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Study On Some Problems of Non-Newtonian Fluid Mechanics

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STUDY ON SOME PROBLEMS OF NON-NEWTONIAN FLUID MECHANICS

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Thesis Dedicated
To My
Beloved Parents

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CERTIFICATE FROM THE SUPERVISOR

Certified that the thesis entitled "**Study On Some Problems of Non-Newtonian Fluid Mechanics**" submitted by Md. Aslam Hossain, for the award of M.Phil degree in Mathematics, University of Rajshahi, is absolutely based on his own work under my supervision and guidance in the Department of Mathematics, University of Rajshahi.

He has fulfilled all the conditions required for the submission of the thesis under the rules of University of Rajshahi. I believe that this research work is an original one and it has not been submitted elsewhere for any degree. In my opinion this work is worthy of consideration for the award of the M.Phil. degree in Mathematics.

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DECLARATION

I do hereby declare that the thesis entitled "**STUDY ON SOME PROBLEMS OF NON-NEWTONIAN FLUID MECHANICS**", submitted to the University of Rajshahi for the award of M.Phil. degree in Science (Mathematics) is based on my research work and has not been submitted to any University/Institute for the award of any degree or diploma.

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CHAPTER-1

Introduction

This is an elaborate discussion of main theme ideas and the condition in "Fluid mechanics", which help us to realized a clear concept of the background and elements of various viscous Newtonian and non-Newtonian flow problems. The basic equation viz, the equation of a visco-elastic fluid of oldroyds model, the equation of continuity, the equation of motion and the equation for isotropic incompressible fluid of Newtonian and non-Newtonian. Here we also have shown the equation the velocity profile in the dimensionless form.

Fluid mechanics is one of the engineering science that form the basis for all engineering like meteorology, oceanography and other subject of physical sciences. The subject branches out into various specialities such as aerodynamics, hydraulic engineering, marine engineering, gas dynamics and rate processes. It deals with the statics, kinematics and dynamics of fluids, since the motion of a fluid is caused by unbalanced forces exerted upon it. Available methods of analysis stem from the application of the following principles concepts and laws, Newton's laws of motion, the first and second laws of thermodynamics, the principle of conservation of mass, equations of state relating fluid properties. Newton's law of viscosity, mixing-lenth concepts and restrictions caused by the presence of boundaries.

In fluid flow calculations, viscosity and density are the fluid properties most generally encountered; they play the principal roles in open-and closed-channel flow and in flow around immersed bodies. Surface tension effects are of importance in the formation of droplets, in flow of small jets and in situations where liquid-gas-solid or liquid-liquid-solid interfaces occurs, as well as in the formation of capillary waves. The property of vapor pressure, accounting for changes of phase from liquid to gas, becomes important when reduced pressures are encountered. In this chapter fluid properties are discussed, as well as units and dimensions and concepts of the continuum.

Fluids may be defined as materials which continue to deform in the, presence of any shearing stress. When the space between two plates is filled with a fluid the plates can be kept moving relative to each other by a force; however small. Generally, the larger the force, the higher the rate of the relative motion. As a contrast, a solid under shearing stresses can do not exceed a certain limit. There are solids that will continue to deform like a fluid when the stress exceeds a certain value. The solid is then said to be in a plastic state. The study of plastic is out side the domain of fluid mechanics.

In fluid mechanics, fluids are considered to be continuous although they like any substance; consist of discrete molecules. This approach is taken not only for the resultant simplicity in analysis, but also because the behavior of the individual molecules is not usually of primary interest in technology. The average, properties of the molecules in a small parcel of fluid are used as the properties of the continuous material. For example; the mass of all the molecules per unit volume of the parcel is called the density of the fluid. For this approach to be successful; the size of the slow system must be much larger than the mean free path of the molecules, so that the properties, such as density, of the fluid can be meaningfully computed. Ordinarily, this requirement presents no difficulties. For example, there are about 2.7×10^{16} molecules in one cubic millimeter of air under atmospheric conditions. However there are cases where this requirement is not satisfied. For example; in the upper atmosphere where air molecules may be, on the average, several fact apart, the term density becomes meaningless of one considers the flow around a satellite of a foot in size.

Fluids dynamics is the science treating the study of fluids in motion. By the term fluid is meant a substance that flows: one which does not is termed a solid. Fluids may be divided into two kinds (i) liquids which are incompressible, i.e. their volumes do not change when the pressure changes and (ii) gases which are compressible fluids suffering change in volume whenever the pressure changes. There are no sharp distinctions between the three states of matter, however, the term hydrodynamics is often applied to the science of moving incompressible fluids.

When matter is subjected to examine on the microscopic or molecular scale, it is found to consist of molecules in random motion and separated from one another by distances, which are at least comparable with molecular size. In the case of gases, the separation distances are great: in the case of liquids, they are less great and in the case of solids even less so.

For the purpose of macroscopic analysis, however, the molecular structure of matter is, in general, of no interest. It is thus more convenient to treat the fluid as having continuous structure so that at each point we can prescribe a unique velocity, a unique pressure, a unique density, etc. moreover, for a continuous or ideal fluid. We can define a fluid particle as the fluid contained within an infinitesimal volume whose size is so small that it may be regarded as a geometrical point.

We conclude this introductory section by mentioning briefly the natures of the different types of forces which are called into lay moving fluids. Suppose two fluid particles, moving at different velocities, have a common boundary. Then across the boundary there will be interchange of momentum. The normal transport of molecules across the boundary will lead to a direct or normal force. In the case of viscous fluids, there is friction between the particles: this will manifest itself in the form of equal and opposite tangential or shearing forces on each particle at the common boundary. In the case of in viscid fluids, however, there is no friction and consequently there are no tangential or shearing forces. All real fluids exhibit viscosity but in many cases. Such as arise when the rates of variation of fluid velocity with distances are small, viscous effects may be ignored.

Fluid mechanics and hydraulics represent that branch of applied mechanics dealing with the behavior of fluids at rest and in motion. In the development of the principles of fluids mechanics, some fluid properties play principal roles others only minor roles or no roles at all.

In fluid static's, specific weight is the important property, where as in fluid flow, density and viscosity are predominant properties. Where appreciable compressibility occurs, principles of thermodynamics must be considered. Vapor pressure becomes important when negative pressures are involved, and surface tension affects static and flow conditions in small passages.

Fluids are substances which are capable of flowing and which conform to the shape of containing vessels. When in equilibrium, fluids cannot sustain tangential or shear forces. All fluids have some degree of compressibility and offer little resistance to change of form.

1.1 Classification of fluids

The fluids can be classified as (i) Ideal fluid and (ii) Real fluid based on its physical properties.

(i) The ideal fluid:

Since it would be difficult to construct equations involving simultaneously all the factors, we have just enumerated and the solution of then when constructed virtually impossible, the fluid is simplified in classical theory as follows

1. The fluid retains the same density throughout.
2. The fluid is incompressible.

These conditions are not synonymous though they have the same effect. In the molecular picture of the processes of locally heating or of mechanically compressing a fluid the former is supposed to increases molecular kinitic energy, the latter to bring the particles closer together. A "hotspot" defuses outwards, a compression travels out as a sound wave ·

3. The fluid is inelastic

4. The fluid has no free surface

These restrictions leave the fluid with mass (or inertia) and viscosity, but further simplification of the mathematics ensues if we suppose the fluid frictionless·

(ii) Real fluid:

A portion of a real fluid is composed of a very large number of molecules each of which has its own mass and velocity. At any instant the several molecules within a given closed surface have a great variety of velocities, since the velocities of the molecules vary both in magnitude and direction from molecule to molecule. If the closed surface has a small but finite volume v it is possible to consider the average mass per unit volume and the average vector velocity within. The surface, these quantities might be regarded as the density and velocity q of the fluid at some point within v though is must be remembered that their values depend upon the size of the small volume considered. In fact it the volume be too small it may contain only one or two particles or even none at all and the quantities then evaluated could hardly be regarded as the density and velocity of the fluid. On the other hand if the volume chosed values and will not give a meaning to density or velocity at a point in the fluid.

The truth of the matter is that the concepts of density and velocity at a point in the fluid pertain only to the idealized notion of a continuous fluid and are not strictly applicable to a real fluid. The mathematical difficulties indicated above arise from the fact a real fluid is a discrete assemblage of a moleclues and is not a continuous fluid.

1.1.1 Newtonian fluids:

All real fluids exhibit internal friction. At solid boundaries, there is practically no relative velocity between the solid and the contacting fluid particles. As a result of there propertied, mechanical energy is dissipated into heat, and there is skin fraction on solid surfaces. Due to the presence of the slow-moving fluid particles near solid surfaces there is a tendency for separation of the flow from the boundary in zones of deceleration. Shere phenomena occur in on frictionless fluid, internal friction must be considered when dissipation, skin friction, separation, and other phenomena related to internal friction are being studied.

It has been found experimentally that most common fluids, including air and water, when tested as shown in the given equation offer shearing resistance proportional to the rate of deformation: $\tau = \frac{F}{A} = \mu \frac{U}{d}$

where τ is the shearing stress, and μ is the absolute, or dynamic, viscosity of the fluid. So Newtonian fluids is called those fluids which obey Newtonians law of viscosity. Water, air and Mercury are the example of Newtonian fluids.

1.1.2 Non-Newtonian fluid:

Fluids which do not obey Newton's law of viscosity are Known as non-Newtonian fluids. Thus for such fluids the shear stress is not proportional to the velocity gradient. Non-Newtonian fluids are those in which the viscosity at a given Pressure and temperature is a function of the velocity gradient. Such fluids as colloidal suspensions emulsions and gels are included in this classification. Non-Newtonian fluids may be further classified according to the manner in which the viscosity varies with the rate of shear. Bingham plastics, sometimes called ideal plastic; can withstand a certain amount of shearing stress. When the shearing stress has reached a certain yield value the material deforms. The ideal plastic has been deformed it viscosity is independent of the velocity gradient and is a function only of the material. The relationship between shearing stress and shearing strain is $\tau - \tau_o = \frac{\mu du}{g_c dy}$

Where τ_o is the yield stress, sewage sludge is a common example of a Bingham plastic. In most real plastic the viscosity does not be come Constant untill fairly high rates of shear are attained. Suspension of clay in water behave like real plastics and are used extensively as drilling mud in the petroleum industry. Pseudoplastic materials ae those in which the viscosity decreases with rate of shear but the material deform as soon as a shearing stress in applied. The viscosity becomes, constant at high shear rates.

Non-Newtonian fluids may be thixotropic or non thixotropic. If the fluid possesses some sort of structure which is broken down when it is subjected to shear, then an removal of the shearing stress the viscosity, insteal of being the same as at zero rate of shear will change with time as the fluid builds up the Structure of had prior being deformed. It a thixtropic fluid is tested in as apparatus in which the rate of shear can be increased the relation ship between the shear stress and rate of shear will be found to be different when the stress is increasing then when the stress is decreasing such Curves for thixotropic pseudoplastic and dilatant materials are illustrated.

1.1.3 Visco-elastic fluid model:

These fluids possess certain degree of elasticity in addition to viscosity. When a visco-elastic fluid is in motion, a certain amount of energy is stored up in the material as strain energy while some energy is lost due to viscous dissipation. In this class of fluids unlike inelastic viscous fluids, one can not neglect the strain, however small it may be as it is responsible for the recovery to the original state and for the possible reverse flow that follows the removal of the stress. During the flow the natural state of the fluid changes constantly and it tries to attain the instantaneous state or the deformed state, but it does never succeed completely. This lag is a measure of the elasticity or the so called "memory" of the fluid. But there are some fluids like soap solution, polymer solution, which have some elastic properties besides having fluid properties. Such fluids are the examples of visco-elastic fluids.

There various models for visco-elastic fluids. Examples are second order (Rivlin-Erickson fluids) Oldroyd fluids, Walters B' fluid and so on. A part from second- order visco-elastic fluid model, there are some class of visco-elastic second grade or third grade fluids.

Coleman and Noll (1960), originally suggested a constitutive equation for the incompressible visco-elastic second grade fluid, based on the postulate of fading memory as

$$T = -PI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2$$

where T is the stress tensor, P is the pressure, μ is the dynamic viscosity α_1 and α_2 are the first and second normal stress coefficients. A_1 and A_2 are the kinematic tensors, expressed as:

$$A_1 = \text{grad}(V) + (\text{grad}(V))^T$$

$$\text{and } A_2 = \frac{d}{dt}A_1 + A_1(\text{grad}V) + (\text{grad}V)^T A_1$$

where V is the velocity and $\frac{d}{dt}$ is the material time derivative.

1.1.4 Power law fluid model:

The mathematical model for describing the mechanistic behaviour of a variety of commonly used non-Newtonian fluids is the Power-law model which is also know as Ostwald-de Waele model. According to Ostwald-de Waele model, the constitutive equation is $\tau = m\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy}$ where m denotes the flow consistency index and in is the flow behaviour index. Viscosity is the ratio of shear stress to the deformation rate. For power law fluid, model it is $m\left|\frac{du}{dy}\right|^{n-1}$, known as apparent fluid viscosity. When n < 1, the model is vaild for pseudoplastic fluids such as gelatine, blood, milk etc. In these types of fluids the apparent fluid viscosity decreases with increasing deformation rate a (n < 1) and are called shear thinning fludis. When n > 1, the model is valid for dilatant fluids, such as sugar in water, aqueous suspension of rice stanch, sand etc.

1.2 The continuum concept

In many cases problems involve systems in which the dimensions are very large compared with molecular distances. One is interested in the statistical average properties and the behavior of large numbers of molecules, and not in that of individual molecules (that is, macroscopic, and not microscopic, properties are of interest).

As induvidual molecules are not being considered, the fluid can be regarded as a continuous substance. A continuum model of the fluid is adopted. Physical quantities such as the mass and momentum of the matter contained in a very small volume are regarded as being spread uniformly throughout that volume.

With normal measuring instruments (transducers, hot-wire anemom-eters), the continuous and smoothly varying properties of fluids are easily demonstrated and support the continuum hypothesis.

The sensitive volume of the instrument is usually chosen so that the property being measured does not change with the volume (the measure-ment is “local”). If the sensitive volume is reduced so much that is contains only a few molecules at the time of observation, then the measurement will vary irregularly from time to time. This is due to the statistical fluctuations in the number and kind of molecules in the sensitive volume.

Under normal conditions, a cubic millimeter of air contains 2.7×10^{16} molecules. One is usually involved with dimensions of 1 cm or more and very little variation in physical and dynamical properties of the fluid occurs over a distance of 10^{-3} cm (except perhaps in a shock wave). Thus, an instrument with a sensitive volume of 10^{-9} cm^3 would still give a measure of a local property. This volume still contains more than 10^{10} molecules of air, say at NTP, and a property average over such a number is independent of the actual number (law of large numbers). In dealing with the structure of shock fromnts, or with the flow of rarefied gases, the continuum approach of classical fluid dynamics and thermodynamics must be abandoned and replaced by the microscopic approach of kinetic theory and statistical mechanics.

In continuum mechanics one assumes that the macroscopic fluid properties, for example mean density, mean pressure and mean viscosity, vary continuously with (a) the size of the lump of fluid considered, (b) the position is the fluid system, and (c) the time. In (a), the variation becomes imperceptible when the element, or lump, is very small but still large enough to satisfy the continuum criterion. Such an element is called a fluid particle. The mean properties of the fluid particle are assigned to a point in space, so that a field representation may be used for continuum properties. Thus, fluid properties, for example density, pressure and velocity are expressed as continuous functions of position and time only. On this basis, it is possible to establish equations governing the motion of a fluid, which are independent, in their form, of the nature of the particle structure. So gases and liquids may be treated together. Consider, as an illustration, the definition of the density of a fluid at a given point, A fluid mass δm in a small volume δv around the point p (x, y, z) in a continuous fluid. The mean density of the fluid in this volume is defined as

$\delta m/\delta v$. As the volume v is allowed to shrink about. It show how $\ell = \delta m/\delta v$ varies with δv^*. When δv is shrunk below δv^*, the mean density starts to fluctuate wildly due to the fluctuation of the small number of molecules in the volume. So one cannot fix a definite value of ℓ when $\delta v \leq \delta v^*$. The density at P is defined as

$$\ell \equiv \lim_{\delta V \to \delta V^*} \frac{\delta m}{\delta V} .$$

The field representation for ℓ is written as

$$\ell = \ell(x, y, z, t) \cdot$$

This, of course, is a scalar density field. There are also vector fields such as velocity, and tensor fields such as stress.

1.3 The equation of continuity

From the continuity equation is derived from the continuum concept. A part from heterogenous and noncontinuous fluids, the equation simply expresses the law of conservation of mass. The quantity of fluid entering a certain volume in space must be balanced by that quantity leaving, unless compression occurs. Let V be an arbitrary volume fixed in space, bounded by a surface s, and containing a fluid of density ℓ. The volume element δV is small enough so that ℓ can be regarded as constant through it.

The time rate of increase of mass in v through part of its bounding surface s and leaves through another part. For an element of surface dS, the outward mass flux is $(\ell v) \cdot n\, ds$, where n is the outward directed normal. The total outward flux is

$$\int_s (\ell v) \cdot n\, ds. \qquad (1.1)$$

The sum of the net outward convection of mass plus the time rate of increase of mass in volume must be zero:

$$\int_S (\ell v)\cdot n\, ds + \frac{\partial}{\partial t}\int_v \ell dv = 0. \qquad (1.2)$$

Using Gauss's theorem on the surface integral, one has

$$\int_v \left[\frac{\partial \ell}{\partial t} + \nabla.(\ell v)\right] dv = 0. \qquad (1.3)$$

Since this is true for arbitrary elementary volumes,

$$\frac{\partial \ell}{\partial t} + \nabla.(\ell v) = 0 \qquad (1.4)$$

or

$$\frac{\partial \ell}{\partial t} + v\nabla.\ell + \ell\nabla.v = 0.$$

Therefore

$$\frac{D\ell}{Dt} + \ell\nabla.v = 0. \qquad (1.5)$$

Where $\nabla .v$ is called the dilation of the fluid at a point in the field.

This is the general continuity equation for non-homogeneous of compressible fluids.

In rectangular Cartesian coordinates, eq. (1.5) has the form

$$\frac{D\ell}{Dt} + \ell\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0. \qquad (1.6)$$

For steady motion, $\partial \ell / \partial t = 0$ and the continuity relation [from eq. (1.6)]is

$$\nabla.(\ell v) = 0. \qquad (1.7)$$

In the case of homogeneous and incompressible fluids, the continuity equation become simple

$$\nabla.v = 0. \qquad (1.8)$$

This covers the case of the ideal fluid, which is defined as being inviscid and incompressible. All real gases are compressible and liquids are slightly so. It is found, however, that as long as the Mach number does not exceed about 0.3, the fluid can be regarded as incompressible to a first approximation.

1.4 Navier-Stokes' equations

The equations of motion are derived from Newton's second law of motion which states that

Rate of change of linear momentum = Total force.

Let us consider a closed surface *S*, as earlier enclosing a volume *V* in the region occupied by the moving fluid. The rate at which momentum entering the element *dS* is $v_i\left(-p\,d\,S v_j n_j\right)$. Therefore, the rate at which the momentum enters the controlled surface *S* is

$$-\int_S v_i\left(p v_i n_i ds\right). \qquad (1.9)$$

Also the rate at which the momentum increases in the enclosed volume *V* is

$$\frac{\partial}{\partial t}\int_V p v_i dV. \qquad (1.10)$$

Hence, from (1.9) and (1.10), the rate of change of linear momentum is given by

$$\frac{\partial}{\partial t}\int_V p v_i\, dV + \int_S v_i\left(p v_i n_i\right)dS. \qquad (1.11)$$

In fluid motion it is necessary to consider the following two classes of forces, (i) forces acting throughout the mass of the body of fluid, such as gravitational forces, known as body forces and (ii) forces acting on the boundary, the fluid stresses, and are known as surface stresses. If f_i denotes the body forces per unit mass and P_i the forces on the boundary per unit area, the equation of motion can be written as

$$\frac{\partial}{\partial t}\int_V p v_i dV \qquad + \qquad \int_S v_i (p v_i n_i) dS.$$

rate at which the momentum increases in the enclosed volume *V*	rate of outflow of momentum through the controlled surface *S*

$$\int_V p f_i dV \qquad + \qquad \int_S p_i dS.$$

body forces acting on the enclosed volume *V*	surface forces acting on the controlled surface *S* (1.12)

Where the stress Vector P_i is given by

$$p_i = v_{ij} n_j, \tag{1.13}$$

and

$$v_{ij} = -p\delta_{ij} + \tau_{ij}. \tag{1.14}$$

Substituting (1.13) and (1.14) in equation (1.12) and changing the surface integrals into volume integrals by Gauss' theorem and noting that *V* is an arbitrary chosen volume, we get the equations of motion as

$$\frac{\partial}{\partial t}(p v_i) + \frac{\partial}{\partial x_j}(p v_i v_j) = p f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}. \tag{1.15}$$

∴ Using the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0,$$

$$\rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}. \tag{1.16}$$

It should be dept in mind that equation (1.16) is valid for any continuous fluid medium.

In order to use these equations to determine velocity distribution, however, we must insert expressions for the viscous stresses in terms of velocity gradients and fluid properties. For isotropic Newtonian fluid these expression are given by the constitutive equation.

$$\tau_{ij} = 2\mu\, e_{ij} - \frac{8}{3}\mu\, e_{kk}\, \delta_{ij}, \tag{1.17}$$

Where $$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{1.18}$$

Substituting (1.17), with (1.18), in equation (1.16), we finally get

$$\rho\left[\frac{\partial v_i}{\partial t} + \rho_i \frac{\partial \rho_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right)\right] \tag{1.19}$$

These are known as *Navier-Stokes equations* for the motion of a viscous compressible fluid and are three in number.

In case of incompressible fluid flow the equation of continuity is

$$\frac{\partial v_k}{\partial x_k} = 0, \tag{1.20}$$

And if μ is also regarded as constant, the equation (1.19) can be further simplified to $$\rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right] = \rho f_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \tag{1.21}$$

Keeping in view that $$\frac{\partial}{\partial x_j}\left(\frac{\partial v_j}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial v_i}{\partial x_j}\right) = 0 \tag{1.22}$$

Equation (1.21) in vector notation can be written as

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{F} - \nabla p + \mu\nabla^2\vec{V}. \tag{1.23}$$

Where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \left(\vec{V}.\nabla\right) \tag{1.24}$$

is known as the 'material derivative'.

1.5 Viscosity

Of all the fluid properties, viscosity requires the greatest consideration in the study of fluid flow. The nature and characteristics of viscosity are discussed in this section as well as dimensions and conversion factors for both absolute and kinematic viscosity. Viscosity in that property of a fluid by virtue of which it offers resistance to shear stress. Newton's law of viscosity states that for a given rate of angular deformation of fluid the shear stress is directly proportional to the viscosity. Molasses and tar are examples of highly viscous liquids; water and air have very small viscosities. The viscosity of a gas increases with temperature, but the viscosity of a liquid decreases with temperature. The variation in temperature trends may be explained upon examination of the causes of viscosity. The resistance of a fluid to shear depends upon its cohesion and upon its rate of transfer of molecular momentum. A liquid, with molecules much more closely spaced than a gas, has cohesive forces much larger than a gas. Cohesion appears to be the predominant cause of viscosity in a liquid and since cohesion decreases with temperature, the viscosity does likewise. A gas, on the other hand, has very small cohesive forces. Most of its resistance to shear stress is the result of the transfer of momentum.

As a rough model of the way in which momentum transfer gives rise to an apparent shear stress, consider two idealized railroad cars loaded with sponges and on parallel tracks, Assume each car has a water tank and pump, arranged so that the water is directed by nozzles at right angles to the track. First, consider A stationary and B moving to the right, with the water from its nozzles striking A and being absorbed by the sponges. Car A will be set in motion owing to the component of the momentum of the jets which is parallel to the tracks, giving rise to an apparent shear stress between A and B. Now if A is pumping water back into B at the same rate, its action tends to slow down B, and equal and opposite apparent shear forces result. When A and B are both stationary or have the same velocity, the pumping does not exert an apparent shear stress on either car.

Within fluid there is always a transfer of molecules back and forth across any fictitious surface drawn in it. When one layer moves relative to an adjacent layer, the molecular transfer of momentum brings momentum from one side to the other so that an apparent shear stress is set up that resists the relative motion and tends to equalize the velocities of adjacent layers in a manner analogous to that of the measure of the motion of one layer relative to an adjacent layer is du/dy.

Molecular activity gives rise to an apparent shear stress in gases which in more important than the cohesive forces, and since molecular activity increases with temperature, the viscosity of a gas also increases with temperature.

For ordinary pressures viscosity is independent of pressure and depends upon temperature only. For very great pressures gases and most liquids have shown erratic variations of viscosity with pressure.

A fluid at rest, or in motion so that no layer moves relative to an adjacent layer, will not have apparent shear forces set up, regardless of the viscosity, because du/dy is zero throughout the fluid. Hence, in the study of fluid static, on shear forces can be considered because they do not occur in a static fluid, and the only stresses remaining are normal stresses, or pressures. This greatly simplifies the study of fluid statics, since any free body of fluid can have only gravity forces and normal surface forces acting on it.

$$\mu = \frac{\tau}{du / dy}$$

1.6 Mammalian

"Tetrapods with young nourish by milk from mammary glands of females; most viperous and covered with hair; only vertebrates with only one bone in each side of lower jaw. The mammals monotremes (echidna and duckbilled platypus), marsupials (e.g. opossum, kangaroo) and placental mammals (e.g. human, whales rodents, dogs cattle, elephants, horses) that means warm blooded animals whose young are nourished by milk from the mammary glands of the female parent.

Pregnancy begins with the fertilization of the ovum and terminates with the birth of the offspring. In mammals fetilization normally occours in the ovarian end of the uterine or Fallopian tube. The fertilized ovum begins at once to undergo cleavage to form a morula mass or soild cluster of cells called blastomeres. While this early development is in progress, the ovum is carried through the uterine tube into the uterus. Cleavage of the egg in marsupial and placental mammals is holoblastic and nearly equal.

The morula of mammals differentiates into the blastocyst or blastodermic vesicle which at the time of implantation consists of an outer capsular layer and an inner central mass. The outer layer of blastomerse is known as the trophoblast, and the inner central group in referred to as the inner cell mass. A median section through the blastocyst shows the inner cell mass adhering to the trophoblast on the side and separated from the remainder of it by a fluid-filled cavity known as the blalstocoel or primitive segmentation cavity. The latter probably never exists in human development. The trophoblast is destined to form a part of one of the fetal membrance, the chorion. The inner cell mass gives rise to the various germ layers of the embryo.

From the unattached side of the inner cell mass, cells are delami- nated to from the entodermal layer. They proliferate along the inner surface of the trophoblast, line the blast cyst, and form a spherical sac, the yolk sac. The remainder of the inner cell mass is now termed the ectodermic layer. Near the region bordering the trophoblast, it hollows out to form the amniotic cavity. The ectoderm of the floor of this cavity and the entoderm of the roof of the yolk sac are now in apposition and form a cellular plate, called the embryonic disc or blastoderm, which gives rise to the embryo proper. In the center of the long axis of the disc appears an elongated thickening known as the primitive streak. Primitive streak to form mesoderm. Extra-embryoinc mesoderm fills in between the yolk sac and the trophoblast. It undergoes a splitting process into two layers, and outer or somatic layer and an inner or splanchnic layer. In primates the extra-embryonic mesoderm has a separate origin from the mesderm of the embeded in the uterine mucosa (now called the decidua) through the activity of the trophoblastic layer of calls which secrete a cytolytic enzyme thet erodes neighboring uterine tissue. Since the developing mammal depends on the mother for oxygen and nutritive substances and for the elimination of carbon dixide and other wastes of fetal metabolism, the fetal membranes (yolk sac, allantois, chorion and amnion) begin to develop at an early date. Considerable diversity exists in the origin, size, and functions of these membranes in various mammals.

1.7 Arteries

Arteries are vessels which convey blood from the heart to the tissues of the body. According to size, they are divisible into the large, the medium-sized, and the small arteries. The large arteries include the aorta, the innominate, the subclavians, the common carotids, the common iliacs, and the pulmonary artery, Nearly all the remaining named arteries are medium-sized. Small arteries are found in the tissues and organs where, for the most part, they are unnamed. According to their structure, arteries are divisible into elastic and muscular arteries.

The arterial wall consists of the coats: an inner coat (tunica intima, or interna), a middle coat (tunica media), and an outer coat (tunica externa, or adventitia). The architecture of a medium-sized artery is first described. The tunica intima consists of three strata, the innermost being a layer of endothelium, the outermost a layer of elastic tissue, the internal elastic membrane, and between the two is a layer of fine collagenic connective tissue. The tunica media, usually the thickest of the elastic tissue, and fibrous connective tissue. The tunica adventitia consists largely of fibrous connective tissue and contains small, nutrient blood vessels, the vasa vasorum. The large artery differs from the medium-sized artery in that it contains an excess of elastic tissue and proportionately les smooth muscle. In the small arteries there is a relative incresase of smooth muscle and a relative decrease of elastic tissue.

The walls of the arterioles consist of three layers, intima, media, and adventitia. Little elastic tissue is present, but smooth muscle is present in a proportionately large amount and accounts for the fact that the arteriole possesses a relatively thicker wall than any other vessel of the arterial system. The smooth muscle in the arteriole has and excellent nerve supply, and hence the central nervous system exercises an exquisite control over the caliber of its lumen.

The pulmonary artery carries venous blood from the right ventricle of the heart to the lungs. It arises from the conus arteriosus and passes almost directly dorsad, lying laterosinistrad of the ascending aorta. Upon reaching the aortic arch, the pulmonary artery is brought into close relation with the concavity of the arch by an arterial ligament, the ligamentum arteriosum the remnant of a canal, the ductus Botalli, which formed, in the fetal life of the cat, a connection between the pulmonary artery and the aorta.

The left branch of the pulmonary artery passes ventrad of the desecending aorta to the left lung. Since the left branch of the pulmonary artery divides at a point craniad of all the lobes of the lung, this lung is regarded to be hyparterial in reference to the pulmonary artery.

The right branch of the pulmonary artery turns dextrad at the conavaity of the aortic arch to run dorsad of the ascending aorta and superior vena cava. As the right branch of the pulmonary artery emerges from beneath the dorsal surface of the superior vena cava, it divides to send a branch to the cranial lobe of the right lung, and then continues caudolaterad toward the second proximal lobe of the right lung. Since the cranial lobe of the right lung lies craniad of the primary division of the rightbranch of the pulmonary artery, it is referred to as being eparterial in reference to the pulmonary artery. The other lobes of the right lung, lying caudad of the primary division of the right pulmonary branch, are hyparterial.

1.8 Blood vessels

These are ting pipes through thick blood circulates, The blood vessels supply the skin with fresh blood, which contains nutrients and oxigen, and carry away waste products, The blood in its passage from the heart to the tissues and back again passes through six principal types of vessels: the elastic arteries, the muscular arteries, the arterioles, the capillaries, the venules, and the veins. This entire system of tubes is lined by a single layer of flattended epithelial cells, the endothelium. Outside the endothelium, in all vessels except the capillaries, several layers of tissue are present. The con struction of the various vessels is related to the function they perform in the circulatory system.

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Capillaries are extremely fine vessels with a minimal luminal diameter in man of approximately eight microns. They form plexuses with one another and connect the arterioles and the venules. The wall of the typical or true capillary consists of plate like endothelial cells held together by an intercellular cement substance. In certain organs of the body such as the liver, the vascular channels which are comparable to the true capillaries in that their wall consists only of endothelium but differ from them in that they have a very wide lumen, are termed sinus oids. The capillaries have an inherent ability to contract and dilate. Capillary caliber is, to a large extent, dependent on chemical stimuli from blood and interstitial fluid; but it is also influenced by nervous stimuli and by blood pressure alterations in the larger blood vessels.

The venous ends of the capillaries converge to from venules. The typical venule consists of an endothelial tybe encased by an incomplete layer of smooth muscle cells circularly arranged and by elastic and col- lagenic fibers among which there are a few fibroblasts. The veins have thinner walls and larger lumina than the arteries, but like them they possess three coats. Veins contain less elastic and muscular tissue than the corresponding arteries. In most veins, white fibrous tissue is more abundant than all the other tissues combined; hence, the tunica adventitia is disproportionately thick. The valves which are present in some of the larger veins are formed by reduplications of the tunica intima. They are abundant in the veins of the extremities. They assist venous return against the influence of gravity. The large veins are equipped with vasa vasorum to supply oxygen and nourishment to their walls.

1.9 Stenosis

Stenosis Any part's of blood vessels are narrowing for any reason or the intravascular plaques aries in the vessels wall, then the blood flow are hamperd for its Natural activities. This process is known as stenosis.

Stenosis refers to the narrowing of blood vessel due to the development of arteriosclerotic plaques or another type of abnormal tissue development. The deposit of cholesterol and proliferation of connective tissues in an arterial wall forms plaques which grow inward into the lumen of the artery and restrict the natural blood flow. The possibility that haemodynamic factors may participate in the genesis and the proliferation of atherosclerosis has fostered increased study during the past decade. Study of suspension flows in stenosed/constricted vessels is of great medical, industrial and physiological significance.

1.10 Myocardial infarction

Although Herrick described acute coronary Thrombosis in 1912, more than half centurg elapsed before there was consensus concerning the pathogenesis of acute myocardial infarction. Only since the 1970s has plaque fissuring with overlying thrombus formation been widely recognized as the process underlying the majority of cases of myocardial infarctions, unstable angina pectoris, and sudden cardiac deth. In 1980, Dewood and coworkers demonstrated the high frequency of theromotic coronary occlusion in the early hours of acute myocardial infarction; this led to the use of intra coronary streptokinase, which was eventually superseded by intravenous Administration of thrombolytic agents to allow wider application and less delay in administration. She use of percutaneous transluminal coronary angioplasty (PTCA) after intracoronary administration of streptokinasein the setting of acute myocardial interaction was reported.

In 1982 by Meyer and coworkers. In 1983, Hartzler and coworkers first described the use of angioplasty in acute myocardial infarction without prior thrombolytic therapy.

After the major thrombolytic trials of the last decade, early reperfusion in considered crucial for reduction of infarctsize and mortality in the management of acute myocardial infraction. Contemporary thrombolytic regimens have notable deficiencies, including failure of clot lysis and incomplete reperfusion, ÔEarly hazardÕ, serious hemorrhagic complications, and frequent presence of contraindications of their use, Mechanical reperfusion strategies are in many respects complementary to the pharmacologic approach and may have much to ofer in the broader context of reperfusion therapy for acute myocardial infraction.

1.11 Cardiovascular diseases

The most common cardiovascular diseases are hypertension and heart disease but the basis for most cardiovascular diseases is atherosclerosis, which is almost universally present in U.S. adults and is manifest clinically as coronary heart disease (CHD), are brovascular disease (stroke), or peripheral arterial disease.

Heart disease and hypertension, respectively, are the third and fourth most common chronic conditions causing limitation of activity. Almost 60 percent of those with hypertension are under 65 years of age, and about 50 percent of persons with heart disease are under that age. The prevalence and mortality from the cardiovascular diseases increase with decreasing levels of family income and education.

The trend in mortality from total cardiovascular disease has been downward since about 1940, with long term declines for the three subgroupsrheumatic, cerebrovascular, and hypertensive diseases-and a decline for CHD since the mid 1960S.

Cardiovascular mortality declined just less than 1 percent per year in the 1650S and 1960S. The decline in cardiovascular mortality, including the steep rise and fall in CHD mortality, indicates that the major cause of mortality is controllable. Whether attributable more to beneficial changes in disease-promoting lifestyle or to better medical care of those already affected, it is clear that cardiovascular disease in most patients is not an inevitable burden of aging or genetic makeup. Although the causes of the decline in cardio vascular mortality are uncertain, the decline has been substantial, sustained and real. The decline has coincided with increased efforts to achieve healthier living habits and with improvements in the ambient burden of cardiovascular risk factors.

Observational studies in populations such as the Framingham study have documented factors that increase the risk of cardiovascular diseases. There include atherogenic attributes such as dyslipidemia, hypertension, glucose, intolerance and clevated fibrinogen; living habits that promote them; indicators of unstable lesions; and signs of compromised circulation e.g measures of subclinecal arterial disease. Risk factors can be classified into the lipids, metabolic factors, hemostatic factors, blood pressure and lifestyle factors. Some are modifiable. They promote cardiovascular disease in both sexes at all ages but with different strengths. Diabetes and high- density lipoprotein (HDL) cholesterol operate with greater power in women. Cigarette smoking is particularly influential in men, is noncumulative, and loses some of its adverse impact shortly after quitting. Some risk factors, such as blood lipids, impaired glucose tolerance, uric acid, and fibrinogen, have smaller risk ratios in advanced age, but this lower relative risk factors remain relevant in the elderly. obesity or weight gain promotes or aggravates all the atherogenic risk factors, and physical indolence worsens some of them and predisposes to cardiovascular events at all ages.

The major modifiable risk factors that contribute powerfully to cardiovascular disease are highly prevalent in the population. Trends in their prevalence and differences in their impact on the various atheroselerotic are noteworthy. Despite 30 years of appreciable decline in the percentage of persons who smoke cigarettes, one fourth of adults, 49 million, still smoke.

An estimated 10 million persons are at increased risk of cardiovascular disease because they have diabetes. Another highly prevalence of over weight, dyslipidemia, and hypertension and, thus, cardiovascular disease. There also are persons under 18 years of age who have one or more modifiable risk factors.

Very early symptomatic cardiovascular disease can be diagnosed by non invasive testing, such as magnetic resonance imaging (MRI) and computed tomographic (C.T) scanning wellestablished clinical indicators include left ventricular hypertroply, audible vascular bruits, a positive exercise, electro cardiogram (ECG), absent arterial. Poses in the limbs and neek, regional wall motion abnormality on the. Echocardiogram, reduced ankle-arm blood. Pressure ratio; sonographic involved of cardiotid wall thickness reduced left venticular ejection fracton, and presence of coronary calcium.

No individual risk factor is essential or sufficient in the causation of cardiovascular disease; causation is multifactorial. Indeed, the risk posed by one factor is generally enhanced in the presence of another thus multivariate risk factor assessment gives the most useful measure of the joint effect of the risk factors. Multivariate analyses help provide a better understanding of he pathogenesis of the disease and guideline for prevention. Based on the absolute, relative and attributable risks imposed by the various risk factors, the older concepts of normal have evolved to optimal values associated with long-term freedom from disease.

1.12 Stroke

Popular term for apoplexy resuling from a vascular accident in the brain, usually resulting in hemiplegia. Heat stroke is hyperpyrexia due to inhibition of heat regulation mechanism is conditions of high temperature or high humidity, or because Sweating is interfered with.

Worldwide, stroke is the second leading cause of death and it is the leading cause of permanent disability. Stroke is also among the most common indications for diagnostic imaging of brain.

The term stroke is most accurately used to describe be a clinical event that consists of the sudden onset of neurologic symptoms and use of the term implies that symptoms are caused by cerebra vascular disease (ie a " cerebravascular accident). Cerebral infraction, by contrast, is a term that describes a lethal tissue level is chemicevent that may or may not cause symptoms. Cerebral infraction accounts for approximately 85% of all strokes. Primary cerebral hemorrhage (eg. subarachnoid hemorrhage and intra parenchymal hemorrhage) account for most of the remainder.

A number of practical topics are related to clinical imaging of ischemic stroke and physicians who interpret imaging studies of the brain should be familiar with them.

Ischemic stroke is most often caused by obstruction of cerebral arteries or cerebral veins, although stroke due to obstruction of cerebral arteries is subs tactically more common than stroke duc to obstruction of cerebral veins. It is useful to consider strokes that are caused by obstruction of large cerebral arteries separately from those that are caused by obstruction of small cerebral arteries because the locations and extents of brain tissue involved by these two types of stroke are different.

A number of medical conditions are associated with atherosclerotic disease and stroke. Hypertention, smoking obesity, hyperlipidemia, diabetes, mellitus, and homocystinemia are all examples of such conditions that are risk factors for atheroselesoasis and stroke. A theroclerosis of the easotid artery is one of the most important conditions that predisposes of stroke.

Knowledge of some of the pathos-physiologic change that occurs in acute stroke can be helpful to understanding the imaging findings present in patients with acute stroke. Likewise, some of the hestopathologic findings that are seen in the first days and weeks after stroke are relevant to an understanding of the imaging findings seen in these patients.

1.13 MRI

Since its introduction as a clinically practicable diagnostic modality in the early 1980s. MRI has rapidly earned recognition as the optimal screening technique for the detection of most intracranial neoplasms. Compared with CT MRI using spin echo, gradient echo and combination spin and gradient echo pulsing sequences before and after intravenous (IV) administration of paramagnetic contrast agents provides inherently greater contrast resolution between structural abnormalities and adjacent brain parenchyma and has proved to be even more sensitive in the detection of focal lesions of the brain. Early experience suggested that 3% to 30% more focal intracranial lesions could be identified on MRI than on CT.

Lesions and tissues with increased water content appear even more conspicuous on T_2- weighted MRI images than on CT images obtained after IV infusion of contrast agent. Delination by MRI of normal and abnormal soft tissue anatomy in the posterior cranial fossa, near the base of the Skull; and in other areas of the brain that lie adjacent to dense bone is considerably better than with CT because MR1s lack of the beam of lesion localization on MRI is enhanced by its direct multiplaner hardening artifacts secondary to absorption of X-rays in bone seen on CT Accuracy capability, which permits acquisition of images in the Cornal and sagittal planes in additions to the axial plane. Conventionally used in CT MRI offers superior contrast resolution, including greater sensitivity for the detection of sub acute and chronic hemorrhage is association with tumors and other structural, lesions of the brain. The ability with MRI to visualeize vessels supplying and draining structural lesions in the brain adds yet another important dimension of information that can contribute to the diagnostic assessment.

Even with currents state of The art equipment utilizing very high magnetic fields and rapidly. Switching gradient coils, MR never the less suffers two disadvantages in comparison with CT in the assessment of in intracranial Structural abnormalities; (1) MRI requires Significantly longer image acquisition times and (2) abnormalities involving cortical bone; intratumoral calcification, and hyperacute hemorrhage are more clearly and accurately assessed with CT Newer multi-slice helical or Spiral, CT seamners are capable of providing highly collimated sub millimeter thickness sectional images in extremely short acquisition times, and thus areas of hyper ostosis or bone distraction, intratumoral calcifications, and early intratumoral or peritumoral hemorrhage are more completely defined with greater certainty on CT Than on MRI. The much faster acquisitions, capability of current CT units strongly favour their use in patients who are critically all or medically ustable. Also and other internal, paramagnetic metallic devices,, the risk of the MRI magnet interacting with such devices may preclude the use of MRI.

Given both higher cost and more restricted availability of MR equipment to date as well as continuing improvements in CT equipment and Scanning techniques that permit shorter examination times with improved spatial and contrast resolution, it is not surprising that CT remains a major imaging technique for the follow up of intracranial mass lesions. In current clinical practice, initial diagnosis know localization of brain lesions are most after accomplished with MRI but the imaging modality of convenience, for follow up studies is often CT.

1.14 Synopsis of the problems worked out in the thesis

In chapter-2 we consider the steady laminar flow of a Non-Newtonian visco- inelastic fluid of Reiner-Rivlin type through an inclined channel. The non-Newtonian parameter of this problem has been assumed to be small and the coefficient of viscosity and cross-viscosity are scalar functions of the flow invariants. The technique of successive approximations has been used to slove the problem. The expressions for the velocity distributions have been obtained and discussed are shown in tabulated form.

In Chapter-3 The oscillatory motion of a visco-elastic fluid of oldroyd model between two co-axial cylinders having deferent amplitude of frequency have been studied. It is seen that both the cylinders execute longitudinal oscillation in their own planes. The nature of velocity of fluid have been shown in tabulated form and the results are discussed.

In chapter-4 A mathematical model has been developed to study the influence of externally applied magnetic field on the blood flow through a mammalian blood vessel with slip velocity in the wall in the presence of a stenosis. Using the momentum integral technique, analytical expressions for the velocity profile, pressure gradient and skin-friction are obtained. The condition for an adverse pressure gradient is also deduced. It is observed that the slip velocity as well as the magnetic field bear the potential to influence the velocity distribution of blood to considerable extent and to reduce remarkably the pressure gradient as well as the skin friction.

CHAPTER-2

Mathematical Analysis of Steady flow of visco-inelastic fluid though an inclined channel

2.1 Introduction

Steady laminar flow of an incompressible visco-inelastic fluid between two porous infinite parallel plates with uniform suction has been discussed by Dutta [1]. Gupta [2] Presented the analysis of head loses for different types of non-Newtonian fluids including Reiner-Rivlin through channels of different cross-sections. Kapur and Gupta [3] carried out investigations about the constitutive relations for such a fluid. A comprehensive review about the works in non-Newtonian fluid can be found in [4] et al.

In the present note, it is proposed to study the steady Laminar flow of an incompressible Visco-inelastic fluid of through an inclined channel. The non-Newtonian parameter involved has been assumed to be small and the effect of this parameter on the velocity has been shown in tabulated form. It is observed that the non-Newtonian character of the fluid increases its velocity.

2.2 Rheological equations

The basic equations for an isotropic incompressible fluid of visco-inclastic type are

$$P_{ij} = p\,\delta_{ij} + 4\mu_c\, e_{ik}e_{kl}, \tag{2.1}$$

$$e_{ij} = \frac{1}{2}(k_{ij} + \omega_{j},i), \tag{2.2}$$

$$\left[\frac{\partial \omega_i}{\partial t} + \omega_j,\ \omega_{i,j}\right] = \frac{1}{\rho} P_{ik,k} + \frac{\partial \Omega}{\partial x_i} \tag{2.3}$$

$$\text{and } I_1 = \omega_{i,i} = o, \tag{2.4}$$

where P_{ij} is the stress tensor, e_{ij} is the strain rate tensor, p is an undetermined isotropic pressure to be determined by the equations of motion, ω_i is the velocity vector, δ_{ij} is the kronecker delta, ρ is the density, Ω is the gravitational potential and the co-efficient μ_c are arbitrary scalar function of the flow invariant. Also

$$I_1 = e_{ii},\ I_2 = \frac{1}{2}\ [(e_{ii})^2 - e_{ik}e_{ik}],\ I_3 = \det(e_{ik}). \tag{2.5}$$

2.3 Formation of the problem

Let us consider the flow of the non-Newtonian fluid under gravity through an inclined channel of inclination β to the horizontal. The walls of the channel are assumed to be of rectangular cross-sections and the distance between the two sides is much smaller than the others. The x-axis is taken along the lower plate and the y-axis is perpendicular to it. Let L be the distance between the two plates.

Than the equations of motion can be written from (2.1), (2.3) and (2.5) as

$$v\frac{\partial \omega}{\partial y} = -\frac{1}{\rho}\frac{\partial p^*}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \omega}{\partial y}\right) + h \sin\beta \tag{2.6}$$

$$v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p^*}{\partial y} + \frac{2}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) - h \cos\beta \tag{2.7}$$

$$\text{where } p^* = p - \mu_c\left(\frac{\partial \omega}{\partial y}\right)^2 \tag{2.8}$$

and the co-efficient μ is an arbitrary scalar function of the flow invariant.

The flow invariants in the problem are

$$I_1 = 0.\ I_2 = -\frac{1}{4}\left(\frac{\partial \omega}{\partial y}\right)^2, I_3 = 0 \tag{2.9}$$

From (2.9) it is clear that μ is a function of $\frac{\partial \omega}{\partial y}$. In the present problem, we discuss about a particular type of fluid characterized by

$$\mu = \mu_o [1 - \alpha \frac{\partial \omega}{\partial y}] \quad (2.10)$$

where μ_o and α are constants. Since the co-efficient of viscosity μ is positive, we have

$$\mu_o \geq 0,\ 1 - \alpha \frac{\partial \omega}{\partial y} \geq 0 \quad (2.11)$$

Now the equation of continuity is

$$\frac{\partial \omega}{\partial y} = o \text{ i. e. } v = \text{constant} = -v_o \quad (2.12)$$

Then from equations (2.6) and (2.7), with the help of the equations (2.10) and (2.12) we get

$$-v_o \frac{\partial \omega}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 \omega}{\partial y^2} - \alpha\ \nu \left(\frac{\partial \omega}{\partial y} \right)^2 + h \sin \beta \quad (2.13)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - h \cos \beta \quad (2.14)$$

where, $\nu = \frac{\mu_o}{\rho}$. In the equation (2.14) the term $\frac{\partial p^*}{\partial y}$ is small. Hence the pressure may be assumed to be constant along any normal and is given by its value outside the boundary layer we can therefore neglect the pressure gradient in (2.13) also.

2.4 Solution of the problem

We now introduce the following non- dimensional quantities

$$y' = \frac{y}{L}, \omega' = \frac{\omega}{U_0}, R = \frac{v_0 L}{v} \text{(Suction Reynold number)}$$

$$\alpha' = \frac{\alpha U_o}{L}, H = \frac{h \sin\beta\, L^2}{U_o} \qquad (2.15)$$

where U_0 is some typical velocity

Substituting the equation (2.15) in the equation (2.13) we get.

$$-\frac{Rv}{L} \times v \frac{\partial^2(\omega' U_0)}{\partial(y'L)} = v\left[\frac{\partial \omega' U_0}{\partial(y'L)^2}\right] - \alpha\, v \frac{\partial}{\partial(y'L)}\left\{\frac{\partial(\omega' U_0)}{\partial(y'L)}\right\}^2 + \frac{v\, HU_0}{L^2 \sin\beta} \sin\beta \text{ or}$$

$$-\frac{RvU_0}{L^2}\frac{\partial\omega'}{\partial y'} = \frac{vU_0}{L^2}\frac{\partial^2\omega'}{\partial y'^2} - \frac{\alpha' L}{U_0}\frac{vU_0^2}{L_3}\frac{\partial}{\partial y'}\left(\frac{\partial\omega'}{\partial y'}\right)^2 + H\frac{vU_0}{L^2}$$

$$\text{or } -R\frac{\partial\omega'}{\partial y'} = \frac{\partial^2\omega'}{\partial y'^2} - \alpha'\frac{\partial}{\partial y'}\left(\frac{\partial\omega'}{\partial y'}\right)^2 + H$$

$$\text{or } -R\frac{\partial\omega'}{\partial y'} = \frac{\partial^2\omega'}{\partial y'^2} + -2\alpha'\frac{\partial\omega'}{\partial y'}, \frac{\partial^2\omega'}{\partial y'^2} + H$$

On dropping the prime from both sides, we get

$$\frac{d^2\omega}{dy^2} + R\frac{d\omega}{dy} - 2\alpha\frac{d\omega}{dy}.\frac{d^2\omega}{dy^2} + H = 0 \qquad (2.16)$$

We also assume the no-slip conditions on the boundaries. i. e.

$$\omega = 0. \text{ at } y = 0, 1 \text{ and } \omega = U \quad \text{at } y = 1 \qquad (2.17)$$

To solve the equation (2.16), we assume that the non-Newtonian parameter α is very small and we expand ωs as a series expression in powers of α i. e.

$$\omega = \omega_0 + \alpha\omega_1 + \alpha^2\omega_2 + \cdots\cdots \qquad (2.18)$$

Substituting the equation (2.18) in equation (2.16), we get

$$\frac{d^2}{dy^2}(\omega_o+\alpha\,\omega_1+\alpha^2\omega_2+\cdots\cdots\cdots)+R\frac{d}{dy}(\omega_0+\alpha\omega_1+\alpha^2\omega_2+\cdots\cdots\cdots)$$

$$-2\alpha\frac{d}{dy}(\omega_o+\alpha\omega_1+\alpha^2\omega_2+\cdots\cdots)\frac{d^2}{dy^2}(\omega_o+\alpha\omega_1+\alpha^2\omega_2+\cdots\cdots)+H=0;$$

$$\text{or } \omega_o''+\alpha\,\omega_1''+\alpha^2\omega_2''+\cdots\cdots+R\omega_0'+\alpha\,R\omega_1'+\alpha^2R\omega_2'+\cdots\cdots$$

$$-2\alpha\Big((\omega_0'+\alpha\,\omega_1'+\alpha^2\,\omega_2'+\cdots\cdots\Big)\Big((\omega_0''+\alpha\,\omega_1''+\alpha^2\,\omega_2''+\cdots\cdots\Big)+H=0$$

Equating the co-efficient of like powers of α, on both sides we get

$$\left.\begin{aligned}&\omega_0''+R\omega_0'+H=0\\&\omega_0''+R\,\omega_1'-2\omega_o'\omega_0''=0\\&\omega_2''+R\omega_2'-2\,(\omega_o'\omega_1''+\omega_0''\omega_1')=0\end{aligned}\right\}\qquad(2.19)$$

and the boundary conditions (2.17) become

$$\left.\begin{aligned}&\omega_i=0\ (i=0,1,2,\cdots\cdots)\text{ at } y=0\\&\omega_i=U\ (i=0,1,2,\cdots\cdots)\text{ at } y=1\end{aligned}\right\}\qquad(2.20)$$

Now solving the equation (2.19) by using the boundary conditions (2.20) as.

$$\omega_0=\frac{H}{R}\left[\frac{e^{-Ry}-1}{e^{-Ry}-1}-y\right]\qquad(2.21)$$

$$\omega_1=\frac{H^2}{R(e^{-R}-1)^2}+\frac{2H.^2e^{-R}}{R(e^{-R}-1)^2}+\frac{H^2(1-e^{-2R})}{R(e^{-R}-1)^3}-\frac{H^2(1-e^{-2R})}{R(e^{-R}-1)^3}-\frac{2H^2.e^{-R}}{R(e^{-R}-1)^2}$$

$$-\frac{H^2e^{-3Ry}}{R(e^{-R}-1)^2}+\frac{2H^2e^{-R}e^{-Ry}}{(e^{-R}-1)^2}\qquad(2.22)$$

$$\omega_2=\frac{H^3\left(e^{-Ry}-e^{-R}\right)}{R^3(e^{-R}-1)^5}\left[\begin{aligned}&\left\{(2e^{-R}+R-1)-e^{-2R}(R+1)\right\}(2R+e^{-R}-1)\\&+(e^{-R}-1)\left\{2R(e^{-R}+R-1)+(e^{-R}-1)(4R+e^{-R}-1)\right\}\end{aligned}\right]$$

$$+\frac{H^3\left(1-e^{-Ry}\right)}{R^3\left(e^{-R}-1\right)^5}\left\{2e^{-R}+R-1)-e^{-2R}(R+1)\right\}\left\{(2R+1)e^{-2R}-e^{-R}\right\}$$

$$+2R(e^{-R}-1)\left\{(R+1)e^{3R}-e^{-2R}\right\}+(e^{-R}-1)^2\left\{(4R+1)e^{-2R}e^{-R}\right\}$$

$$+\frac{H^3}{R^3(e^{-2}-1)^4}\left\{(2e^{-R}+R-1)-e^{-2R}(R+1)\right\}\left\{2Re^{-2Ry}+(e^{-R}-1)e^{-Ry}\right\}$$

$$+\frac{H^3}{R^3\left(e^{-R}-1\right)^3}\begin{bmatrix}2R\left\{Re^{-3Ry}+(e^{-R}-1)e^{-2Ry}\right\}+(e^{-R}-1)\\ \left\{4Re^{-2Ry}+(e^{-R}-1)e^{-Ry}\right\}\end{bmatrix} \tag{2.23}$$

2.5 Numerical calculation and discussion

The difference values of ω obtain from equation (2.18) are shown for different values of α and y by the following table (where U =1, R=.3 and H =3)

α \ y	0	0.2	0.4	0.6	0.8	1.0
0	0	0	0	0	0	0
0.002	0	0.56	0.67	0.81	0.89	1
0.005	0	0.60	0.74	0.77	0.93	1
0.01	0	0.67	0.84	0.95	0.99	1

2.6 Conclusion

In order to discuss the behavior of velocity of the fluid numerically and to show the effect of non-Newtonian parameter α on it, we are to take H = 3, R =0.3 and U = 1. The nature of velocity is represented in the above table. It shows that the variations of velocity for different values of α and y. It is seen that the velocity of the fluid increases with the increases of α and y. It is observed that the non-Newtonian parameter increases the velocity of fluid near the end of the channel.

CHAPTER-3

A not eon the oscillatory motion of Visco-elastic fluid between two Co-axial circular Cylinder

3.1 Introduction

Siddappa [1] investigated oscillatory motion of a flat plate in visco-elastic fluid. Crane [2], Vleggaar [3] and Gupta and Gupta [4] also Studied the motion of an incompressible viscus fluid bounded by two infinite plates, the upper one fixed and the other performing a simple harmonic motion on its own plane. Siddappa and Khapate [5] for a special class of non-Newtonian fluids known as second-order fluids which are visco-elastic in nature. Seen [6] Studied the oscillatory motion of Rivlin-Ericksen fluid between two-Co-axial circular Cylinders. All the aspects of velocity field in the case of the flow of incompressible, viscus and electrically conducting, or non-conducting, Newtonian fluids were discussed in the above references.

In technological fields, another important class of fluids called non-Newtonian fluids, are also being Studied. The oscillatory motion in non-Newtonian fluids has also been studied by a number of research workers.

3.2 Mathematical analysis

In the present paper the oscillatory motion of a visco-elastic fluid of oldroyd model between two co-axial circular cylinders, both of which execute simple harmonic motion but have different amplitude as frequency has been discussed.

The constitutive equation of a visco-elastic fluid of oldroyds model are-

$$P_{ik} = -P\,\partial_{ik} + P'_{ik} \tag{3.1}$$

$$P'_{ik} + \lambda_1 \frac{D}{Dt} P'_{ik} + \mu_0 P'_{ij}\, e_{ik} - \mu_1 \left(P'_{ij}\, e_{jk} + P'_{jk} e_{ij} \right) + \nu_1 P'_{ji}\, e_{ji} \partial_{ik}$$

$$= 2\eta_0 \left[e_{ik} + \lambda_2 \frac{D}{Dt} e_{ik} - 2\mu_2 e_{ij} e_{jk} + \nu_2 e_{ji} e_{ij} \partial_{jk} \right] \tag{3.2}$$

where

$$\frac{D}{Dt}b_{ik} = \frac{\partial}{\partial t}b_{ik} + v_j b_{ik},j + \omega_{ij} b_{ik} + \omega_{kj} b_{ij} \tag{3.3}$$

$$e_{ik} = \frac{1}{2}\left(v_{k,j} + v_{ik}\right),\ \omega_{ik} = \frac{1}{2}\left(v_{k,i} - v_{ik}\right) \tag{3.4}$$

the equation of continuity is-

$$C_{ii} = 0 \tag{3.5}$$

where ∂_{ik} is the kroneker delta. e_{ik} the rate of strain tensor, P_{ik} the stress thensor, λ_1 the relaxation time, λ_2 the retardation time, μ_0, μ_1, μ_2, v_1 and v_2 are material constants and η_0 the co-efficient of viscosity.

The equation of motion in absence of external force is given by

$$\rho\left[\frac{\partial v_i}{\partial t} + v_{i,j}v_j\right] = P_{ji} + P'_{i,j,j}. \tag{3.6}$$

3.3 Formulation of the problem

Let $(r_1\theta,z)$ be the cylindrical polar Co- ordinates and (u,v,w) be the components of velocity along the direction of r, θ and z respectively. let the common axis of the cylinders coincide with the Z-axis. It is assumed from the nature of the problems that all entities depends on radial co-ordinate r and time t only due to infinite lenth of the cylinders having radius a and b, (b > a).

The only equation of motion can be written in the form-

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\frac{\partial w}{\partial t} = \nu\left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{3.7}$$

The boundary conditions are-

$$\left.\begin{array}{ll} w = u_1 e^{i\omega t} & \text{at } r = a \\ w = u_2 e^{i\Omega t} & \text{at } r = b \end{array}\right\} \tag{3.8}$$

where (u_1, ω) and (u_2,Ω) are respectively the amplitude and frequency of two cylinders.

We assume the solution of the equation (3.7) as

$$w = u_1 f(r)\, e^{i\omega t} + u_2\, g(r) e^{i\Omega t} \tag{3.9}$$

Which is evidently separable as periodic in t Now with the help of (3.9) and (3.7) we get

$$f'' + \frac{1}{r} f' - m^2 f = 0 \tag{3.10}$$

$$g'' + \frac{1}{r} g' - n^2 g = 0 \tag{3.11}$$

where $m^2 = \dfrac{i\omega(1+\lambda_1 i\omega)}{\nu(1+\lambda_2 i\omega)}$ and $n^2 = \dfrac{i\Omega(1+\lambda_1 i\omega)}{\nu(1+\lambda_2 i\omega)}$

The transformed boundary Conditions are

$$\left.\begin{array}{ll} f(r) = 1 & \text{at } r = a \\ f(r) = 0 & \text{at } r = b \end{array}\right\} \tag{3.12}$$

and

$$\left.\begin{array}{ll} g(r) = 0 & \text{at } r = a \\ g(r) = 1 & \text{at } r = b \end{array}\right\} \tag{3.13}$$

The solution of (3.10) Subject to the boundary condition (3.12) is

$$f(r) = \frac{I_0(mr)K_0(mb) - K_0(mr)I_0(mb)}{I_0(ma)K_0(mb) - K_0(ma)I_0(mb)}$$

Again the solution of (3.11) subject to the boundary condition (3.13) is

$$g(r) = \frac{I_0(nr)K_0(na) - K_0(nr)I_0(na)}{I_0(nb)K_0(na) - K_0(nb)I_0(na)}$$

Thus the solution of the given problem is

$$W = \frac{u_1\{I_0(mr)K_0(mb) - K_0(mr)I_0(mb)\}}{I_0(ma)K_0(mb) - K_0(ma)I_0(mb)} e^{i\omega t}$$

$$+ \frac{u_2\{I_0(nr)K_0(na) - K_0(na)I_0(mb)\}}{I_0(nb)K_0(na) - K_0(nb)I_0(na)} e^{i\Omega t}$$

For small frequency:

ω is small e.i m is small I_0 (mr) K_0 (mb) = I_0 (mr)

$$\left[-\left\{In\left(\frac{1}{2}mb\right)+\nu\right\}I_0(mb)+\frac{1}{4}m^2b^2+\dots\dots\dots\dots\right]$$

$$= -\left\{In\left(\frac{1}{2}mb\right)+\nu\right\}I_0(mb)I_0(mr)+\frac{1}{4}m^2b^2I_0(mr)$$

Where $K_0(mb) = -\left\{In\left(\frac{1}{2}mb\right)+\nu\right\}I_0(mb)+\left(\frac{1}{2}mb\right)^2+\dots\dots\dots\dots$

Similarly $I_0(mb)K_0(mr) = -\left\{In\left(\frac{1}{2}mr\right)+\nu\right\}I_0(mb)I_0(mr)+\frac{1}{4}m^2r^2I_0(mb)$

$\therefore$ I_0 (mr) K_0 (mb) – I_0 (mb) K_0 (mr)

$$= I_0(mb)I_0(mr)In\left(\frac{r}{b}\right)+\frac{1}{4}m^2\{b^2I_0(mr)-r^2I_0(mb)\} \tag{3.14}$$

Let $I_0(z) = \sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}z\right)^{2r}}{r\,\rceil r+1} = 1+\frac{\left(\frac{1}{2}z\right)^2}{\rceil 2}+\dots\dots\dots\dots\dots\dots\dots\dots$

$$= 1+\frac{1}{4}z^2+\dots\dots\dots\dots\dots\dots\dots$$

$$\therefore I_0(mr)I_0(mb) = \left(1+\frac{1}{4}m^2r^2+\ldots\ldots\ldots\ldots\right)\left(1+\frac{1}{4}m^2b^2+\ldots\ldots\ldots\ldots\ldots\right)$$

$$= \left[1+\frac{1}{4}m^2\left(b^2+r^2\right)+\ldots\ldots\ldots\ldots\ldots\right]$$

and $\frac{1}{4}m^2\left\{b^2I_0(mr)-r^2I_0(mb)\right\}$

$$= \frac{1}{4}m^2\left\{b^2\left(1+\frac{1}{4}m^2r^2+\ldots\ldots\ldots\ldots\ldots\right)-r^2\left(1+\frac{1}{4}m^2b^2+\ldots\ldots\ldots\ldots\ldots\right)\right\}$$

$$= \frac{1}{4}m^2\left(b^2-r^2\right)+\ldots\ldots\ldots\ldots\ldots \text{(neglecting the higher powers of m)}$$

From (3.14) We get

$$\therefore I_0(mr)K_0(mb)-I_0(mb)K_0(mr)$$

$$= \left\{1+\frac{1}{4}m^2\left(b^2+r^2\right)\right\}In\left(\frac{r}{b}\right)+\frac{1}{4}m^2\left(b^2-r^2\right)$$

Similarly we have

$$\therefore I_0\,(ma)\;K_0\,(mb)\text{-}I_0\,(mb)\;K_0\,(ma)$$

$$= \left\{1+\frac{1}{4}m^2\left(b^2+a^2\right)\right\}In\left(\frac{a}{b}\right)+\frac{1}{4}m^2\left(b^2-a^2\right)$$

Now

$$\frac{I_0(mr)K_0(mb)-I_0(mb)K_0(mr)}{I_0(ma)K_0(mb)-I_0(mb)K_0(ma)}e^{i\omega t}$$

$$= \frac{\left\{1+\frac{1}{4}m^2\left(b^2+r^2\right)\right\}In\left(\frac{r}{a}\right)+\frac{1}{4}m^2\left(b^2-r^2\right)}{\left\{1+\frac{1}{4}m^2\left(b^2+a^2\right)\right\}In\left(\frac{a}{b}\right)+\frac{1}{4}m^2\left(b^2-a^2\right)}e^{i\alpha t} \qquad (3.15)$$

similarly we have

$$\frac{I_0(nr)K_0(na)-k_0(nr)I_0(na)}{I_0(nb)K_0(na)-k_0(nb)I_0(na)}e^{i\Omega t}$$

$$=\frac{\left\{1+\frac{1}{4}n^2\left(a^2+r^2\right)\right\}\text{In}\left(\frac{r}{a}\right)+\frac{1}{4}n^2\left(a^2-r^2\right)}{\left\{1+\frac{1}{4}n^2\left(a^2+r^2\right)\right\}\text{In}\left(\frac{b}{a}\right)+\frac{1}{4}n^2\left(a^2-b^2\right)}e^{i\Omega t} \tag{3.16}$$

From the equation (3.15) we get,

$$\frac{\left\{1+\frac{1}{4}m^2\left(b^2+r^2\right)\right\}\text{In}\left(\frac{r}{b}\right)+\frac{1}{4}m^2\left(b^2-r^2\right)}{\left\{1+\frac{1}{4}m^2\left(b^2+a^2\right)\right\}\text{In}\left(\frac{a}{b}\right)+\frac{1}{4}m^2\left(b^2-a^2\right)}$$

$$=\frac{\left\{1+\frac{1}{4}m^2\left(b^2+r^2\right)\right\}\text{In}\left(\frac{r}{b}\right)+\frac{1}{4}m^2\left(b^2-r^2\right)}{\text{In}\left(\frac{a}{b}\right)\left[1+\frac{1}{4}m^2\left(b^2+a^2\right)+\frac{1}{4}m^2\left(b^2-a^2\right)\frac{1}{\text{In}\left(\frac{a}{b}\right)}\right]}$$

$$=\left[\left\{1+\frac{1}{4}m^2\left(b^2+r^2\right)\right\}\frac{\text{In}\left(\frac{r}{b}\right)}{\text{In}\left(\frac{a}{b}\right)}+\frac{1}{4}m^2\left(b^2-r^2\right)\frac{1}{\text{In}\left(\frac{a}{b}\right)}\right]\times$$

$$\left[1+\frac{1}{4}m^2\left(b^2+a^2\right)+\frac{1}{4}m^2\left(b^2-a^2\right)\frac{1}{\text{In}\left(\frac{a}{b}\right)}\right]^{-1}$$

$$=\left[\left\{1+\frac{1}{4}m^2\left(b^2+r^2\right)\right\}\frac{\ln\left(r/b\right)}{\ln\left(a/b\right)}+\frac{1}{4}m^2\left(b^2-r^2\right)\frac{1}{\ln\left(\frac{a}{b}\right)}\right]$$

$$\left[1-\frac{1}{4}m^2\left(b^2+a^2\right)-\frac{1}{4}m^2\left(b^2-a^2\right)\frac{1}{\ln\left(\frac{a}{b}\right)}\cdots\cdots\cdots\right]$$

$$=\left\{1+\frac{1}{4}m^2\left(b^2+r^2\right)\right\}\frac{\ln\left(r/b\right)}{\ln\left(a/b\right)}+\frac{1}{4}m^2\left(b^2-r^2\right)\frac{1}{\ln\left(\frac{a}{b}\right)}$$

$$-\frac{1}{4}m^2\left(b^2+a^2\right)\frac{\ln\left(r/b\right)}{\ln\left(a/b\right)}-\frac{1}{4}m^2\left(b^2-r^2\right)\frac{\ln\left(r/b\right)}{\ln\left(a/b\right)^2}$$

$$=\left\{1+\frac{1}{4}m^2\left(r^2-a^2\right)\right\}\frac{\ln\left(r/b\right)}{\ln\left(a/b\right)}+\frac{1}{4}m^2\left(b^2-r^2\right)\frac{1}{\ln\left(\frac{a}{b}\right)}-\frac{1}{4}m^2\left(b^2-a^2\right)\frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)^2}$$

(3.17)

∴ Again from (3.15) with the help of the equation (3.17)

$$\frac{I_0(mr)K_0(mb)-K_0(mr)I_0(mb)}{I_0(ma)K_0(mb)-K_0(ma)I_0(mb)}e^{i\omega t}$$

$$=\left[\left\{1+\frac{1}{4}m^2\left(r^2-a^2\right)\right\}\frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)}+\frac{1}{4}m^2\left(b^2-r^2\right)\frac{1}{\ln\left(\frac{a}{b}\right)}-\frac{1}{4}m^2\left(b^2-a^2\right)\frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)^2}\right]e^{i\omega t}$$

Let $\frac{I_0(mr)K_0(mb)-K_0(mr)I_0(mb)}{I_0(ma)K_0(mb)-K_0(ma)I_0(mb)}e^{i\omega t}=F(m)$

F(m)

$$= \left[\left\{ 1 + \frac{1}{4} m^2 \left(r^2 - a^2 \right) \right\} \frac{\operatorname{In}\left(b/r \right)}{\operatorname{In}\left(b/a \right)} - \frac{1}{4} m^2 \left(b^2 - r^2 \right) \frac{1}{\operatorname{In}\left(\frac{b}{a} \right)} + \frac{1}{4} m^2 \left(b^2 - a^2 \right) \frac{\operatorname{In}\left(b/r \right)}{\operatorname{In}\left(b/a \right)^2} \right] e^{i\omega t}$$

Similarly from (3.16) we have

G (n)

$$= \left[\left\{ 1 + \frac{1}{4} n^2 \left(r^2 - b^2 \right) \right\} \frac{\operatorname{In}\left(a/r \right)}{\operatorname{In}\left(a/b \right)} - \frac{1}{4} n^2 \left(a^2 - r^2 \right) \frac{1}{\operatorname{In}\left(\frac{a}{b} \right)} + \frac{1}{4} n^2 \left(a^2 - b^2 \right) \frac{\operatorname{In}\left(a/r \right)}{\operatorname{In}\left(a/b \right)^2} \right] e^{i\Omega t}$$

where $G\,(n) = \dfrac{I_0(nr)K_0(na) - K_0(nr)I_0(na)}{I_0(nb)K_0(na) - K_0(nb)I_0(na)} e^{i\omega t}$

For Large frequency

ω is large, i.e. m is large

$$I_0(z) = \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 + \frac{1}{8z} + \frac{9}{128z} + \dots\dots\dots\dots \right\}$$

$$K_0'(z) = \left(\frac{\pi}{2z} \right)^{1/2} e^{-z} \left\{ 1 - \frac{1}{8z} + \frac{9}{128z^2} - \dots\dots\dots\dots \right\}$$

$$\therefore I_0(mr)K_0(mb) = \frac{e^{mr}}{(2\pi mr)^{1/2}} \left\{ 1 + \frac{1}{8mr} + \frac{9}{128m^2r^2} + \dots\dots \right\} \times$$

$$\left\{ \left(\frac{\pi}{2mb} \right)^{1/2} e^{-mb} \left(1 - \frac{1}{8mb} + \dots\dots \right) \right\}$$

$$= \frac{e^{m(r-b)}}{2m\sqrt{br}} \left\{ 1 + \frac{1}{8m} \left(\frac{1}{r} - \frac{1}{b} \right) + \dots\dots\dots\dots \right\}$$

$$= \frac{e^{-m(b-r)}}{2m\sqrt{br}}\left(1+\frac{b-r}{8mbr}\text{....................}\right)$$

$$\therefore I_0(mb)K_0(mr) = \frac{e^{m(b-r)}}{2m\sqrt{br}}\left\{1+\frac{r-b}{8mbr}+\text{............}\right\}$$

$$\therefore I_0(mr)K_0(mb) - I_0(mb)K_0(mr)$$

$$= \left[e^{-m(b-r)}\left\{1+\frac{b-r}{8mbr}\right\} - e^{m(b-r)}\left\{1+\frac{r-b}{8mbr}\right\}\right]\frac{1}{2m\sqrt{br}}$$

Similarly, we have

$$I_0(ma)K_0(mb) - I_0(mb)K_0(ma)$$

$$= \left[e^{-m(b-a)}\left\{1+\frac{b-a}{8ba}\right\} - e^{m(b-a)}\left\{1+\frac{a-b}{8mba}\right\}\right]\frac{1}{2m\sqrt{ba}}$$

$$\therefore \frac{I_0(mr)K_0(mb) - I_0(mb)K_0(mr)}{I_0(ma)K_0(mb) - I_0(mb)K_0(ma)}$$

$$= \frac{\left[e^{-m(b-r)}\left(1+\frac{b-r}{8mbr}\right) - e^{m(b-r)}\left(1-\frac{b-r}{8mbr}\right)\right]\frac{1}{2m\sqrt{br}}}{\left[e^{-m(b-r)}\left(1+\frac{b-a}{8mba}\right) - e^{m(b-a)}\left(1-\frac{b-a}{8mba}\right)\right]\frac{1}{2m\sqrt{ba}}}$$

$$= \sqrt{\frac{a}{r}}\,\frac{e^{m(b-r)}\left\{1-\frac{b-r}{8mbr}\right\}\left[1-e^{-2m(b-r)}\left(1+\frac{b-r}{8mbr}\right)\left(1+\frac{b-r}{8mbr}\right)\right]}{e^{m(b-r)}\left\{1-\frac{b-a}{8mba}\right\}\left[1-e^{-2m(b-a)}\left(1+\frac{b-a}{8mba}\right)\left(1+\frac{b-a}{8mba}\right)\right]}$$

$$= \sqrt{\frac{a}{r}}\ e^{-m(r-a)}\left(1-\frac{b-r}{8mbr}\right)\left(1+\frac{b-a}{8mba}\right)\times$$

$$\left[1-e^{-2m(b-r)}\left(1+\frac{b-r}{8mbr}\right)^2\right]\left[1+e^{-2m(b-a)}\left(1+\frac{b-a}{8mba}\right)^2\right]$$

$$= \sqrt{\frac{a}{r}}\ e^{-m(r-a)}\left(1+\frac{r-a}{8mar}\right)\left[1-e^{-2m(b-r)}-\frac{b-r}{4mbr}e^{-2m(b-r)}\right]$$

$$\left[1+e^{-2m(b-a)}+\frac{b-a}{4mba}e^{-2m(b-a)}\right]$$

(Negating the higher powers of m)

$$\therefore F(m) = \sqrt{\frac{a}{r}}\ e^{-m(r-a)}\left(1+\frac{r-a}{8mar}\right)\left[1+e^{-2m(b-a)}-e^{-2m(b-r)}-e^{-2m(2b-r-a)}\right]e^{i\omega t}$$

$$F(m) = \sqrt{\frac{a}{r}}\ e^{-m(r-a)}\left[1+e^{-2m(b-a)}-e^{-2m(b-r)}-e^{-2m(2b-r-a)}\right]e^{i\omega t}$$

(Sine m is large $\therefore \frac{1}{m}$ is small)

Similarly we have

$$G(n) = \sqrt{\frac{b}{r}}\ e^{-n(r-b)}\left[1+e^{-2n(a-b)}-e^{-2n(a-r)}-e^{-2n(2a-r-b)}\right]e^{i\omega t}$$

We have

$$m^2 = \frac{i\omega(1+\lambda_1 i\omega)}{\nu(1+\lambda_2 i\omega)} = \frac{(i\omega+\lambda_1\omega^2)(1-\lambda_1 i\omega)}{\nu(1+\lambda_2^2\omega^2)}$$

$$= \frac{(\lambda_2-\lambda_1)\omega^2 + i(1+\lambda_1\lambda_2\omega^2)\omega}{\nu(1+\lambda_2^2\omega^2)}$$

$$= \frac{(\lambda_2-\lambda_1)\omega^2}{\nu(1+\lambda_2^2\omega^2)} + i\frac{(1+\lambda_1\lambda_2\omega^2)\omega}{\nu(1+\lambda_2^2\omega^2)}$$

$$\therefore m = \left\{ \frac{(\lambda_2 - \lambda_1)}{\nu\left(1+\lambda_2^2\omega^2\right)} + i\frac{\left(1+\lambda_1\lambda_2\omega^2\right)\omega}{\nu\left(1+\lambda_2^2\omega^2\right)} \right\}^{\frac{1}{2}} \quad \therefore\ m = m_1 + im_2 \text{ (Say)}$$

Similarly we have

$$n = \left\{ \frac{(\lambda_2 - \lambda_1)\Omega^2}{\nu\left(1+\lambda_2^2\Omega^2\right)} + i\frac{(1+\lambda_1\lambda_2\Omega)}{\nu\left(1+\lambda_2^2\Omega^2\right)} \right\}^{\frac{1}{2}} = n_1 + in_2 \text{ (Say)}$$

$$\text{Let} \left[\frac{(\lambda_2 - \lambda_1)\omega^2}{\nu\left(1+\lambda_2^2\omega^2\right)} + i\frac{(1+\lambda_1\lambda_2\omega^2)\omega}{\nu\left(1+\lambda_2^2\omega^2\right)} \right]^{\frac{1}{2}} = m_1 + im_2$$

$$\therefore \frac{(\lambda_2 - \lambda_1)\omega^2}{\nu\left(1+\lambda_2^2\omega^2\right)} + i\frac{(1+\lambda_1\lambda_2\omega^2)\omega}{\nu\left(1+\lambda_2^2\omega^2\right)} = m_1^2 - m_2^2 + i_2 m_1 m_2$$

$$m_1^2 - m_2^2 = \frac{\left(\lambda_2 - \lambda_1\right)\omega^2}{\nu\left(1+\lambda_2^2\omega^2\right)} = \alpha \text{ (say)} \tag{3.18}$$

$$2m_1m_2 = \frac{\left(1+\lambda_1 - \lambda_2\omega^2\right)\omega}{\nu\left(1+\lambda_2^2\omega^2\right)} = \beta \text{ (say)} \tag{3.19}$$

$$\therefore \left(m_1^2 + m_2^2\right)^2 = \left(m_1^2 - m_2^2\right) + 4m_1^2m_2^2 = \alpha^2 + \beta^2$$

$$\therefore\ m^2 + m_2^2 = \sqrt{\alpha^2 + \beta^2} \tag{3.20}$$

From (3.18) and (3.19) we have

$$m_1^2 = \frac{\sqrt{\alpha^2 + \beta^2} + \alpha}{2}$$

$$\therefore\ m_1 = \sqrt{\frac{1}{2}\left(\sqrt{\alpha^2 + \beta^2} + \alpha\right)}$$

$$\text{and } m_2 = \sqrt{\frac{1}{2}\left(\sqrt{\alpha^2 + \beta^2} - \alpha\right)}$$

$$\text{i.e.} m_1 m_2 = \left[\frac{1}{2}\left\{\sqrt{\alpha^2+\beta^2} \pm \alpha\right\}\right]^{\frac{1}{2}} \quad \text{(Putting the value of } \alpha \text{ and } \beta)$$

$$\therefore\ m_1 m_2 = \left[\left\{\sqrt{\left(1+\lambda_1^2\omega^2\right)\left(1+\lambda_2^2\omega^2\right)} \pm (\lambda_1 - \lambda_1)\omega\right\}\omega/2\nu\left(1+\lambda_2^2\omega^2\right)\right]^{\frac{1}{2}}$$

Similarly we have

$$n_1 n_2 = \left[\left\{\frac{\sqrt{\left(1+\lambda_1^2\Omega^2\right)\left(1+\lambda_2^2\Omega^2\right)} \pm (\lambda_2 - 1)\Omega}{2\nu\left(1+\lambda_2^2\Omega^2\right)}\right\}^{\Omega}\right]^{\frac{1}{2}}$$

For Small Frequency

We have

$$F(m) = \left[\left\{1 + \frac{1}{4}\left\langle \frac{(\lambda_2 - \lambda_1)\omega^2}{\nu\left(1+\lambda_2^2\omega^2\right)} + \frac{\left(1+\lambda_1\lambda_2\omega^2\right)\omega}{\nu\left(1+\lambda_2^2\omega^2\right)} \right\rangle\right\}\right.$$

$$\left(r^2 - a^2\right)\frac{\ln\frac{b}{r}}{\ln\frac{b}{r}} - \frac{1}{4}\left\langle \frac{(\lambda_2 - \lambda_1)\omega^2}{\nu\left(1+\lambda_2^2\omega^2\right)} + i\frac{\left(1+\lambda_1\lambda_2\omega^2\right)\omega}{\nu\left(1+\lambda_2^2\omega^2\right)} \right\rangle$$

$$\left(b^2 - r^2\right)\frac{1}{\ln b/a} + \frac{1}{4}\left\langle \frac{(\lambda_2 - \lambda_1)\omega^2}{\upsilon\left(1+\lambda_2^2\omega^2\right)} + i\frac{\left(1+\lambda_1\lambda_2\omega^2\right)\omega}{\upsilon\left(1+\lambda_2^2\omega^2\right)} \right\rangle$$

$$\left.\left(b^2 - a^2\right)\frac{\ln b/r}{\left(\ln b/a\right)^2}\right] (\cos\omega t + i\sin\omega t)$$

Taking real part only, we get

$$F(m) = \cos\omega t\left[1 - \frac{(\lambda_2 - \lambda_1)\omega^2}{\nu(1+\lambda_2^2\omega^2)}\frac{\left(r^2 - a^2\right)}{4}\frac{\ln\frac{b}{r}}{\ln\frac{b}{r}} - \frac{(\lambda_2 - \lambda_1)\omega^2}{\nu\left(1+\lambda_2^2\omega^2\right)}\frac{\left(r^2 - a^2\right)}{4}\frac{1}{\ln\frac{b}{r}} + \right.$$

$$\frac{(\lambda_2-\lambda_1)\omega^2}{\nu(1+\lambda_2^2\omega^2)}\frac{(b^2-a^2)}{4}\frac{\ln\frac{b}{a}}{\left(\ln\frac{b}{a}\right)^2}\Bigg] - \operatorname{Sin}\omega t\left[\frac{(1+\lambda_1\lambda_2\omega^2)\omega}{\nu(1+\lambda_2^2\omega^2)}\frac{(b^2-a^2)}{4}\frac{\ln b/r}{\ln b/a} -\right.$$

$$\left.\frac{(1+\lambda_1\lambda_2\omega^2)\omega}{\upsilon(1+\lambda_2^2\omega^2)}\frac{(b^2-r^2)}{4}\frac{1}{\ln\frac{b}{a}} - \frac{(1+\lambda_1\lambda_2\omega^2)\omega}{\nu(1+\lambda_2^2\omega^2)}\frac{(b^2-a^2)}{4}\frac{\ln b/r}{(\ln b/a)^2}\right]$$

$$G(n) = \operatorname{Cos}\Omega t\left[1+\frac{(\lambda_2-\lambda_1)\Omega^2}{\nu(1+\lambda_2^2\Omega^2)}\frac{(r^2-b^2)}{4}\frac{\ln a/r}{\ln a/b} - \frac{(\lambda_2-\lambda_1)\Omega^2}{\nu(1+\lambda_2^2\Omega^2)}\frac{(a^2-r^2)}{4}\frac{1}{\ln\frac{a}{b}} +\right.$$

$$\left.\frac{(\lambda_2-\lambda_1)\Omega^2}{\nu(1+\lambda_2^2\Omega^2)}\frac{(a^2-b^2)}{4}\frac{\ln a/r}{(\ln a/b)^2}\right] - \operatorname{Sin}\Omega t\left[\frac{(1-\lambda_1\lambda_2\Omega^2)\Omega}{\nu(1+\lambda_2^2\Omega^2)}\frac{(r^2-b^2)}{4}\frac{\ln a/r}{\ln a/b} -\right.$$

$$\left.\frac{(1+\lambda_1\lambda_2\Omega^2)w}{\upsilon(1+\lambda_2^2\Omega^2)}\frac{(a^2-r^2)}{4}\frac{1}{\ln a/b} + \frac{(1+\lambda_1\lambda_2\Omega^2)\Omega}{\upsilon(1+\lambda_2^2\Omega^2)}\frac{(a^2-b^2)}{4}\frac{\ln a/r}{(\ln a/b)^2}\right]$$

$$\therefore\ W = u_1\ F(m) + u_2\ G(n)$$

For Large frequency:

$$F(m) = \sqrt{\frac{a}{r}}\ e^{-m(r-a)}\left[1+e^{-2m(b-a)} - e^{-2m(b-r)} - e^{-2m(2b-r-a)}\right]e^{iwt}$$

$$\therefore F(m) = \sqrt{\frac{a}{r}}\ e^{-(m_1+im_2)(r-a)}\left[1+e^{-2(m_1+im_2)(b-a)} - e^{-2(m_1+im_2)(b-a)}\right.$$

$$\left.-e^{-2(m_1+im_2)(2b-r-a)}\right]$$

$$(\operatorname{Cos}\omega t + i\ \operatorname{Sin}\omega t)$$

$$= \sqrt{\frac{a}{r}}\; e^{-m_1(r-a)}\{\text{Cos}m_1(r-a) - \text{Sin}m_2(r-a)\}\Big[1 + e^{-2m_1(b-a)}\{\text{Cos}2m_2(b-a) -$$

$$i\sin 2m_2(b-a)\} - e^{-2m_2(b-r)}\{\text{Cos}2m_2(b-r) - i\,\text{Sin}2m_2(b-r)\} -$$

$$e^{-2m_1(2b-r-a)}\{\text{Cos}2m_2(2b-r-a) - i\,\text{Sin}2m_2(2b-r-a)\}\Big](\text{Cos}\omega t + i\,\text{Sin}\omega t)$$

3.4 Numerical calculation

For small frequency:

$$F(m) = \text{Cos}\,\omega t\left[1 + \frac{1}{4}\frac{(\lambda_2-\lambda_1)\omega^2}{\nu(1+\lambda_2^2\omega^2)}\left\{(r^2\text{-}a^2)\frac{In\,b/r}{In\,b/a} - \frac{(b^2-r^2)}{In\,b/a} + (b^2-a^2)\frac{In\,b/r}{\left(In\,b/a\right)^2}\right\}\right]$$

$$-\text{Sin}\omega t\left[\frac{(1+\lambda_1\lambda_2\omega^2)\omega^2}{4\nu(1+\lambda_2^2\omega^2)}\left\{(r^2-a^2)\frac{In\,b/r}{In\,b/a} - \frac{(b^2-r^2)}{In\,b/a} + (b^2-a^2)\frac{In\,b/r}{\left(In\,b/a\right)^2}\right\}\right]$$

Put $\omega t = \dfrac{\pi}{2}$

Let $\dfrac{b}{a} = C$ and $\dfrac{r}{a} = R$ $\qquad \therefore \dfrac{r}{b} = \dfrac{r}{a} = \dfrac{a}{b} = \dfrac{R}{C}$

$$\therefore F(m) = \text{-}\frac{1}{4}\frac{(1+\lambda_1\lambda_2\omega^2)\omega}{\nu(1+\lambda_2^2\omega^2)}\left[(r^2-a^2)\frac{In\,b/r}{In\,b/a} - \frac{(b^2-r^2)}{In\,b/a} + (b^2-a^2)\frac{In\,b/r}{\left(In\,b/a\right)^2}\right]$$

$$= -\frac{1}{4}\frac{\omega a^2}{\nu}\frac{(1+\lambda_1\lambda_2\omega^2)}{(1+\lambda_2^2\omega^2)}\left[(R^2-1)\frac{In\,C/R}{In\,C} - \frac{(C^2-R^2)}{In\,C} + (C^2-1)\frac{In\,C/R}{(In\,C)^2}\right] \quad 1 \le R \le 2$$

Put $C = 2$, $\dfrac{\omega a^2}{\nu} = 1$ and $\omega = 10$

$$\therefore F(m) = \text{-}\frac{1}{4}\frac{(1+\lambda_1\lambda_2 100)}{(1+\lambda_2^2 100)}\left[(R^2-1)\frac{In\,R/2}{In\,2} - \frac{(4-R^2)}{In\,2} - \frac{In\,R/2}{(In\,2)^2}\right]$$

$$F(m) = -\frac{1}{4}\frac{(1+100\lambda_1\lambda_2)}{(1+\lambda_2^2 100)} f(m)$$

where $f(m) = -(R^2-1)\frac{\text{In}\, R/2}{\text{In}\,2} - \frac{(4-R^2)}{\text{In}\,2} - \frac{3\text{In}\, R/2}{(\text{In}2)^2}$

$$= \frac{1}{(\text{In}2)^2}\left\{-(R^2-1)\text{In}\, R/2\ \text{In}\,2 - (+R^2)\text{In}\,2 - 3\text{In}\, R/2\right\}$$

when R = 1 f(m) = 0

R = 1.2 f(m) = -.1793846193

R = 1.4 f(m) = -.1819404329

R = 1.6 f(m) = -.1819404329

R = 1.8 f(m) = -.09807897429

R = 2 f(m) = 0

Table-1

R	1	1.2	1.4	1.6	1.8	2	
$\lambda_1 = 1$ $\lambda_2 = 1$	0	0.04484615483	0.05549778565	0.04548520823	0.02451974357	0	← F(m)
$\lambda_1 = 1$ $\lambda_2 = 5$	0	0.001811060231	0.002241214055	0.001836867665	0.0009902015592	0	← F(m)
$\lambda_1 = 1$ $\lambda_2 = 10$	0	0.0004529008736	0.005554772866	0.004552614082	0.002454180914	0	← F(m)
$\lambda_1 = 5$ $\lambda_2 = 1$	0	0.2224546888	0.2752909961	0.2256246467	0.1216276389	0	← F(m)
$\lambda_1 = 10$ $\lambda_2 = 1$	0	0.0444653562	0.00488651233	0.005554772866	0.002454180914	0	← F(m)

Again

$$G(n) = -\left[\frac{\left(1+\lambda_1\lambda_2\Omega^2\right)}{\nu\left(1+\lambda_2^2\Omega^2\right)}\Omega\frac{r^2-b^2}{4}\frac{\ln a/r}{\ln a/b} - \frac{\left(1+\lambda_1\lambda_2\Omega^2\right)}{\nu\left(1+\lambda_2^2\Omega^2\right)}\Omega\frac{a^2-r^2}{4}\frac{1}{\ln a/b} + \right.$$

$$\left.\frac{\left(1+\lambda_1\lambda_2\Omega^2\right)}{\nu\left(1+\lambda_2^2\Omega^2\right)}\Omega\frac{a^2-b^2}{4}\frac{\ln a/r}{\left(\ln a/b\right)^2}\right]$$

$$= -\frac{1}{4}\frac{\left(1+\lambda_1\lambda_2\Omega^2\right)\Omega b^2}{\nu\left(1+\lambda_2^2\Omega^2\right)}\left[\left(\frac{r^2}{b^2}-1\right)\frac{\ln a/r}{\ln a/b} - \left(\frac{a^2}{b^2}-\frac{r^2}{b^2}\right)\frac{1}{\ln a/b} + \left(\frac{a^2}{b^2}-1\right)\frac{\ln a/r}{\left(\ln a/b\right)^2}\right]$$

$$\text{Let } \frac{a}{b} = C \text{ and } \frac{r}{b} = R \;\therefore\; \frac{b}{a} = \frac{R}{c}, \frac{r}{a} = \frac{r}{b}$$

$$\text{Let } g(n) = \left[\left(R^2-1\right)\frac{\ln C/R}{\ln C} - \frac{\left(C^2-R^2\right)}{\ln C} + \left(C^2-1\right)\frac{\ln C/R}{(\ln C)^2}\right]$$

$$= \left[\left(R^2-1\right)\frac{\ln\left(R/C\right)^{-1}}{\ln C} - \frac{\left(C^2-R^2\right)}{\ln C} + \left(C^2-1\right)\frac{\ln\left(R/C\right)^{-1}}{(\ln C)^2}\right]$$

$$= \left[-\left(R^2-1\right)\frac{\ln R/C}{\ln C} - \frac{\left(C^2-R^2\right)}{\ln C} - \left(C^2-1\right)\frac{\ln R/C}{(\ln C)^2}\right]$$

$$\text{Put } C = 2,\ \frac{\Omega b^2}{\nu} = 1 \text{ and } \Omega = 10$$

$$G(n) = -\frac{1}{4}\frac{\left(1+\lambda_1\lambda_2 100\right)}{\nu\left(1+\lambda_2^2 100\right)}\left[-\left(R^2-1\right)\frac{\ln R/2}{\ln 2} - \frac{\left(4-R^2\right)}{\ln 2} - 3\frac{\ln\left(R/2\right)}{(\ln 2)^2}\right]$$

$$\therefore g(n) \left(R^2 - 1\right)\frac{\text{In}\left(R/2\right)^{-1}}{\text{In}2} - \frac{\left(4 - R^2\right)}{\text{In}2} - \frac{3\text{In}\left(R/2\right)}{(\text{In}2)^2}$$

$$= \frac{1}{(\text{In}2)^2}\left[-\left(R^2 - 1\right)\text{In}2.\text{In}\,R/2 - \left(4 - R^2\right)\text{In}2 - 3\text{In}\,R/2\right]$$

$$G(n) = -\frac{1}{4}\frac{1(1 + \lambda_1\lambda_2 100)}{\nu\left(1 + \lambda_2^2 100\right)}\left[-\left(R^2 - 1\right)\frac{\text{In}\,R/2}{\text{In}\,2} - \frac{\left(4 - R^2\right)}{\text{In}2} - 3\frac{\text{In}\left(R/2\right)}{(\text{In}2)^2}\right]$$

$$g(n) = -\left(R^2 - 1\right)\frac{\text{In}\,R/2}{\text{In}2} - \frac{\left(4 - R^2\right)}{\text{In}2} - \frac{3\text{In}\,R/2}{(\text{In}2)^2}$$

$$= \frac{1}{(\text{In}2)^2}\left[-\left(R^2 - 1\right)\text{In}2.\text{In}\,R/2 - \left(4 - R^2\right)\text{In}2 - 3\text{In}\,R/2\right]$$

When $R = 1$ $g(n) = 0$,

$R = 1.2$, $g(n) = -0.1793846193$

$R = 1.4$, $g(n) = -0.22199114226$

$R = 1.6$, $g(n) = -0.1819408329$

$R = 1.8$, $g(n) = -0.09807847429$

$R = 2$ $g(n) = 0$

Table-2

R	1	1.2	1.4	1.6	1.8	2	
$\lambda_1 = 1$ $\lambda_2 = 1$	0	0.04484615483	0.05549778565	0.04548520823	0.02451974357	0	$\overleftarrow{G}(n)$
$\lambda_1 = 1$ $\lambda_2 = 5$	0	0.001811060231	0.002241214055	0.001836867665	0.0009902015592	0	$\overleftarrow{G}(n)$
$\lambda_1 = 1$ $\lambda_2 = 10$	0	0.0004529008736	0.005554772866	0.004552614082	0.002454180914	0	$\overleftarrow{G}(n)$
$\lambda_1 = 5$ $\lambda_2 = 1$	0	0.2224546888	0.2752909961	0.2256246467	0.1216276389	0	$\overleftarrow{G}(n)$
$\lambda_1 = 10$ $\lambda_2 = 1$	0	0.0444653562	0.00488651233	0.005554772866	0.002454180914	0	$\overleftarrow{G}(n)$

3.5 Conclusion

It is seen that with the help of the table-1 and table-2 in numerical calculation for small frequency ω, the parts of velocity, F(m) and G(n) show the same result. There is no change in velocity. So we can say that the fluid moves with constant velocity in both cylinders.

CHAPTER-4

The impact of magnetic field and slip velocity on conducting flow for the artery with mild stenosis

4.1 Introduction

In medical science stenosis means the localisation of narrowing in a blood vessel. In mammalian arteries many cardiovascular diseases are closely related to the nature of blood position and behaviour of blood vessel. This type of diseases may lead to morbidity and mortality. Although the correct theory of stenosis in the lumen of artery is clearly unknown to us. Various Scholars [1,2] emphasized that some of the major factors which developed the vascular disease are due to the formation of intra-vascular plaques and the impingement of ligaments and spurs on wall of the blood vessel. It has been mentioned that the blood flow characteristics may be altered and many abnormalities arise in the flow pattern. Some experimental investigations of arterial stenosis have been carried out by Young and Tsai [3] and it was noted that the changed characteristics of the blood flow may have a connection impact on the further development of the vascular disease. Various investigators. [4-6] pointed out that the study of different hydrodynamic factors such as skin-friction and pressure under normal physiological conditions and in pathological states provide useful information's for better understanding of the pathogenesis and proper treatment of various arterials diseases like myocardial infarction, stock etc.

Different mathematical models studied by several researchers [7-12] were investigated to consider blood flow through stenosed blood vessels. Among the researchers Young's [7] work may be considered as one of the earliest works of major importance. Lee and Fung [10] employed numerical techniques to study the blood flow through a stenosed tube.

It also may be pointed out that although blood is non-Newtonian suspension of cells in plasma Mcdonald [13] observed that for vessels at radious greater than 0.25mm blood may be considered as a homogeneous Newtonian fluid. At lower shear rates blood exhibits non-Newtonian behaviour

[14], but in larger arteries where the shear rate is high, blood may be considered as Newtonian [15].

It is worthwhile to mention that most of the aforementioned studies are based on the usual assumption of the no-slip condition at the vessel wall. But Benneth [16] on the basis of his in-vitro experiments to study the behaviour of red cells during blood flow, suggested that there might exist the possibility of the red cells to have a slip-velocity at the wall under certain conditions. Subsequently, several investigators [17-20] also indicated the possibility of slip-velocity at the inner surface of the wall.

On the other hand, Barnothy[21] reported that biological systems, in general, are affected by the application of an external magnetic field. In a recent paper, Halder and Ghosh[22] investigated the effect of magnetic field on blood flow through an indented tube in the presence of erythrocytes.

In the present investigation, a mathematical model has been developed to study the effect of externally applied uniform magnetic field on the characteristics of blood flow through stenosed vessels, by accounting for the slip velocity at the endothelium of the blood vessel. the analytical expressions are computed numerically in order to quantiate of the extent to which the slip velocity and the magnetic field can influence the blood flow pattern of a given stenosed blood vessel in a specific situation. Momentum integral techniques has been employed to solve the problem. The effects of an external magnetic field may have some consequences in these type of situations, for example, during MRI scanning.

4.2 The stenosis model

Let us consider an axially symmetric steady, laminar flow of blood through an artery in which a mild stenosis has been developed and the fluid is acted on by an externally applied uniform magnetic field B. The geometry of the stenosis is described as [19].

$$\frac{R(z)}{R_1} = 1 - \frac{\delta}{R_1} \exp\left(-m^2k^2z^2/R_1^{\,2}\right) \tag{4.1}$$

In which R(z) is the radius of the artery in the stenosed portion; R_1 denotes the radius of the artery outside the stenosis; δ and m are the height and slope of the stenosis where it intersets the vessel wall; $k = \frac{R_1}{L_1}$ is the relative length of the stenosed portion; z represents the axial distance and $2L_1$ is the length of the stenosed segment. Stenosis geometry described by equation (1) can be written alternatively in the form.

$$\frac{R(z)}{R_1} = 1 - \gamma \exp\left(-m^2x^2/m_0^{\,2}\right) \tag{4.2}$$

where $\gamma = \frac{\delta}{R_1}$, $x = \frac{z}{L}$, $m_0 = \frac{L_1}{L}$ and 2L is the length of the artery.

In biological systems and particularly in case of problems of blood flow through artery, the condition of steady flow in general may not be valid. But the consideration of a steady laminar flow is meaningful in certain situations as discussed below:

Blood flow in large arteries is pulsatile in nature, the frequency parameter β being given by $\beta = R_1\sqrt{2\pi f/\nu}$, where R_1 is the radius of the artery, f is the frequency of the pulsation and ν is the coefficient of kinematic viscosity of blood. The flow may be treated as quasi-steady for $\beta > 0$ in smaller arteries. McDonald [13] pointed out that for several blood vessels, e.g. the human femoral artery for which $2.5 < \beta < 3.5$, the quasi-steady condition remains valid and it is also likely to be valid in arteries much smaller than the human femoral artery. It may also be possible that such a quasi-steady flow exists in some larger arteries due to an acquired constriction in a major artery [8,23]. Thus, the assumption of steady laminar flow is justified in that part of the arterial tree where the flow is nearly steady.

Moreover, when a stenosis develops in an artery, an immediate effect is hardening of the walls due to complex physiological changes. For this reason, the stenosed portion of the arterial wall may also be treated as rigid.

4.3 Governing equations

Let us take the artery to be a long cylindrical tube with the axis coinciding with z-axis and the motion is axially symmetric. Assuming quasi-steady condition and the azimuthal dependence because of the rotatinoal symmetry of the stenosis, the basic equations of motion in the cylindrical co-ordinate system (r, θ,z) are given by.

$$u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}(\frac{\partial p}{\partial z})+\nu(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial z^2})-\frac{\sigma B^2}{\rho}u. \tag{4.3}$$

$$u\frac{\partial v}{\partial z}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}(\frac{\partial p}{\partial r})+\nu(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}-\frac{v}{r^2}) \tag{4.4}$$

The continuity equation is

$$\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(vr)=0 \tag{4.5}$$

In the above equations, u and v represent the axial and radial velocity components respectively; ρ is the density, p is the pressure; ν is the kinematic viscosity coefficient of blood, σ is the conductivity of the fluid and B is the applied external uniform transverse magnetic field.

Due to the presence of the nonlinear terms representing convective acceleration, an analytical solution of the above system of equations seems to be difficult and hence an attempt has been made to consider an approximate solution of the problem, by preserving the principal considerations regarding the stenosis geometry.

For a mild stenosis δ/L_1 is considerably small compared to unity and the normal stress gradient $\frac{\partial^2 u}{\partial z^2}$ is negligible compared to the shear stress $\frac{\partial^2 u}{\partial r^2}$. Also if δ/L_1 is sufficiently small compared to unity, the radial variation of pressure, i.e. $\frac{\partial p}{\partial r}$ may be neglected. Thus the differential equation determining the flow past a mild stenosis may be approximated as

$$u\frac{\partial u}{\partial z}+v\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)-\frac{\sigma B^2}{\rho}u. \tag{4.6}$$

and $\frac{\partial p}{\partial r}=0$ (4.7)

Now integrating equation (4.6) over the cross-section of the vessel and using the continuity equation (4.5), we obtain the momemtum integral equation as

$$\frac{\partial}{\partial z}\int_0^R ru^2 dr=-\frac{1}{\rho}\frac{R^2}{2}\frac{dp}{dz}+\nu R\left(\frac{\partial u}{\partial r}\right)_{r=R}-\frac{\sigma B^2}{\rho}\int_0^R ru\, dr. \tag{4.8}$$

where we have used the boundary conditions u = W (the velocity slip condition) and v = 0 at r = R.

Integrating the continuity equation (4.5), the volume flux Q is obtained as

$$Q=\pi R^2\,\overline{U}=\pi\int_0^R ru\, dr \tag{4.9}$$

where $\overline{U}$ is the mean velocity at any given cross-section with radius R.

In the present analysis, we take the velocity constraints as

$$u = U \quad \text{at } r = 0 \tag{4.10a}$$

$$u = W \quad \text{at } r = R \tag{4.10b}$$

$$\frac{\partial u}{\partial r} = 0 \qquad \text{at } r = 0 \qquad (4.10c)$$

$$\frac{\partial^2 u}{\partial r^2} = -\frac{2U}{R^2} \qquad \text{at } r = 0 \qquad (4.10d)$$

and $$\frac{dp}{dz} = \frac{\mu}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) - \sigma B^2 u \qquad \text{at } r = R \qquad (4.10e)$$

In the above, the first condition defines the centre line velocity, the second is the condition of slip velocity on the artery wall, the third is the regularity condition and is deduced by considering the forces on a cylindrical fluid element in the following way: If the pressure and the inetial forces are to be infinite as the radius of the element tends to zero, the viscous force that is proportional to $\frac{\partial u}{\partial r}$ must tend to zero. Assuming the velocity profile to be nearly parabolic at the axis, as represented by the Poiseull's profile $\frac{u}{U} = 1 - (\frac{r}{R})^2$, the second radial derivative of u at r = 0 may be approximate by the fourth condition. Finally, the fifth condition represents the validity of equation (6) at r = R.

4.4 Solutions

We choose the velocity profile in the dimensionless form as

$$\hat{u} = \frac{u}{U} = A_1 + A_2\eta + A_3\eta^2 + A_4\eta^3 + A_5\eta^4 \qquad (4.11a)$$

$$\text{where } \eta = \frac{R - r}{R}, \qquad (4.11b)$$

U being the centre line velocity and A_1, A_2, A_3, A_4, A_5 are constants to be determined from the velocity constraints. Using equations (4.11a) and (4.11b) the volume flux given in (4.9) may be re-written as

$$Q = 2\pi R^2 U \int_0^1 (1-\eta)\hat{u}\,d\eta \qquad (4.11c)$$

The velocity constraints in terms of η are given by

$$\hat{u} = 1 \qquad \text{at } \eta = 1 \qquad (4.12a)$$

$$\hat{u} = \frac{W}{U} \qquad \text{at } \eta = 0 \qquad (4.12b)$$

$$\frac{\partial \hat{u}}{\partial \eta} = 0 \qquad \text{at } \eta = 1 \qquad (4.12c)$$

$$\frac{\partial^2 \hat{u}}{\partial \eta^2} = 0 \qquad \text{at } \eta = 1 \qquad (4.12d)$$

$$\frac{dp}{dz} = \frac{\mu U}{R^2(1-\eta)}[(1-\eta)\frac{\partial^2 \hat{u}}{\partial \eta^2} - \frac{\partial \hat{u}}{\partial \eta}] - \sigma B^2 U \hat{u} \quad \text{at } \eta = 0 \qquad (4.12e)$$

Applying the conditions (4.12a) to (4.12e), the velocity profile $\hat{u}$ is evaluated in the form.

$$u = A_1 + \frac{1}{7}(-\lambda+10-12A_1)\eta + \frac{1}{7}(3\lambda+5-6A_1)\eta^2 + \frac{1}{7}(-3\lambda-12+20A_1)\eta^3$$

$$+ \frac{1}{7}(\lambda+4-9A_1)\eta^4 \qquad (4.13)$$

$$\text{in which } \lambda = \frac{R^2}{\mu U}[\frac{dp}{dz} + \sigma B^2 W], \; A_1 = \frac{W}{U} \qquad (4.14)$$

From (4.13), it is clear that when A_1 is known, the velocity profile becomes a function of a single parameter λ which is a function of the pressure gradient $\frac{dp}{dz}$ and the magnetic field strength B.

Substituting (4.13) into the equation (4.11c) and then integrating we obtain.

$$U = \frac{210}{97}\frac{Q}{\pi R^2} + \frac{2}{97}\frac{R^2}{\mu}\frac{dp}{dz} + \frac{2}{97}\frac{R^2}{\mu}\sigma B^2 W - \frac{102}{97}W \qquad (4.15)$$

The parameter λ can be determined from the integral equation (4.8) as

$$\lambda = \frac{4}{5}(6A_1 - 5) + \frac{7R^2}{5\mu}\sigma B^2 \frac{W}{U} - \frac{14}{5\nu U}[\frac{\partial}{\partial z}\{U^2R^2 \int_0^1 (1-\eta)\hat{u}d\eta\}]$$

$$-\frac{14R^2}{5\mu}\sigma B^2 - \int_0^1 (1-\eta)\hat{u}d\eta \qquad (4.16)$$

The subsequent part of the analysis will be carried out by neglecting higher than two in the velocity profile and retaining only the poiseuille profile [19].

$$u = 2\bar{U}[1-(\frac{r}{R})^2] \qquad (4.17)$$

where $\bar{U} = -(\frac{R^2}{8\mu})\frac{dp}{dz}$

is the average velocity at any given cross – section and $\frac{\partial p}{\partial z} < 0$.

Now substituting the value of u obtained from equation (4.17) into the momentum integral equation (4.8) we have

$$\frac{d}{dz}(\frac{2}{3}R^2\bar{U}^2) = -\frac{1}{\rho}\frac{R^2}{2}\frac{dp}{dz} + \frac{\nu}{7}(\lambda U - 10U + 12A_1U)$$

$$-\frac{\sigma B^2 R^2}{\rho}(-\frac{\lambda U}{210} + \frac{97U}{420} + \frac{17A_1U}{70}). \qquad (4.18)$$

In this equation if we substitute $\bar{U} = \frac{Q}{\pi R^2}$ and combine the resulting equation with (4.15), the pressure gradient is obtained in the form.

$$\frac{dp}{dz} = \frac{776}{225}(\frac{\rho Q^2}{\pi^2 R^5})\frac{dR}{dz} - \frac{8\mu Q}{\pi R^4} + \frac{624}{75}\frac{W\mu}{R^2} + \frac{22}{75}\sigma B^2 W - \frac{97}{75}\frac{Q}{\pi R^2}\sigma B^2 \qquad (4.19)$$

The first term on the right hand side of equation (4.19) is due to the inertia of blood, the second term is due to the viscous shearing stress, the third term is due to the slip velocity, the fourth and fifth terms represent the influence of magnetic field on the pressure gradient.

In non-dimensional form, the equation (4.19) is reduced to

$$\left(\frac{R_1}{\rho\overline{U}_0^2}\right)\frac{dp}{dz}=\frac{776}{225}\left(\frac{R_1}{R}\right)^5\frac{dR}{dz}-\frac{16}{R_e}\left(\frac{R_1}{R}\right)^4+\frac{1248}{75}\frac{1}{R_e}\left(\frac{R_1}{R}\right)^2\left(\frac{W}{\overline{U}_0}\right)$$

$$+\frac{22}{75}\left(\frac{\sigma B^2}{\rho}\right)\left(\frac{W}{\overline{U}}\right)\left(\frac{R_1}{\overline{U}_0}\right)-\frac{97}{75}\left(\frac{\sigma B^2}{\rho}\right)\left(\frac{R_1}{R}\right)^2\left(\frac{R_1}{\overline{U}_0}\right) \qquad (4.20)$$

where $R_e=\frac{2\rho R_1\overline{U}_0}{\mu}$ is the Reynolds number upstream from the stenosis, $\overline{U}_0$ being the average velocity at a cross-section of the artery.

The condition for an average pressure gradient to develop

(when $\frac{dp}{dz}>0$) is

$$R_e\left(\frac{R_1}{R}\right)\frac{dR}{dz}\geq 4.64-4.82\left(\frac{W}{\overline{U}_0}\right)\left(\frac{R_1}{R}\right)^{-2}-0.09\frac{\sigma B^2}{\rho}\left(\frac{W}{\overline{U}_0}\right)$$

$$\left(\frac{R_1}{\overline{U}_0}\right)\left(\frac{R_1}{R}\right)^{-5}R_e+0.38\frac{\sigma B^2}{\rho}\left(\frac{R_1}{R}\right)^{-3}\left(\frac{R_1}{\overline{U}_0}\right)R_e \qquad (4.21)$$

Using equations (4.15) and (4.18), the velocity distribution u is obtained from (4.13) as a function of r and z in the form

$$\frac{u}{\overline{U}_0}=R_e\left(\frac{R_1}{R}\right)^3\frac{dR}{dz}f(\eta)+2\left(\frac{R_1}{R}\right)^2(2\eta-\eta^2)+\frac{W}{\overline{U}_0}g(\eta)$$

$$-\left(\frac{R_1}{R}\right)^2M^2\frac{W}{\overline{U}_0}\varphi(\eta)+M^2\varphi(\eta) \qquad (4.22)$$

where

$$f(\eta)=0.2\eta+0.76\eta^2-0.8\eta^3+0.24\eta^4 \qquad (4.23a)$$

$$g(\eta)=1-4.16\eta+2.08\eta^2+0.8\eta^3-0.6\eta^4 \qquad (4.23b)$$

$$\varphi(\eta)=0.15\eta-0.57\eta^2+0.6\eta^3-0.2\eta^4 \qquad (4.23c)$$

and $M = B_1 R_1 \sqrt{\frac{\sigma}{\mu}}$ = Hartmann number.

The skin – friction τ_ω is given by

$$\tau_\omega = \mu \left(\frac{\partial u}{\partial r}\right)_{r=R} \tag{4.24}$$

which in non-dimensional form is obtained with the help of (4.22) as

$$\frac{\tau_\omega}{\rho \bar{U}_0^2} = 0.4\left(\frac{R_1}{R}\right)^4 \frac{dR}{dz} - \frac{8}{R_e}\left(\frac{R_1}{R}\right)^3 + \frac{8.32}{R_e}\left(\frac{R_1}{R}\right)\left(\frac{W}{\bar{U}_0}\right)$$

$$+ \frac{0.3}{R_e}\left(\frac{R_1}{R}\right) M^2 \frac{W}{\bar{U}_0} - \frac{0.3}{R_e}\left(\frac{R_1}{R}\right) M^2 \tag{4.25}$$

At the separation and reattachment points, the skin friction must vanish, so that we have

$$R_e\left(\frac{R_1}{R}\right)\frac{dR}{dz} = 20 - 20.8\left(\frac{R_1}{R}\right)^{-2}\left(\frac{W}{\bar{U}_0}\right) - 0.75\left(\frac{R_1}{R}\right)^{-4}\frac{W}{\bar{U}_0} M^2$$

$$+ 0.75\left(\frac{R_1}{R}\right)^{-2} M^2 \tag{4.26}$$

In the case of incipient separation for which the Reynolds number is just enough to cause separation, the separation location in the diverging section of the stenosis is given by the condition that $\frac{1}{R}\left(\frac{dp}{dz}\right)$ is maximum which demands that

$$R\left(\frac{d^2R}{dz^2}\right) = \left(\frac{dR}{dz}\right)^2 \tag{4.27}$$

For the stenosis geometry defined by equation (4.1), the location $\left(\frac{Z}{L_1}\right)$ of the initial point of separation is given by the relation.

$$e^{(m^2z^2/L_1^2)}(1-2m^2z^2L_1^2)=\frac{\delta}{R_1}\frac{2m^2z^2}{L_1^2} \tag{4.28}$$

yelding

$$(\frac{z}{L_1})^2 \approx \frac{1}{4m^2}[\sqrt{(9+4\frac{\delta}{R_1})}-(1+2\frac{\delta}{R_1})]. \tag{4.29}$$

4.5 Results and discussions

The analytical expressions derived in the previous section have been computed numerically for different Reynolds and Hartmann numbers. The aim of the computational work is to quantify the influence of the magnetic field and the slip velocity at the wall on the velocity distribution. The computation has been carried out at the location defined by $z = 0.06$ for three different values of Reynolds number R_e = 100, 300, 500 and Hartmann number M given by M^2 = 0,9,18. The slip velocity has been taken to be equal to 10% of the average velocity of blood in a normal artery [19]. The length of the stenosis has been taken to be 20 mm while the maximum depth of the stenosis is assumed to be 0.2 mm. Figures 2,3 and 4 illustrate the variation of the non-dimensional axial velocity of blood flow in the stenosed arterial segment for different Hartmann number. It may be observed that the magnetic field increases the blood velocity near the wall but decreases it near the central axis of the artery. Figures 5,6 and 7 predict the same behaviour of blood without slip velocity. The variation of blood flow with and without slip velocity has been shown in figure 8 for R_e = 500 and M^2 =9. It is noted that the slip velocity increases the flow very near to the wall but decreases it as we pass on to the centre. In figure 9, the effect of Reynolds number on the blood velocity has been shown and the influence is to reduce the velocity with increasing Reynolds number near the wall and then to increase.

4.6 Conclusions

Although the present investigation of the mathematical model of blood flow through a stenosed segment of the artery is based on some approximations, it bears the potential to reveal some characteristics of the problem. The model firmly establishes the fact that the velocity slip at the wall of the arterial segment as well as the magnetic field enhance the axial velocity of the blood.

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