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Study of Structures in Some Branches of Mathematics

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STUDY OF STRUCTURES IN SOME BRANCHES OF MATHEMATICS

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by

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To

my beloved parents,

my dearest wife

and

to my affectionate daughter

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Declaration

Certified that the thesis entitled **Study of Structures in Some Branches of Mathematics** submitted by Mohd. Altab Hossain in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi, Bangladesh has been completed under my supervision. I believe that this research work is an original one and that it has not been submitted elsewhere for any degree.

Subrata Majumdar
(Dr. Subrata Majumdar)
Supervisor

Statement of Originality

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any university, and to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

09, 12, 08

(Mohd. Altab Hossain)

Author

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Abstract

Classification is a deep and fundamental problem in every branch of Mathematics. Although an uphill task, this target has been achiev -ed in a number of cases in different fields. In general, the strategy for this grand program consists of identification of relatively simple types of entities in a particular branch of Mathematics, getting a thorough knowledge about these, and then describing other entities in terms of these fundamental and basic objects tagged with one another in a well known manner. We call these basic entities **building blocks** and the method of tagging these together **glues**. For a given object in a particular branch, the building blocks, the glues and the manner of their application determine the structure of the object. The situation is comparable to the architecture of a building.

Our objective in this thesis is to study the structures or the architectural designs of mathematical objects in a number of branches of mathematics.

In the first chapter, known structures of important objects in algebra, geometry, topology and some other areas related to physics have been described briefly to provide a glimpse of the area of our interest mentioned above.

The second chapter is a study of a particular type of commutative semigroups which have been termed **special**. A number of structure

theorems have been proved in this context.

The third chapter deals with the determination of the structure of the centraliser of an endomapping of a finite set X in the full transformation semigroup $F(X)$. This has been done by representing the endomapping by a directed graph and then determining the structure of all endomorphisms of the relevant graph.

In the fourth chapter, some **groups of morphisms** in certain categories have been considered and studied. Structures have been established in certain cases.

In the fifth and the last chapter, a number of **sums** of topological spaces have been considered and their fundamental gruops and homologies have been determined using the Seifert Van Kampen theorem and the Mayer-Vietoris sequence for homology.

CHAPTER-1

SURVEY OF STRUCTURES IN ALGEBRA, TOPOLOGY, GEOMETRY AND RELEVANT BRANCHES

1.1 Introduction

We shall describe how interesting mathematical objects in various branches of mathematics have been built up with the use of well known simple objects and well known methods of attaching these objects with one another. The objects and the methods of *gluing* these together are different for different branches. In this chapter we shall describe structures of mathematical objects mainly in algebra, topology and geometry. Structures concerning objects such as sheaves, Riemann surfaces and some entities related to string theory have also been briefly described. The classifications of finite simple groups and three dimensional geometries of manifolds have also been included.

1.2 Algebraic structures

1.2.1 Structure of Groups

In this section, we shall include our necessary objects and gluing operations for obtaining the structures of groups. We have considered the following as the well-known objects for groups.

(i) *Cyclic groups,* i.e., groups with one generator. These may be either infinite or finite. In the former case, the group is isomorphic to the additive group of integers and is denoted by $\mathbb{Z}$ or C_∞, and in the latter the group is isomorphic to the additive group of the residue classes of the integers modulo some positive integer, say n, and is denoted by $\mathbb{Z}_n$ or C_n.

(ii) $\mathbb{Q}$, the additive group of rationals;

(iii) $\mathbb{Z}(p^\infty)$, the *quasi-cyclic group,* given by generators:

$$x_1,\ x_2,\ x_3,\ \cdots$$

and relations:

$$px_1 = 0,\ px_2 = x_1,\ \cdots\ ,\ px_{n+1} = x_n,\ \cdots\ ,$$

where p is a prime. $\mathbb{Z}(p^\infty)$ is isomorphic to additive group $\frac{A}{\mathbb{Z}}$, where A is the additive group of all rational numbers of the form $\frac{a}{p^n}$ $(a \in \mathbb{Z},\ n \in \mathbb{N})$.

(iv) S_n, the *symmetric group* of degree n, being the group of all permutations on n symbols under composition of mappings.

(v) A_n, the *alternating group* of degree n. This is the subgroup of S_n consisting of all even permutations on n symbols.

Comment: Sometimes Sylow subgroups of a finite group or a periodic abelian group are considered as building blocks for the relevant group.

The different methods of glueing groups with one another that have been used here are as follows.

(i) Direct Product: If G_1 and G_2 are two groups, the *direct product* of G_1 with G_2 is

$$G_1 \times G_2 = \{(g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2\}$$

with multiplication given by

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2') \, .$$

If G_1, G_2 are additive abelian groups, direct product is replaced by direct sum and in that case we write $G_1 \oplus G_2$ for $G_1 \times G_2$.

(ii) Semidirect Product: Given two groups H and K and for every element $h \in H$ an automorphism of K, $k \rightleftharpoons k^h$ all $k \in K$, such that $(k^{h_1})^{h_2} = k^{h_1 h_2}$, h_1, $h_2 \in H$. Then the symbols $[h, k]$, $h \in H$, $k \in K$ form a group under the product rule $[h_1, k_1].[h_2, k_2] = [h_1 h_2, {k_1}^{h_2} k_2]$, called the *semi-direct product* of K by H.

Comments: (1) $S_3 =< x, \, y \mid x^3 = y^2 = (xy)^2 = 1 >$. Here $< x >= \{x$, x^2, $x^3 = 1\} = K$, say. Then, $K \cong C_3$, $K \triangleleft S_3$. Also, $< y >= \{y$, y^2, $= 1\} = H$, say. Then, $H \subseteq S_3$. In this case, S_3 is the semidirect product of K with H.

(2) The dihedral group, $D_{2n} =< x, \, y \mid x^n = y^2 = (xy)^2 = 1 >$. Here $K =< x >= \{x, \, x^2, \, x^3, \, \cdots, \, x^k = 1\} \triangleleft D_{2n}$ and $H =< y >= \{y, \, y^2 = 1\} \subseteq D_{2n}$. In this case, D_{2n} is the semidirect product of K with H.

(iii) Free Product: A group G is said to be the *free product* of its subgroups A_α (α ranges over some index set) if the subgroups A_α generate G, that is, if every element g of G is the product of a finite number of the elements of the A_α,

$g = a_1 a_2 \cdots a_n, \quad a_i \in A_{\alpha_i}, \ (i = 1, 2, \cdots, n) \qquad (1)$

and if every element g of G, $g \neq 1$, has a unique representation in the form (1) subject to the condition that all the elements a_i are different from the unit element and that in (1) no two adjacent elements are in the same subgroup A_α – although the product (1) may in general, contain several factors from one and the same subgroup. The free product is denoted by the symbol $G = \prod_\alpha * A_\alpha$, and if G is the free product of a finite number of subgroups $A_1, A_2, \cdots, A_k$, the free product is denoted by the symbol $G = A_1 * A_2 * \cdots * A_k$.

Comment: Every free group is a free product of groups each isomorphic to C_∞. Also, no group can be decomposable both into a free product and into a direct product.

Theorem 1.1

If

$$G = \prod_\alpha * A_\alpha$$

and if H is an arbitrary subgroup of G, then there exists a free decomposition

$$H = F^* \prod_\beta * B_\beta$$

of H, where F is a free group and every B_β is conjugate in G to subgroup of one of the free factors A_α.

(iv) Wreath Product: Let G and H be permutation groups on sets A and B, respectively. The *wreath product* of G by H, written $G \wr H$, is defined in the following way:
$G \wr H$ is the group of all permutations θ on $A \times B$ such that

$$\theta(a, b) = (\gamma_b a, \eta b), \quad a \in A, \; b \in B\,,$$

where $n \in H$ and for each $b \in B$, γ_b is a permutation of G on A, but for different b's the choice of the permutations γ_b are independent. The permutations θ with $\eta = 1$ form a normal subgroup G^* isomorphic to the direct product of n copies of G, where n is the number of letters in the set B.

We shall now describe the structures of a class of groups using the objects and glues mentioned above.

Structure for finitely generated abelian groups and finite abelian groups are given by the following theorems:

Theorem 1.2

If G is finitely generated abelian group then

$$G \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r \quad terms} \oplus \underbrace{\mathbb{Z} \oplus \mathbb{Z}_{m_k}}_{k \quad terms},$$

when G has a basis i-th r elements of infinite order and k elements with finite order of $m_1, \cdots, m_k$.

Theorem 1.3

A finite abelian group G of order $n = p_1^{e_1} \cdots p_r^{e_r}$ is the direct product of Sylow subgroups $S(p_1), \cdots S(p_r)$ *i.e.,* $G \cong S(p_1) \times \cdots \times S(p_r)$.

Here $S(p_i)$ is of order $p_i^{e_i}$ and is the direct product of cyclic groups of orders $p_i^{e_{i_1}}, \cdots p_i^{e_{i_s}}$, where $e_{i_1} + \cdots + e_{i_s} = e_i$.

An abelian group G is called *free* if there exists a subset X of G such that each element $g \in G$ can be written uniquely as $g = n_1x_1 + \cdots + n_rx_r$ for some r, where $x_i \in X$ and $n_i \in \mathbb{Z}$. The structure of a free abelian group is as follows:

Theorem 1.4

If G is a free abelian group, then $G \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$. The cardinality of the set of copies of $\mathbb{Z}$ is known as the rank of G.

An abelian group A is called *divisible* if for each $a \in A$ and for each $0 \neq n \in \mathbb{Z}$, there exists $a' \in A$ such that $a = na'$.
For examples, the additive groups $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are divisible.

The structure of divisible groups in terms of direct sum of copies of $\mathbb{Q}$ and quasi-cyclic groups $\mathbb{Z}(p^\infty)$ with generators and relations: $< x_1, x_2, \cdots / px_1 = 0, px_2 = x_1, \cdots, px_{n+1} = x_n, \cdots >$ are given as follows:

Theorem 1.5

If D is a divisible group then
$D \cong (\mathbb{Q} \oplus \cdots \oplus \mathbb{Q} \oplus \cdots) \oplus (\mathbb{Z}(p_1^\infty) \oplus \cdots \oplus \mathbb{Z}(p_r^\infty) \oplus \cdots)$.

Theorem 1.6

A periodic abelian group is the direct product of its Sylow subgroups $S(p)$ i.e., if A is a periodic abelian group, then $A = \prod_p S(p), S(p)$ Sylow p-group of A.

Let A be an abelian group and n an integer such that $n > 1$. Let D be a divisible hull of A. Define $A_0^{(D)} = A$, $A_r^{(D)} = \{x \in D/\, nx = y,\ for\ some\ y \in A_{r-1}^{(D)}\}$, $r = 1, 2, \cdots$, and $A_{[n]}^{(D)} = \cup_0^{\infty} A_r^{(D)}$. Then each $A_r^{(D)}$ is a subgroup of D and $A_{[n]}^{(D)}$ is the smallest n-divisible subgroup of D which contains A. Define $A_{(D)}^{[n]} = A_{[n]}^{(D)}/A$. The structures for the groups $A_{[n]}^{(D)}$ and $A_{(D)}^{[n]}$ for $\mathbb{Z}$, $\mathbb{Z}_r$ and J_p are given through the following theorems (see [44]).

Theorem 1.7

Let p be a prime, $\mathbb{Q}^p$ the additive group of rationals with denominators powers of p and $\mathbb{Z}_{(p)}^{\infty}$ then the following results hold good:

(i) $\mathbb{Z}_{[p]} \cong \mathbb{Q}^p$,

(ii) $\mathbb{Z}_{[p]} \cong \mathbb{Z}_{(p^\infty)}$,

(iii) $(\mathbb{Z}_r)_{[p]} \cong \mathbb{Z}_{(p^\infty)}$, *and*

(iv) $(\mathbb{Z}_r)^{[p]} \cong \mathbb{Z}_{(p^\infty)}$.

Theorem 1.8

Let J_p be the additive abelian group of p-adic integers where p is a prime. Then

(i) $(J_p)_{[q]} = J_p$ *and* $(J_p)^{[q]} = 0$, *for each prime* $q \neq p$.

(ii) $(J_p)_{[p]} \cong K^p$, *the additive abelian group of p-adic numbers.*

$\cong \oplus_{2\aleph_0}\mathbb{Q}$, *and*

(iii) $(J_p)^{[p]} \cong \oplus_{2\aleph_0}\mathbb{Z}(p^\infty)$.

Suzuki [69] has studied structures of M-groups as well as UM-groups. A group whose subgroups form a modular lattice is called an *M-group.* A lattice is termed *upper semi-modular* if $a \vee b$ covers a whenever $a \wedge b$ is maximal in b. A group is called an *UM-group* if its subgroup lattice is upper semi-modular. We shall now state a few structures of M-groups and UM-groups.

Theorem 1.9 ([32], p.727)

Let G be an M-group and let E be the normal subgroup of G consisting of all elements of finite order in G. If the abelian group G/E is of rank 1. Then G has the following structure:
$G = \{E, z_1, z_2, \cdots\}$ where z_1 is of finite order, $z_{i-1}^{p_i} = z_i e_i$, $z_{i-1} z_{i-1}^{-1} = e_i^{\beta_i}$ (p_i a prime number, $e_i \in E$), and for any element a of the p-component E_p of E, $z_i a z_i^{-1} = a^{\alpha_i(p)}$ where $\alpha_i(p)$ is a p-adic number, uniquely determined modulo the exponent p^n of E_p and $\alpha_i(p) \equiv 1 \,(mod\, p)$ $\quad (\alpha_i(2) \equiv 1\, mod\, 4))$, $\alpha_{i-1}^{p_i}(p) \equiv \alpha_i(p)\,(mod\, p^n)$.

The structure for the UM- groups due to Sato [63] is given by:

Theorem 1.10

A finite group G is an UM-group if and only if G is a direct product of groups H_i such that the orders of H_i and H_j ($i \neq j$) are relatively prime and each group H_i is eiter a modular p-group or a group of the following type:
$H = (P_1 \times \cdots \times P_r) \cup Q$ where each P_i is a p_i-Sylow subgroup and Q is a q-Sylow subgroup, and moreover
(1) each P_i is elementary abelian,

(2) Q is cyclic: $Q = \{b\}$,

(3) $ba_ib^{-1} = a_i^{r_i}$ *for every element* $a_i \in P_i$, *where* $r_i \not\equiv 1, r_i^{q^{\beta_i}} \equiv 1 \,(mod\, p)$, *and*

(4) if β_i *is chosen as small as possible in (3), then* $\beta_i \neq \beta_j \,(i \neq j)$.

Classification of Finite Simple Groups

For about a century the simple group problem captured the attention of leading algebraists. The problem was *the classification of all finite simple groups.*

There remained a gap concerning the existence and uniqueness of some sporadic simple groups, notably the Monster. With the constructions of the Lyons group and the Baby Monster group as a permutation group, Sims had developed the finite simple groups as permutation groups. However, it appeared in 1980 that one could claim the following theorem as the classification theorem of the finite simple groups. The final completion came about in 2003.

Theorem 1.11 ([66], p.341)

Let G be a finite simple group. Then G is either

(a) a cyclic group of prime order;

(b) an alternating group of degree $n \geq 5$*;*

(c) a finite simple group of Lie type; or

(d) one of 26 *sporadic finite simple groups: the five Mathieu groups, the four Janko groups, the three Conway groups, the three Fischer*

groups, HS, Mc, Suz, Ru, He, Ly, ON, HN,Th, BM and M.

It is actually the resultant of a number of research works. It is an important and one of the most remarkable results of mathematical research in the twentieth century.

1.2.2 Vector Spaces

If V is a vector space over a field F, then F, regarded as a vector space over itself, is the standard object yeilding the structure of the space V. The glueing operations here are direct sum and tensor product.

(i) Direct sum: If V and W are two vector spaces over F then the direct sum of V and W, written $V \oplus W$, is the vecor space over F which consists of all ordered pairs(v, w) with $v \in V$, $w \in W$, where addition and scalar multiplication are component-wise. If V and W are finite dimensional then $V \oplus W$ too is finite dimensional and $dim(V \oplus W) = dim\, V + dim\, W$.

(ii) Tensor Product: Let V and W be two vector spaces over a field F. Then the tensor product $V \otimes_F W$ is the vector space over F generated by all symbols of the form $v \otimes w$ with relations

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w\,,$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2\,,$$

$$(v\lambda) \otimes w = v \otimes w\,,$$

for all $v,\ v_1,\ v_2 \in V,\ w,\ w_1,\ w_2 \in W,\ \lambda \in F$.

Theorem 1.12

For finite dimensional vector spaces V, W with $\dim V = m$, $\dim W = n$, we have

(a) $V \cong F_1 \oplus \cdots \oplus F_m$, $W \cong F_1' \oplus \cdots \oplus F_n'$, *where each* F_i *and each* F_j' *are isomorphic to* F *as vector spaces over* F.

(b) $V \oplus W \cong (F_1 \oplus \cdots \oplus F_m) \oplus (F_1' \oplus \cdots \oplus F_n')$, *and*

(c)
$$\begin{aligned} V \otimes W &= (F_1 \oplus \cdots \oplus F_m) \otimes (F_1' \oplus \cdots \oplus F_n') \\ &\cong \textstyle\sum_{1 \le j \le n}^{1 \le i \le m} \oplus \, (F_i \otimes F_j') \\ &\cong \textstyle\sum_{1 \le j \le n}^{1 \le i \le m} F_{ij}\,, \end{aligned}$$
where $F_{ij} = F_i \otimes F_j' \cong F$.

1.2.3 Semisimple Rings

In this case the fundamental objects are

(i) The ring of all $n \times n$ matrices over a skew-field.

(ii) Simple rings, i.e., rings R whose ideals are R and 0 only.

(iii) Prime rings, i.e., rings R such that for each pair of ideals I, J of R, $I\,J = 0$ implies $I = 0$ or $J = 0$.

(iv) *Primitive rings*, i.e., rings of $n \times n$ matrices over a skew field which are dense in $M^{n \times n}$.

(v) Subdirectly irreducible rings with idempotent hearts: a ring R is *subdirectly irreducible* if the intersection of all ideals of R is non-zero. This intersection is a simple ring and is called the *heart* of R.

The glues in case of rings will be the following:

(i) Direct Sum: If R and S are rings, *the direct sum* $R \oplus S = \{(r, s) \,|\, r \in R, s \in S\}$ together with component-wise addition and multiplication. Direct sum of any finite number of rings is defined in a similar way.

(ii) Subdirect Sum: Let $\{R_\alpha\}$ be a non-empty class of rings. A ring S will be called a *subdirect sum* of $\{R_\alpha\}$ if, for each α, there is an ideal I_α of S such that $\cap_\alpha I_\alpha = 0$ and $\frac{S}{I_\alpha} \cong R_\alpha$, for each α.

We shall describe the structure of semisimple rings with descending chain condition or d.c.c. and also those without any chain condition.

We recall that a ring R is said to have *descending chain condition* or d.c.c. on left ideals if every descending chain $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ of left ideals of R terminates after a finite number of terms, i.e., $L_n = L_{n+r}$, for each $r = 1, 2, \cdots$. The *radical* of a ring A with d.c.c., written $R(A)$, is the sum of all nilpotents ideals of A and is the same as the sum of all nil ideals of A. A is *semisimple* if $R(A) = 0$.

Theorem 1.13 ([18], p.168)

Let R be a semi-simple ring with d.c.c., and let L be a minimal left ideal of R. The sum B_L of all the minimal left ideals of R which are isomorphic to L is a simple ring and a two-sided ideal of R. Furthermore, R is the direct sum of all the ideals B_L obtained by letting L ranges over a full set of non-isomorphic minimal left ideals of R, where $B_iB_j = 0$ for $i \neq j$, so that $R = B_1 \oplus \cdots \oplus B_m$, where the $\{B_i\}$

are subrings of R which annihilate each other.

Theorem 1.14 (Wedderburn)

Let A be a simple semisimple ring with descending chain condition. Then $A \cong Hom_D(M, M)$ for some finite-dimensional right vector space M over a skewfield D. The dimension $(M : D)$ and the skewfield D are uniquely determined by A. Thus $A \cong \mathcal{M}^{n \times n}$, where $n = dim(M : D)$ and $\mathcal{M}^{n \times n}$ is the ring of all $n \times n$ matrices over D.

For our convenience, we recall some necessary definitions. An ideal I of a ring is called a *nil ideal* for each $x \in I$, $x^n = 0$ for some positive integer n. An element a of a ring A is called *right quasi-regular* if, there exists an element b such that $a + b + ab = 0$. And the set of all right quasi-regular elements of a ring A form an ideal of A which is called the *Jacobson radical* of A. An ideal P of a ring R is a *prime ideal* if, for any two ideals I, J of A, $I\,J \subseteq P$ implies $I \subseteq P$ or$J \subseteq P$. Let R be a ring and let $a \in R$. Let $G(a) = \{ar + r + \sum(x_i a y_i + x_i y_i)$, where r, x_i, y_i range over R, and the summation is finite. Then a is called *G-regular* if $a \in G(a)$. The set of all G-regular elements of R is an ideal of R. This ideal is called the *G-radical* of R. It is also called *Brown-McCoy* radical. R is termed as *G-semi-simple* if the G-radical of R is zero.

We see that in the case when the ring A is without d.c.c., the radical of A can be described in a number of ways to obtain different radicals:

(1) The sum of nil ideals of A is the nil radical $N(A)$.

(2) The sum of right quasi-regular ideals of A is the Jacobson radical $J(A)$.

(3) The intersection of prime ideals of A is the Baer-lower radical $\beta(A)$.

A is nil-semisimple, J-semisimple or β-semisimple if $N(A) = 0$, $J(A) = 0$, or $\beta(A) = 0$.

The structure of the above radicals can be expressed in terms of the subdirect sum and the structure is given by the following theorem:

Theorem 1.15

The following three hold good:

(i) A is β-semisimple $\Leftrightarrow$ A is a subdirect sum of prime rings.

(ii) A is J-semisimple $\Leftrightarrow$ A is a subdirect sum of primitive rings.

(iii) Any G-semisimple ring is isomorphic to a subdirect sum of simple rings with unity.

1.3 Topological Structures

1.3.1 Surfaces and Manifolds

Assume that n is a positive integer. An *n-manifold* is a Hausdorf space (i.e., a space that satisfies the T_2 separation axiom) such that each point has an open neighborhood homeomorphic to the open n-dimensional disc U^n $(= \{x \in \mathbb{R}^n :| x |< 1\})$. For examples, Euclidean n-space $\mathbb{R}^n$ is a n-manifold, $n$dimensional sphere $S^n = \{x \in \mathbb{R}^{n+1}$:

$| x |= 1\}$ is also an n-manifold, 2-sphere S^2 is a 2-manifold etc. A connected 2-manifold is defined to be *orientable* if every closed path is orientation preserving whereas a connected 2 manifold is *nonorientable* if there is at least one orientation-reversing path. For examples, sphere and torus are orientable surfaces wheras Mobius strip, real projective plane,Klein Bottle are nonorientable surfaces.

We shall now describe a very special way to form a new surface from given more by *cut and paste* method. Let S_1 and S_2 be disjoint surfaces. We form $S_1 \# S_2$ by cutting a small circular hole in each surface, and then gluing the two surfaces together along the boundaries of the holes. To be, precise, we choose subsets $D_1 \subset S_1$ and $D_2 \subset S_2$ such that D_1 and D_2 are closed discs (i.e., homeomorphic to E^2). Let $S^{i'}$ denote the complement of the interior of D_i in S_i for $i = 1$ *and* 2. Choose a homeomorphism h of the boundary circle of D_1 onto the boundary of D_2. Then $S_1 \# S_2$ is the quotient space of $S_1' \cup S_2'$ obtained by identifying the points x and $h(x)$ for all points x in the boundary of D_1. It is clear that $S_1 \# S_2$ is a surface known as *connected sum* of surfaces S_1 and S_2.

The classification of compact surfaces (orientable or nonorientable) can be given by the following structure theorem:

Theorem 1.16 ([53], p.29)

Any compact orientable surface is either homeomorphic to a sphere, or to a connected sum of tori.Any compact nonorientable surface is homeomorphic to the connected sum of either a projective plane or Klein Bottle and a compact orientable surface.

1.3.2 Triangulation of compact surfaces

A *triangulation* of a compact surface S consists of a finite family of closed subsets $\{T_1, \cdots, T_n\}$ that cover S, and a family of homeomorphisms $\varphi : T_i' \longrightarrow T_i$, $i = 1, \cdots, n$, where each T_i' is a triangle in the plane $\mathbb{R}^2$. The subsets T_i are called triangles. The subsets of T_i that are the images of the vertices and edges of the triangle T_i' under φ_i are also called vertices and edges respectively. Finally, it is required that any two distinct triangles, T_i and T_j, either be disjoint, have a single vertex in common, or have one entire edge in common.

The *Euler characteristic, of a triangulated compact surface M with triangulation $\{T_1, \cdots, T_n\}$, denoted by $\chi(M)$, as* $\chi(M) = v - e + t$ *where*

$v =$ *total number of vertices of M,*

$e =$ *total number of edges of M and*

$t =$ *total number of triangles (in this case, $t = n$).*

The Euler characteristic give the uniqueness of the structure of surfaces obtained by forming connected sums by the following :

Proposition 1.17

Let S_1 and S_2 be compact surfaces. The Euler characteristics of S_1 and S_2 and their connected sum, $S_1\#S_2$, are related by the formula $\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2.$

Using the above proposion we obtain the following values for the Euler characteristics of the various possible compact surfaces :

Surface	*Euler characteristic*
Sphere	2
Connected sum of n tori	$2-2n$
Connected sum of n projective planes	$2-n$
Connected sum of projective plane and n tori	$1-2n$
Connected sum of Klein Bottle and n tori	$-2n$.

We note that the characteristic of an orientable surface is always even, whereas for a nonorientable surface it may either be odd or even.

The structure of a connected, simply connected and complete Riemannian manifold is given in the following theorem (see Kobayashi and Nomizu [39], p-192):

Theorem 1.18

A connected, simply connected and complete Riemannian manifold M is isometric to the direct product $M_0 \times M_1 \times \cdots \times M_k$, where M_0 is a Eclidean space(possibly of dimension 0) and $M_1, M_2, \cdots, M_k$ are all simply connected, complete, irreducible Riemannian manifolds. Such a decomposition is unique upto an order.

1.3.3 Fundamental group

Sometimes a group, called the *fundamental group*, is associated with a topological space. It is topological invariant and serves to classify spaces to some degree. Let X be a topological spaces and $x_0 \in X$. A *loop* at x_0 is a continuous map $F : I \longrightarrow X$ such that $f(0) = f(1) = x_0$. Two loops f, g at x_0 are said to be homotopic, written $f \simeq g$, if there exists a continuous map $F : I \times I \longrightarrow X$ such that $F((0,t)) = f(t)$ and $F((1,t)) = g(t)$ for each $t \in I$. Then $\simeq$ is an equivalence relation on the set of loops at x_0. We denote the equivalence class of f by [f]. A multiplication is defined on these equivalence classes by $[g][f] = [g * f]$, where $(g * f)(t) = 2t$ when $0 \leq t \leq 1/2$ and $(g * f)(t) = 1 - 2t$ when $1/2 \leq t \leq 1$. This multiplication is well defined and is associative. If 1 denotes the constant map $1 : I \longrightarrow X$ with $1(t) = x_0$, for each $t \in I$, then $[1][f] = [f][1] = [f]$, and if $g : I \longrightarrow X$ is the loop given by $g(t) = f(1-t)$, then $[f][g] = [g][f] = [1]$. Thus the equivalence classes of the loops at x_0 form a group. This group is called the *fundamental group* of X with base point at x_0 and it is usually denoted by $\pi(X, x_0)$.

X is called *path connected* if for each $x, y \in X$, there is a path from x to y i.e., there is a continuous map $f : I \longrightarrow X$ such that $f(0) = x$, $f(1) = y$. If X is path connected , then $\pi(X, x_0) \cong \pi(X, x_1)$ for every pair of points x_0 and x_1 in X. In this situation, we write $\pi(X)$ to denote the fundamental group of X without mentioning the base point.

Example: If $X = S^1$, the unit circle, then $\pi(X) = \mathbb{Z}$.

The fundamental group of product space is given below:

Theorem 1.19

The fundamental group of the product of two path-connected spaces is isomorphic to the direct product of their fundamental groups; in symbols,

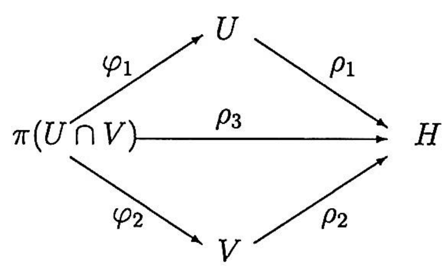
$\pi(X \times Y) \approx \pi(X) \times \pi(Y)$.

As an application of this theorem, we see that the fundamental group of a torus $T = S^1 \times S^1$ is $\pi(T) \cong \mathbb{Z} \oplus \mathbb{Z}$.

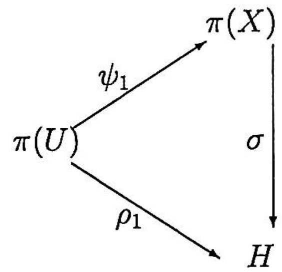
For determination of the structure of the fundamental group of a space the following theorem of Seifert and Van Kampen is very useful.

Theorem 1.20 ([53], p.114)

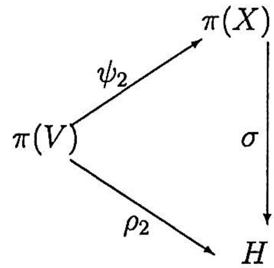
Let H be any group, and ρ_1, ρ_2, and ρ_3 any three homomorphisms such that the following diagram is commutative:



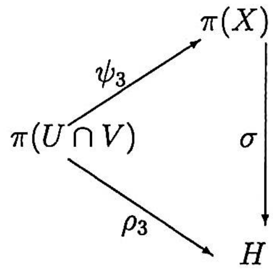
Then, there exists a unique homomorphism $\sigma : \pi(X) \longrightarrow H$ such that the following three diagrams are commutative:



,



and



1.3.4 Homology Group $H_n(X)$

One of the useful tools in classifying topological spaces is the concept of homology. To a topological space X, an abelian group $H_n(X)$, called the homology group of X, is associated for each positive integer n. These give us some informations about the topological nature of the space X. In particular, if $H_n(X) \ncong H_n(Y)$ for some n, then X and Y are not homeomorphic. Hence the study of the homology groups $H_n(X)$ for a topological space is useful for classifying spaces.

The homology groups are in general expressed as direct sums of cyclic groups. If

$$H_n(X) = (\underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n \text{ terms}}) \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_r},$$

then n is called the Betti number and $m_1, \cdots, m_r$ are called the torsion coefficients of X. These are topological invariants.

There is a famous theorem in the homology theory of simplicial complexes and also in general theories like Czech homology which expresses the homology of a union of subspaces in terms of the homologies of the subspaces and their intersections. This result is expressed in terms of a long exact sequence of abelian groups.

Theorem 1.21 (Mayer-Vietoris sequence)

Let K_1 and K_2 be two subcomplexes of a complex K. Then the following sequence is exact :

$$\cdots\cdots\cdots \xrightarrow{s_*} H_{p+1}(K_1 \cup K_2) \xrightarrow{v_*} H_p(K_1 \cap K_2) \xrightarrow{j_*} H_p(K_1) \oplus H_p(K_2)$$
$$\xrightarrow{s_*} H_p(K1 \cup K_2) \xrightarrow{v_*} H_{p-1}(K_1 \cap K_2) \xrightarrow{j_*} \cdots\cdots\cdots .$$

We do not describe the maps s_*, v_*, j_* here. For detail description of the maps s_*, v_*, j_* we refer the reader to *Hilton and Wiley ([28], p-290)*.

In particular, if $K_1 \cap K_2$ is a singleton, then $H_n(K_1 \cap K_2) = 0, \quad n > 0$. In this case, the structure of $H_n(K_1 \cup K_2)$ becomes easier to describe. In fact, $H_n(K_1 \cup K_2) \cong H_n(K_1) \oplus H_n(K_2)$. If $X = T^2$, the torus, then $H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$. In fact, if X is a sphere with g handles, then $H_1(X) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2g \text{ terms}}$. From here we obtain:

Corollary 1.22

If $K_1 \cap K_2$ is a singleton, then $H_n(K_1 \cap K_2) \cong H_n(K_1) \oplus H_n(K_2)$.

1.3.5 The De Rham Complex

In this context, Ω^* is the algebra over $\mathbb{R}$, generated by $dx_1, \cdots, dx_n$, with the relations

$$\begin{cases} (dx_i)^2 = 0 \\ dx_i\, dx_j = -dx_j\, dx_i, \ i \neq j\,. \end{cases}$$

As a vector space over $\mathbb{R}$, Ω^* has basis

$$1,\ dx_i,\ dx_i\, dx_j\ (i < j),\ dx_i\, dx_j\, dx_k\ (i < j < k),\ \cdots, dx_1 \cdots dx_n\,.$$

The C^∞ differential forms on $\mathbb{R}^n$ are elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{functions on } \mathbb{R}^n\} \otimes_{\mathbb{R}} \Omega^*\,.$$

Thus, if ω is such a form, then ω can be uniquely written as $\sum f_{i_1 \cdots i_q} dx_{i_1} \cdots dx_{i_q}$ where the coefficients $f_{i_1 \cdots i_q}$ are C^∞ functions. We also write $\omega = \sum f_I\, dx_I$. The algebra $\Omega^*(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$ is graded, where $\Omega^q(\mathbb{R}^n)$ consists of the C^∞ q-forms on $\mathbb{R}^n$. There is a differential operator $d : \Omega^q(\mathbb{R}^n) \longrightarrow \Omega^{q+1}(\mathbb{R}^n)$, defined as follows:

i) if $f \in \Omega^0(\mathbb{R}^n)$, then $df = \sum \partial f / \partial x_i\, dx_i$

ii) if $\omega = \sum f_I\, dx_I$, then $d\omega = \sum df_I\, dx_I$.

The Mayer Vietoris sequence for De Rham cohomology will now be considered. We first reformulate the Mayer-Vietoris sequence for two open sets as follows. Let $\mathcal{U}$ be the open cover $\{U, V\}$. Consider the double complex $C^*(\mathcal{U},\Omega^*) = \oplus K^{p,q} = \oplus C^p(\mathcal{U}, \Omega^q)$ where,

$K^{0,q} = C^0(\mathcal{U},\Omega^q) = \Omega^q(U) \oplus \Omega^q(v)$,

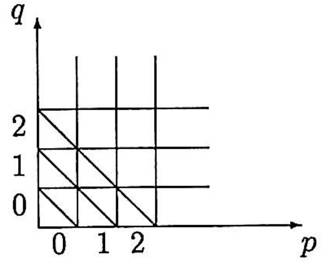
$K^{1,q} = C^1(\mathcal{U},\Omega^q) = \Omega^q(U \cap V)$,

$K^{p,q} = 0,\ p \geq 2.$

This double complex is two differential operators, the exterior derivative d and the difference operator δ. Since the operators d and δ are independent, they commute.

In general given a doubly graded complex $K^{*,*}$ with commuting differentials d and δ, one can form a singly graded complex K^* by summing along the antidiagonal lines

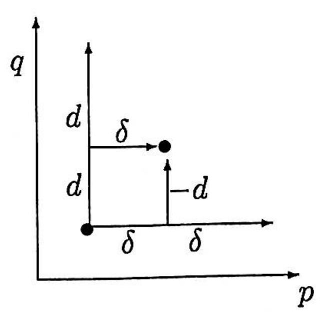
$K^n = \oplus_{p+q=n} K^{p,q}$



and defining the differential operator to be

$D = D' + D''$ *with* $D' = \delta,\ D'' = (-1)^p d$ on $K^{p,q}$.

Remark:



If D were naively defined as $\bar{D} = d{+}\delta$, it would not be a differential operator since $\bar{D}^2 = 2\,d\,\delta \neq 0$. However, if we alternate the sign of d from one column to the next, then as is apparent from the diagram above,
$D^2 = d^2 + \delta d - d\delta + \delta^2 = 0.$

Theorem 1.23

The double complex $C^*(\mathcal{U}, \Omega^*)$ computes the de Rham cohomology of M:

$$H_D\{C^*(\mathcal{U}, \Omega^*)\} \simeq \mathcal{H}^*_{\mathcal{DR}}(\mathcal{M}).$$

The Mayer-Vietoris sequence will be generalized from two open sets to countably many open sets. The result of this generalization yields:

Theorem 1.24 (Generalized Mayer-Vietoris Principle)

The double complex $C^*(\mathcal{U}, \Omega^*)$ computes the de Rham cohomology of M; more precisely, the restriction map $r : \Omega^*(M) \longrightarrow C^*(\mathcal{U}, \Omega^*)$ induces an isomorphism in cohomology:

$$r^* : H^*_{DR}(M) \longrightarrow H_D\{C^*(\mathcal{U},\, \Omega^*)\}\,.$$

Thus using the conclusion of the above theorem the Kunneth formula can be stated as below:
If M and N are two manifolds and F has finite-dimensional cohomology, then the de Rham cohomology of the product $M \times F$ is
$H^*(M \times F) = H^*(M) \otimes H^*(F)$.

Let G be a topological group which acts effectively on a space F on the left. A surjection $\pi : E \longrightarrow B$ between topological spaces is a *fibre bundle* with fibre F and structure group G if B has an open cover $\{U_\alpha\}$ such that there are fibre-preserving homeomorphisms $\phi_\alpha : E|_{U_\alpha} \simeq U_\alpha \times F$ and the transition functions are continuous functions with values in G:

$$g_{\alpha\beta}(x) = \phi_\alpha \phi_{\beta^{-1}} |\{x\} \times F \in G.$$

sometimes the total space E is refered to as the fibre bundle.

Using spectral sequences one has the following:

Theorem 1.25 (Leray's Theorem for De Rham Cohomology)

Given a fiber bundle $\pi : E \longrightarrow M$ with fiber F over a manifold M and a good cover $\mathcal{U}$ of M, there is a spectral sequence $\{E_r\}$ converging to the cohomology of the total space $H^*(E)$ with E_2 term $E_2^{p,q} = H^p(\mathcal{U}, \mathcal{H}^q)$, where $\mathcal{H}^q$ is the locally constant presheaf $\mathcal{H}^q(U) = H^q(\pi^{-1}U)$ on $\mathcal{U}$. If M is simply connected and $H^q(F)$ is finite-dimensional, then $E_2^{p,q} = H^p(M) \otimes H^q(F)$.

In the above theorem, a cover U_α of an n-manifold M is a *good cover* if every finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is diffeomorphic to $\mathbb{R}^n$.

We shall state the Leray's theorem for singular cohomology with coefficients in a commutative ring and the Kunneth formula for singuar cohomology respectively in the following:

Theorem 1.26 (see Bott and Loring [6])

Let $\pi : E \longrightarrow X$ be a fiber bundle with fiber F over a topological space X and $\mathcal{U}$ an open cover of X. Then there is a spectral sequence converging to $H^*(E;\ A)$ with E_2 term $E_2^{p,q} = H^p(\mathcal{U},\mathcal{H}^q(F;\ A))$.
Each E_r in the spectral sequence can be given a product structure relative to which the differential d_r is an antiderivation. If X is simply connected and has a good cover, then $E_2^{p,q} = H^p(X, H^q(F;\ A))$. If in addition $H^*(F;\ A)$ is finitely generated free A-module, then $E_2 = H^*(X;\ A) \otimes H^*(F;\ A)$ algebras over A.

Theorem 1.27

If X is a space having a good cover, e.g., a triangularizable space, and Y is any topological space, prove using the spectral sequence of the fiber bundle $\pi : X \times Y \longrightarrow X$ that $H^n(X \times Y) = \oplus_{p+q=n} H^p(X,\ H^q(Y))$.

We shall now state the universal coefficient theorem ([6], p.194) which is as below:

Theorem 1.28 (Universal Coefficient Theorems)

For any space X and abelian group G,
(a) the homology of X with coefficients in G has a splitting: $H_q(X;\ G) \simeq H_q(X) \otimes G \oplus Tor(H_{q-1}(X), G)$;
(b) the cohomology of X with coefficients in G has also a splitting: $H^q(X;\ G) \simeq Hom(H_q(X), G) \oplus Ext(H_{q-1}(X),\ G)$.
Applying part (b) with $G = \mathbb{Z}$ yields the following formula for the integer cohomology in terms of the integer homology.

Corollary 1.29

For any space X for which $H_q(X)$ and $H_{q-1}(X)$ are finitely generated $\mathbb{Z}$-modules, $H^q(X) \simeq F_q \oplus T_{q-1}$, where F_q is the free part of $H_q(X)$ and T_{q-1} is the torsion part of H_{q-1}.

1.4 Geometrical Structures

1.4.1 Classifications of 2D and 3D Geometries

The geometry of 2-dimensions have been classified long ago. There are three such geometries:

(i) Euclidean geometry,

(ii) Riemannian geometry, and

(iii) Hyperbolic geometry.

The classification of 3-dimensional geometries have been accomplished by William Thurston (see [70], [71], [72], [73]). It is one of the most remarkable mathematical discoveries of the twentieth century. The classification is described in the following theorem:

Theorem 1.30 (Scott [65], p.474)

Any maximal, simply connected, 3-dimensional geometry which admits a compact quotient is equivalent to one of the geometries $(X, Isom\,X)$ where X is one of E^3, H^3, S^3, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL_2 \mathbb{R}}$, Nil or Sol.

In the above theorem, the 3-dimension Lie group of all 2×2 real matrices with determinant 1 is $SL_2\mathbb{R}$ and $\widetilde{SL_2 \mathbb{R}}$ is its universal covering. $SL_2\mathbb{R}$ is called a *special linear group* and $\widetilde{SL_2 \mathbb{R}}$ is just a line bundle over H^2.

Nil is the 3-dimensional Lie group which consists of all 3×3 real upper triangular matrices of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ under multiplication. It is also called the *Heigenberg group.* Nil is just a line bundle over E^2.

Sol is defined as the split extension of $\mathbb{R}^2$ by $\mathbb{R}$. One has an exact sequence

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow Sol \longrightarrow \mathbb{R} \longrightarrow 0 .$$

If we identify *Sol* with $\mathbb{R}^3$ so that the xy-plane corresponds to the subgroup $\mathbb{R}^2$, we can write down the multiplication of *Sol*, and an invariant metric.

The multiplication is given by

$$(x,\, y\,, z)\,(x',\, y',\, z') = (x + e^{-z}x',\, y + e^{z}y',\, z + z')$$

A left invariant metric on $\mathbb{R}^3$ is given by the formula

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2 \;.$$

Sol has the isometry group consisting of the eight maps $(x,\, y,\, z) \longrightarrow (\pm x,\, \pm y,\, z)\,,\; (x,\, y,\, z) \longrightarrow (\pm y,\, \pm x,\, -z)$. This is precisely the dihedral group $D(4)$.

1.4.2 Differential Geometry

We shall state here a few structural results in differential geometry with a brief background.

Let $M_p^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemann ian manifold of index p and of constant curvature c, which is called as an *indefinite space form* of index p. The standard models of indefinite space forms are given as follows. In an $(n+p)$-dimensional real vector space $\mathbf{R}^{n+p}$ with the standard basis, the scalar product $\langle , \rangle$ is given by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i - \sum_{j=n+1}^{n+p} x_j y_j,$$

where $x = (x_1, x_2, \cdots, x_{n+p})$ and $y = (y_1, y_2, \cdots, y_{n+p})$. Then $(\mathbf{R}^{n+p}$ $, \langle , \rangle)$ is an indefinite Euclidean space, which is denoted by $\mathbf{R}_p^{n+p}$.

Let $S_p^{n+p}(c)$ for $c > 0$ be the hypersurface in $\mathbf{R}_p^{n+p+1}$ given as $\langle x, x\rangle = \frac{1}{c} =: r_0^2$. Then $S_p^{n+p}(c)$ inherits an indefinite Riemannian metric induced through $\mathbf{R}_p^{n+p+1}$ and has constant curvature c. This is called a de Sitter space of constant curvature c with index p. On the other hand, let $\mathbf{H}_p^{n+p}(c)$ for $c < 0$ be the hypersurface in $\mathbf{R}_{p+1}^{n+p+1}$ given as $\langle x, x\rangle = \frac{1}{c} =: -r_0^2$.

T. Ishihara [31] prove the result which is as below:

Let M^n be an n-dimensional complete maximal spacelike submanifold in $\mathbf{H}_p^{n+p}(c)$, then $S \leq -npc$ and $S = -npc$ if and only if $M^n = \mathbf{H}_1^n(\frac{nc}{n_1}) \times \cdots \times \mathbf{H}_{p+1}^n(\frac{nc}{n_{p+1}})$, where S is the squared norm of the second fundamental form of M^n.

In the consequence of the above result Cheng proved in [13] that the following result holds:

Theorem 1.31

Let M^2 be a complete maximal spacelike surface of an anti-de Sitter space $\mathbf{H}_2^4(c)$ with constant scalar curvature, then $S = 0$, $S = \frac{-10c}{11}$, $S =$

$\frac{-4c}{3}$ *or* $S = -2c$, *where* S *is the squared norm of the second fundamental form of* M^2. *And*

(a) $S = 0$ *if and only if* M^2 *is the totally geodesic surface* $\mathbf{H}^2(c)$;

(b) $S = \frac{-4c}{3}$ *if and only if* M^2 *is the hyperbolic Veronese surface;*

(c) $S = -2c$ *if and only if* M^2 *is the hyperbolic cylinder of the totally geodesic surface* $\mathbf{H}_1^3$ *of* $\mathbf{H}_2^4$.

Remark: It is still open for the author whether there exists complete maximal spacelike surfaces of the anti-de Sitter space $\mathbf{H}_2^4$ with $S = \frac{-10c}{11}$. In the above theorem, $\mathbf{H}^{n_i}(c_i)$ is an n_i-dimensional hyperbolic space of constant curvature c_i, called the *hyperbolic Veronese surface.*

Hopf proved that compact surfaces with constant mean curvature and with genus zero are the standard spheres. Hopf's result was extended to complete surfaces in E^3 by Klotz and Osserman in [38] as following:

Theorem 1.32 (T. Klotz and Osserman)

Let M *be a complete and connected surface with constant mean curvature in* E^3. *If the Gauss curvature of* M *is nonnegative, then* M *is the plane* E^2, *the sphere* $S^2(c)$ *or the cylinder* $S^1(c) \times E^1$.

By using the above theorem Cheng and Nonaka ([15], p.353) proved the following:

Theorem 1.33

Let M *be an* n*-dimensional complete connected submanifold with parallel mean curvature vector* H *in* $E^n + p$, $n \geq 3$. *If the second fundamental form* h *of* M*satisfies* $\langle h \rangle^2 \leq \frac{n^2|H|^2}{n-1}$, *then* M *is the totally* *geodesic Euclidean space E^n, the totally umbilical sphere $S^n(c)$ or the generalized cylinder $S^{n-1}(c) \times E^1$ in E^{n+1}.*

The metric structure on conformally flat Riemannian manifold M^n will be specified under certain restrictions on the behavior of Ricci curvature tensor of M^n for the classification of them. Throughout the following two theorems, let M^n be a connected complete and conformally flat Riemannian manifold of constant scalar curvature r without boundary. Then one has the following:

Theorem 1.34 ([14], p.210)

Let M^3 be a 3-dimensional complete conformally flat Riemannian man -ifold with constant curvature and constant squared norm of the Ricci curvature tensor. Then we have
(i) If the scalar curvature r is nonnegative, M^3 is either isometric to a space form or else the Riemannian product $M^2(c) \times N^1 \quad (c \geq 0)$.
(ii)If the scalar curvature r is negative, either M^3 is isometric to a space form or else the squared norm of the Ricci curvature tensor of M^3 lies in $(r^2/3,\ r^2/2]$.

In addition to the assumption in the above theorem, if $r \geq 0$, then M^3 is either isometric to a space form or the Riemannian product $M^2 \times N^1$, where M^2 and N^1 are of constant curvature with dimension 2 and 1 respectively.

Let M be an n-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. Let S denote the squared norm of the second fundamental form of M. When the scalar curvature of

M is constant, Yang and cheng proved in [77] that if $n \leq S \leq n + \frac{n}{3}$, then M is isometric to a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$.

1.5 Structures of the 10D Manifold in String Theory

Before giving the structures , we need to recall the **Kahler manifold.** A Riemannian manifold (M_n, g) is said to be **Kahler** if there exists a tensor $f^i{}_j$ (the complex structure) on M which satisfies

$f^i{}_k f^k{}_j = -\delta^i{}_j$

$g_{ij} f^i{}_k f^j{}_l = g_{kl}$

$\nabla_i f^j{}_k = 0.$

On a Kahler manifold, we can consider forms which are p-times holomorphic (contain p dz's) and q times antiholomorphic (q dz's) so that *the space of r-forms* $\bigwedge^r$ *splits naturally into* $\bigwedge^r = \bigwedge^{(r,0)} \oplus \bigwedge^{(r-1,1)} \oplus \cdots \oplus \bigwedge^{(0,r)}$.

Similarly, the exterior derivative d and its adjoint $\delta = d^*$ can be split as $d = \partial + \bar{\partial}$, $\delta = \partial^* + \bar{\partial}^*$ $(\partial^* = - * \bar{\partial} *$, $\bar{\partial}^* = * \partial *$, $\bar{\partial}^* = - * \partial *)$, so that $\partial : \bigwedge^{p,q} \longrightarrow \bigwedge^{p+1,q}$, $\bar{\partial} : \bigwedge^{p,q} \longrightarrow \bigwedge^{p,q+1}$, $\delta^* : \bigwedge^{(p,q)} \longrightarrow \bigwedge^{(p-1,q)}$, and $\bar{\partial}^* : \bigwedge^{(p,q)} \longrightarrow \bigwedge^{(p,q-1)}$. It also follows from the covariant constancy of the complex structure that the Laplacian on forms satisfies $\diamond \equiv d\delta + \delta d = 2(\partial\partial^* + \partial^*\partial) = 2(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$, and that $\partial\bar{\partial}^* + \bar{\partial}^*\partial = \bar{\partial}\partial^* + \partial^*\bar{\partial} = 0$. Since the generators of the cohomology groups are the harmonic forms,*we have* $H^r = H^{(r,0)} \oplus H^{(r-1,1)} \oplus \cdots \oplus H^{(0,r)}$, where $H^{(p,q)}$ is generated by harmonic forms of type (p, q).

On a Kahler manifold the standard formulae of Riemannian geometry simplify dramatically. The curvature tensor becomes

$R^{\alpha}{}_{\beta\bar{\rho}\gamma} = \partial_{\bar{\rho}}\Gamma^{\alpha}_{\beta\gamma}$, where $\Gamma^{\alpha}_{\beta\gamma} = g^{\alpha\bar{\rho}}\partial_{\beta}g_{\gamma\bar{\rho}}$.

and the cyclic and Bianchi identites reduce to

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}},\ R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\delta}\gamma\bar{\beta}}$$

$$\nabla_{\lambda}R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla_{\alpha}R_{\lambda\bar{\beta}\gamma\bar{\delta}},\ \nabla_{\bar{\lambda}}R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla_{\bar{\beta}}R_{\alpha\bar{\lambda}\gamma\bar{\delta}} \quad .$$

The Ricci tensor on a Kahler manifold turns out to take the simple form

$$R_{\alpha\bar{\beta}} = g^{\mu\bar{\lambda}}R_{\bar{\lambda}\mu\alpha\bar{\beta}} = -\partial_{\alpha}\partial_{\bar{\beta}}\, ln\, det(g) \quad .$$

$R_{\alpha\bar{\beta}}$ is a Kahler tensor, and generates a non-trivial cohomology class in general because in going from a coordinate patch z^{α} to another z'^{α}, $log\, det\, g \longrightarrow log\, det\, g + h(z) + \bar{h}(\bar{z})$, where $h(z) = log\, det\,(\frac{\partial z^{\alpha}}{\partial z'^{\lambda}})$. In fact the Ricci form

$$\sum = i\, R_{\alpha\bar{\beta}}dz^{\alpha} \wedge d\bar{z}^{\beta}\,,$$

can be shown to represent the first **Chern class** on any Kahler manifold.

We shall give the structure here in the following manner:

The ten dimensional manifold on which string theories are described has the structure $M^4 \times K$, where M^4 is the four dimensional Minkowski space and K is a three dimensional complex manifold - the Ricci flat Kahler manifold (see [78], [79]).

The requirement that K be a three complex dimensional Ricci flat Kahler manifold is quite a restrictive one. We describe this in the following:

Recall that a six dimensional real manifold can be viewed as patches of $\mathbb{R}^6$ which are glued together at the edges by identifying

points of one patch with those of another in a smooth (i.e., $\mathbb{C}^\infty$) manner. Similarly, a three dimensional complex manifold can be viewed as patches of $\mathbb{C}^3$ glued together in a holomorphic i.e., complex analytic manner. Here glueing mathematically means considering suitable identification spaces. Since holomorphic functions are always smooth, it is clear that every complex three manifold can be viewed as a real six manifold. However, the converse is not always the case.
For example, although it is not obvious, the familiar manifolds S^6 and $S^2 \times S^4$ cannot be viewed as complex manifolds, although $S^3 \times S^3$ and $S^1 \times S^5$ can.

To understand the Kahler condition we must discuss the metric. The analog of a positive definite metric for a real manifold is a hermitian metric for a complex manifold. Given a hermitian metric one can define a unique (torsion free, metric compatible) covariant derivative. Now consider a vector v such that for any function f depending only on the complex coordinates z^{-i} and not on z^i we have $v^i \nabla_i f = 0$. Such a vector is called holomorphic. In most cases, if one starts with a holomorphic vector and parallel transports it along a curve, then the vector will not remain holomorphic.

However, there are special metrics for which this difficulty doesn't occur, namely those with $U(3)$ holonomy. These special metrics are called **Kahler**. Not every complex manifold admits a Kahler metric. For example, no metric on $S^3 \times S^3$ or $S^1 \times S^5$ can be Kahler, although $S^2 \times S^2 \times S^2$ does admit Kahler metrics. One can view a Kahler manifold as the nicest type of complex manifold in that the metric structure and the complex structure are compatible in the above sense.

1.6 Lie Groups

1.6.1 Preliminaries

Consider spaces which behave locally like $\mathbb{R}^n$. Thus consider a topological space T and let W be a non-empty open subspace of T which is homeomorphic to an open subspace X of $\mathbb{R}^n$. If $\sigma : W \longrightarrow X$ denotes a homeomorphism of W onto X, we call σ *a chart* in T, or, more precisely, on W.

A topological space T is said to be *locally Euclidean at a point p,* if there exists a chart σ on a neighbourhood of p; we then say that σ is *a chart at p.* A Hausdorff space which is locally Euclidean at each point is a manifold. Thus in a manifold M each point has a chart defined on some neibourhood so that the family of all charts in M covers M.

Let M be a Hausdorff space. Then an *analytic structure* on M is a family $\mathcal{F}$ of charts defined in M such that

(i) at each point of M there is a chart which belong to $\mathcal{F}$,

(ii) any two charts of $\mathcal{F}$ are analytically related,

(iii) any chart in M which is analytically related to every chart of $\mathcal{F}$ itself belong to $\mathcal{F}$.

We shall express (ii) and (iii) by saying that $\mathcal{F}$ is *analytic and maximal,* respectively. Thus an analytic structure on M is a maximal analytic family of charts covering M. It is clear that a Hausdorff space with an analytic structure is necessarily a manifold, and the space, together with this structure, is called an *analytic manifold.*

Let M and N be any manifolds, and consider the mapping $\Phi : p \longrightarrow \Phi(p)$ of M into N. For each function f defined in N we can define a function f^* in M by the rule $f^*(p) = f(\Phi(p)) \quad (p \in M)$ and this f^* will be denoted by $\{\Phi\}f$. Suppose now that $\mathcal{M}$ and $\mathcal{N}$ are analytic manifolds. Then the mapping $\Phi : \mathcal{M} \longrightarrow \mathcal{N}$ is said to be *analytic*, if $\{\Phi\}f \in \mathcal{A}(\mathcal{M})$ whenever $f \in \mathcal{A}(\mathcal{N})$, where $\mathcal{A}(\mathcal{M})$ and $\mathcal{A}(\mathcal{N})$ denote the set of analytic functions in $\mathcal{M}$ and $\mathcal{N}$ respectively. Thus every analytic mapping Φ of $\mathcal{M}$ into $\mathcal{N}$ induces a mapping $\{\Phi\}$ of $\mathcal{A}(\mathcal{M})$ into $\mathcal{A}(\mathcal{N})$.

Topological groups and Lie groups are such objects which carry a two-fold structure such as algebraic and topological or algebraic and differential. A set G whose members can be subjected to a group operation as well as can be considered as points of a topological space is called a *topological group* provided the group and the topological structures can be combined together by asking that:

(a) The map $\phi : G \times G \longrightarrow G$ defined as $(\sigma, \Gamma) \longrightarrow \sigma\Gamma^{-1}$ be continuous.

If we replace the topological space by a topological manifold with differential structure and condition of continuity in (a) by C^∞-differenti ability, then then what we obtain is *Lie group.* Thus, a Lie group G is a group which is also a differentiable manifold and for which the mapping ϕ of $G \times G \longrightarrow G$ given by :

(a') $(\sigma, \Gamma) \longrightarrow \sigma\Gamma^{-1}$ is differentiable for all pairs $(\sigma, \Gamma) \in G \times G$.

1.6.2 Classification of Lie Groups

In addition to the Lie groups of translations in n-dimensional space, there are four series of lie groups:

(A) Unitary transformations in n-dimensional complex space;

(B) Rotations in odd-dimensional real space;

(C) Transformations in n-dimensional quaternion space; and

(D) Rotations in even-dimensional real space.

The (B) and (D) are real rotations, denoted spin$(2n+1)$ and spin$(2n)$, and are called *spin groups*, the double covers of special orthogonal groups; the (A) are complex generalized rotations, denoted SU$(n+1)$, and are called special *unitary groups*; and the (C) are quaternionic generalized rotations, denoted Sp(n), and are called *symplectic groups.*

The only other Lie groups that exist are 5 exceptional ones:

$$G2,\ F4,\ E6,\ E7,\ and\ E8.$$

One should not be surprised about the two facts:
the exeptional Lie groups are all related to the octonions; and they do not form an infinite series because the non-associativity of the octonions terminates the series.

$G2$ is the automorphism group of the octonions, that is, the group of operations on the octonions that preserve the octonion product. $G2$ is 14-dimensional and its smallest non-trivial representation is 7-dimensional.

$F4$ is the automorphism group of 3×3 matrices of octonions

$$\begin{matrix} o11 & o12 & o13 \\ o21 & o22 & o23 \\ o31 & o32 & o33 \end{matrix}$$

such that $o11$, $o22$, and $o33$ are real (have no imazinary part), and $o12$, $o13$, $o23$ are the octonion conjugates of $o21$, $o31$, $o32$ respectively (*such matrices are called Hermitian matrices*). $F4$ is 52-dimen sional and its smallest non-trivial representation is 26-dimensional.

$E6$ is in some sense $F4$ expanded by the complex numbers. $E6$ is 78-dimensional and its smallest non-trivial representation is 27-dimensional.

$E7$ is in some sense $F4$ expanded by the quaternions. $E7$ is 133-dimensional and its smallest non-trivial representation is 56-dimensi onal.

$E8$ is in some sense $F4$ expanded by the octonions. $E8$ is 248-dimensional and its smallest non-trivial representation is also 248-dimensional.

These mentioned above are all the Lie groups that exist.

The octonions referred to the above are described below:

1.6.3 Quaternions and Octonions

A *division algebra* is a vector space with multiplication of vectors such that the system is a skew field. There are four division algebras over $\mathbb{R}$ (kervaire [37], Bott-Milnor [5]). These are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, where H is the algebra of quaternions with a basis $\{i, i, j, k\}$ satisfying multiplicative relations $i^2 = j^2 = k^2 = 1$, $ij = -ji = k$ (Cayley [12], Hamilton [26]), and $\mathbb{O}$ is the algebra of octonions with a basis $\{1, e_1, \cdots, e_7\}$ with multiplication table (Baej [4])

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

The octonions are very much used in modern physics specially in string theory (see [11],[64]).

1.7 Sheaves and Schemes in Algebraic Geometry

The idea of schemes in connection with algebraic geometry was developed by Grothendieck. The concept of a sheaf provides a systematic way of keeping track of local algebraic data on a topological space. Sheaves are essential in the study of schemes. In fact, we can not even define a scheme without sheaves.

1.6.1 Presheaf

Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups on X consists of the data

(a) for every open subset $U \subseteq X$, an abelian group $\mathcal{F}(\mathcal{U})$, and
(b) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{uv} : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(\mathcal{V})$, subject to the conditions
(i) $\mathcal{F}(\phi) = \prime$, where ϕ is the empty set,
(ii) ρ_{uv} is the identity map $\mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(\mathcal{U})$, and
(iii) if $W \subseteq U \subseteq V$ are three open subsets, then $\rho_{uw} = \rho_{vw} \circ \rho_{uv}$.

We define a *presheaf of rings, a presheaf of sets*, or a presheaf with values in any fixed catagory $\mathcal{C}$, by replacing the words abelian group in the definition by ring, sets, or object of $\mathcal{C}$ respectively. We will stick to the case of abelian groups in this section, and let the reader make the necessary modification for the case of rings, sets, etc.

1.6.2 Sheaf

A presheaf $\mathcal{F}$ on a topological space X is a *sheaf* if it satisfies the following supplementary conditions:
(a) if U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(\mathcal{U})$ is an element such that $s\,|_{V_i} = 0$ for all i, then $s = 0$;
(b) if U is an open set, if $\{V_i\}$ is an open covering of U, and if we have elements $s_i \in \mathcal{F}(V_i)$ for each i, with the property that for each i, j, $s_i\,|_{V_i \cap V_j} = s_j\,|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(\mathcal{U})$ such that $s\,|_{v_i} = s_i$ for each i. Note that (a)implies that s is unique.

Direct Sum of Sheaves: Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X. Then the presheaf $U \longmapsto \mathcal{F}(\mathcal{U}) \oplus \mathcal{G}(\mathcal{U})$ is a sheaf. It is called the direct sum of $\mathcal{F}$ and $\mathcal{G}$, and is denoted by $\mathcal{F} \oplus \mathcal{G}$. In fact, it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.

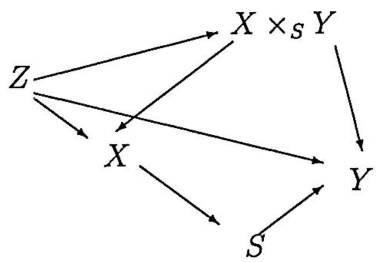
Gluing Sheaves: Let X be a topological space, let $\mathcal{U}= \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j an isomorphish $\varphi_{ij} :\mathcal{F}|_{U_i \cap U_j} \longrightarrow \mathcal{F}_j \ |_{U_i \cap U_j}$ such that (1) for each i, $\varphi_{ii} = id$, and (2) for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i :\mathcal{F}|_{U_i} \longrightarrow \mathcal{F}_i$ such that for each i, j, $\varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathcal{F}$ is obtained by glueing the sheaves $\mathcal{F}_i$ via the isomorphisms φ_{ij}.

1.6.3 Scheme

An *affine scheme* is a locally ringed space $(X, \mathcal{O}_{\mathcal{X}})$ which is isomorphic (as a locally ringed space) to the spectrum of some ring (for the ringed space and spectrum of a ring, we refer the reader to R. Hartshorne [80]. A *scheme* is a locally ringed space $(X, \mathcal{O}_{\mathcal{X}})$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_{\mathcal{X}} \ |_{\mathcal{U}}$, is an affine scheme. We call X the underlying topological space of the scheme $(X, \mathcal{O}_{\mathcal{X}})$, and $\mathcal{O}_{\mathcal{X}}$ its structure sheaf. We will open write simply X for the scheme $(X, \mathcal{O}_{\mathcal{X}})$. A morphism of schemes is a morphism as locally ringed space.

Example: If K is a field, Spec$k$ is an affine scheme whose topological space consists of one point, and whose structure sheaf consists of the field k.

Fibred Product: Let S be a scheme, and let X, Y be schemes over S, i.e., schemes with morphisms to S. We define the fibred product of X and Y over S, denoted by $X \times_S Y$, to be a scheme, together with morphisms $p_1 : X \times_S Y \longrightarrow X$ and $p_2 : X \times_S Y \longrightarrow Y$, which make a commutative diagram with the given morphisms $X \longrightarrow S$ and $Y \longrightarrow S$, such that given any scheme Z over S, and given morphisms $f : Z \longrightarrow X$ and $g : Z \longrightarrow Y$ which make commutative diagram with the given morphisms $X \longrightarrow S$ and $Y \longrightarrow S$, then there exists a unique morphism $\theta : Z \longrightarrow X \times_S Y$ such that $f = p_1 \circ \theta$, and $g = p_2 \circ \theta$. The morphisms p_1 and p_2 are called the projection morphisms of the fibred product onto its factors.



Theorem 1.35

For any two schemes X and Y over a scheme S, the fibred froduct $X \times_S Y$ exists, and is unique up to unique isomorphism.

1.8 Riemannian Surface

Let X be a two dimensional manifold. A *complex chart* on X is a homeomorphism $\varphi : U \longrightarrow V$ of an open subset $U \subset X$ onto an open subset $V \subset \mathbb{C}$. Two complex charts $\varphi_i : U_i \longrightarrow V_i$, $i = 1, 2$ are said to be *holomorphically compatible* if the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi(U_1 \cap U_2)$$

is biholomorphic (see Fig.1 in the next page).

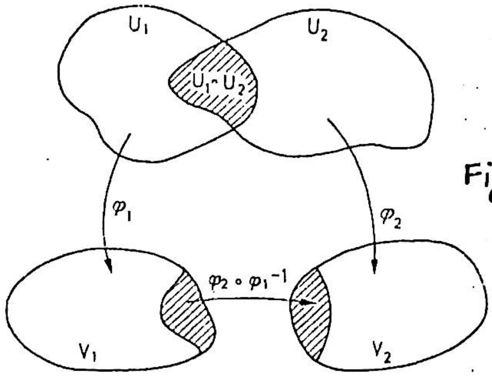


Fig. 1

A *complex atlas* on X is a system $\mathcal{U} = \{\varphi_i : U_i \longrightarrow V_i,\ i \in I\}$ of charts which are holomorphically compatible and which cover X. Two complex atlases $\mathcal{U}$ and $\mathcal{U}'$ are called *analytically equivalent* if every chart of $\mathcal{U}$ is holomorphically compatible with every chart of $\mathcal{U}'$. By a *complex structure* on a two dimensional manifold X we mean an equivalence class of analytically equivalent atlases on X.

A *Riemann surface* X is a connected two dimensional manifold together with a complex structure on X. If X is a Riemann surface, then by a chart on X we always mean a complex chart belonging to the maximal atlas of the complex structure on X.

Examples:

(a) *The complex plane* $\mathbb{C}$ is a Riemann surface. Its complex structure is defined by the atlas whose only chart is the idntity map $\mathbb{C} \longrightarrow \mathbb{C}$.

(b) *The Riemann sphere* $\mathbb{P}^1$ is a Riemann surface. Let $\mathbb{P} = \mathbb{C} \cup \{\infty\}$, where ∞ is a sybbol not contained in $\mathbb{C}$. Introduce the following topology on $\mathbb{P}^1$: the open sets are the usual open sets $U \subset \mathbb{C}$ together with sets of the form $V \cup \{\infty\}$, where $V \subset \mathbb{C}$ is the complement of a compact set $K \subset \mathbb{C}$. With this topology $\mathbb{P}$ is a compact hausdorff topological space, homeomorphic to the 2-sphere S^2. Set $U_1 = \mathbb{P}^1 - \{\infty\} = \mathbb{C}$ and $U_2 = \mathbb{P}^1 - \{0\} = \mathbb{C}^* \cup \{\infty\}$. Define maps $\varphi_i : U_i \longrightarrow \mathbb{C}$, $i = 1, 2$, as follows. φ_1 is the identity map and

$$\varphi_2(z) = \begin{cases} \frac{1}{z} & \text{for } z \in \mathbb{C} \\ 0 & \text{for } z = \infty . \end{cases}$$

Clearly these maps are homeomorphisms and thus $\mathbb{P}^1$ is a two dimensional manifold. Since U_1 and U_2 are connected and have non-empty intersection, $\mathbb{P}^1$ is also connected. The complex structure on $\mathbb{P}^1$ is now defined by the atlas consisting of the charts $\varphi_i : U_i \longrightarrow \mathbb{C}$, $i = 1, 2$.

Theorem 1.36

Suppose X is a topolgical space .and

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

is a short exact sequence of sheaves on X. Then the induced sequence of cohomology groups

$$\begin{gathered} 0 \longrightarrow H^0(X,\mathcal{F}) \xrightarrow{\alpha^0} H^o(X,\mathcal{G}) \xrightarrow{\beta^0} H^0(X,\mathcal{H}) \xrightarrow{\partial^*} \\ H^1(X,\mathcal{F}) \xrightarrow{\alpha^1} H^1(X,\mathcal{G}) \xrightarrow{\beta^1} H^1(X,\mathcal{H}) \end{gathered}$$

is exact.

For the details of the induced homomorphisms α^0, α^1, β^0, β^1 and the connecting homomorphisms ∂^* in the above theorem, we refer to Forster [20].

Dolbeault's Theorem 1.37

Let X be a Riemann surface. Then there are isomorphisms
(a) $H^1(X,\mathcal{O}) \cong \mathcal{E}^{0,1}(X)/d''\mathcal{E}(X)$,
(b)$H^1(X,\Omega) \cong \mathcal{E}^{(2)}(X)/d\,\mathcal{E}^{1,0}(X)$.

On every Riemannian surface X, every exact 1-form is closed but every closed form is not necessarily exact. Consequently one is interested in the quotient group

$$\mathbf{Rh}^1 = \frac{Ker(\mathcal{E}^{(1)}(X) \xrightarrow{d} \mathcal{E}^{(2)}(X))}{IM(\mathcal{E}(X) \xrightarrow{d} \mathcal{E}^{(1)}(X))}$$

of closed 1-forms modulo exact 1-forms. Two closed differential forms which determine the same element in $\mathbf{Rh}^1(X)$ is called the *1st deRham group* of X.

DeRham Theorem 1.38

Let X be a Riemann surface. Then $H^1(X,\mathbb{C}) \cong \mathbf{Rh}^1(X)$.

If X is a compact Riemann surface, then $dim\,H^1(X,\mathcal{O})$ is finite and it is called the *genus* of X, and hence one has the following:

Theorem 1.39

Every Riemann surface of genus zero is isomorphic to the Riemann sphere.

CHAPTER-2

SPECIAL SEMIGROUPS

2.1 Introduction

S. Majumdar and A. K. Mallick [49] studied a class of abelian groups which they termed special abelian groups. These groups have a set of generators such that each non-zero element of the group can be expressed uniquely in terms of the generators using each generator or its inverse, but not both, at most once. The structures of these abelian groups were determined in certain cases also. Following them and using Majumdar ideas, we have introduced in this chapter a class of commutative semigroups termed special semigroups and studied their properties. These semigroups have a set of generators similar to those for special abelian groups. These generators have been termed special generators. Some characterization theorems have been derived and the structures of these semigroups have been determined in certain cases. The automorphism groups and the endomorphism semigroups for these semigroups have been also discussed in this chapter.

2.2 Special Semiroups

Consider a type of semigroups which have a set of generators such that each non-zero element of the group can be expressed uniquely in terms of the generators using each generator at most once. These semigroups will be called the *special semigroups* and the generators will be known as the *special generators.* These kind of semigroups occur in abundance. $\mathbb{Z}^+$, $\mathbb{Q}^+$, $\mathbb{N}(2)$ are a few examples of such special semigroups. Here $\mathbb{Z}^+$ is the additive semigroup of all positive integers, $\mathbb{Q}^+$ is the additive semigroup of all positive rational numbers, and $\mathbb{N}(2)$ is the additive semigroup of all positive rational numbers which have denominators of the form 2^x, x being a non-negative integer.

We shall prove below that the above three semigroups $\mathbb{Z}^+$, $\mathbb{Q}^+$ and $\mathbb{N}(2)$ are indeed special.

Theorem 2.1

Every infinite cyclic semigroup is special.

Proof:

Suppose S is an infinite cyclic semigroup with generator x, and let $A = \{2^r x \ / \ r = 0, 1, 2, 3\}$. We shall prove that A is a special set of generators of S. Suppose that s is any element in S. Then $s = nx$, for some positive integer n. Since n has a unique expression in the binary system , i.e., $n = \sum_{k=0}^{\infty} n_k 2^k$, with $n_k = 0$ or 1 with only a finite number of $n_k \neq 0$ (or equivalently, as $n = \{n_k\}$ with n_k's as before), A is a special set of generators of S. Therefore S is special.

Clearly $\mathbb{Z}^+$ is an infinite cyclic semigroup with generator 1.

2.3 Binary Representation for the Elements of $\mathbb{Q}^+$

Let $\mathbb{Q}^+$ denote the additive group of all positive rational numbers. We shall represent each element x of $\mathbb{Q}^+$ as

$$x = a_1 \cdots a_l . b_1 \cdots b_m \overline{c_1 \cdots c_n},$$

where each a_i, b_j and c_k is either 0 or 1. This is the usual representation of a positive rational number relative to the base 2. Here $a_1 \cdots a_l$ is the integral part of x and $b_1 \cdots b_m \overline{c_1 \cdots c_n}$ is the fractional part, the bar denoting recurrence of the finite sequence $c_1, \cdots, c_n$. Here we are writting

$$a_1 \cdots a_l . b_1 \cdots b_m \overline{0 \cdots 0}$$

for the usual expression of the rational number

$$a_1 \cdots a_l . b_1 \cdots b_m .$$

Since $0 \neq \mathbb{Q}^+$ we see that at least one of the a_i's, b_j's and c_k's is non-zero. To have a unique such expression we may assume that at least one of the c_k's is zero; for, otherwise, we can always replace the expression by the sum

$$a_1 \cdots a_l . b_1 \cdots b_m \underbrace{\overline{0 \cdots 0}}_{n\,places} + 0.\underbrace{0 \cdots 0 1}_{m\,places} \underbrace{\overline{0 \cdots 0}}_{n\,places}$$

which after addition is of the required form again.

We recall that to add two such expressions

$$x = a_1 \cdots a_l . b_1 \cdots b_m \overline{c_1 \cdots c_n}$$

and

$$y = a'_1 \cdots a'_{l'} . b'_1 \cdots b'_{m'} \overline{c'_1 \cdots c'_{n'}} \quad ,$$

where $m' \geq m$ with $m = m' + r\,, r \geq 0$,

we write

$$x = a_1 \cdots a_l . b_1 \cdots b_m \underbrace{\overline{c_1 \cdots c_n\, c_1 \cdots c_n \cdots c_1 \cdots c_n}}_{nn'\ terms}$$

and

$$y = a'_1 \cdots a'_{l'} . b'_1 \cdots b'_{m'}\, c'_1 \cdots c'_r$$

$$\underbrace{\overline{c'_{r+1} \cdots c'_{n'}\, c'_1 \cdots c'_r\, c'_{r+1} \cdots c'_{n'}\, c'_1 \cdots c'_r \cdots c'_{r+1} \cdots c'_n\, c'_1 \cdots c'_r}}_{nn'\ terms}$$

and then add in the usual manner.

Theorem 2.2

$\mathbb{Q}^+$ is a special semigroup.

Proof:

It is clear that the set S of all expressions

$$e_1 \cdots e_l . f_1 \cdots f_m \overline{g_1 \cdots g_n}$$

in which only one of the e's, f's and g's taken together is 1 and the others are 0 is a special generators for $\mathbb{Q}^+$. Therefore $\mathbb{Q}^+$ is a special semigroup.

For the following result, we use the following notation. We denote by $N(2)$ the set of all positive rational numbers of the form

$$a_1 \cdots a_l . b_1 \cdots b_m$$

relative to the base 2, i.e., the set of all those rationals which have a trivial recurring part in the base-2 expression. Clearly $N(2)$ is a subsemigroup of $\mathbb{Q}^+$.

Theorem 2.3

$N(2)$ is a Special Semigroup.

Proof:

We see as before that the set of all elements

$$x = a_1 \cdots a_l . b_1 \cdots b_m$$

of $N(2)$ in which only one of the a's and b's taken together is 1 and the others are 0 is a special set of generators. Thus $N(2)$ is special.

2.4 Closure Properties of Speciality

We shall now study the closure properties of the class of special semigroups under various operations. The result will show the abundance of special semigroups.

Theorem 2.4

(i) *Let $A_1, \cdots, A_n$ be a family of additive commutative semigroups and let A be the direct product of the family, i.e.,*

$$A = \{(a_1, \cdots, a_n) \,/\, a_i \in A_i\}$$

with componentwise addition. If A is special then each A_i is special.

(ii) *Let$\bar{A}_\alpha$ denote the monoid obtained by adjoining an identity element e_α to A_α and let $\bar{A}$ denote the direct product of $\bar{A}_1, \cdots \bar{A}_n$. Then $\bar{A}-\{\bar{0}\}$ is special if each A_α is special, where $\bar{0}$ is the identity of $\bar{A}$.*

Proof:

(i) Let A be special, and let S be a special set of generators of A. Let π, the projection maps $\pi_\alpha(s_1, \cdots, s_n) = s_\alpha$. Then, $S_\alpha = \{\pi_\alpha(s)/ \quad s \in S\}$ is a special set of generators of A_α. To see this , let $a_\alpha \in A_\alpha$. Then $a_\alpha = \pi_\alpha(a)$, for some $a \in A$. Since S is a special set of generators of A, there exist a unique finite set of elements $s^{(1)}, \cdots, s^{(r)} \in S$ such that $a = s^{(1)} + \cdots + s^{(r)}$. Then, $a_\alpha = \pi_\alpha(a) = \pi_\alpha(s^{(1)}) + \cdots + \pi_\alpha(s^{(r)})$ – a unique expression.

(ii) Let each A_α be special and for each α, let S_α denote a set of special generators of A_α. Let $\bar{\iota_\alpha}$ denote the injection $\bar{\iota_\alpha} : \bar{A}_\alpha \longrightarrow \bar{A}$. Then , $S = \cup_{\alpha=1}^{n} \{\bar{\iota_\alpha}(s_\alpha/ \quad s_\alpha \in S_\alpha\}$ is a special set of generators of $\bar{A} - \{\bar{0}\}$, where $\bar{0}$ is the identity element of $\bar{A}$. Thus, if $S_\alpha = \{s_{\alpha,1}, \cdots, s_{\alpha,r_\alpha}\}$, then

$$\begin{aligned}&\{(s_{1,1}, 0, \cdots, 0), \cdots, (s_{1,r_1}, 0, \cdots, 0),\\ &(0, s_{2,1}, 0, \cdots, 0), \cdots, (0, s_{2,r_2}, 0, \cdots, 0),\\ &\quad\vdots\\ &(0, \cdots, 0, s_{n,1}), \cdots, (0, \cdots, 0, s_{n,r_n})\}\end{aligned}$$

is a special set of generators of $\bar{A} - \{\bar{0}\}$. Hence $\bar{A} - \{0\}$ is special.

Theorem 2.5

Let $\{A_\alpha\}$ be a non empty family of special semigroups such that the direct product $\prod_\alpha \bar{A}_\alpha$, where $\bar{A}_\alpha$ is the monoid obtained by adjoining an identity element α_0 to A_α, is special. Then each A_α is special.

Proof:

Let S be a special set of generators of A, and let

$$S_\alpha = \{\pi_\alpha(s) / \quad s \in S\},$$

where $\pi_\alpha : A \longrightarrow A_\alpha$ is the projection homomorphism. Let $a_\alpha \in A_\alpha$ and let a be the element of A such that $\pi_\alpha(a) = a_\alpha$ and $\pi_\beta(a) = 0 \in A_\beta, \quad \beta \neq \alpha$. But we have $a = s_1 + \cdots + s_r$, for some unique $s_1, \cdots, s_r \in S$. Then, $a_\alpha = \pi_\alpha(s_1) + \cdots + \pi_\alpha(s_r)$. Hence S_α is a special set generators of A_α. Hence the theorem.

2.5 Sum and Direct Sum

We shall now define the sum and the direct sum of a family of subsemigroups of a semigroup and examine the closure of speciality for sums and direct sums. Let $\{A_\alpha\}$ be a family of subsemigroups of an additive commutative semigroup A. The **sum** $\sum_\alpha A_\alpha$ of $\{A_\alpha\}$ is the subsemigroup of A consisting of all finite sums $a_{\alpha_1} + \cdots + a_{\alpha_n}$, $a_{\alpha_i} \in A_{\alpha_i}$. Clearly, $\sum_\alpha A_\alpha$ is a subsemigroup of A.

Let $\hat{A_\beta}$ denote the sum $\sum_{\alpha \neq \beta} A_\alpha$. If for each, $A_\alpha \cap \hat{A_\beta} = \Phi$, when $\beta \neq \alpha$, then $\sum_\alpha A_\alpha$ will called the **direct sum** of the family $\{A_\alpha\}$ and is written $\sum_\alpha \oplus A_\alpha$.

If A and B are two semigroups with $A \cap B = \Phi$, and if $\bar{A}$, $\bar{B}$ denote the the monoids obtained from A, B by adjoining identity elements to $A$and B, then the subsemigroup $\iota_{\bar{A}}(A) + \iota_{\bar{B}}(B)$ of the direct product $\bar{A} \times \bar{B}$ is a direct sum where $\iota_{\bar{A}}$, $\iota_{\bar{B}}$ are the injection homomorphisms.

Theorem 2.6

Let $\{A_\alpha\}$ be a family of subsemigroups of an additive commutative semigroup A such that $A = \sum_\alpha \oplus A_\alpha$. Then A is special if and only if each A_α is special.

Proof:

First suppose that each A_α is special, and let S_α be a special set of generators of A_α. Let $S = \cup_\alpha S_\alpha$. Then S is a special set of generators of A. For, let $a \in A$. Then,

$a = a_{\alpha_1} + \cdots + a_{\alpha_n}$, for some $a_{\alpha_r} \in A_{\alpha_r}$; $r = 1, \cdots n$.

Now $a_{\alpha_r} = s_{r,1} + \cdots + s_{r,t_r}$ for some unique $s_{r,1}, \cdots, s_{r,t_r} \in S_{\alpha_r}$. Hence $a = \sum_{r=1}^{n} \sum_{k=1}^{t_r} s_{r,k}$, and this expression is unique. Thus A is special.

Conversely, suppose that A is special. Let S be a special set of generators of A. It is then clear from the definition of direct sum that $S_\alpha = S \cap A_\alpha$ is a special set of generators of A_α so that each A_α is special. The proof is therefore complete.

2.6 Free Commutative Semigroups

An additive commutative semigroup S is said to be *free* on a non empty subset X of S, if each element of S can be expressed uniquely as a finite sum

$$n_1 x_1 + \cdots + n_r x_r,$$

where $n_i \in \mathbb{N}$ and $x_i \in X$. S is called *free* if S is free on some $X \subseteq S$. It is obvious from the definition of direct sum that $S = \sum_{x \in X} \oplus S_x$, where S_x is the infinite cyclic semigroup grnerated by x, i.e.,

$$S_x = \{x, 2x, 3x, \cdots, nx, \cdots\}.$$

Clearly, each $S_x \cong \mathbb{N}$, the additive semigroup of all positive integers. It thus follows from the theorem 2.6 that

Theorem 2.7

A free semigroup is special.

Theorem 2.8

Let S be a commutative semigroup. Then $\bar{S} = S - \{0\}$ is special if

$$\bar{S} = \sum_{\alpha} {}_{\oplus}\bar{A}_{\alpha} \oplus \sum_{\beta} {}_{\oplus}\bar{B}_{\beta} \oplus \sum_{\gamma} {}_{\oplus}\bar{C}_{\gamma},$$

where each $A_\alpha \cong \mathbb{Q}^+$, each $B_\beta \cong \mathbb{N}(2)$ and each $C_\gamma \cong \mathbb{N}^+$.

Proof:

This again is a consequence of the theorem 2.2, theorem 2.3, theorem 2.4 and theorem 2.6.

2.7 Divisibility

An additive commutative semigroup S will be called **divisible** if, for each $s \in S$ and for each positive integers n, there exists an element $s' \in S$ such that $s = n\, s'$.

Obviously, $\mathbb{Q}^+$ is a divisible semigroup. Also, if $\{S_\alpha\}$ is a family of divisible semigroups, then

$$\sum_{\alpha} {}_{\oplus}S_{\alpha}$$

is divisible. Clearly $\mathbb{N}^+$ and $\mathbb{N}(2)$ are not divisible. It therefore follows from the theorem 2.8 that

Theorem 2.9

A commutative semigroup S is divisible and special if

$$\bar{S} = \sum_{\alpha} \oplus \bar{A}_\alpha,$$

where $\bar{A}_\alpha \cong \bar{\mathbb{Q}}^+$, for each α.

Proof:

The proof of this theorem is analogous to theorem 10 of Majumdar and Mallick [49].

We may note from the above theorems that special semigroups occur in abundance, and that in many situations, their structures are expressible in terms of direct sums and direct products of $\mathbb{N}^+$, $\mathbb{Q}^+$ and $\mathbb{N}(2)$.

2.8 Automorphism Groups and Endomorphism Semigroups

The structures of the automorphism groups and the endomorphism semigroups of the special semigroups $\mathbb{N}^+$, $\mathbb{Q}^+$ and $\mathbb{N}(2)$ can be determined through solution of the following problems:

(1) determination of the automorphism group / the endomorphism semigroup of

$$\sum_{\alpha} \oplus S_\alpha ,$$

when $Aut\, S_\alpha$ / $End\, S_\alpha$ are known for each α;

(2) determination of $Aut\, S_\alpha$ / $End\, S_\alpha$, where S_α is $\mathbb{N}^+$, $\mathbb{Q}^+$ or $\mathbb{N}(2)$.

We have solved the second problem:

Theorem 2.10

The followings are hold good:

(i) $Aut\, \mathbb{N}^+ \cong \{1\}$, *the group with one element;*

(ii) $End\, \mathbb{N}^+ \cong \mathbb{N}^+$,

(iii) $Aut\, \mathbb{Q}^+ = End\, \mathbb{Q}^+ \cong \mathbb{Q}^+$,

(iv) $End\, \mathbb{N}(2) \cong \mathbb{N}(2)$; $Aut\, \mathbb{N}(2) \cong \{1\}$, *the group with one element.*

Proof:

(i) Let $\varphi \in Aut\, \mathbb{N}^+$, and let $\varphi(1) = x$. If $x \neq 1$ $1 \notin Im\varphi$, and so φ is not onto — a contradiction. Hence $x = 1$, and so $Aut\, \mathbb{N}^+ = \{1_\mathbb{N}\} = \{1\}$, the group with one element.

(ii) clearly $\varphi : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ given by $\varphi(1) = n$ is an endomorphism of $\mathbb{N}^+$ for each n. Here $\varphi(r) = rn$, for each $r \in \mathbb{N}^+$. So, $\varphi(r+s) = (r+s)n = rn + sn + \varphi(r) + \varphi(s)$ i.e., φ is an endomorphism.
Also, if φ is an endomorphim of $\mathbb{N}^+$, let $\varphi(1) = n$. Then $\varphi(r) = rn$, for all $r \in \mathbb{N}^+$. Thus, $\varphi \xrightarrow{f} \varphi(1)$ is an isomorphism of $End\, \mathbb{N}^+$ onto $\mathbb{N}^+$.

(iii) Let $\varphi \in Aut\, \mathbb{Q}^+$, then for each $\frac{m}{n} \in \mathbb{Q}^+$, $m,\ n \in \mathbb{N}^+$, $\varphi(\frac{m}{n}) = \frac{m}{n}x$, where $x = \varphi(1)$. Also, for each $y \in \mathbb{Q}^+$, and a fixed $y_0 \in \mathbb{Q}^+$, $\varphi_{y_0} : y \longrightarrow y_0 y$ is an automorphism of $\mathbb{Q}^+$. Clearly, $y \longrightarrow \varphi_{y_0}$ is an

isomorphism of $\mathbb{Q}^+$ onto $Aut\,\mathbb{Q}^+$. Its inverse is the automorphism f of $Aut\,\mathbb{Q}^+$ onto $\mathbb{Q}^+$ given by $\varphi \xrightarrow{f} \varphi(1)$.
As in (ii) it can be proved that every endomorphism of $\mathbb{Q}^+$ too is given by the maps $x \longrightarrow rx$, for a fixed $r \in \mathbb{Q}^+$. Thus, $End\,\mathbb{Q}^+ = Aut\,\mathbb{Q}^+ \cong \mathbb{Q}^+$.

(iv) Since each element of $\mathbb{N}(2)$ is a rational number of the form $\frac{m}{2^r}$ $(r \geq 0)$, it follows from (iii) that a map $\varphi : \mathbb{N}(2) \longrightarrow \mathbb{N}(2)$ is an endomorphism if and only if $\varphi(x) = x.\varphi(1)$, for each $x \in \mathbb{N}(2)$. Also $\varphi(1)$ may be any element of $\mathbb{N}(2)$. Hence $\varphi \xrightarrow{f} \varphi(1)$ gives an isomorphism of $End\,\mathbb{N}(2)$ onto $\mathbb{N}(2)$.

However, the only automorphism of $\mathbb{N}(2)$ is the identity map. For, otherwise, if $\varphi \in Aut\,\mathbb{N}(2)$ $(\varphi \neq 1_{\mathbb{N}(2)})$, and $\varphi(1) = \frac{m}{2^n}$, m may be assumed to be odd without loss of generality. Also, m may be assumed to be 3; for, otherwise φ may be replaced by $\varphi + 1_{\mathbb{N}(2)}$. So, $\varphi^{-1}(1) = \frac{2^n}{m}$, $m \geq 3$. This is of the form $a_1 \cdots a_r.b_1 \cdots b_s\,\overline{c_1 \cdots c_t}$, with non-trivial recurring part $\overline{c_1 \cdots c_t}$, relative to the base 2.
Thus, $Aut\,\mathbb{N}(2) \cong \{1\}$, the group with one element.

CHAPTER-3

THE ENDOMORPHISM SEMIGROUP OF AN ENDOMAPPING OF A FINITE SET

3.1 Introduction

Let X be a finite set and let $F(X)$ denote the full transformation semigroup on X, i.e., the semigroup of all mappings of X into itself. Let $f \in F(X)$ and $End\,(f) = \{g \in F(X) \mid gf = fg\}$. Then $End\,(f)$ is a subsemigroup of $F(X)$ and is the *centralizer of* f *in* $F(X)$. We call $End\,(f)$ the *endomorphism semigroup* of f and an element of $End\,(f)$ an *endomorphism* of f. The name is justified since each endomorphism of f is precisely a map $X \longrightarrow X$ which maps the directed graph representation of f into itself in such a way that a vertex is mapped onto a vertex and a directed edge (u, v) is mapped onto the directed edge $(f(u), f(v))$.

We shall determine the structure of $End\,(f)$ for certain maps f. The subgroup of $End\,(f)$ consisting of all permutations π on X with $\pi f = f\pi$ is called the *automorphism group* of f and is denoted by $Aut\,(f)$. For arbitrary f, the structure of $Aut\,(f)$ has been determined by Majumdar [52].

3.2 Representation of an endomapping by a directed graph

3.2.1 Let X be a nonempty finite set. Let $f : X \longrightarrow X$ be a mapping. Then

$$X \supseteq f(X) \supseteq f^2(X) \supseteq \cdots \supseteq f^n(X) \supseteq \cdots$$

is a decreasing sequence of subsets of X which terminates at some stage, i.e., for some r,

$$f^r(X) = f^{r+1}(X).$$

Then, $f|_{f^r(X)}$ is a permutation, and hence, a product of cycles.

We obtain a directed graph (X, f) with X as the set of vertices and $\{(x, f(x)) \mid x \in X\}$ as the set of directed edges. The above discussion shows that the subgraph of (X, f) with the vertex-set $f^r(X)$ is a union of disjoint directed cycles. And (X, f) consists of these directed cycles at each vertex of which there is a directed tree with root at the vertex, the edges of the trees being directed towards the corresponding vertices.

This is illustrated through the following example:

Example: Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and let $f : X \longrightarrow X$ be given by $f(1) = 3,\ f(2) = 3,\ f(3) = 5,\ f(4) = 3,\ f(5) = 2,\ f(6) = 7,\ f(7) = 6,\ f(8) = 6,\ f(9) = 10,\ f(10) = 10$. Then the directed graph representation of f, i.e., (X, f) is

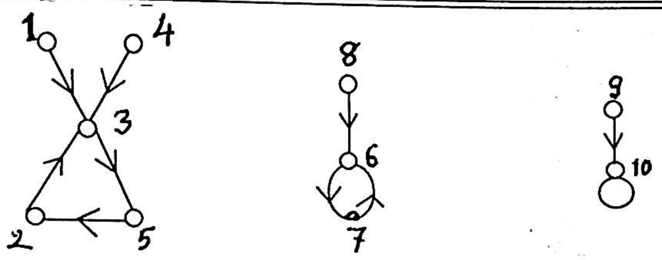


Fig. 1

3.2.2 It follows from the definition of the directed graph (X, f) that $g \in End\,(f)$ if and only if the induced map on (X, f) maps the edge $(x, f(x))$ onto the corresponding edge $(g(x), f(g(x)))$. Such maps are called *endomorphisms* of the directed graph. Hence the name endomorphism semigroup of f.

For determination of the structure of $End\,(f)$ we shall actually study the structure of $End\,(X, f)$, the endomorphism semigroup of (X, f).

Throughout our discussion in this chapter X will denote a non-empty finite set and f a mapping of X into itself. (X, f) will denote the directed graph determined by f.

3.2.3 For study of the structure of $End\,(X, f)$ we need two kinds of products of transformation semigroups: the direct product and the wreath product. We shall give the definitions here. A more detailed study of these products appear in [51]

3.3 Transformation semigroups and Their Products

3.3.1 Transformation Semigroup

Let Y be a non-empty set. A semigroup S is called a *transformation semigroup* on Y and written (S, Y) if each s in S is a map

$s : Y \longrightarrow Y$ such that$(s_1, s_2)(y) = s_1(s_2(y))$ for each $y \in Y$. If S has an identity element 1, then $1(y) = y$, for each $y \in Y$.

The $F(Y)$, the full transformation semigroup on Y, is obviously a transformation semigroup. In fact, every transformation semigroup (S, Y) is a subsemigroup of $F(Y)$.

3.3.2 Direct Product

Let (S_1, Y_1) and (S_2, Y_2) be two transformation semigroups where Y_1 and Y_2 are disjoint. Then the *direct product* $(S_1 \times S_2, Y_1 \cup Y_2)$ is defined as the semigroup of all ordered pairs

$$(s_1, s_2) \quad s_1 \in S_1, \; s_2 \in S_2,$$

with component-wise multiplication and with

$$(s_1, s_2)(y_1) = s_1(y_1),$$

and

$$(s_1, s_2)(y_2) = s_2(y_2),$$

for each $y_1 \in Y, \; y_2 \in Y_2$.

3.3.3 Wreath Product

Let (S_1, Y_1) and (S_2, Y_2) be two transformation semigroups. The **wreath product**

$$(S_1 \wr S_2, Y_1 \times Y_2)$$

is the set of all mappings

$$\theta : Y_1 \times Y_2 \longrightarrow Y_1 \times Y_2$$

such that

$$\theta\,(y_1,\, y_2) = (s_{\,1,y_2}(y_1),\, s_2(y_2))$$

where $s_2 \in S_2$ and $s_{\,1,y_2}$ is an element of S_1 which depends on y_2 and in general is different for different y_2.

If

$$(S_1, Y_1),\, \cdots,\, (S_n,\, Y_n)$$

are n transformation semigroups where $S_i \cong S_j$, for each i, j and each Y_i has the same number of elements and $Y_i \cap Y_j \neq \Phi$, then the semisubgroup of those elements of

$$(S_1 \times\, \cdots\, \times S_n,\, Y_1 \cup\, \cdots\, \cup Y_n)$$

which map an S_i into S_j may be regarded as identical with

$$(S \wr F_n,\, Y \times \{1,\, 2,\, \cdots,\, n\}) \text{ and denote it by } (1)$$

where $S \cong S_i$, for each i, Y is in 1–1 correspondence with each Y_i, and F_n is the semigroup of all endomappings of

$$\{1,\, 2,\, \cdots,\, n\}$$

i.e., the full transformation semigroup on

$$\{1,\, 2,\, \cdots,\, n\}.$$

The generalisation of direct products and wreath products to semigroups and their applications have been made using Majumdar's ideas. We shall use these products for description of structures of $End\,(X, f)$

for a few situations. Application of direct products and wreath products of transformation groups for determination of the structures of the automorphism groups of directed graphs and lattices are to be found in [51] and [52] respectively. An account of our findings here appear in [47].

3.4 The Structure of *End* (X, f)

3.4.1 Structure of *End* (X, F) for Case I

We first consider the case when $f^r(X) = \{x_0\}$ a singleton subset of X. A typical such case is given by the directed graph in the fig.2, i.e., (X, F) is given by the fig.2.

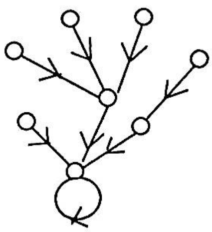


Fig. 2

(i) To study such cases, we begin with the simplest situations arranged in the order of increasing complexity:

First we consider $(X, F) =$ 

Here *End* $(X, f) = \{1\}$, the group with one element.

(ii) In this step, we consider (X, F) is given by fig.3.

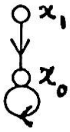


Fig. 3

Here *End* $(X, f) = \{\sigma_0, \sigma_1\}$, where

$$\sigma_0 = \begin{pmatrix} x_0 & x_1 \\ x_0 & x_1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} x_0 & x_1 \\ x_0 & x_0 \end{pmatrix}.$$

The multiplication is given by the table

$\bullet$	σ_0	σ_1
σ_0	σ_0	σ_1
σ_1	σ_1	σ_1

This is the semigroup $\{1, 0\}$, being the group $\{1\}$ adjoined with 0, with $\sigma_0 = 1$, $\sigma_1 = 0$. It is also the 0 semigroup with 1. We denote this semigroup by $E(1)$.

(iii) Here we have $(X, F) =$fig.4. In this case $End\,(X, f) = \{\sigma_0, \sigma_1, \sigma_2\}$, where

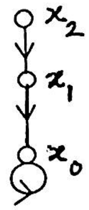


Fig.4

$$\sigma_0 = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \end{pmatrix}, \sigma_1 = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_0 & x_1 \end{pmatrix},$$

and $\sigma_2 = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_0 & x_0 \end{pmatrix}$.

The multiplication is given by the table

$\bullet$	σ_0	σ_1	σ_2
σ_0	σ_0	σ_1	σ_2
σ_1	σ_1	σ_2	σ_2
σ_2	σ_2	σ_2	σ_2

It is clear from the table that $End\,(X, f)$ is the 2-element semigroup $\{\sigma_1, \sigma_2\}$ with zero multiplication (σ_2 being the zero element) adjoined with $1 = \sigma_0$.

(iv) Let $(X, f) =$ fig.5. In this case, $End\,(X, f) = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, and the multiplication is given by the table

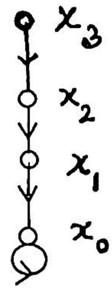


Fig.5

•	σ_0	σ_1	σ_2	σ_3
σ_0	σ_0	σ_1	σ_2	σ_3
σ_1	σ_1	σ_2	σ_3	σ_3
σ_2	σ_2	σ_3	σ_3	σ_3
σ_3	σ_3	σ_3	σ_3	σ_3

It is clear from the table that $End\,(X, f)$ is the 3-element semigroup $\{\sigma_1, \sigma_2, \sigma_3\}$ with zero multiplication (σ_3 being the zero element) adjoined with $1 = \sigma_0$.

(v) We can conclude that in general if $(X, F) =$ fig.6 then, $End\,(X, f) = \{\sigma_0, \sigma_1, \sigma_2, \cdots, \sigma_n\}$ and the multiplication is given by the table

•	σ_0	σ_1	σ_2	$\cdots\cdots$	σ_{n-1}	σ_n
σ_0	σ_0	σ_1	σ_2	$\cdots\cdots$	σ_{n-1}	σ_n
σ_1	σ_1	σ_2	σ_3	$\cdots\cdots$	σ_n	σ_n
σ_2	σ_2	σ_3	σ_4	$\cdots\ \sigma_n$	σ_n	σ_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
σ_{n-1}	σ_{n-1}	σ_n	σ_n	$\cdots\cdots$	σ_n	σ_n
σ_n	σ_n	σ_n	σ_n	$\cdots\cdots$	σ_n	σ_n

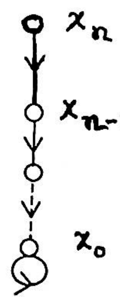


Fig.6

We write this semigroup as E(n) and denote it by (2).

(vi) We now consider graphs with more than trees of height 1 with root x_0: Thus, let $(X, f) =$ fig.7 . Here we encounter a situation where the concept of wreath product can be applied. There

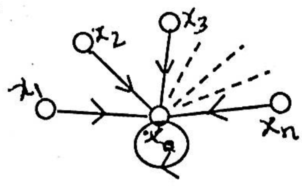


Fig.7

are n rooted trees they are to mapped onto themselves such that each rooted tree is mapped into some rooted tree. Since all the tree have the same root, the endomorphisms of a tree with vertices, say x_i, x_0, are precisely the maps $\sigma_0 = \begin{pmatrix} x_0 & x_i \\ x_0 & x_i \end{pmatrix}$ *and* $\sigma_i = \begin{pmatrix} x_0 & x_i \\ x_0 & x_0 \end{pmatrix}$. Thus , for each such tree the endomorphism semigroup is isomorphic to E(1). The trees may be designated as $T_1, \cdots, T_n$, each T_i being isomorphic to the graph $T =$, so that $T_1 \cup \cdots \cup T_n$ may be regarded as $T \times \{1, 2, \cdots, n\}$.

Hence by the discussion following the definition of the wreath product we have

$$(3) \qquad End\,(X, f) \cong E(1) \wr F_n \ ,$$

where F_n and $E(1)$ are given by (1) and (2) respectively.

(vii) We now consider the directed graph in the fig.8 consisting of r chains (rooted trees with a trivial cycle at the root) each of length m with root at x_0.

Arguing as before we obtain

$$End\,(X, f) = E(m) \wr F_r \qquad (4)$$

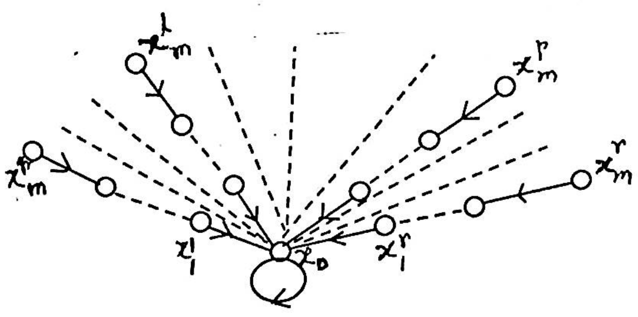


Fig. 8

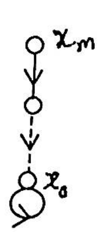


Fig. 9.1

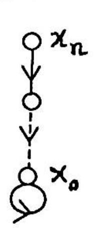


Fig. 9.2

(viii) We now consider (X, f) given by fig.9 with two chains of length m and n rooted at x_0, with $m > n$. We denote the subgraph (given by fig: 9.1) by T_1 and the subgraph (given by fig: 9.2) by T_2.

Then $End\,(X, f)$ will consist of maps of the following kind:

(a) $T_1 \longrightarrow T_1$, $T_2 \longrightarrow T_2$,

(b) $T_1 \longrightarrow T_1$, $T_2 \longrightarrow T_1$,

(c) $T_1 \longrightarrow T_2$, $T_2 \longrightarrow T_1$,

(d) $T_1 \longrightarrow T_2$, $T_2 \longrightarrow T_2$.

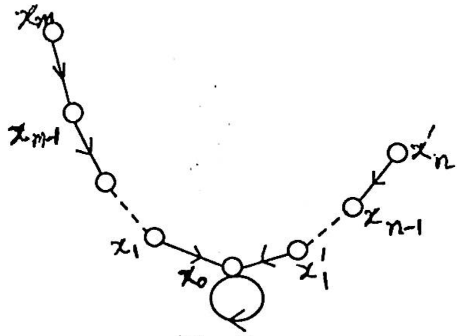


Fig.9

Hence

$$End\,(X, f) = (End\,T_1 \times End\,T_2) \cup (End\,T_1 \times Hom(T_2, T_1)) \cup$$

$$(Hom(T_1, T_2) \times Hom(T_2, T_1)) \cup (Hom(T_1, T_2) \times End\,T_2) \qquad (A)$$

with

$$(B) \begin{cases} (End\,T_1)\,(Hom(T_2, T_1)) \subseteq Hom(T_2, T_1) \,, \\ (Hom(T_1, T_2))\,(End\,T_1) \subseteq Hom(T_1, T_2) \,, \\ (End\,T_2)\,(Hom(T_1, T_2)) \subseteq Hom(T_1, T_2) \,, \\ (Hom(T_2, T_1))\,(End\,T_2) \subseteq Hom(T_2, T_1) \,, \\ (Hom(T_1, T_2))\,(Hom(T_2, T_1)) \subseteq End\,T_2 \,, \\ (Hom(T_2, T_1))\,(Hom(T_1, T_2)) \subseteq End\,T_1 \,. \end{cases}$$

We note that

$$Hom(T_1, T_2) = (End\,T_2)\,f\ , \quad where$$

$$f = \begin{pmatrix} x_0 & \cdots & x_{m-n} & x_{m-n+1} & \cdots & x_m \\ x'_0 & \cdots & x'_0 & x'_1 & \cdots & x'_n \end{pmatrix}\ ,$$

and

$$Hom(T_2, T_1) = (End\,T_1)\,g\ , \quad where$$

$$g = \begin{pmatrix} x'_0 & \cdots\cdots & x'_n \\ x_0 & \cdots\cdots & x_n \end{pmatrix}\ .$$

We may then have definite expressions:

$$(C)\begin{cases} (End\,T_1)\,(Hom(T_2, T_1)) = (End\,T_1)^2\,g\ , \\ (Hom(T_1, T_2))\,(End\,T_1) = (End\,T_2)\,f\,(End\,T_1)\ , \\ (End\,T_2)\,(Hom(T_1, T_2)) = (End\,T_2)^2\,f\ , \\ (Hom(T_2, T_1))\,(End\,T_2) = (End\,T_1)\,g\,(End\,T_2)\ , \\ (Hom(T_1, T_2))\,(Hom(T_2, T_1)) = (End\,T_2)\,f\,(End\,T_1)\,g\ , \\ (Hom(T_2, T_1))\,(Hom(T_1, T_2)) = (End\,T_1)\,g\,(End\,T_2)\,f\ . \end{cases}$$

(A) and (C) together completely express the semigroup structure of (X, f) in terms of $End\,T_1$ and $End\,T_2$.

(ix) We next let (X, f) be given by a general form of (C): fig.10

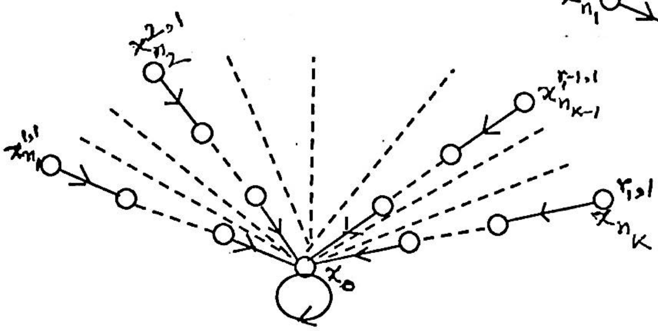


Fig.10

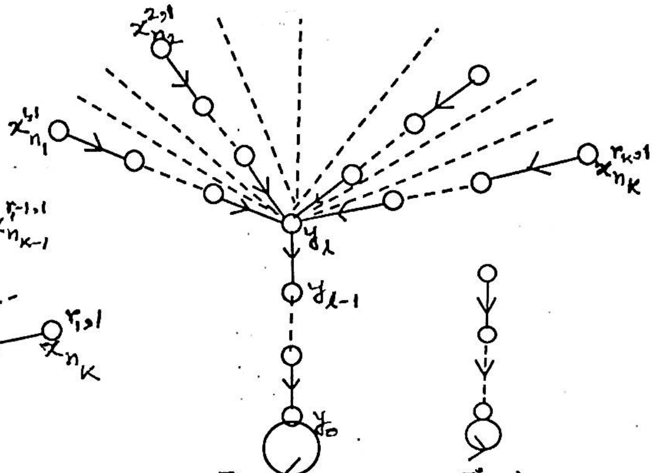


Fig.11

Fig.A

with r_1 chains $T'_1, \cdots, T'_{r_1}$ of length $n_1, \cdots, r_k$ and chains $T^k_1, \cdots, T^k_{r_k}$ of length n_k all being trees rooted at x_0.

Let T^i denote the tree combining $T^i_1, ,\cdots T_{r_i}$. Then, as before,

$$
(D)\begin{cases}
End(X,f) = \\
\bigcup[End\,T^{i_1} \times \cdots \times End\,T^{i_u} \times (\bigtimes^{v\neq v'}_{u<v,\,v'<k} \;\; Hom(T^{i_v}, T^{i_{v'}}))], \\
\\
\text{the union being taken over all permutations} \\
\\
\begin{pmatrix} 1 & 2 & 3\cdots\cdots k \\ i_1 & i_2 & i_3\cdots\cdots i_k \end{pmatrix}.
\end{cases}
$$

where

$$
(E)\begin{cases}
End\,T^i = End\,T^{i,\alpha} \wr F_{r_i} \qquad (\alpha \in \{1, 2, \cdots, i_r\}), \\
\\
Hom(T^{i_v}, T^{i_{v'}}) = \bigtimes^{1\leq\alpha\leq r_{i_v}}_{1\leq\beta\leq r_{i_{v'}}} \; Hom(T^{i_v}_\alpha, T^{i_{v'}}_\beta))
\end{cases}
$$

and the products of $End T^{i,\alpha}$ with themselves and with $Hom(T^i_\alpha, T^j_\beta)$ as well as the products of $Hom(T^i_\alpha, T^j_\beta)$; among themselves are given by (C). Here $End T^{i,\alpha}$'s are isomorphic to one another, since $T^{i,\alpha}$'s are chains of the same length. Thus the structure of $End(X, f)$ has been determined.

(x) Now let (X, f) be given by the directed graph in fig.11 .Then a map $g : X \longrightarrow X$ is in $End\,(X, f)$ if and only if g induces an endomorphism of chain C [fig.A] as well as the graph in (ix) and two such endomorphisms combine to yeild an element of $End\,(X, f)$. Thus, if we denote $End\,(X, f)$ in (ix) by E, then $End\,(X, f)$ in (x) is given by

$$End\,(X, f) \cong E \times E(l) \qquad (direct\ product)$$

by (v).

Comment: We note that the endomorphism semigroup of the graph of (X, f) in (x) and that of the directed rooted tree with root at y_0 obtained by removing the loop at y_0 are isomorphic to each other.

(xi) Let (X, f) be now given by a more general form fig. 12

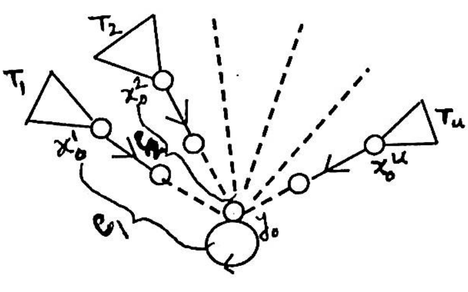
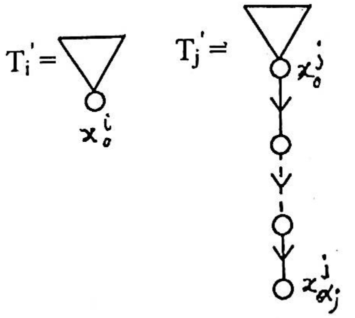


Fig. 12



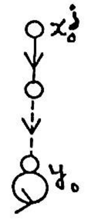


Fig. C

where $T_1, \cdots, T_u$ are graphs of the form in (x) which are isomorphic to directed trees each with root at y_0.

An endomorphism of (X,f) maps every tree T_i into a tree T_j (i, j not necessarily distinct) in such a way that

(a) y_0 is mapped onto y_0,

(b) x_0^i is mapped onto a point in C, where C is shown in the fig.C.

(c) the subtree T_i' of T_i into the subtree T_j', where $x^j_{\alpha_j} = f(x_0^i)$ so that the directed edges are mapped onto the corresponding directed edges.

Thus,

$$(D)\begin{cases} End(X,f) = \\ \bigcup[End\,T^{i_1} \times \cdots \times End\,T^{i_u} \times (\times^{v\neq v'}_{u<v,\,v'<k} \;\; Hom(T^{i_v}, T^{i_{v'}}))]\,, \\ \text{the union being taken over all permutations} \\ \begin{pmatrix} 1 & 2 & 3 \cdots\cdots k \\ i_1 & i_2 & i_3 \cdots\cdots i_k \end{pmatrix}. \end{cases}$$

Here, it is easily verified that

(α) if the length l_i of C_1 is greater than or equal to the length l_j of C_2 (C_1 and C_2 as in the fig.), then

$$Hom(T_i, T_j) = \{(\sigma_{j,2}, \sigma_{j,1})(f_{ij}, \sigma_{i,1})\}\,,$$

$$\text{where } f_{ij} = \begin{pmatrix} x_0^i & \cdots & x^i_{l_1.2} & \cdots & x^i_{l_i.s} & y_0 \\ x_0^j & \cdots & x_0^j & \cdots & x^j_{l_j} & y_0 \end{pmatrix},$$

$$\text{where } f_{ij} = \begin{pmatrix} x_0^i & \cdots & x_{l_1.2}^i & \cdots & x_{l_i.s}^i & y_0 \\ x_0^j & \cdots & x_0^j & \cdots & x_{l_j}^j & y_0 \end{pmatrix} ,$$

$\sigma_{i,1} \in End\,(T_i')$, $\sigma_{j,2} \in End\,(C)$, $\sigma_{j,2} \in End\,(T_j')$ (C as in the fig.C).

(β) if $l_i \leq l_j$, then

$Hom(T_i, T_j) = \{(\sigma_{j,2}, \sigma_{j,1})(f_{ij}', \sigma_{i,j}')\}$.

In the above, $f_{ij}' : T_i \longrightarrow T_j$ maps every subgraph of T_j, which is a chain or tree of length $\leq l_j$ with root x_0^i onto the subgraph C (as in the fig.C) of T_j isomorphically.

Also, $\sigma_{i,1}' \in End\,T_i''$, where T_i'' is the directed rooted tree obtained from T_i' by (a) deleting the maximal chain-subtrees in T_i with root x_0^i and length$<$ l_j and (b) collapsing the chain-subtrees of T_i with root x_0^i and length $\geq l_j$ by identifying all of $x_0^i, x_i^i, \cdots, x_{l_j}^i$ so that they represent the same point while the directed edges $x_{\alpha+1}x_\alpha \quad (0 \leq \alpha \leq l_{j_1})$ vanish.

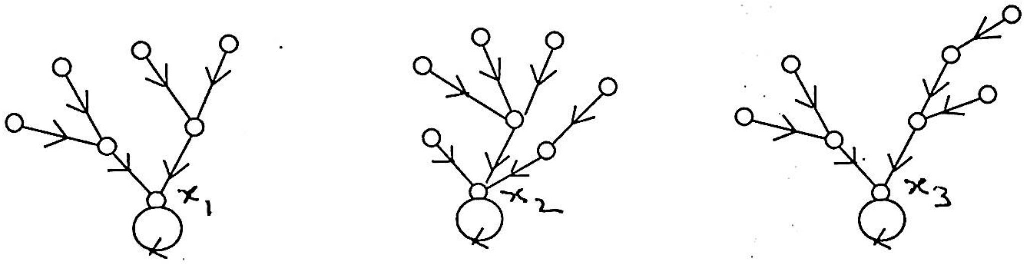
Multiplication of the elements of $Hom(T_i, T_j)$ with those of $Home(T_j, T_k)$ is the composition of the maps representing the elements.

Multiplication of the elements of $End\,T_1 \times \cdots \times End\,T_n$ with the elements of $Hom(T_i, T_j)$ is defined similarly.

3.4.2 Structure of $End(X, F)$ for Case II

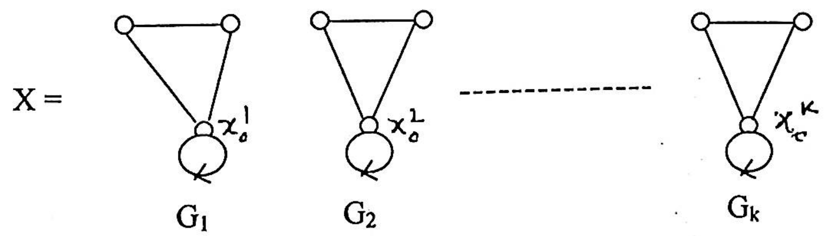
We next consider the case when, for some positive integer r, f, restricted to $f^r(X)$, is the identity map, i.e., $f \mid f^r(X) = 1_{f^r(X)}$.

A typical such case is shown in the following figure:



Here $f^3(X) = \{x_1, x_2, x_3\}$ and $f(x_1) = x_1$, $f(x_2) = x_2$, $f(x_3) = x_3$.

In general, (X, f) will now be given by



where each G_i is a directed graph of the type shown in (xi) of **Case I.**

The endomorphism semigroup for X, $End(X, f)$, will be determined by the endomorphism semigroups $End(G_i)$, where $G_1, \cdots, G_k$ are the components of X. Under f, the whole of G_i will mapped into itself or into another component G_j in such a way that x_0^i is mapped onto x_0^j .

Thus,

$$(I)\begin{cases} End(X,f) \cong \\ (End\, G_{i_1} \times \cdots \times End\, G_{k_u}) \times (\bigtimes_{u<v,v'<k}^{v\neq v'} Hom(G_{i_v}, Gj'_v)) \end{cases},$$

where $\begin{pmatrix} 1 & 2 \cdots k \\ i_1 & i_2 \cdots i_k \end{pmatrix}$ is a permutation on $\{1, 2, \cdots, k\}$.

Also, each $Hom(G_i, G_j)$ may be written as

$$Hom(G_i, G_j) = \bigtimes_{\alpha_i,\beta_j} Hom(T_i^{\alpha_i}, T_j^{\beta_j}), \text{ (Cartesian product)}$$

where $T_i^{\alpha_i}$ and $T_j^{\beta_j}$ are the directed trees constituting the graphs G_i and G_j respectively as in (xi), since each element of X, f maps G_i into G_j in such a manner that x_0^i is mapped onto x_0^j and each $T_i^{\alpha_i}$ is mapped into some $T_j^{\beta_j}$.

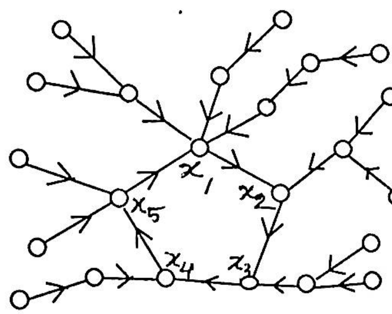
Considerations similar to those given in (xi) about multiplicaion of the elements of $Hom(T_i, T_j)$ with those of $Hom(T_j, T_k)$ as well as multiplication the elements of $End\, T_i$ with $Hom(T_i, T_j)$ hold here also.

This shows that $End\,(X, f)$ has been determined.

3.4.3 Structure of $End\,(X, F)$ for Case III

We now consider the situation when , for some positive integer r, f is a cyclic permutation on $f^r(X)$.

A typical example is given by the following figure:

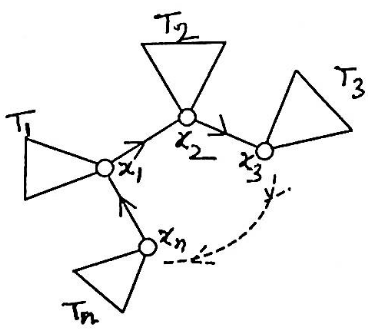


In this example, $f^4(X) = \{x_1, x_2, x_3, x_4, x_5\} = X'$,say, and f induces a cyclic permutation on X' and

$$f(x_1) = x_2, \cdots\cdots, f(x_4) = x_5, f(x_5) = x_1 .$$

We note that under the action of any element of $End(X, f)$ the cycle $C = (x_1, x_2, x_3, x_4, x_5)$ is mapped onto itself. Thus $End\, C = C_5$, the cyclic group of order 5.

In general, let (X, f) be given by



where $T_1, \cdots\cdots, T_n$ are directed rooted trees with roots at $x_1, \cdots\cdots, x_n$ respectively, the trees being of the form described in **(xi)** of **Case I.**

Then, under the action of an element α in $End(X, f)$, the cycle $\{x_1, \cdots\cdots, x_n\}$ on the set of the roots of the directed trees

$T_1, \cdots\cdots, T_n$ is endomorphically mapped onto itself and the tree T_1 is mapped endomorphically into the tree $T_{\alpha(1)}$ with the root of T_1 being mapped onto the root of $T_{\alpha()}1$. Thus,

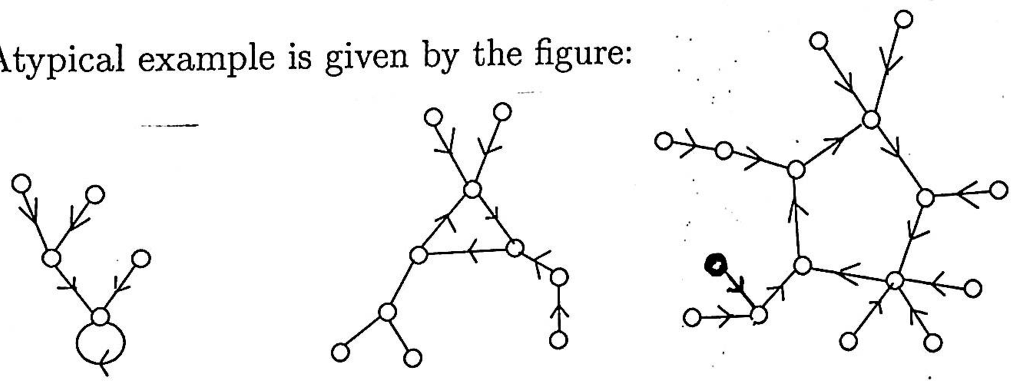
$$End(X,f) = \bigcup_{\pi \in C_n} \left(\times\, Hom(T_i, T_{\pi(i)}) \right),$$

C_n being the cyclic group of permutation on the set $Hom(T_i, T_{\pi(1)})$ and $\{x_1, \cdots\cdots, x_n\}$ of the roots of $T_1, \cdots\cdots, T_n$. The products of the elements of $Hom(T_i, T_{\pi(i)})$ and $Hom(T_{\pi(i), T_{\pi\pi'(i)}})$ are defined as in **(ix)** of **Case I** and **Case II**.

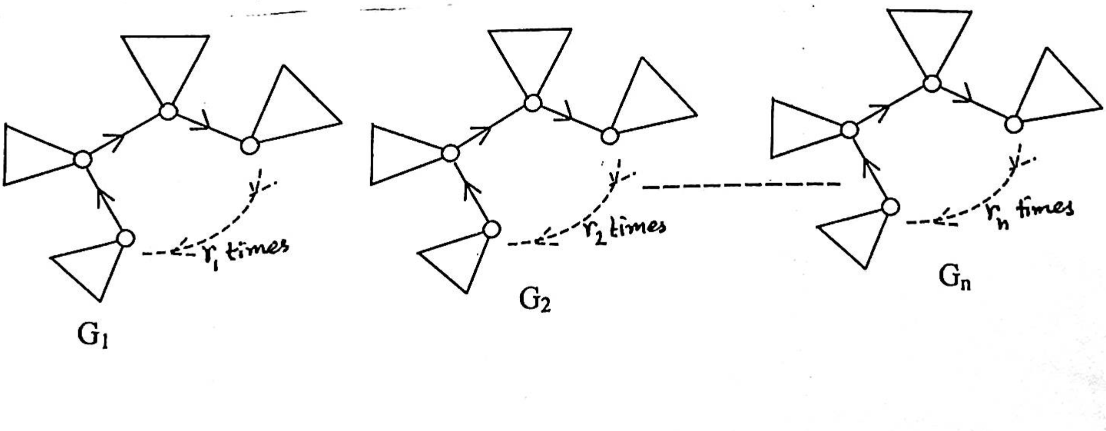
3.4.4 Structure of *End* (X, F) for Case IV

Finally, let us have the most general situation of an arbitrary endomapping f of X.

Atypical example is given by the figure:



In general, (X,f) will be given by a finite number of cycles at each vertex of which there is a directed tree with root at the vertex. Thus,



where each G_i is a directed graph of the form given in Case III.

In this case, we can see easily that

$End(X, f) =$

$$(End\, G_{i_1} \times \cdots \times G_{i_u}) \times (\bigtimes_{u<v,v'<n}^{v \neq v'} Home(G_{i_v}, G_{i'_v})) \;,$$

where $\begin{pmatrix} 1 & 2 \cdots n \\ i_1 & i_2 \cdots i_n \end{pmatrix}$ is a permutation on $\{1, 2, \cdots, n\}$.

The multiplication of the elements of $End\, G_{i_\alpha}$ with the elements of $Hom\,(G_{i_\alpha}, G_{i_\beta})$ and the multiplication of the elements of $Hom(G_{i_\alpha}, G_{i_\beta})$ with the elements of $Hom(G_{i_\beta}, G_{i_\gamma})$ are defined in the manner as in (ix) Case I, Case II and Case III.

The determination of the structure of $End\,(X, f)$ for an endomapping f of X is thus complete.

CHAPTER-4

GROUPS OF MORPHISMS

4.1 Introduction

In this chapter we have introduced and characterized a class of groups named *groups of morphisms.* In some categories, for some groups of morphisms $A \longrightarrow A$, certain subgroups of $Hom(A, A)$ are interesting. We have studied some such groups for the category of vector spaces, groups and topological spaces. The structures of the automorphism groups, i.e., the set of all morphisms which form a subgroup of $Hom(A, A)$, have been determined in certain cases.

4.2 Group of Morphisms and Automorphism group

We start with a category $\mathcal{C}$, and let A be an object of $\mathcal{C}$. If a subset $G(A)$ of $Hom(A, A)$ is a group under the composition of morphisms of $\mathcal{C}$ with 1_A as the identity element, $G(A)$ will be called a *group of morphisms of A*. Clearly, $G(A)$ is a group if and only if, for each $f \in G(A)$, there exists $f' \in G(A)$ such that $ff' = 1_A = f'f$. Clearly such f' is unique and may be denoted by f^{-1}.

The subset of $Hom(A, A)$ consisting of all $f \in Hom(A, A)$ for which the inverse exists is the largest group of morphisms of A and contains all groups of morphisms of A. This group will be called the *automorphism group of* A and denoted by $Aut\, A$.

Now we shall describe and study $Aut\, A$ and some of its subgroups for certain categories. An account of this discussion appears in Majumdar, Hossain and Akhter [48].

4.3 Vector spaces

Let F be a field and V a finite dimensional vector space over F. Let $dim\, V = n$. Then $Aut\, V =$ The group of all invertible linear operators of V. Thus, $Aut\, V \cong M^{n \times n}(F)$, the group of all $n \times n$ non-singular matrices with entries in F. For $\mathbb{R}$ and $\mathbb{C}$, these are called the general linear groups and are denoted by $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ respectively.

Several subgroups of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are important and widely used. These are

(i) $O(n) =$ the group of all $n \times n$ orthogonal matrices, i.e., $n \times n$ real matrices O such that $O' = O^{-1}$;

(ii) $SO(n) =$ the group of all orthogonal matrices M with $|M| = 1$;

(iii) $U(n) =$ the group of all $n \times n$ unitary matrices, i.e., all $n \times n$ complex matrices U with $\bar{U}' = U^{-1}$;

(iv) $SU(n) =$ the group of all $n \times n$ special unitary matrices, i.e., all $n \times n$ unitary matrices M with $|M| = 1$.

These are groups of symmetry and are very useful in theoretical physi -cs. These groups are topological groups too. In fact these are Lie groups and that aspect is important for application in physics. Unified field theories in theoretical physics use these groups and their Lie algebras, and sometimes tensor products of these groups, regarded as Lie algebras (see for example, Michio Kaku [35]). For example,

- $O(4) \cong SO(2) \otimes SU(2)$ (p-552)

Also,

- $O(4) \cong SU(2) \otimes SU(2)$ (p-568)
- $O(3) \cong SU(2)$ (p-552)
- $O(2) \cong U(1)$ (p-551)

4.4 Groups

For a group G, $Aut\,G$ is the group of all automorphisms of G, i.e., the group of all 1–1 homomorphisms of G onto itself.

It is easy to see that if G is infinite cyclic with generator x, then the only automorphisms of G are given by the maps $x \xrightarrow{\alpha} x$ and $x \xrightarrow{\beta} x^{-1}$. Hence $Aut\,G$ is the cyclic group of order 2 generated by β, i.e., $Aut\,G = C_2(\beta)$.

For a finite cyclic group G, the situation is more complex, if x is a generator of G, then the automorphisms of G are precisely the homomorphisms of G given by the maps $f_i : x \longrightarrow x^i$, where i is any integer relatively prime to n. Thus, the order of $Aut\,G$ is $\phi(n)$, where n is the order of G and ϕ is the Euler function.

To complete our programme we now determine the structure of $Aut\, C_n$. The structure is well known (see, for example, Scott [81], Zassenhaus [27]). However, we shall determine the structure of $Aut\, C_n$ in a different manner after Majumdar [82]:

Consider the ring $\mathbb{Z}_n$ of the residue classes of the integers modulo n $(n \geq 2)$. Then the set of elements $\bar{r} \in \mathbb{Z}_n$, with $(r, n) = 1$ is a group under multiplication. For, since there are integers a, b such that $ar + bn = 1$ and so, we have $\bar{a}\bar{r} = \bar{1}$, $\bar{a}$ is the multiplicative inverse of $\bar{r}$, the bar denoting the residue class modulo n. We denote this group by $[n]$. Clearly the order of $[n]$ is $\phi(n)$, where ϕ is the Euler function.

Theorem 4.1

$Aut\, C_n \cong [n]$.

Proof:

The map $\psi : \mathrm{Aut}\, C_n \longrightarrow [n]$ given by $\psi(f) = \bar{r}$, where $f(x) = x^r$, is an isomorphism. If f, g $\in \mathrm{Aut}\, C_n$ with $f(x) = x^r$, $g(x) = x^s$, then $(fg)(x) = f(g(x)) = (x^s)^r = x^{sr}$. So, then $\psi(fg) = \bar{s}\bar{r} = \bar{r}\bar{s} = \psi(f)\psi(g)$. Thus ψ is a homomorphism. Clearly ψ is both 1–1 and onto.

Therefore, if it will suffice to determine the structure of the group $[n]$, we do so using Majumdar's method [82]. The result will be established through the following three theorems:

Theorem 4.2

If a and b are two relatively prime integers, then

$$[a\,b] \cong [a] \times [b] \ \ (\textit{direct product}).$$

Proof:

The elements of $[a\,b]$ are $\{\overline{a\,q_{j(r_i)} + r_i}\}$, where $\{r_k\}$ is the set of all positive integers less than a and relatively prime to a, and for a fixed r_i, $\{a\,q_{j(r_i)} + r_i\}$ is the set of integers in

$$\{a + r_i,\ 2a + r_i,\ \cdots\ ,\ (b-1)a + r_i\}$$

which are relatively prime to b.

Define $\psi :\ [a\,b] \longrightarrow [a]$ by $\psi(\overline{a\,q_{j(r_i)} + r_i}) = \bar{\bar{r}}_i \in [a]$. Then, ψ is an onto homomorphism and $\operatorname{Ker}\psi = \{aq_{j(r_i)} + \bar{1}\}$. Now, $\bar{\psi} : \operatorname{Ker}\psi \longrightarrow [b]$ given by

$$\bar{\psi}(\overline{a\,q_{j(r_i)} + r_i}) = \overline{\overline{\overline{a\,q_{j(r_i)} + r_i}}} \in [b]$$

is an isomorphism.

Hence the sequence of abelian multiplicative groups and group homomorphisms

$$(A) \qquad\qquad 1 \longrightarrow [b] \xrightarrow{\bar{\psi}^{-1}} [a\,b] \xrightarrow{\psi} [a] \longrightarrow 1$$

is exact.

Now, $\psi^* :\ [a\,b] \longrightarrow [b]$ given by

$$\psi^*(\overline{a\,q_{j(r_i)} + r_i}) = \overline{\overline{\overline{a\,q_{j(r_i)} + r_i}}} \in [b]$$

is a well defined homomorphism and $\psi^*\bar{\psi}^{-1} = 1_{[b]}$. Thus the sequence (A) splits. Hence $[a\,b] \cong [a] \times [b]$.

Theorem 4.3

(i) If p is an odd prime, then $[p^n] \cong C_{\phi(p^n)}$ *for for each positive integers* n.

(ii)$[2^n] \cong C_2 \times C_{2^{n-2}}$, *for each* $n \geq 2$.

Proof:

(i) The order of $[p^n] = \phi(p^n)$

$= p^{n-1}(p-1)$. The element $\bar{2}$ of $[p^n]$ must have order exactly $p^{n-1}(p-1)$. Hence $\bar{2}$ is a generator of $[p^n]$. Thus, $[p^n] \cong C_{\phi(p^n)}$.

(ii) For $n = 2$ and 3, the result is easily verified; for $[2^2] =< \bar{3} >$ and $[2^3] =< \bar{3} > \times < \bar{5} >$.

We first note that for each $\bar{a} \in [2^n]$, $\bar{a}^{2^{n-2}} = \bar{1}$, the identity element of $[2^n]$. For $n = 2,\ 3,\ 4,\ 5$, this is true. Let it be true for $n \geq 2$, let a be any integer then $a^{2^{n-2}} = k2^n + 1$, for some integer k, squaring both sides, $a^{2^{n-1}} = k^2 2^{2n} + k2^{n+1} + 1 = l2^n + 1$, where $l = k^2 2^n + 2k$. Hence $\bar{a}^{2^{n-1}} = \bar{\bar{1}}$, where $\bar{\bar{1}}$ is the identity element of $[2^{n+1}]$.

We shall now prove that the order of $\bar{3}$ in $[2^n]$ is exactly 2^{n-2}.

All we shall have to do is to show that for all $n \geq 4$, $\bar{3}^{2n-3} \neq \bar{1}$ in $[2^n]$, i.e., $3^{2^{n-3}} \neq k2^n + 1$ for any integer k.

By the above paragraph, there exists an integer l such that $3^{2^{n-3}} =$

$l2^{n-1} + 1$. Hence we have to show that l is odd.

We prove this by induction on n. This is seen to be true for $n = 4$. Assume that for $n > 4$, $3^{2^{n-3}} = l2^{n-1} + 1$, where l is odd. Squaring both sides, $3^{2^{n-2}} = l^2 2^{2n-2} + l2^n + 1 = (l^2 2^{n-2} + l)2^n + 1 = l'2^n + 1$, where l' is odd.

Hence $\bar{3}$ has order 2^{n-2} in $[2^n]$ i.e., $< \bar{3} >$, the cyclic subgroup generated by $\bar{3}$ in $[2^n]$ has order 2^{n-2}.

Let $y \in [2^n]$, but $y \notin < \bar{3} >$. Then, $y^2 \in < \bar{3} >$, since order of $[2^n]$ is 2^{n-1}. y^2 cannot be equal to an odd power of $\bar{3}$, for then y will be of order 2^{n-1}, which is impossible by the second paragraph of our proof. Hence $y^2 = \bar{3}^{2r}$, for some non-negative integer r. Then $\bar{3}^r y^{-1}$ has order 2 and it does not belong to $< \bar{3} >$. Therefore $[2^n] =< \bar{3} > \times < \bar{3}^r y^{-1} >$, the internal direct product. Thus, $[2^n] \cong C_{2^{n-2}} \times C_2$.

The structure of $Aut\, C_n$ for an arbitrary positive integer $n \geq 2$ follows from the above three theorems :

Theorem 4.4

Let n be a positive integer $n = p_1^{e_1} \cdots p_r^{e_r}$ where $p_1, \cdots, p_r$ are prime numbers with $p_1 < p_2 < \cdots < p_r$ and $e_1, \cdots, e_r$ are positive integers. Then

$$Aut\, C_n \cong [n]$$
$$\cong \begin{cases} C_{2^{e_1-2}} \times C_2 \times C_{\phi(p_2^{e_2})} \times \cdots \times C_{\phi(p_r^{e_r})} & \text{if } p_1 = 2, \\ C_{\phi(p_1^{e_1})} \times \cdots \times C_{\phi(p_r^{e_r})} & \text{if } p_1 \neq 2. \end{cases}$$

Proof:

See theorem 4.1 (Majumdar[82], Scott[81],p.120, Zassenhaus [27] , p.145-146)

For a positive integer $n \geq 2$, $Aut\, C_n$, the group of automorphisms of a cyclic groups of order n, has the following structure:

(i) $Aut\, C_n = C_{\phi(p_1)\cdots\phi(p_r)}$, *if* $n = p_1^{e_1} \cdots p_r^{e_r}$, *where each* p_i *is an odd prime,* $i = 1, \cdots, r$;

(ii) $Aut\, C_n \cong C_2 \times C_{2^{r-2}} \times C_{\phi(p_1)\cdots\phi(p_s)}$, *if* $n = 2^r p_1^{e_1} \cdots p_s^{e_s}$, $r \geq 2$, *where* $p_1, \cdots, p_s$ *are odd primes.*

3.5 The Automorphism Groups of $\mathbb{Q}$ and $\mathbb{R}$

The following propositions describe the structure of $Aut\,\mathbb{Q}$ and $Aut\,\mathbb{R}$, where $\mathbb{Q}$ is the additive group of rational numbers and $\mathbb{R}$ is the additive group of real numbers:

Proposition 4.5

$Aut\,\mathbb{Q} \cong \mathbb{Q}^*$, where $\mathbb{Q}^*$ is the multiplicative group of all non-zero rationals.

Proof:

Let $f \in \mathrm{Aut}\,\mathbb{Q}$ and let $f(1) = x$. Then,$x \neq 0$. Let $m,\ n \in \mathbb{Z},\ n \neq 0$. Then, $f(m) = mx$. Let $f(\frac{m}{n}) = y$, then $f(m) = f(n\frac{m}{n}) = nf(\frac{m}{n}) = ny$. So, $f(\frac{m}{n}) = y = \frac{m}{n}x = \frac{m}{n}f(1)$.

Conversely, let $x \neq 0$ be any element of $\mathbb{Q}$. Define $f : \mathbb{Q} \longrightarrow \mathbb{Q}$ as follows:

for each $\frac{m}{n}$, m, $n \in \mathbb{Z}$, $n \neq 0$, $f(\frac{m}{n}) = \frac{m}{n}x$. Then, $f(1) = x$, and so, $f(\frac{m}{n}) = \frac{m}{n}f(1)$. Clearly f is an automorphism of $\mathbb{Q}$. Here $f^{-1}(\frac{m}{n}) = \frac{m}{n}\frac{1}{x}$.

Consider the map $\phi : \text{Aut}\,\mathbb{Q} \longrightarrow \mathbb{Q}^*$ given by $\phi(f) = f(1)$. Clearly $f(1) \neq 0$, so that $f(1) \in \mathbb{Q}^*$. We see that, if f, $g \in \text{Aut}\,\mathbb{Q}$, then $\phi(fg) = (fg)(1) = f(g(1)) = g(1)f(1) = f(1)g(1) = \phi(f)\phi(g)$. Thus ϕ is a homomorphism.

Next consider the map $\psi : \mathbb{Q}^* \longrightarrow \text{Aut}\,\mathbb{Q}$ given by $\psi(x) = f$, where, for each $y \in \mathbb{Q}$, $f(y) = yx$. For x_1, $x_2 \in \mathbb{Q}^*$, let $\psi(x_1) = f_1$, $\psi(x_2) = f_2$, then $\psi(x_1x_2) = f$, wrere, for each $y \in \mathbb{Q}$, $f(y) = y(x_1x_2) = y(x_2x_1) = (yx_2)x_1 = f_1(yx_2) = f_1(f_2(y)) = (f_1f_2)(y)$. Thus $f = f_1f_2$, i.e., $\psi(x_1x_2) = \psi(x_1)\psi(x_2)$. Hence ψ is also a homomorphism.

Now, for each $x \in \mathbb{Q}^*$, $(\phi\psi)(x) = \phi(\psi(x)) = (\psi(x))(1) = x$, by the definitions of ϕ and ψ. Hence $\phi\psi = 1_{\mathbb{Q}^*}$.

Also, for each $f \in Aut\,\mathbb{Q}$, $(\psi\phi)(f) = \psi(\phi(f)) = \psi(f(1)) = g \in Aut\,\mathbb{Q}$, where, for each $y \in Q$, $g(y) = yf(1) = f(y)$ so that $g = f$. Thus, $(\psi\phi)(f) = f$. Therefore $\psi\phi = 1_{Aut\,\mathbb{Q}}$.

Therefore ϕ and ψ are isomorphism so that $\text{Aut}\,\mathbb{Q} \cong \mathbb{Q}^*$.

We note from the above proposition that $Hom(\mathbb{Q}, \mathbb{Q})$, the additive group of all additive endomorphisms of $\mathbb{Q}$, is given by

$$Hom(\mathbb{Q}, \mathbb{Q}) = \{f : \mathbb{Q} \longrightarrow \mathbb{Q} /\ f(x) = ax,\ a \in \mathbb{Q}\} \cong \mathbb{Q}\ .$$

Here the isomorphism is given by $f \longleftrightarrow f(1)$.

$Aut\,\mathbb{R}$, the group of automorphisms of $\mathbb{R}$ where $\mathbb{R}$ is the additive group of real numbers, will now be determined.

Proposition 4.6

$Aut\,\mathbb{R} \cong \mathbb{R}^*$, where $\mathbb{R}^*$ is the multiplicative group of all non-zero real numbers.

Proof:

The map $\phi : Aut\,\mathbb{Q} \longrightarrow \mathbb{R}^*$ given by $\phi(f) = f(1)$ gives the required isomorphism of the proposition. The arguments similar to those in the proof of proposition 3.5 proves the statement.

If $\mathbb{R}$ is the additive group of all real numbers and $\mathbb{R}^+$ is the multiplicative group of all positive real numbers, then the map $\phi : \mathbb{R} \longrightarrow \mathbb{R}^+$ given by $\phi(x) = e^x$ is an isomorphism of $\mathbb{R}$ onto $\mathbb{R}^+$. So, $(\mathbb{R}, +) \cong (\mathbb{R}^+, \times)$. Since $(\mathbb{R}^+, \times)$ is a subgroup of index 2 in the group $(\mathbb{R}^*, \times)$, Aut $(\mathbb{R}, +)$ contains an isomorphic copy of $(\mathbb{R}, +)$ as a subgroup of index 2.

3.6 Topological Spaces

We shall now determine the structure of the automorphism group of the topological space $\mathbb{R}$ with the usual metric topology. It is the group of all homeomorphisms of the real line, i.e., $\mathbb{R}$ with the topology induced by the metric d where $d(x, y) = | x - y |$. We denote this group by $Aut\,\mathbb{R}$.

We note that the functions $x \longrightarrow cx$ (magnification), $x \longrightarrow x^n$, and $x \longrightarrow x^{\frac{1}{n}}$, where c is any non-zero real number and n is any odd positive integer, are homeomorphisms of $\mathbb{R}$ onto itself, i.e., automorphisms of the real line. Also, $x \longrightarrow x + a$ (translation) is an automorphism of this space for each real number a. The maps $x \longrightarrow x^{2n}$ is not a homeomorphism since it is not 1–1.

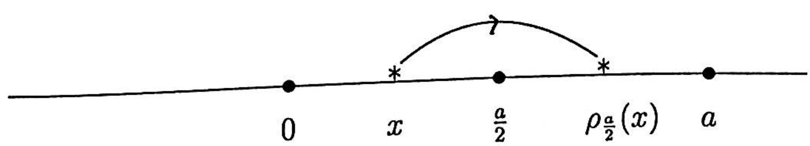
The trigonometric functions are not 1–1, the exponential functions are not onto and the logarithmic function is not defined on the whole of $\mathbb{R}$.

The map $x \longrightarrow cx$ includes as a particular case the reflexion $x \longrightarrow -x$ at the point 0. It is clear that the reflexion about any point is a homeomorphism.

Therefore, the homeomorphisms of the real line consist of all reflections, all magnifications, all translations and all maps $x \longrightarrow x^a$ (a an odd integer or its reciprocal), and their compositions.

Now, every translation is a composition of two reflections. For, if we consider the translation $\tau_a : x \longrightarrow x + a$, we can verify that $\tau_a = \rho_{\frac{a}{2}} \rho_0$, where $\rho_{\frac{a}{2}}$ and ρ_0 are reflections at the points $\frac{a}{2}$ and 0 respectively. The verification is clear from the following:

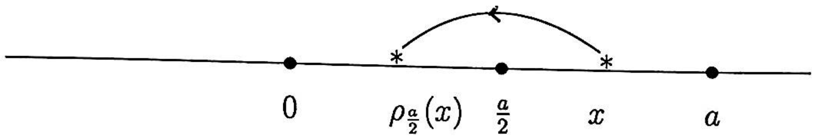
(i) if $x > 0$, $a > 0$ and $x < \frac{a}{2}$, then we have the figure:



and so,

$$\rho_{\frac{a}{2}}(x) = \frac{a}{2} + (\frac{a}{2} - x) = a - x\ .$$

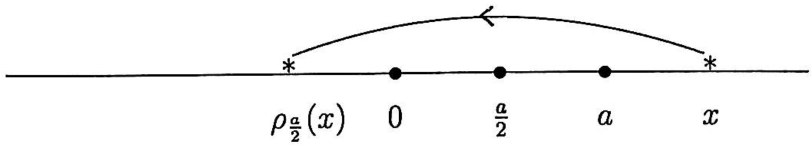
(ii) if $x > 0$, $a > 0$ and $\frac{a}{2} < x < a$, then we have the figure:



and so,

$$\rho_{\frac{a}{2}}(x) = \frac{a}{2} - (x - \frac{a}{2}) = a - x\ .$$

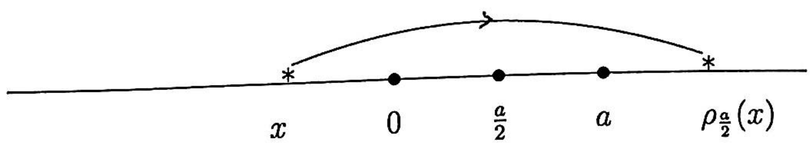
(iii) if $a > 0$ and $x > a$, then we have the figure:



and so,

$$\rho_{\frac{a}{2}}(x) = -(x - a) = a - x\ .$$

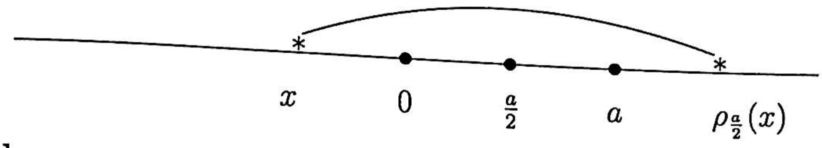
(iv) if $x > 0$ and $a < 0$, then we have the figure:



and so,

$$\rho_{\frac{a}{2}}(x) = \frac{a}{2} + \frac{a}{2} + (-x) = a - x\ .$$

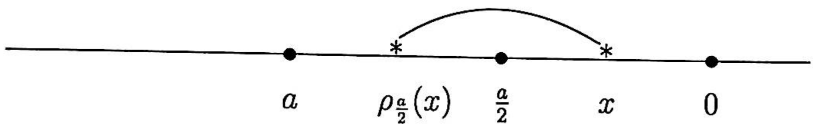
(v) if $x > 0$ and $a < 0$, then we have the figure:



and so,

$$\rho_{\frac{a}{2}}(x) = a + (-x) = a - x \ .$$

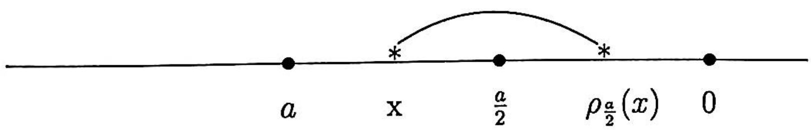
(vi) if $x < 0$, $a < 0$ and $\frac{a}{2} < x$, then we have the figure:



and so,

$$\rho_{\frac{a}{2}}(x) = \frac{a}{2} - (x - \frac{a}{2}) = a - x \ .$$

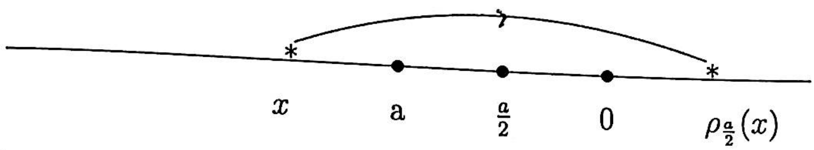
(vii) if $x < 0$, $a < 0$ and $a > x > \frac{a}{2}$, then we have the figure:



and so,

$$\rho_{\frac{a}{2}}(x) = \frac{a}{2} + (\frac{a}{2} - x) = a - x \ .$$

(viii) if $x < 0$, $a < 0$ and $x < a$, then we have the figure:



and so,

$$\rho_{\frac{a}{2}}(x) = a - x \ .$$

Thus in all possible cases, $\rho_{\frac{a}{2}}(x) = a - x$. Also for all $x \in \mathbb{R}$, $\rho_0(x) = -x$. Hence always we have

$$(\rho_{\frac{a}{2}}\rho_0)(x) = \rho_{\frac{a}{2}}(-x) = a + x = \tau_a(x) \ .$$

Therefore, we have $\tau_a = \rho_{\frac{a}{2}}\rho_0$.

In fact, $Aut\,\mathbb{R}$ is generated by

(i) all maps $\mu_c : x \longrightarrow cx$, where c is any non-zero real number ,

(ii) all maps $\pi_{2m+1} : x \longrightarrow x^{2m+1}$ and $\pi'_{2n+1} : x \longrightarrow x^{\frac{1}{2n+1}}$, where m, n are any positive integers, and

(iii) all maps $\rho_a : x \longrightarrow x + a$, $a \in \mathbb{R}$.

Now, it can be easily verified that

$$M = \{\mu_c\}_{c\in\mathbb{R}^*} \cong \mathbb{R}^* \ ,$$

$\mathbb{R}^*$ being the multiplicative group og all nonzero real numbers, and

$$P = \{\pi_{2m+1},\ \pi'_{2n+1}\}_{m,\ n\in\mathbb{N}} \cong \mathbb{Q}_{odd} \ ,$$

where $\mathbb{Q}_{odd}$ is the multiplicative group of all rational numbers of the form $\frac{2m+1}{2n+1}$.

Hence $R = \{\rho_a\}_{a\in\mathbb{R}} \cong \mathbb{R}$, the additive group of all real numbers and we have thus the following proposition:

Proposition 4.7

$Aut\,\mathbb{R} \cong \mathbb{R}^* \star \mathbb{Q}_{odd} \star \mathbb{R}$, where the $\star$ denote the free product.

CHAPTER-5

FUNDAMENTAL GROUPS AND HOMOLOGY OF CERTAIN SUMS

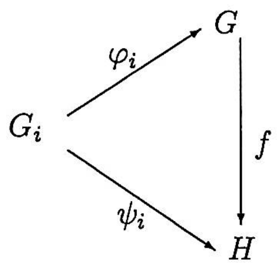
5.1 Introduction

S. Majumdar and Asaduzzaman studied *sums of topological spaces* in [45] and they derived there some characterization theorems for their defined sum. In this chapter, we shall study the fundamental group and the homology groups of Majumdar's sum, connected sum and external sum of topological spaces. To do this we shall use the two famous mathematical tools: (i) Seifert-Van Campen theorem and, (ii) Mayer-Vietoris sequence.

We have already come across the concept of fundamental group, the Siefert-Van Campen theorem, homology group and the Mayer-Vietoris sequence in the first chapter. In this chapter, We need also the concept of free product of groups of the following kind and the concept of free products with an amalgamated subgroup and that of simplicial complex.

Let $\{G_i : i \in I\}$ be a collection of groups, and assume there is given for each index i a homomorphism φ_i of G_i into a fixed group G.

We say that G is the *free product* of the groups G_i with respect to the homomorphisms φ_i if and only if the following condition holds: For any group H and any homomorphisms $\varphi_i : G_i \longrightarrow H, \quad i \in I$, the following diagram is commutative:



Let A_α be groups, where α ranges over a set of indices, and let a proper subgroup B_α be chosen in every A_α such that all these subgroups are isomorphic to a fixed group B. By φ_α we denote a specific isomorphic mapping of B_α onto B; then $\psi_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1}$ is an isomorphic mapping of B_α onto B_β.

The *free product of the groups A_α with the amalgamated subgroup B* is defined as the factor group G of the free product of the groups A_α with respect to the normal subgroup generated by all elements of the form $b_\alpha b_\beta^{-1}$, where $b_\beta = b_\alpha \psi_{\alpha\beta}$, where b_α ranges over the whole subgroup B_α, and where α and β are all possible index paires. In other words, if every group A_α is given by a system of generators $\mathcal{M}_\alpha$ and a system of defining relations Φ_α between these generators, then G has as a system of generators the union of all sets $\mathcal{M}_\alpha$, as a system of defining relations the union of the sets Φ_α, and in addition, all relations obtained by identifying those elements of differentsubgroups B_α and B_β which are mapped by the isomorphisms φ_α and φ_β onto one and same element of B. The subgroups B_α are amalgamated, as it were, in accordance with the isomorphisms $\psi_{\alpha\beta}$.

We need the concept of simplicial complex which is used very widely in algebraic topology. Let $a^0, \cdots, a^p$ be $p+1$ points in $\mathbb{R}^n$. These are called *independent* if for real numbers $\lambda_0, \cdots, \lambda_p$, $\sum_{i=0}^{p} \lambda_i a_i = 0$, and $\sum_{i=0}^{p} \lambda_i = 0$ together imply $\lambda_1 = \cdots = \lambda_p = 0$. A point $b \in \mathbb{R}^n$ is said to be *dependent* on $a^0, \cdots, a^p$, if $b = \sum_{i=0}^{p} \lambda_i a_i$ and $\sum_{i=0}^{p} \lambda_i = 1$, for some $\lambda_0, \cdots, \lambda_p \in \mathbb{R}$.

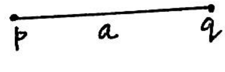
The set of points dependent on $a^0, \cdots, a^p$ is a subspace of $\mathbb{R}^n$, and if, $a^0, \cdots, a^p$ are independent, then each such dependent point is uniquely determined by the co-ordinates $(\lambda_0, \cdots, \lambda_p)$ where the points is $\sum_{i=0}^{p} \lambda_i a_i$. The co-ordinates $(\lambda_0, \cdots, \lambda_p)$ are called the *bary centric co-ordinates.* The *rectilinear p-simplex* s_p with vertices $a^0, \cdots, a^p$ is the set of points dependent on $a^0, \cdots, a^p$ whose barycentric co-ordinates satisfy $\lambda_i > 0$, $i = 0, 1, \cdots, p$. A 0-simplex is a point a^0, a 1-simplex is part of the line segment  obtained by removing the points a^0, a^1 and so on.

A *finite geometric simplicial complex* in R^n is a finite collection K of simplexes s_p^i of $\mathbb{R}^n$ subject to the conditions:

(a) if $s_p \in K$ and $s_q \prec s_p$, then $s_q \in K$ and

(b) distinct simplexes are disjoint.

Here $s_q \prec s_p$ means that s_q is a face of s_p i.e., s_q is a simplex spanned by a subset of the set of vertices of s_p. sometime the empty set is called a simplex and is denoted by s_1.

Thus,  is a simplicial complex $K = \{p, q, a\}$, p, q

being the vertices and a the open segment.

5.2 Majumdar's Sum in [45]

Given two topological spaces (X, T_1) and Y, T_2 such that $X \cap Y$ is open in both X and Y, $X \cup Y$ is a topological space with topology $T = \{U \cup V \mid U \in T_1,\ V \in T_2\}$. $X \cup Y$ is called the sum of X and Y and is denoted by $X + Y$. In this situation X and Y are said to be compatible with each other. The definition is due to Majumdar and Assaduzzaman [45]. An almost similar definition occurs in Bourbaki [8].

If $X \cap Y = \Phi$, $X + Y$ is called a *direct sum* and is denoted by $X \oplus Y$. For a detailed study of sum and direct sum, we refer to [46] and [50]. Clearly, the topologies T_1, T_2 on X and Y are the same as the topologies as subspaces of $X + Y$.

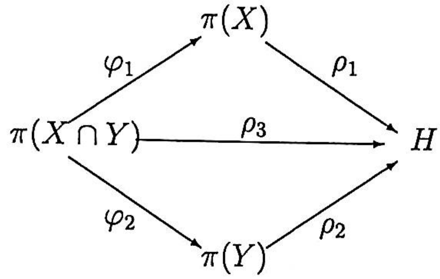
5.2.1 Fundamental group of $X + Y$

The fundamental groups are very important in classification of spaces as much as two spaces with non-isomorphic fundamental groups are non-homeomorphic. We shall now study with the application of Seifert and Van Kampen theorem to express the fundamental groups of a sum of two compatible spaces in terms of those of the summands. We shall verify that the following results on the fundamental groups of the sum $X + Y$ of two compatible spaces X and Y hold good:

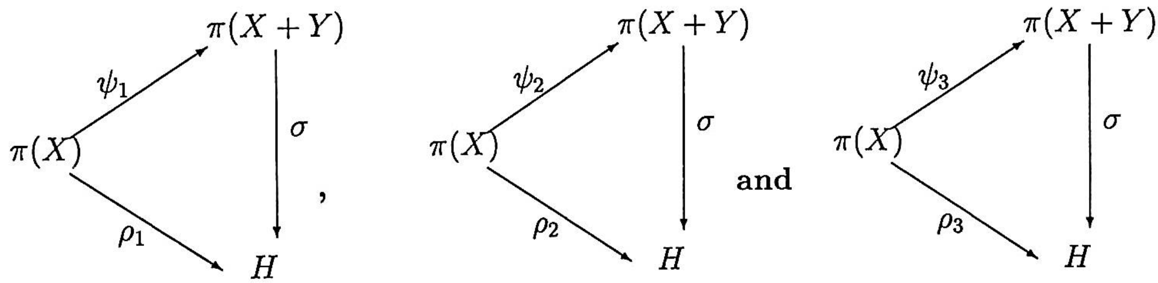
Theorem 5.1

Let X and Y be path connected compatible spaces such that $X \cap Y \neq \phi$

and path connected. If H be a group and $\rho_1,\ \rho_2,\ \rho_3$ are any homomorphisms such that the diagram



is commutative, where the homomorphisms ϕ_1 and ϕ_2 are induced by the inclusion maps. Then there exists a unique homomorphism $\sigma : \pi(X+Y) \longrightarrow H$ such that the following diagrams are commutative:



where $\psi 1,\ \psi_2,\ \psi_3$ are also homomorphisms induced by inclusion maps.

Proof:

Since X and Y are open subsets of $X+Y$, the results follows immediately from the famous theorem of Seifert and Van Kampen.

For the above results, we thus get the following corollary:

Corollary 5.2

If $\pi(X \cap Y) = 1$ i.e., if $X \cap Y$ is simply connected, then $\pi(X+Y)$ is the free product of $\pi(X)$ and $\pi(Y)$ under the homomorphisms $\psi_1,\ \psi_2$

i.e., $\pi(X+Y) = \pi(X) * \pi(Y)$, where $*$ denotes the free product.

5.2.2 Homology Group $H_n(X+Y)$

To the topological space $X+Y$, we have always an abelian group $H_n(X+Y)$, the homology group of $X+Y$, is associated for each positive integer n. Using Mayer-vietoris sequence of complexes in chapter-1, the homology of $X+Y$ of subspaces in terms of the homologies of the subspaces X and Y and that of their intersections is given by the following:

Theorem 5.3

Let X and Y be two subspaces of the sum $X+Y$. Then the following sequence is exact :

$$\cdots\cdots\cdots \xrightarrow{s_*} H_{p+1}(X+Y) \xrightarrow{v_*} H_p(X \cap Y) \xrightarrow{j_*} H_p(X) \oplus H_p(Y) \xrightarrow{s_*}$$
$$H_p(X+Y) \xrightarrow{v_*} H_{p-1}(X \cap Y) \xrightarrow{j_*} \cdots\cdots\cdots.$$

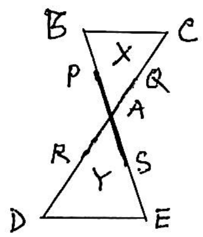
We do not describe the maps s_*, v_*, j_* here. For detail description of the maps s_*, v_*, j_* we refer the reader again to Hilton and Wiley ([28], p.290).

We see from the theorem that the homology groups $H_n(X+Y)$ of the sum is determined by the homology groups $H_n(X)$, $H_n(Y)$ and $H_n(X \cap Y)$. In particular, if $X \cap Y$ is a singleton, then $H_n(X \cap Y) = 0$, $n > 0$, i.e., $X \cap Y$ has a trivial homology in each dimension. In this case, we have $H_n(X+Y) \cong H_n(X) \oplus H_n(Y)$.

If X and Y are simplicial complexes with $X \cap Y$ a singleton set, then we may consider X' and Y' for X and Y where X' are Y' are obtained from X and Y by adding to X and Y open subsets A and B

of $X \cup Y$ so that $X' \cap Y' = (X \cap Y)(A \cup B)$ and this is open. We further assume that X' and Y' are contractible of X and Y respectively. Then $H_n(X') = H_n(X)$, $H_n(Y') = H_n(Y)$ and $H_n(X' \cap Y') = H_n(X \cap Y) = 0$. Hence using the Mayer-Vietoris sequence for X', Y', we again obtain $H_n(X'+Y') \cong H_n(X') \oplus H_n(Y')$. But this means that $H_n(X+Y) \cong H_n(X) \oplus H_n(Y)$.

This situation can be described by the following diagrams:



Here, $X = \{AB, BC, CA, A, B, C\}$, $Y = \{AD, AE, DE, A, D, E\}$, $X' = ABCRS - \{R, S\}$, $Y' = DEAQP - \{P, Q\}$, $\{A\} = X \cap Y$ and $X' \cap Y' = (PS \cup RQ) - \{P, Q, R, S\}$.

5.3 Connected Sum

Let(X, T) and (Y, T') be two topological spaces such that $X \cap Y = \Phi$. Suppose that there exists non-empty closed sets F and F' of X and Y respectively such that $b(F)$ is homeomorphic to $b(F')$. Let $f : b(F) \longrightarrow b(F')$ be a homeomorphism. and let $b(F) = B$) and $b(F') = B'$), i.e., $B = F - Int(F)$ and $B' = F' - Int(F')$. Suppose $Z = (X - Int(F)) \cup (Y - Int(F'))$ and define a relation $\sim$ on Z as follows :

(i) if $z \in Z - (b(F) \cup b(F'))$, then $z \sim z$ and $z \not\sim z$, for any $z' \in Z$, (ii) if $z \in b(F)$, then $z \sim z$ and $z \sim f(z)$, and (iii) if $z \in b(F')$, then $z \sim z$ and $z \sim f(z)$ .Then $\sim$ is an equivalence relation on Z. Under identification topology, $\frac{Z}{\sim}$ $(= \bar{Z})$ is a topological space and is termed the *connected sum* of X and Y with reference to F and F'. We can regard $\bar{Z}$ as $(X \cup Y) - (Int(F) \cup Int(F'))$ under the identification of x on B with $f(x)$ on B' and so $B = B' = (X - Int(F)) \cap (Y - Int(F'))$. It is convenient to denote the connected sum (in this case) of two topological spaces X and Y by $X \#_F Y$ or $X \#_{F'} Y$. But for our convenience, sometimes we denote this sum simply by $X \# Y$.

Comment: It is clear that the connected sum of compact surfaces that has been defined in chapter-1 is a particular case of this sum.

5.3.1 Fundemental group of $X \# Y$

Theorem 5.4

Let X and Y be two topological spaces. Then the following results hold good:

(i) *The connected sum $X \#_F Y$ is connected if and only if $X - Int(F)$ and $Y - Int(F')$ are connected.*

(ii) *If X and Y are path connected, then $X \#_F Y$ is path connected, and*

(iii) *If $X \#_F Y$ is path connected, then $X - Int(F)$ and $Y - Int(F')$ are path connected.*

Proof:

(i) From S. Majumdar and Asaduzzaman [45] we know a lemma which states that:

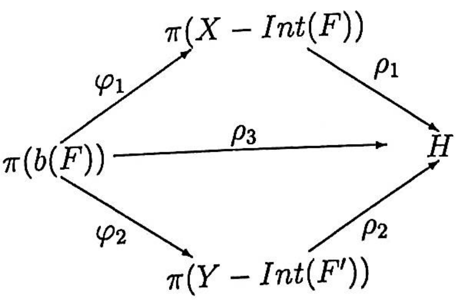
$X \cup Y$ is connected if and only if both X and Y are connected and $X \cap Y \neq \Phi$.

Since $X\#_F Y$ is $(X - IntF) \cup (Y - IntF')$ and $(X - IntF) \cap (Y - IntF') = bd(F) = bd(F')$ is a non-empty subspace of $X\#_F Y$, thus (i) follows from the lemma stated above.

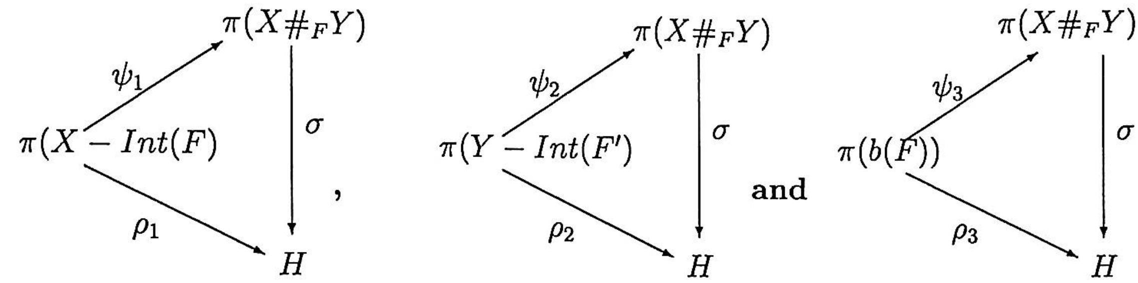
(ii) Let X and Y be path connected. Let $z_1,\ z_2 \in X\#_F Y$. Since $X\#_F Y = (X - IntF) \cup (Y - IntF')$, if $z_1,\ z_2$ both belong to $X - IntF$ or $Y - IntF'$ then there exists a path from $z_1,\ z_2$. So, let $z_1 \in X - Int\,F$ and $z_2 \in Y - Int\,F'$. Since $b(F) \neq \Phi$, we can take a point $z \in b(f)$. Then $z \in X \cap Y$. So there are paths f from z_1 to z and g from z to z_2. Then $g * f$ is a path from z_1 to z_2, where $g * f : [0,1] \longrightarrow (X - Int\,F) \cup (Y - Int\,F')$ is given by $(g * f)(t) = f(2t)$, whenever $0 \leq t \leq \frac{1}{2}$ and $g * f(t) = g(2t - 1)$, whenever $\frac{1}{2} \leq t \leq 1$. Hence $X\#_F Y$ is path connected.

(iii) It is obvious since $X\#_F Y = (X - IntF) \cup (Y - IntF')$.

We now express the fundamental group $\pi(x\#_F Y)$ of $X\#_F Y$ in terms of $\pi(X),\ \pi(Y)$ and $\pi(b(F))$ by the following way:
Let X and Y be path connected spaces and let F be homeomorphic to F' such that both are closed (but not open) subsets of X and Y respectively. Let H be any group and let $\rho_1,\ \rho_2,\ \rho_3$ be homomorphisms such that the diagram:



is commutative, where ϕ_1, ϕ_2 *are induced by inclusion maps. Then there exists a unique homomorphism* $\sigma : \pi(X\#_F Y) \longrightarrow H$ *such that the following diagrams:*



are commutative, where ψ_1, ψ_2, ψ_3 *are also induced by inclusion maps.*

5.4 External sum

Let X_1 and X_2 be two disjoint topological spaces and let there be two non-empty closed sets F_1 and F_2 in X_1 and X_2 respectively such that F_1 and F_2 are homeomorphic. Let$f : F_1 \longrightarrow F_2$ be a homeomorphism. We now define a relation R on the direct sum $X_1 \oplus X_2$ (direct sums studied in [50] and also occur in Dugundji [83]) as follows:

in such a way that

(i) for each $x_1 \in X_1 - F_1$, $x_1 \, R \, z$ and $z \, R \, x_1$ if and only if $z = x_1$,

(ii) for each $x_2 \in X_2 - F_2$, $x_2 \, R \, z$ and $z \, R \, x_2$ if and only if $z = x_2$,

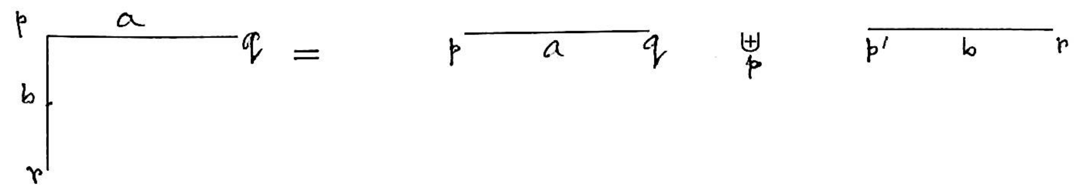
(iii) for each $x_1 \in F_1$, $x_1 \, R \, z$ and $z \, R \, x_1$ if and only if $z = x_1$ or $z = f(x_1)$,

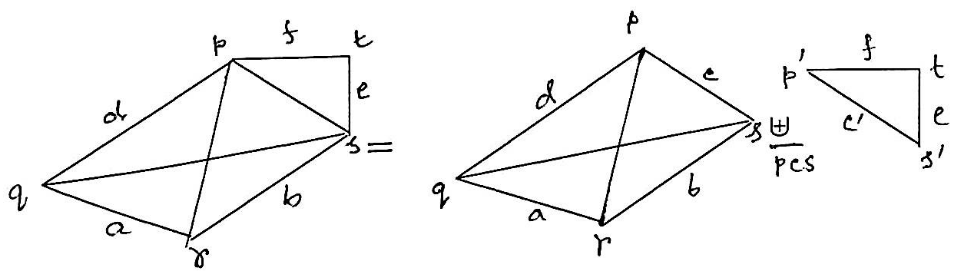
(iv) for each $x_2 \in F_2$, $x_2 \, R \, z$ and $z \, R \, x_2$ if and only if $z = x_2$ or $z = f^{-1}(x_2)$.

Then R is an equivalence relation. The quotient space $\frac{X_1 \oplus X_2}{R}$ will be called an *external sum* and will be denoted by $X_1 \uplus_F X_2$ or simply by $X \uplus Y$ where $F = F_1 = F_2$ (after identification). A study of such products has been made in [50].

If X and Y are subspaces of a topological space Z, then we may choose the subspace topology on $X \cup Y$ and obtain a space which we will call the *usual sum* and write $X +_Z Y$. If X and Y are disjoint, we may define the external sum $X \uplus_Z Y$ as before and call it the *usual external sum.*

Simplicial complexes are given as examples of external sums of subcomplexes, and ultimately, of 1-simplexes shown in the following diagrams:





The above examples show how structures of simplicial complexes are expressed using ⊎.

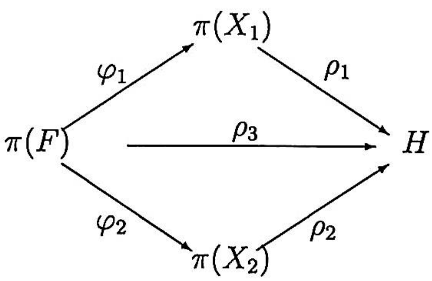
5.4.1 Fundamental group of $X \uplus Y$

As in the case of sum $X + Y$ and connected sum $X \# Y$, fundamental groups of X and Y (when they are path connected) and the homology groups of these spaces together with the fundamental group and the homology group of F give information about the corresponding groups of $X \uplus Y$.

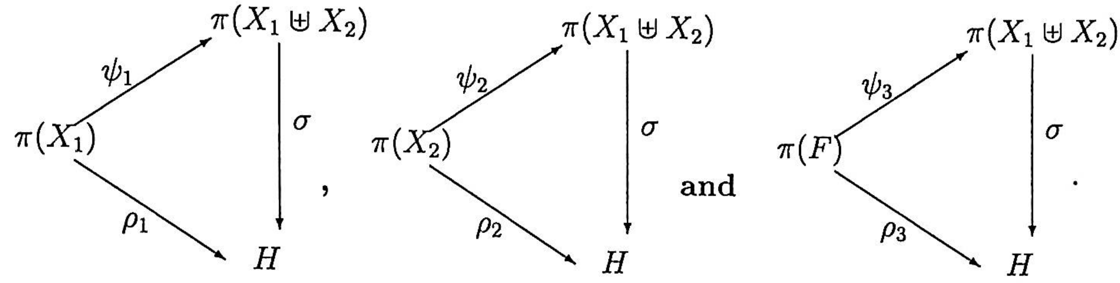
We regard $X \uplus Y$ as XUY with $X \cap Y = F$. Then X and Y are open subsets of $X \uplus Y$. If X and Y are are path-connected, then for any base point $z \in F$, we have the following consequence of Seifert-Van Kampen theorem:

Theorem 5.5

Let H be a group, and ρ_1, ρ_2, ρ_3 any three homomorphisms such that the following diagram:



is commutative where φ_1, φ_2 are induced by the inclusion map $F \longrightarrow X$ and $F \longrightarrow Y$. Then, there exists a unique homomorphism $\sigma : \pi(X_1 \uplus X_2) \longrightarrow H$ such that the following diagrams are commutative:



Here again $\psi 1$, ψ_2, ψ_3 are homomorphisms induced by inclusion maps.

The statement of the above theorem is equivalent to saying:

$\pi(X_1 \uplus X_2)$ is the free product of $\pi(X_1)$ and $\pi(X_2)$ with $\lambda_1\pi(F)$ amalgamated to $\lambda_2\pi(F)$ where λ_1, λ_2 are induced by inclusions $F \longrightarrow X_1$ and $F \longrightarrow X_2$.

Hence as a particular case we have the following:

Corollary 5.6

If F_1 and F_2, and hence F, are simply connected, then $\pi(X_1 \uplus X_2) \cong \pi(X_1) \star \pi(X_2)$ ($\star$ denotes the free product).

we consider two disjint connected simplicial complexes K_1 and K_2 such that K_1 has a subcomplex K_1' and K_2 has a subcomplex K_2' so that K_1' and K_2' are homeomorphic. Then $K_1 \uplus_F K_2$, where K' stands for K_1' , or equivalently, K_2', is the free product of $\pi(K_1)$ and $\pi(K_1)$ with $\lambda_1\pi(K')$ amalgamated to $\lambda_2\pi(K')$ with λ_1, λ_2 induced by inclusion.

5.4.2 Homology group $H_n(X_1 \uplus X_2)$

The Mayer-Vietoris sequence for homology can be applied to spa -ces X_1, X_2, $X_1 \uplus X_2$ and F to obtain the exact sequence:

$$\cdots\cdots\cdots \longrightarrow H_n(F) \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow$$
$$H_n(X_1 \uplus X_2) \longrightarrow H_{n-1}(F) \longrightarrow \cdots\cdots\cdots.$$

In particular, if $H_n(F) = 0$, for each n, then $H_n(X_1 \uplus X_2) \cong H_n(X_1) \oplus H_n(X_2)$, for each n.

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