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# Eulerian Method for Third Order Linear Systems

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# EULERIAN METHOD FOR THIRD ORDER LINEAR SYSTEMS

THESIS SUBMITTED FOR THE DEGREE OF MASTER OF PHILOSOPHY

IN

**MATHEMATICS**

BY

**MD. ABDUL HAQUE**

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF RAJSHAHI

RAJSHAHI-6205, BANGLADESH

2003

# CERTIFICATE

It is certified that the thesis entitled "Eulerian Method for Third Order Linear Systems" submitted by Md. Abdul Haque to the Department of Mathematics, University of Rajshahi, Rajshahi, Bangladesh for the degree of Master of Philosophy in Mathematics has been completed under my supervision. I believe that this research work is original and it has not been submitted elsewhere for any degree.

Supervisor

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# STATEMENT OF ORIGINALITY

I declare that the contents in my M. Phil. thesis entitled "Eulerian Method for Third Order Linear Systems" is original and accurate to the best of my knowledge. I also certify that the materials contained in my research work have not been previously published or written by any person for a degree or diploma.

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# ABSTRACT

In this thesis, we have studied the solutions of the various types of third order linear systems of ordinary differential equations. In chapter 1, we have considered the third order linear homogeneous system with constant coefficients and developed Eulerian methods for all possible cases of the characteristic roots of the variational matrix. In chapter 2, we have considered the third order linear nonhomogeneous system with constant coefficients. The method covers all the cases when the roots of the characteristic equation of the corresponding homogeneous linear system are real and distinct, real and equal and complex. In finding particular solutions for the nonhomogeneous system of equations we have used the method of variation of parameters. Finally, we have obtained solutions of this system with the help of Crammer's rule. In chapter 3, we have considered the generalized form of third order linear nonhomogeneous system with constant coefficients. We have extended the Eulerian method and developed new techniques for obtaining solutions of this system. In chapter 4, we have discussed the third order linear nonhomogeneous system with variable coefficients. This problem is very difficult to solve, so we have examined a special case of this problem. By using a suitable

transformation, we have reduced it to a third order linear nonhomogeneous system with constant coefficients and have found solutions by the method of Chapter 2. We have illustrated all the methods by several suitable examples.

# INTRODUCTION

In natural phenomena, many mathematical relations are described by differential equations. The differential equation may be ordinary or partial, linear or nonlinear, homogeneous or nonhomogeneous and with constant coefficients or with variable coefficients. It is noted that the system of linear ordinary differential equations with constant coefficients may be solved by various methods including the method of elimination, Eulerian method and matrix method. But most of the nonlinear differential equations can not be solved. Even the linear differential equations with variable coefficients, in the general case, remain unsolved. Euler developed a method to solve the second order system of linear differential equations with constant coefficients. To the best of our knowledge we have not seen any paper or book providing theorems and solutions of the third order linear homogeneous systems with constant coefficients by the Eulerian method. In this thesis we have extended the Eulerian method to solve the third order system of linear differential equations with constant coefficients and a special case of the third order system of linear differential equations with variable coefficients.

In solving the third order linear homogeneous system with constant coefficients, we have discussed four types of solutions relating to four types of characteristic roots. In solving the third order linear nonhomogeneous system with constant coefficients, the process of finding the general solution of homogeneous part is the same as the general solution of the third order liner homogeneous equation with constant coefficients as in Chapter 1. For this we have discussed only the process for finding the solution of nonhomogeneous part. We have found the particular integral of the nonhomogeneous part by the help of the method of variation of parameters. Chapter 3 contains the generalized third order linear nonhomogeneous system with constant coefficients, where the dependent variables x, y, z and their derivatives $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ appear in all the equations of the system. This system is an extension of the system given in Chapter 2. Chapter 4 deals with the third order linear nonhomogeneous system with variable coefficients. This is a very difficult problem and can not be solved easily. However, a special case of such a second order system have been solved by Cauchy and Euler separately. They have used a suitable transformation that reduces the system with variable coefficients to a system with constant coefficients. We have extended their method and solved a special case of the third order linear nonhomogeneous system with variable coefficients.

Although most of the differential equations involving physical problems are nonlinear, we can impose some restrictions to linearize the system and then investigate the existence, uniqueness, stability and oscillatory nature of the solutions [1, 2, 5, 6, 8, 11]. The Purpose of this thesis is to develop Eulerian method for the solution of third order linear differential systems. This result may be used in Mathematical Physics, Population Dynamics, Fluid Mechanics and various branches of Engineering.

# CHAPTER 1

# Eulerian Method for Third Order Linear Homogeneous Systems with Constant Coefficients

## 1.1 Introduction

System of linear homogeneous differential equations with constant coefficients can be solved by the method of elimination, Eulerian method and the matrix method. Second order linear homogeneous systems with constant coefficients have been solved by the method of elimination in Braun [3] and by the Eulerian method and the matrix method in Ross [9]. Third order linear homogeneous systems with constant coefficients have been solved by the method of elimination in Spiegel [12] and by the matrix method in Ross [9]. In this chapter we have considered a third order linear homogeneous system with constant coefficients and provided theorems for distinct, complex and repeated eigen values of the variational matrix. The method is supported by the solution of several examples.

## 1.2 The Method

We consider

$$\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y + c_1z\\ \frac{dy}{dt} &= a_2x + b_2y + c_2z \qquad (1.1)\\ \frac{dz}{dt} &= a_3x + b_3y + c_3z\end{aligned}$$

where the coefficients $a_1$, $b_1$, $c_1$; $a_2$ ,$b_2$, $c_2$ and $a_3$, $b_3$, $c_3$ are real constants.

According to the Eulerian method, we assume a solution of the system (1.1) of the form

$$
\begin{aligned}
x &= \alpha\, e^{\lambda t} \\
y &= \beta\, e^{\lambda t} \\
z &= \gamma\, e^{\lambda t}
\end{aligned}
\tag{1.2}
$$

where $\alpha$, $\beta$, $\gamma$ and $\lambda$ are unknown constants.

Substituting (1.2) into (1.1), we obtain the algebraic system

$$
\begin{aligned}
(a_1 - \lambda)\, \alpha + b_1\, \beta + c_1\, \gamma &= 0 \\
a_2\, \alpha + ( b_2 - \lambda ) + c_2\, \gamma &= 0 \\
a_3\, \alpha + b_3\, \beta + (c_3 - \lambda)\, \gamma &= 0
\end{aligned}
\tag{1.3}
$$

We seek a nontrivial solution of the system (1.3). A necessary and sufficient condition that system (1.3) have a nontrivial solution is that the determinant

$$
\begin{vmatrix}
a_1 - \lambda & b_1 & c_1 \\
a_2 & b_2 - \lambda & c_2 \\
a_3 & b_3 & c_3 - \lambda
\end{vmatrix} = 0
$$

or, $$\lambda^3 + 3\lambda^2 k_1 + 3\lambda k_2 + k_3 = 0 \tag{1.4}$$

where

$$k_1 = \frac{-(a_1 + b_2 + c_3)}{3}$$

$$k_2 = \frac{a_1 b_2 + b_2 c_3 + c_3 a_1 - a_2 b_1 - b_3 c_2 - c_1 a_3}{3}$$

$$k_3 = a_1 b_3 c_2 - a_1 b_2 c_3 - a_3 b_1 c_2 + a_2 b_1 c_3 - a_2 b_3 c_1 + a_3 b_2 c_1$$

Equation (1.4) is called the characteristic equation associated with the system (1.1) and its roots, say $\mu, \nu$ and $\eta$ are called the characteristic roots. If (1.2) is to be a solution of the system (1.1), then $\lambda$ in (1.2) must be one of these roots. Suppose $\lambda = \mu$. Then substituting $\lambda = \mu$ into the algebraic equation (1.3), we may obtain a nontrivial solution $\alpha_1$, $\beta_1$ and $\gamma_1$ of this algebraic system. With these values $\alpha_1$, $\beta_1$ and $\gamma_1$, we obtain the nontrivial solution

$$x = \alpha_1 e^{\mu t}$$

$$y = \beta_1 e^{\mu t}$$

$$z = \gamma_1 e^{\mu t}.$$

Substituting $\lambda = y + h$ in (1.4), we obtain

$$y^3 + 3(h+k_1)y^2 + 3(h^2+2k_1h+k_2)y + (h^3+3k_1h^2+3k_2h+k_3) = 0. \quad (1.5)$$

Choosing $h+k_1=0$, the transformed equation is

$$y^3 + 3Hy + G = 0 \quad (1.6)$$

where $G = 2k_1^3 - 3k_1k_2 + k_3$

$$H = k_2 - k_1^2.$$

The substitution $y=u+v$ reduces to

$$y^3 - 3uvy - (u^3+v^3) = 0. \quad (1.7)$$

Comparing (1.6) and (1.7), we get

$$uv = -H, \text{ or } u^3v^3 = -H^3$$

and $u^3+v^3 = -G$.

Hence $u^3$ and $v^3$ are the roots of the quadratic equation

$$t^2+Gt-H^3=0. \tag{1.8}$$

Suppose

$$u^3=\frac{-G+\sqrt{G^2+4H^3}}{2} \tag{1.9}$$

$$v^3=\frac{-G-\sqrt{G^2+4H^3}}{2}. \tag{1.10}$$

From the equation (1.9) and (1.10) we see that there are three values of u and three values of v. Since we have considered $y = u+v$, the possible values of y will be $u_1+v_1$, $u_1+v_2$, $u_1+v_3$; $u_2+v_1$, $u_2+v_2$, $u_2+v_3$ and $u_3+v_1$, $u_3+v_2$, $u_3+v_3$. But $uv = -H$, so we are to take those values of u and v whose product is $-H$.

If the roots of u and v be u, uw, $uw^2$ and v, vw, $vw^2$; then the roots of equation (1.6) will be $u+v$, $uw+vw^2$ and $uw^2+vw$

where

$$w=\frac{1}{2}\left\{-1+\sqrt{-3}\right\}$$

$$w^2=\frac{1}{2}\left\{-1-\sqrt{-3}\right\}.$$

The expression $G^2+4H^3$ is called the discriminant of the cubic equation (1.6). The nature of the roots of (1.6) is dependent on the sign of the expression $G^2+4H^3$.

**Nature of the roots of the cubic equation**

**Case I**

If $G^2+4H^3< 0$, then $u^3$ and $v^3$ are complex conjugates and so u and v are complex conjugates. Thus the roots of the cubic equation (1.6) are real and distinct.

**Case II**

If $G^2+4H^3> 0$, then u and v are real. So, one root of the cubic equation is real and the other two roots are complex conjugate to each other.

**Case III**

If $G^2+4H^3= 0$, then u=v. Therefore the three roots of the cubic equation are real of which two are equal and the other is distinct.

**Case IV**

If G=0, H=0; then the three roots of the cubic equation are real and equal.

**Four cases must now be considered**

**Case I**

The roots of the characteristic equation (1.4) are real and distinct when

$$G^2 + 4H^3< 0.$$

If the roots $\mu$, $\nu$ and $\eta$ of the characteristic equation (1.4) are real and distinct, then we should expect three distinct linearly independent solutions of the form (1.2), one corresponding to each of the three distinct roots. We summarize this case in the following theorem:

### 1.3 The Theorem 1

If the roots μ, ν and η of the characteristic equation (1.4) associated with the system (1.1) are real and distinct, then the system (1.1) has three nontrivial linearly independent solutions of the form

$$x = \alpha_1 e^{\mu t}$$
$$y = \beta_1 e^{\mu}$$
$$z = \gamma_1 e^{\mu t},$$

$$x = \alpha_2 e^{\nu t}$$
$$y = \beta_2 e^{\nu t}$$
$$z = \gamma_2 e^{\nu t}$$

and

$$x = \alpha_3 e^{\eta t}$$
$$y = \beta_3 e^{\eta t}$$
$$z = \gamma_3 e^{\eta t}$$

where $\alpha_1$, $\beta_1$, $\gamma_1$; $\alpha_2$, $\beta_2$, $\gamma_2$ and $\alpha_3$, $\beta_3$. $\gamma_3$ are definite constants. The general solution of the system (1.1) may thus be written as

$$x = c_1 \alpha_1 e^{\mu t} + c_2 \alpha_2 e^{\nu t} + c_3 \alpha_3 e^{\eta t}$$

$$y = c_1 \beta_1 e^{\mu t} + c_2 \beta_2 e^{\nu t} + c_3 \beta_3 e^{\eta t}$$

$$z = c_1 \gamma_1 e^{\mu t} + c_2 \gamma_2 e^{\nu t} + c_3 \gamma_3 e^{\eta t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

**The Example**

$$\frac{dx}{dt} = 7x - y + 6z$$

$$\frac{dy}{dt} = -10x + 4y - 12z \qquad (1.11)$$

$$\frac{dz}{dt} = -2x + y - z.$$

According to the Eulerian method, we assume a solution of the system (1.11) of the form (1.2); i. e.,

$$x = \alpha\, e^{\lambda t}$$

$$y = \beta\, e^{\lambda t} \qquad (1.12)$$

$$z = \gamma\, e^{\lambda t}.$$

Substituting (1.12) into (1.11), we obtain the algebraic system

$$(7-\lambda)\ \alpha - \beta + 6\ \gamma = 0$$

$$-10\ \alpha + (4-\lambda)\ \beta - 12\ \gamma = 0 \qquad (1.13)$$

$$-2\alpha + \beta - (1 + \lambda)\ \gamma = 0$$

in the unknown $\lambda$ . For nontrivial solutions of the system (1.13), we must have

$$\begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0.$$

or $\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0.$ (1.14)

Comparing (1.14) with (1.4) we observe that

$$k_1 = \frac{-10}{3},\ k_2 = \frac{31}{3},\ k_3 = -30.$$

So, $G = \frac{-20}{27},\ H = \frac{-7}{9}.$

Thus, $G^2 + 4H^3 = \frac{-4}{3} < 0.$

Hence the characteristic equation (1.14) has three distinct real roots.

Solving (1.14), we find that the roots of this equation are $\lambda = 2, 3, 5$.

Setting $\lambda = 2, 3, 5$ into (1.13), we obtain simple nontrivial solutions

$$\alpha = 1,\ \beta = -1,\ \gamma = -1;$$

$$\alpha = 1,\ \beta = -2,\ \gamma = -1$$

and $\alpha = 3,\ \beta = -6,\ \gamma = -2$

respectively.

With these values of $\alpha$, $\beta$, $\gamma$ and $\lambda$, we find the following three sets of solutions :

$$\begin{aligned} x &= e^{2t} \\ y &= -e^{2t} \\ z &= -e^{2t}, \end{aligned} \qquad (1.15)$$

$$\begin{aligned} x &= e^{3t} \\ y &= -2e^{3t} \\ z &= -e^{3t} \end{aligned} \qquad (1.16)$$



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and

$$\begin{aligned} x &= 3e^{5t} \\ y &= -6e^{5t} \\ z &= -2e^{5t} \end{aligned} \qquad (1.17)$$

respectively.

Further, the Wronskian W(t) of the functions (1.15), (1.16) and (1.17) is

$$W(t) = e^{10t} \begin{vmatrix} 1 & 1 & 3 \\ -1 & -2 & -6 \\ -1 & -1 & -2 \end{vmatrix}$$

$$= -e^{10t}$$

$$\neq 0.$$

Thus the solutions are linearly independent.

Hence the general solution of the system (1.11) is

$$x = c_1e^{2t} + c_2e^{3t} + 3\, c_3e^{5t}$$

$$y = -c_1e^{2t} - 2c_2e^{3t} - 6c_3e^{5t}$$

$$z = -c_1e^{2t} - c_2e^{3t} - 2c_3e^{5t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

**Case II**

Two roots of the characteristic equation (1.4) are complex and the other root is real when

$$G^2 + 4H^3 > 0.$$

If two roots $\mu$ and $\nu$ of the characteristic equation (1.4) are the complex numbers $a + ib$ and $a - ib$, and the other root be $\lambda$, then we obtain three distinct solutions.

$$\begin{aligned} x &= \alpha_1^* e^{(a+ib)t} \\ y &= \beta_1^* e^{(a+ib)t} \\ z &= \gamma_1^* e^{(a+ib)t}, \end{aligned}$$

$$\begin{aligned} x &= \alpha_2^* e^{(a-ib)t} \\ y &= \beta_2^* e^{(a-ib)t} \\ z &= \gamma_2^* e^{(a-ib)t} \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} x &= \alpha e^{\lambda t} \\ y &= \beta e^{\lambda t} \\ z &= \gamma e^{\lambda t} \end{aligned}$$

where $\alpha_1^*$ $\beta_1^*$ and $\gamma_1^*$ are complex constants. The solutions (1.18) are complex solutions. In order to obtain real solutions in this case we consider the first of the two solutions (1.18) and proceed as follows:

We first express the complex constants $\alpha_1^*$, $\beta_1^*$ and $\gamma_1^*$in this solution in the forms $\alpha_1^* = \alpha_1 + i\alpha_2$, $\beta_1^* = \beta_1 + i\beta_2$ and $\gamma_1^* = \gamma_1 + i\gamma_2$, where $\alpha_1$, $\beta_1$, $\gamma_1$; $\alpha_2$, $\beta_2$, $\gamma_2$ are real. We then apply Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ and express the first solution (1.18) in the form

$$x = (\alpha_1 + i\alpha_2)\, e^{at} (\cos bt + i \sin bt)$$

$$y = (\beta_1 + i\beta_2)\, e^{at} (\cos bt + i \sin bt)$$

$$z = (\gamma_1 + i\gamma_2)\, e^{at} (\cos bt + i \sin bt) .$$

Rewriting this we obtain

$$x = e^{at} [(\alpha_1 \cos bt - \alpha_2 \sin bt) + i (\alpha_2 \cos bt + \alpha_1 \sin bt)]$$

$$y = e^{at} [(\beta_1 \cos bt - \beta_2 \sin bt) + i (\beta_2 \cos bt + \beta_1 \sin bt)] \qquad (1.19)$$

$$z = e^{at} [(\gamma_1 \cos bt - \gamma_2 \sin bt) + i (\gamma_2 \cos bt + \gamma_1 \sin bt)] .$$

It can be shown that a pair $[f_1(t) + if_2(t),\ g_1(t) + ig_2(t),\ h_1(t) + ih_2(t)]$ of complex functions is a solution of the system (1.1 ) if and only if both the pair $[f_1(t), g_1(t), h_1(t)]$ consisting of their real parts and the pair $[f_2(t), g_2(t), h_2(t)]$ consisting of their imaginary parts are solutions of (1.1 ) separately .

Thus both the real part

$$x = e^{at} (\alpha_1 \cos bt - \alpha_2 \sin bt)$$

$$y = e^{at} (\beta_1 \cos bt - \beta_2 \sin bt)) \qquad (1.20)$$

$$z = e^{at} (\gamma_1 \cos bt - \gamma_2 \sin bt)$$

and the imaginary part

$$x = e^{at} (\alpha_2 \cos bt + \alpha_1 \sin bt)$$

$$y = e^{at} (\beta_2 \cos bt + \beta_1 \sin bt) \qquad (1.21)$$

$$z = e^{at} (\gamma_2 \cos bt + \gamma_1 \sin bt)$$

of the solution (1.19) are also solutions of (1.1). The Wronskian for these solutions is

$$W(t) = \begin{vmatrix} e^{at}(\alpha_1 \cos bt - \alpha_2 \sin bt) & e^{at}(\alpha_2 \cos bt + \alpha_1 \sin bt) & \alpha e^{at} \\ e^{at}(\beta_1 \cos bt - \beta_2 \sin bt) & e^{at}(\beta_2 \cos bt + \beta_1 \sin bt) & \beta e^{at} \\ e^{at}(\gamma_1 \cos bt - \gamma_2 \sin bt) & e^{at}(\gamma_2 \cos bt + \gamma_1 \sin bt) & \gamma e^{at} \end{vmatrix}$$

$$= e^{(2a+\lambda)t} \begin{vmatrix} \alpha_1 \cos bt - \alpha_2 \sin bt & \alpha_2 \cos bt + \alpha_1 \sin bt & \alpha \\ \beta_1 \cos bt - \beta_2 \sin bt & \beta_2 \cos bt + \beta_1 \sin bt & \beta \\ \gamma_1 \cos bt - \gamma_2 \sin bt & \gamma_2 \cos bt + \gamma_1 \sin bt & \gamma \end{vmatrix}$$

$$= e^{(2a+\lambda)t}\,[\alpha\,(\beta_1\gamma_2 - \beta_2\gamma_1) - \beta\,(\alpha_1\gamma_2 - \alpha_2\gamma_1) + \gamma\,(\alpha_1\beta_2 - \alpha_2\beta_1)]\,.$$

Now the constants $\alpha_1^*$, $\beta_1^*$ and $\gamma_1^*$ are assumed to be non real multiples of each other. If we suppose that

$$\alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) = 0,$$

then it follows that $\alpha_1^*$, $\beta_1^*$ and $\gamma_1^*$ are real multiples of each other, which contradicts our assumption.

Thus

$$\alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta\,(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma\,(\alpha_1\beta_2 - \alpha_2\beta_1) \neq 0$$

and so the Wronskian W(t) is unequal to zero. Thus the solutions (1.20) and (1.21) are linearly independent. Hence a linear combination of these three real solutions provides the general solution of the system (1.1). We summarize the above result in the following theorem:

**1.4 The Theorem 2**

If the two roots μ and ν of the characteristic equation (1.4) associated with the system (1.1) are complex numbers $a \pm ib$ and the other root be λ, then the system (1.1) has three real linearly independent solutions of the forms

$$x = e^{at} (\alpha_1 \cos bt - \alpha_2 \sin bt)$$

$$y = e^{at} (\beta_1 \cos bt - \beta_2 \sin bt)$$

$$z = e^{at} (\gamma_1 \cos bt - \gamma_2 \sin bt),$$

$$x = e^{at} (\alpha_2 \cos bt + \alpha_1 \sin bt)$$

$$y = e^{at} (\beta_2 \cos bt + \beta_1 \sin bt)$$

$$z = e^{at} (\gamma_2 \cos bt + \gamma_1 \sin bt)$$

and

$$x = \alpha e^{\lambda t}$$
$$y = \beta e^{\lambda t}$$
$$z = \gamma e^{\lambda t}$$

where α. β. γ ; $\alpha_1$, $\beta_1$, $\gamma_1$ and $\alpha_2$, $\beta_2$, $\gamma_2$ are definite real constants. Hence the general solution of the system (1.1) may be written as

$$x = e^{at} [c_1 (\alpha_1 \cos bt - \alpha_2 \sin bt) + c_2(\alpha_2 \cos bt + \alpha_1 \sin bt] + c_3 \alpha e^{\lambda t}$$

$$y = e^{at} [c_1 (\beta_1 \cos bt - \beta_2 \sin bt) + c_2(\beta_2 \cos bt + \beta_1 \sin bt] + c_3 \beta e^{\lambda t}$$

$$z = e^{at} [c_1 (\gamma_1 \cos bt - \gamma_2 \sin bt) + c_2(\gamma_2 \cos bt + \gamma_1 \sin bt)] + c_3 \gamma e^{\lambda t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

**The Example**

$$\frac{dx}{dt} = x + y$$
$$\frac{dy}{dt} = x - 2z \qquad (1.22)$$
$$\frac{dz}{dt} = y + z$$

According to the Eulerian method, we assume a solution of the system (1.22) of the form (1.2); i.e.,

$$x = \alpha e^{\lambda t}$$
$$y = \beta e^{\lambda t} \qquad (1.23)$$
$$z = \gamma e^{\lambda t}$$

Substituting (1.23) into (1.22), we obtain the algebaric system

$$(1-\lambda)\,\alpha+\beta = 0$$
$$\alpha - \lambda\beta - 2\gamma = 0 \qquad (1.24)$$
$$\beta + (1-\lambda)\,\gamma = 0$$

in the unknown $\lambda$. For nontrivial solution of the system (1.24) we must have

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & -2 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0.$$

or, $\lambda^3 - 2\lambda^2 + 2\lambda - 1 = 0.$ (1.25)

Comparing (1.25) with (1.4) we observe that

$$k_1 = \frac{-2}{3},\ k_2 = \frac{2}{3},\ k_3 = -1.$$

So, $G = \frac{-7}{27},\ H = \frac{2}{9}.$

Thus, $G^2 + 4H^3 = \frac{1}{9} > 0.$

Hence the characteristic equation (1.25) has a real and two complex conjugates roots.

Solving (1.25) we find that the roots of this equation are $\lambda = 1$, $(1 + i\sqrt{3})/2$, $(1 - i\sqrt{3})/2$.

Setting $\lambda = (1 - i\sqrt{3})/2$ into (1.24), we obtain a simple nontrivial solution

$$\alpha = \gamma = 2,\ \beta = -1 - \sqrt{3}i\ .$$

Using these values in (1.23), we obtain complex solutions. Since both the real and imaginary parts of this solution are themselves solutions of (1.22), we thus obtain the following two real solutions:

$$x = 2e^{t/2} \cos \frac{\sqrt{3}}{2}t$$

$$y = -e^{t/2} (\cos \frac{\sqrt{3}}{2}t + \sqrt{3} \sin \frac{\sqrt{3}}{2}t) \quad (1.26)$$

$$z = 2\, e^{t/2} \cos\frac{\sqrt{3}}{2}t$$

and

$$x = 2e^{t/2} \sin\frac{\sqrt{3}}{2}t$$

$$y = e^{t/2} (\sqrt{3}\cos \frac{\sqrt{3}}{2}t + \sin\frac{\sqrt{3}}{2}t) \qquad (1.27)$$

$$z = 2\, e^{t/2}\sin\frac{\sqrt{3}}{2}t .$$

Setting $\lambda=1$, in (1.24) a simple nontrivial solution of this system is

$$\alpha = 2,\ \beta = 0,\ \gamma=1$$

and putting these in (1.23) a solution of (1.22) is

$$x = 2e^{t}$$

$$y = 0 \qquad (1.28)$$

$$z = e^{t}.$$

Further, the Wronskian W (t) of the functions (1.26), (1.27) and (1.28) is

$$W(t)=e^{2t} \begin{vmatrix} 2\cos\frac{\sqrt{3}}{2}t & 2\sin\frac{\sqrt{3}}{2}t & 2 \\ -(\cos\frac{\sqrt{3}}{2}t + \sqrt{3}\sin\frac{\sqrt{3}}{2}t) & \sqrt{3}\cos\frac{\sqrt{3}}{2}t + \sin\frac{\sqrt{3}}{2}t & 0 \\ 2\cos\frac{\sqrt{3}}{2}t & 2\sin\frac{\sqrt{3}}{2}t & 1 \end{vmatrix}$$

$$= -2\sqrt{3} - 2\sin\sqrt{3}t$$

$$\neq 0.$$

Thus the solutions are linearly independent.

Hence the general solution of the system (1.22) is

$$x = 2c_1 e^{\frac{t}{2}} \left(c_1 \cos \frac{\sqrt{3}}{2}t + c_2 \sin \frac{\sqrt{3}}{2}t\right) + 2c_3 e^{t}$$

$$y = -c_1 e^{t/2} \left(\cos \frac{\sqrt{3}}{2}t + \sqrt{3} \sin \frac{\sqrt{3}}{2}t\right) + c_2 e^{t/2} \left(\sqrt{3} \cos \frac{\sqrt{3}}{2}t + \sin \frac{\sqrt{3}}{2}t\right)$$

$$z = 2c_1 e^{t/2} \cos \frac{\sqrt{3}}{2}t + 2c_2 e^{t/2} \sin \frac{\sqrt{3}}{2}t + c_3 e^{t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

**Case III**

Two of the three real roots of the characteristic equation are equal and the other is distinct when

$$G^2 + 4H^3 = 0.$$

If the two roots of the characteristic equation (1.4) are equal, it would appear that we could find only one solution of the form (1.2) corresponding to the two equal roots. Now if

$$\begin{aligned} x &= \alpha e^{\lambda t} \\ y &= \beta e^{\lambda t} \\ z &= \gamma e^{\lambda t} \end{aligned} \qquad (1.29)$$

be a solution for one of the equal roots, then a second solution for the corresponding equal root has the form

$$x = (\alpha t + \alpha_1)\, e^{\lambda t}$$
$$y = (\beta t + \beta_1)\, e^{\lambda t} \qquad (1.30)$$
$$z = (\gamma t + \gamma_1)\, e^{\lambda t}.$$

where $\alpha_1$, $\beta_1$, $\gamma_1$ are constants and not all zero.

The third solution for the unequal root has the form

$$x = \alpha_2 e^{\nu t}$$
$$y = \beta_2 e^{\nu t} \qquad (1.31)$$
$$z = \gamma_2 e^{\nu t}.$$

We now summarize the result in the following theorem:

## 1.5 The Theorem 3

If the roots $\mu=\nu=\lambda$ of the characteristic equation (1.4) associated with (1.1) are real and equal and $\eta$ is distinct, then the system (1.1) has three linearly independent solutions of the form

$$x = \alpha\, e^{\lambda t}$$
$$y = \beta\, e^{\lambda t}$$
$$z = \gamma\, e^{\lambda t},$$

$$x = (\alpha t + \alpha_1)\, e^{\lambda t}$$
$$y = (\beta t + \beta_1) e^{\lambda t}$$
$$z = (\gamma\, t + \gamma_1)\, e^{\lambda t}$$

and

$$x = \alpha_2 \, e^{\eta t}$$

$$y = \beta_2 \, e^{\eta t}$$

$$z = \gamma_2 \, e^{\eta t}$$

where $\alpha, \beta, \gamma$ ; $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$ and $\alpha_3, \beta_3, \gamma_3$ are definite real constants and $\alpha_1, \beta_1, \gamma_1$ are not all zero.

The general solution may thus be written as

$$x = c_1 \alpha \, e^{\lambda t} + c_2 \, (\alpha \, t + \alpha_1) \, e^{\lambda t} + c_3 \, \alpha_2 \, e^{\mu t}$$

$$y = c_1 \, \beta \, e^{\lambda t} + c_2 (\beta \, t + \beta_1) \, e^{\lambda t} + c_3 \beta_2 \, e^{\mu t}$$

$$z = c_1 \gamma \, e^{\lambda t} + c_2 (\gamma \, t + \gamma_1) \, e^{\lambda t} + c_3 \, \gamma_2 \, e^{\mu t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

**The Example**

$$\frac{dx}{dt} = 7x + 4y + 4z$$
$$\frac{dy}{dt} = -6x - 4y - 7z \qquad (1.32)$$
$$\frac{dz}{dt} = -2x - y + 2z$$

According to the Eulerian method, we assume a solution of the system (1.32) of the form (1.2); i.e.,

$$x = \alpha \, e^{\lambda t}$$
$$y = \beta \, e^{\lambda t} \qquad (1.33)$$
$$z = \gamma \, e^{\lambda t}.$$

Substituting (1.33) into (1.32), we obtain the algebraic system

$$(7-\lambda)\,\alpha + 4\beta + 4\gamma = 0$$
$$-6\alpha + (-4-\lambda)\beta - 7\gamma = 0 \qquad (1.34)$$
$$-2\alpha - \beta + (2-\lambda)\,\gamma = 0$$

in the unknown $\lambda$. For a nontrivial solution of the system (1.34), we must have

$$\begin{vmatrix} 7-\lambda & 4 & 4 \\ -6 & -4-\lambda & -7 \\ -2 & -1 & 2-\lambda \end{vmatrix} = 0.$$

or, $\lambda^3 - 5\lambda^2 + 3\lambda + 9 = 0.$ (1.35)

Comparing (1.35) with (1.4) we observe that

$$k_1 = \frac{-5}{3},\ k_2 = 1,\ k_3 = 9.$$

So, $G = \dfrac{128}{27},\ H = \dfrac{-16}{9}.$

Thus, $G^2 + 4H^3 = 0$.

Hence the characteristic equation (1.35) has two equal and a distinct real roots.

Solving (1.35) we find that the roots of this equation are $\lambda = -1, 3, 3$.

Setting $\lambda = 3$ into (1.34), we obtain a simple nontrivial solution

$$\alpha = 0,\ \beta = 1,\ \gamma = -1.$$

Thus a solution of (1.32) has the form

$$x = 0$$

$$y = e^{3t} \tag{1.36}$$

$$z = -e^{3t}.$$

Since the two roots of the characteristic equation are equal to 3, we must seek a second solution of (1.32) of the form

$$x = (\alpha_1 t + \alpha_2)\, e^{3t}$$

$$y = (\beta_1 t + \beta_2)\, e^{3t} \tag{1.37}$$

$$z = (\gamma_1 t + \gamma_2)\, e^{3t}.$$

Substituting (1.37) into (1.32), we obtain

$$\alpha_1 = 0,\ \beta_1 = 1,\ \gamma_1 = -1$$

and $\alpha_2 = 1,\ \beta_2 = -1,\ \gamma_2 = 0.$

Thus a second solution of (1.32) has the form

$$x = e^{3t}$$

$$y = (t-1)\, e^{3t} \tag{1.38}$$

$$z = -t\, e^{3t}$$

Setting $\lambda = -1$ into (1.34), we find

$$\alpha = 1,\ \beta = -2,\ \gamma = 0.$$

Thus a third solution of (1.32) has the form

$$x = e^{-t}$$

$$y = -2e^{-t} \qquad (1.39)$$

$$z = 0 .$$

Further, the Wronskian W (t) of the functions (1.36), (1.38) and (1.39) is

$$W(t) = e^{5t} \begin{vmatrix} 0 & 1 & 1 \\ 1 & t-1 & -2 \\ -1 & -t & 0 \end{vmatrix}$$

$$= e^{5t}$$

$$\neq 0.$$

Thus the solutions are linearly independent.

Hence the general solution of (1.32) is

$$x = c_2 e^{3t} + c_3 e^{-t}.$$

$$y = c_1 e^{3t} + c_2 (t-1) e^{3t} - 2 c_3 e^{-t},$$

$$z = -c_1 e^{3t} - c_2 te^{3t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

**Case IV**

Three roots of the characteristic equation (1.4) are real and equal when

$$G= 0, H = 0.$$

If the three roots of the characteristic equation are real and equal it would appear that we could find only one solution of the form (1.2). In this case if

$$x = \alpha e^{\lambda t}$$
$$y = \beta e^{\lambda t} \qquad (1.40)$$
$$z = \gamma e^{\lambda t}$$

is a solution of (1.1), then a second solution for one of the equal roots has of the form

$$x = (\alpha t + \alpha_1) e^{\lambda t}$$
$$y = (\beta t + \alpha_2 e^{\lambda t}) \qquad (1.41)$$
$$z = (\gamma t + \alpha_3) e^{\lambda t}$$

and a third solution for the other equal root has the form

$$x = (\alpha t^2 + \beta_1 t + \gamma_1) e^{\lambda t}$$
$$y = (\beta t^2 + \beta_2 t + \gamma_2) e^{\lambda t} \qquad (1.42)$$
$$z = (\gamma t^2 + \beta_3 t + \gamma_3) e^{\lambda t}.$$

This result is summed up in the following theorem:

## 1.6 The Theorem 4

If the three roots of the characteristic equation (1.4), associated with the system (1.1) are real and equal, then the system (1.1), has three linearly independent solutions of the form

$$x = \alpha e^{\lambda t}$$
$$y = \beta e^{\lambda t}$$
$$z = \gamma e^{\lambda t},$$

$$x = (\alpha t + \alpha_1)\, e^{\lambda t}$$

$$y = (\beta t + \alpha_2)\, e^{\lambda t}$$

$$z = (\gamma t + \alpha_3)\, e^{\lambda t}$$

and

$$x = (\alpha t^2 + \beta_1 t + \gamma_1)\, e^{\lambda t}$$

$$y = (\beta t^2 + \beta_2 t + \gamma_2)\, e^{\lambda t}$$

$$z = (\gamma t^2 + \beta_3 t + \gamma_3)\, e^{\lambda t}$$

where $\alpha, \beta, \gamma$ ; $\alpha_1, \beta_1, \gamma_1$ ; $\alpha_2, \beta_2, \gamma_2$ and $\alpha_3, \beta_3, \gamma_3$ are definite constants and $\alpha, \beta, \gamma$ are not all zero.

The general solution (1.1) may thus be written as

$$x = c_1\, \alpha e^{\lambda t} + c_2 (\alpha t + \alpha_1) e^{\lambda t} + c_3\, (\alpha t^2 + \beta_1 t + \gamma_1)\, c^{\lambda t}$$

$$y = c_1\, \beta e^{\lambda t} + c_2 (\beta t + \alpha_2) e^{\lambda t} + c_3\, (\beta t^2 + \beta_2 t + \gamma_2)\, c^{\lambda t}$$

$$z = c_1\, \gamma e^{\lambda t} + c_2 (\gamma t + \alpha_3) e^{\lambda t} + c_3\, (\gamma t^2 + \beta_3 t + \gamma_3)\, c^{\lambda t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

**The Example**

$$\frac{dx}{dt} = 8x + 12y - 2z$$

$$\frac{dy}{dt} = -3x - 4y + z \qquad (1.43)$$

$$\frac{dz}{dt} = -x - 2y + 2z$$

According to the Eulerian method, we assume a solution of the system (1.43) of the form (1.2); i.e.,

$$x = \alpha\, e^{\lambda t}$$

$$y = \beta\, e^{\lambda t} \qquad (1.44)$$

$$z = \gamma\, e^{\lambda t}$$

Substituting (1.44) into (1.43), we obtain the algebraic system

$$(8-\lambda)\alpha + 12\,\beta - 2\,\gamma = 0$$

$$-3\,\alpha - (4+\lambda)\,\beta + \gamma = 0 \qquad (1.45)$$

$$-\alpha - 2\,\beta + (2-\lambda)\,\gamma = 0$$

in the unknown $\lambda$. For a nontrivial solution of the system (1.45) we must have

$$\begin{vmatrix} 8-\lambda & 12 & -2 \\ -3 & -4-\lambda & 1 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$

or, $\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0.$ (1.46)

Comparing (1.46) with (1.4) we observe that

$k_1 = -2,\ k_2 = 4,\ k_3 = -8.$

So, $G = 0,\ H = 0.$

Hence the characteristic equation (1.46) has three real and equal roots.

Solving (1.46), we find that the roots of this equation are $\lambda = 2, 2, 2$.

Setting $\lambda=2$ into (1.45) we obtain a simple non trivial solution

$$\alpha = 2,\ \beta = -1,\ \gamma = 0.$$

Using these values we obtain the solution

$$
\begin{aligned}
x &= 2\, e^{2t} \\
y &= -\, e^{2t} \\
z &= 0\,.
\end{aligned}
\tag{1.47}
$$

Since the roots of the characteristic equation are equal to 2, we must seek second and third solutions of the form (1.41) and (1.42) respectively. Thus we must determine $\alpha, \beta, \gamma$ ; $\alpha_1, \beta_1 . \gamma_1$ ; $\alpha_2, \beta_2, \gamma_2$ and $\alpha_3, \beta_3, \gamma_3$ such that

$$
\begin{aligned}
x &= (\alpha t + \alpha_1)\, e^{2t} \\
y &= (\beta t + \alpha_2)\, e^{2t} \\
z &= (\gamma t + \alpha_3)\, e^{2t}
\end{aligned}
\tag{1.48}
$$

and

$$
\begin{aligned}
x &= (\alpha t^2 + \beta_1 t + \gamma_1)\, e^{2t} \\
y &= (\beta t^2 + \beta_2 t + \gamma_2)\, e^{2t} \\
z &= (\gamma t^2 + \beta_3 t + \gamma_3)\, e^{2.t}
\end{aligned}
\tag{1.49}
$$

respectively.

Substituting (1.48) and (1.49) into (1.43), we obtain the following simple nontrivial solutions

$$\alpha = 2,\ \beta = -1, \gamma = 0$$

$$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = -1$$

and $\alpha = 2, \beta = -1, \quad \gamma = 0$

$$\beta_1 = 0, \beta_2 = 0, \beta_3 = -2$$

$$\gamma_1 = 2, \gamma_2 = 0, \gamma_3 = 6$$

respectively.

Setting these values, we obtain

$$x = 2t\, e^{2t}$$

$$y = -t\, e^{2t} \qquad (1.50)$$

$$z = -\, e^{2t}$$

and

$$x = (2t^2 + 2)\, e^{2t}$$

$$y = -\, t^2\, e^{2t} \qquad (1.51)$$

$$z = (-2t + 6)\, e^{2t}$$

respectively.

Further, the Wronskian W (t) of the functions (1.47), (1.50) and (1.51) is

$$W(t) = e^{6t} \begin{vmatrix} 2 & 2t & 2t^2+2 \\ -1 & -t & -t^2 \\ 0 & -1 & -2t+6 \end{vmatrix}$$

$$= 2e^{6t}$$

$$\neq 0.$$

Thus the solutions are linearly independent.

Hence the general solution of the system (1.43) is

$$x = 2c_1 e^{2t} + 2c_2 t e^{2t} + c_3 (t^2 + 1) e^{2t}$$

$$y = - c_1 e^{2t} - c_2 t e^{2t} - 1/2 \, c_3 t^2 e^{2t}$$

$$z = -c_2 e^{2t} - c_3 (t - 3) e^{2t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

# CHAPTER-2

## Eulerian Method for Third Order Linear Nonhomogeneous Systems with Constant Coefficients

### 2.1 Introduction

In this chapter a third order linear nonhomogeneous system with constant coefficients has been considered. The method of variation of parameters is used to find the particular integral of the nonhomogeneous part. We have developed Eulerian method for the systems and provided a theorem for this case. The method is illustrated by an example.

### 2.2 The Method

We consider

$$\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y + c_1z + \xi_1(t)\\ \frac{dy}{dt} &= a_2x + b_2y + c_2z + \xi_2(t)\\ \frac{dz}{dt} &= a_3x + b_3y + c_3z + \xi_3(t)\end{aligned} \tag{2.1}$$

where the coefficients $a_1$, $b_1$, $c_1$; $a_2$, $b_2$ $c_2$ and $a_3$, $b_3$, $c_3$ are real constants.

The corresponding homogeneous part of (2.1) is

$$\begin{aligned}\frac{dx}{dt} &= a_1x + b_1y + c_1z\\ \frac{dy}{dt} &= a_2x + b_2y + c_2z\\ \frac{dz}{dt} &= a_3x + b_3y + c_3z.\end{aligned} \tag{2.2}$$

According to the Eulerian method, we assume a solution of this system (2.2) of the form

$$x=\alpha e^{\lambda t}$$
$$y=\beta e^{\lambda t} \qquad (2.3)$$
$$z=\gamma e^{\lambda t}$$

where $\alpha$, $\beta$, $\gamma$ and $\lambda$ are unknown constants. Substituting (2.3) into (2.2), we obtain the algebraic system

$$(a_1-\lambda)\alpha + b_1\beta + c_1\gamma = 0$$
$$a_2\alpha + (b_2-\lambda)\beta + c_2\gamma = 0 \qquad (2.4)$$
$$a_3\alpha + b_3\beta + (c_3-\lambda)\gamma = 0.$$

We seek a nontrivial solution of the system (2.4). A necessary and sufficient condition that the system (2.4) has a nontrivial solution is that the determinant

$$\begin{vmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{vmatrix} = 0.$$

This gives

$$\lambda^3+3\lambda^2k_1+3\lambda k_2+k_3=0 \qquad (2.5)$$

where

$$k_1=\frac{-(a_1+b_2+c_3)}{3}$$

$$k_2=\frac{a_1b_2+b_2c_3+c_3a_1-a_2b_1-b_3c_2-c_1a_3}{3}$$

$$k_3=a_1b_3c_2-a_1b_2c_3-a_3b_1c_2+a_2b_1c_3-a_2b_3c_1+a_3b_2c_1.$$

The equation (2.5), called the characteristic equation associated with the system (2.2), has three characteristic roots.

Let

$$x = \mu_1(t)$$
$$y = \nu_1(t)$$
$$z = \eta_1(t),$$

$$x = \mu_2(t)$$
$$y = \nu_2(t) \qquad (2.6)$$
$$z = \eta_2(t)$$

and

$$x = \mu_3(t)$$
$$y = \nu_3(t)$$
$$z = \eta_3(t)$$

be the three sets of linearly independent solutions of (2.2) for the three corresponding characteristic roots of (2.5).

Then the general solution of (2.2) i.e. the complementary function of (2.1), is given by

$$x_c = c_1\mu_1(t)+c_2\mu_2(t)+c_3\mu_3(t)$$
$$y_c = c_1\nu_1(t)+c_2\nu_2(t)+c_3\nu_3(t)$$
$$z_c = c_1\eta_1(t)+c_2\eta_2(t)+c_3\eta_3(t)$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

Now the method of variation of parameters can be used to find the particular integral of (2.1). Thus we assume a particular integral of (2.1) of the form

$$x_p = \psi_1(t)\mu_1(t)+\psi_2(t)\mu_2(t)+\psi_3(t)\mu_3(t)$$
$$y_p = \psi_1(t)v_1(t)+\psi_2(t)v_2(t)+\psi_3(t)v_3(t) \qquad (2.7)$$
$$z_p = \psi_1(t)\eta_1(t)+\psi_2(t)\eta_2(t)+\psi_3(t)\eta_3(t)$$

where the arbitrary constants $c_1$, $c_2$ and $c_3$ in the complementary function have been replaced by the unknown functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ to be determined.

Since (2.6) satisfies (2.2), we obtain

$$\mu_1'(t) = a_1\mu_1(t) + b_1v_1(t) + c_1\eta_1(t)$$
$$v_1'(t) = a_2\mu_1(t) + b_2v_1(t) + c_2\eta_1(t)$$
$$\eta_1'(t) = a_3\mu_1(t) + b_3v_1(t) + c_3\eta_1(t),$$

$$\mu_2'(t) = a_1\mu_2(t) + b_1v_2(t) + c_1\eta_2(t)$$
$$v_2'(t) = a_2\mu_2(t) + b_2v_2(t) + c_2\eta_2(t)$$
$$\eta_2'(t) = a_3\mu_2(t) + b_3v_2(t) + c_3\eta_2(t)$$

and $$\mu_3'(t) = a_1\mu_3(t) + b_1v_3(t) + c_1\eta_3(t)$$
$$v_3'(t) = a_2\mu_3(t) + b_2v_3(t) + c_2\eta_3(t)$$
$$\eta_3'(t) = a_3\mu_3(t) + b_3v_3(t) + c_3\eta_3(t).$$

Again (2.7) satisfies (2.1)

$$\therefore \Psi'(t)\mu_1(t) + \Psi'_2(t)\mu_2(t) + \Psi'_3(t)\mu_3(t) + \Psi_1(t)\mu'_1(t) + \Psi_2(t)\mu'_2(t) + \Psi_3(t)\mu'_3(t) =$$

$\psi_1(t)$ $\{a_1\mu_1(t)+b_1v_1(t)+c_1\eta_1(t)\}+$ $\psi_2(t)$ $\{a_1\mu_2(t)+b_1v_2(t)+c_1\eta_2(t)\}+$ $\psi_3(t)$ $\{a_1\mu_3(t)+b_1v_3(t)+c_1\eta_3(t)\}+\xi_1$,

$$\Psi'(t)\nu_1(t) + \Psi'_2(t)\nu_2(t) + \Psi'_3(t)\nu_3(t) + \Psi_1(t)\nu'_1(t) + \Psi_2(t)\nu'_2(t) + \Psi_3(t)\nu'_3(t) =$$

$\psi_1(t)$ $\{a_2\mu_1(t)+b_2v_1(t)+c_2\eta_1(t)\}+$ $\psi_2(t)$ $\{a_2\mu_2(t)+b_2v_2(t)+c_2\eta_2(t)\}+$ $\psi_3(t)$ $\{a_2\mu_3(t)+b_2v_3(t)+c_2\eta_3(t)\}+\xi_2$

and

$$\Psi'_1(t)\eta_1(t) + \Psi'_2(t)\eta_2(t) + \Psi'_3(t)\eta_3(t) + \Psi_1(t)\eta'_1(t) + \Psi_2(t)\eta'_2(t) + \Psi_3(t)\eta'_3(t) =$$

$\psi_1(t)$ $\{a_3\mu_1(t)+b_3v_1(t)+c_3\eta_1(t)\}+$ $\psi_2(t)$ $\{a_3\mu_2(t)+b_3v_2(t)+c_3\eta_2(t)\}+$ $\psi_3(t)$ $\{a_3\mu_3(t)+b_3v_3(t)+c_3\eta_3(t)\}+\xi_3$.

or,

$$\begin{aligned} \psi'_1(t)\,\mu_1(t)+\psi'_2(t)\,\mu_2(t)+\psi'_3(t)\,\mu_3(t) &= \xi_1(t) \\ \psi'_1(t)\,v_1(t)+\psi'_2(t)\,v_2(t)+\psi'_3(t)\,v_3(t) &= \xi_2(t) \\ \psi'_1(t)\,\eta_1(t)+\psi'_2(t)\,\eta_2(t)+\psi'_3(t)\,\eta_3(t) &= \xi_3(t). \end{aligned} \tag{2.8}$$

Solution of (2.8) by Crammer's rule yields

$$\Psi'_1(t) = \frac{\Delta_1}{\Delta}$$

$$\Psi'_2(t) = \frac{\Delta_2}{\Delta}$$

$$\Psi'_3(t) = \frac{\Delta_3}{\Delta}$$

where

$$\Delta = \begin{vmatrix} \mu_1(t) & \mu_2(t) & \mu_3(t) \\ \nu_1(t) & \nu_2(t) & \nu_3(t) \\ \eta_1(t) & \eta_2(t) & \eta_3(t) \end{vmatrix}, \qquad \Delta_1 = \begin{vmatrix} \xi_1(t) & \mu_2(t) & \mu_3(t) \\ \xi_2(t) & \nu_2(t) & \nu_3(t) \\ \xi_3(t) & \eta_2(t) & \eta_3(t) \end{vmatrix},$$

$$\Delta_2 = \begin{vmatrix} \mu_1(t) & \xi_1(t) & \mu_3(t) \\ \nu_1(t) & \xi_2(t) & \nu_3(t) \\ \eta_1(t) & \xi_3(t) & \eta_3(t) \end{vmatrix}, \qquad \Delta_3 = \begin{vmatrix} \mu_1(t) & \mu_2(t) & \xi_1(t) \\ \nu_1(t) & \nu_2(t) & \xi_2(t) \\ \eta_1(t) & \eta_2(t) & \xi_3(t) \end{vmatrix}.$$

Thus we obtain the functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$, defined by

$$\psi_1(t) = \int \frac{\Delta_1}{\Delta} dt$$

$$\psi_2(t) = \int \frac{\Delta_2}{\Delta} dt \qquad (2.9)$$

$$\psi_3(t) = \int \frac{\Delta_3}{\Delta} dt .$$

Therefore, a particular integral of (2.1) is obtained from (2.7) by using (2.9).

Hence the general solution of (2.1) is given by

$$x = \{c_1+\psi_1(t)\}\mu_1(t) + \{c_2+\psi_2(t)\}\mu_2(t) + \{c_3+\psi_3(t)\}\mu_3(t)$$

$$y = \{c_1+\psi_1(t)\}\nu_1(t) + \{c_2+\psi_2(t)\}\nu_2(t) + \{c_3+\psi_3(t)\}\nu_3(t)$$

$$z = \{c_1+\psi_1(t)\}\eta_1(t) + \{c_2+\psi_2(t)\}\eta_2(t) + \{c_3+\psi_3(t)\}\eta_3(t)$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants and the functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are obtained from (2.9).

Summarizing the results, we have the following:

## 2.3 The Theorem 5

The general solution of the third order linear nonhomogeneous system with constant coefficients of the form

$$\frac{dx}{dt} = a_1 x + b_1 y + c_1 z + \xi_1(t)$$

$$\frac{dy}{dt} = a_2 x + b_2 y + c_2 z + \xi_2(t)$$

$$\frac{dz}{dt} = a_3 x + b_3 y + c_3 z + \xi_3(t)$$

is given by

$$x = \{c_1 + \psi_1(t)\}\mu_1(t) + \{c_2 + \psi_2(t)\}\mu_2(t) + \{c_3 + \psi_3(t)\}\mu_3(t)$$

$$y = \{c_1 + \psi_1(t)\}\nu_1(t) + \{c_2 + \psi_2(t)\}\nu_2(t) + \{c_3 + \psi_3(t)\}\nu_3(t)$$

$$z = \{c_1 + \psi_1(t)\}\eta_1(t) + \{c_2 + \psi_2(t)\}\eta_2(t) + \{c_3 + \psi_3(t)\}\eta_3(t)$$

where $\mu_1(t)$, $\mu_2(t)$, $\mu_3(t)$; $\nu_1(t)$, $\nu_2(t)$, $\nu_3(t)$ and $\eta_1(t)$, $\eta_2(t)$, $\eta_3(t)$ are the solutions of the corresponding homogeneous part and $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are the variations of the parameters $c_1$, $c_2$ and $c_3$ of the above system.

## 2.4 The Example

$$\begin{aligned}
\frac{dx}{dt} &= 7x - y + 6z - 5t - 6 \\
\frac{dy}{dt} &= -10x + 4y - 12z - 4t + 23 \\
\frac{dz}{dt} &= -2x + y - z + 2.
\end{aligned} \tag{2.10}$$

The corresponding homogeneous part of (2.10) is

$$\frac{dx}{dt} = 7x - y + 6z$$

$$\frac{dy}{dt} = -10x + 4y - 12z \tag{2.11}$$

$$\frac{dz}{dt} = -2x + y - z.$$

According to the Eulerian method, we assume a solution of the system (2.11) of the form

$$x = \alpha e^{\lambda t}$$

$$y = \beta e^{\lambda t} \tag{2.12}$$

$$z = \gamma e^{\lambda t}$$

where $\alpha$, $\beta$, $\gamma$ and $\lambda$ are unknown real constants.

Substituting (2.12) into (2.11), we obtain the algebraic system

$$(7-\lambda)\alpha - \beta + 6\gamma = 0$$

$$-10\alpha + (4-\lambda)\beta - 12\gamma = 0 \tag{2.13}$$

$$-2\alpha + \beta + (-1-\lambda)\gamma = 0.$$

For nontrivial solution of the system (2.13), we must have

$$\begin{vmatrix} 7-\lambda & -1 & 6 \\ -10 & 4-\lambda & -12 \\ -2 & 1 & -1-\lambda \end{vmatrix} = 0,$$

which simplifies to

$$\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0. \tag{2.14}$$

Solving (2.14) we find that the roots of this equation are $\lambda = 2, 3, 5$.

Setting $\lambda = 2, 3, 5$ into (2.13), we obtain simple nontrivial solutions

$$\alpha = 1, \quad \beta = -1, \gamma = -1;$$

$$\alpha = 1, \beta = -2, \gamma = 1$$

and $\alpha = 3, \beta = -6, \gamma = -6$

respectively.

With these values of $\alpha$, $\beta$, $\gamma$ and $\lambda$; we find the following three sets of solutions:

$$\begin{aligned} x &= e^{2t} \\ y &= -e^{2t} \\ z &= -e^{2t}, \end{aligned} \qquad (2.15)$$

$$\begin{aligned} x &= e^{3t} \\ y &= -2e^{3t} \\ z &= -e^{3t} \end{aligned} \qquad (2.16)$$

and

$$\begin{aligned} x &= 3e^{5t} \\ y &= -6e^{5t} \\ z &= -2e^{5t}. \end{aligned} \qquad (2.17)$$

Further, the Wronskian W (t) of the functions (2.15), (2.16) and (2.17) is

$$W(t) = e^{10t} \begin{vmatrix} 1 & 1 & 3 \\ -1 & -2 & -6 \\ -1 & -1 & -2 \end{vmatrix}$$

$$= -e^{10t}$$
$$\neq 0.$$

Thus the solutions are linearly independent.

Therefore the complementary function of (2.10) is

$$\begin{aligned} x_c &= c_1e^{2t} + c_2e^{3t} + 3c_3e^{5t} \\ y_c &= -c_1e^{2t} - 2c_2e^{3t} - 6c_3e^{5t} \\ z_c &= -c_1e^{2t} - c_2e^{3t} - 2c_3e^{5t} \end{aligned} \qquad (2.18)$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

Now by using the method of variation of parameters, we assume a particular integral of (2.10) of the form

$$\begin{aligned} x_p &= \psi_1(t)e^{2t} + \psi_2(t)e^{3t} + 3\psi_3(t)e^{5t} \\ y_p &= -\psi_1(t)e^{2t} - 2\psi_2(t)e^{3t} - 6\psi_3(t)e^{5t} \\ z_p &= -\psi_1(t)e^{2t} - \psi_2(t)e^{3t} - 2\psi_3(t)e^{5t} \end{aligned} \qquad (2.19)$$

where the arbitrary functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are such that

$$\begin{aligned} \psi_1'(t)e^{2t} + \psi_2'(t)e^{3t} + 3\psi_3'(t)e^{5t} &= -5t - 6 \\ -\psi_1'(t)e^{2t} - 2\psi_2'(t)e^{3t} - 6\psi_3'(t)e^{5t} &= -4t + 23 \\ -\psi_1'(t)e^{2t} - \psi_2'(t)e^{3t} - 2\psi_3'(t)e^{5t} &= 2. \end{aligned} \qquad (2.20)$$

Solving (2.20) by Crammer's rule, we obtain

$$\psi_1(t) = (7t-2)e^{-2t},$$

$$\psi_2(t) = -(8t+1)e^{-3t}$$

$$\psi_3(t) = (t+1)\, e^{-5t}.$$

Thus a particular integral of (2.10) is

$$x_p = 2t$$

$$y_p = 3t - 2$$

$$z_p = -t + 1.$$

Hence the general solution of (2.10) is

$$x = c_1e^{2t} + c_2e^{3t} + 3c_3e^{5t} + 2t$$

$$y = -c_1e^{2t} - 2c_2e^{3t} - 6c_3e^{5t} + 3t - 2$$

$$z = -2c_1e^{2t} - c_2e^{3t} - 2c_3e^{5t} - t + 1$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

# CHAPTER-3

## Eulerian Method for General Third Order Linear Nonhomogeneous Systems with Constant Coefficients

### 3.1 Introduction

In this chapter, we have considered the general system of equations, each equation contains the terms of x, y, z & $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$. The number of independent arbitrary constants appearing in the general solution of this system is equal to the degree in $\lambda$ of the determinant of the coefficient matrix, provided that the determinant does not vanish identically. If the determinant is equal to zero, then the system is dependent, such systems will not be considered here.

### 3.2 The Method

We consider the most general third order linear nonhomogeneous system with constant coefficients

$$
\begin{aligned}
&a_1\frac{dx}{dt}+b_1\frac{dy}{dt}+c_1\frac{dz}{dt}+d_1x+e_1y+f_1z=P_1(t)\\
&a_2\frac{dx}{dt}+b_2\frac{dy}{dt}+c_2\frac{dz}{dt}+d_2x+e_2y+f_2z=P_2(t)\\
&a_3\frac{dx}{dt}+b_3\frac{dy}{dt}+c_3\frac{dz}{dt}+d_3x+e_3y+f_3z=P_3(t)
\end{aligned}
\tag{3.1}
$$

where $a_1$, $b_1$, $c_1$, $d_1$, $e_1$, $f_1$; $a_2$, $b_2$, $c_2$, $d_2$, $e_2$, $f_2$ and $a_3$, $b_3$, $c_3$, $d_3$, $e_3$, $f_3$ are real constants. Here the dependent variables x, y, z and their derivatives

$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ appear in all the three equations of (3.1). Thus the system (2.1) is a special case of the system (3.1).

The corresponding homogeneous part of (3.1) is

$$
\begin{aligned}
&a_1\frac{dx}{dt}+b_1\frac{dy}{dt}+c_1\frac{dz}{dt}+d_1x+e_1y+f_1z=0 \\
&a_2\frac{dx}{dt}+b_2\frac{dy}{dt}+c_2\frac{dz}{dt}+d_2x+e_2y+f_2z=0 \\
&a_3\frac{dx}{dt}+b_3\frac{dy}{dt}+c_3\frac{dz}{dt}+d_3x+e_3y+f_3z=0.
\end{aligned}
\tag{3.2}
$$

According to the Eulerian method, we assume a solution of the system (3.2) of the form

$$
\begin{aligned}
&x=\alpha e^{\lambda t} \\
&y=\beta\, e^{\lambda t} \\
&z=\gamma\, e^{\lambda t}
\end{aligned}
\tag{3.3}
$$

where $\alpha$, $\beta$, $\gamma$ and $\lambda$ are unknown constants. Substituting (3.3) into (3.2), we obtain the algebraic system

$$
\begin{aligned}
&\alpha\,(a_1\lambda+d_1)+\beta\,(b_1\lambda+e_1)+\gamma(c_1\lambda+f_1)=0 \\
&\alpha\,(a_2\lambda+d_2)+\beta\,(b_2\lambda+e_2)+\gamma(c_2\lambda+f_2)=0 \\
&\alpha\,(a_3\lambda+d_3)+\beta\,(b_3\lambda+e_3)+\gamma(c_3\lambda+f_3)=0.
\end{aligned}
\tag{3.4}
$$

We seek a nontrivial solution of this system. A necessary and sufficient condition that system (3.4) have a nontrivial solution is that the determinant

$$\Delta = \begin{vmatrix} a_1\lambda + d_1 & b_1\lambda + e_1 & c_1\lambda + f_1 \\ a_2\lambda + d_2 & b_2\lambda + e_2 & c_2\lambda + f_2 \\ a_3\lambda + d_3 & b_3\lambda + e_3 & c_3\lambda + f_3 \end{vmatrix} = 0$$

or, $k_1\lambda^3 + k_2\lambda^2 + k_3\lambda + k_4 = 0$ (3.5)

where,

$k_1 = a_1(b_2c_3 - b_3c_2) + b_1(a_3c_2 - a_2c_3) + c_1(a_2b_3 - a_3b_2)$

$k_2 = a_1(c_3e_2 - c_2e_3 + b_2f_3 - b_3f_2) + d_1(b_2c_3 - b_3c_2) + b_1(c_2d_3 - c_3d_2 + a_3f_2 - a_2f_3) + e_1(a_3c_2 - a_2c_3) + c_1(b_3d_2 - b_2d_3 + a_2e_3 - a_3e_2) + f_1(a_2b_3 - a_3b_2)$

$k_3 = a_1(e_2f_3 - e_3f_2) + d_1(c_3e_2 - c_2e_3 + b_2f_3 - b_3f_2) + b_1(d_3f_2 - d_2f_3) + e_1(c_2d_3 - c_3d_2 + a_3f_2 - a_2f_3) + c_1(d_2e_3 - d_3e_2) + f_1(b_3d_2 - b_2d_3 + a_2e_3 - a_3e_2)$

$k_4 = d_1(e_2f_3 - e_3f_2) + e_1(d_3f_2 - d_2f_3) + f_1(d_2e_3 - d_3e_2)$.

The equation (3.5) is called the characteristic equation associated with the system (3.2).

If the roots of the equation (3.5) be $\lambda_1$, $\lambda_2$, $\lambda_3$ then the system (3.2) has three nontrivial linearly independent solutions of the form

$$x = \alpha_1 e^{\lambda_1 t}$$
$$y = \beta_1 e^{\lambda_1 t}$$
$$z = \gamma_1 e^{\lambda_1 t}$$

$$x = \alpha_2 e^{\lambda_2 t}$$
$$y = \beta_2 e^{\lambda_2 t} \qquad (3.6)$$
$$z = \gamma_2 e^{\lambda_2 t}$$

and

$$x = \alpha_3 e^{\lambda_3 t}$$
$$y = \beta_3 e^{\lambda_3 t}$$
$$z = \gamma_3 e^{\lambda_3 t}$$

where $\alpha_1$, $\beta_1$, $\gamma_1$; $\alpha_2$, $\beta_2$, $\gamma_2$ and $\alpha_3$, $\beta_3$, $\gamma_3$ are definite constants. The general solution of the system (3.2), i.e. the complimentary function of (3.1) may thus be written as

$$
\begin{aligned}
x_c &= \bar{c}_1\alpha_1 e^{\lambda_1 t} + \bar{c}_2\alpha_2 e^{\lambda_2 t} + \bar{c}_3\alpha_3 e^{\lambda_3 t} \\
y_c &= \bar{c}_1\beta_1 e^{\lambda_1 t} + \bar{c}_2\beta_2 e^{\lambda_2 t} + \bar{c}_3\beta_3 e^{\lambda_3 t} \\
z_c &= \bar{c}_1\gamma_1 e^{\lambda_1 t} + \bar{c}_2\gamma_2 e^{\lambda_2 t} + \bar{c}_3\gamma_3 e^{\lambda_3 t}
\end{aligned}
\qquad (3.7)
$$

where $\bar{c}_1$, $\bar{c}_2$ and $\bar{c}_3$ are arbitrary constants.

Now the method of variation of parameters can be used to find the particular integral of (3.1). Thus we assume a particular integral of (3.1) of the form

$$
\begin{aligned}
x_p &= \Psi_1(t)\alpha_1 e^{\lambda_1 t} + \Psi_2(t)\alpha_2 e^{\lambda_2 t} + \Psi_3(t)\alpha_3 e^{\lambda_3 t} \\
y_p &= \Psi_1(t)\beta_1 e^{\lambda_1 t} + \Psi_2(t)\beta_2 e^{\lambda_2 t} + \Psi_3(t)\beta_3 e^{\lambda_3 t} \\
z_p &= \Psi_1(t)\gamma_1 e^{\lambda_1 t} + \Psi_2(t)\gamma_2 e^{\lambda_2 t} + \Psi_3(t)\gamma_3 e^{\lambda_3 t}
\end{aligned}
\qquad (3.8)
$$

where the arbitrary constants $\bar{c}_1$, $\bar{c}_2$ and $\bar{c}_3$ in the complementary function (3.7) have been replaced by the unknown functions $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ to be determined.

Since (3.8) satisfies (3.1), we obtain a system of equations in $\Psi'_1(t), \Psi'_2(t)$ and $\Psi'_3(t)$. If we apply Crammer's rule, we obtain $\Psi'_1(t), \Psi'_2(t)$ and $\Psi'_3(t)$. Finally, by integrating, we obtain $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$.

Therefore, a particular integral of (3.1) is obtained from (3.8) by using $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$.

Hence the general solution is given by

$$x= \{c_1+\psi_1(t)\}\alpha_1\ e^{\lambda_1 t} + \{c_2+\psi_2(t)\}\alpha_2\ e^{\lambda_2 t} + \{c_3+\psi_3(t)\}\alpha_3\ e^{\lambda_3 t}$$

$$y= \{c_1+\psi_1(t)\}\beta_1\ e^{\lambda_1 t} + \{c_2+\psi_2(t)\}\beta_2\ e^{\lambda_2 t} + \{c_3+\psi_3(t)\}\beta_3\ e^{\lambda_3 t}$$

$$z= \{c_1+\psi_1(t)\}\gamma_1\ e^{\lambda_1 t} + \{c_2+\psi_2(t)\}\gamma_2\ e^{\lambda_2 t} + \{c_3+\psi_3(t)\}\gamma_3\ e^{\lambda_3 t}.$$

Summaring this discussion, we have the following theorem:

### 3.3 The Theorem 6

The general solution of the third order linear nonhomogeneous system with constant coefficients of the form

$$a_1\frac{dx}{dt}+b_1\frac{dy}{dt}+c_1\frac{dz}{dt}+d_1x+e_1y+f_1z=P_1(t)$$

$$a_2\frac{dx}{dt}+b_2\frac{dy}{dt}+c_2\frac{dz}{dt}+d_2x+e_2y+f_2z=P_2(t)$$

$$a_3\frac{dx}{dt}+b_3\frac{dy}{dt}+c_3\frac{dz}{dt}+d_3x+e_3y+f_3z=P_3(t)$$

is given by

$$x = \{c_1+\psi_1(t)\}\alpha_1\ e^{\lambda_1 t} + \{c_2+\psi_2(t)\}\alpha_2\ e^{\lambda_2 t} + \{c_3+\psi_3(t)\}\alpha_3\ e^{\lambda_3 t}$$

$$y = \{c_1+\psi_1(t)\}\beta_1\ e^{\lambda_1 t} + \{c_2+\psi_2(t)\}\beta_2\ e^{\lambda_2 t} + \{c_3+\psi_3(t)\}\beta_3\ e^{\lambda_3 t}$$

$$z = \{c_1+\psi_1(t)\}\gamma_1\ e^{\lambda_1 t} + \{c_2+\psi_2(t)\}\gamma_2\ e^{\lambda_2 t} + \{c_3+\psi_3(t)\}\gamma_3\ e^{\lambda_3 t}.$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants and $\psi_1$, $\psi_2$ and $\psi_3$ are variation of parameters.

### 3.4 The Example

$$\frac{dx}{dt} - 2\frac{dy}{dt} + \frac{dz}{dt} + 2x - y - 3z = 3e^{t} - 12e^{3t}$$

$$2\frac{dx}{dt} - \frac{dy}{dt} - \frac{dz}{dt} + x - y - z = 2e^{t} + 5e^{2t} - 3e^{3t} \qquad (3.9)$$

$$\frac{dx}{dt} + \frac{dy}{dt} + 2\frac{dz}{dt} - x + 2y - 2z = -3e^{t} + 3e^{2t} - 5e^{3t}.$$

The corresponding homogeneous part of (3.9) is

$$\frac{dx}{dt} - 2\frac{dy}{dt} + \frac{dz}{dt} + 2x - y - 3z = 0$$

$$2\frac{dx}{dt} - \frac{dy}{dt} - \frac{dz}{dt} + x - y - z = 0 \qquad (3.10)$$

$$\frac{dx}{dt} + \frac{dy}{dt} + 2\frac{dz}{dt} - x + 2y - 2z = 0.$$

According to the Eulerian method, we assume a solution of the system (3.10) of the form

$$x = \alpha\, e^{\lambda t}$$

$$y = \beta\ e^{\lambda t} \qquad (3.11)$$

$$z = \gamma\ e^{\lambda t}$$

where $\alpha$, $\beta$, $\gamma$ and $\lambda$ are real constants.

Substituting (3.11) into (3.10), we obtain the algebraic system

$$\begin{aligned}(\lambda+2)\,\alpha-(2\lambda+1)\,\beta+(\lambda-3)\,\gamma&=0\\(2\lambda+1)\,\alpha-(\lambda+1)\,\beta-(\lambda+1)\,\gamma&=0\\(\lambda-1)\,\alpha+(\lambda+2)\,\beta+(2\lambda-2)\,\gamma&=0.\end{aligned}\tag{3.12}$$

For a nontrivial solution of the system (3.12), we must have

$$\begin{vmatrix}\lambda+2 & -2\lambda-1 & \lambda-3\\ 2\lambda+1 & -\lambda-1 & -\lambda-1\\ \lambda-1 & \lambda+2 & 2\lambda-2\end{vmatrix}=0.$$

which simplifies to

$$6\lambda^3-\lambda^2-6\lambda+1=0. \tag{3.13}$$

Solving (3.13), we find the roots of the characteristic equation $\lambda = 1, -1, \frac{1}{6}$.

Setting $\lambda = 1, -1, \frac{1}{6}$ into (3.12), we obtain simple nontrivial solutions

$\alpha=2,\ \beta=0,\ \gamma=3,$

$\alpha=0,\ \beta=4,\ \gamma=1$

and $\alpha=7,\ \beta=5,\ \gamma=3$

respectively.

With these values of $\alpha$, $\beta$, $\gamma$ and $\lambda$; we find the following three sets of solutions

$$\begin{aligned}x&=2e^t\\y&=0\\z&=3e^t,\end{aligned}\tag{3.14}$$

$$x = 0$$
$$y = 4e^{-t}$$
$$z = e^{-t} \qquad (3.15)$$

and

$$x = 7e^{\frac{1}{6}t}$$
$$y = 5e^{\frac{1}{6}t}$$
$$z = 3e^{\frac{1}{6}t}. \qquad (3.16)$$

Further, the Wronskian W (t) of the functions (3.14), (3.15) and (3.16) is

$$W(t) = e^{\frac{1}{6}t} \begin{vmatrix} 2 & 0 & 7 \\ 0 & 4 & 5 \\ 3 & 1 & 3 \end{vmatrix}$$
$$= -70e^{\frac{1}{6}t}$$
$$\neq 0.$$

Thus the solutions are linearly independent.

Hence the general solution of the system (3.10) is

$$x_c = 2c_1e^{t} + 7c_3 e^{\frac{1}{6}t}$$
$$y_c = 4c_2e^{-t} + 5c_3 e^{\frac{1}{6}t} \qquad (3.17)$$
$$z_c = 3c_1e^{t} + c_2e^{-t} + 3c_3 e^{\frac{1}{6}t}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

Now by the method of variation of parameters, we assume a particular integral of (3.9) of the form

$$x_p = 2\psi_1(t)\, e^{t} + 7\psi_3(t)\, e^{\frac{1}{6}t}$$

$$y_p = 4\psi_2(t)\, e^{-t} + 5\psi_3(t)\, e^{\frac{1}{6}t} \qquad (3.18)$$

$$z_p = 3\psi_1 e^{t} + \psi_2 e^{-t} + 3\psi_3\, e^{\frac{1}{6}t}$$

where $\psi_1(t)$, $\psi_2(t)$ and $\psi_3(t)$ are such that

$$5\Psi_1'(t)e^{t} - 7\Psi_2'(t)e^{-t} = 3e^{t} - 12e^{3t}$$

$$\Psi_1'(t)e^{t} - 5\Psi_2'(t)e^{-t} + 6\Psi_3'(t)e^{\frac{1}{6}t} = 2e^{t} + 5e^{2t} - 3e^{3t} \qquad (3.19)$$

$$8\Psi_1'(t)e^{t} + 6\Psi_2'(t)e^{-t} + 18\Psi_3'(t)e^{\frac{1}{6}t} = -3e^{t} + 3e^{2t} - 5e^{3t}.$$

Solving (3.19) by Crammer's rule, we obtain

$$\psi_1(t) = -\frac{1}{5}(3e^{t} + 4e^{2t})$$

$$\psi_2(t) = -\frac{1}{7}\left(\frac{3}{2}e^{2t} + e^{3t} - e^{4t}\right)$$

$$\psi_3(t) = -\frac{1}{35}\left(e^{\frac{5}{6}t} - 11e^{\frac{11}{6}t} - 3e^{\frac{17}{6}t}\right).$$

Thus a particular integral of (3.9) is

$$x_p = -\frac{1}{5}e^{t} + e^{2t} - e^{3t}$$

$$y_p = -e^{t} + e^{2t} + e^{3t}$$

$$z_p = -\frac{3}{10}e^{t} - e^{2t} - 2e^{3t}.$$

Hence the general solution of (3.9) is

$$x = 2c_1e^{t} + 7c_3e^{\frac{1}{6}t} - \frac{1}{5}e^{t} + e^{2t} - e^{3t}$$

$$y = 4c_2e^{-t} + 5c_3e^{\frac{1}{6}t} - e^{t} + e^{2t} + e^{3t}$$

$$z = 3c_1e^{t} + c_2e^{-t} + 3c_3e^{\frac{1}{6}t} - \frac{3}{10}e^{t} - e^{2t} - 2e^{3t}.$$

# CHAPTER-4

## Eulerian Method for Third Order Linear Nonhomogeneous Systems with Variable Coefficients

### 4 .1 Introduction

In the preceding chapters we have shown how to obtain the general solution of the third order linear systems with constant coefficients. We have seen that in such cases the complementary function and the particular integral may readily be determined. In this chapter a third order linear nonhomogeneous system with variable coefficients is considered. This is a hard problem and very difficult to solve. However, only in certain special cases the complementary function can be obtained explicitly in closed form. One special case of considerable importance was solved by Cauchy and Euler [9]. They solved such a second order system. We have extended their technique to the third order system. In this special case, we use a transformation which reduces the system with variable coefficients to the system with constant coefficients. Then the procedure of Chapter 2 is applied to solve this problem. The method is supported by an example.

### 4.2 The Method

Consider

$$\begin{aligned}
\xi_1(t)\frac{dx}{dt} &= a_1x+b_1y+c_1z+\eta_1(t)\\
\xi_2(t)\frac{dy}{dt} &= a_2x+b_2y+c_2z+\eta_2(t)\\
\xi_3(t)\frac{dz}{dt} &= a_3x+b_3y+c_3z+\eta_3(t)
\end{aligned} \tag{4.1}$$

where the coefficients $a_1$, $b_1$, $c_1$; $a_2$, $b_2$ $c_2$ and $a_3$, $b_3$, $c_3$ are real constants. As the general case (4.1) is difficult to solve, we choose $\xi_1(t)=\xi_2(t)=\xi_3(t)=t$ and consider a special case of (4.1) of the form

$$t\frac{dx}{dt}=a_1x+b_1y+c_1z+\eta_1(t)$$

$$t\frac{dy}{dt}=a_2x+b_2y+c_2z+\eta_2(t) \qquad (4.2)$$

$$t\frac{dz}{dt}=a_3x+b_3y+c_3z+\eta_3(t).$$

This is also a third order nonhomogeneous system with variable coefficients. In solving (4.2) we use the transformation

$$t=e^u, \text{ i.e., } u=\ln t \qquad (4.3)$$

where $t \geq 0$, and obtain

$$t\frac{dx}{dt}=\frac{dx}{du},$$

$$t\frac{dy}{dt}=\frac{dy}{du}$$

and $$t\frac{dz}{dt}=\frac{dz}{du}$$

The transformation (4.3) reduces the system (4.2) to the form.

$$\frac{dx}{du} = a_1 x + b_1 y + c_1 z + \zeta_1(u)$$
$$\frac{dy}{du} = a_2 x + b_2 y + c_2 z + \zeta_2(u) \qquad (4.4)$$
$$\frac{dz}{du} = a_3 x + b_3 y + c_3 z + \zeta_3(u).$$

This is a third order linear nonhomogeneous system with constant coefficients. The system (4.4) can be solved by the Eulerian method discussed in Chapter 2. Thus we have the following theorem:

## 4.3 The Theorem 7

The transformation $t = e^u$ reduces the third order linear system with variable coefficients

$$t\frac{dx}{dt} = a_1 x + b_1 y + c_1 z + \eta_1(t)$$
$$t\frac{dy}{dt} = a_2 x + b_2 y + c_2 z + \eta_2(t)$$
$$t\frac{dz}{dt} = a_3 x + b_3 y + c_3 z + \eta_3(t)$$

to a third order linear system with constant coefficients

$$\frac{dx}{du} = a_1 x + b_1 y + c_1 z + \zeta_1(u)$$

$$\frac{dy}{du} = a_2 x + b_2 y + c_2 z + \zeta_2(u)$$

$$\frac{dz}{du} = a_3 x + b_3 y + c_3 z + \zeta_3(u).$$

### 4.4 The Example

$$t\frac{dx}{dt} = 7x - y + 6z - 5t - 6$$

$$t\frac{dy}{dt} = -10x + 4y - 12z - 4t + 23 \qquad (4.5)$$

$$t\frac{dz}{dt} = -2x + y - z + 2$$

Let $t = e^u$. Then (4.5) becomes

$$\frac{dx}{du} = 7x - y + 6z - 5e^u - 6$$

$$\frac{dy}{du} = -10x + 4y - 12z - 4e^u + 23 \qquad (4.6)$$

$$\frac{dz}{du} = -2x + y - z + 2.$$

By the same process of Example (2.10), the complementary solution of (4.6) is given by

$$x_c = c_1e^{2u} + c_2e^{3u} + 3c_3e^{5u}$$

$$y_c = -c_1e^{2u} - 2c_2e^{3u} - 6c_3e^{5u} \qquad (4.7)$$

$$z_c = -c_1e^{2u} - c_2e^{3u} - 2c_3e^{5u}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

Using the method of variation of parameters, we assume

$$x_p = \Psi_1(t)\, e^{2u} + \Psi_2(t)\, e^{3u} + 3\Psi_3(t)\, e^{5u}$$

$$y_p = -\Psi_1(t)\, e^{2u} - 2\Psi_2(t)\, e^{3u} - 6\Psi_3(t)\, e^{5u}$$

$$z_p = -\Psi_1(t)\, e^{2u} - \Psi_2(t)\, e^{3u} - 2\Psi_3(t)\, e^{5u}.$$

Following Example (2.10), we obtain

$$\Psi_1(u) = 14e^{-u} - \frac{11}{2}e^{-2u}$$

$$\Psi_2(u) = -12e^{-2u} + \frac{5}{3}e^{-3u}$$

$$\Psi_3(u) = \frac{5}{4}e^{-4u} + \frac{4}{5}e^{-5u}.$$

So the particular integral of (4.6) is

$$x_p = \frac{23}{4}e^{u} - \frac{43}{30}$$

$$y_p = \frac{5}{2}e^{u} - \frac{79}{30}$$

$$z_p = -\frac{9}{2}e^{u} + \frac{67}{30}.$$

Thus the general solution of (4.6) is

$$x = c_1e^{2u} + c_2e^{3u} + 3c_3e^{5u} + \frac{23}{4}e^{u} - \frac{43}{30}$$

$$y = -c_1e^{2u} - 2c_2e^{3u} - 6c_3e^{5u} + \frac{5}{2}e^{u} - \frac{79}{30}$$

$$z = -c_1e^{2u} - c_2e^{3u} - 2c_3e^{5u} - \frac{9}{2}e^{u} + \frac{67}{30}.$$

Hence the general solution of (4.5) is

$$x = c_1t^2 + c_2t^3 + 3c_3t^5 + \frac{23}{4}t - \frac{43}{30}$$

$$y = -c_1t^2 - 2c_2t^3 - 6c_3t^5 + \frac{5}{2}t - \frac{79}{30}$$

$$z = -c_1t^2 - c_2t^3 - 2c_3t^5 - \frac{9}{2}t + \frac{67}{30}$$

where $c_1$, $c_2$ and $c_3$ are arbitrary constants.

# CONCLUSION

We have considered systems of third order linear homogeneous and nonhomogeneous differential equations with constant and variable coefficients and developed the techniques of finding solutions by extending the Eulerian method. The method has been illustrated by giving various examples. Firstly, an extended Eulerian method is developed to find the solution of the homogeneous part of a third order linear system. Here we have discussed various cases. Secondly, the method is used to find the solution of the third order nonhomogeneous system with constant coefficients. Further, we develop the procedure of solving the general third order nonhomogeneous system with constant coefficients. Although this system is similar to the system mentioned in Chapter 2, the procedure of finding the solution is different. Finally, we have considered the third order nonhomogeneous system with variable coefficients. Since the general case is very different to solve, we have examined a special case. A suitable transformation reduces this system with variable coefficients to a system with constant coefficients which is analogous to the system mentioned in Chapter 2. These systems of equations have extensive applications in Population Dynamics, Fluid Dynamics, Mathematical Physics etc. We hope that the method discussed in the thesis will help the researchers working in the field of differential equations and their applications.

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