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KBM Asymptotic Method for third order Nonlinear Oscillations

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KBM ASYMPTOTIC METHOD FOR THIRD ORDER NONLINEAR OSCILLATIONS

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Declaration

Certified that the thesis 'KBM Asymptotic Method for Third Order Nonlinear Oscillations' by Mr. Md. Shamsul Alam in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, University of Rajshahi, Rajshahi, has been completed under my supervision. I believe that this research work is an original one and it has not been submitted elsewhere for any degree.

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Abstract

In most treatments of nonlinear oscillations by perturbation method, only periodic oscillations are treated; transients are not considered. Krylov and Bogoliubov have used a perturbation method to discuss transients in the second order autonomous systems with small nonlinearities. The method is well known as an 'averaging method' in the theory of nonlinear oscillations. Later the method has been amplified and justified by Bogoliubov and Mitropolskii. In this dissertation, we investigate some third order nonlinear oscillations based on the work of Krylov-Bogoliubov-Mitroplskii (KBM).

First, nonlinear oscillation described by a third order ordinary autonomous differential equation is considered and a new perturbation technique is developed. Then a method has been developed to find asymptotic solution of a damped nonlinear system. The method is a generalization of Bogoliubov's asymptotic method and covers both under-damped and overdamped systems. Later damped oscillations including critically damped motion have also been investigated in presence of more significant damping forces.

Third order nonlinear oscillations with damping and time delay, and with varying coefficients have been investigated separately. Moreover, a simple overdamped solution has been found for the third order weakly nonlinear systems.

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Introduction

Many physical, biological and engineering, and some biomedical (epidemics), chemical and economical laws and relations appear mathematically in the form of differential or difference-differential equations. It may be noted that certain linear differential equations have only exact analytic solutions for some initial or boundary conditions. Most of the nonlinear differential equations or linear differential equations with variable coefficients and nonlinear boundary conditions have not such kind of analytic solutions. Even if the exact solution of a differential equation can be found explicitly, it may be useless for mathematical and physical interpretation and numerical evaluations. Examples of such problems are Bessel functions of large argument and large order, and doubly periodic functions. Thus in order to obtain information about solutions of differential equations, we are forced to resort to approximations, numerical solutions, or combinations of both. Foremost among the approximation techniques are the method of perturbations, *i.e.*, asymptotic expansions in terms of a small or a large parameter or coordinate. According to these techniques, the solutions are presented by the first few terms of an asymptotic expansion, usually the first two terms. Although these perturbation expansions may be divergent, they can be more useful for a qualitative as well as a quantitative representation than expansions that are uniformly and absolutely convergent.

In this dissertation, we shall discuss problems on oscillations that can be described by the dynamical systems of third order nonlinear autonomous differential equations with small nonlinearities by the Krylov-Bogoliubov-Mitropolskii (KBM) method. An important approach to the study of such nonlinear oscillations is the small parameter expansion. Two widely spread methods in this theory are mainly used in the literature; the averaging asymptotic method of KBM and the multiscale method.

The method of KBM starts with the solution sometimes called the generating solution of the linear equation, assuming that, in the nonlinear case, the amplitude and the phase in the solution of a linear equation are time-dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the solution. The method was originally developed for obtaining analytic solutions of oscillatory-type of second order nonlinear differential equations. Today, this method is being used in both oscillatory and non-oscillatory systems of second and third order nonlinear differential equations.

The purpose of this thesis is to study the effects of small parameter perturbation on some third order nonlinear oscillatory and non-oscillatory systems in the sense of the KBM method. The results may be used in various oscillatory and non-oscillatory processes in mechanics, physics, chemistry, circuit and control theory, economics and population dynamics.

Chapter 1

1.1 The Survey

In most treatments of nonlinear oscillations by perturbation methods, *e.g.*, Lindsteadt's [31] method, Poincare's [46] method, etc. only periodic oscillations are treated; transients are not considered. Krylov and Bogoliubov [27] have introduced a new perturbation method to discuss transients in the equation

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}), \tag{1.1}$$

where ε is a small parameter. In this equation the damping terms are small. But in particular it gives those periodic solutions obtained by Poincare [46]. We may note that, Poincare's method is a well known perturbation method for obtaining periodic solutions of nonlinear differential equations with small nonlinearities.

When $\varepsilon = 0$, the solution of the equation (1.1) can be written as

$$x = b\cos(\omega_0 t + \vartheta). \tag{1.2}$$

where b and ϑ are arbitrary constants to be determined from the initial conditions. Sometimes the solution (1.2) is called the generating solution of (1.1).

Now to determine an approximate solution of (1.1) for ε small but different from zero, Krylov and Bogoliubov assumed that the solution is still given by (1.2) with time-varying b and ϑ, and subject to the conditions

$$\dot{x} = -b\omega_0 \sin\varphi, \quad \varphi = \omega_0 t + \vartheta. \tag{1.3}$$

Differentiating (1.2) with respect to t gives

$$\dot{x} = -b\omega_0 \sin\varphi + \dot{b}\cos\varphi - b\dot{\vartheta}\sin\varphi . \tag{1.4}$$

Hence

$$\dot{b}\cos\varphi - b\dot{\vartheta}\sin\varphi = 0 , \tag{1.5}$$

on account of (1.3). Differentiating (1.3) with respect to t gives

$$\ddot{x} = -\omega_0^2 b\cos\varphi - \omega_0\dot{b}\sin\varphi - \omega_0 b\dot{\vartheta}\cos\varphi . \tag{1.6}$$

Substituting (1.6) into (1.1) and using (1.2) and (1.3), one obtains

$$\omega_0\dot{b}\sin\varphi + \omega_0 b\dot{\vartheta}\cos\varphi = -f\left(b\cos\varphi, -\omega_0 b\sin\varphi\right). \tag{1.7}$$

Solving (1.5) and (1.7) for $\dot{b}$ and $\dot{\vartheta}$ yields

$$\begin{aligned} \dot{b} &= -\frac{\varepsilon}{\omega_0}\sin\varphi . f\left(b\cos\varphi, -\omega_0 b\sin\varphi\right), \\ \dot{\vartheta} &= -\frac{\varepsilon}{\omega_0 b}\cos\varphi . f\left(b\cos\varphi, -\omega_0 b\sin\varphi\right). \end{aligned} \tag{1.8}$$

Thus according to Krylov and Bogoliubov's technique the original second order differential equation (1.1) for x has been replaced by the two first order differential equations (1.8) for the amplitude b and the phase ϑ. It is obvious that, the solution is periodic with constant amplitude and period $\frac{2\pi}{\omega}$ as the limit $\varepsilon \to 0$. But one can not tell about the amplitude and the periodicity of oscillations when ε is small, rather than sufficiently small.

Expressing f in a Fourier series in the total phase φ and assuming that the parameter ε is small, so that the amplitude b and the phase ϑ change very slowly during one period of the oscillation, *i.e.*,

$$\frac{\dot{b}}{b} << \omega_0, \quad \frac{\dot{\vartheta}}{\vartheta} << \omega_0, \tag{1.9}$$

one obtains to the first order of ε by averaging (1.8) over one period

$$\left\langle \dot{b} \right\rangle = -\frac{\varepsilon}{2\pi\omega_0} \int_0^{2\pi} \sin\varphi . f\left(b\cos\varphi, -\omega_0 b \sin\varphi\right) d\varphi,$$
$$\left\langle \dot{\vartheta} \right\rangle = -\frac{\varepsilon}{2\pi\omega_0 b} \int_0^{2\pi} \cos\varphi . f\left(b\cos\varphi, -\omega_0 b \sin\varphi\right) d\varphi. \tag{1.10}$$

where b and ϑ are assumed to be time independent under the integrals. Higher order effects were obtained by Volosov [60,61], Musen [42] and Zebreiko [62]. The equations (1.10) are the differential equations of the first approximation in the form in which they were originally obtained by Krylov and Bogoliubov [27] and in which they are generally used in applications.

This method, though it is restricted to differential equations of the type (1.1), has been used extensively in Plasma physics, theory of oscillations, control theory. Kruskal [26] has extended this method to solve equations of the type

$$\ddot{x} = F(x, \dot{x}, \varepsilon) . \tag{1.11}$$

The solutions of these fully nonlinear equations are based on recurrent relations and are given in the forms of power series of the small parameter ε. Cap [17] has investigated some nonlinear systems of the type

$$\ddot{x} + \omega_0^2 . f(x) = \varepsilon F(x, \dot{x}) , \tag{1.12}$$

using elliptic functions in the sense of the Krylov and Bogoliubov method.

Later this technique has been modified and justified by Bogoliubov and Mitropolskii [3] and extended to non-stationary vibrations by Mitropolskii [35]. They have assumed a solution of the nonlinear differential equation (1.1) of the form

$$x = b\cos\varphi + \varepsilon u_1(b, \varphi) + \varepsilon^2 \ldots + \varepsilon^n u_n(b, \varphi) + O\left(\varepsilon^{n+1}\right), \tag{1.13}$$

where each u_k, $k = 1, 2, \cdots n$ is a periodic function of φ with period 2π, and b and φ vary with time t according to

$$\dot{b} = \varepsilon B_1(b) + \varepsilon^2 \ldots + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}),$$
$$\dot{\varphi} = \omega_0 + \varepsilon C_1(b) + \varepsilon^2 \ldots + \varepsilon^n C_n(b) + O(\varepsilon^{n+1}) \tag{1.14}$$

where the functions u_k, B_k and C_k are chosen such that (1.13) and (1.14) satisfy the differential equation (1.1). In order to determine B_k and C_k uniquely, it is assumed that no u_k contains $\cos\varphi$ and $\sin\varphi$. This assumption results in the conditions

$$\int_0^{2\pi} u_k(b,\varphi)\cos\varphi \, d\varphi = 0,$$
$$\int_0^{2\pi} u_k(b,\varphi)\sin\varphi \, d\varphi = 0, \quad k = 1, 2, \cdots n. \tag{1.15}$$

Differentiating (1.13) twice with respective to t, utilizing relations (1.14), substituting (1.13) and the derivatives $\dot{x}$, $\ddot{x}$ in the original equation (1.1), and equating the coefficients of $\varepsilon^k, k = 1, 2, \cdots, n$ one obtains a recursive systems

$$\omega_0^2\left(\frac{\partial^2 u_k}{\partial\varphi^2} + u_k\right) = f^{(k-1)}(b,\varphi) + 2\omega_0\left(bC_k\cos\varphi + B_k\sin\varphi\right), \tag{1.16}$$

where

$$f^{(0)}(b,\varphi) = f(b\cos\varphi, -\omega_0 b\sin\varphi),$$

$$f^{(1)}(b,\varphi) = u_1 f_x(b\cos\varphi, -\omega_0 b\sin\varphi) + \left(B_1\cos\varphi - bC_1\sin\varphi + \omega_0\frac{\partial u_1}{\partial\varphi}\right) \tag{1.17}$$
$$\times f_{\dot{x}}(b\cos\varphi, -\omega_0 b\sin\varphi) + \left(bC_1^2 - B_1\frac{dB_1}{db}\right)\cos\varphi + \left(2B_1C_1 + bB_1\frac{dC_1}{db}\right)\sin\varphi$$
$$-2\omega_0\left(B_1\frac{\partial^2 u_1}{\partial b\,\partial\varphi} + C_1\frac{\partial^2 u_1}{\partial\varphi^2}\right),$$

etc.

It is clear that $f^{(k-1)}$ is a periodic function of φ with period 2π depending also on the amplitude b. Thus $f^{(k-1)}$ as well as u_k can be expanded in Fourier series as

$$f^{(k-1)}(b,\varphi) = g_0^{(k-1)}(b) + \sum_{n=1}^{\infty}\left[g_n^{(k-1)}(b)\cos n\varphi + h_n^{(k-1)}(b)\sin n\varphi\right],$$
$$u_k(b,\varphi) = v_0^{(k-1)}(b) + \sum_{n=2}^{\infty}\left[v_n^{(k-1)}(b)\cos n\varphi + w_n^{(k-1)}(b)\sin n\varphi\right], \tag{1.18}$$

where

$$g_0^{(k-1)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}\left(b\cos\varphi, -\omega_0 b\sin\varphi\right)d\varphi,$$
$$g_n^{(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}\left(b\cos\varphi, -\omega_0 b\sin\varphi\right)\cos n\varphi\, d\varphi, \tag{1.19}$$
$$h_n^{(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}\left(b\cos\varphi, -\omega_0 b\sin\varphi\right)\sin n\varphi\, d\varphi, \quad n \ge 1.$$

Here $v_1^{(k-1)} = w_1^{(k-1)} = 0$ for all values of k, since both integrals of (1.15) vanish.

Substituting these values into the equations (1.16), we obtain

$$\omega_0^2 v_0^{(k-1)}(b) + \sum_{n=1}^{\infty}\omega_0^2(1-n^2)\left[v_n^{(k-1)}(b)\cos n\varphi + w_n^{(k-1)}(b)\sin n\varphi\right] = g_0^{(k-1)}(b)$$
$$+\left(g_1^{(k-1)}(b) + 2\omega_0 bC_k\right)\cos\varphi + \left(h_1^{(k-1)}(b) + 2\omega_0 B_k\right)\sin\varphi \tag{1.20}$$
$$+\sum_{n=2}^{\infty}\left[v_n^{(k-1)}(b)\cos n\varphi + w_n^{(k-1)}(b)\sin n\varphi\right].$$

Now equating the coefficients of harmonics of the same order, one obtains

$$g_1^{(k-1)}(b) + 2\omega_0 bC_k = 0, \quad h_1^{(k-1)}(b) + 2\omega_0 B_k = 0,$$
$$v_0^{(k-1)}(b) = \frac{g_0^{(k-1)}(b)}{\omega_0^2}, \quad v_n^{(k-1)}(b) = \frac{g_n^{(k-1)}(b)}{\omega_0^2(1-n^2)}, \quad w_n^{(k-1)}(b) = \frac{h_n^{(k-1)}(b)}{\omega_0^2(1-n^2)}, \quad n \ge 1. \tag{1.21}$$

These are sufficient conditions to obtain the desired order of approximation. For the first order approximation, one has

$$B_1 = -\frac{h_1^{(1)}(b)}{2\omega_0} = -\frac{1}{2\pi\omega_0}\int_0^{2\pi} f(b\cos\varphi, -\omega_0 b\sin\varphi)\sin\varphi\, d\varphi,$$

$$C_1 = -\frac{g_1^{(1)}(b)}{2\omega_0 b} = -\frac{1}{2\pi\omega_0 b}\int_0^{2\pi} f(b\cos\varphi, -\omega_0 b\sin\varphi)\cos\varphi\, d\varphi. \quad (1.22)$$

Hence the variational equations (1.14) become

$$\dot{b} = -\frac{\varepsilon}{2\pi\omega_0}\int_0^{2\pi} f(b\cos\varphi, -\omega_0 b\sin\varphi)\sin\varphi\, d\varphi,$$

$$\dot{\varphi} = \omega_0 - \frac{\varepsilon}{2\pi\omega_0 b}\int_0^{2\pi} f(b\cos\varphi, -\omega_0 b\sin\varphi)\cos\varphi\, d\varphi. \quad (1.23)$$

Note that the equations (1.23) are similar to equations (1.10). Thus the 'first order solution' obtained by Bogoliubov and Mitropolskii [3] method is identical to the original solution obtained by Krylov and Bogoliubov [27]. In the second method, higher order solution can be found easily. The unknown function u_1, called the first order correction term, is obtained from (1.21) as

$$u_1 = \frac{g_0^{(1)}(b)}{\omega_0^2} + \sum_{n=2}^{\infty} \frac{g_n^{(1)}(b)\cos n\varphi + h_n^{(1)}(b)\cos n\varphi}{\omega_0^2(1-n^2)}. \quad (1.24)$$

The solution (1.13) together with u_1 is known as the 'first order improved solution' in which b and φ are given by (1.23). When the values of the functions A_1 and B_1 are substituted from (1.22) in the second relation of (1.17), one obtains the function $f^{(1)}$, and in a similar way one can find the unknown functions u_2, A_2 and B_2. Hence the determination of the higher order approximations is sufficiently clear.

Summing up, the conditions (1.15) eliminate the fundamental harmonic in the unknown functions $u_k, k = 1, 2\cdots$ and this, in turn, guarantees the absence of *secular-terms* in all successive approximations.

This new derivation due to Bogoliubov and Mitroplskii represents a considerable improvement as compared to the early Krylov and Bogoliubov derivation in which the first approximation was established by a direct argument and the higher order approximations were introduced owing to an additional procedure resembling the Lindstedt's method.

The authors (Krylov and Bogoliubov) call this method asymptotic in the sense of the small parameter ε. Approximate solutions of differential equations in the form of an asymptotic series were introduced by Liouville [33] (most probably initiated by Poissons around 1830). Although the series is not convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as the small parameter ε tends to zero. It should be noted that the 'asymptotic' in the theory of oscillations is frequently used also in the sense of $\varepsilon \to \infty$, in which case the mathematical approach is entirely different.

Later the asymptotic method of Krylov-Bogoliubov-Mitropolskii has been extended by Popov [47] to damped nonlinear systems

$$\ddot{x} + 2k\,\dot{x} + \omega_0^2\, x = \varepsilon\, f(x, \dot{x}), \tag{1.25}$$

where $-2k\,\dot{x}$, $0 < k < \omega_0$, is a linear damping force. Mendelson [34] has rediscovered the Popov's results. In the case of damped nonlinear systems the first equation of (1.14) has been replaced by

$$\dot{b} = -kb + \varepsilon\, B_1(b) + \varepsilon^2 \ldots + \varepsilon^n\, B_n(b) + O\left(\varepsilon^{n+1}\right). \tag{1.14a}$$

On the other hand, Murty, Deekshatulu and Krisna [39] have developed an asymptotic method in the sense of the KBM method to obtain approximate solution for an overdamped system represented by the nonlinear differential equation (1.25) *i.e.*, when $k > \omega_0$. They have used hyperbolic function, $\cosh\varphi$ or $\sinh\varphi$ instead of the harmonic function, $\cos\varphi$, which have been used in [3,27,34,47]. It is noted that, for an oscillatory or a damped oscillatory system, $\cos\varphi$ may be used arbitrarily for all kinds of initial conditions. But for a non-oscillatory system either $\cosh\varphi$ or $\sinh\varphi$ should be used depending on a given set of initial conditions [14,39,41]. Murty and Deekshatulu [40] have developed another asymptotic method obtaining simple analytic solutions of the overdamped systems represented by the same equation (1.25). Murty [41] has also presented a unified KBM method solving the equation (1.25). Bojadziev and Edwards [14] have found some oscillatory and non-oscillatory solutions of (1.25) when k and ω_0 vary slowly with time t. Sattar [51] has developed an asymptotic method to obtain approximate solution of a critically damped system represented by the nonlinear differential equation (1.25) when $k = \omega_0$. He has used a simple linear function in φ in the form

$$x = b(1+\varphi) + \varepsilon\, u_1(b,\varphi) + \varepsilon^2 \ldots + \varepsilon^n\, u_n(b,\varphi) + O(\varepsilon^{n+1}), \tag{1.26}$$

where b is defined by the equation (1.14a) and φ is defined by

$$\dot{\varphi} = 1 + \varepsilon\, C_1(b) + \varepsilon^2 \ldots + \varepsilon^n\, C_n(b) + O(\varepsilon^{n+1}), \tag{1.14b}$$

instead of the second equation of (1.14).

Bojadziev [4] has investigated nonlinear damped oscillatory systems with small time delay using the KBM method. Bojadziev [10] has also found damped forced nonlinear oscillations with small time delay. Rubanic [50], Mitropolskii and Martinyuk [37], and Lardner and Bojadziev [28] have used the KBM method to obtain approximate solutions to second order nonlinear differential equations with damping and large time delay. Bojadziev [11,13] has used the KBM method to investigate certain biological and biochemical nonlinear systems. Bojadziev and Chan [12] have used the same method to biological systems with significant damping and time delay. Lin and Khan [30] have also used KBM method to some biological problems.

Proskurjakov [48], and Bojadziev, Lardner and Arya [5] have found periodic solutions for several nonlinear systems using the KBM thechnique. They have also compared the periodic solutions obtained by KBM method to Poincare's solutions [46].

Mitropolskii and Moseenkov [36] have extended the KBM method to nonlinear partial differential equations with small nonlinearities. Bojadziev and Lardner [6,7] have studied monofrequent oscillations in mechanical systems governed by second order hyperbolic differential equations. Bojadziev and Lardner [8] have also studied hyperbolic differential equations with damping and large time delay. Arya and Bojadziev [1] has examined a system of second order nonlinear hyperbolic differential equations with varying coefficients.

Firstly, Osiniski [43] has extended the KBM method to a third order nonlinear differential equation

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x = \varepsilon f(x,\dot{x},\ddot{x}), \qquad (1.27)$$

where ε is a small parameter. He has found an asymptotic solution in the form

$$x = a + b\cos\varphi + \varepsilon\, u_1(a,b,\varphi) + \varepsilon^2 \ldots + \varepsilon^n\, u_n(a,b,\varphi) + O\left(\varepsilon^{n+1}\right), \qquad (1.28)$$

where each u_k, $k = 1, 2, \cdots n$ is again a periodic function of φ with period 2π, and a, b and φ vary with time t according to

$$\dot{a} = -\lambda a + \varepsilon A_1(a) + \varepsilon^2 \ldots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}),$$
$$\dot{b} = -\mu b + \varepsilon B_1(b) + \varepsilon^2 \ldots + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}), \quad (1.29)$$
$$\dot{\varphi} = \omega + \varepsilon C_1(b) + \varepsilon^2 \ldots + \varepsilon^n C_n(b) + O(\varepsilon^{n+1}),$$

where $-\lambda$, $-\mu \pm \omega$ are the characteristic roots of the equation (1.27) when $\varepsilon = 0$, and the functions u_k, A_k, B_k and C_k are chosen such that (1.28) and (1.29) satisfy the differential equation (1.27). Secondly, Osiniski [44] has extended the KBM method to a third order nonlinear partial differential equation. Later, Lardner and Bojadziev [29] have also investigated nonlinear damped oscillations governed by a third order partial differential equation. They have introduced a concept of 'couple-amplitudes' such that the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Bojadziev [15], and Bojadziev and Hung [16] have used the KBM method to investigate a 3-dimensional nonlinear mechanical elastic system. Shamsul and Sattar [55] have presented a unified KBM method for solving third order nonlinear differential equations. Then a forced nonlinear oscillation modeling by a third order differential equation with small nonlinearities has been studied by these authors [53]. Shamsul and Sattar [54] have also developed an asymptotic method for a third order nonlinear critically damped system in the sense of the KBM method. Except Osiniski's [43,44] papers, in all the above papers, Lardner and Bojadziev's concept of 'couple-amplitudes' has been considered. But this concept precludes a simple analytic solution for a third order nonlinear differential equation. In general, the equations (1.29) are solved numerically when A_k, B_k and C_k are functions of both amplitudes *i.e.*, a and b. Mulholand [38] have used the KBM method to investigate only the oscillatory part of a third order nonlinear differential equation, like a Vander Pol's [59] equation used in control theory.

1.2 The Proposal

We propose a perturbation system of a third order nonlinear differential equation

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x = \varepsilon f(x, \dot{x}, \ddot{x}), \tag{1.16}$$

where ε is a small parameter and f is a given nonlinear function.

In **Chapter 2**, nonlinear oscillations described by a third order differential equation are investigated. Nonlinear oscillations in presence of linear damping forces are examined in **Chapter 3**. In **Chapter 4**, nonlinear oscillations in presence of more significant damping forces are investigated. Third order nonlinear oscillations with retardation effects are studied in **Chapter 5**, nonlinear oscillations described by a third order differential equation with slowly varying coefficients are investigated in **Chapter 6** and lastly, an over-damped nonlinear system is considered in **Chapter 7**.

Chapter 2

Third Order Nonlinear Oscillations

2.1 Introduction

Most of the differential equations involving physical problems are nonlinear and the solutions of these equations are more complicated. Generally, these equations can be linearized by imposing certain restrictions and then they are solved in simple approaches. In vibrating processes many problems are solved by linearzing such differential equations when the amplitude of oscillations are small. Increasing with the amplitudes, the nonlinearity of the governing equations also increases. When the amplitudes are not small enough, the linear solutions are not sufficient to describe the vibration. In these cases, KBM perturbation, an asymptotic expansion, is a widely used technique.

Third order nonlinear oscillations have been studied earlier by some authors, *e.g.*, Friedrichs [22], Rauch [49] etc. Mulholand [38] has found a KBM solution of a third order nonlinear differential equation, which is similar to the Vander Pol's [59] equation. But he did not consider the non-oscillatory part of the solution. Shamsul and Sattar [55] have studied by the unified KBM method adding non-oscillatory part to the solution. However the method developed in [55] is not a simple analytic one; numerical integration has been partly used. Now a new technique is presented here to find a simple analytic solution of a third order nonlinear differential equation with small nonlinearities.

2.2 The Method

Let us consider a third order nonlinear differential equation

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x = \varepsilon f(\ddot{x},\ \dot{x},\ x), \qquad (2.1)$$

where the dots denote differentiation with respect to t, ε is a small parameter, k_1, k_2 and k_3 are constants. Let equation (2.1) has three characteristic roots $-\lambda,\ \pm i\omega,\ \lambda,\ \omega > 0$, when $\varepsilon = 0$. Then equation (2.1) has the solution

$$x = a_0 e^{-\lambda t} + b_0 \cos\omega t + c_0 \sin\omega t, \tag{2.2}$$

where a_0, b_0 and c_0 are arbitrary constants to be determined from the given initial conditions $[x(0), \dot{x}(0), \ddot{x}(0)]$.

We seek a solution of the differential equation (2.1) in the form of an asymptotic expansion that reduces to (2.2) as the limit $\varepsilon \to 0$. Let us consider

$$x = a e^{-\lambda t} + b \cos\omega t + c \sin\omega t + \varepsilon u_1(a,b,c,t) + \varepsilon^2 \cdots, \tag{2.3}$$

where a, b and c satisfy the first order differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,b,c,t) + \varepsilon^2 \cdots, \\ \dot{b} &= \varepsilon B_1(a,b,c,t) + \varepsilon^2 \cdots, \\ \dot{c} &= \varepsilon C_1(a,b,c,t) + \varepsilon^2 \cdots. \end{aligned} \tag{2.4}$$

It is noted that, for some particular and important cases the equations (2.4) may be solved in a simple analytic approach, and sometimes it may be solved by an approximation method independent of the numerical techniques, and thus the solution (2.3) does not depend on the numerical method.

Differentiating (2.3) three times with respect to t, using relations (2.4), substituting (2.3) and the derivatives $\dot{x}, \ddot{x}, \dddot{x}$ in the original equation (2.1), and comparing the coefficients of various powers of ε, we get for the coefficient of ε :

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)\left(A_1 e^{-\lambda t}\right)+\left(\left(\frac{\partial^2}{\partial t^2}+\lambda\frac{\partial}{\partial t}-2\omega^2\right)B_1+\omega\left(3\frac{\partial}{\partial t}+2\lambda\right)C_1\right)\cos\omega t$$

$$+\left(-\omega\left(3\frac{\partial}{\partial t}+2\lambda\right)B_1+\left(\frac{\partial^2}{\partial t^2}+\lambda\frac{\partial}{\partial t}-2\omega^2\right)C_1\right)\sin\omega t \qquad (2.5)$$

$$+\left(\frac{\partial}{\partial t}+\lambda\right)\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)u_1=f^{(0)}(a,b,c,t),$$

where $f^{(0)}=f(x_0,\dot{x}_0,\ddot{x}_0)$ and $x_0=a\,e^{-\lambda t}+b\cos\omega t+c\sin\omega t$.

Let the function $f^{(0)}$ be expanded in a Fourier series

$$f^{(0)}=\sum_{n=0}\left(F_n(a,b,c,t)\cos n\omega t+G_n(a,b,c,t)\sin n\omega t\right). \qquad (2.6)$$

To solve the equation (2.5) for u_1, A_1, B_1 and C_1, it is assumed according to KBM method, that the function u_1 does not contain *secular terms* (*i.e.*, terms containing the variable t out side the sign of the trigonometric functions), since they grow up indefinitely when $t\to\infty$.

The appearance of *secular terms* in the perturbation theory is a serious difficulty, since it restricts to obtain a uniformly valid expansion. For an example, in an expansion with two terms, if *secular terms* of the form $t(\cos t,\sin t)$ appear in the second term, the solution breaks down when $t=O(\varepsilon^{-1})$. Several authors have used different techniques, which prevent the appearance of *secular terms* in the solution. Krylov, Bogoliubov and Mitropolskii allowed to vary slowly both the amplitude and the phase for second order nonlinear systems in such a manner that each correction term u_i, $i=1,2\cdots$ exclude the terms with first harmonic. Thus no *secular terms* appear in the KBM solution. But for a third order nonlinear differential equation, like the *secular terms*, another term of the form $t\exp(-t)$ may appear. We may call such a term as a *secular-like* *term*. The solutions together with the *secular-like term* do not break down, even if t is large, since these terms vanish as the limit $t \to \infty$. The appearance of the *secular-like term* resists only to obtain the desired perturbation solution compared to those solutions obtained by the numerical method. So we have to eliminate the terms with $t\exp(-t)$ from u_1. Substituting $f^{(0)}$ from (2.6) into (2.5) and equating the coefficients of $\cos 0\omega t$, $\cos\omega t$ and $\sin\omega t$, we obtain

$$\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)\left(A_1 e^{-\lambda t}\right)=F_0, \tag{2.7}$$

$$\left(\frac{\partial^2}{\partial t^2}+\lambda\frac{\partial}{\partial t}-2\omega^2\right)B_1+\omega\left(3\frac{\partial}{\partial t}+2\lambda\right)C_1=F_1, \tag{2.8}$$

$$-\omega\left(3\frac{\partial}{\partial t}+2\lambda\right)B_1+\left(\frac{\partial^2}{\partial t^2}+\lambda\frac{\partial}{\partial t}-2\omega^2\right)C_1=G_1, \tag{2.9}$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)\left(\frac{\partial^2}{\partial t^2}+\omega^2\right)u_1=\sum_{n=2}\left(F_n\cos n\omega t+G_n\sin n\omega t\right). \tag{2.10}$$

The particular solutions of (2.7)-(2.9) give the three unknown functions A_1, B_1 and C_1. It is obvious that the change of the variables *a*, *b* and *c* are small. When F_0, F_1 and G_1 are given we may easily solve the equations (2.7)-(2.9) assuming that *a*, *b* and *c* are constants. Substituting the values of A_1, B_1 and C_1 into (2.4) and then solving them, we obtain the first approximate solution of the nonlinear differential equation. The procedure can be carried to higher orders in the same way.

2.2.1 Determination of the first order correction term u_1

The equation (2.10) is a third order nonhomogeneous linear partial differential equation and its particular solution gives the first order correction term u_1. When the nonlinear function f of the equation (2.1) is given, F_n and G_n, $n \geq 2$ are specified. Then substituting the values of F_n and G_n in (2.10), we may solve it assuming again that a, b and c are constants. Thus the first order correction term u_1 is found and we obtain the first improved solution of the equation (2.1).

2.3 Example

Now consider $f = x^3$. So,

$$f^{(0)} = a^3 e^{-3\lambda t} + \tfrac{3}{2} a e^{-\lambda t}\left(b^2 + c^2\right) + 3\left(a^2 e^{-2\lambda t} + \tfrac{1}{4}\left(b^2 + c^2\right)\right)\left(b \cos \omega t + c \sin \omega t\right)$$
$$+ \tfrac{3}{2} a e^{-\lambda t}\left(\left(b^2 - c^2\right)\cos 2\omega t + 2bc \sin 2\omega t\right) + \tfrac{1}{4}\left(b\left(b^2 - 3c^2\right)\cos 3\omega t + c\left(3b^2 - c^2\right)\sin 3\omega t\right)$$

Therefore the non zero coefficients of F_n and G_n are

$$F_0 = a^3 e^{-3\lambda t} + \tfrac{3}{2} a e^{-\lambda t}\left(b^2 + c^2\right),\ F_1 = 3b\left(a^2 e^{-2\lambda t} + \tfrac{1}{4}\left(b^2 + c^2\right)\right),$$
$$G_1 = 3c\left(a^2 e^{-2\lambda t} + \tfrac{1}{4}\left(b^2 + c^2\right)\right),\ F_2 = \tfrac{3}{2} a\left(b^2 - c^2\right)e^{-\lambda t},\ G_2 = 3abce^{-\lambda t}, \tag{2.11}$$
$$F_3 = \tfrac{1}{4} b\left(b^2 - 3c^2\right),\ G_3 = \tfrac{1}{4} c\left(3b^2 - c^2\right).$$

Substituting the values of F_0, F_1 and G_1 from (2.11) into the equations (2.7)-(2.9) and solving them, we obtain

$$A_1 = l_1 a^3 e^{-2\lambda t} + l_2 a\left(b^2 + c^2\right),$$
$$B_1 = a^2\left(m_1 b + n_1 c\right)e^{-2\lambda t} + \left(m_2 b + n_2 c\right)\left(b^2 + c^2\right), \tag{2.12}$$
$$C_1 = a^2\left(-n_1 b + m_1 c\right)e^{-2\lambda t} + \left(-n_2 b + m_2 c\right)\left(b^2 + c^2\right),$$

where

$$l_1 = \frac{1}{9\lambda^2 + \omega^2}, \quad l_2 = \frac{3}{2(\lambda^2 + \omega^2)},$$

$$m_1 = \frac{3(\lambda^2 - \omega^2)}{2(\lambda^2 + \omega^2)(\lambda^2 + \omega^2)}, \quad m_2 = \frac{-3}{8(\lambda^2 + \omega^2)}, \tag{2.13}$$

$$n_1 = \frac{3\lambda}{(\lambda^2 + \omega^2)(\lambda^2 + \omega^2)}, \quad n_2 = \frac{-3\lambda}{8\omega(\lambda^2 + \omega^2)}.$$

Substituting the values of A_1, B_1 and C_1 from (2.12) into the equations (2.4) we obtain

$$\begin{aligned}
\dot{a} &= \varepsilon\left(l_1 a^3 e^{-2\lambda t} + l_2 a(b^2 + c^2)\right), \\
\dot{b} &= \varepsilon\left(a^2(m_1 b + n_1 c)e^{-2\lambda t} + (m_2 b + n_2 c)(b^2 + c^2)\right), \\
\dot{c} &= \varepsilon\left(a^2(-n_1 b + m_1 c)e^{-2\lambda t} + (-n_2 b + m_2 c)(b^2 + c^2)\right)
\end{aligned} \tag{2.14}$$

Replacing a, b and c by their respective values obtained in the linear case, and then integrating with respect to t, we obtain

$$\begin{aligned}
a &= a_0 + \varepsilon\left(l_1 a_0^3(1 - e^{-2\lambda t})/(2\lambda) + l_2 a_0(b_0^2 + c_0^2)t\right), \\
b &= b_0 + \varepsilon\left(a_0^2(m_1 b_0 + n_1 c_0)(1 - e^{-2\lambda t})/(2\lambda) + (m_2 b_0 + n_2 c_0)(b_0^2 + c_0^2)t\right), \\
c &= c_0 + \varepsilon\left(a_0^2(-n_1 b_0 + m_1 c_0)(1 - e^{-2\lambda t})/(2\lambda) + (-n_2 b_0 + m_2 c_0)(b_0^2 + c_0^2)t\right)
\end{aligned} \tag{2.15}$$

Hence the first order solution of (2.1) is

$$x = ae^{-\lambda t} + b\cos\omega t + c\sin\omega t, \tag{2.16}$$

where a, b and c are given by (2.15).

Now substituting the values of F_2, F_3, G_2 and G_3 from (2.11) into the equation (2.10) and then solving it, we obtain

$$\begin{aligned}
u_1 &= ae^{-\lambda t}\left((b^2 - c^2)(c_2\cos 2\omega t + d_2\sin 2\omega t) + 2bc(-d_2\cos 2\omega t + c_2\sin 2\omega t)\right) \\
&\quad + \left(b(b^2 - 3c^2)(c_3\cos 3\omega t + d_3\sin 3\omega t) + c(3b^2 - c^2)(-d_3\cos 3\omega t + c_3\sin 3\omega t)\right)
\end{aligned} \tag{2.17}$$

where

$$c_2 = \frac{3\lambda}{(\lambda^2+\omega^2)(\lambda^2+9\omega^2)},$$

$$d_2 = \frac{3(\lambda^2-\omega^2)}{4\omega(\lambda^2+\omega^2)(\lambda^2+9\omega^2)},$$

$$c_3 = \frac{-\lambda}{32\omega^2(\lambda^2+9\omega^2)},$$

$$d_3 = \frac{-3}{32\omega(\lambda^2+9\omega^2)}. \tag{2.18}$$

Therefore, the first improved solution of (2.1) is

$$x = a\,e^{-\lambda t} + b\cos\omega\, t + c\sin\omega\, t + \varepsilon\, u_1, \tag{2.19}$$

where a, b and c are given by (2.15) and u_1 is given by (2.17).

2.4. Discussion

In general, the straightforward expansion of Poincare [46] type such as

$$x(t,\varepsilon) \sim \sum_{m=0}^{\infty} \delta_m(\varepsilon) x_m(t),$$

where $\delta_m(\varepsilon)$ is an asymptotic sequence in the term of the parameter ε, is non-uniformly valid and breaks down in the regions called regions of non-uniformity. One of the main sources of non-uniformities is the 'infinite domain'. In the case of nonlinear oscillations the non-uniformity manifests itself in the presence of *secular terms* such as $t^m \cos t$ and $t^m \sin t$, which make $\frac{x_m(t)}{x_{m-1}(t)}$ unbounded as t approaches to infinity. In the method of KBM such type of non-uniformity is prevented.

A type of non-uniformity has occurred when a first order approximate solution of the set of equations (2.14) is found in the form (2.15). It is obvious that, in the solution (2.15) a, b and c increases monotonously as t increases and they tend to infinity as t approaches to infinity. So the second and third terms of (2.19) where b and c are given by (2.15) oscillate with large amplitude when t is large. Thus the solution (2.19) is not a uniformly valid solution. It is noted that, when the equations (2.14) are solved numerically, the function x in the solution (2.19) oscillates with decay and vanishes as $t \to \infty$. Therefore, one should obtain an approximate solution of (2.1) by solving the equations (2.14) numerically, rather than by choosing the analytic solution (2.15) when t is large. However, in this case, the present method facilitates only the numerical method. The variables a, b and c change slowly with time. So, it requires the numerical calculation of a few number of points (*i.e.*, one may compute it by choosing normal step size). Contrary, a direct attempt to solve the equation (2.1) dealing with harmonic terms in the solution (2.3), namely $b\cos\omega t + c\sin\omega t$, requires the numerical calculation of many number of points (*i.e.*, one must choose small step size) and it needs more computing time.

The present solution (2.15) is fairly used until $t = \frac{1}{2}\varepsilon^{-1}$. Therefore, one can know the nature of oscillation when ε is very small (see **Fig 2.1**). If one obtains a second order solution of (2.1) in the form (2.3), one can use it until $t = \frac{1}{2}\varepsilon^{-2}$ and knows the complete picture of oscillations, since the function x oscillates with decay.

It is also noted that, for most of the physical problems the linear terms dominate the system. The nonlinear term εx^3 decays the oscillatory terms, while the linear decay-term is absent in the oscillatory part. If we add a small linear decay-term in the oscillatory part of the system described by the nonlinear equation (2.1), similar solutions for a, b and c to (2.15) are found and the solution of (2.1) is uniformly valid. Let us add the small damping forces $-2\varepsilon(\ddot{x}+\lambda\dot{x})$ to the right hand side of the equation (2.1) with the same nonlinearity εx^3. Therefore $f=-2(\ddot{x}+\lambda\dot{x})+x^3$.

For this reason the set of equations (2.14) become

$$\begin{aligned} \dot{a} &= \varepsilon\left(l_1 a^3 e^{-2\lambda t} + l_2 a\left(b^2+c^2\right)\right), \\ \dot{b} &= \varepsilon\left(-b + a^2\left(m_1 b + n_1 c\right)e^{-2\lambda t} + \left(m_2 b + n_2 c\right)\left(b^2+c^2\right)\right), \\ \dot{c} &= \varepsilon\left(-c + a^2\left(-n_1 b + m_1 c\right)e^{-2\lambda t} + \left(-n_2 b + m_2 c\right)\left(b^2+c^2\right)\right) \end{aligned} \tag{2.20}$$

If we replace b and c respectively by $b e^{-\varepsilon t}$ and $c e^{-\varepsilon t}$ in the second and third equations of (2.20), they become

$$\begin{aligned} \dot{b} &= \varepsilon\left(a^2\left(m_1 b + n_1 c\right)e^{-2\lambda t} + \left(m_2 b + n_2 c\right)\left(b^2+c^2\right)e^{-2\varepsilon t}\right), \\ \dot{c} &= \varepsilon\left(a^2\left(-n_1 b + m_1 c\right)e^{-2\lambda t} + \left(-n_2 b + m_2 c\right)\left(b^2+c^2\right)e^{-2\varepsilon t}\right) \end{aligned} \tag{2.20b}$$

The solutions of the equations (2.20b) together with the first equation of (2.20) are

$$\begin{aligned} a &= a_0 + \varepsilon l_1 a_0^3\left(1-e^{-2\lambda t}\right)/(2\lambda) + l_2 a_0\left(b_0^2+c_0^2\right)\left(1-e^{-2\varepsilon t}\right)/2, \\ b &= b_0 + \varepsilon a_0^2\left(m_1 b_0 + n_1 c_0\right)\left(1-e^{-2\lambda t}\right)/(2\lambda) + \left(m_2 b_0 + n_2 c_0\right)\left(b_0^2+c_0^2\right)\left(1-e^{-2\varepsilon t}\right)/2, \\ c &= c_0 + \varepsilon a_0^2\left(-n_1 b_0 + m_1 c_0\right)\left(1-e^{-2\lambda t}\right)/(2\lambda) + \left(-n_2 b_0 + m_2 c_0\right)\left(b_0^2+c_0^2\right)\left(1-e^{-2\varepsilon t}\right)/2. \end{aligned} \tag{2.21}$$

In this case, the solution (2.19) reduces to

$$x = a e^{-\lambda t} + e^{-\varepsilon t}\left(b\cos\omega t + c\sin\omega t\right) + \varepsilon u_1, \tag{2.22}$$

where

$$\begin{aligned} u_1 &= a e^{-(\lambda+2\varepsilon)t}\left(\left(b^2-c^2\right)\left(c_2\cos 2\omega t + d_2\sin 2\omega t\right) + 2bc\left(-d_2\cos 2\omega t + c_2\sin 2\omega t\right)\right) \\ &+ e^{-3\varepsilon t}\left(b\left(b^2-3c^2\right)\left(c_3\cos 3\omega t + d_3\sin 3\omega t\right) + c\left(3b^2-c^2\right)\left(-d_3\cos 3\omega t + c_3\sin 3\omega t\right)\right) \end{aligned} \tag{2.23}$$

For small values of ε the solution (2.22) where a, b, c and u_1 are given by (2.21) and (2.23) respectively agree with the numerical solution (see **Fig. 2.2**).

When $\lambda = O(1)$ the non-oscillatory part in the solution (2.3) dies-out quickly. On the other hand, the coefficient m_1 becomes small when both $\lambda = O(1)$ and $\omega = O(1)$. In this case, we may obtain an approximate solution of (2.14). Let $a = \alpha, b = \beta\cos\varphi, c = \beta\sin\varphi$. Here β and φ are respectively the amplitude and the total phase of the oscillatory part of the solution. So the equations (2.14) transform to

$$\begin{aligned}
\dot{\alpha} &= \varepsilon\left(l_1\alpha^3 e^{-2\lambda t} + l_2\alpha\beta^2\right),\\
\dot{\beta} &= \varepsilon\left(m_1\alpha^2\beta e^{-2\lambda t} + m_2\beta^3\right),\\
\dot{\varphi} &= -\varepsilon\left(n_1\alpha^2 e^{-2\lambda t} + n_2\beta^2\right).
\end{aligned} \tag{2.24}$$

An approximate solution of (2.24) may be found. The first and second relation of (2.24) may be written as

$$\begin{aligned}
\frac{d\alpha}{\alpha} &= \varepsilon\left(l_1\alpha^2 e^{-2\lambda t} + l_2\beta^2\right),\\
\frac{d\beta}{\beta^3} &= \varepsilon\left(m_1\left(\frac{\alpha}{\beta}\right)^2 e^{-2\lambda t} + m_2\right).
\end{aligned} \tag{2.24a}$$

Since m_1 is small, we may replace α and β by their respective linear values in the right hand side of the second relation of (2.24a). Then integrating with respect to t, we obtain

$$\beta = \frac{\beta_0}{\sqrt{\varepsilon l_1\alpha_0^2\left(1-\exp(-\lambda t)\right)/\lambda + z}}, \tag{2.25b}$$

where $z = 1 - 2\varepsilon m_2\beta_0^2 t$.

Now substituting the values of β in the first relation of (2.24a) and in the third relation of (2.24) and integrating with respect to t, we obtain

$$\alpha = \alpha_0 e^{-\lambda t}\left(1+\varepsilon l_1 \alpha_0^2 (1-\exp(-\lambda t))/(2\lambda) - l_2 \ln(z)/(2m_2)\right),$$
$$\varphi = \varphi_0 + \varepsilon n_1 \alpha_0^2 (1-\exp(-\lambda t))/(2\lambda) - n_2 \ln(z)/(2m_2). \tag{2.25}$$

In this case, the solution (2.19) becomes

$$x = \alpha + \beta\cos(\omega t - \varphi) + \varepsilon u_1, \tag{2.26}$$

where

$$u_1 = \alpha\beta^2\left(c_2 \cos 2(\omega t - \varphi) + d_2 \sin 2(\omega t - \varphi)\right) + \beta^3\left(c_3 \cos 3(\omega t - \varphi) + d_3 \sin 3(\omega t - \varphi)\right). \tag{2.27}$$

The solution (2.26) with the approximate solution (2.25) and (2.25b) for α, β and φ also agree with the numerical solution (see **Fig. 2.3**).

To obtain the numerical solution, the initial conditions $[x(0), \dot{x}(0), \ddot{x}(0)]$ are computed from

$$x(0) = a_0 + b_0 + \varepsilon\left(c_2 a_0 (b_0^2 - c_0^2) - 2d_2 a_0 b_0 c_0 + c_3 (b_0^3 - 3b_0 c_0^2) - d_3 (3b_0^2 c_0 - c_0^3)\right),$$

$$\begin{aligned}\dot{x}(0) = &-\lambda a_0 + \omega c_0 + \varepsilon\left(l_1 a_0^3 + l_2 a_0 (b_0^2 + c_0^2) + a_0^2 (m_1 b_0 + n_1 c_0) + (m_2 b_0 + n_2 c_0)(b_0^2 + c_0^2)\right)\\ &+\varepsilon\big((-\lambda c_2 + 2\omega d_2) a_0 (b_0^2 - c_0^2) + 2(2\omega c_2 + \lambda d_2) a_0 b_0 c_0\\ &+3\omega d_3 (b_0^3 - 3b_0 c_0^2) + 3\omega c_3 (3b_0^2 c_0 - c_0^3)\big),\end{aligned} \tag{2.28}$$

$$\begin{aligned}\ddot{x}(0) = &\lambda^2 a_0 - \omega^2 b_0\\ &-2\varepsilon\big(2\lambda l_1 a_0^3 + \lambda l_2 a_0 (b_0^2 + c_0^2) + a_0^2 ((\lambda m_1 + \omega n_1) b_0 + (-\omega m_1 + \lambda n_1) c_0)\\ &+2(\omega n_2 b_0 - \omega m_2 c_0)(b_0^2 + c_0^2)\big)\\ &+\varepsilon\big((\lambda^2 - 4\omega^2) c_2 - 4\lambda\omega d_2) a_0 (b_0^2 - c_0^2) - 2(4\lambda\omega c_2 + (\lambda^2 - 4\omega^2) d_2) a_0 b_0 c_0\\ &-9\omega^2 c_3 (b_0^3 - 3b_0 c_0^2) + 9\omega^2 d_3 (3b_0^2 c_0 - c_0^3)\big).\end{aligned}$$

Usually, in a problem the initial conditions $[x(0), \dot{x}(0), \ddot{x}(0)]$ are specified. Then one has to solve the nonlinear algebraic equations in order to determine the arbitrary constants a_0, b_0 and c_0 that appear in the solution. In those cases, the equations (2.28) are solved for a_0, b_0 and c_0 by a numerical formula.

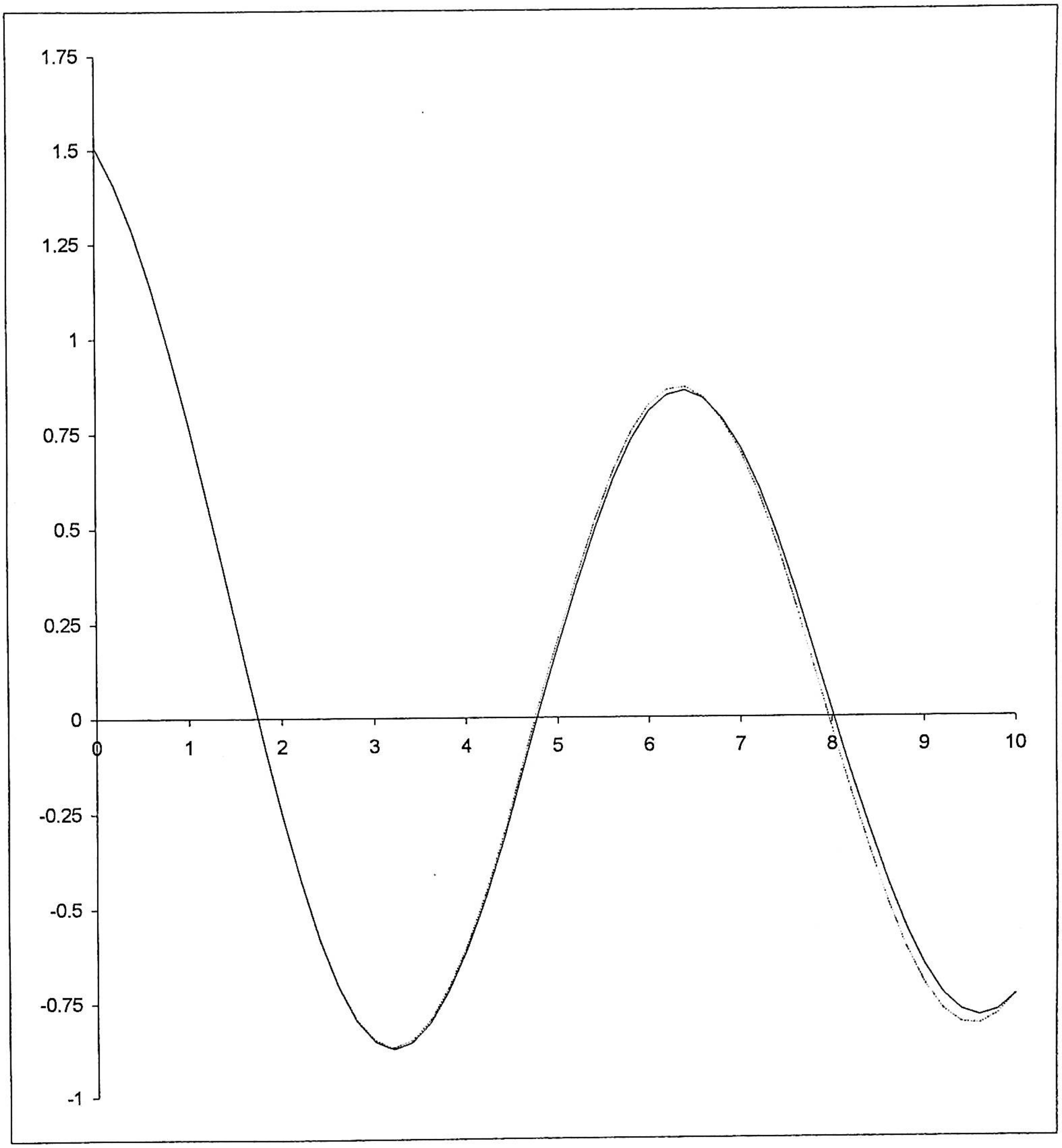

Fig. 2.1 Solution of equation (2.1) obtained by perturbation theory (solid lone) in which a, b and c are evaluated by (2.15), and numerical integration (doted line) for λ=0.8, μ=0, ω=1.0, ε=0.1 and $f=x^3$ with the initial conditions a_0 =0.5, b_0 =1.0 and c_0 =0 or, [$x(0)$=1.50733, $\dot{x}(0)$= −0.40049, $\ddot{x}(0)$= −0.76434].

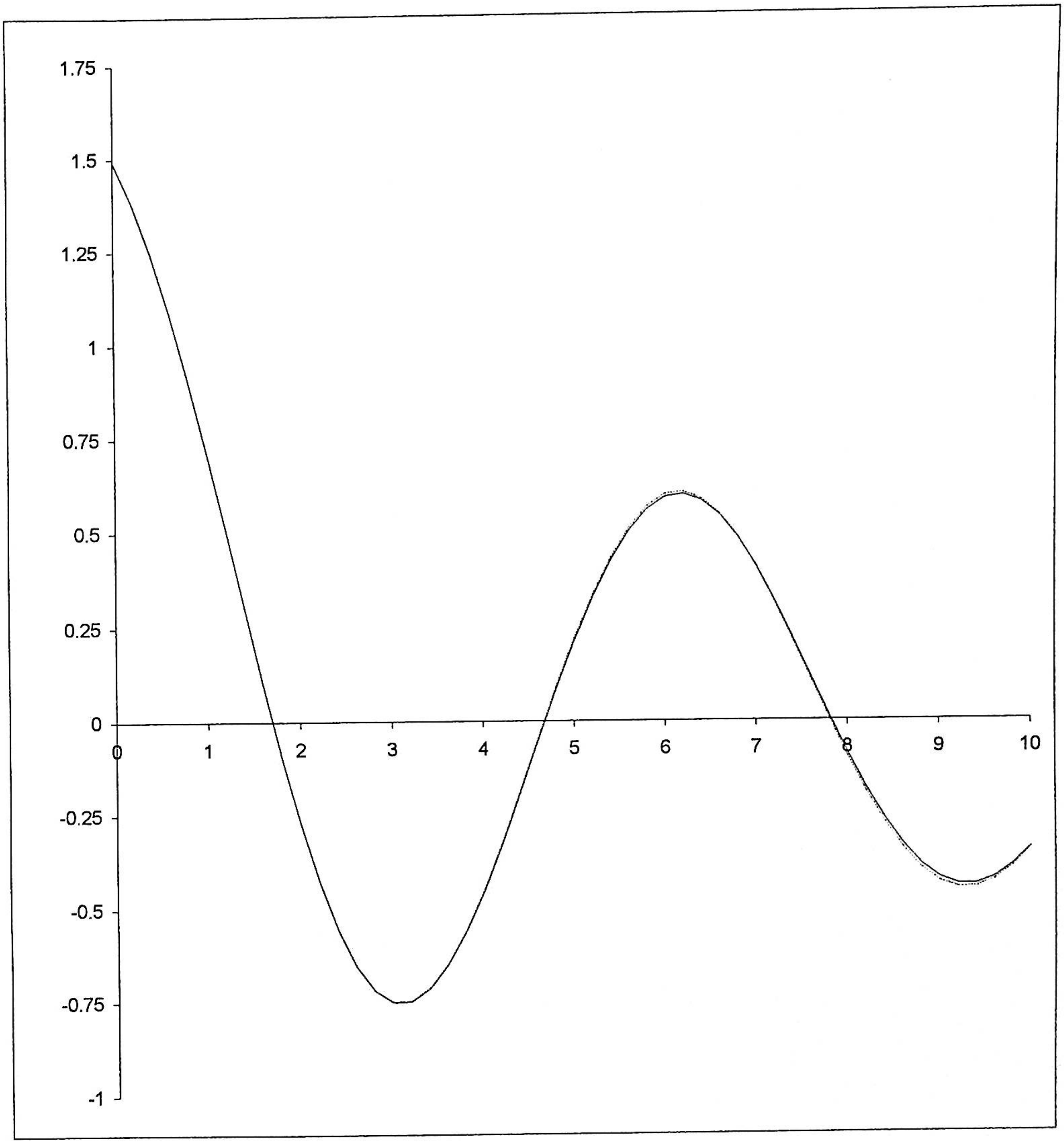

Fig. 2.2 Solution of equation (2.1) obtained by perturbation theory (solid lone) in which a, b and c are evaluated by (2.21), and numerical integration (doted line) for λ=0.8, ω=1.0, $\varepsilon=-0.1$ and $f=-2(x+\lambda x)+x^3$ with the initial conditions a_0 =0.5, b_0 =1.0 and c_0 =0 or, [$x(0)$=1.49203, $\dot{x}(0)$= −0.49000, $\ddot{x}(0)$= −0.58204].

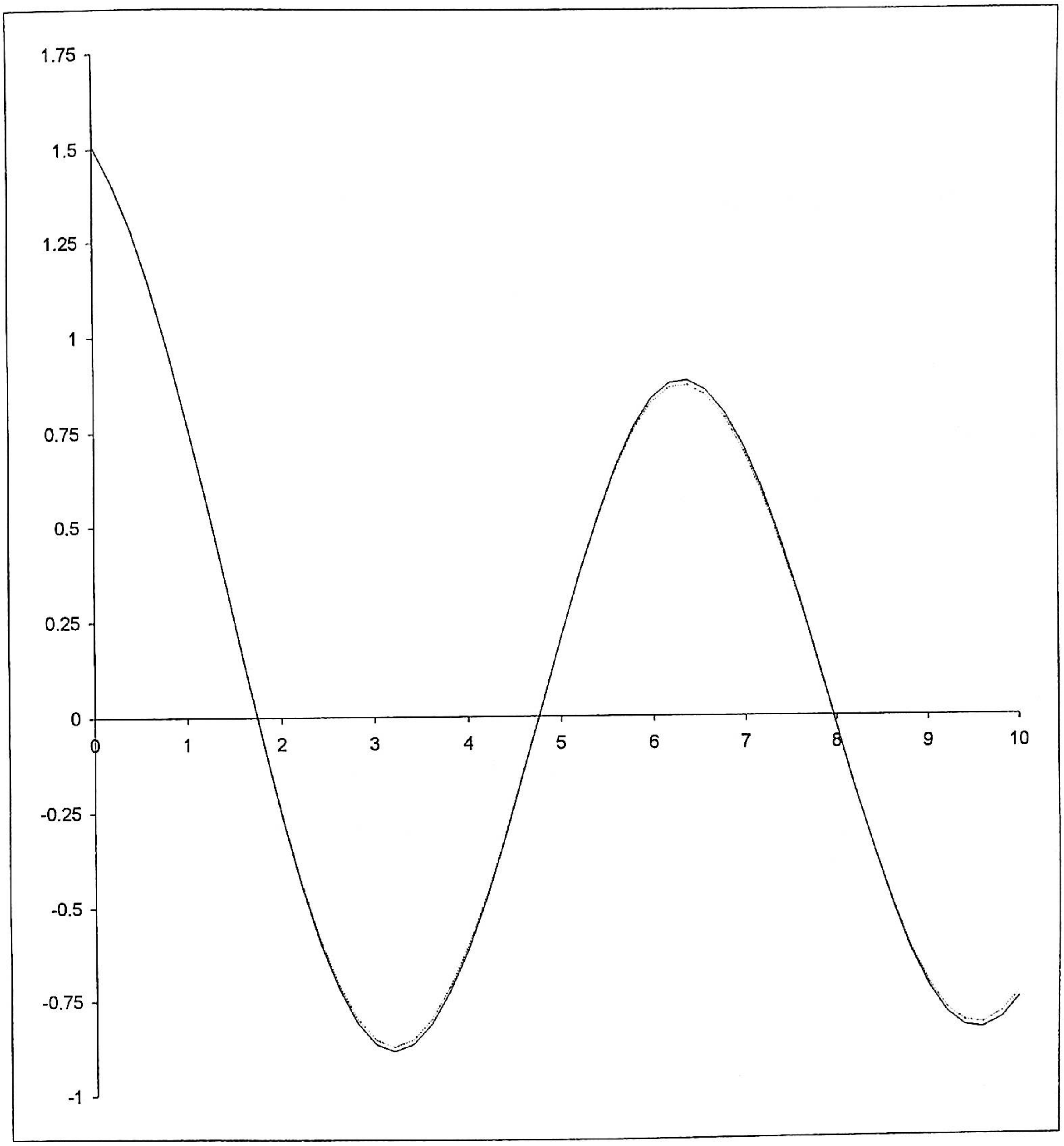

Fig. 2.3 Solution of equation (2.1) obtained by perturbation theory (solid lone) in which α, β and φ are evaluated by (2.25) and (2.25b), and numerical integration (doted line) for λ=0.8, μ=0, ω=1.0, $\varepsilon = 0.1$ and $f = x^3$ with the initial conditions α_0 =0.5, β_0 =1.0 and φ_0 =0 or, [$x(0)$=1.50733, $\dot{x}(0)$= –0.40049, $\ddot{x}(0)$= –0.76434].

Chapter 3

Third Order Nonlinear Oscillations with Damping Forces

3.1 Introduction

Popov [47] has extended the KBM method to second order nonlinear damped oscillatory systems. Using the same purturbation method, Osiniski [43] and Bojadziev [15] have studied a nonlinear mechanical system with internal friction and relaxation governed by

$$m\ddot{x} + \sigma = 0,$$
$$\sigma + c\dot{\sigma} = ax + b\dot{x} + ex^3, \tag{3.1}$$

where x is the deformation, m is the mass of the system, and c, a, b and e are positive constants. The terms with the coefficients a and e (small) represent respectively the linear and nonlinear elasticity, the term with the coefficient b corresponds to linear viscous damping, and the term with the coefficient c reflects the linear relaxation.

It is obvious that the above equation can be written as

$$\dddot{x} + c^{-1}\ddot{x} + bc^{-1}m^{-1}\dot{x} + ac^{-1}m^{-1}x + \varepsilon x^3 = 0, \quad \varepsilon = ec^{-1}. \tag{3.2}$$

which is the form of the equation (2.1) in chapter 2. For stability, the characteristic roots of the linear equation of (3.2) must be negative or complex with non-positive real parts. Osiniski [43] has found an asymptotic solution of the form

$$x = a + b\cos\varphi + \varepsilon\, u_1(a, b, \varphi) + \varepsilon^2 \cdots, \tag{3.3}$$

where the amplitudes a, b and the phase φ satisfy the first order differential equations

$$\dot{a} = -\lambda a + \varepsilon A_1(a) + \varepsilon^2 \cdots,$$
$$\dot{b} = -\mu b + \varepsilon B_1(b) + \varepsilon^2 \cdots, \tag{3.4}$$
$$\dot{\varphi} = \omega + \varepsilon C_1(b) + \varepsilon^2 \cdots.$$

Since Osiniski's solution does not always agree with the numerical solution, later Bojadziev [15] has found another solution of (3.1) in the form (3.3) with (3.4) where the unknown functions A_1, B_1 and C_1 are functions of both the amplitudes a and b, *i.e.*,

$$\dot{a} = -\lambda a + \varepsilon A_1(a,b) + \varepsilon^2 \cdots,$$
$$\dot{b} = -\mu b + \varepsilon B_1(a,b) + \varepsilon^2 \cdots, \tag{3.4b}$$
$$\dot{\varphi} = \omega + \varepsilon C_1(a,b) + \varepsilon^2 \cdots.$$

But Bojadziev's solution is not a simple analytic solution; the equations (3.4b) for amplitudes a and b and phase φ are solved generally by a numerical method. However, the method of Bojadziev facilitates numerical analysis, since it is less costly to use computer in solving the truncated equation (3.4b) instead of solving directly (3.1). A direct attempt is not justified to solve numerically the equation (3.1) because it leads to dealing with a harmonic term in the solution (3.3), namely $b\cos\varphi$, which requires the numerical calculation of a great number of points, and also is not practical [15].

In this chapter, we obtain a new asymptotic solution of a third order nonlinear autonomous differential equation which is fully independent of the numerical method. The solutions obtained for different initial conditions are in good agreement with those obtained by the numerical method.

3.2 The Method

Let us consider the third order nonlinear differential equation

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x = \varepsilon f(\ddot{x}, \dot{x}, x), \tag{3.5}$$

with the generating solution

$$x = a_0 e^{-\lambda t} + e^{-\mu t}\left(b_0 \cos\omega t + c_0 \sin\omega t\right), \tag{3.6}$$

where $-\lambda$, $-\mu \pm i\omega$, λ, μ, $\omega > 0$ are three characteristic roots of (3.5) when $\varepsilon = 0$, and a_0, b_0 and c_0 are arbitrary constants.

Now we seek a solution of the differential equation (3.5) in the form of an asymptotic expansion

$$x = a\, e^{-\lambda t} + e^{-\mu t}\left(b \cos\omega t + c \sin\omega t\right) + \varepsilon\, u_1(a, b, c, t) + \varepsilon^2 \cdots, \tag{3.7}$$

where a, b and c satisfy the differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a, b, c, t) + \varepsilon^2 \cdots, \\ \dot{b} &= \varepsilon B_1(a, b, c, t) + \varepsilon^2 \cdots, \\ \dot{c} &= \varepsilon C_1(a, b, c, t) + \varepsilon^2 \cdots. \end{aligned} \tag{3.8}$$

Differentiating (3.7) three times with respect to t, using relations (3.8), substituting (3.7) and the derivatives $\dot{x}$, $\ddot{x}$, $\dddot{x}$ in the original equation (3.5), and comparing the coefficients of various powers of ε, we get for the coefficient of ε :

$$\begin{aligned} &\left(\left(\frac{\partial}{\partial t} + \mu\right)^2 + \omega^2\right)\left(A_1 e^{-\lambda t}\right) + e^{-\mu t}\left(\left(\left(\frac{\partial^2}{\partial t^2} + (\lambda - \mu)\frac{\partial}{\partial t} - 2\omega^2\right)B_1\right.\right. \\ &\left. + \omega\left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right)C_1\right)\cos\omega t + \left(-\omega\left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right)B_1\right. \\ &\left.\left. + \left(\left(\frac{\partial^2}{\partial t^2} + (\lambda - \mu)\frac{\partial}{\partial t} - 2\omega^2\right)\right)C_1\right)\sin\omega t + \left(\frac{\partial}{\partial t} + \lambda\right)\left(\left(\frac{\partial}{\partial t} + \mu\right)^2 + \omega^2\right)u_1 = f^{(0)}(a, b, c, t). \end{aligned} \tag{3.9}$$

where $f^{(0)} = f\left(x_0, \dot{x}_0, \ddot{x}_0\right)$ and $x_0 = a\, e^{-\lambda t} + e^{-\mu t}\left(b \cos\omega t + c \sin\omega t\right)$.

Let the function $f^{(0)}$ be expanded in a Fourier series

$$f^{(0)} = \sum_{n=0}^{\infty} \left(F_n(a,b,c,t)\cos n\omega t + G_n(a,b,c,t)\sin n\omega t\right). \tag{3.10}$$

To solve the equation (3.9) for u_1, A_1, B_1 and C_1, it is assumed, according to KBM method, that the function u_1 does not contain *secular terms* as the limit $\mu \to 0$. This assumption is allowed according to Popov [47]. We also assume that like a *secular term*, u_1 does not contain a term of the form $t\exp(-t)$ as the limit $\mu \to 0$. Substituting (3.10) in (3.8) and equating the coefficients of $\cos 0\omega t$, $\cos\omega t$ and $\sin\omega t$ we obtain

$$\left(\left(\frac{\partial}{\partial t} + \mu\right)^2 + \omega^2\right)\left(A_1 e^{-\lambda t}\right) = F_0, \tag{3.11}$$

$$\left(\frac{\partial^2}{\partial t^2} + (\lambda - \mu)\frac{\partial}{\partial t} - 2\omega^2\right)B_1 + \omega\left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right)C_1 = F_1, \tag{3.12}$$

$$-\omega\left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right)B_1 + \left(\frac{\partial^2}{\partial t^2} + (\lambda - \mu)\frac{\partial}{\partial t} - 2\omega^2\right)C_1 = G_1, \tag{3.13}$$

and

$$\left(\frac{\partial}{\partial t} + \lambda\right)\left(\left(\frac{\partial}{\partial t} + \mu\right)^2 + \omega^2\right)u_1 = \sum_{n=2}\left(F_n \cos n\omega t + G_n \sin n\omega t\right). \tag{3.14}$$

The particular solutions of (3.11)-(3.13) give the three unknown functions A_1, B_1 and C_1. It is obvious that the change of the variables a, b and c are small. When F_0, F_1 and G_1 are given, we may easily solve the equations (3.11)-(3.13), assuming that a, b and c are constants. Substituting the values of A_1, B_1 and C_1 into (3.6) and then solving them, we obtain the first approximate solution of the nonlinear differential equation. The procedure can be applied to higher orders in the same way.

3.2.1 Determination of the first order correction term u_1

The particular solution of (3.14) gives the first order correction term u_1. When the nonlinear function f of the equation (3.5) is given, F_n and G_n, $n \geq 2$ are specified. Then substituting the values of F_n and G_n in (3.14), we may solve it by assuming again that a, b and c are constants. Thus the correction term u_1 is found and we obtain the first improved solution of the equation (3.8).

3.3 Example

Now consider $f = x^3$. So,

$$f^{(0)} = a^3 e^{-3\lambda t} + \tfrac{3}{2} a e^{-(\lambda+2\mu)t}\left(b^2 + c^2\right) + 3\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} e^{-3\mu t}\left(b^2 + c^2\right)\right)\left(b\cos\omega t + c\sin\omega t\right)$$

$$+ \tfrac{3}{2} a e^{-(\lambda+2\mu)t}\left(\left(b^2 - c^2\right)\cos 2\omega t + 2bc\sin 2\omega t\right) + \frac{e^{-3\mu t}}{4}\left(b\left(b^2 - 3c^2\right)\cos 3\omega t + c\left(3b^2 - c^2\right)\sin 3\omega t\right)$$

Therefore the non zero coefficients of F_n and G_n are

$$\begin{aligned} &F_0 = a^3 e^{-3\lambda t} + \tfrac{3}{2} a e^{-(\lambda+2\mu)t}\left(b^2 + c^2\right),\ F_1 = 3b\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} e^{-3\mu t}\left(b^2 + c^2\right)\right), \\ &G_1 = 3c\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} e^{-3\mu t}\left(b^2 + c^2\right)\right),\ F_2 = \tfrac{3}{2} a\left(b^2 - c^2\right)e^{-(\lambda+2\mu)t}, \\ &G_2 = 3abce^{-(\lambda+2\mu)t},\ F_3 = \tfrac{1}{4} b\left(b^2 - 3c^2\right)e^{-3\mu t},\ G_3 = \tfrac{1}{4} c\left(3b^2 - c^2\right)e^{-3\mu t}. \end{aligned} \tag{3.15}$$

Substituting the values of F_0, F_1 and G_1 from (3.15) into the equations (3.11)-(3.13) and solving them we obtain

$$\begin{aligned} &A_1 = l_1 a^3 e^{-2\lambda t} + l_2 a\left(b^2 + c^2\right)e^{-2\mu t}, \\ &B_1 = a^2\left(m_1 b + n_1 c\right)e^{-2\lambda t} + \left(m_2 b + n_2 c\right)\left(b^2 + c^2\right)e^{-2\mu t}, \\ &C_1 = a^2\left(-n_1 b + m_1 c\right)e^{-2\lambda t} + \left(-n_2 b + m_2 c\right)\left(b^2 + c^2\right)e^{-2\mu t}, \end{aligned} \tag{3.16}$$

where

$$l_1 = \frac{1}{(3\lambda-\mu)^2+\omega^2}, \quad l_2 = \frac{3}{2\left((\lambda+\mu)^2+\omega^2\right)},$$

$$m_1 = \frac{3\left(\lambda^2+\lambda\mu-\omega^2\right)}{2\left(\lambda^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)}, \quad m_2 = \frac{-3\left(\mu(\lambda-3\mu)+\omega^2\right)}{8\left(\mu^2+\omega^2\right)\left((\lambda-3\mu)^2+\omega^2\right)}, \tag{3.17}$$

$$n_1 = \frac{3\omega(2\lambda+\mu)}{2\left(\lambda^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)}, \quad n_2 = \frac{3\omega(-\lambda+4\mu)}{8\left(\mu^2+\omega^2\right)\left((\lambda-3\mu)^2+\omega^2\right)}.$$

Substituting the values of A_1, B_1 and C_1 from (3.16) into the equation (3.8), we obtain

$$\begin{aligned} \dot{a} &= \varepsilon\left(l_1 a^3 e^{-2\lambda t} + l_2 a\left(b^2+c^2\right)e^{-2\mu t}\right), \\ \dot{b} &= \varepsilon\left(a^2(m_1 b + n_1 c)e^{-2\lambda t} + (m_2 b + n_2 c)\left(b^2+c^2\right)e^{-2\mu t}\right), \\ \dot{c} &= \varepsilon\left(a^2(-n_1 b + m_1 c)e^{-2\lambda t} + (-n_2 b + m_2 c)\left(b^2+c^2\right)e^{-2\mu t}\right) \end{aligned} \tag{3.18}$$

Replacing a, b and c by their respective values obtained in the linear case, and then integrating with respect to t, we obtain

$$\begin{aligned} a &= a_0 + \varepsilon\left(l_1 a_0^3\left(1-e^{-2\lambda t}\right)/\lambda + l_2 a_0\left(b_0^2+c_0^2\right)\left(1-e^{-2\mu t}\right)/\mu\right)/2, \\ b &= b_0 + \varepsilon\left(a_0^2(m_1 b_0 + n_1 c_0)\left(1-e^{-2\lambda t}\right)/\lambda + (m_2 b_0 + n_2 c_0)\left(b_0^2+c_0^2\right)\left(1-e^{-2\mu t}\right)/\mu\right)/2, \\ c &= c_0 + \varepsilon\left(a_0^2(-n_1 b_0 + m_1 c_0)\left(1-e^{-2\lambda t}\right)/\lambda + (-n_2 b_0 + m_2 c_0)\left(b_0^2+c_0^2\right)\left(1-e^{-2\mu t}\right)/\mu\right)/2. \end{aligned} \tag{3.19}$$

Hence the first order solution of (3.5) is

$$x = ae^{-\lambda t} + e^{-\mu t}(b\cos\omega t + c\sin\omega t), \tag{3.20}$$

where a, b and c are given by (3.19).

Now substituting the values of F_2, F_3, G_2 and G_3 from (3.15) into the equation (3.14) and then solving it we obtain

$$\begin{aligned} u_1 &= ae^{-(\lambda+2\mu)t}\left(\left(b^2-c^2\right)(c_2\cos 2\omega t + d_2\sin 2\omega t) + 2bc(-d_2\cos 2\omega t + c_2\sin 2\omega t)\right) \\ &\quad + e^{-3\mu t}\left(b\left(b^2-3c^2\right)(c_3\cos 3\omega t + d_3\sin 3\omega t) + c\left(3b^2-c^2\right)(-d_3\cos 3\omega t + c_3\sin 3\omega t)\right) \end{aligned} \tag{3.21}$$

where

$$c_2 = \frac{3\left(-\mu(\lambda+\mu)^2 + (4\lambda+7\mu)\,\omega^2\right)}{4\left(\mu^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)\left((\lambda+\mu)^2+9\omega^2\right)},$$

$$d_2 = \frac{3\left((\lambda+\mu)(\lambda+5\mu)-\omega^2\right)\omega}{4\left(\mu^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)\left((\lambda+\mu)^2+9\omega^2\right)},$$

$$c_3 = \frac{\mu^2(\lambda-3\mu)+(-2\lambda+15\mu)\,\omega^2}{16\left(\mu^2+\omega^2\right)\left(\mu^2+4\omega^2\right)\left((\lambda-3\mu)^2+9\omega^2\right)}, \tag{3.22}$$

$$d_3 = \frac{-3\left(\mu(\lambda-4\mu)+2\omega^2\right)\omega}{16\left(\mu^2+\omega^2\right)\left(\mu^2+4\omega^2\right)\left((\lambda-3\mu)^2+9\omega^2\right)}.$$

Therefore, the first improved solution of (3.5) is

$$x = a\,e^{-\lambda t} + e^{-\mu t}\left(b\cos\omega\,t + c\sin\omega\,t\right) + \varepsilon\,u_1, \tag{3.23}$$

where a, b and c are given by (3.19) and u_1 is given by (3.21).

3.4 Discussion

A simple analytical method has been developed to obtain the time response of a third order nonlinear differential equation with small nonlinearities when the damping forces are significant. The method is independent of the numerical techniques. Bojadziev [15] has obtained a solution of a 3-dimensional system, equivalent to a third order equation, which depends partly on the numerical method.

One can find the solution, obtained in **Chapter 2**, if one takes the limit $\mu \to 0$ in the present solution (3.23). Thus the present method is a generalized method for all the undamped or significantly damped oscillatory systems described by the third order nonlinear differential equations. Moreover, the solution (3.20) or (3.23) with (3.19), (3.17), (3.21) and (3.22) may be used in the case of an over-damped system replacing simply ω by iw, $\cos i\omega$ by $\cosh w$ and $\sin i\omega$ by $i\sinh w$.

As a check on the solution (3.23) obtained in **section 3.3**, a second solution was obtained by numerical integration by using a fourth-order *Runge-Kutta* formula. The results are plotted in **Fig. 3.1** for $\lambda = 0.8, \mu = 0.1, \omega = 1, \varepsilon = -0.1$ with the initial conditions $a_0 = 0.5,\ b_0 = 1,\ c_0 = 0$. The two curves agree very closely in the region where the function is changing rapidly. However, in the case of strong damping forces, *i.e.*, when μ is increased, the two curves almost coincide even in the region where the function is changing slowly (see **Fig. 3.2**).

The solution (3.23) is not expressed in terms of amplitudes and phase. But one can easily transform it to various usual forms [15,55]. One can transform the solution (3.23) together with (3.18) and (3.21) as

$$x = \alpha e^{-\lambda t} + \beta e^{-\mu t}\cos(\omega t - \varphi) + \varepsilon u_1, \tag{3.24}$$

$$\begin{aligned} \dot{\alpha} &= \varepsilon\left(l_1\alpha^3 e^{-2\lambda t} + l_2\alpha\beta^2 e^{-2\mu t}\right), \\ \dot{\beta} &= \varepsilon\left(m_1\alpha^2\beta e^{-2\lambda t} + m_2\beta^3 e^{-2\mu t}\right), \\ \dot{\varphi} &= -\varepsilon\left(n_1\alpha^2 e^{-2\lambda t} + n_2\beta^2 e^{-2\mu t}\right), \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} u_1 &= \alpha\beta^2 e^{-(\lambda+2\mu)t}\left(c_2\cos 2(\omega t - \varphi) + d_2\sin 2(\omega t - \varphi)\right) \\ &\quad + \beta^3 e^{-3\mu t}\left(c_3\cos 3(\omega t - \varphi) + d_3\sin 3(\omega t - \varphi)\right), \end{aligned} \tag{3.26}$$

under the transformations

$$\begin{aligned} a &= \alpha, \\ b &= \beta\cos\varphi, \\ c &= \beta\sin\varphi. \end{aligned} \tag{3.27}$$

The equations (3.25) may be integrated by assuming that α and β are constants in the right hand side as

$$
\begin{aligned}
\alpha &= \alpha_0 + \varepsilon\left(l_1\alpha_0^3\left(1-e^{-2\lambda t}\right)/\lambda + l_2\alpha_0\beta_0^2\left(1-e^{-2\mu t}\right)/\mu\right)/2, \\
\beta &= \beta_0 + \varepsilon\left(m_1\alpha_0^2\beta_0\left(1-e^{-2\lambda t}\right)/\lambda + m_2\beta_0^3\left(1-e^{-2\mu t}\right)/\mu\right)/2, \qquad (3.28)\\
\varphi &= \varphi_0 - \varepsilon\left(n_1\alpha_0^2\left(1-e^{-2\lambda t}\right)/\lambda + n_2\beta_0^2\left(1-e^{-2\mu t}\right)/\mu\right)/2.
\end{aligned}
$$

But when the amplitudes α and β and the phase φ are computed by (3.28), the solution (3.23) gives more accurate results than the solution (3.24). The solution (3.24) is given in the **Fig. 3.3** for the same values of λ, μ, ω and ε, and as well as the same initial values of a_0, b_0 and c_0. Comparing the two figures **3.1** and **3.3**, we may say that the solution (3.23) together with (3.19) and (3.21) is better than (3.24) with (3.28) and (3.26).

Now one can obtain Bojadziev's form of the solution, if one transforms again the equations (3.24), (3.25) and (3.26) under the transformations

$$
\begin{aligned}
\alpha e^{-\lambda t} &= a, \\
\beta e^{-\mu t} &= b, \qquad (3.29)\\
\varphi &= \omega t - \psi.
\end{aligned}
$$

It is obvious that, under these transformations, the solution (3.24) becomes

$$
x = a + b\cos\psi + \varepsilon u_1, \qquad (3.30)
$$

where a, b and ψ satisfy the equations

$$
\begin{aligned}
\dot{a} &= -\lambda a + \varepsilon\left(l_1 a^3 + l_2 a b^2\right), \\
\dot{b} &= -\mu b + \varepsilon\left(m_1 a^2 b + m_2 b^3\right), \qquad (3.31)\\
\dot{\psi} &= \omega + \varepsilon\left(n_1 a^2 + n_2 b^2\right),
\end{aligned}
$$

and

$$
u_1 = ab^2\left(c_2\cos 2\psi + d_2\sin 2\psi\right) + b^3\left(c_3\cos 3\psi + d_3\sin 3\psi\right). \qquad (3.32)
$$

Note that, the solution (3.30) together with (3.31) and (3.32) is identical to Bojadziev's solution [15]; since the set of equations (3.31) for a, b and ψ is a type of (3.4b). It has been mentioned above that, Bojadziev solved the equations (3.31) by a numerical method.

On the other hand, if one transforms the equations (3.24), (3.25) and (3.26) under the transformations

$$\begin{aligned} \alpha e^{-\lambda t} &= e^{\xi}, \\ \beta e^{-\mu t} &= e^{\eta}, \\ \varphi &= \omega t - \phi, \end{aligned} \tag{3.33}$$

one obtains

$$x = e^{\xi} + e^{\eta} \cos\phi + \varepsilon u_1, \tag{3.34}$$

$$\begin{aligned} \dot{\xi} &= -\lambda + \varepsilon\left(l_1 e^{2\xi} + l_2 e^{2\eta}\right), \\ \dot{\eta} &= -\mu + \varepsilon\left(m_1 e^{2\xi} + m_2 e^{2\eta}\right), \\ \dot{\phi} &= \omega + \varepsilon\left(n_1 e^{2\xi} + n_2 e^{2\eta}\right), \end{aligned} \tag{3.35}$$

and

$$u_1 = e^{\xi+2\eta}\left(c_2 \cos 2\phi + d_2 \sin 2\phi\right) + e^{3\eta}\left(c_3 \cos 3\phi + d_3 \sin 3\phi\right). \tag{3.36}$$

The above form of the solution was obtained by the unified KBM method in [55] in the case of an under-damped system. It is noted that, Bojadziev's solution is always identical to the under-damped solution obtained by the unified KBM method. In the unified method, the under-damped solution has been used as an over-damped solution simply by transforming harmonic functions to their corresponding hyperbolic functions. It is also noted that, in the unified method the equations (3.35) are solved by a fourth-order *Runge-Kutta* formula.

To obtain the corresponding numerical solution, the initial conditions $[x(0), \dot{x}(0), \ddot{x}(0)]$ are computed from

$$x(0) = a_0 + b_0 + \varepsilon\left(c_2 a_0 (b_0^2 - c_0^2) - 2d_2 a_0 b_0 c_0 + c_3 (b_0^3 - 3b_0 c_0^2) - d_3 (3b_0^2 c_0 - c_0^3)\right),$$

$$\begin{aligned}
\dot{x}(0) = & -\lambda a_0 - \mu b_0 + \omega c_0 \\
& + \varepsilon\left(l_1 a_0^3 + l_2 a_0 (b_0^2 + c_0^2) + a_0^2 (m_1 b_0 + n_1 c_0) + (m_2 b_0 + n_2 c_0)(b_0^2 + c_0^2)\right) \\
& + \varepsilon\left((-(\lambda + 2\mu)c_2 + 2\omega d_2) a_0 (b_0^2 - c_0^2) + 2(2\omega c_2 + (\lambda + 2\mu) d_2) a_0 b_0 c_0\right) \\
& + 3(-\mu c_3 + \omega d_3)(b_0^3 - 3b_0 c_0^2) + 3(\omega c_3 + \mu d_3)(3b_0^2 c_0 - c_0^3)\Big),
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
\ddot{x}(0) = & \lambda^2 a_0 + (\mu^2 - \omega^2) b_0 - 2\mu\omega c_0 \\
& - 2\varepsilon\Big(2\lambda l_1 a_0^3 + (\lambda + \mu) l_2 a_0 (b_0^2 + c_0^2) + a_0^2 \left(((\lambda + \mu) m_1 + \omega n_1) b_0 + (-\omega m_1 + (\lambda + \mu) n_1) c_0\right) \\
& + 2\left((\mu m_2 + \omega n_2) b_0 + (-\omega m_2 + \mu n_2) c_0\right)(b_0^2 + c_0^2)\Big) \\
& + \varepsilon\Big(\left((\lambda + 2\mu)^2 - 4\omega^2\right) c_2 - 4(\lambda + 2\mu)\omega d_2\Big) a_0 (b_0^2 - c_0^2) \\
& - 2\left(4(\lambda + 2\mu)\omega c_2 + \left((\lambda + 2\mu)^2 - 4\omega^2\right) d_2\right) a_0 b_0 c_0 \\
& + 9\left((\mu^2 - \omega^2) c_3 - 2\mu\omega d_3\right)(b_0^3 - 3b_0 c_0^2) - 9\left(2\mu\omega c_3 + (\mu^2 - \omega^2) d_3\right)(3b_0^2 c_0 - c_0^3)\Big)
\end{aligned}$$

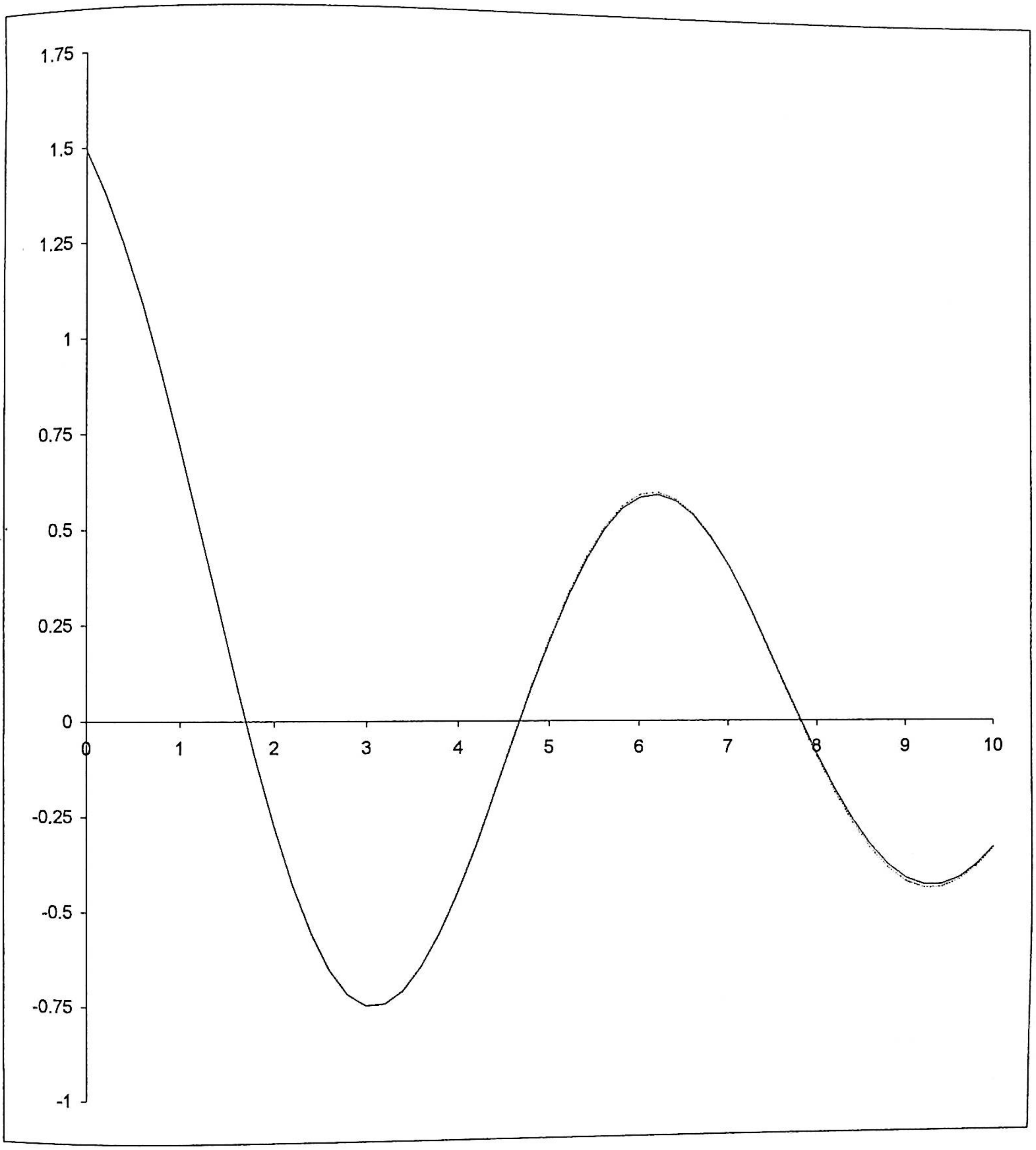

Fig. 3.1 Solution of equation (3.5) obtained by perturbation theory (solid lone) in which a, b and c are evaluated by (3.19), and numerical integration (doted line) for $\lambda=0.8$, $\mu=0.1$, $\omega=1.0$, $\varepsilon=-0.1$ and $f=x^3$ with the initial conditions $a_0=0.5$, $b_0=1.0$ and $c_0=0$ or, $[x(0)=1.49203,\ \dot{x}(0)=-0.49000,\ \ddot{x}(0)=-0.58204]$.

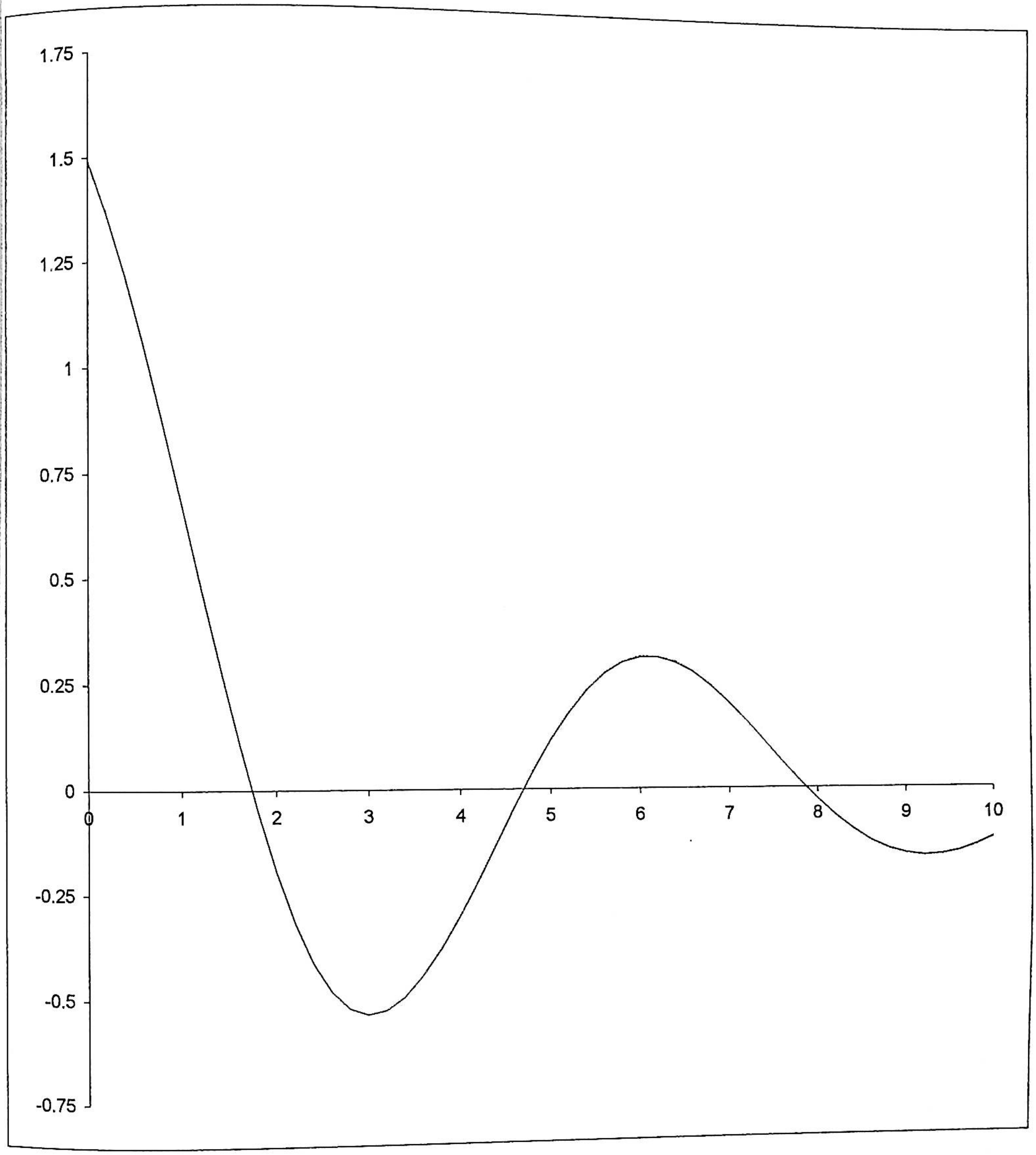

Fig. 3.2 Solution of equation (3.5) obtained by perturbation theory (solid lone) in which a, b and c are evaluated by (3.19), and numerical integration (doted line) for λ=0.8, μ=0.2, ω=1.0, $\varepsilon=-0.1$ and $f=x^3$ with the initial conditions $a_0=0.5$, $b_0=1.0$ and $c_0=0$ or, [$x(0)=1.49184$, $\dot{x}(0)=-0.58435$, $\ddot{x}(0)=-0.54204$].

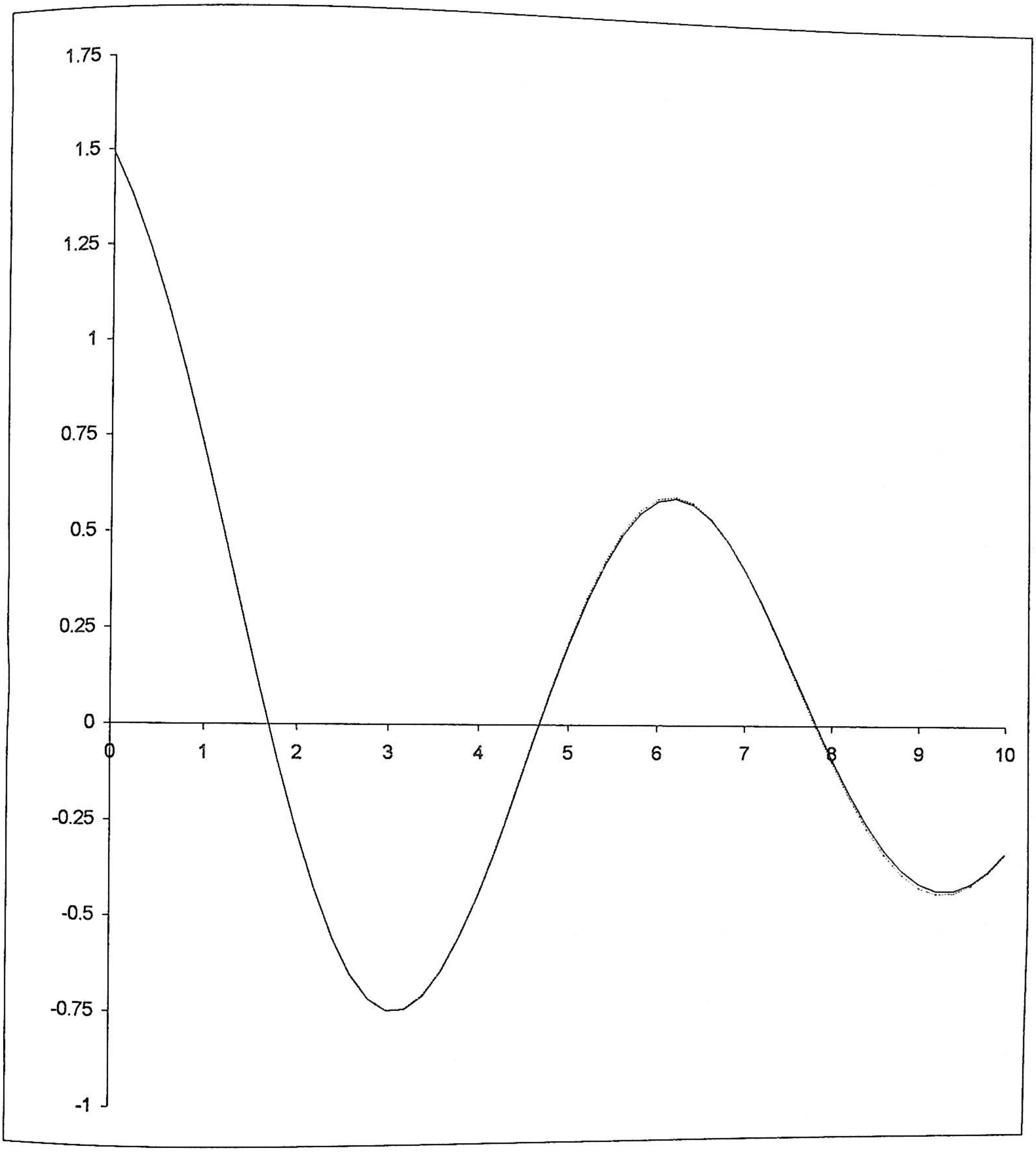

Fig. 3.3 Solution of equation (3.5) obtained by perturbation theory (solid lone) in which α, β and φ are evaluated by (3.28), and numerical integration (doted line) for λ=0.8, μ=0.1, ω=1.0, $\varepsilon = -0.1$ and $f = x^3$ with the initial conditions α_0 =0.5, β_0 =1.0 and φ_0 =0 or, [$x(0)$=1.49203, $\dot{x}(0)$=−0.49000, $\ddot{x}(0)$=−0.58204].

Chapter 4

Third Order Nonlinear Oscillations with More Significant Damping Forces

4.1 Introduction

Murty [41] has presented a unified KBM method for solving second order nonlinear differential equations. The method of Murty is a generalization of Bogoliubov's asymptotic method and covers three cases when the roots of the linear equation are real and unequal, complex conjugate and purely imaginary. Thus the unified KBM method of Murty can not be used when the roots of the linear equation are real and equal, which represents the critical damping. Moreover, Murty's overdamped solutions do not always agree with those solutions obtained by the numerical method. On the other hand, Murty's under-damped solutions are similar to Popov's [47] or Mendelsons's [34] solutions and become the original KBM [3,27] solutions when the coefficient of linear damping force vanishes. However, Popov's [47] and Mendelsons's [34] solutions, and Murty's [41] under-damped solutions for different initial conditions are in good agreement with those obtained by the numerical method when the damping forces are significant. But when the damping forces are more significant, *i.e.*, when the discriminant of the linear equation is much smaller than unity (not only $O(\varepsilon)$), the solutions obtained in [34,41,47] do not agree with the numerical solutions. In these cases errors occur more than $O\left(\varepsilon^{n+1}\right)$ for an *n*-th order approximation. Recently, Shamsul and Sattar [57] have obtained an asymptotic solution of a second order nonlinear damped systems with more significant damping forces. The solution may be used when the corresponding linear system is critically damped or overdamped. Thus this is a more generalized method than the unified method of Murty [41].

The method of Bojadziev [15] and the unified KBM method of Shamsul and Sattar [55] are not sufficient to obtain desired results of a third order nonlinear differential equation, when the damping forces are too strong that the discriminant of the linear equations are much smaller than unity. In this chapter, we have developed a generalized asymptotic method for solving a third order nonlinear autonomous differential equation when the damping forces are more significant. The method can also be used for a critically damped system as a limit.

4.2 The Method

Let us consider the third order nonlinear differential equation

$$\dddot{x} + k_1 \ddot{x} + k_2 \dot{x} + k_3 x = \varepsilon f(\ddot{x}, \dot{x}, x), \tag{4.1}$$

with the generating solution

$$x = a_0 e^{-\lambda t} + e^{-\mu t}\left(b_0 \cos\omega t + c_0 \sin\omega t\right), \tag{4.2}$$

where $-\lambda$, $-\mu \pm i\omega$, $\lambda > 0$, $\mu = O(1)$, $\omega < 1$ are three characteristic roots of (4.1) when $\varepsilon = 0$, and a_0, b_0 and c_0 are arbitrary constants.

Now we seek a solution of the differential equation (4.1) in the form of an asymptotic expansion

$$x = a\, e^{-\lambda t} + e^{-\mu t}\left(b \cos\omega t + c \sin\omega t\right) + \varepsilon\, u_1(a,b,c,t) + \varepsilon^2 \cdots, \tag{4.3}$$

where a, b and c satisfy the differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,b,c,t) + \varepsilon^2 \cdots, \\ \dot{b} &= \varepsilon B_1(a,b,c,t) + \varepsilon^2 \cdots, \\ \dot{c} &= \varepsilon C_1(a,b,c,t) + \varepsilon^2 \cdots. \end{aligned} \tag{4.4}$$

Differentiating (4.3) three times with respect to t, using relations (4.4), substituting (4.3) and the derivatives $\dot{x}, \ddot{x}, \dddot{x}$ in the original equation (4.1), and comparing the coefficients of various powers of ε, we get for the coefficient of ε :

$$\left(\left(\frac{\partial}{\partial t}+\mu\right)^2+\omega^2\right)\left(A_1 e^{-\lambda t}\right)+e^{-\mu t}\left(\left(\left(\frac{\partial^2}{\partial t^2}+(\lambda-\mu)\frac{\partial}{\partial t}-2\omega^2\right)B_1\right.\right.$$
$$\left.+\omega\left(3\frac{\partial}{\partial t}+2\lambda-2\mu\right)C_1\right)\cos\omega t+\left(-\omega\left(3\frac{\partial}{\partial t}+2\lambda-2\mu\right)B_1\right. \tag{4.5}$$
$$\left.+\left(\left(\frac{\partial^2}{\partial t^2}+(\lambda-\mu)\frac{\partial}{\partial t}-2\omega^2\right)\right)C_1\right)\sin\omega t+\left(\frac{\partial}{\partial t}+\lambda\right)\left(\left(\frac{\partial}{\partial t}+\mu\right)^2+\omega^2\right)u_1=f^{(0)}(a,b,c,t),$$

where $f^{(0)}=f\left(x_0,\dot{x}_0,\ddot{x}_0\right)$ and $x_0=a\,e^{-\lambda t}+e^{-\mu t}\left(b\cos\omega t+c\sin\omega t\right)$.

To solve the equation (4.5) for u_1, A_1, B_1 and C_1 it is assumed that the function u_1 does not contain the first harmonic terms which are produced from $(\cos\omega t)^r(\sin\omega t)^s$, $r\geq 1, s=1,0$ only. We also assume that u_1 does not contain a term of the form $t\exp(-t)$. In this case all the terms of $(\cos\omega t)^r$, $r\geq 1$ of $f^{(0)}$ can be expanded in various harmonic terms, *e.g.*,

$$f^{(0)}=F_0+F_1\cos\omega t+G_1\sin\omega t+F_2\cos 2\omega t+\cdots+\sum_{r=1,\,s=2}^{\infty,\,\infty} g_{r,s}(\cos\omega t)^r(\sin\omega t)^s, \tag{4.6}$$

while the higher order terms of $(\cos\omega t)^r(\sin\omega t)^s$, $r\geq 1$, $s\geq 2$ have remained unchanged.

Substituting (4.6) in (4.5) and equating the coefficients of $\cos 0\omega t$, $\cos\omega t$ and $\sin\omega t$, we obtain

$$\left(\left(\frac{\partial}{\partial t}+\mu\right)^2+\omega^2\right)\left(A_1 e^{-\lambda t}\right)=F_0, \tag{4.7}$$

$$\left(\frac{\partial^2}{\partial t^2}+(\lambda-\mu)\frac{\partial}{\partial t}-2\omega^2\right)B_1+\omega\left(3\frac{\partial}{\partial t}+2\lambda-2\mu\right)C_1=F_1, \tag{4.8}$$

$$-\omega\left(3\frac{\partial}{\partial t}+2\lambda-2\mu\right)B_1+\left(\frac{\partial^2}{\partial t^2}+(\lambda-\mu)\frac{\partial}{\partial t}-2\omega^2\right)C_1=G_1, \tag{4.9}$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)\left(\left(\frac{\partial}{\partial t}+\mu\right)^2+\omega^2\right)u_1=F_2\cos 2\omega t+\cdots+\sum_{r=1,\,s=2}^{\infty,\,\infty} g_{r,s}(\cos\omega t)^r(\sin\omega t)^s. \tag{4.10}$$

The particular solutions of (4.7)-(4.9) give the three unknown functions A_1, B_1 and C_1. It is obvious that the change of the variables *a*, *b* and *c* are small. When F_0, F_1 and G_1 are given we may easily solve the equations (4.7)-(4.9) by assuming that *a*, *b* and *c* are constants. Substituting the values of A_1, B_1 and C_1 into (4.4) and then solving them, we obtain the first approximate solution of the nonlinear differential equation. The procedure can be carried to higher orders in the same way.

4.2.1 Determination of the first order correction term u_1

The particular solution of (4.10) gives the first order correction term u_1. When the nonlinear function f of the equation (4.1) is given, F_2, $F_3,\cdots$ and G_2, $G_3,\cdots$ are specified. Then substituting the values of F_2, $F_3,\cdots$ and G_2, $G_3,\cdots$ in (4.10), we may solve it by assuming again that *a*, *b* and *c* are constants. Thus the correction term u_1 is found and we obtain the first improved solution of the equation (4.1).

4.3 Example

Now consider $f=x^3$. So,

$$
\begin{aligned}
f^{(0)} &= a^3 e^{-3\lambda t} + \tfrac{3}{2} a b^2 e^{-(\lambda+2\mu)t} + 3\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} b^2 e^{-3\mu t}\right)\left(b\cos\omega t + c\sin\omega t\right) \\
&+ 3a e^{-(\lambda+2\mu)t}\left(\tfrac{1}{2} b^2 \cos 2\omega t + bc\sin 2\omega t + c^2 \sin^2 \omega t\right) \\
&+ e^{-3\mu t}\left(\tfrac{1}{4} b^3 \cos 3\omega t + \tfrac{3}{4} b^2 c \sin 3\omega t + 3bc^2 \cos\omega t \sin^2 \omega t + c^3 \sin^3 \omega t\right).
\end{aligned}
$$

Therefore the non zero coefficients of $F_0, F_1, \cdots$ and $G_1, G_2, \cdots$ are

$$
\begin{aligned}
&F_0 = a^3 e^{-3\lambda t} + \tfrac{3}{2} a b^2 e^{-(\lambda+2\mu)t}, \quad F_1 = 3b\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} b^2 e^{-3\mu t}\right), \\
&G_1 = 3c\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} b^2 e^{-3\mu t}\right), \quad F_2 = \tfrac{3}{2} a b^2 e^{-(\lambda+2\mu)t}, \\
&G_2 = 3abce^{-(\lambda+2\mu)t}, \quad F_3 = \frac{b^3 e^{-3\mu t}}{4}, \quad G_3 = \frac{3b^3 e^{-3\mu t}}{4}.
\end{aligned}
\tag{4.11}
$$

Substituting the values of F_0, F_1 and G_1 from (4.11) into the equations (4.7)-(4.9) and solving them we obtain

$$
\begin{aligned}
A_1 &= l_1 a^3 e^{-2\lambda t} + l_2 a b^2 e^{-2\mu t}, \\
B_1 &= a^2 (m_1 b + n_1 c) e^{-2\lambda t} + (m_2 b + n_2 c) b^2 e^{-2\mu t}, \\
B_1 &= a^2 (-n_1 b + m_1 c) e^{-2\lambda t} + (-n_2 b + m_2 c) b^2 e^{-2\mu t},
\end{aligned}
\tag{4.12}
$$

where

$$
\begin{aligned}
&l_1 = \frac{1}{(3\lambda-\mu)^2 + \omega^2}, \quad l_2 = \frac{3}{2\left((\lambda+\mu)^2 + \omega^2\right)}, \\
&m_1 = \frac{3\left(\lambda^2 + \lambda\mu - \omega^2\right)}{2\left(\lambda^2 + \omega^2\right)\left((\lambda+\mu)^2 + \omega^2\right)}, \quad m_2 = \frac{-3\left(\mu(\lambda - 3\mu) + \omega^2\right)}{8\left(\mu^2 + \omega^2\right)\left((\lambda - 3\mu)^2 + \omega^2\right)}, \\
&n_1 = \frac{3\omega(2\lambda + \mu)}{2\left(\lambda^2 + \omega^2\right)\left((\lambda+\mu)^2 + \omega^2\right)}, \quad n_2 = \frac{3\omega(-\lambda + 4\mu)}{8\left(\mu^2 + \omega^2\right)\left((\lambda - 3\mu)^2 + \omega^2\right)}.
\end{aligned}
\tag{4.13}
$$

Substituting the values of A_1, B_1 and C_1 from (4.12) into the equations (4.4), we obtain

$$\dot{a} = \varepsilon\left(l_1 a^3 e^{-2\lambda t} + l_2 a b^2 e^{-2\mu t}\right),$$
$$\dot{b} = \varepsilon\left(a^2 (m_1 b + n_1 c) e^{-2\lambda t} + (m_2 b + n_2 c) b^2 e^{-2\mu t}\right),$$
$$\dot{c} = \varepsilon\left(a^2 (-n_1 b + m_1 c) e^{-2\lambda t} + (-n_2 b + m_2 c) b^2 e^{-2\mu t}\right). \tag{4.14}$$

Replacing a, b and c by their respective values obtained in the linear case, and then integrating with respect to t, we obtain

$$a = a_0 + \varepsilon\left(l_1 a_0^3 \left(1 - e^{-2\lambda t}\right)/\lambda + l_2 a_0 b_0^2 \left(1 - e^{-2\mu t}\right)/\mu\right)/2,$$
$$b = b_0 + \varepsilon\left(a_0^2 (m_1 b_0 + n_1 c_0)\left(1 - e^{-2\lambda t}\right)/\lambda + b_0^2 (m_2 b_0 + n_2 c_0)\left(1 - e^{-2\mu t}\right)/\mu\right)/2, \tag{4.15}$$
$$c = c_0 + \varepsilon\left(a_0^2 (-n_1 b_0 + m_1 c_0)\left(1 - e^{-2\lambda t}\right)/\lambda + b_0^2 (-n_2 b_0 + m_2 c_0)\left(1 - e^{-2\mu t}\right)/\mu\right)/2.$$

Hence the first order solution of (4.1) is

$$x = a\, e^{-\lambda t} + e^{-\mu t}\left(b \cos \omega t + c \sin \omega t\right), \tag{4.16}$$

where a, b and c are given by (4.15).

Now substituting the values of F_2, F_3, G_2 and G_3 into the equation (4.10) and then solving it, we obtain

$$u_1 = a e^{-(\lambda + 2\mu)t}\left(c^{(0)} c^2 + (b^2 - c^2)(c_2 \cos 2\omega t + d_2 \sin 2\omega t)\right.$$
$$\left.+ 2bc(-d_2 \cos 2\omega t + c_2 \sin 2\omega t)\right)$$
$$+ e^{-3\mu t}\left(b(c_1 \cos \omega t + d_1 \sin \omega t) + c(-d_1 \cos \omega t + c_1 \sin \omega t)\right. \tag{4.17}$$
$$\left.+ b\left(b^2 - 3c^2\right)(c_3 \cos 3\omega t + d_3 \sin 3\omega t) + c\left(3b^2 - c^2\right)(-d_3 \cos 3\omega t + c_3 \sin 3\omega t)\right),$$

where

$$c^{(0)} = -\frac{3}{4\mu\left((\lambda+\mu)^2+\omega^2\right)},$$

$$c_1 = \frac{3\left(\mu(\lambda-3\mu)+\omega^2\right)}{16\mu\left(\mu^2+\omega^2\right)\left((\lambda-3\mu)^2+\omega^2\right)},$$

$$d_1 = -\frac{3\omega(\lambda-4\mu)}{16\mu\left(\mu^2+\omega^2\right)\left((\lambda-3\mu)^2+\omega^2\right)},$$

$$c_2 = \frac{3\left(-\mu(\lambda+\mu)^2+(4\lambda+7\mu)\,\omega^2\right)}{4\left(\mu^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)\left((\lambda+\mu)^2+9\omega^2\right)},$$

$$d_2 = \frac{3\left((\lambda+\mu)(\lambda+5\mu)-\omega^2\right)\omega}{4\left(\mu^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)\left((\lambda+\mu)^2+9\omega^2\right)},$$

$$c_3 = \frac{\mu^2(\lambda-3\mu)+(-2\lambda+15\mu)\,\omega^2}{16\left(\mu^2+\omega^2\right)\left(\mu^2+4\omega^2\right)\left((\lambda-3\mu)^2+9\omega^2\right)},$$

$$d_3 = \frac{-3\left(\mu(\lambda-4\mu)+2\omega^2\right)\omega}{16\left(\mu^2+\omega^2\right)\left(\mu^2+4\omega^2\right)\left((\lambda-3\mu)^2+9\omega^2\right)}. \tag{4.18}$$

Therefore, the first improved solution of (4.1) is

$$x = a\,e^{-\lambda t} + e^{-\mu t}\left(b\cos\omega\,t + c\sin\omega\,t\right) + \varepsilon\,u_1, \tag{4.19}$$

where a, b, c are given by (4.15) and u_1 is given by (4.17). The solution (4.19) may be used even when ω is very small. When ω is sufficiently small, *i.e.*, in the case of critical damping forces, similar solution may be found from the equation (4.5). Let

$$\begin{aligned} b\cos\omega t &= b^*(t), \\ c\sin\omega t &= c^*(t)\,t, \end{aligned} \tag{4.20}$$

where b^* and c^* satisfy the differential equations

$$\begin{aligned} \dot{b}^* &= \varepsilon B_1^*(a, b^*, c^*, t) + \varepsilon^2 \cdots, \\ \dot{c}^* &= \varepsilon C_1^*(a, b^*, c^*, t) + \varepsilon^2 \cdots. \end{aligned} \tag{4.21}$$

Differentiating (4.20) three times with respect to t, using second and third relations of (4.4) and the relations (4.20), and then comparing the coefficients of ε we obtain the following relations for the unknown functions B_1 and C_1, and B_1^* and C_1^* as

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial t^2} - 2\omega^2\right)B_1 \cos\omega t - 3\omega\frac{\partial B_1}{\partial t}\sin\omega t = \omega^2 B_1^* + \frac{\partial^2 B_1^*}{\partial t^2},\\
&\frac{\partial B_1}{\partial t}\cos\omega t - 2\omega B_1 \sin\omega t = \varepsilon^{-1}\omega^2 b^* + \frac{\partial B_1^*}{\partial t},\\
&3\omega\frac{\partial C_1}{\partial t}\cos\omega t + \left(\frac{\partial^2}{\partial t^2} - 2\omega^2\right)C_1 \sin\omega t = -\varepsilon^{-1}\omega^2 c^* + \omega^2 C_1^* + 3\frac{\partial C_1^*}{\partial t} + t\frac{\partial^2 C_1^*}{\partial t^2},\\
&2\omega C_1 \cos\omega t + \frac{\partial B_1}{\partial t}\sin\omega t = -\varepsilon^{-1}\omega^2 c^* t + 2C_1^* + t\frac{\partial C_1^*}{\partial t}.
\end{aligned} \tag{4.22}$$

By (4.22) we can easily eliminate the functions B_1 and C_1 from the equation (4.5) and obtain

$$\begin{aligned}
&e^{-\lambda t}\left(\frac{\partial}{\partial t} - \lambda + \mu\right)^2 A_1 + e^{-\mu t}\left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)B_1^* + \omega^2 B_1^* + \left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right)C_1^*\right.\\
&\left.+ t\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)C_1^*\right) + \left(\frac{\partial}{\partial t} + \lambda\right)\left(\left(\frac{\partial}{\partial t} + \mu\right)^2 + \omega^2\right)u_1\\
&= \varepsilon^{-1}\omega^2 e^{-\mu t}\left(b^* - c^* - c^* t\right) + f^{(0)}(a,b,c,t).
\end{aligned} \tag{4.23}$$

As the limit $\omega \to 0$, the above equation (4.23) reduces to

$$\begin{aligned}
&e^{-\lambda t}\left(\frac{\partial}{\partial t} - \lambda + \mu\right)^2 A_1 + e^{-\mu t}\left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)B_1^* + \left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right)C_1^*\right.\\
&\left.+ t\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)C_1^*\right) + \left(\frac{\partial}{\partial t} + \lambda\right)\left(\frac{\partial}{\partial t} + \mu\right)^2 u_1 = \bar{f}^{(0)}(a,b^*,c^*,t),
\end{aligned} \tag{4.24}$$

where $\bar{f}^{(0)} = \lim\limits_{\omega \to 0} f^{(0)}(a,b,c,t)$.

Here we assume that the function $\bar{f}^{(0)}$ can be expanded in a Macluarin series as

$$\bar{f}^{(0)} = g_0\left(a, b^*\right) + \sum_{r=1} g_r\left(a, b^*, c^*\right)t^r. \tag{4.25}$$

Substituting $\bar{f}^{(0)}$ from (4.25) into equation (4.24) and equating the coefficients of various powers of t, we obtain

$$e^{-\lambda t}\left(\frac{\partial}{\partial t} - \lambda + \mu\right)^2 A_1 + e^{-\mu t}\left(\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)B_1^* + \left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right)C_1^*\right) = g_0, \tag{4.26}$$

$$e^{-\mu t}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)C_1^* = g_1, \tag{4.27}$$

$$\left(\frac{\partial}{\partial t} + \lambda\right)\left(\frac{\partial}{\partial t} + \mu\right)^2 u_1 = \sum_{r=2} g_r\left(a, b^*, c^*\right)t^r. \tag{4.28}$$

We also assume that the coefficient g_0 can be written as

$$g_0 = ae^{-\lambda t}h_1\left(a, b^*, t\right) + b^* e^{-\mu t}h_2\left(a, b^*, t\right). \tag{4.29}$$

Now substituting g_0 from (4.29) into the equation (4.26) and equating the coefficients of $e^{-\lambda t}$ and $e^{-\mu t}$, we obtain

$$\left(\frac{\partial}{\partial t} - \lambda + \mu\right)^2 A_1 = ah_1, \tag{4.30}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right)B_1^* + \left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right)C_1^* = b^* h_2. \tag{4.31}$$

Solving equations (4.30), (4.31) and (4.27) we obtain the functions A_1, B_1^* and C_1^*. It is noted that the function A_1 is similar to its previous value obtained in **Sec. 4.2.** In the case of critical damping ω vanishes only. While the functions B_1^* and C_1^* are different form B_1 and C_1. One or more terms may be inserted for the cause of critical damping. To illustrate it, we consider again the nonlinear function $f = x^3$. Therefore

$$\bar{f}^{(0)} = a^3 e^{-3\lambda t} + 3a^2 b^* e^{-(2\lambda+\mu)t} + 3a\left(b^*\right)^2 e^{-(\lambda+2\mu)t} + \left(b^*\right)^2 e^{-3\mu t}$$
$$+ \left(3a^2 c^* e^{-(2\lambda+\mu)t} + 6ab^* c^* e^{-(\lambda+2\mu)t} + 3\left(b^*\right)^2 c^* e^{-3\mu t}\right) + 3a\left(c^*\right)^2 t^2 e^{-(\lambda+2\mu)t}$$
$$+ e^{-3\mu t}\left(3b^*\left(c^*\right)^2 t^2 + \left(c^*\right)^3 t^3\right).$$

So, the nonzero coefficients of g_r are

$$\begin{aligned} g_0 &= a^3 e^{-3\lambda t} + 3a^2 b^* e^{-(2\lambda+\mu)t} + 3a\left(b^*\right)^2 e^{-(\lambda+2\mu)t} + \left(b^*\right)^2 e^{-3\mu t}, \\ g_1 &= 3a^2 c^* e^{-(2\lambda+\mu)t} + 6ab^* c^* e^{-(\lambda+2\mu)t} + 3\left(b^*\right)^2 c^* e^{-3\mu t}, \\ g_2 &= 3a\left(c^*\right)^2 e^{-(\lambda+2\mu)t} + 3b^*\left(c^*\right)^2 e^{-3\mu t}, \\ g_3 &= e^{-3\mu t}\left(c^*\right)^3. \end{aligned} \tag{4.32}$$

From the expression of g_0 by (4.32) we easily obtain

$$h_1 = a^3 e^{-2\lambda t} + 3a\left(b^*\right)^2 e^{-2\mu t}, \quad h_2 = 3a^2 b^* e^{-2\lambda t} + \left(b^*\right)^3 e^{-2\mu t}.$$

Hence the equations (4.30), (4.31), (4.27) and (4.28) become respectively

$$\left(\frac{\partial}{\partial t} - \lambda + \mu\right)^2 A_1 = a^3 e^{-2\lambda t} + 3a\left(b^*\right)^2 e^{-2\mu t}, \tag{4.33}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right) B_1^* + \left(3\frac{\partial}{\partial t} + 2\lambda - 2\mu\right) C_1^* = 3a^2 b^* e^{-2\lambda t} + \left(b^*\right)^2 e^{-2\mu t}, \tag{4.34}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + \lambda - \mu\right) C_1^* = 3a^2 c^* e^{-2\lambda t} + 6ab^* c^* e^{-(\lambda+\mu)t} + 3\left(b^*\right)^2 c^* e^{-2\mu t}, \tag{4.35}$$

and

$$\left(\frac{\partial}{\partial t} + \lambda\right)\left(\frac{\partial}{\partial t} + \mu\right)^2 u_1 = 3a\left(c^*\right)^2 t^2 e^{-(\lambda+2\mu)t} + e^{-3\mu t}\left(3b^*\left(c^*\right)^2 t^2 + \left(c^*\right)^3 t^3\right). \tag{4.36}$$

Solving the above four equations, we obtain

$$A_1 = \frac{a^3 e^{-2\lambda t}}{(3\lambda - \mu)^2} + \frac{3a\left(b^*\right)^2 e^{-2\mu t}}{2(\lambda + \mu)^2},$$

$$B_1^* = \tfrac{3}{2} a^2 \left(\frac{b^*}{\lambda(\lambda + \mu)} + \frac{(2\lambda + \mu)c^*}{\lambda^2 (\lambda + \mu)^2} \right) e^{-2\lambda t} + \frac{3(\lambda + 5\mu)ab^* c^*}{2\mu^2 (\lambda + \mu)^2} e^{-(\lambda+\mu)t}$$

$$+ \tfrac{1}{2}\left(b^*\right)^2 \left(\frac{b^*}{\mu(-\lambda + 3\mu)} + \frac{(-\lambda + 4\mu)c^*}{\mu^2 (-\lambda + 3\mu)^2} \right) e^{-2\mu t}, \qquad (4.37)$$

$$C_1^* = 3c^* \left(\frac{a^2 e^{-2\lambda t}}{2\lambda(\lambda + \mu)} + \frac{ab^* e^{-(\lambda+\mu)t}}{\mu(\lambda + \mu)} + \frac{\left(b^*\right)^2 e^{-2\mu t}}{\mu(-\lambda + 3\mu)} \right),$$

and

$$u_1 = a\left(c^*\right)^2 e^{-(\lambda+2\mu)t}\left(d_0 + d_1 t + d_2 t^2\right) + e^{-3\mu t}\left(6\left(c^*\right)^2\left(e_1 b^* + e_0 c^*\right)\right.$$

$$\left. + 6\left(c^*\right)^2\left(e_2 b^* + e_1 c^*\right)t + 3\left(c^*\right)^2\left(e_3 b^* + e_2 c^*\right)t^2 + e_3\left(c^*\right)^3 t^3\right). \qquad (4.38)$$

where

$$d_0 = -\frac{3\left(\lambda^2 + 6\lambda\mu + 17\mu^2\right)}{4\mu^3 (\lambda + \mu)^4}, \quad d_1 = \frac{-3(\lambda + 5\mu)}{2\mu^2 (\lambda + \mu)^3}, \quad d_2 = \frac{-3}{2\mu(\lambda + \mu)^2},$$

$$e_0 = -\frac{2\lambda^3 - 21\lambda^2\mu + 76\lambda\mu^2 - 97\mu^3}{16\mu^5 (\lambda - 3\mu)^4}, \quad e_1 = \frac{3\lambda^2 - 22\lambda\mu + 43\mu^2}{16\mu^2 (\lambda - 3\mu)^3}, \qquad (4.39)$$

$$e_2 = \frac{\lambda - 4\mu}{4\mu^3 (\lambda - 3\mu)^2}, \quad e_3 = \frac{1}{4\mu^2 (\lambda - 3\mu)}, \quad \lambda \neq 3\mu.$$

Substituting the values of A_1, B_1^* and C_1^* in the first equation of (4.4) and in (4.21) and then integrating with respect to t, we obtain the solutions similar to (4.15).

Thus the first order critically damped solution of the nonlinear equation (4.1) is

$$x = ae^{-\lambda t} + e^{-\mu t}\left(b^* + c^* t\right) + \varepsilon u_1. \qquad (4.40)$$

where u_1 is given by (4.38).

4.4 Discussion

An analytical method has been developed to obtain the time response of a third order nonlinear differential equation with small nonlinearities when the damping forces are more significant. The solution (4.19) can also be used when the damping forces are near critical damping forces. Similar to the solution (4.19), the critically damped solution (4.38) is found.

The solution (3.23) in the previous **Chapter 3** is used for the significant damping forces where $0 < \mu < \omega$ and $\omega = O(1)$. If the linear damping forces are increased in such a manner that $\mu \geq \omega$ and $\omega < 1$, then the solution (4.19) gives better results than the solution (3.23) when compared to the numerical solution. One can easily verify it by computing x by both the formulae (3.23) and (4.19) for $a_0 = 0$ and $b_0 = c_0 = 1$ (see **Table 4.1**).

The solution (4.40) has been applied in nonlinear mechanical elastic system with internal friction and relaxation described in **Chapter 3** by the equation (3.1), in the case of critical damping forces [54]. In [54] the variables a, b^* and c^* have been transformed by

$$\begin{aligned} a &= e^{\xi}, \\ b^* &= e^{\eta}, \\ c^* t &= e^{\eta}\varphi(t), \end{aligned} \tag{4.41}$$

where $c_0^* \neq 0$. For small values of c_0^*, the solution in terms of φ does not give correct results. However, the solution (4.40) are used for all values of c_0^*. When $\lambda = \mu$, *i.e.*, in the case of three equal characteristic roots, the solution (4.40) can not be used; since the generating solution takes the form

$$x(t,0) = e^{-\lambda t}(a_0^* t^2 + b_0^* + c_0^* t). \tag{4.42}$$

which is not of the form of (4.40). Shamsul and Sattar [58] have developed an asymptotic method for a critically damped nonlinear system when the three characteristic roots are equal.

To obtain the corresponding numerical solution of (4.19), the initial conditions $[x(0), \dot{x}(0), \ddot{x}(0)]$ are computed from

$$
\begin{aligned}
x(0) &= a_0 + b_0 + \varepsilon\left(c^{(0)} a_0 c_0^2 + c_1 b_0 c_0^2 - d_1 c_0^3\right) \\
&\quad + \varepsilon\left(c_2 a_0 (b_0^2 - c_0^2) - 2d_2 a_0 b_0 c_0 + c_3 (b_0^3 - 3b_0 c_0^2) - d_3 (3b_0^2 c_0 - c_0^3)\right), \\
\dot{x}(0) &= -\lambda a_0 - \mu b_0 + \omega c_0 + \varepsilon\left(-(\lambda + 2\mu) c^{(0)} a_0 c_0^2 + b_0 c_0^2 (-3\mu c_1 + \omega d_1) + c_0^3 (\omega c_1 + 3\mu d_1)\right) \\
&\quad + \varepsilon\left(l_1 a_0^3 + l_2 a_0 b_0^2 + a_0^2 (m_1 b_0 + n_1 c_0) + b_0^2 (m_2 b_0 + n_2 c_0)\right) \\
&\quad + \varepsilon\left((-(\lambda + 2\mu) c_2 + 2\omega d_2) a_0 (b_0^2 - c_0^2) + 2(2\omega c_2 + (\lambda + 2\mu) d_2) a_0 b_0 c_0\right) \\
&\quad + 3(-\mu c_3 + \omega d_3)(b_0^3 - 3b_0 c_0^2) + 3(\omega c_3 + \mu d_3)(3b_0^2 c_0 - c_0^3)\big), \\
\ddot{x}(0) &= \lambda^2 a_0 + (\mu^2 - \omega^2) b_0 - 2\mu\omega c_0 \\
&\quad + \varepsilon\left((\lambda + 2\mu)^2 c^{(0)} a_0 c_0^2 + b_0 c_0^2 \left((9\mu^2 - \omega^2) c_1 - 6\mu\omega d_1\right) - c_0^3 \left(6\mu\omega c_1 + (9\mu^2 - \omega^2) d_1\right)\right) \\
&\quad - 2\varepsilon\left(2\lambda l_1 a_0^3 + (\lambda + \mu) l_2 a_0 b_0^2 + a_0^2 \left(((\lambda + \mu) m_1 + \omega n_1) b_0 + (-\omega m_1 + (\lambda + \mu) n_1) c_0\right)\right. \\
&\quad \left. + 2b_0^2 \left((\mu m_2 + \omega n_2) b_0 + (-\omega m_2 + \mu n_2) c_0\right)\right) \\
&\quad + \varepsilon\left(\left((\lambda + 2\mu)^2 - 4\omega^2\right) c_2 - 4(\lambda + 2\mu)\omega d_2\right) a_0 (b_0^2 - c_0^2) \\
&\quad - 2\left(4(\lambda + 2\mu)\omega c_2 + \left((\lambda + 2\mu)^2 - 4\omega^2\right) d_2\right) a_0 b_0 c_0 \\
&\quad + 9\left((\mu^2 - \omega^2) c_3 - 2\mu\omega d_3\right)(b_0^3 - 3b_0 c_0^2) - 9\left(2\mu\omega c_3 + (\mu^2 - \omega^2) d_3\right)(3b_0^2 c_0 - c_0^3)\big).
\end{aligned}
\tag{4.43}
$$

t	x	x_{nu}	|Error| (%)
0.0	0.989375	0.989375	0.00000
0.5	0.823955	0.823906	0.00598
1.0	0.507640	0.507598	0.00827
1.5	0.236597	0.236555	0.01775
2.0	0.065116	0.065116	0.05530
2.5	-0.017699	-0.017723	0.13542
3.0	-0.042788	-0.042802	0.03271
3.5	-0.039037	-0.039043	0.01537
4.0	-0.025816	-0.025817	0.00387
4.5	-0.013121	-0.013121	0.00000
5.0	-0.004471	-0.004470	0.02237

Table 4.1(a) Solution (4.19) for $\lambda = 2$, $\mu = \omega = 1$ and $\varepsilon = 0.1$ with initial conditions $a_0 = 0$ and $b_0 = c_0 = 1$ or $[x(0) = 0.989375, \dot{x}(0) = 0.063750, \ddot{x}(0) = -2.26625]$ are given in the second column. In the third column corresponding numerical results (considered to be exact) are given. In the fourth column the percentage errors are given.

t	x	x_{nu}	|Error| (%)
0.0	0.998750	0.998750	0.00000
0.5	0.826195	0.826198	0.00036
1.0	0.506641	0.506598	0.00849
1.5	0.234718	0.234659	0.02510
2.0	0.063484	0.063435	0.07718
2.5	-0.018745	-0.018775	0.16004
3.0	-0.043301	-0.043316	0.03460
3.5	-0.039199	-0.039203	0.01020
4.0	-0.025799	-0.025798	0.00388
4.5	-0.013044	-0.013041	0.02300
5.0	-0.004395	-0.004393	0.04551

Table 4.1(b) Solution (3.23) for $\lambda = 2$, $\mu = \omega = 1$ and $\varepsilon = 0.1$ with initial conditions $a_0 = 0$ and $b_0 = c_0 = 1$ or $[x(0) = 0.998750, \dot{x}(0) = 0.045000, \ddot{x}(0) = -2.247500]$ are given in the second column. In the third column corresponding numerical results are given. In the fourth column the percentage errors are given.

Remark : The errors in **Table 4.1(b)** oscillate with more amplitude than errors in **Table 4.1(a)**.

Chapter 5

Third Order Nonlinear Oscillations with Damping Forces and Delay

5.1 Introduction

Many physical systems possess the feature of having a delayed response, so that the rate at which processes occur depends not only on the current state of the system but also on past states. Mathematical models of such processes commonly result in differential equations with time delay. Examples of such models arise in a wide variety of fields, from the vibrations of mechanical systems and electric circuit theory to certain bio-medical phenomena and the theory of economic dynamics. Besides involving time delay these systems often possess feature which can not be modeled by linear equations and the resulting difference-differential equations are nonlinear.

The asymptotic solutions of such nonlinear difference-differential equations have previously been investigated by Rubanic [50], Mitropolskii and Martinyuk [37], Lardner and Bojadziev [28], Bojadziev and Lardner [9], Linkens [32] for second order systems. Some authors, *e.g.*, Pavlidis [45], Dutt, Ghosh and Karmakar [18], Lin and Khan [30], Bojadziev and Chan [12] have used KBM method in the neighborhood of the equilibrium of 2-dimensional biological systems with time delay. Pavlidis [45] has also investigated *n*-dimensional nonlinear systems. On the other hand some authors, *e.g.*, Goel, Mitra and Montrol [23], Bojadziev [4,10] have used KBM method in nonlinear systems with small time delay. A system with one degree of freedom, small time lag and significant damping has been studied by Kan [24,25], by the means of successive transformations.

Freedman, Rao and Lakshmi [20], and Freedman and Ruan [21] etc. have studied 3-dimensional biological systems with time delay. However, they dealt mainly with stability problems. In the present paper, a different approach based on the small parameter expansion is applied to a third order nonlinear system with significant time delay.

5.2 The method

Let us consider a third order nonlinear difference-differential equation

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x + \kappa_3 x_\Delta = \varepsilon f(\ddot{x}, \ddot{x}_\Delta, \dot{x}, \dot{x}_\Delta, x, x_\Delta), \tag{5.1}$$

where $x_\Delta = x(t - \Delta)$, $\Delta \geq 0$. If $\Delta = 0$, the difference-differential equation becomes an ordinary differential equation. It is obvious that, the unperturbed equation of (5.1) has a 'non-retarded' restoring force $k_3 x$ and a 'retarded' restoring force $\kappa_3 x$. The term 'non-retarded' does not require further explanation, inasmuch as this is the usual significance of terms $\dddot{x}(t), \ddot{x}(t), \dot{x}(t), x(t)$ encountered in the ordinary differential equation. As to the 'retarded' terms, they are often encountered in the control problems where a certain control action is produced artificially. In such a case, in view of inevitable time-lags in a control system, this action does not relate to the instant t at which it is supposed to be exerted, but to a past instant $t - \Delta$.

A differential equation, for instance with constant coefficients, has the property that the degree of its characteristic equation is always equal to the order of the differential equation. The fundamental property of a difference-differential equation is that its characteristic equation is always of an infinite degree whatever its order may be. One readily ascertains this peculiarity of difference-differential equation if one develops a 'retarded' quantity as a Taylor series in terms of the corresponding 'non-retarded' quantities. Thus, for instance

$$x_\Delta = x(t-\Delta) = x(t) - \frac{\Delta}{1!}\dot{x}(t) + \frac{\Delta^2}{2!}\ddot{x}(t) - \ldots$$

$$= x(t)\left(1 - \frac{\Delta}{1!}\frac{\dot{x}}{x} + \frac{\Delta^2}{2!}\frac{\ddot{x}}{x} - \ldots\right), \tag{5.2}$$

if one tries to satisfy a difference-differential equation by a solution of the form $x = x_0\, e^{zt}$, it is clear that $\frac{\dot{x}}{x} = z$, $\frac{\ddot{x}}{x} = z^2, \cdots$ and (5.2) becomes

$$x_\Delta = x\left(1 - \frac{\Delta z}{1!} + \frac{\Delta^2 z^2}{2!} - \cdots\right) \tag{5.3}$$

$$= x\, e^{-\Delta z}.$$

If one substitutes this expression of x_Δ into the unperturbed equation of (5.1), one obtains an *algebraico-transcendental* characteristic equation

$$\Omega(z,\Delta) \equiv z^3 + k_1 z^2 + k_2 z + k_3 + \kappa_3 e^{-\Delta z} = 0, \tag{5.4}$$

and the problem consists in determining the zeros of the entire function $\Omega(z,\Delta)$. Since the degree of (5.4) is infinity, it has infinite number of roots. In a similar fashion one can show that, every linear equation with 'retarded' terms always has infinite number of characteristic roots. So, the solution of difference-differential equation with constant coefficients is not simple even if it is linear. However, in the case of monofrequent oscillation, two roots with non-positive real parts, namely $-\mu \pm i\omega$, where μ is the smallest, are considered to obtain the solution, since other modes of vibrations will die-out relatively quickly. Here, we consider a real root $-\lambda$, where λ is the smallest, assuming that as $\Delta \to 0$ the solution reduces to the solution (3.7). Therefore, the generating solution of (5.1) takes the form

$$x = a_0 e^{-\lambda t} + e^{-\mu t}(b_0 \cos\omega t + c_0 \sin\omega t), \tag{5.5}$$

which is similar to the generating solution of the nonlinear differential equation (3.5) (see Sec. 3.2).

Now we seek a solution of the difference-differential equation (5.1) in the form of an asymptotic expansion

$$x = a\,e^{-\lambda t} + e^{-\mu t}\left(b\cos\omega t + c\sin\omega t\right) + \varepsilon\, u_1(a,b,c,t) + \varepsilon^2\cdots, \tag{5.6}$$

where a, b and c satisfy the differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,b,c,t) + \varepsilon^2\cdots, \\ \dot{b} &= \varepsilon B_1(a,b,c,t) + \varepsilon^2\cdots, \\ \dot{c} &= \varepsilon C_1(a,b,c,t) + \varepsilon^2\cdots. \end{aligned} \tag{5.7}$$

Differentiating (5.6) three times with respect to t, using relations (5.7), substituting (5.6) and the derivatives $\dot{x}$, $\ddot{x}$, $\dddot{x}$ in the original equation (5.1), and comparing the coefficients of various powers of ε, we get for the coefficient of ε :

$$\begin{aligned} & e^{-\lambda t}\left(\frac{\partial^2 A_1}{\partial t^2} - 3\lambda\frac{\partial A_1}{\partial t} + 3\lambda^2 A_1 + k_1\left(\frac{\partial A_1}{\partial t} - 2\lambda A_1\right) + k_2 A_1\right) \\ & + e^{-\mu t}\Bigg(\Bigg(\frac{\partial^2 B_1}{\partial t^2} - 3\mu\frac{\partial B_1}{\partial t} + 3(\mu^2-\omega^2)B_1 + 3\omega\left(\frac{\partial C_1}{\partial t} - 2\mu C_1\right) \\ & \qquad + k_1\left(\frac{\partial B_1}{\partial t} - 2\mu B_1 + 2\omega C_1\right) + k_2 B_1\Bigg)\cos\omega t \\ & \qquad + \Bigg(-3\omega\left(\frac{\partial B_1}{\partial t} - 2\mu B_1\right) + \frac{\partial^2 C_1}{\partial t^2} - 3\mu\frac{\partial C_1}{\partial t} + 3(\mu^2-\omega^2)C_1 \\ & \qquad + k_1\left(-2\omega B_1 + \frac{\partial C_1}{\partial t} - 2\mu C_1\right) + k_2 C_1\Bigg)\sin\omega t \\ & \qquad + \frac{\partial^3 u_1}{\partial t^3} + k_1\frac{\partial^2 u_1}{\partial t^2} + k_2\frac{\partial u_1}{\partial t} + k_3 u_1 + \kappa_3 u_1(t-\Delta) = f^0(a,b,c,t) \end{aligned} \tag{5.8}$$

where $f^{(0)} = f\left(x_0, \dot{x}_0, \ddot{x}_0\right)$ and $x_0 = a\,e^{-\lambda t} + e^{-\mu t}\left(b\cos\omega t + c\sin\omega t\right)$. When $\Delta = 0$, one can easily transform the equation (5.8) to the equation (3.5) obtained in Chapter 3 by utilizing the relations between roots and coefficients of the characteristic equation (5.4). Note that in this case the equation (5.4) has exactly three roots.

Let the function $f^{(0)}$ be expanded in a Fourier series

$$f^{(0)} = \sum_{n=0}^{\infty} \left(F_n(a,b,c,t)\cos n\omega t + G_n(a,b,c,t)\sin n\omega t \right). \tag{5.9}$$

To solve the equation (5.8) for u_1, A_1, B_1 and C_1, it is assumed that the function u_1 does not contain *secular terms* as well as a term with $t\exp(-t)$. Substituting (5.9) in (5.8) and equating the coefficients of $\cos 0\omega t$, $\cos\omega t$ and $\sin\omega t$, we obtain

$$e^{-\lambda t}\left(\frac{\partial^2 A_1}{\partial t^2} - 3\lambda \frac{\partial A_1}{\partial t} + 3\lambda^2 A_1 + k_1\left(\frac{\partial A_1}{\partial t} - 2\lambda A_1 \right) + k_2 A_1 \right) = F_0, \tag{5.10}$$

$$\begin{aligned} e^{-\mu t}\Bigg(& \frac{\partial^2 B_1}{\partial t^2} - 3\mu \frac{\partial B_1}{\partial t} + 3(\mu^2 - \omega^2)B_1 + 3\omega\left(\frac{\partial C_1}{\partial t} - 2\mu C_1 \right) \\ & + k_1\left(\frac{\partial B_1}{\partial t} - 2\mu B_1 + 2\omega C_1 \right) + k_2 B_1 = F_1, \end{aligned} \tag{5.11}$$

$$\begin{aligned} e^{-\mu t}\Bigg(& -3\omega\left(\frac{\partial B_1}{\partial t} - 2\mu B_1 \right) + \frac{\partial^2 C_1}{\partial t^2} - 3\mu \frac{\partial C_1}{\partial t} + 3(\mu^2 - \omega^2)C_1 \\ & + k_1\left(-2\omega B_1 + \frac{\partial C_1}{\partial t} - 2\mu C_1 \right) + k_2 C_1 \Bigg) = G_1, \end{aligned} \tag{5.12}$$

and

$$\frac{\partial^3 u_1}{\partial t^3} + k_1 \frac{\partial^2 u_1}{\partial t^2} + k_2 \frac{\partial u_1}{\partial t} + k_3 u_1 + \kappa_3 u_1(t-\Delta) = \sum_{n=2}^{\infty} \left(F_n \cos n\omega t + G_n \sin n\omega t \right). \tag{5.13}$$

The particular solutions of (5.10)-(5.12) give the three unknown functions A_1, B_1 and C_1. When F_0, F_1 and G_1 are given, we may easily solve the equations. Substituting the values of A_1, B_1 and C_1 into (5.7) and then solving them, we obtain the first approximate solution of the nonlinear difference-differential equation (5.1). The procedure can be applied to higher orders in the same way.

5.2.1 Determination of the first order correction term u_1

The particular solution of (5.13) gives the first order correction term u_1. When the nonlinear function f of the equation (5.1) is given, F_n and G_n, $n \geq 2$ are specified. Then substituting the values of F_n and G_n in (5.13), we may solve this equation. Thus the correction term u_1 is found and we obtain the first improved solution of the equation (5.1).

5.3 Example

Now consider $f = x^3$. So

$$f^{(0)} = a^3 e^{-3\lambda t} + \tfrac{3}{2} a e^{-(\lambda+2\mu)t}\left(b^2 + c^2\right) + 3\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} e^{-3\mu t}\left(b^2 + c^2\right)\right)\left(b\cos\omega t + c\sin\omega t\right)$$

$$+ \tfrac{3}{2} a e^{-(\lambda+2\mu)t}\left(\left(b^2 - c^2\right)\cos 2\omega t + 2bc\sin 2\omega t\right) + \frac{e^{-3\mu t}}{4}\left(b\left(b^2 - 3c^2\right)\cos 3\omega t + c\left(3b^2 - c^2\right)\sin 3\omega t\right)$$

Therefore the non zero coefficients of F_n and G_n are

$$F_0 = a^3 e^{-3\lambda t} + \tfrac{3}{2} a e^{-(\lambda+2\mu)t}\left(b^2 + c^2\right),\ F_1 = 3b\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} e^{-3\mu t}\left(b^2 + c^2\right)\right),$$
$$G_1 = 3c\left(a^2 e^{-(2\lambda+\mu)t} + \tfrac{1}{4} e^{-3\mu t}\left(b^2 + c^2\right)\right),\ F_2 = \tfrac{3}{2} a\left(b^2 - c^2\right)e^{-(\lambda+2\mu)t}, \tag{5.14}$$
$$G_2 = 3abce^{-(\lambda+2\mu)t},\ F_3 = \tfrac{1}{4} b\left(b^2 - 3c^2\right)e^{-3\mu t},\ G_3 = \tfrac{1}{4} c\left(3b^2 - c^2\right)e^{-3\mu t}.$$

Substituting the values of F_0, F_1 and G_1 from (5.14) into the equations (5.10)-(5.12) and solving them, we obtain

$$A_1 = l_1 a^3 e^{-2\lambda t} + l_2 a\left(b^2 + c^2\right)e^{-2\mu t},$$
$$B_1 = a^2\left(m_1 b + n_1 c\right)e^{-2\lambda t} + \left(m_2 b + n_2 c\right)\left(b^2 + c^2\right)e^{-2\mu t}, \tag{5.15}$$
$$C_1 = a^2\left(-n_1 b + m_1 c\right)e^{-2\lambda t} + \left(-n_2 b + m_2 c\right)\left(b^2 + c^2\right)e^{-2\mu t},$$

where

$$l_1 = \frac{1}{13\lambda^2 - 4k_1\lambda + k_2}, \quad l_2 = \frac{3}{2\left(3\lambda^2 + 6\lambda\mu + 4\mu^2 - 2k_1(\lambda+\mu) + k_2\right)},$$

$$m_1 = \frac{3p_1}{2\left(p_1^2 + q_1^2\right)}, \quad m_2 = \frac{3p_2}{8\left(p_2^2 + q_2^2\right)}, \tag{5.16}$$

$$n_1 = \frac{3q_1}{2\left(p_1^2 + q_1^2\right)}, \quad n_2 = \frac{3q_2}{8\left(p_2^2 + q_2^2\right)}.$$

and

$$\begin{aligned} p_1 &= 4\lambda^2 + 6\lambda\mu + 3\mu^2 - 3\omega^2 - 2k_1(\lambda+\mu), \\ q_1 &= 6\omega(\lambda+\mu) - 2k_1\omega, \\ p_2 &= 13\mu^2 - 3\omega^2 - 4k_1\mu, \\ q_2 &= 12\mu\omega - 2k_1\omega. \end{aligned} \tag{5.16a}$$

Substituting the values of A_1, B_1 and C_1 from (5.15) into the equation (5.7), we obtain

$$\begin{aligned} \dot{a} &= \varepsilon\left(l_1 a^3 e^{-2\lambda t} + l_2 a\left(b^2 + c^2\right)e^{-2\mu t}\right), \\ \dot{b} &= \varepsilon\left(a^2(m_1 b + n_1 c)e^{-2\lambda t} + (m_2 b + n_2 c)\left(b^2 + c^2\right)e^{-2\mu t}\right), \\ \dot{c} &= \varepsilon\left(a^2(-n_1 b + m_1 c)e^{-2\lambda t} + (-n_2 b + m_2 c)\left(b^2 + c^2\right)e^{-2\mu t}\right). \end{aligned} \tag{5.17}$$

Replacing a, b and c by their respective values obtained in the linear case, and then integrating with respect to t, we obtain

$$\begin{aligned} a &= a_0 + \varepsilon\left(l_1 a_0^3\left(1 - e^{-2\lambda t}\right)/\lambda + l_2 a_0\left(b_0^2 + c_0^2\right)\left(1 - e^{-2\mu t}\right)/\mu\right)/2, \\ b &= b_0 + \varepsilon\left(a_0^2(m_1 b_0 + n_1 c_0)\left(1 - e^{-2\lambda t}\right)/\lambda + (m_2 b_0 + n_2 c_0)\left(b_0^2 + c_0^2\right)\left(1 - e^{-2\mu t}\right)/\mu\right)/2, \\ c &= c_0 + \varepsilon\left(a_0^2(-n_1 b_0 + m_1 c_0)\left(1 - e^{-2\lambda t}\right)/\lambda + (-n_2 b_0 + m_2 c_0)\left(b_0^2 + c_0^2\right)\left(1 - e^{-2\mu t}\right)/\mu\right)/2. \end{aligned} \tag{5.18}$$

Hence the first order solution of (5.1) is

$$x = a e^{-\lambda t} + e^{-\mu t}(b\cos\omega t + c\sin\omega t), \tag{5.19}$$

where a, b and c are given by (5.18).

Now substituting the values of F_2, F_3, G_2 and G_3 from (5.14) into the equation (5.13) and then solving it, we obtain

$$u_1 = ae^{-(\lambda+2\mu)t}\left(\left(b^2 - c^2\right)\left(c_2\cos 2\omega t + d_2 \sin 2\omega t\right) + 2bc\left(-d_2\cos 2\omega t + c_2 \sin 2\omega t\right)\right)$$
$$+ e^{-3\mu t}\left(b\left(b^2 - 3c^2\right)\left(c_3\cos 3\omega t + d_3\sin 3\omega t\right) + c\left(3b^2 - c^2\right)\left(-d_3\cos 3\omega t + c_3\sin 3\omega t\right)\right). \quad (5.20)$$

where

$$c_2 = \frac{3r_2}{2\left(r_2^2 + s_2^2\right)}, \quad d_2 = \frac{3s_2}{2\left(r_2^2 + s_2^2\right)},$$
$$c_3 = \frac{r_3}{4\left(r_3^2 + s_3^2\right)}, \quad d_3 = \frac{s_3}{4\left(r_3^2 + s_3^2\right)}. \quad (5.21)$$

and

$$r_2 = -(\lambda + 2\mu)^3 + 12\omega^2(\lambda + 2\mu) + k_1\left((\lambda + 2\mu)^2 - 4\omega^2\right) - k_2(\lambda + 2\mu) + k_3$$
$$+ \kappa_3 e^{(\lambda+2\mu)\Delta}\cos 2\omega\Delta,$$
$$s_2 = 6\omega(\lambda + 2\mu)^2 - 8\omega^3 - 4k_1\omega(\lambda + 2\mu) + k_2\omega - \kappa_3 e^{(\lambda+2\mu)\Delta}\sin 2\omega\Delta, \quad (5.21a)$$
$$r_3 = -27\mu^3 + 81\mu\omega^2 + 9k_1(\mu^2 - \omega^2) - 3k_2\mu + k_3 + \kappa_3 e^{3\mu\Delta}\cos 3\omega\Delta,$$
$$s_3 = 81\mu^2\omega - 27\omega^3 - 18k_1\mu\omega + 3k_2\omega - \kappa_3 e^{3\mu\Delta}\sin 3\omega\Delta.$$

Therefore, the first improved solution of (5.1) is

$$x = a\,e^{-\lambda t} + e^{-\mu t}\left(b\cos\omega t + c\sin\omega t\right) + \varepsilon u_1, \quad (5.22)$$

where a, b and c are given by (5.18) and u_1 is given by (5.20).

5.4 Discussion

An asymptotic solution has been found to obtain the time response of a third order weakly nonlinear difference-differential equation on the basis of the extended KBM method. While the solution of the type (5.22) is not the most general ones, they are important in various oscillating systems in mechanics, electrical circuits and control theory.

The principal difficulty in the studies of the difference-differential equations is in the linear problem itself, which is of a special *transcendental* character. Generally, a linear problem has a simple solution. Here, on the contrary, the linear problem leads always to an infinite spectrum of frequencies with which such a system can oscillate. The determination of this spectrum requires a corresponding determination of zeros of certain analytic functions $\Omega(z,\Delta)$. Some roots of the characteristic equation (5.4) are given in **Table 5.1** for different values of Δ. When $\Delta \to 0$ the solution (5.22) is identical to the solution (3.23) obtained in **Chapter 3**. Moreover, when $\Delta = O(\varepsilon)$, *i.e.*, in the case of small retardation effects, one can find a simple solution similar to (3.7) by transforming the difference-differential equation (5.1) to the differential equation

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + (k_3 + \kappa_3)x = \varepsilon\big(f(\ddot{x},\dot{x},x) + \kappa_3\Delta_1\dot{x}\big), \tag{5.23}$$

where $\Delta = \varepsilon\Delta_1$, $\Delta_1 = O(1)$ and x_Δ is expanded in a Taylor series as

$$x_\Delta = x(t - \varepsilon\Delta_1) = x - \frac{\varepsilon\Delta_1\dot{x}}{1!} + O(\varepsilon^2). \tag{5.24}$$

For the case of retarded damping forces, similar solutions can be found, but it becomes more complicated and laborious.

Δ	Real root or roots	Complex roots
0	-2.20409..	-0.89795.. ± i1.11145..
0.5	-2.53021.., -18.42054.., ...	-0.79703.. ± i1.08401.., -24.86568.. ± i44.18732.., ...
0.75	-2.95135.., -9.17368.., ...	-0.74681.. ± i1.05272.., -11.96837.. ± i11.70767.., ...

Table 5.1 Roots of the *algebraico-transcendental* characteristic equation (5.4) are given for different values of Δ when $k_1 = k_3 = 4$, $k_2 = 6$ and $\kappa_3 = 0.5$.

Chapter 6

Third Order Nonlinear Oscillations with Varying Coefficients

6.1 Introduction

From the beginning of modern developments in the theory of oscillations the asymptotic method has been neglected to the investigations of nonstationary phenomena, meant all the cases in which the coefficients of differential equations are varying slowly with time. The oscillations of this nature are not necessarily periodic. Mitropolskii [35] has first used asymptotic method to investigate nonstationary solutions of the second order nonlinear systems. Bojadziev and Edwards [14] have studied some damped-oscillatory and non-oscillatory second order systems with slowly varying coefficients. Feshchenko, Shkil and Nikolenko [19] have used asymptotic method to linear differential equations with slowly varying coefficients. Arya and Bojadziev [2] have studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Arya and Bojadziev [1] have also studied a system of second order nonlinear hyperbolic differential equations with slowly varying coefficients. However, the more difficult and no less important case of third order nonlinear differential equations with damping and slowly varying coefficients has remained almost untouched. The aim of the present work is in part to fill that gap.

6.2 The method

Consider the third order nonlinear differential equation with slowly varying coefficients

$$\dddot{x} + k_1(\tau)\ddot{x} + k_2(\tau)\dot{x} + k_3(\tau)x = \varepsilon f(\tau, \ddot{x}, \dot{x}, x), \tag{6.1}$$

where $\varepsilon << 1$, and $\tau = \varepsilon t$ is the slowly varying time. It is assumed that, for a certain interval $(0 \le t \le T)$, the slow time τ is in the interval $(0 \le t \le T/\varepsilon)$ and in this interval the coefficients $k_s(\tau)$, $s = 1, 2, 3$ are continuously differentiable for an infinite number of times for all finite values of their arguments.

Let the unperturbed equation of (6.1) has three roots $-\lambda$, $-\mu \pm i\omega$, where λ, μ and ω are constants, but when $\varepsilon \ne 0$, λ, μ and ω are functions of τ. Therefore the generating solution of equation (6.1) is

$$x = a_0 e^{-\lambda t} + b_0 e^{-\mu t} \cos(\omega t + \varphi_0), \tag{6.2}$$

where a_0, b_0 and φ_0 are arbitrary constants. The generating solution (6.2) is identical to (3.6) (discussed in **Chapter 3**). We look for a solution of (6.1) in the form

$$x = a + b\cos\varphi + \varepsilon u_1(a, b, \varphi, \tau) + \varepsilon^2 \cdots, \tag{6.3}$$

where a, b and φ satisfy the first order differential equations

$$\begin{aligned} \dot{a} &= -\lambda(\tau)a + \varepsilon A_1(a, b, \tau) + \varepsilon^2 \cdots, \\ \dot{b} &= -\mu(\tau)b + \varepsilon B_1(a, b, \tau) + \varepsilon^2 \cdots, \\ \dot{\varphi} &= \omega(\tau) + \varepsilon C_1(a, b, \tau) + \varepsilon^2 \cdots. \end{aligned} \tag{6.4}$$

Differentiating (6.3) three times with respect to t, using relations (6.4), substituting (6.3) and the derivatives $\dot{x}, \ddot{x}, \dddot{x}$ in the original equation (6.1), and comparing the coefficients of various powers of ε, we get for the coefficient of ε :

$$
\begin{aligned}
&2\lambda(\lambda-\mu)\lambda' a+\left(\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\mu\right)^2+\omega^2\right)A_1+\left(-\lambda\mu'+\mu'\mu-3\omega\omega'\right)b\cos\varphi\\
&+\left(\left(\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\lambda\right)\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\mu\right)-2\omega^2\right)B_1\right.\\
&\left.+\omega b\left(3\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}\right)-2\lambda+2\mu\right)C_1\right)\cos\varphi+\left(3\mu'\omega-\lambda\omega'+\mu\omega'\right)b\sin\varphi\\
&+\left(\omega\left(3\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\lambda\right)-2\lambda-\mu\right)B_1\right.\\
&\left.+b\left(-\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\lambda\right)^2+(\lambda-\mu)\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\lambda\right)+2\omega^2\right)C_1\right)\sin\varphi\\
&+\left(-\lambda a\frac{\partial}{\partial a}-\mu b\frac{\partial}{\partial b}+\omega\frac{\partial}{\partial\varphi}+\lambda\right)\left(\left(-\lambda a\frac{\partial}{\partial a}-\mu b\frac{\partial}{\partial b}+\omega\frac{\partial}{\partial\varphi}\right)^2+\omega^2\right)u_1=f^{(0)}(a,b,\varphi,\tau),
\end{aligned}
\tag{6.5}
$$

where $f^{(0)}=f\left(x_0,\dot{x}_0,\ddot{x}_0,\tau\right)$ and $x_0=a+b\cos\varphi$.

Let the function $f^{(0)}$ be expanded in a Fourier series

$$
f^{(0)}=F_0(a,b,\tau)+F_1(a,b,\tau)\cos\varphi+G_1(a,b,\tau)\sin\varphi+F_2(a,b,\tau)\cos 2\varphi+\cdots. \tag{6.6}
$$

To solve the equation (6.5) for u_1, A_1, B_1 and C_1 it is assumed that the function u_1 does not contain *secular terms* as well as the terms with $t\,e^{-t}$. Substituting (6.6) into (6.5) and equating the coefficients of $\cos 0\varphi$, $\cos\varphi$ and $\sin\varphi$, we obtain

$$
2\lambda(\lambda-\mu)\lambda' a+\left(\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\mu\right)^2+\omega^2\right)A_1=F_0, \tag{6.7}
$$

$$
\begin{aligned}
&\left(-\lambda\mu'+\mu'\mu-3\omega\omega'\right)b+\left(\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\lambda\right)\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\mu\right)-2\omega^2\right)B_1\\
&+\omega b\left(3\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}\right)-2\lambda+2\mu\right)C_1=F_1,
\end{aligned}
\tag{6.8}
$$

$$\left(3\mu'\omega-\lambda\omega'+\mu\omega'\right)b+\omega\left(3\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}\right)-2\lambda-\mu\right)B_1$$

$$+b\left(-\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}\right)^2+(\lambda-\mu)\left(\lambda a\frac{\partial}{\partial a}+\mu b\frac{\partial}{\partial b}-\lambda\right)+2\omega^2\right)C_1=G_1, \tag{6.9}$$

and

$$\left(-\lambda a\frac{\partial}{\partial a}-\mu b\frac{\partial}{\partial b}+\omega\frac{\partial}{\partial\varphi}+\lambda\right)\left(\left(-\lambda a\frac{\partial}{\partial a}-\mu b\frac{\partial}{\partial b}+\omega\frac{\partial}{\partial\varphi}\right)^2+\omega^2\right)u_1$$
$$=\cos 2\varphi+\sin 2\varphi+\cdots. \tag{6.10}$$

The particular solutions of (6.7)-(6.9) give the unknown functions A_1, B_1 and C_1. Substituting these values into (6.4) and integrating numerically we obtain the first approximate solution of the equation (6.1). The procedure can be carried to higher orders in the same way.

6.2.1 Determination of the correction term u_1

The equation (6.10) is a third order non-homogeneous partial differential equation. Its particular solution gives the first order correction term u_1. Thus the first order improved solution of (6.1) can be found.

6.3 Example

Now consider $f=x^3$. So,

$$f^{(0)}=a^3+\tfrac{3}{2}ab^2+\left(3a^2b+\tfrac{3}{4}b^3\right)\cos\varphi+\tfrac{3}{2}ab^2\cos 2\varphi+\tfrac{1}{4}b^3\cos 3\varphi\,.$$

Therefore the nonzero coefficients of F_n and G_n, $n=0,1,\cdots$ are

$$F_0=a^3+\tfrac{3}{2}ab^2,\ F_1=3a^2b+\tfrac{3}{4}b^3,\ F_2=\tfrac{3}{2}ab^2 \text{ and } F_3=\tfrac{1}{4}b^3.$$

Substituting the values of F_0, F_1 and $G_1=0$ in (6.7)-(6.9) and then solving them, we obtain

$$A_1 = \frac{2(\lambda-\mu)\lambda' a}{(\lambda-\mu)^2+\omega^2} + l_1 a^3 + l_2 a b^2,$$

$$B_1 = \frac{-2(\lambda-\mu)\mu'\omega + \left((\lambda-\mu)^2+3\omega^2\right)\omega'}{2\omega\left((\lambda-\mu)^2+\omega^2\right)} b + m_1 a^2 b + m_2 b^3, \tag{6.11}$$

$$C_1 = -\frac{\left((\lambda-\mu)^2+3\omega^2\right)\mu' + 2(\lambda-\mu)\omega\omega'}{2\omega\left((\lambda-\mu)^2+\omega^2\right)} + n_1 a^2 + n_2 b^2,$$

where

$$l_1 = \frac{1}{(3\lambda-\mu)^2+\omega^2}, \quad l_2 = \frac{3}{2\left((\lambda+\mu)^2+\omega^2\right)},$$

$$m_1 = -\frac{3\left(\lambda^2+\lambda\mu-\omega^2\right)}{2\left(\lambda^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)}, \quad m_2 = -\frac{3\left(\mu(\lambda-3\mu)+\omega^2\right)}{8\left(\mu^2+\omega^2\right)\left((\lambda-3\mu)^2+\omega^2\right)}, \tag{6.12}$$

$$n_1 = -\frac{3\omega(2\lambda+\mu)}{2\left(\lambda^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)}, \quad n_2 = \frac{3\omega(-\lambda+4\mu)}{8\left(\mu^2+\omega^2\right)\left((\lambda-3\mu)^2+\omega^2\right)}.$$

Substituting these values of A_1, B_1 and C_1 from (6.11) into (6.4) we obtain

$$\dot{a} = -\lambda a + \varepsilon\left(\frac{2(\lambda-\mu)\lambda' a}{(\lambda-\mu)^2+\omega^2} + l_1 a^3 + l_2 a b^2\right),$$

$$\dot{b} = -\mu b + \varepsilon\left(\frac{-2(\lambda-\mu)\mu'\omega + \left((\lambda-\mu)^2+3\omega^2\right)\omega'}{2\omega\left((\lambda-\mu)^2+\omega^2\right)} b + m_1 a^2 b + m_2 b^3\right), \tag{6.13}$$

$$\dot{\varphi} = \omega + \varepsilon\left(-\frac{\left((\lambda-\mu)^2+3\omega^2\right)\mu' + 2(\lambda-\mu)\omega\omega'}{2\omega\left((\lambda-\mu)^2+\omega^2\right)} + n_1 a^2 + n_2 b^2\right).$$

In general, equations (6.13) are solved in a numerical procedure.

Thus the first approximate solution of (6.1) is

$$x = a + b\cos\varphi, \tag{6.14}$$

where a, b and φ are the solutions of (6.13).

Substituting the values of F_2 and F_3 in (6.10) and then solving it, we obtain

$$u_1 = ab^2\left(c_2 \cos 2\varphi + d_2 \sin 2\varphi\right) + b^3\left(c_3 \cos 3\varphi + d_3 \sin 3\varphi\right), \tag{6.15}$$

where

$$\begin{aligned}
c_2 &= \frac{3\left(-\mu(\lambda+\mu)^2 + (4\lambda+7\mu)\,\omega^2\right)}{4\left(\mu^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)\left((\lambda+\mu)^2+9\omega^2\right)},\\
d_2 &= \frac{3\omega\left(\left((\lambda+\mu)(\lambda+5\mu)-3\omega^2\right)\right)}{4\left(\mu^2+\omega^2\right)\left((\lambda+\mu)^2+\omega^2\right)\left((\lambda+\mu)^2+9\omega^2\right)},\\
c_2 &= \frac{\mu^2(\lambda-3\mu)+(-2\lambda+15\mu)\,\omega^2}{16\left(\mu^2+\omega^2\right)\left(\mu^2+4\omega^2\right)\left((\lambda-3\mu)^2+9\omega^2\right)},\\
c_3 &= \frac{-3\omega\left(\mu(\lambda-3\mu)+2\omega^2\right)}{16\left(\mu^2+\omega^2\right)\left(\mu^2+4\omega^2\right)\left((\lambda-3\mu)^2+9\omega^2\right)}.
\end{aligned} \tag{6.16}$$

Thus the first improved solution of (6.1) is

$$x = a + b\cos\varphi + \varepsilon\, u_1, \tag{6.17}$$

where a, b and φ are the solutions of (6.13) and u_1 is given by (6.15).

6.4 Discussion

For certain special cases, a simple analytic method has been developed to obtain the time response of a third order nonlinear differential equation with small nonlinearities. The method is not always independent of the numerical techniques. Yet in these cases the method essentially replaces the task of solving the nonlinear equations (6.1) by much simpler task of solving the truncated equations (6.4), in particular equations (6.13). Often one is not interested in only the oscillating processes itself, i.e., finding the x in terms of t, but mainly in the behavior of the amplitudes a, b and the phase φ, which as t increases characterize the oscillating processes. This gives another merit to dealing with equations (6.13) instead of equation (6.1). It is important to note the following interpretation of equations (6.13) expressing the rate of change of the amplitudes a, b and the phase φ as sum of the four terms. For $\varepsilon = 0$, the unperturbed case, only first terms are presented and (6.13) reduces to three linear equations of a, b and φ. The second, third and fourth terms with the factor ε perturb this simple situation and show that the linear parts of (6.13) change slowly in time. Hence in order to have a complete portrait of the oscillating processes it is sufficient to find numerically only few points. On the contrary, a direct attempt to solve numerically equation (6.1) leads to dealing with a harmonic term in the solution (6.3), namely, $b\cos\varphi$. This requires the numerical calculation of a great number of points, and also is not practical.

Chapter 7

Third Order Overdamped Nonlinear Systems

7.1 Introduction

Though the KBM method was originally developed to obtain oscillatory-type solutions of weakly nonlinear differential equations; nowadays it is also useful in different types of non-oscillatory systems. Murty and Deekshatulu [40], Murty, Deekshatulu and Krishna [39], Murty [41], Bojadziev and Edwards [14], Sattar [51,52], Shamsul and Sattar [54,55,56,58] have found asymptotic solutions for various overdamped and critically damped nonlinear systems based on the work of KBM.

In this chapter, we obtain a very simple overdamped solution of a third order nonlinear differential equation followed by Murty and Deekshatulu [40]. The solutions obtained for different initial conditions are also in good agreement with those obtained by the numerical method and sometimes give more accurate results than the unified KBM method [55].

7.2 The Method

Let us consider the third order nonlinear differential equation

$$\dddot{x} + k_1 \ddot{x} + k_2 \dot{x} + k_3 x = \varepsilon f(\ddot{x}, \dot{x}, x), \tag{7.1}$$

with the generating solution

$$x = a_0 e^{-\lambda_1 t} + b_0 e^{-\lambda_2 t} + c_0 e^{-\lambda_3 t}, \tag{7.2}$$

where $-\lambda_1, -\lambda_2, -\lambda_3$ are three characteristic roots of (7.1) when $\varepsilon = 0$, and a_0, b_0 and c_0 are arbitrary constants.

Now we seek a solution of the differential equation (7.1) in the form of an asymptotic expansion

$$x = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + ce^{-\lambda_3 t} + \varepsilon u_1(a,b,c,t) + \varepsilon^2 \cdots, \tag{7.3}$$

where a, b and c satisfy the differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,b,c,t) + \varepsilon^2 \cdots, \\ \dot{b} &= \varepsilon B_1(a,b,c,t) + \varepsilon^2 \cdots, \\ \dot{c} &= \varepsilon C_1(a,b,c,t) + \varepsilon^2 \cdots. \end{aligned} \tag{7.4}$$

Differentiating (7.3) three times with respect to t, using relations (7.4), substituting (7.3) and the derivatives $\dot{x}$, $\ddot{x}$, $\dddot{x}$ in the original equation (7.1), and equating the coefficients of ε, we obtain

$$\begin{aligned} & e^{-\lambda_1 t}\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3\right)A_1 + e^{-\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_2 + \lambda_3\right)B_1 \\ & + e^{-\lambda_3 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_3\right)C_1 \\ & + \left(\frac{\partial}{\partial t} + \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_2\right)\left(\frac{\partial}{\partial t} + \lambda_3\right)u_1 = f^{(0)}(a,b,c,t), \end{aligned} \tag{7.5}$$

where $f^{(0)} = f(x_0, \dot{x}_0, \ddot{x}_0)$ and $x_0 = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + ce^{-\lambda_3 t}$.

Let the function $f^{(0)}$ be expanded in a Taylor series

$$\begin{aligned} f^{(0)} &= g_0 + g_1^{(1)}(b,c)ae^{-\lambda_1 t} + g_1^{(2)}(a,c)be^{-\lambda_2 t} + g_1^{(3)}(a,b)ce^{-\lambda_3 t} \\ &\quad + g_1^{(2)}(b,c)a^2 e^{-2\lambda_1 t} + \cdots, \end{aligned} \tag{7.6}$$

To solve the equation (7.5) for u_1, A_1, B_1 and C_1, it is assumed, according to Murty and Deekshatulu [40], that the function u_1 does not contain the *fundamental terms*, since these are already included in the first three terms of the solution (7.3). Substituting (7.6) in (7.5) and equating the coefficients of $e^{-\lambda_1 t}$, $e^{-\lambda_2 t}$ and $e^{-\lambda_3 t}$ we obtain

$$\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_3\right)A_1 = g_1^{(1)}a, \tag{7.7}$$

$$\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)\left(\frac{\partial}{\partial t}-\lambda_2+\lambda_3\right)B_1=g_1^{(2)}b, \tag{7.8}$$

$$\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_3\right)C_1=g_1^{(3)}c, \tag{7.9}$$

$$\left(\frac{\partial}{\partial t}+\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_3\right)u_1=g_2^{(1)}(b,c)a^2e^{-2\lambda_1 t}+\cdots. \tag{7.10}$$

The particular solutions of (7.7)-(7.9) give the three unknown functions A_1, B_1 and C_1. It is obvious that the change of the variables a, b and c are small. When $g_1^{(1)}$, $g_1^{(2)}$ and $g_1^{(3)}$ are given, we may easily solve the equations (5.7)-(5.9) by assuming that a, b and c are constants. Substituting the values of A_1, B_1 and C_1 into (7.4) and then solving them, we obtain the first approximate solution of the nonlinear differential equation. The procedure can be applied to higher orders in the same way.

7.2.1 Determination of the first order correction term u_1

The particular solution of (7.10) gives the first order correction term u_1. When the nonlinear function f of the equation (7.1) is given, $g_2^{(1)},\cdots$ are specified. Then substituting the values of $g_2^{(1)},\cdots$ in (7.10), we may solve it by assuming again that a, b and c are constants. Thus the correction term u_1 is found and we obtain the first improved solution of the equation (7.1).

7.3 Example

Now consider $f=x^3$. So,

$$\begin{aligned}f^{(0)}&=ae^{-\lambda_1 t}\left(3b^2e^{-2\lambda_2 t}+2bce^{-(\lambda_2+\lambda_3)t}+3c^2e^{-2\lambda_3 t}\right)\\&+be^{-\lambda_2 t}\left(3a^2e^{-2\lambda_1 t}+2ace^{-(\lambda_1+\lambda_3)t}+3c^2e^{-2\lambda_3 t}\right)+ce^{-\lambda_3 t}\left(3a^2e^{-2\lambda_1 t}+2abe^{-(\lambda_1+\lambda_2)t}+3b^2e^{-2\lambda_2 t}\right)\\&+a^3e^{-3\lambda_1 t}+b^3e^{-3\lambda_2 t}+c^3e^{-3\lambda_3 t}.\end{aligned}$$

Therefore the non zero coefficients of $g_i^{(j)}$, $i = 1, 2 \cdots, j = 1, 2, 3$ are

$$g_1^{(1)} = 3b^2 e^{-2\lambda_2 t} + 2bce^{-(\lambda_2+\lambda_3)t} + 3c^2 e^{-2\lambda_3 t},$$

$$g_1^{(2)} = 3a^2 e^{-2\lambda_1 t} + 2ace^{-(\lambda_1+\lambda_3)t} + 3c^2 e^{-2\lambda_3 t}, \tag{7.11}$$

$$g_1^{(3)} = 3a^2 e^{-2\lambda_1 t} + 2abe^{-(\lambda_1+\lambda_2)t} + 3b^2 e^{-2\lambda_2 t}, \quad g_3^{(1)} = g_3^{(2)} = g_3^{(3)} = 1.$$

Substituting the values of $g_1^{(1)}$, $g_1^{(2)}$ and $g_1^{(3)}$ from (7.11) into the equations (7.7)-(7.9) and solving them we obtain

$$\begin{aligned} A_1 &= a\left(l_1 b^2 e^{-2\lambda_2 t} + l_2 bce^{-(\lambda_2+\lambda_3)t} + l_3 c^2 e^{-2\lambda_3 t}\right), \\ B_1 &= b\left(m_1 a^2 e^{-2\lambda_1 t} + m_2 ace^{-(\lambda_1+\lambda_3)t} + m_3 c^2 e^{-2\lambda_3 t}\right), \\ C_1 &= c\left(n_1 a^2 e^{-2\lambda_1 t} + n_2 abe^{-(\lambda_1+\lambda_2)t} + l_3 b^2 e^{-2\lambda_2 t}\right), \end{aligned} \tag{7.12}$$

where

$$\begin{aligned} & l_1 = 3((\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3))^{-1}, \quad l_2 = 2((\lambda_1+\lambda_2)(\lambda_1+\lambda_3))^{-1}, \\ & l_3 = 3((\lambda_1+\lambda_3)(\lambda_1-\lambda_2+2\lambda_3))^{-1}, \quad m_1 = 3((\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3))^{-1}, \\ & m_2 = 2((\lambda_1+\lambda_2)(\lambda_2+\lambda_3))^{-1}, \quad m_3 = 3((\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+2\lambda_3))^{-1}, \\ & n_1 = 3((\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3))^{-1}, \quad n_2 = 2((\lambda_1+\lambda_3)(\lambda_2+\lambda_3))^{-1}, \\ & n_3 = 3((\lambda_2+\lambda_3)(-\lambda_1+2\lambda_2+\lambda_3))^{-1}, \end{aligned} \tag{7.13}$$

provided that $\lambda_i + 2\lambda_j \neq \lambda_k$, $i, j, k = 1, 2, 3$.

Substituting the values of A_1, B_1 and C_1 from (7.12) into the equation (7.4) we obtain

$$\begin{aligned} \dot{a} &= a\left(l_1 b^2 e^{-2\lambda_2 t} + l_2 bce^{-(\lambda_2+\lambda_3)t} + l_3 c^2 e^{-2\lambda_3 t}\right), \\ \dot{b} &= b\left(m_1 a^2 e^{-2\lambda_1 t} + m_2 ace^{-(\lambda_1+\lambda_3)t} + m_3 c^2 e^{-2\lambda_3 t}\right), \\ \dot{c} &= c\left(n_1 a^2 e^{-2\lambda_1 t} + n_2 abe^{-(\lambda_1+\lambda_2)t} + n_3 b^2 e^{-2\lambda_2 t}\right). \end{aligned} \tag{7.14}$$

Replacing a, b and c by their respective values obtained in the linear case, and then integrating with respect to t, we obtain

$$\begin{aligned} a &= a_0 + \varepsilon a_0\Big(l_1 b_0^2\left(1-e^{-2\lambda_2 t}\right)/(2\lambda_2) + l_2 b_0 c_0\left(1-e^{-(\lambda_2+\lambda_3)t}\right)/(\lambda_2+\lambda_3) \\ &\qquad + l_3 c_0^2\left(1-e^{-2\lambda_3 t}\right)/(2\lambda_3)\Big) \\ b &= b_0 + \varepsilon b_0\Big(m_1 a_0^2\left(1-e^{-2\lambda_1 t}\right)/(2\lambda_1) + m_2 a_0 c_0\left(1-e^{-(\lambda_1+\lambda_3)t}\right)/(\lambda_1+\lambda_3) \\ &\qquad + m_3 c_0^2\left(1-e^{-2\lambda_3 t}\right)/(2\lambda_3)\Big) \\ c &= c_0 + \varepsilon c_0\Big(n_1 a_0^2\left(1-e^{-2\lambda_1 t}\right)/(2\lambda_1) + n_2 a_0 c_0\left(1-e^{-(\lambda_1+\lambda_3)t}\right)/(\lambda_1+\lambda_3) \\ &\qquad + n_3 b_0^2\left(1-e^{-2\lambda_2 t}\right)/(2\lambda_2)\Big) \end{aligned} \tag{7.15}$$

Hence the first order solution of (7.1) is

$$x = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + ce^{-\lambda_3 t}, \tag{7.16}$$

where a, b and c are given by (7.15).

Now substituting the values of $g_3^{(1)}$, $g_3^{(2)}$ and $g_3^{(3)}$ from (7.11) into the equation (7.10) and then solving it, we obtain

$$u_1 = c_1 e^{-3\lambda_1 t} + c_2 e^{-3\lambda_2 t} + c_3 e^{-3\lambda_3 t}, \tag{7.17}$$

where

$$\begin{aligned} c_1 &= -\left(2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)\right)^{-1}, \\ c_2 &= -\left(2\lambda_2(-\lambda_1+3\lambda_2)(3\lambda_2-\lambda_3)\right)^{-1}, \\ c_3 &= -\left(2\lambda_3(-\lambda_1+3\lambda_3)(-\lambda_2+3\lambda_3)\right)^{-1}, \end{aligned} \tag{7.18}$$

provided that $\lambda_i \neq 3\lambda_j$, $i, j = 1, 2, 3$.

Therefore, the first improved solution of (7.1) is

$$x = ae^{-\lambda_1 t} + be^{-\lambda_2 t} + ce^{-\lambda_3 t} + \varepsilon u_1, \tag{7.19}$$

where a, b and c are given by (7.15) and u_1 is given by (7.17).

7.4 Discussion

A simple analytical method has been developed to obtain the time response of a third order overdamped weakly nonlinear system. In **Chapter 3**, we have already discussed that the solution of an under-damped system may be used as an overdamped solution replacing harmonic functions by their respective hyperbolic functions [55]. But the procedure to obtain the present overdamped solution is simpler than the method developed in **Chapter 3**.

As a check on the solution (7.19) obtained in **section 7.3**, a second solution and a third solution were obtained by numerical integration using a fourth-order Runge-Kutta formula and by the unified KBM method respectively. The results are given in **Table 7.1** and **Table 7.2** for two sets of the initial conditions when $\lambda_1 = 2, \lambda_2 = 1.2, \lambda_3 = 0.8$. In both the cases the present method gives more accurate results than the unified KBM method.

To obtain the corresponding numerical solution, the initial conditions $[x(0), \dot{x}(0), \ddot{x}(0)]$ are computed from

$$
\begin{aligned}
x(0) &= a_0 + b_0 + c_0 + \varepsilon\left(c_1 a_0^3 + c_2 b_0^3 + c_3 c_0^3\right), \\
\dot{x}(0) &= -\lambda_1 a_0 - \lambda_2 b_0 - \lambda_3 c_0 + \varepsilon a_0\left(l_1 b_0^2 + l_2 b_0 c_0 + l_3 c_0^2\right) \\
&\quad + \varepsilon b_0\left(m_1 a_0^2 + m_2 a_0 c_0 + m_3 c_0^2\right) + \varepsilon c_0\left(n_1 a_0^2 + n_2 a_0 a_0 + n_3 b_0^2\right), \\
&\quad - 3\varepsilon\left(\lambda_1 c_1 a_0^3 + \lambda_2 c_2 b_0^3 + \lambda_3 c_3 c_0^3\right), \\
\ddot{x}(0) &= -\lambda_1^2 a_0 - \lambda_1^2 b_0 - \lambda_1^2 c_0 \\
&\quad - \varepsilon a_0\left(2(\lambda_1 + \lambda_2) l_1 b_0^2 + (2\lambda_1 + \lambda_2 + \lambda_3) l_2 b_0 c_0 + 2(\lambda_1 + \lambda_3) l_3 c_0^2\right) \\
&\quad - \varepsilon b_0\left(2(\lambda_1 + \lambda_2) m_1 a_0^2 + (\lambda_1 + 2\lambda_2 + \lambda_3) m_2 a_0 c_0 + 2(\lambda_2 + \lambda_3) m_3 c_0^2\right) \\
&\quad - \varepsilon c_0\left(2(\lambda_1 + \lambda_3) n_1 a_0^2 + (\lambda_1 + \lambda_2 + 2\lambda_3) n_2 a_0 a_0 + 2(\lambda_2 + \lambda_2) n_3 b_0^2\right) \\
&\quad + 9\varepsilon\left(\lambda_1^2 c_1 a_0^3 + \lambda_1^2 c_2 b_0^3 + \lambda_1^2 c_3 c_0^3\right).
\end{aligned}
\qquad (7.20)
$$

t	x	x_{nu}	x_{unf}
0.0	1.395643	1.395643	1.395643
0.5	0.707096	0.707082	0.707320
1.0	0.372571	0.372558	0.372633
1.5	0.203633	0.203655	0.203629
2.0	0.114817	0.114864	0.114807
2.5	0.066416	0.066471	0.066416
3.0	0.039231	0.039284	0.039239
3.5	0.023580	0.023624	0.023591
4.0	0.014383	0.014418	0.014394

Table 7.1 Solution (7.19) for $\lambda_1 = 2, \lambda_2 = 1.2, \lambda_3 = 0.8$ and $\varepsilon = 0.1$ with initial conditions $a_0 = 0.5, b_0 = 0.7, c_0 = 0.2$ or $[\,x(0) = 1.395643,\ \dot{x}(0) = -1.949025,\ \ddot{x}(0) = 2.903369\,]$ is given in the second column. In the third and fourth columns corresponding numerical solution and the perturbation solution obtained by the unified KBM method [55] are given.

t	x	x_{nu}	x_{unf}
0.0	0.966930	0.966930	0.966930
0.5	0.572551	0.572569	0.572306
1.0	0.346370	0.346010	0.346470
1.5	0.213924	0.213446	0.214130
2.0	0.134408	0.133958	0.134605
2.5	0.085588	0.085216	0.085745
3.0	0.055065	0.054778	0.055182
3.5	0.035711	0.035498	0.035794
4.0	0.023304	0.023149	0.023362

Table 7.2 Solution (7.19) for $\lambda_1 = 2, \lambda_2 = 1.2, \lambda_3 = 0.8$ and $\varepsilon = 0.2$ with initial conditions $a_0 = 0.2, b_0 = 0.3, c_0 = 0.5$ or $[\,x(0) = 0.9966930,\ \dot{x}(0) = -1.028135,\ \ddot{x}(0) = 1.130903\,]$ is given in the second column. In the third and fourth columns corresponding numerical solution and the perturbation solution obtained by the unified KBM method are given.

Remark : The solution (7.19) is closer (on the average) to the numerical solution.

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