b_1, b_2, \ldots b_s >_n$ in $[< a_1, a_2, \ldots a_r >_n, < c_1, c_2, \ldots c_k >_n]$.

Therefore, $F_n(S)$ is relatively pseudocomplemented.

(ii) ⇒ (iii). Suppose (ii) holds. Let $n \leq a \leq b \leq c$. Then $< c >_n \subseteq < b >_n \subseteq < a >_n$. So $[n, a] \subseteq [n, b] \subseteq [n, c]$. Since (ii) holds, so there exists $< t >_n$ such that $< t >_n$ is the relative pseudocomplement of $< b >_n$ in $[< c >_n, < a >_n]$. Now, $< c >_n \subseteq < t >_n \subseteq < a >_n$ implies $c \leq t \leq a$.

So, $[n, b] \cap [n, t] = [n, a]$ which implies $b \wedge t = a$.

Let $d \in S$ with $a \leq d \leq c$ such that $b \wedge d = a$. This implies that $< b >_n \cap < d >_n = < a >_n$, since $< b >_n$ is the relative pseudocomplement of $< b >_n$.

So, $< d >_n \subseteq < t >_n$. That is $[n, d] \subseteq [n, t]$ and so, $d \leq t$.

Therefore, t is the relative pseudocomplement of b in [a, c] and so [a, c] is pseudocomplemented and hence [n) is relatively pseudocomplemented.

A dual Proof of (ii) $\Rightarrow$(iii) shows that $(n]^d$ is relatively dual pseudocomplemented.

(iii) $\Rightarrow$(ii). Suppose (iii) holds. Let $\{n\} \subseteq \langle a \rangle_n \subseteq \langle b \rangle_n \subseteq \langle c \rangle_n$.

Then $c \wedge n \le b \wedge n \le a \wedge n \le n \le a \vee n \le b \vee n \le c \vee n$. By (iii) , [n) is relatively pseudocomplemented , so $b \vee n$ has a relative pseudocomplement t in $[a \vee n, c \vee n]$. Thus $t \wedge (b \vee n) = a \vee n$.

Again $c \wedge n \le b \wedge n \le a \wedge n$. Since (n] is relative dual pseudocomplement. so, there exists the relative dual pseudocomplements of $b \wedge n$ in $[c \wedge n, a \wedge n]$.

Then $s \vee (b \wedge n) = a \wedge n$ and $\langle s, t \rangle_n \cap \langle b \rangle_n = [s, t] \cap [b \wedge n, b \vee n]$

$$= [s \vee (b \wedge n), t \wedge (b \vee n)] = \langle a \rangle_n$$

Moreover, $\langle s, t \rangle_n \subseteq \langle c \rangle_n$ implies $\langle s, t \rangle_n$ is a principal n-ideal by Corollary 1.4.5.

Let there exists $\langle p \rangle_n$ in $[\langle a \rangle_n, \langle c \rangle_n]$ such that $\langle b \rangle_n \cap \langle p \rangle_n = \langle a \rangle_n$. This implies $[b \wedge n, b \vee n] \cap [p \wedge n, p \vee n] = \langle a \rangle_n$. That is, $[b \wedge n \vee (p \wedge n), (b \vee n) \wedge (p \vee n)] = \langle a \rangle_n$.

This implies $(b \vee n) \wedge (p \vee n) = a \vee n$ and $(b \wedge n) \vee (p \wedge n) = a \wedge n$.

But $p \vee n \le t$ and $p \wedge n \ge s$ as t and s are relative pseudocomplement and relative dual pseudocomplement. Thus $\langle p \rangle_n \subseteq [s, t]$.

Therefore, [s, t] is the relative pseudocomplement of $\langle b \rangle_n$ in $[\langle a \rangle_n, \langle c \rangle_n]$.

Therefore, $[\langle a \rangle_n, \langle c \rangle_n]$ is pseudocomplemented and so, $P_n(S)$ is relatively pseudocomplemented. ●

Theorem 4.4.9. *Let* S *be a distributive nearlattice with an upper element n. Then the following conditions are equivalent.*

(i) $F_n(S)$ *is relatively Stone.*

(ii) $P_n(S)$ *is relatively Stone.*

(iii) $(n]$ *is relatively dual Stone and* $[n)$ *is relatively Stone.*

Proof: (i) $\Rightarrow$(ii) . Suppose (i) holds. So , $P_n(S)$ is relatively pseudocomplemented by 4.4.8. Consider $<a>_n \subseteq <b>_n \subseteq <c>_n$. Since $F_n(S)$ is generalized Stone , so the interval $[<a>_n, <c>_n]$ is Stone. Thus $<b>_n^{\circ} \vee <b>_n^{\circ\circ} = <c>_n$, where $<b>_n^{\circ}$ is the relative pseudocomplement of $<b>_n$ in $[<a>_n, <c>_n]$. This implies $P_n(S)$ is relatively Stone.

(ii) $\Rightarrow$ (i).Suppose (ii) holds .Then by Theorem 4.4.8, F_n (S) is relatively pseudocomplemented.

Let $<a_1, a_2, \ldots, a_r>_n \subseteq <b_1, b_2, \ldots, b_s>_n \subseteq <c_1, c_2, \ldots, c_k>_n$. If for each t = 1, 2, …,k, $<e_t>_n$ is the relative pseudocomplement of $<b_1, b_2, \ldots, b_s>_n \cap <c_t>_n$ in $[<a_1, a_2, \ldots, a_r>_n \cap <c_t>_n , <c_t>_n]$ and $<f_t>_n$ is the double relative pseudocomplement of $<b_1, b_2, \ldots, b_s>_n \cap <c_t>_n$ in $[<a_1, a_2, \ldots, a_r>_n \cap <c_t>_n , <c_t>_n]$, then $[<b_1, b_2, \ldots, b_s>_n^{\circ} \vee <b_1, b_2, \ldots, b_s>_n^{\circ\circ}] \cap <c_t>_n$

$= (<b_1, b_2, \ldots, b_s>_n^{\circ} \cap <c_t>_n) \vee (<b_1, b_2, \ldots, b_s>_n^{\circ\circ} \cap <c_t>_n)$

$= <e_t>_n \vee <f_t>_n$ by lemma 4.2.4.

$= <c_t>_n$ as $P_n(S)$ is relatively Stone.

Therefore, $<b_1, b_2, \ldots, b_s>_n^{\circ} \vee <b_1, b_2, \ldots, b_s>_n^{\circ\circ}$

$= (<b_1, b_2, \ldots, b_s>_n^{\circ} \vee <b_1, b_2, \ldots, b_s>_n^{\circ\circ}) \cap <c_1, c_2, \ldots, c_k>_n$

$= <c_1>_n \vee <c_2>_n \vee \ldots\ldots \vee <c_k>_n$

$= <c_1, c_2, \ldots c_k>_n$.

Hence $[<a_1, a_2, \ldots, a_r>_n , <c_1, c_2, \ldots, c_k>_n]$ is Stone , and so $F_n(S)$ is relatively Stone.

(ii) ⇒ (iii). Let $n \leq a \leq b \leq c$. So , $< a >_n \subseteq < b >_n \subseteq < c >_n$. Since $P_n(S)$ is relatively Stone, so the interval $[< a >_n , < c >_n]$ is Stone. Thus

$$< b >_n^{\circ} \vee < b >_n^{\circ\circ} = < c >_n .$$

That is , $[n, b]^{\circ} \vee [n, b]^{\circ\circ} = [n, c]$ and so by Threorem 4.2.1,

$[c, b^{\circ}] \vee [n, b^{\circ\circ}] = [n, c]$,which implies

$[n, b^{\circ} \vee b^{\circ\circ}] = [n, c]$,

and so $b^{\circ} \vee b^{\circ\circ} = c$ .Thus by 4.4.8, [n) is relative Stone.

A dual proof of (ii) ⇒ (iii) shows that (n] is relatively dual Stone.

(iii) ⇒ (ii). Suppose (iii) holds. Let $< a >_n \subseteq < b >_n \subseteq < c >_n$.

Then $c\wedge n \leq b\wedge n \leq a\wedge n \leq n \leq a\vee n \leq b\vee n \leq c\vee n$.

Consider, $a\vee n \leq b\vee n \leq c\vee n$ .By (iii) , $(b\vee n)^{\circ} \vee (b\vee n)^{\circ\circ} = c\vee n$, and $(b\wedge n)^{\circ d} \wedge (b\wedge n)^{\circ d} = c\wedge n$.

Therefore , by 4.2.1, $< b >_n^{\circ} \vee < b >_n^{\circ\circ}$

$$= [b\wedge n , b\vee n]^{\circ} \vee [b\wedge n , b\vee n]^{\circ\circ}$$

$$= [(b\wedge n)^{\circ d}, (b\vee n)^{\circ}] \vee [(b\wedge n)^{\circ\circ d} , (b\vee n)^{\circ\circ}]$$

$$= [(b\wedge n)^{\circ d} \wedge (b\wedge n)^{\circ\circ d} , (b\vee n)^{\circ} \vee (b\vee n)^{\circ\circ}]$$

$$= [c\wedge n , c\vee n]$$

$$= < c >_n.$$

Therefore, by Th. 4.4.8, $P_n(S)$ is relatively Stone. ●

Following result has been proved by [2] when n is central and S is a distributributive medial nearlattice. We prove it in a general distributive nearlattice with n as only an upper element .

Theorem 4.4.10. *Let* n *be an upper element of a distributive nearlattice and* $F_n(S)$ *be relatively pseudocomplemented. Suppose* A , B *are two n-ideals of* S. *Then for all* a , b , c ∈ S, *the following conditions are equivalent.*

(i) $F_n(S)$ *is relatively Stone.*

(ii) $<<a>_n, <b>_n> \vee <<b>_n, <a>_n> = S$.

(iii) $<<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n> \vee <<c>_n, <b>_n>$, *whenever* $a \vee b$ *exists.*

(iv) $<<c>_n, A \vee B> = <<c>_n, A> \vee <<c>_n, B>$.

(v) $<m(a, n, b)>_n, <c>_n> = <<a>_n \vee <c>_n> \vee <<b>_n, <c>_n>$.

Proof: (i) ⇒ (ii) . Let (i) holds. So $P_n(S)$ is relatively Stone by theorem 4.4.9. Let $z \in S$. Consider the interval

$I = [<a>_n \cap <b>_n \cap <z>_n, <z>_n]$ in $F_n(S)$. Then $<a>_n \cap <b>_n \cap <z>_n$ is the smallest element of the interval I.

By (i) , I is Stone .Then by theorem 4.4.5, there exist principal n-ideals $<p>_n, <q>_n \in I$ such that,

$$<a>_n \cap <z>_n \cap <p>_n = <a>_n \cap <b>_n \cap <z>_n$$
$$= <b>_n \cap <z>_n \cap <q>_n \text{ and } <z>_n = <p>_n \vee <q>_n.$$

Now, $<a>_n \cap <p>_n = <a>_n \cap <p>_n \cap <z>_n$
$= <a>_n \cap <b>_n \cap <z>_n \subseteq <b>_n$ implies $<p>_n \subseteq <<a>_n, <b>_n>$.

Also, $<b>_n \subseteq <q>_n = <b>_n \cap <z>_n \cap <q>_n$
$= <a>_n \cap <b>_n \cap <z>_n \subseteq <a>_n$ implies

$<q>_n \subseteq <<b>_n, <a>_n>$.

Thus $<z>_n \subseteq <<a>_n, <b>_n> \vee <<b>_n, <a>_n>$ and so

$z \in <<a>_n, <b>_n> \vee <<b>_n, <a>_n>$.

Hence, $<<a>_n, <b>_n> \vee <<b>_n, <a>_n> = S$.

(ii) ⇒ (iii) . Suppose (ii) holds and $a \vee b$ exists.

For (iii), R.H.S. ⊆ L.H.S. is obvious.

Now, let $z \in < <c>_n \;, <a>_n \vee <b>_n >$.

Then $z \vee n \in << c>_n , <a>_n \vee <b>_n>$ and so $m(z\vee n, n, c) \in <a>_n \vee <b>_n$.

That is , $m(z\vee n, n, c) \in [\; a\wedge b\wedge n,\; a\vee b\vee n]$.

This implies $(z\vee n) \wedge (c\vee n) \leq a\vee b\vee n$.

Now by (ii) , $z\vee n \in < <a>_n \;, <b>_n > \vee < <b>_n \;, <a>_n >$. Then by Theorem 1.4.7, $z\vee n \leq (p\vee n) \vee (q\vee n)$ for some $p\vee n \in < <a>_n,\; <b>_n >$ and $q\vee n \in < <b>_n , <a>_n >$. Hence, $z\vee n = (z\vee n) \wedge (p\vee n)) \vee ((z\vee n) \wedge (q\vee n)$ $= r\vee t$ (say).

Now, $m(p\vee n, n, a) = (p\vee n) \wedge (a\vee n) \leq b\vee n$. So $b\wedge n \leq r\wedge (a\vee n) \leq b\vee n$.

Hence, $r\wedge (c\vee n) = r\wedge (z\vee n) \wedge (c\vee n)$

$$\leq r \wedge (a\vee b\vee n)$$

$$= r\wedge (a\vee n) \vee (r\wedge (b\vee n))$$

$$\leq (b\vee n).$$

This implies $r \in << c>_n , <b>_n$. Similarly ,$t \in << c>_n , <a>_n >$.

Hence, $z\vee n \in << c>_n , \; <a>_n \; \vee << c>_n, \; <b>_n>$.

Again $z \in << c>_n , \; <a>_n \; \vee <b>_n>$

implies $z\wedge n \in << c>_n , \; <a>_n \; \vee <b>_n >$

Then a dual calculation of above shows that

$a\wedge n \in << c>_n , <a>_n > \vee \; << c>_n , <b>_n >$. Thus by convexity,

$z \in \; << c>_n , <a>_n > \vee \; < <c>_n , <b>_n >$ and so L.H.S $\subseteq$ R.H.S.

Hence (iii) holds.

(iii) $\Rightarrow$ (iv).Suppose (iii) holds. In (iv), R.H.S. $\subseteq$ L.H.S. is obvious.

Now let , $x \in \; << c>_n \;, A\vee B>$. Then $x\vee n << c>_n , A\vee B >$.

Thus $m(x\vee n \;, n \;, c) \in A\vee B$. Now $m(x\vee n \;, n \;, c) = (x\vee n) \wedge (n\vee c) \geq n$ implies $m(x\vee n, n, c) \in (A\vee B) \cap [\, n)$.

Hence by Theorem 4.4.1. (ii), $x \vee n \in << c>_n \;, (A\cap [\,n) \vee (B\cap [n)>$.

$= \vee_{r\in (A\cap [n))\vee(B\cap [n))} << c>_n \;, <r>_n>$.

But by theorem 1.4.7, $r \in (A \cap [n) \vee (B \cap [n))$ implies

$r = s \vee t$ for some $s \in A$, $t \in B$ and $s, t \geq n$.

Then by (iii) $<<c>_n, <r>_n> = <<c>_n, <s \vee t>_n>$

$$= <<c>_n, <s>_n \vee <t>_n>$$

$$= <<c>_n, <s>_n> \vee <<c>_n, <t>_n>$$

$$\subseteq <<c>_n, A> \vee <<c>_n, B>.$$

Hence $x \vee n \in <<c>_n, A> \vee <<c>_n, B>$.

Also, $x \in <<c>_n, A \vee B>$ implies $x \wedge n \in <<c>_n, A \vee B>$.

Since $m(x \wedge n, n, c) = (x \wedge n) \vee (n \wedge c) \leq n$,

So, $x \wedge n \in <<c>_n, (A \vee B) \cap (n]>$. Then by theorem 4.4.4(ii),

$x \wedge n \in <<c>_n, (A \cap (n]) \vee (B \cap (n])>$

$$= \vee_{i \in (A \cap [n)) \vee (B \cap [n))} <<c>_n, <i>_n>.$$

Again, using theorem 1.4.7, we see that $i = p \wedge q$ where $p \in A$, $q \in B$ and $p, q \leq n$.

Then by (iii), $<<c>_n, <i>_n> = <<c>_n, <p \wedge q>_n>$

$$= <<c>_n, <p>_n \vee <q>_n>$$

$$= <<c>_n, <p>_n> \vee <<c>_n, <q>_n>$$

$$\subseteq <<c>_n, A> \vee <<c>_n, B>$$

Hence $x \wedge n \in <<c>_n, A> \vee <<c>_n, B>$. Therefore by convexity,

$x \in <<c>_n, A> \vee <<c>_n, B>$ and so, L.H.S. $\subseteq$ R.H.S. Thus (iv) holds.

(iv) $\Rightarrow$ (iii) is trivial.

(ii) $\Rightarrow$ (v). Suppose (ii) holds.

In (v), R.H.S. $\subseteq$ L.H.S. is obvious.

Now let $z \in << m(a, n, b)>_n <c>_n>$.

which implies $z \vee n \in << m(a, n, b)>_n, <c>_n>$. By (ii),

$z \vee n \in <<a>_n, <b>_n> <<b>_n, <a>_n>$. Then by theorem 1.4.7, $z \vee n = x \vee y$ for some $x \in <<a>_n, <b>_n>$ and $y \in <<b>_n, <a>_n>$ and $x, y \geq n$.

Thus $< x >_n \cap < a >_n \subseteq < b >_n$ and so,

$< x >_n \cap < a >_n = < x >_n \cap < a >_n \cap < b >_n \subseteq < z \vee n >_n \cap < a >_n \cap < b >_n$

$= < z \vee n >_n \cap < m(a , n , b) >_n \subseteq < c >_n$.

This implies $x \in << a >_n , < c >_n >$. Similarly $y \in << b >_n , < c >_n >$ and so

$z \vee n \in << a >_n , < c >_n > \vee << b >_n , < c >_n >$.

Similarly, a dual calculation of above shows that

$z \wedge n \in << a >_n , < c >_n > \vee << b >_n , < c >_n >$.

Thus by convexity,

$z \in << a >_n , < c >_n > \vee << b >_n , < c >_n >$ and so

L.H.S. $\subseteq$ R.H.S. Hence (v) holds

(v) $\Rightarrow$ (i). Suppose (v) holds. Let a, b, c $\geq$ n.

By (v), $<< m(a, n, b) >, < c >_n >$.

$= << a >_n , < c >_n > \vee << b >_n , < c >_n >$.

But by lemma 4.4.3.(i), this is equivalent to $< a \wedge b , c > = < a, c > \vee < b, c >$.

Then by [56, Th.3.3.5], this shows that [n) is a relatively Stone.

Similarly, for a , b , c $\leq$ n, using the Lemma 4.4.3 (ii) and Theorem 4.4.7, we find that (n] is relatively dual Stone .

Therefore, by theorem 4.4.9, $F_n(S)$ is relatively Stone. Finally, we need to prove that (iii) $\Rightarrow$ (i).

Suppose (iii) holds .Let a, b, c $\in$ S $\cap$ [n).

By (iii), $< c >_n >, < a >_n \vee < b >_n$

$= << c >_n , < a >_n > \vee << c >_n , < b >_n >$.

But by Lemma 4.4.4(i) ,this is equivalent to $< c , a \vee b > = < c, a > \vee < c, b >$.

Then by [56 , Th.3.3.5],this showsl that [n) is relatively Stone .

Similarly, for a, b, c $\leq$ n, using the lemma 4.4.4(ii) and theorem 4.4.7, we find that (n] is relatively dual Stone. Therefore, by theorem $F_n(S)$ is relatively Stone. ●

By [9] , [38] and [14] we know that a lattice is relatively Stone if and only if any two incomparable prime ideals are co-maximal. [49] extend this result for nearlattices. We conclude this chapter by proving the following result, which is a generalization of [49, Th.1.5]

Theorem 4.4.11. *Let* S *be a distributive medial nearlattice and* $F_n(S)$ *be relatively pseudocomplemented. If* n *is an upper element in* S *then the following conditions are equivalent.*

(i) $F_n(S)$ *is relatively Stone.*

(ii) *Any two incomparable prime* n*-ideals* P and Q *are co-maximal, That is,* $P \vee Q = S$.

Proof: (i) $\Rightarrow$(ii). Suppose $F_n(S)$ is relatively Stone . Then by 4.4.9, $P_n(S)$ is relatively Stone. Let P and Q be two incomparable prime n-ideals of S. Then there exist a , b $\in$ S such that $a \in P - Q$ and $b \in Q - P$.

Then $< a >_n \subseteq P - Q$ and $< b >_n \subseteq Q - P$. Since by (i) , $F_n(S)$ is relatively Stone, so by theorem 4.4.8 , $<< a >_n , < b >_n > \vee << b >_n , < a >_n > = S$.

But as P , Q are prime , so it is easy to see that , $<< a >_n , < b >_n > \subseteq Q$ and $<< b >_n , < a >_n > \subseteq P$.

Therefore, $S \subseteq P \vee Q$ and $P \vee Q = S$.

Thus (ii) holds.

(ii)$\Rightarrow$(i). Suppose (ii) holds.

Let P_1 and Q_1 be two incomparable prime ideals of [n) .Then by Lemma 2.2.5, there exist two incomparable prime ideals P and Q of S such that $P_1 = P \cap [n)$ and $Q_1 = Q \cap [n)$.

Since $n \in P_1$ and $n \in Q_1$, So by Lemma 2.1.3 , P and Q are in fact two incomparable prime n-ideals of S.

Then by (ii), $P \vee Q = S$. Therefore, $P_1 \vee Q_1 = (P \vee Q) \cap [n) = S \cap [n) = [n)$.

Thus by [49, Th. 1.5], [n) is relatively Stone.

Similarly, considering two prime filters of (n] and proceeding as above and using the dual result of [49, Th.1.5], we find that (n] is relatively dual Stone. Therefore by Theorem 4.4.9, $F_n(S)$ is relatively Stone. ●

Chapter 5

CHAPTER 5

CHARACTERIZATIONS OF PRINCIPAL n-IDEALS WHICH ARE SECTIONALLY IN B_m AND RELATIVELY IN B_m

Introduction

Lee in [36], also see Lakser [33] has determined the lattice of all equational subclasses of the class of all pseudocomplemented distributive lattices. They are given by $B_{-1} \subset B_0 \subset B_1 \subset \ldots\ldots\ldots \subset B_m \subset \ldots\ldots \subset B_\omega$, where all the inclusions are proper and B_ω is the class of all pseudocomplemented distributive lattices, B_{-1} consists of all one element algebra, B_0 is the variety of Boolean algebras while B_m, for $-1 \leq m < \omega$ consists of all algebrassatisfying the equation $(x_1 \wedge x_2 \wedge x_3 \wedge \ldots \wedge x_m)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge x_2 \wedge \ldots \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \ldots \wedge x_m)^* = 1$ where x^* denotes the pseudocomplement of x. Thus B_1 consists of all Stone algebras.

Cornish in [9] and Mandelker in [38] have studied distributive latices analogues to B_1-lattices and relatively B_1-lattices. Cornish [9], Beazer [4] Davey [14] have each independently given several characterizations of (sectionally) B_m and relatively B_m-lattices. Moreover, Gratzer and Lakser in [19] and [20] have obtained some results on this topic.

Cornish in [10] have studied distributive lattices (without pseudocomplementation) analogues to B_m-lattices and relatively B_m-lattices. These are known as m-normal and relatively m-normal lattices.

A sectionally pseudocomplemented distributive nearlattice S is called sectionally in B_m if for each $x \in S$, $[0, x]$ is in B_m.

A relatively pseudocomplemented distributive nearlattice is called relatively in B_m if for all $x, y \in S$ with $x < y$, the interval $[x, y]$ is in B_m.

In section 1, we will study principal n-ideals which are sectionally in B_m. We will include several characterizations which generalize several results of [41] and [56]. We shall show that if for a central element $n \in S$, $P_n(S)$ is a sectionally pseudocomple- mented distributive nearlattice , then $P_n(S)$ is sectionally in B_m if and only if for any $x_0, x_1, \ldots, x_m \in S$, $<x_0>_n^+ \vee <x_1>_n^+ \vee \ldots \vee <x_m>_n^+ = S$, where $m(x_i, n, x_j) = n$, which is also equivalent to the condition that for any m+1 distinct minimal prime n-ideals $P_0, \ldots, P_m$ of S, $P_0 \vee P_1 \vee \ldots \vee P_m = S$.

In section 2, we will study those $P_n(S)$ which are relatively in B_m. Here we will include a number of characterizations of those $P_n(S)$ which are relatively in B_m. We shall show that for a central element n , if $P_n(S)$ is relatively pseudocomplemented , then $P_n(S)$ is relatively in B_m if and only if for any m+1 pair wise incomparable prime n-ideals $P_0, \ldots, P_m$ of S, $P_0 \vee P_1 \vee \ldots \vee P_m = S$.

5.1. Nearlattices whose $P_n(S)$ are sectionally in B_m

The following result is due to [14, Lemma 2.2]. This follows from the corresponding result for commutative semi-groups due to Kist [32]. This is also true in case of a distributive nearlattice.

Lemma 5.1.1. *Let* M *be a prime ideal containing an ideal* J *in a distributive nearlattice . Then* M *is a minimal prime ideal belonging to* J *if and only if for all* $x \in M$, *there exists* $x' \notin M$ such that $x \wedge x' \in J$. ●

Now we generalize this result for n-ideals.

Lemma 5.1.2. *Let n be a medial element and* M *be a prime n-ideal containing an n-ideal* J . *Then* M *is a minimal prime n-ideal belonging to* J *if and only if for all* $x \in M$, *there exists* $x' \notin M$ such that $m(x, n, x') \in J$.

Proof: Let M be a minimal prime n-ideal belonging to J and $x \in M$. Then by theorem 4.3.7, $<<x>_n, J> \not\subseteq M$. So there exists x' with $m(x, n, x') \in J$ such that $x' \notin M$.Conversely , suppose $x \in M$, then there exists $x' \notin M$ such that $m(x, n, x') \in J$. This implies $x' \notin M$, but $x' \in <<x>_n, J>$, that is $<<x>_n, J> \not\subseteq M$. Hence by Theorem 4.3.7, M is a prime n-ideal belonging to J. ●

Davey in [14, Corollary 2.3] used the following result in proving several equivalent conditions on B_m-lattices. On the other hand, Cornish in [9] has used this result in studying m-normal lattices. We omit the proof as it is trivial.

Proposition 5.1.3. *Let* $M_0, M_1, \ldots, M_n$ be n+1 distinct minimal prime ideals belonging to J. Then there exists $a_0, a_1, \ldots a_n \in S$ such that $a_i \wedge a_j \in J$ $(i \neq j)$ and $a_j \notin M_j$, j= 0 , 1,.......n. ●

The following result is a generalization of above result in terms of n-ideals.

Proposition 5.1.4. *Let* S *be a distributive nearlattice and* $n\in S$ is medial. Suppose M_0, $M_1, \ldots, M_m$ *be* m+1 *distinct minimal prime* n-*ideals containing* n-*ideal* J. *Then there exists* $a_0, a_1, \ldots a_m \in S$ such that $m(a_i, n, a_j) \in J$ $(i \neq j)$ and $a_j \notin M_j$ $(j = 0, 1, \ldots m)$.

Proof: For n = 1. Let $x_0 \in M_1 - M_0$ and $x_1 \in M_0 - M_1$. Then by lemma 5.1.1, there exists $x_1' \notin M_0$ such that $m(x_0, n, x_1') \in J$.

Hence $a_1 = x_1$, $a_0 = m(x_0, n, x_1')$ are the required elements.

Observe that $m(a_0, n, a_1) = m(m(x_0, n, x_1'), n, x_1)$

$$= (x_0 \wedge x_1 \wedge x_1') \vee (x_0 \wedge n) \vee (x_1 \wedge n) \vee (x_1' \wedge n)$$

$$= (x_0 \wedge m(x_1, n, x_1')) \vee (x_0 \wedge n) \vee (m(x_1, n, x_1') \wedge n)$$

$$= m(x_0, n, m(x_1, n, x_1'))$$

Now, $m(x_1, n, x_1') \wedge n \leq m(x_0, n, m(x_1, n, x_1'))$

$$\leq m(x_1, n, x_1') \vee n$$

and $m(x_1, n, x_1') \in J$, so by convexity $m(a_0, n, a_1) \in J$. Assume that, the result is true for n = m-1, and let $M_0, \ldots M_m$ be m+1 distinct minimal prime n-ideals. Let b_j $(j = 0, 1, \ldots m-1)$ satisfy $m(b_i, n, b_j) \in J$ $(i \neq j)$ and $b_j \notin M_j$. Now choose $b_m \in M_m - \bigcup_{j=0}^{m-1} M_j$ and by Lemma 5.1.2, let $b_{m'}$ satisfy $b_{m'} \notin M_m$ and $m(b_m, n, b_{m'}) \in J$.

Clearly, $a_j = m(b_j, n, b_m)$ $(j = 0, \ldots m-1)$ and $a_m = b_{m'}$, establish the result. ●

Let J be an n-ideal of a distributive lattice L. A set of elements $x_0, \ldots, x_n \in L$ is said to be pair wise in J if $m(x_i, n, x_j) = n$ for all $i \neq j$.

The next result is due to [56, Lemma 3.4.1]. In case of lattices it was proved by [10, Lemma2.3] which was suggested by Hindman in [26, Th.1.8].

Lemma 5.1.5. *Let* J *be an ideal in a distributive nearlattice* S. *For a given positive integer* $n \geq 2$, *the following conditions are equivalent.*

(i) *For any* $x_1, x_2, \ldots, x_n \in S$ *which are 'pairwise in* J', that is $x_i \wedge x_j \in J$ *for any* $(i \neq j)$, *there exists* k *such that* $x_k \in J$.

(ii) For any ideals $J_1, \ldots, J_n \in S$ *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$, *there exists k such that* $J_k \subseteq J$.

(iii) J *is the intersection of at most m-1 distinct prime ideals.* ●

Our next result is a generalization of above result. This result will be needed in proving the next theorem which is the main result of this section. In fact, the following lemma is very useful in studying those $P_n(S)$, which are sectionally in B_m.

Lemma 5.1.6. *Let* J *be an n-ideal in a distributive nearlattice S and* $n \in S$ *is medial. For a given positive integer* $m \geq 2$, *the following conditions are equivalent.*

(i) *For any* $x_1, x_2, \ldots x_n \in S$ *with* $m(x_i, n, x_j) \in J$ (*that is, they are pair wise in* J) *for any* $i \neq j$, *there exists* k *such that* $x_k \in J$.

(ii) *For any* n-*ideals* $J_1, \ldots, J_m$ in S *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$, *there exists k such that* $J_k \subseteq J$.

(iii) J *is the intersection of at most m-1 distinct prime* n-*ideals.*

Proof: (i) and (ii) are easily seen to be equivalent.

(iii) $\Rightarrow$ (i). Suppose $P_1, P_2, \ldots P_k$ are $(1 \leq k \leq m\text{-}1)$ distinct prime n-ideals such that $J = P_1 \cap P_2 \cap \ldots \cap P_k$. Let $x_1, x_2, \ldots x_m \in S$ be such that $m(x_i, n, x_j) \in J$ for all $i \neq j$.

Suppose no element x_i is a member of J. Then for each r $(1 \leq r \leq k)$ there is at most one i $(1 \leq i \leq m)$ such that $x_i \in P_r$. Since $k < m$, there is some i such that $x_i \in P_1 \cap P_2 \cap \ldots \cap P_k$.

We need to show (i) $\Rightarrow$ (iii). Suppose (i) holds for m = 2, Then it implies that J is a prime n-ideal. Then (iii) is trivially true. Thus we may assume that there is a largest integer t with $2 \leq t < m$ such that the condition (i) does not hold for J (consequently condition (i) holds for t+1 , t+2 , , m). Then for t < m, we may suppose that there exist elements $a_1, a_2, ..., a_t \in S$ such that $m(a_i, n, a_j) \in J$ for $i \neq j$, $i = 1,2, ...t$, $j = 1,2,...,t$ yet $a_2, ..., a_t \notin J$.

As S is a distributive nearlattice, $< <a_i>_n , J >$ is an n-ideal for any $i \in \{1,2,...,t\}$. Each $< <a_i>_n , J >$ is in fact a prime n-ideal. Firstly $< <a_i>_n , J > \neq S$, since $a_i \notin J$. Secondly, suppose that b and c are in S and $m(b, n, c) \in < <a_i>_n , J >$. Consider the set of t+1 elements $\{ a_1, ..., a_{i-1}, m(b , n , a_i) , m(c , n , a_i), a_{i+1}, ..., a_t \}$. This set is pair wise in J and so, either $m(b , n , a_i) \in J$ or $m(c , n , a_i) \in J$. Since condition (i) holds for t+1.

That is, $b \in < <a_i>_n , J >$ or $c \in < <a_i>_n , J >$ and so $< <a_i>_n , J >$ is prime. Clearly, $J \subseteq \bigcap_{1 \leq i \leq t} < <a_i>_n , J >$. If $w \in < <a_i>_n , J >$, then w, $a_1, ..., a_t$ are pair wise in J and so, $w \in J$ .Hence $J = \bigcap_{1 \leq i \leq t} < < a >_n , J >$ is the intersection of t < m prime n-ideals. ●

An ideal $J \neq S$ satisfying the equivalent conditions of Lemma 5.1.5 is called an m-prime ideal.

Similarly, an n-ideal $J \neq S$ satisfying the equivalent conditions of Lemma 5.1.7, is called an m-prime n-ideal.

Now we generalize a result of Davey in [14 , Proposition 3.1.] .

Therem 5.1.7. *Let* J *be an* n-*ideal of a distributive nearlattice* S *and* n *be a central element of* S. *Then the following conditions are equivalent.*

(i) *For any* m+1 *distinct prime* n-*ideals* $P_0, ..., P_m$ *belonging to* J, $P_0 \vee P_1 \vee ..... \vee P_m = S$.

(ii) Every prime n-ideal containing J contains at most m distinct minimal prime n-ideals belonging to J.

(iii) *If* $a_0, \ldots, a_m \in S$ *with* $m(a_i, n, a_j) \in J$ $(i \neq j)$ *then* $\vee_j < \langle a_j \rangle_n, J > = S$.

Proof: (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Assume $a_0, a_1, \ldots, a_m \in S$ with $m(a_i, n, a_j) \in J$ and $\vee_j < \langle a_j \rangle_n, J > \neq S$. It follows that $a_j \notin J$, for all j. Then by Theorem 2.1.7, there exists a prime n-ideal P such that $\vee_j < \langle a_j \rangle_n, J > \subseteq P$. But by Theorem 2.1.2, we know that P is either a prime ideal or a prime filter.

Suppose P is a prime ideal. For each j, let $F_j = \{ x \wedge y : x \geq a_j, x, y \geq n, y \notin P\}$. Let $x_1 \wedge y_1, x_2 \wedge y_2 \in F_j$. Then $(x_1 \wedge y_1) \wedge (x_2 \wedge y_2) = (x_1 \wedge x_2) \wedge (y_1 \wedge y_2)$. Now $x_1 \wedge x_2 \geq a_j$ and $y_1 \wedge y_2 = m(y_1, n, y_2)$. So, $t \geq x \wedge y$ implies $t = (t \vee x) \wedge (t \vee y)$. Since $y \notin P$, so $t \vee y \notin P$. Hence $t \in F_j$. and so F_j is a dual ideal.

We now show that $F_j \cap J = \Phi$, for all $j = 0, 1, .., m$. If not, let $b \in F_j \cap J$, then $b = x \wedge y$, $x \geq a_j$, $x, y \geq n$, $y \notin P$. Hence $m(a_j, n, y) = (a_j \wedge n) \vee n \vee (a_j \wedge y) = (a_j \wedge y) \vee n = (a_j \vee n) \wedge (y \vee n)$. But $(a_j \vee n) \wedge (y \vee n) \in F_j$ and $n \leq (a_j \wedge y) \vee n \leq b$ implies $m(a_j, n, y) \in J$. Therefore $m(a_j, n, y) \in F_j \cap J$. Again, $m(a_j, n, y) \in J$ with $y \notin P$ implies $< \langle a_j \rangle_n, J > \not\subseteq P$, which is a contradiction. Hence $F_j \cap J = \Phi$ for all j.

For each j, let P_j be a minimal prime n-ideal belonging to J and $F_j \cap P_j = \Phi$. Let $y \in P_j$. If $y \notin P$, then $y \vee n \notin P$. Then $m(a_j, n, y \vee n) = (a_j \vee n) \wedge (y \vee n) \in F_j$. But $m(a_j, n, y \vee n) \in \langle y \vee n \rangle_n \subseteq \langle y \rangle_n \subseteq P_j$, which is a contradiction. So $y \in P$. Therefore, $P_j \subseteq P$, and $a_j \notin P_j$. For if $a_j \in P_j$, then $a_j \vee n \in P_j$. Now, $a_j \vee n = (a_j \vee n) \wedge (a_j \vee n \vee y) \in F_j$ for any $y \notin P$. This implies $P_j \cap F_j \neq \Phi$, which is a contradiction. So $a_j \notin P_j$. But $m(a_i, n, a_j) \in J \subseteq P_j$ $(i \neq j)$ which implies $a_i \in P_j$ $(i \neq j)$ as P_j is prime. It follows that P_j form a set of m+1 distinct minimal prime n-ideals belonging to J and contained in P. This contradicts (ii). Therefore $\vee_j < \langle a_j \rangle_n, J > = S$.

Similarly, if P is filter, then a dual proof of above also shows that.

$\underset{j}{\vee} < <a_j>_n , J > = S$, and hence (iii) holds.

Finally, we need to show that (iii) $\Rightarrow$ (i). Let $P_0, P_1, \ldots, P_m$ be m+1 distinct minimal prime n-ideals belonging to J. Then by poposition 5.1.4, there exists $a_0, a_1, \ldots, a_m \in S$ such that $m(a_i , n , a_j) \in J$ $(i \neq j)$ and $a_j \notin P_j$. This implies $< <a_j>_n , J > \subseteq P_j$ for all j. Then by (iii), $< <a_0>_n , J > \vee < <a_1>_n , J > \vee \ldots\ldots\ldots \vee < <a_m>_n , J > \subseteq P_0 \vee P_1 \vee \ldots\ldots \vee P_m$, which implies $P_0 \vee P_1 \vee \ldots\ldots \vee P_m = S$. ●

We have already mentioned that Lee [36] and Lakser [33] have shown That the equational classes of pseudocomplemented distributive lattices form a chain $B_{-1} \subset B_0 \subset B_1 \subset \ldots\ldots \subset B_m \subset \ldots\ldots \subset B_\omega$, where B_{-1} is the trivial class, B_0 is the class of Boolean algebras and B_1 is the class of Stone lattices. Cornish in [9] and Mandelker in [38] considered distributive lattices analogues to B_1-lattices and relative B_1-lattices.The following result is due to [56 , Th. 3.4.2] , also see [4] , which is a generalization of a result in [10] . For lattices this result characterizes the distributive lattices analogues to B_m-lattices.

Beazer [4] , Davey [14] have each independently obtained a version of this result . Gratzer and Lakser in [20] (also see [16 Lemma 2 page 169]) have shown that condition (iii) of the theorem is equivalent to Lee's condition which characterize the nth variety for $0<n<\omega$, of pseudocomplemented distributive lattices. Thus this theorem should be compared with Lee's Theorem 2 of [36]. Recall that for a prime ideal P of a distributive nearlattice S, $0(P) = \{x: x \wedge y = 0$ for some $y \in S - P$
Ph.D. Thesis, Rajshahi University, Rajshahi, which is an ideal contained in P.

Theorem 5.1.8. *Let* S *be a distributive nearlattice with* 0. *Then the following conditions are equivalent.*

(i) *For any* m+1 *distinct minimal prime ideals* $P_0, P_1, \ldots, P_m$, $P_0 \vee P_1 \vee \ldots\ldots\ldots \vee P_m = S$.

(ii) *Every prime ideals contains at most m minimal prime ideals.*

(iii) *For any* $x_0, x_1, \ldots, x_m \in S$ *such that* $x_i \wedge x_j = 0$ for $(i \neq j)$, $i = 0, 1, 2, \ldots, m$, $j = 0, 1, 2, \ldots, m$, $(x_0]^* \vee (x_1]^* \vee \ldots\ldots\ldots\ldots \vee (x_m]^* = S$.

(iv) *For each prime ideal* P, 0(P) is m+1 *prime.* ●

Our next result is a nice extension of above result in terms of n-ideals. Recall that for a prime n-ideal P of S, $n(P) = \{ x \in S: m(x, n, y) = n \text{ for some } y \in S-P\}$. Of course, n(P) is an ideal and $n(P) \subseteq P$.

Theorem 5.1.9. *Let* S *be a distributive nearlattice with a central element* n. *Then the following conditions are equivalent.*

(i) *For any* m+1 *distinct minimal prime* n-*ideals* $P_0, P_1, \ldots, P_m$, $P_0 \vee P_1 \vee \ldots\ldots\ldots \vee P_m = S$.

(ii) *Every prime* n-*ideals contains at most* m *minimal prime* n-*ideals.*

(iii) *For any* $a_0, a_1, \ldots, a_m \in S$ *with* $m(a_i, n, a_j) = n$ for $(i \neq j)$, $i = 0, 1, 2, \ldots, m$, $j = 0, 1, 2, \ldots, m$, $\langle a_0 \rangle_n^+ \vee \langle a_1 \rangle_n^+ \vee \ldots\ldots\ldots\ldots \vee \langle a_m \rangle_n^+ = S$.

(iv) *For each prime* n-*ideal* P, n(P) is an m+1 *prime n-ideal.*

Proof: (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) easily hold by theorem 5.1.8, replacing J by { n}.

To complete the proof we need to show that (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (iii). Suppose (iv) holds and $x_0, x_1, ..., x_m$ are m+1 elements of S such that $m(x_i, n, x_j) = n$ for $(i \neq j)$. Suppose $<x_0>_n^+ \vee <x_1>_n^+ \vee .... \vee <x_m>_n^+ \neq S$. Then by Theorem 2.1.7, there is a prime n-ideal P such that $<x_0>_n^+ \vee <x_1>_n^+ \vee .... \vee <x_m>_n^+ \subseteq P$. Hence $x_0, x_1, ..., x_m \in S - n(P)$.

This contradicts (iv) by lemma 5.1.6 , since $m(x_i, n, x_j) = n \in n(P)$ for $i \neq j$.

Thus (iii) holds.

(ii) $\Rightarrow$ (iv). This follows immediately from Proposition 4.2.9 and Lemma 5.1.6. above. ●

Following result is due to [56, Th.3.4.5]

Proposition 5.1.10. *Let* S *be a distributive nearlattice with* 0. *If the equivalent conditions of Theorem* 5.1.8. *hold , then for any* m+1 *elements* $x_0, x_1,..., x_m$, $(x_0 \wedge x_1 \wedge .... \wedge x_m]^* = \bigvee_{0 \leq i \leq n} (x_0 \wedge x_1 \wedge ..... \wedge x_{i-1} \wedge x_{i+1} \wedge .... \wedge x_m]^*$ ●

Proposition 5.1.11. *Let* S *be distributive nearlattice and* $n \in S$ *be a central element. If the equivalent conditions of Theorem* 5.1.10 *hold , then for any* m+1 *elements* $x_0, x_1,...,_m$, $(<x_0>_n \cap <x_1>_n \cap ... \cap <x_m>_n)^+ = \bigvee_{0 \leq i \leq n} (<x_0>_n \cap <x_1>_n \cap .... \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap ... \cap <x_m>_n)^+$.

Proof: Let $<b_i>_n = <x_0>_n \cap <x_1>_n \cap ..... \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap .... \cap <x_m>_n$ for each $0 \leq i \leq m$. Suppose $x \in (<x_0>_n \cap <x_1>_n \cap .... \cap <x_m>_n)^+$. Then $<x>_n \cap <x_0>_n \cap <x_1>_n \cap .... \cap <x_m>_n = \{n\}$. For all $i \neq j$, $(<x>_n \cap <b_i>_n) \cap (<x>_n \cap <b_j>_n) = \{n\}$. So, $(<x>_n \cap <b_0>_n)^+ \vee ...... \vee (<x>_n \cap <b_m>_n)^+ = S$. Thus $x \in (<x>_n \cap <b_0>_n)^+ \vee ...... \vee (<x>_n \cap <b_m>_n)^+$. Hence by the Theorem1.4.7, $x \vee n = a_0 \vee a_1 \vee .... \vee a_m$

where $a_i \in (<x>_n \cap <b_i>_n)^+$ and $a_i \geq n$ for $i = 0, 1, \ldots, m$. Then $x \vee n = (a_0 \wedge (x \vee n)) \vee (a_1 \wedge (x \vee n)) \vee \ldots \vee (a_m \wedge (x \vee n))$. Now $a_i \in (<x>_n \cap <b_i>_n)^+$ implies $<a_i>_n \cap <x>_n \cap <b_i>_n = \{n\}$. Then by routine calculation we find that $(a_i \wedge x \wedge b_i) \vee n = n$. Thus, $a_i \wedge (x \vee n) \in <b_i>_n^+$ and so, $< a_i \wedge (x \vee n) >_n \cap < b_i >_n = [\, n\,,\ (a_i \wedge x \wedge b_i) \vee n\,] = \{n\}$ implies that $x \vee n \in <b_0>_n^+ \vee <b_1>_n^+ \vee \ldots \vee <b_m>_n^+$. By a dual proof of above and using Theorem 1.4.7, we can easily show that

$x \wedge n \in <b_0>_n^+ \vee <b_1>_n^+ \vee \ldots \vee <b_m>_n^+$. Thus by convexity,

$x \in \in <b_0>_n^+ \vee <b_1>_n^+ \vee \ldots \vee <b_m>_n^+$. This proves that L.H.S. $\subseteq$ R.H.S. The reverse inclusion is trivial. ●

Theorem 5.1.12. *Let* S *be a distributive nearlattice and* $n \in S$ *is central. If* $P_n(S)$ *is sectionally pseudocomplemented ,then the following conditions are equivalent.*

(i) $P_n(S)$ *is sectionally in* B_m.

(ii) *Every prime* n-*ideal contains at most* m *minimal prime* n-*ideals.*

(iii) *For any* m+1 *distinct minimal prime* n-*ideals* $P_0, P_1, \ldots, P_m$,

$P_0 \vee P_1 \vee \ldots \vee P_m = S$.

(iv) *If* $m(a_i\,, n\,, a_j) = n$, *this implies* $<a_0>_n^+ \vee <a_1>_n^+ \vee \ldots \vee <a_m>_n^+ = S$.

(v) *For each prime* n-*ideal* P, n(P) *is an* m+1 *prime* n-*ideal.*

Proof: (i) $\Rightarrow$ (ii). Let $P_n(S)$ be sectionally in B_m , since n is central, so by Theorem 2.2.2, both $(n]^d$ and $[n)$ are sectionally in B_m. Suppose P is any prime n-ideal of S. Then by Theorem 2.1.1, either $P \supseteq (n]^d$ or $P \supseteq [n)$. Without loss of generality, suppose $P \supseteq [n)$. Then by Theorem 2.1.2, P is prime ideal of S. Hence by Lemma 2.2.5, $P_1 = P \cap [n)$ is a prime ideal of $[n)$. Since $[n)$ is sectionally in B_m, so by definition P_1 contains at most m minimal prime ideals $R_1, R_2, \ldots R_m$ of $[n)$.

Therefore, P contains at most m minimal prime ideals $T_1, \ldots, T_m$ of S where $R_1 = T_1 \cap [n)$, $R_2 = T_2 \cap [n)$, ..., $R_m = T_m \cap [n)$. Since $n \in R_1, R_2, \ldots R_m$, $n \in T_1, \ldots, T_m$, hence $T_1, \ldots, T_m$ are minimal prime n-ideals of S Thus (ii) holds.

(ii) $\Rightarrow$ (i). Suppose (ii) holds. Let P_1 be a prime ideal in [n). Then by Lemma 2.2.5, $P_1 = P \cap [n)$ for some prime ideal P of S. Since $n \in P_1 \subseteq P$, so P is prime n-ideal. Therefore, P contains at most m minimal prime n-ideals $R_1, R_2, \ldots R_m$ of S . Thus by lemma 2.2.5, P_1 contains at most m minimal prime n-ideals $T_1 = R_1 \cap [n)$, $T_2 = R_2 \cap [n)$,.........., $T_m = R_m \cap [n)$ of [n).

Hence by definition , [n) is sectionally in B_m. Similarly , we can prove that $(n]^d$ is also sectionally dual in B_m. Hence by Theorem 2.2.2, $P_n(S)$ is sectionally in B_m.

(ii) $\Rightarrow$ **(iii)** easily hold by Theorem 5.1.7 replacing J by {n}. Other conditions follow from Theorem 5.1.9. ●

5.2. Generalizations of some results on nearlattices which are relatively in B_m

Several characterizations on relative B_m-lattices have been given by Davey in [14]. Also Cornish have studied these lattices in [10] under the name of relatively in B_m lattices. Then [56] have given the concept of relatively m-normal nearlattices. Following result give some characterizations of $P_n(S)$ which are relatively in B_m. This also generalizes a result in [14].

Theorem 5.2.1. *Let* S *be a distributive nearlattice with* n *as a central element of* S. *Suppose* $P_n(S)$ *is relatively pseudocomplemented. Then the following conditions are equivalent.*

(i) $P_n(S)$ *relatively in* B_m.

(ii) *For all* $x_0, x_1, \ldots, x_m \in S$

$$<<x_1>_n \cap <x_2>_n \cap \ldots\ldots \cap <x_m>_n, <x_0>_n> \vee < <x_0>_n \cap <x_2>_n \cap \ldots\ldots \cap <x_m>_n, <x_1>_n> \vee \ldots\ldots \vee < <x_0>_n \cap \ldots\ldots\ldots \cap <x_{m-1}>_n, <x_m>_n> = S.$$

(iii) $x_0, x_1, \ldots. x_m, z \in S$,

$$<<x_0>_n \cap <x_1>_n \cap \ldots\ldots\ldots \cap <x_m>_n, <z>_n> = < <x_1>_n \cap <x_2>_n \cap \ldots. \cap <x_m>_n, <z>_n> \vee <<x_0>_n \cap \ldots.. \cap <x_m>_n, <z>_n> \vee \ldots.. \vee <x_0>_n \cap <x_1>_n \cap \ldots.. \cap <x_{m-1}>_n, <z>_n>.$$

(vi) *For any* m+1 *pair wise incomparable pime* n-*ideals* $P_0, P_1, \ldots\ldots, P_m$,

$$P_0 \vee P_1 \vee \ldots\ldots\ldots \vee P_m = S.$$

(iv) *Any prime* n-*ideal contains at most* m *mutually incomparable prime* n-*ideals.*

Proof: (i) $\Rightarrow$ (ii). Let $z \in S$, consider the interval

$$I = [<< x_0 >_n \cap < x_1 >_n \cap \ldots\ldots\ldots. \cap < x_m >_n \cap < z >_n, < z >_n>] \text{ in } P_n(S).$$

Then $< x_0 >_n \cap < x_1 >_n \cap \dots\dots\dots \cap < x_m >_n \cap < z >_n$ is the smallest element of the interval I. For $0 \leq i < m$, the set of elements $< t_i >_n = < x_0 >_n \cap < x_1 >_n \cap \dots\dots \cap < x_{i-1} >_n \cap < x_{i+1} >_n \cap \dots\dots \cap < x_m >_n \cap < z >_n >$ are obviously pairwise disjoint in the interval I.Since I is in B_m, so by Theorem 5.1.12, $< t_0 >_n{}^{\circ} \vee \dots\dots\dots\dots \vee < t_m >_n{}^{\circ} = < z >_n$. So by Theorem 1.4.7, $z \vee n = p_0 \vee p_1 \vee \dots\dots\dots \vee p_m$ where $p_i \geq n$.

Thus , $< p_0 >_n \cap < t_0 >_n = < p_1 >_n \cap < t_1 >_n = \dots\dots\dots = < p_m >_n \cap < t_m >_n =$ The smallest element of I $= < x_0 >_n \cap < x_1 >_n \cap \dots\dots\dots \cap < x_m >_n \cap < z >_n$.

Now , $< p_0 >_n \cap < t_0 >_n = < x_0 >_n \cap < x_1 >_n \cap \dots\dots\dots \cap < x_m >_n \cap < z >_n$ which implies $< p_0 >_n \cap < t_0 >_n \subseteq < x_0 >_n$.
Again, $< p_0 >_n \cap < t_0 >_n = < p_0 >_n \cap < x_1 >_n \cap \dots\dots\dots \cap < x_m >_n \cap < z >_n = < p_0 >_n \cap < x_1 >_n \cap \dots\dots\dots \cap < x_m >_n$, as $< p_0 >_n \subseteq < z >_n >$

This implies $< p_0 >_n \cap < x_1 >_n \cap \dots\dots\dots \cap < x_m >_n \subseteq < x_0 >_n$ and so,

$< p_0 >_n \in << x_1 >_n \cap \dots\dots\dots \cap < x_m >_n , < x_0 >_n >$

$< p_1 >_n \in << x_0 >_n \cap < x_2 >_n \cap \dots\dots\dots \cap < x_m >_n , < x_1 >_n >$

....

....

$< p_m >_n \in << x_0 >_n \cap < x_1 >_n \cap \dots\dots\dots \cap < x_{m-1} >_n , < x_m >_n >$.

Therefore, $z \vee n \subseteq << x_1 >_n \cap < x_2 >_n \cap \dots\dots\dots \cap < x_m >_n , < x_0 >_n >$

$\vee << x_0 >_n \cap < x_2 >_n \cap \dots\dots\dots \cap < x_m >_n , < x_1 >_n >$

$\vee \dots\dots \vee << x_0 >_n \cap < x_1 >_n \cap \dots\dots\dots \cap < x_{m-1} >_n , < x_m >_n >$.

By a dual proof of above we can easily show that,

$z \wedge n \subseteq << x_1 >_n \cap < x_2 >_n \cap \dots\dots\dots \cap < x_m >_n , < x_0 >_n >$

$\vee << x_0 >_n \cap < x_2 >_n \cap \dots\dots\dots \cap < x_m >_n , < x_1 >_n >$

$\vee \dots\dots\dots \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n , \langle x_m\rangle_n \rangle$.

Hence by convexity,

$z \in \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n , \langle x_0\rangle_n \rangle$

$\vee \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n , \langle x_1\rangle_n \rangle$

$\vee \dots\dots\dots \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n , \langle x_m\rangle_n \rangle$.

This implies (ii) holds.

(ii) $\Rightarrow$ (iii). Suppose $b \in \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n , \langle z\rangle_n \rangle$. Then by (ii) and the Theorem 1.4.7, $b \vee n = s_0 \vee s_1 \dots\dots \vee s_m$ for some

$s_0 \in \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n , \langle x_0\rangle_n \rangle$

$s_1 \in \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n , \langle x_1\rangle_n \rangle$

....

....

$s_m \in \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n , \langle x_m\rangle_n \rangle$ and so $s_i \geq n$, $i = 0,1,2, \dots,m$.

Thus $\langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n \cap \langle s_0\rangle_n \subseteq \langle x_0\rangle_n$

$\langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n \cap \langle s_1\rangle_n \subseteq \langle x_1\rangle_n$

....

....

$\langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n \cap \langle s_m\rangle_n \subseteq \langle x_m\rangle_n$.

This implies $\langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n \cap \langle s_0\rangle_n$

$= \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n \cap \langle s_0\rangle_n$

$\subseteq \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n \cap \langle b \vee n\rangle_n \subseteq \langle z \rangle_n$.

Hence so, $s_0 \in \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n , \langle z \rangle_n \rangle$

Similarly , $s_1 \in \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n , \langle z \rangle_n \rangle$

....

....

$s_m \in \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n, \langle z\rangle_n\rangle$.

Therefore, $b \vee n \subseteq \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle$

$\vee \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle$

$\vee \dots\dots \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n, \langle z\rangle_n\rangle$.

The dual proof of above gives,

$b \wedge n \subseteq \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle$

$\vee \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle$

$\vee \dots\dots\dots \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n, \langle z\rangle_n\rangle$.

Thus by convexity,

$b \in \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle$

$\vee \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle$

$\vee \dots\dots\dots \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n, \langle z\rangle_n\rangle$.

Therefore,

$\langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle \subseteq$

$\langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle$

$\vee \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \dots\dots\dots \cap \langle x_m\rangle_n, \langle z\rangle_n\rangle$

$\vee \dots\dots \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \dots\dots\dots \cap \langle x_{m-1}\rangle_n, \langle z\rangle_n\rangle$.

Since the reverse inequality always holds, so (iii) holds.

(iii) $\Rightarrow$ (i). Suppose $n \leq b \leq d$. Let $x_0, x_1, \dots, x_m \in [b, d]$ such that $x_i \wedge x_j = b$, for all $i \neq j$.

Let $t_0 = x_1 \vee x_2 \vee \dots\dots\dots \vee x_m$

$t_1 = x_0 \vee x_2 \vee \ldots\ldots\ldots\ldots \vee x_m$

....

....

$t_m = x_0 \vee x_1 \vee \ldots\ldots\ldots\ldots \vee x_{m-1}.$

Clearly, $n \leq b \leq t_i \leq d$ and

$x_0 = t_1 \vee t_2 \vee \ldots\ldots\ldots\ldots \vee t_m$

$x_1 = t_0 \vee t_2 \vee \ldots\ldots\ldots\ldots \vee t_m$

....

....

$x_m = t_1 \vee t_2 \vee \ldots\ldots\ldots\ldots \vee t_{m-1}.$

Then $[b, d] \cap \{ <<x_0>_n, <b>_n> \vee \ldots\ldots\ldots\ldots \vee <<x_m>_n, <b>_n> \}$

$= [b, d] \cap \{ <<t_1>_n \cap <t_2>_n \cap \ldots\ldots\ldots\ldots \cap <t_m>_n, <b>_n >$

$\vee <<t_0>_n \cap <t_2>_n \cap \ldots\ldots\ldots\ldots \cap <t_m>_n, <b>_n >$

$\vee \ldots\ldots\ldots\ldots \vee <<t_1>_n \cap <t_2>_n \cap \ldots\ldots\ldots\ldots \cap <t_{m-1}>_n, <b>_n >$

$= [b, d] \cap \{ <<t_0>_n \cap <t_1>_n \cap \ldots\ldots\ldots\ldots \cap <t_m>_n, <b>_n >\}$

$= [b, d] \cap < <b>_n, <b>_n$

$= [b, d] \cap S.$

$= [b, d].$

That is $[b, d]$ is in B_m. Hence $[n)$ is relatively in B_m. A dual proof of above shows that $(n]$ is relatively in dual B_m. Hence by Theorem 2.2.2, $P_n(S)$ is relatively in B_m.

(ii) $\Rightarrow$ (iv). Suppose (ii) holds. Let $P_0, P_1, \ldots\ldots P_m$ be m+1 pairwise incomparable prime n-ideals. Then there exists $x_0, x_1, \ldots\ldots x_m \in S$ such that

$x_i \in P_j - \bigcup_{i \neq 1}^{n} P_j$. Then by (ii),

$<<x_1>_n \cap <x_2>_n \cap .... \cap <x_m>_n , <x_0>_n >$

$\vee <<x_0>_n \cap <x_2>_n \cap .... \cap <x_m>_n , <x_1>_n >$

$\vee .... \vee <<x_0>_n \cap <x_1>_n \cap .... \cap <x_{m-1}>_n , <x_m>_n > = S.$

Let $t_0 \in <<x_1>_n \cap <x_2>_n \cap .... \cap <x_m>_n , <x_0>_n >$, then

$<<t_0>_n \cap <x_1>_n \cap .... \cap <x_m>_n \subseteq <x_0>_n \subseteq P_0.$

Now, $x_i \notin P_0$ for $i = 1, 2, ...m$.

Thus $<<x_1>_n \cap <x_2>_n \cap .... \cap <x_m>_n \not\subset P_0$ as P_0 is prime . This implies $< t_0 >_n \subseteq P_0,$ and so, $t_0 \in P_0.$

Therefore,

$<<x_1>_n \cap <x_2>_n \cap .... \cap <x_m>_n , <x_0>_n > \subseteq P_0$

$<<x_0>_n \cap <x_2>_n \cap .... \cap <x_m>_n , <x_1>_n > \subseteq P_1$

$<<x_1>_n \cap <x_2>_n \cap .... \cap <x_m>_n , <x_0>_n > \subseteq P_0$

$<<x_0>_n \cap <x_1>_n \cap .... \cap <x_m>_n , <x_2>_n > \subseteq P_2$

....

....

$<<x_0>_n \cap <x_1>_n \cap .... \cap <x_{m-1}>_n , <x_m>_n > \subseteq P_m.$

Hence , $P_0 \vee P_1 \vee .... \vee P_m = S.$

(iv) $\Rightarrow$ (v) is trivially by Stone's seperation theorem.

(iv) $\Rightarrow$ (i). Let any m+1 pair wise incomparable prime n-ideals of S are co-maximal. Consider the interval [b , d] in S with $d \geq n$, let P_0' , P_1', ...,P_m' be m+1 distinct minimal prime ideals of [b , d] . Then by lemma 2.2.5 there exists prime ideals $P_0,$ ...., P_m of S such that $P_0' = P_0 \cap [b, d]$, ..., $P_m' = P_m \cap [b, d]$.

Since P_i is an ideal , so $b \in P_i$. Moreover, $n \leq b$ implies that $n \in P_i$. Therefore each Pi is a prime n-ideal by lemma 1.4.3, i = 1, 2,....,m. Since P_0' , P_1', ...,P_m' are incomparable , so P_0 , P_1, ...,P_m are also incomparable . Now by (iv), $P_0 \vee P_1 \vee \dots\dots\vee P_m = S$.

Hence $P_0' \vee P_1' \vee \dots\dots\dots \vee P_m' = (P_0 \vee P_1 \vee \dots\dots\vee P_m) \cap [b, d]$

$$= S \cap [b, d]$$

$$= [b, d].$$

Therefore, by Theorem 4.1.2, [b, d] is in B_m. Hence [n) is relatively in B_m. A dual proof of above shows that (n] is relatively in dual B_m. Since by 2.2.2,

$P_n(S) \cong (n]^d \times [\,n\,)$, so $P_n(S)$ is relatively in B_m. ●

We conclude the thesis with the following result which is also a generalization of [14, Theorem 3.4].

Theorem 5.2.2. *Let* S *be a distributive nearlattice with* n $\in$ S *as an upper element . Then the following conditions are equivalent.*

(i) $P_n(S)$ *is relatively in* B_m .

(ii) If b , a_0 , ..., $a_m \in S$ *with* $m(a_i, n, a_j) \in \langle b\rangle_n (i \neq j)$, *then*

$$\langle \langle a_0\rangle_n , \langle b\rangle_n\rangle \vee \dots\dots\dots \vee \langle \langle a_m\rangle_n , \langle b\rangle_n\rangle = S.$$

Proof: (i) $\Rightarrow$ (ii). By Theorem 5.2.1. (v), any prime n-ideal containing b contains at most m minimal prime n-ideals belonging to $\langle b\rangle_n$. Hence by Theorem 5.1.7, with $J = \langle b\rangle_n$, we have ,

$\langle \langle a_0\rangle_n , \langle b\rangle_n\rangle \vee \dots\dots\dots \vee \langle \langle a_m\rangle_n , \langle b\rangle_n\rangle = S$. Thus (ii) holds.

(ii) $\Rightarrow$ (i). Consider $b \in [\,n\,)$ with $b \leq c$. Let a_0 , ..., $a_m \in [\,b, c\,]$ with

$a_i \wedge a_j = b$ $(i \neq j)$, then by $m(a_i, n, a_j) = b \in \langle b\rangle_n$. Then by (ii),

$<<a_0>_n, <b>_n> \vee \ldots\ldots\ldots\ldots \vee << a_m>_n, <b>_n> = S.$

So, $[b, c] = (<<a_0>_n, <b>_n> \cap [b, c]) \vee \ldots\ldots \vee << a_m>_n, <b>_n> \cap [b, c]$

$$= < a_0, b >_{[b,c]} \vee \ldots\ldots\ldots\ldots\ldots\ldots \vee < a_m, b >_{[b,c]} .$$

Hence by the Theorem 5.1.8, [b, c] is in B_m. Therefore, [n) is relatively in B_m.

A dual proof of above shows that (n] is relatively in dual B_m. Therefore by Theorem 2.2.2, $P_n(S)$ is relatively in B_m. ●

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