1998

Sectionally Pseudocomplemented Distributive Nearlattices

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SECTIONALLY PSEUDOCOMPLEMENTED DISTRIBUTIVE NEARLATTICES

A THESIS

PRESENTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

by

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July, 1998.

DEDICATED TO
MY PARENTS,

WHO HAVE PROFOUNDLY INFLUENCED MY LIFE.

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SUMMARY

This thesis studies the nature of a sectionally pseudocomplemented distributive nearlattice. By a nearlattice S we will always mean a meet semilattice together with the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman in their paper [14] referred this property as the upper bound property, and a semilattice of this nature as a semilattice with the upper bound property. Cornish and Noor in [15] preferred to call these semilattices as nearlattices, as the behaviour of such a semilattice is closer to that of a lattice than an ordinary semilattice. Of course a nearlattice with a largest element is a lattice. So the idea of pseudocomplementation is not possible in case of a general nearlattice. But for a nearlattice with a smallest element we can talk about sectionally pseudocomplemented nearlattice. Moreover, we can discuss on relatively pseudocomplemented nearlattices. In this thesis, we give several results on sectionally (relatively) pseudocomplemented nearlattices which certainly extend and generalize many results in lattice theory.

Chapter one discusses ideals, congruences, semiboolean algebras, sectionally (relatively)

pseudocomplemented nearlattices and many other results on nearlattices which are basic to this thesis. We also include a treatment of localization.

In chapter two we give a description of generalized Stone nearlattices. We have also studied normal nearlattices and distributive quasi-complimented nearlattices. Generalized Stone lattices have been studied by Katrinak [27], Cornish [13] and many other authors. Here we extend several results of [13] and [27] to nearlattices. We have given a characterization of minimal prime ideals of a sectionally pseudocomplemented distributive nearlattice. Then we show that a distributive nearlattice S with 0 is generalized Stone if and only if it is both normal and sectionally quasi-complemented.

Chapter three introduces the concept of relative annihilators in nearlattices. Relative annihilators in lattices were studied by several authors including Mandelker [33] and Varlet [57]. For a, b $\in$ S, we define $<a, b> = \{x \in S \ / \ a \wedge x \leq b\}$. According to Mandelker [33], $< a, b >$ is known as an annihilator of a relative to b or simply a relative annihilator. Cornish [13] has used the annihilators in studying relatively normal lattices. Here we have studied the relative annihilators in nearlattices. In terms of relative annihilators ,we have characterized modular and distributive nearlattices. Then we have generalized some of the results of [33]. We have shown that in a distributive nearlattice S, $<a, b> \vee <b, a> = S$ for all $a, b \in S$ if and only if the filters containing any given prime filter form a chain. Relatively Stone lattices have been studied by several authors including Mandelker [33], Varlet [58] and Grätzer and Schmidt [23]. Since then a little attention has been paid in this topic. Here we have given several characterizations of relatively Stone nearlattices which are certainly the generalizations of above authors work. We also show that for a distributive nearlattice S in which every closed interval is pseudocomplemented is relatively Stone if and only if any two incomparable prime ideals of S are comaximal.

Chapter four is concerned with sectionally B_n-nearlattices and relatively B_n - nearlattices. Cornish in [8] have studied n-normal lattices. Then Noor in [41] has extended the idea to nearlattices and generalized some results of [8]. By [41], a distributive nearlattice S with 0 is called n-normal if every prime ideal contains at most n minimal prime ideals. Sectionally B_n-lattices and relatively B_n- lattices have been studied by Davey in [16] . In this chapter we have given several characterizations for sectionally B_n-nearlattices and generalized many works of Davey [16] and Cornish [8]. We show that a distributive nearlattice is sectionally in B_n if and only if it is n-normal and sectionally pseudocomplemented. We a distributive nearlattice S is relatively in B_n if and only if any n+1 pairwise incomparable prime ideals are comaximal.

A very interesting type of ideals called the α-ideals have been studied in chapter five . Annulets and α-ideals in lattices were studied extensively by Cornish in [11] . Here we discuss annulets of a distributive nearlattice with 0 and α-ideals in a distributive nearlattice. Then we generalize and extend a number of results of [11]. We conclude the thesis with the result that a distributive nearlattice S with 0 is generalized Stone if and only if each prime ideal contains a unique prime α - ideal.

STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University or Institute, and to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

A.K.M. Sirajul Islam
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ACKNOWLEDGMENTS

I like to express my sincere gratitude to my respectable teacher and supervisor **Dr. A. S. A. Noor**, Professor, Department of Mathematics, Rajshahi University, Rajshahi, Bangladesh, for his constant guidance, suggestions, valuable criticism and encouragement during my research work and for the preparation of this thesis.

I heartily express my gratefulness to the University Grants Commission of Bangladesh for awarding me fellowship and Ministry of Education, Bangladesh for granting me deputation for persuing the Ph.D. Program in the Department of Mathematics, Rajshahi University, Rajshahi, Bangladesh.

I was fortunate in obtaining the expert services of Engg. Md. Badiur Rahman, (Proprietor of Agrazatra Computers) for the computer typing of this thesis. His patience, co-operation and careful attention to detail are deeply appreciated.

The acknowledgement will remain incomplete if I do not mention the inspiration that I received from my father Md. Aminul Islam and my wife Shilpi during the course of this work. I also deeply regret for my sons to have deprived them from my love and affection during my research work .My love and thanks to them.

I am also thankful to the Department of Mathematics, Rajshahi University for extending me all co-operations during my stay in the department.

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CHAPTER ONE

IDEALS AND CONGRUENCES

1. Preliminaries

The intention of this section is to outline and fix the notation for some of the concepts of nearlattices which are basic to this thesis. We also formulate some results on arbitrary nearlattices for later use. For the background material in lattice theory we refer the reader to the text of G. Birkhoff [7],G.Grätzer [19],[20] and D.E. Rutherford [48].

By a *nearlattice* S we will always mean a (lower) semilattice which has the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman, in their paper [14], referred this property as the *upper bound property*, and a semilattice of this nature as *a semilattice with the upper bound property.* We shall see later, the behaviour of such a semilattice is closer to that of a lattice than an ordinary semilattice. For the sake of brevity, we prefer to use the term nearlattice in place of *semilattice with the upper bound property.*

Of course, a nearlattice with a largest element is a lattice. Since any semilattice satisfying the descending chain condition has the upper bound property, all finite semilattices are nearlattices.

Now we give an example of a meet semilattice which is not a nearlattice.

Example : In $\mathbf{R}^2$ consider the set

$S = \{(0, 0)\} \cup \{(I, 0)\} \cup \{(0, 1)\} \cup \{(1, y) / y > 1\}$

shown by the figure 1.1.

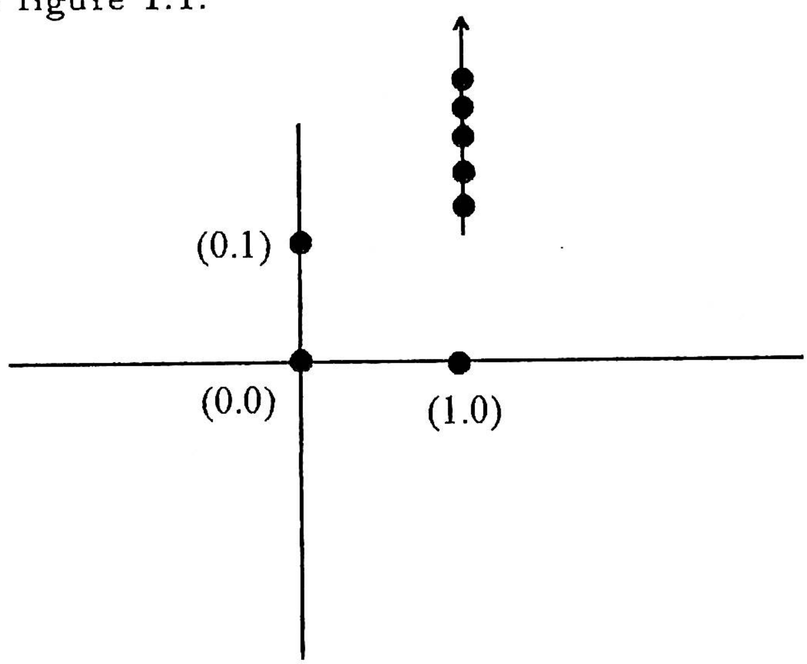


Figure 1.1

Define the partial ordering $\leq$ on S by $(x, y) \leq (x_1, y_1)$ if and only if $x \leq x_1$ and $y \leq y_1$. Observe that $(S; \leq)$ is a meet semilattice. Both (1, 0) and (0,1) have common upper bounds. In fact $\{(1, y) / y > 1\}$ are common upper bounds of them. But the supremum of (1, 0) and (0, 1) does not exist. Therefore $(S ; \leq)$ is not a nearlattice.

The upper bound property appears in Grätzer and Lakser [21], while Rozen [47] shows that it is the result of placing certain associativity conditions on the partial join operation. Moreover, Evans in a more recent paper [17] referred nearlattices as *conditional lattices.* By a conditional lattice he means a (lower) semilattice S with the condition that for each $x \in S$, $\{y \in S \mid y \leq x\}$ is a lattice; and it is very easy to check that this condition is equivalent to the upper bound property of S. Also Nieminen refers to nearlattices as *"Partial lattices"* in his paper [39].

Whenever a nearlattice has a least element we will denote it by 0. If $x_1, x_2, \text{---------------}, x_n$ are elements of a nearlattice then by $x_1 \vee x_2 \vee \text{----------------} \vee x_n$, we mean that the supremum of $x_1, x_2, \text{------------}, x_n$ exists and $x_1 \vee x_2 \vee \text{------------------} \vee x_n$, is the symbol denoting this supremum.

A non- empty subset K of a nearlattice S is called a *subnearlattice* of S if for any $a, b \in K$, both $a \wedge b$ and $a \vee b$ (whenever it exists in S) belong to K ($\wedge$ and $\vee$ are taken in S), and the $\wedge$ and $\vee$ of K are the restrictions of the $\wedge$ and $\vee$ of S to K. Moreover, a subnearlattice K of a nearlattice S is called a *sublattice* of S if $a \vee b \in K$ for all $a, b \in K$.

A nearlattice S is called *modular* if for any $a, b, c \in S$ with $c \leq a$, $a \wedge (b \vee c) = (a \wedge b) \vee c$ whenever $b \vee c$ exists.

A nearlattice S is called *distributive* if for any

$x, x_1, x_2, \text{---------}, x_n, x \wedge (x_1 \vee x_2 \vee \text{--------------} \vee x_n)$

$= (x \wedge x_1) \vee (x \wedge x_2) \vee \text{--------------------} \vee (x \wedge x_n)$

whenever $x_1 \vee x_2 \vee \text{-------------------} \vee x_n$ exists. Notice that the right hand expression always exists by the upper bound property of S.

Lemma 1.1.1. *A nearlattice* S *is distributive (modular) if and only if* $(x] = \{y \in S \,/\, y \leq x\}$ *is a distributive (modular) lattice for each* $x \in S$.□

Consider the following lattices:

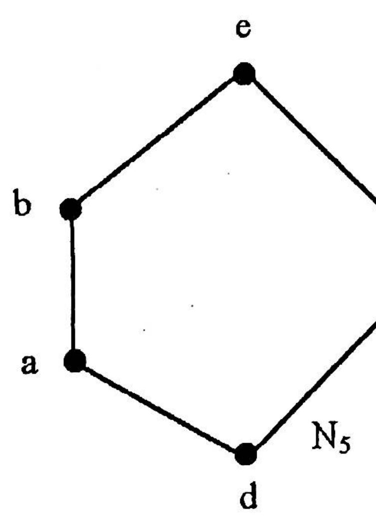


Figure 1.2

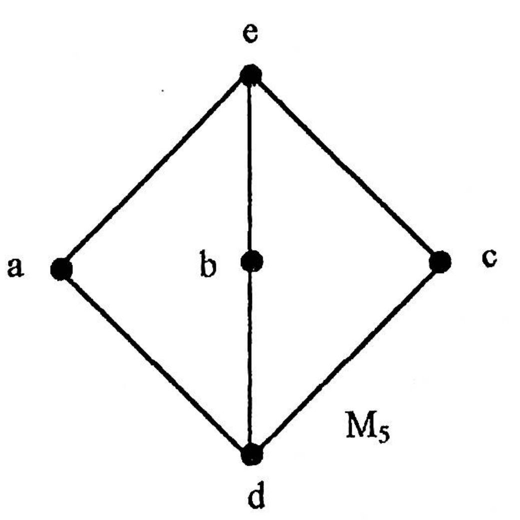


Figure 1.3

Hickman in [25] has given the following extension of a very fundamental result of lattice theory .

Theorem 1.1.2. *A nearlattice* S *is distributive if and only if* S *does not contain a sublattice isomorphic to* N_5 or M_5. □

Now we give another extension of a fundamental result of Lattice Theory.

Theorem 1.1.3. *A nearlattice* S *is modular if and only if* S *does not contain a sublattice isomorphic to* N_5 .

Proof : Suppose S does not contain any sublattice isomorphic to N_5 . Then (x] does not contain any sublattice isomorphic to N_5 for each $x \in S$. Thus a fundamental result of lattice theory says that (x] is modular for each $x \in S$ as (x] is a sublattice of S. Hence S is modular by Lemma 1.1.1.

Conversely, let S be modular. If S contains a sublattice isomorphic to N_5, then letting e as the largest element of the sublattice, we see that (e] is not modular [by Lattice Theory]. Thus by Lemma 1.1.1, S is not modular and this gives a contradiction. This completes the proof. □

In this context it should be mentioned that many Lattice theorists e.g. R. Balbes [4], J. C. Varlet [55], R. C. Hickman [24] and K. P. Shum [51] have worked with a class

of semilattices S which has the property that for each $x, a_1, a_2, \text{---------------------} a_r \in S$,
if $a_1 \vee a_2 \vee \text{-----------------------} \vee a_r$ exists then
$(x \wedge a_1) \vee (x \wedge a_2) \vee \text{---------------} \vee (x \wedge a_r)$ exists and equals $x \wedge (a_1 \vee a_2 \vee \text{--------------} \vee a_r)$. [4] called them as *prime semilattices* while [51] referred them as *weakly distributive semilattices.*

Hickman in [24] has defined a ternary operation j by $j(x, y, z) = (x \wedge y) \vee (y \wedge z)$, on a nearlattice S (which exists by the upper bound property of S).In fact he has shown that (also see Lyndon [32], Theorem 4]), the resulting algebras of the type (S ; j) form a variety, which is referred to as the variety of join algebras and following are its defining identities.

(i) $j(x, x, x) = x$

(ii) $j(x, y, x) = j(y, x, y)$

(iii) $j(j(x, y, x), z, j(x, y, x)) = j(x, j(y, z, y), x)$

(iv) $j(x, y, z) = j(z, y, x)$

(v) $j(j(x, y, z), j(x, y, x), j(x, y, z)) = j(x, y, x)$

(vi) $j(j(x, y, x), y, z) = j(x, y, z)$

(vii) $j(x, y, j(x, z, x)) = j(x, y, x)$

(viii) $j(j(x,y,j(w,y,z)),j(x,y,z), j(x, y, j(x, y, z)))$
$= j(x, y, z)$.

We do not want to elaborate it further as it is beyond the scope of this thesis.

We call a nearlattice S a *medial nearlattice* if for all $x, y, z \in S$, $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists. For a (lower) semilattice S, if $m(x, y, z)$ exists for all $x, y, z \in S$, then it is not hard to see that S has the upper bound property and hence is a nearlattice. Distributive medial nearlattices were first studied by Sholander in [49] and [50], and recently by Evans in [17]. Sholander preferred to call these as *median semilattices.* There he showed that every medial nearlattice S can be characterized by means of an algebra $(S ; m)$ of type $<3>$, known as *median algebra,* satisfying the following two identities:

(i) $m(a, a, b) = a$

(ii) $m(m(a,b,c), m(a, b, d), e) = m(m(c,d,e), a, b)$.

A nearlattice S is said to have the *three* property if for any $a, b, c \in S$, $a \vee b \vee c$ exists whenever $a \vee b$, $b \vee c$ and $c \vee a$ exist. Nearlattices with the *three* property were discussed by Evans in [17], where he referred it as *strong conditional lattices.*

The equivalence of (i) and (iii) of the following lemma is trivial, while the proof of (i) <=> (ii) is inductive.

Lemma 1.1.4. [Evans [17]] . *For a nearlattice* S *the following conditions are equivalent :*

(i) S *has the three property.*

(ii) *Every pair of a finite number* n (≥ 3) *of elements of* S *possess a supremum ensures the existence of the supremum of all the n elements.*

(iii) S *is medial* .□

A family $\mathcal{A}$ of subsets of a set A is called a *closure system* on A if

(i) $A \in \mathcal{A}$ and

(ii) $\mathcal{A}$ is closed under arbitrary intersection.

Suppose ß is a subfamily of $\mathcal{A}$. ß is called a *directed system* if for any X, Y ∈ ß there exists Z in ß such that X, Y ⊆ Z.

If $\cup \{ X : X \in ß \} \in \mathcal{A}$ for every directed system ß contained in the closure system $\mathcal{A}$, then $\mathcal{A}$ is called *algebraic.* When ordered by set inclusion, an algebraic closure system forms an *algebraic lattice.*

A non- empty subset H of a nearlattice S is called *hereditary* if for any $x \in S$ and $y \in H$, $x \leq y$ implies $x \in H$. The set H (S) of all hereditary subsets of S is a complete distributive lattice when partially ordered by set- inclusion, where the meet and join in H (S) are given by set theoretic intersection and union respectively. The largest element of H (S) is S, while the smallest element is the empty set Φ.

For the following 3-element nearlattice S(Figure 1.4), $H(S) = \{\Phi, \{0\}, \{0, a\}, \{0, b\}, \{0, a, b\}\}$

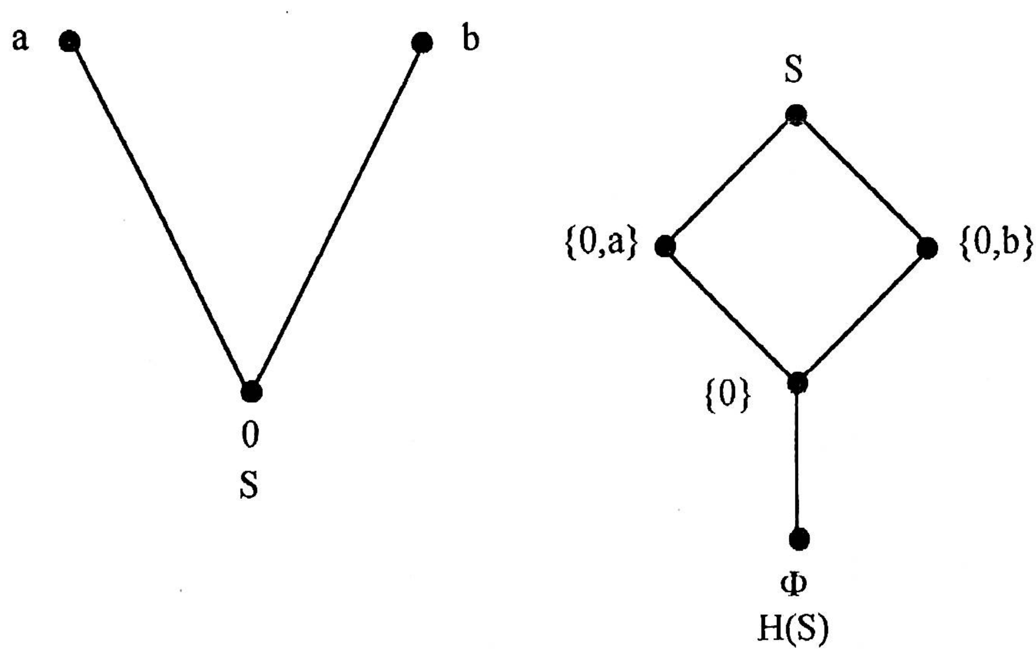


Figure 1.4

2. Ideals of Nearlattices.

A non- empty subset I of a nearlattice S is called an *ideal* if it is hereditary and closed under existent finite suprema. We denote the set of all ideals of S by I(S). If S has a smallest element 0 then I(S) is an algebraic closure system on S, and is consequently an algebraic lattice. However, if S does not possess smallest element then we can only assert that $I(S) \cup \{\Phi\}$ is an algebraic closure system.

For any subset K of a nearlattice S, (K] denotes the ideal generated by K.

Infimum of two ideals of a nearlattice is their set theoretic intersection. Supremum of two ideals I and J in a lattice L is given by

$I \vee J = \{x \in L \,/\, x \leq i \vee j \text{ for some } i \in I, j \in J\}$. Cornish and Hickman in [14] showed that in a distributive nearlattice S for two ideals I and J,

$I \vee J = \{i \vee j \,/\, i \in I, j \in J \text{ where } i \vee j \text{ exists}\}$. But in a general nearlattice the formula for the supremum of two ideals is not very easy. We start this section with the following lemma which gives the formula for the supremum of two ideals. It is in fact exercise 22 of Grätzer [19, p 54] for partial lattice.

Lemma 1.2.1. *Let* I *and* J *be ideals of a nearlattice* S. *Let* $A_0 = I \cup J$, $A_n = \{ x \in S / x \leq y \vee z$; $y \vee z$ *exists and* $y, z \in A_{n-1}\}$ *for* $n = 1, 2, \text{--------}$, *and* $K = \bigcup_{n=0}^{\infty} A_n$. *Then* $K = I \vee J$.

Proof : Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq \text{----------} \subseteq A_n \subseteq \text{------}$,K is an ideal containing I and J. Suppose H is any ideal containing I and J. Of course, $A_0 \subseteq H$. We proceed by induction. Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$ and let $x \in A_n$. Then $x \leq y \vee z$ with $y, z \in A_{n-1}$. Since $A_{n-1} \subseteq H$ and H is an ideal, $y \vee z \in H$ and $x \in H$. That is $A_n \subseteq H$ for every n. Thus $K = I \vee J$. □

Lemma 1.2.2. and corollary 1.2.3. were suggested to the author by his supervisor professor A. S. A. Noor.

Lemma 1.2.2. *Let* K *be a non empty subset of a nearlattice* S. *Then* $(K] = \bigcup_{n=0}^{\infty} \{A_n / n \geq 0\}$, *where* $A_0 = \{t \in S / t = j(k_1, t, k_2)$ *for some* $k_1, k_2 \in K\}$ *and* $A_n = \{t \in S / t = j(a_1, t, a_2)$ *for some* $a_1, a_2 \in A_{n-1}\}$ for $n \geq 1$.

Proof : For any $k \in K$ clearly $k = j(k, k, k)$ and so$K \subseteq A_0$. Similarly, for any $a \in A_{n-1}$, $a = j(a, a, a)$ implies that $A_{n-1} \subseteq A_n$. Thus,

$K \subseteq A_0 \subseteq A_1 \subseteq \cdots\cdots \subseteq A_{n-1} \subseteq A_n \subseteq \cdots\cdots$. Let $t \in \bigcup_{n=0}^{\infty} A_n$; $n = 0,1,2,\ldots\ldots$, and $t_1 \in S$ such that $t_1 \leq t$. Then $t \in A_m$ for some $m \geq 0$. Clearly, $t_1 = j\,(t, t_1, t)$ and so $t_1 \in A_{m+1}$. Thus, $\bigcup_{n=0}^{\infty} A_n$ is hereditary. Now, suppose, $t_1, t_2 \in \bigcup_{n=0}^{\infty} A_n$, and $t_1 \vee t_2$ exists. Let $t_1 \in A_r$ and $t_2 \in A_s$ for some $r, s \geq 0$ with $r \leq s$ (say). Then, $t_1, t_2 \in A_s$ and $t_1 \vee t_2 = j\,(t_1, t_1 \vee t_2, t_2)$ says $t_1 \vee t_2 \in A_{s+1}$.

Finally, suppose H is an ideal containing K. If $x \in A_0$, then $x = j\,(k_1, x, k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1, k_2 \in K$. As $K \subseteq H$ and H is an ideal, $k_1 \wedge x, k_2 \wedge x \in H$ and so $x \in H$. Again we use the induction. Suppose $A_{n-1} \subseteq H$ for some $n \geq 1$. Let $x \in A_n$ so that $x = j\,(a_1, x, a_2)$ for some $a_1, a_2 \in A_{n-1}$. Then $x \in H$ as $a_1, a_2 \in H$ and $x = (a_1 \wedge x) \vee (a_2 \wedge x)$.□

Corollary 1.2.3. *A non empty subset* K *of a nearlattice* S *is an ideal if and only if* $x \in K$ *whenever* x *is an element of* S *such that* $x = j\,(k_1, x, k_2)$ *for same* $k_1, k_2 \in K$.

Proof : Since the only if part is obvious, suppose $x \in K$ whenever x is an element of S and $x = j\,(k_1, x, k_2)$ for some $k_1, k_2 \in K$. Then clearly, A_0 (of Lemma 1.2.2)$\subseteq K$. Now for any $x \in A_1$, $x = j\,(a_1, x, a_2)$ for some $a_1, a_2 \in A_0 \subseteq K$. Thus,

$x \in K$ and so $A_1 \subseteq K$. Hence, using induction we obtain that $(k] = \bigcup_{n=0}^{\infty} A_n \subseteq K$; i.e. $K = (k]$. Therefore K is an ideal.□

We now give an alternative formula for the supremum of two ideals in an arbitrary nearlattice .

Lemma 1.2.4. *For any two ideals* K_1 *and* K_2 ,

$K_1 \vee K_2 = \bigcup_{n=0}^{\infty} B_n$ *where* $B_0 = \{x \in S \,/\, x = j\,(k_1 , x, k_2),$ $k_i \in K_i\}$ *and* $B_n = \{x \in S \,/\, x = j\,(b_1 , x, b_2);\ b_1, b_2 \in B_{n-1}\}$ *and* n = 0, 1, 2, ---------------.

Proof : Clearly,

$K_1, K_2, \subseteq B_0 \subseteq B_1 \subseteq$ ---------- $\subseteq B_{n-1} \subseteq B_n \subseteq$ -----------.

Suppose $b \in \bigcup_{n=0}^{\infty} B_n$ and $b_1 \leq b$; $b_1 \in S$. Then $b \in B_m$ for some $m \geq 0$. Also, $b_1 = j\,(b, b_1, b)$ and so $b_1 \in B_{m+1}$. Thus, $\bigcup_{n=0}^{\infty} B_n$ is hereditary. Now, suppose $t_1, t_2 \in \bigcup_{n=0}^{\infty} B_n$ such that $t_1 \vee t_2$ exists. Then there exists $r, s \geq 0$ such that $t_1 \in B_r$ and $t_2 \in B_s$. If $r \leq s$ then $t_1, t_2 \in B_s$, and

$t_1 \vee t_2 = j\,(t_1, t_1 \vee t_2, t_2)$ implies that $t_1 \vee t_2 \in B_{s+1}$. Hence, $\bigcup_{n=0}^{\infty} B_n$ is an ideal.

Finally, suppose H is an ideal containing K_1 and K_2. If $x \in B_0$ then $x = j\,(k_1, x, k_2) = (k_1 \wedge x) \vee (k_2 \wedge x)$ for some $k_1 \in K_1$ and $k_2 \in K_2$. Since H is an ideal and $K_1, K_2 \subseteq H$, clearly $x \in H$. Then using the induction on n it is very easy to see that $H \supseteq B_n$ for each n.□

Theorem 1.2.5. *Cornish and Hickman* [14, *Theorem* 1.1]]. *The following conditions on a nearlattice* S *are equivalent* :

(i) S *is distributive.*

(ii) *For any* $H \in H(S)$,

$(H] = \{h_1 \vee \text{---------} \vee h_n / h_1, \text{-------}, h_n \in H\}$

(iii) *For any* $I, J \in J(S)$,
$I \vee J = \{a_1 \vee \text{-------} \vee a_n / a_1, \text{-----}, a_n \in I \vee J\}$
(iv) $J(S)$ *is a distributive lattice.*

(v) *The map* $H \text{------->} (H]$ *is a lattice homomorphism of* $H(S)$ *onto* $J(S)$ (*which preserves arbitrary suprema*) . □

Observe here that (iii) of above could easily be improved by 1. 2. 4 to (iii)$'$. For any $I, J \in J(S)$,
$I \vee J = \{i \vee j / i \in I, j \in J\}$.

Let $J_f(S)$ from henceforth denote the set of all *finitely generated ideals* of a nearlattice S. Of course $J_f(S)$ is an upper subsemilattice of $J(S)$. Also for any
$x_1, x_2, \text{----------------}, x_m \in S$, $(x_1, x_2, \text{----------}, x_m]$ is clearly the supremum of $(x_1] \vee (x_2] \vee \text{---------} \vee (x_m]$. When S is distributive, $(x_1, x_2, \text{----------}, x_m] \cap (y_1, y_2, \text{------}, y_n]$
$= ((x_1] \vee (x_2] \vee \text{---------} \vee (x_m]) \cap ((y_1] \vee (y_2] \vee \text{------} \vee (y_n])$

$= \bigvee_{i,j} (x_i \wedge y_j]$ for any $x_1, x_2, \text{------} x_m, y_1, y_2, \text{------} y_n \in S$ (by 1.2.5) and so $J_f(S)$ is a distributive sublattice of $J(S)$.
c. f. Cornish and Hickman [14].

A nearlattice S is said to be *finitely smooth* if the intersection of two finitely generated ideals is itself finitely generated. For example, (i) distributive nearlattices, (ii) finite nearlattices, (iii) lattices, are finitely smooth. Hickman in [24] exhibited a nearlattice which is not finitely smooth.

By Cornish and Hickman [14], we know that a nearlattice S is distributive if and only if I(S) is so. Our next result shows that the case is not the same with the modularity.

Theorem 1.2.6. *Let* S *be a nearlattice. If* I(S) *is modular then* S *is also modular but the converse is not necessarily true.*

Proof : Suppose I(S) is modular. Let a, b, c $\in$ S with $c \leq a$ and $b \vee c$ exists. Then $(c] \subseteq (a]$. Since I(S) is modular. So,
$(a \wedge (b \vee c)] = (a] \wedge ((b] \vee (c])$
$= ((a] \wedge (b]) \vee (c] = ((a \wedge b) \vee c]$. This implies that
$a \wedge (b \vee c) = (a \wedge b) \vee c$, and so S is modular.

Nearlattice S of figure 1.5 shows that the converse of this result is not true.

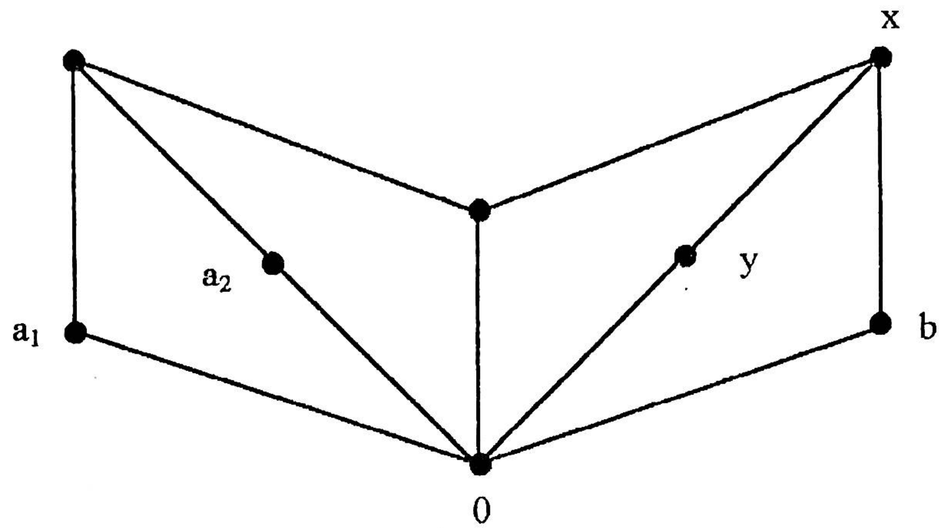


Fig 1.5

Notice that $(r]$ is modular for each $r \in S$. But in $I(S)$, clearly $\{(0], (a_1], (a_1,y],(a_2,b],S\}$ is a pentagonal sublattice. □

A *filter* F of a nearlattice S is a non empty subset of S such that if f_1, $f_2 \in F$ and $x \in S$ with $f_1 \leq x$, then both $f_1 \wedge f_2$ and x are in F. A filter G is called a *prime filter* if $G \neq S$ and at least one of x_1, x_2, ------ x_n is in G whenever $x_1 \vee x_2 \vee$ ------------- $\vee\, x_n$ exists and is in G.

An ideal P in a nearlattice S is called a *prime ideal* if $P \neq S$ and $x \wedge y \in P$ implies $x \in P$ or $y \in P$. It is not hard to see that a filter F of a nearlattice S is prime if and only if $S - F$ is a prime ideal.

The set of filters of a nearlattice is an upper semilattice; yet it is not a lattice in general, as there is no guarantee that the intersection of two filters is non empty. The join $F_1 \vee F_2$ of two filters is given by
$F_1 \vee F_2 = \{ s \in S \,/\, s \geq f_1 \wedge f_2$ for some $f_1 \in F_1, f_2 \in F_2\}$.
The smallest filter containing a subsemilattice H of S is $\{s \in S \,/\, s \geq h$ for some $h \in H\}$ and is denoted by $[H)$. Moreover, the description of the join of filters shows that for all $a,b \in S$, $[a) \vee [b) = [a \wedge b)$.

Following theorem and corollary is due to Noor and Rahman [45] which is an extension of a well known theorem of Lattice theory due to M. H. Stone; c. f. [19, Theorem 15, pp 74].

Theorem 1.2.7. *Let* S *be a nearlattice. The following conditions are equivalent*:

(i) S *is distributive.*

(ii) *For any ideal* I *and any filter* F *of* S, *such that* $I \cap F = \Phi$, *there exists a prime ideal* $P \supseteq I$ *and disjoint from* F. □

Corollary 1.2.8. *A nearlattice* S *is distributive if and only if every ideal is the intersection of all prime ideals containing it.* □

3. Congruences.

An equivalence relation Θ of a nearlattice S is called a *congruence relation* if $x_i \equiv y_i\ (\Theta)$ for $i=1,2$ $(x_i, y_i \in S)$, then

(i) $x_1 \wedge x_2 \equiv y_1 \wedge y_2\ (\Theta)$, and

(ii) $x_1 \vee x_2 \equiv y_1 \vee y_2\ (\Theta)$ provided $x_1 \vee x_2$ and $y_1 \vee y_2$ exist.

It can be easily shown that for an equivalence relation Θ on S ,the above conditions are equivalent to the conditions that for $x,y \in S$ if $x \equiv y\ (\Theta)$, then

(i$'$) $x \wedge t \equiv y \wedge t\ (\Theta)$ for all $t \in S$ and

(ii$'$) $x \vee t \equiv y \vee t\ (\Theta)$ for all $t \in S$ provided both $x \vee t$ and $y \vee t$ exist.

The set C(S) of all congruences on S is an algebraic closure system on $S \times S$ and hence, when ordered by set inclusion, is an algebraic lattice.

Cornish and Hickman [14] showed that for an ideal I of a distributive nearlattice S, the relation Θ (I), defined by $x \equiv y\ (\Theta(I))$ if and only if $(x] \vee I = (y] \vee I$ is the smallest congruence having I as a congruence class. Moreover the equivalence relation R(I), defined by $x \equiv y\ (R(I))$ if and only if for any $s \in S$, $x \wedge s \in I$ is equivalent to $y \wedge s \in I$, is the largest congruence having I as a congruence class.

Suppose S is a distributive nearlattice and $x \in S$, we will use Θ_x as an abbreviation for $\Theta((x])$. Moreover Ψ_x denotes the congruence, defined by $a \equiv b\ (\Psi_x)$ if and only if $a \wedge x = b \wedge x$.

Cornish and Hickman [14] also showed that for any two elements a, b of a distributive nearlattice S with $a \leq b$, the smallest congruence identifying a and b is equal to $\Psi_a \cap \Theta_b$ and we denote it by $\Theta(a,b)$. Also, in a distributive nearlattice S, they observed that if S has a smallest element 0, then clearly $\Theta_x = \Theta(0, x)$ for any $x \in S$. Moreover, it is easy to see that :

(i) $\Theta_a \vee \Psi_a = \iota$, the largest congruence of S.

(ii) $\Theta_a \cap \Psi_a = \omega$, the smallest congruence of S and

(iii) $\Theta(a, b)' = \Theta_a \vee \Psi_b$ where $a \leq b$ and '/' denotes the complement.

Now suppose S is an arbitrary nearlattice and E(S) denotes its lattice of equivalence relations . For $\Phi_1, \Phi_2 \in E(S)$, $\Phi_1 \vee \Phi_2$ denotes their supremum; $x \equiv y\ (\Phi_1 \vee \Phi_2)$ if and only if there exists $x = z_0, z_1, ----------, z_n = y$ such that $z_{i-1} \equiv z_i\ (\Phi_1 \text{ or } \Phi_2)$ for $i = 1,2, ----- n$.

The following result was stated by Grätzer and Lakser in[21] without proof and a proof, different than given below, appears in Cornish and Hickman [14] ; also see Hickman [24] and [25].

Theorem 1.3.1. *For any nearlattice* S, C(S) *is a distributive (complete) sublattice of* E(S).

Proof : Suppose Θ, $\Phi \in C(S)$. Define Ψ to be the supremum of Θ and Φ in the lattice of equivalence relations E(S) on S. Let $x \equiv y$ (Ψ). Then there exists $x = z_0, z_1,$ -------- $z_n = y$ such that $z_{i-1} \equiv z_i$ (Θ or Φ). Thus, for any $t \in S$,
$z_{i-1} \wedge t \equiv z_i \wedge t$ (Θ or Φ) as Θ, $\Phi \in C(S)$.

Hence $x \wedge t \equiv y \wedge t$ (Ψ) and consequently Ψ is a semilattice congruence. Then, in particular $x \wedge y \equiv x$ (Ψ) and $x \wedge y \equiv y$ (ψ). To show that ψ is a congruence, let
$x \equiv y$ (ψ), with $x \leq y$, and choose any $t \in S$ such that both $x \vee t$ and $y \vee t$ exist. Then there exists $z_0, z_1, z_2,$-------- z_n such that $x = z_0$, $z_n = y$ and $z_{i-1} \equiv z_i$ (Θ or Φ). Put
$w_i = z_i \wedge y$ for all $i = 0, 1,$ ------, n. Then $x = w_0$, $w_n = y$,
$w_{i-1} \equiv w_i$ (Θ or Φ). Hence by the upper bound property,
$w_i \vee t$ exists for all $i = 0, 1,$ ----, n (as $w_i, t \leq y \vee t$) and
$w_{i-1} \vee t \equiv w_i \vee t$ (Θ or Φ) for all $i = 1, 2,$ -----------, n (as Θ, $\Phi \in C(S)$), i. e. $x \vee t \equiv y \vee t$ (Ψ). Then by [15,

Lemma 2.3] Ψ is a congruence on S . Therefore, C(S) is a sublattice of the lattice E(S).

To show the distributivity of C(S), let $x \equiv y\ (\Theta \cap (\Theta_1 \vee \Theta_2))$. Then $x \wedge y \equiv y\ (\Theta)$ and $((\Theta_1 \vee \Theta_2)$. Also, $x \wedge y \equiv x\ (\Theta)$ and $(\Theta_1 \vee \Theta_2)$.

Since $x \wedge y \equiv y\ (\Theta_1 \vee \Theta_2)$, there exists $t_0, t_1, \text{-------}\ t_n$ such that (as we have seen in the proof of the first part), $x \wedge y = t_{0,}\ t_n = y,\ t_{i-1} \equiv t_i\ (\Theta_1$ or $\Theta_2)$ and $x \wedge y = t_0 \leq t_i \leq y$ for each $i = 0, 1,$ -----------, n. Hence, $t_{i-1} \equiv t_i\ (\Theta)$ for all $i = 1,2,$ ----------, n, and so $t_{i-1} \equiv t_i\ (\Theta \cap \Theta_1)$ or $(\Theta \cap \Theta_2)$. Thus, $x \wedge y \equiv y\ ((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$. By symmetry, $x \wedge y \equiv x\ ((\Theta \cap \Theta_1) \vee (\Theta \cap \Theta_2))$ and the proof completes by transitivity of the congruences.□

In lattice theory it is well known that a lattice is distributive if and only if every ideal is a class of some congruence. Following theorem gives a generalization of this result in case of nearlattices.

This also characterizes the distributivity of a near lattice , which is an extension of [14, Theorem 3.1].

Theorem 1.3.2. S *is distributive if and only if every ideal is a class of some congruence.*

Proof : Suppose S is distributive . Then by [14, Theorem 3.1] for each ideal I of S, Θ (I) is the smallest congruence containing I as a class.

To prove the converse, let each ideal of S be a congruence class with respect to some congruence on S. Suppose S is not distributive. Then by Theorem 1.1.2, we have either N_5 (Figure 1.2) or M_5 (Figure 1.3) as a sublattice of S. In both cases consider I = (a] and suppose I is a congruence class with respect to Θ. Since $d \in I$, $d \equiv a$ (Θ). Now $b = b \wedge c = b \wedge (a \vee c) \equiv b \wedge (d \vee c) = b \wedge c = d$ (Θ) i. e, $b \equiv d$ (Θ) and this implies $b \in I$, i. e, $b \leq a$ which is a contradiction. Thus S is distributive. □

Following theorem is due to Noor and Rahman [45].

Theorem 1.3.3. *For a distributive nearlattice* S, *the mapping* I -----→ Θ (I) *is an embedding from the lattice of ideals to the lattice of congruences.* □

4. Semiboolean algebras.

An interesting class of distributive nearlattices is provided by those semilattices in which each principal ideal is a Boolean algebra. These semilattices have been studied by Abbott [1], [2], [3] under the name of semiboolean algebras and mainly from the view of Abbott's implication algebras [an implication algebra is a groupoid (I ; .) satisfying :

(i) $(ab)\,a = a$,

(ii) $(ab)\,b = (ba)\,a$,

(iii) $a(bc) = b(ac)$]

Abbott shows in [1,pp.227-236] that each implication algebra determines a semiboolean algebra and conversely each semiboolean algebra determines an implication algebra.

Following result gives a characterization of semiboolean algebras which is due to Cornish and Hickman in their paper of weakly distributive semilattices [14] (such semilattices were first studied by Balbes [4] under the name of prime semilattices).

Theorem 1.4.1. *[Cornish and Hickman [14, Theorem 2.2]]. A semilattice* S *is a semiboolean algebra if and only if the following conditions are satisfied :*

(i) S *has the upper bound property.*

(ii) S *is distributive.*

(iii) S *has a* 0 *and for any* $x \in S$,

$(x]^* = \{y \in S / y \wedge x = 0\}$ *is an ideal and* $(x] \vee (x]^* = S$.□

A nearlattice S is *relatively complemented* if each interval $[x,y]$ in S is complemented. That is, for $x \leq t \leq y$, there exists t' in $[x, y]$ such that $t \wedge t' = x$ and $t \vee t' = y$.

A nearlattice S with 0 is called *sectionally complemented* if $[0, x]$ is complemented for each $x \in S$. Of course every relatively complemented nearlattice S with 0 is sectionally complemented. It is not hard to see that S is semiboolean if and only if it is sectionally complemented and distributive. We denote the set of all prime ideals of S by P(S).

There is a well known result in Lattice Theory due to Nachbin in 1947, c. f. [19, Theorem 22, pp-76] that a distributive lattice is Boolean if and only if its prime ideals are unordered. Following theorem is a generalization to this result which is due to Cornish and Hickman in [14].

Theorem 1.4.2. *For a distributive nearlattice* S *with* 0, *the following conditions are equivalent :*

(i) S *is semiboolean.*

(ii) $J_f(S)$ *is a generalized Boolean algebra.*

(iii) P(S), *the set of all prime ideals is unordered by set inclusion.*□

Noor and Rahman [45] has proved the following theorem which is an extension of above result.

Theorem 1.4.3. *Let* S *be a distributive nearlattice.* S *is relatively complemented if and only if* P(S) *is unordered.*□

We conclude this section with the following result which is due to [14, Theorem 3.6]. This generalizes a well known result of Hashimoto in Lattice Theory [19, Theorem 8, pp-91].

Theorem 1.4.4. *For a nearlattice* S *with* 0 *the following conditions are equivalent.*

(i) S *is semiboolean.*

(ii) I(S) *is isomorphic to* C(S).

(iii) *For all ideals* I, Θ (I) = R(I).□

5. Principle of localization.

Principle of Localizations are extensions of lecture notes of my supervisor **Dr. Noor** on localization. For some ideas on localization see section 5 of Cornish[12]. This will be needed for the developement of this thesis.

Let F be a filter of a distributive nearlattice S. It can be easily shown that the relation ψ_F on S, defined by $x \equiv y\ (\psi_F)\ (x, y \in S)$ if and only if $x \wedge f = y \wedge f$, for some $f \in F$ is a congruence on S. Let us denote $S/\psi(F)$ by S_F (the quotient lattice). Then $\psi_F : S \dashrightarrow S_F$ is the natural epimorphism.

Lemma 1.5.1. S_F *is a distributive lattice* .

Proof : Clearly S_F is a lower semilattice. Now, let $p, q \in S_F$. Then there exist $x, y \in S$ such that $p = \psi_F(x)$, $q = \psi_F(y)$, as ψ_F is an epimorphism. Clearly $x \equiv x \wedge f\ (\psi_F)$ and $y \equiv y \wedge f\ (\psi_F)$ for any $f \in F$.

So, $\psi_F(x) = \psi_F(x \wedge f)$ and $\psi_F(y) = \psi_F(y \wedge f)$. Now, $(x \wedge f) \vee (y \wedge f)$ always exists in S, due to the upper bound property of S. Thus , $p \vee q$ exists. Moreover,

$p \vee q = \psi_F(x \wedge f) \vee \psi_F(y \wedge f) = \psi_F((x \wedge f) \vee (y \wedge f))$. Hence S_F is a lattice. The distributivity of S_F clearly follows from the distributivity of S. □

Lemma 1.5.2. *Let* F *be any filter of a distributive nearlattice* S. *For any ideals* I *and* J *of* S, *the following hold* :

(i) $\psi_F(I)$ *is an ideal of* S_F.

(ii) $\psi_F(I)$ *is a proper ideal* (i. e, ≠ *whole lattice*) *if and only if* $I \cap F = \Phi$.

(iii) $\psi_F(I) \vee \psi_F(J) = \psi_F(I \vee J)$.

(vi) $\psi_F(I) \cap \psi_F(J) = \psi_F(I \cap J)$.

Proof : (i) For $i, j \in I$, $\psi_F(i) \vee \psi_F(j)$ $= \psi_F(i \wedge f) \vee \psi_F(j \wedge f) = \psi_F((i \wedge f) \vee (j \wedge f))$ for any $f \in F$. Thus, $\psi_F(I)$ is closed under finite supremum. Now, suppose $t \in S_F$ and $t \leq \psi_F(i)$ for some $i \in I$. Then $t = \psi_F(x)$ for some $x \in S$, and $t = \psi_F(x) \wedge \psi_F(i) = \psi_F(x \wedge i) \in \psi_F(I)$. Therefore, $\psi_F(I)$ is an ideal of S_F.

(ii) If $\psi_F(I)$ is proper, then there exists $x \in S$, such that $\psi_F(x)$ does not belong to $\psi_F(I)$. Suppose $I \cap F \neq \Phi$ and $r \in I \cap F$. Since $r \in F$, $x \equiv x \wedge r\ (\psi_F)$. But $x \wedge r \in I$, and this implies $\psi_F(x) \in \psi_F(I)$, which is a contradiction. Hence $I \cap F = \Phi$.

Conversely, if ψ_F (I) is not proper, then for any $f \in F$, $\psi_F(f) \in \psi_F(I)$. Thus, $\psi_F(f) = \psi_F(i)$ for some $i \in I$. Then, $f \wedge f_1 = i \wedge f_1$ for some $f_1 \in F$ and this implies $f \wedge f_1 \in I \cap F$ and so $I \cap F \neq \Phi$.

(iii) and (iv) are trivial. □

Let J be an ideal of a nearlattice S . We say that a prime ideal P of S is a *prime divisor* of J if $J \subseteq P$. A prime divisor P of J is called *minimal prime divisor* if it does not properly contain another prime divisor of J . If $0 \in S$, then a minimal prime divisor of (0] is called minimal prime ideal. For the theory of minimal prime ideals in a general setting see Cornish [13].

Theorem 1.5.3. *Suppose* F *is a filter of a distributive nearlattice* S. *Then for any ideal* J *of* S, $\psi_F^{-1}\psi_F(J) = \{x \in S \,/\, x \wedge f \in J$ *for some* $f \in F\} = \cap\, \{P \,/\, P$ *is a (minimal) prime divisor of* J *in* S *such that* $P \cap F = \Phi\}$.

Proof: $\psi_F^{-1}\psi_F(J) = \{y \in S \,/\, \psi_F(y) \in \psi_F(J)\}$
$=\{y \in S / y \equiv x\ (\psi_F)$ for some $x \in J\} = \{y \in S / y \wedge f = x \wedge f$ for some $f \in F$, $x \in J) = \{ y \in S \,/\, y \wedge f \in J$ for some $f \in F\}$. Now we consider two cases :

Case 1. Let $J \cap F \neq \Phi$. Then there exists $x \in J \cap F$ and for any prime divisor P of J, $P \cap F \neq \Phi$. Thus,
$\{P / P$ is a prime divisor of J and $P \cap F = \Phi\} = \Phi$, and so
$\cap \{P / P$ is a prime divisor of J and $P \cap F = \Phi\} = S$
$= \{y \in S / y \wedge x \in J, x \in J \cap F\}$.

Case 2. Suppose $J \cap F = \Phi$. Clearly, $\{y \in S / y \wedge f \in J$ for some $f \in F\} \subseteq \cap \{P / P$ is a prime divisor of J and $P \cap F = \Phi\}$. Let $x \in S$ be such that $x \wedge f \notin J$ for all $f \in F$, and let $G = [x) \vee F$. If $J \cap G \neq \Phi$, then there exists $t \in J$ and $t \geq x_1 \wedge f$ for some $x_1 \geq x$ and for some $f \in F$. This implies $x \wedge f \leq x_1 \wedge f \leq t$ and consequently $x \wedge f \in J$, which is a contradiction. Thus, $J \cap G = \Phi$. Then by Theorem 1.2.7, there exists a prime ideal P of S such that $J \subseteq P$ and $G \cap P = \Phi$. In effect, $x \notin P$ and $F \cap P = \Phi$ as $F \subseteq G$. This completes the proof. □

Theorem 1.5.4. *Suppose* F *is a filter of a distributive nearlattice* S. *Also, suppose* $\mathcal{Q} = \{P/P$ *is a prime ideal of* S, *such that* $P \cap F = \Phi\}$ *and* $\mathcal{P} = \{\bar{P} \,/\, \bar{P}$ *is a prime ideal of* $S_F\}$. *Then* $\mathcal{Q}$ *and* $\mathcal{P}$ *are order isomorphic posets.*

Proof : Let $P \in \mathcal{Q}$. Then $\psi_F(P) \neq S_F$ by 1.5.2. Also,

$\psi_F(x) \wedge \psi_F(y) \in \psi_F(P)$ implies $\psi_F(x \wedge y) = \psi_F(q)$ for some $q \in P$. Then, $x \wedge y \wedge f = q \wedge f$ for some $f \in F$ and so either $x \in P$ or $y \in P$. Hence, $\psi_F(x) \in \psi_F(P)$ or $\psi_F(y) \in \psi_F(P)$, showing that $\psi_F(P)$ is a prime ideal of S_F. Thus, ψ_F is a map from $\mathcal{Q}$ to $\mathcal{P}$ and it is clearly isotone. Again, for any $\overline{P} \in \mathcal{P}$ it is very easy to show that $\psi_F^{-1}(\overline{P}) \in \mathcal{Q}$ and $\psi_F^{-1} : \mathcal{P} \dashrightarrow \mathcal{Q}$ is obviously isotone. As $\psi_F : S \dashrightarrow S_F$ is onto, $\psi_F \psi_F^{-1} = I_{\mathcal{P}}$. Moreover, by Theorem 1.5.3, $\psi_F^{-1}\psi_F(Q) = Q$ for any $Q \in \mathcal{Q}$, and hence $\psi_F^{-1}\psi_F = I_{\mathcal{Q}}$. Therefore $\mathcal{P}$ and $\mathcal{Q}$ are order isomorphic .□

In the above theorem, $S - P \supseteq F$ for all $P \in \mathcal{Q}$. Of course in any nearlattice S, the map $P \dashrightarrow S - P$ is an order reversing isomorphism between the poset of prime ideals and the poset of prime filters of S. Thus, we have the following important corollary which is an immediate consequence of above theorem.

Corollary 1.5.5. *For a distributive nearlattice* S, *the set of prime filters of* S *containing a given filter* F *of* S *is order isomorphic to the set of prime filters of* S_F. □

Theorem 1.5.6. *Let* S *be a distributive nearlattice . Then for each ideal* J *of* S, $J = \bigcap_F (\psi_F^{-1}\psi_F(J))$ *where F ranges over the prime filters of* S.

Hence for any ideals I *and* J of S, $\psi_F(I) = \psi_F(J)$ *for all prime filters* F *of* S *implies* I = J.

Proof : For any filter F of S, Clearly $\psi_F^{-1}\psi_F(J) \supseteq J$. Hence $J \subseteq \bigcap_F (\psi_F^{-1}\psi_F(J))$ where F ranges over the prime filters F of S. Now, let $x \in \bigcap_F (\psi_F^{-1}\psi_F(J))$. Then, $x \in (\psi_F^{-1}\psi_F(J))$ for all prime filters F of S. But, for any filter F of S, $\psi_F^{-1}\psi_F(J) = \{ y \in S \,/\, y \wedge f \in J$ for some $f \in F\}$ by Theorem 1.5.3. Thus, for any prime filter F of S, $x \wedge f_1 \in J$ for some $f_1 \in F$. If $x \notin J$, then by Theorem 1.2.7, there is a prime ideal Q of S such that x does not belong to Q and $J \subseteq Q$. Then for any $f \in S - Q$, $x \wedge f$ does not belong to $J \subseteq Q$ which is a contradiction as Q is a prime ideal of S . Hence $x \in J$.□

Suppose S is a distributive nearlattice . For any $x, y \in S$, we define $< x, y > = \{ s \in S \,/\, s \wedge x \leq y\}$ and $< x, J > = \{s \in S \,/\, s \wedge x \in J\}$ for any ideal J of S. It is easily seen that $<x, y>$ and $<x, J>$ are ideals of S. Moreover, $< x, y >$ is known as the relative annihilator ideal c. f. Mandelker [33]. For any x in a nearlattice S with 0, we denote $(x]^* = \{y \in S/y \wedge x = 0.\}$

The following proposition is needed for the further development of this thesis. We omit the proof as it is easily verifiable.

Proposition 1.5.7. *Suppose* F *is a filter of a distributive nearlattice* S *with* 0. *Then the following condition hold :*

(i) $\psi_F((x]) = (\psi_F(x)]$.

(ii) *For any ideal* J *of* S, $\psi_F(<x, J>) = <\psi_F(x), \psi_F(J)>$

(iii) $\psi_F((x]^*) = (\psi_F(x)]^*$

(iv) $\psi_F(<x, y>) = <\psi_F(x), \psi_F(y)>$.□

6. Sectionally pseudocomplemented nearlattices.

Pseudocomplemented lattices have been studied by several authors [23], [26], [28], [29], [30], [31].

On the other hand [9], [34], extended the notion of pseudocomplementation for meet semilattices.

Let L be a lattice with 0 and 1. For an element $x \in L$ element $x^* \in L$ is called *pseudocomplement* of x if $x \wedge x^* = 0$ and $x \wedge y = 0$ ($y \in L$) implies $y \leq x^*$.

A lattice is called *pseudocomplemented* if its every element has a pseudocomplement.

For a nearlattice S if $1 \in S$ then S becomes a lattice. So the idea of pseudocomplementation is not possible in case of a general nearlattice. But for a nearlattice S with 0, we can talk about sectionally pseudocomplemented nearlattices.

A nearlattice S with 0 is called *sectionally pseudocomplemented* if interval $[0, x]$ for each $x \in S$ is pseudocomplemented . Of course every finite distributive nearlattice is sectionally pseudocomplemented. Sectionally pseudocomplemented nearlattices have also been studied by [46]. Following Figure 1.6 gives an example of a distributive nearlattice with 0 which is not sectionally pseudocomplemented.

In $\mathbf{R}^2$ consider the set :

$E = \{ (0, y) \ / \ 0 \le y < 5\} \cup \{ (2, y) \ / \ 0 \le y < 5\} \cup \{(3, 5), (4, 5), (3, 6)\}$.

Define the partial ordering $\le$ on E by $(x, y) \le (x_1, y_1)$ if and only if $x \le x_1$ and $y \le y_1$ .Here E is clearly a distributive nearlattice. This is not a lattice as the supremum of (3, 6) and (4, 5) does not exist. Consider the interval [0, P]. Observe that in this interval (2, 0) has no relative pseudocomplement. So, $(E, \le)$ is not sectionally pseudocomplemented .

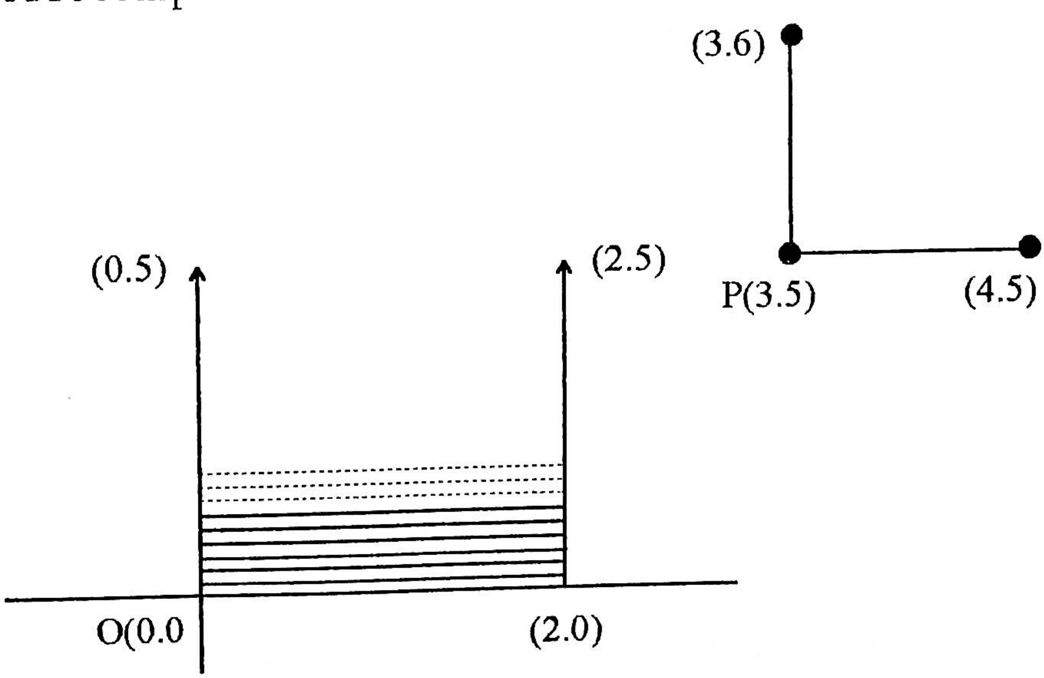


Fig. 1.6

Though we can not talk about pseudocomplementation in a distributive nearlattice S with 0, I(S) the lattice of ideals of S is pseudocomplemented as it is a distributive algebraic lattice.

A nearlattice S is called *relatively pseudocomplemented* if interval $[a, b]$ for each $a, b \in S$, $a < b$ is pseudocomplemented .

Theorem 1.6.1. *If* S *is a distributive sectionally pseudocomplemented nearlattice, then* S_F *is a distributive pseudocomplemented lattice.*

Proof: Suppose S is sectionally pseudocomplemented. By Lemma 1.5.1, S_F is a distributive lattice. Let $[x] \in S_F$, then $[0] \subseteq [x] \subseteq F$. Now $0 \leq x \wedge f \leq f$, for all $f \in F$.

Let y be the pseudocomplement of $x \wedge f$ in $[0, f]$, then $y \wedge x \wedge f = 0$ implies $[y \wedge f] \wedge [x] = [0]$, that is $[y] \wedge [x] = [0]$.

Suppose $[z] \wedge [x] = [0]$, for some $[z] \in S_F$, then $z \wedge x \equiv 0\ (\psi_F)$. This implies $z \wedge x \wedge f' = 0$ -------- (i) for some $f' \in F$. Since $z \equiv z \wedge f\ (\psi_F)$, so $z \wedge f'' = z \wedge f \wedge f''$----(ii) for some $f'' \in F$. From (i) and (ii) we get $z \wedge x \wedge f' \wedge f'' = 0$ and $z \wedge f' \wedge f'' = z \wedge f \wedge f' \wedge f''$. Setting $g = f' \wedge f''$ we have $z \wedge g = z \wedge g \wedge f$, which implies

$z \wedge g \leq f$ and $z \wedge g \wedge x \wedge f=0$.So, $0 \leq z \wedge g \leq f$ and $z \wedge g \leq y$.

Hence, $[z \wedge g] \subseteq [y]$. But $[z] = [z \wedge g]$ as $g \in F$. Therefore, $[z] \subseteq [y]$, and so S_F is a pseudocomplemented distributive lattice .□

We conclude this chapter with the following theorem. To prove this we need the following lemma:

Lemma 1.6.2. *Let* S *be a distributive relatively pseudocomplemented nearlattice. Let* $x \leq y \leq z$ *in* S *and* s *be the relative pseudocomplement of* y *in* $[x , z]$.*Then for any* $r \in S$, $s \wedge r$ *is the relative pseudocomplement of* $y \wedge r$ *in* $[x \wedge r, z \wedge r]$.

Proof: Suppose $t \wedge r$ is the relative pseudocomplement of $y \wedge r$ in $[x \wedge r, z \wedge r]$. Since s is the relative pseudocomplement of y in $[x , z]$,so $s \wedge y = x$. Thus , $(s \wedge r) \wedge (y \wedge r) = x \wedge r$. This implies $s \wedge r \leq t \wedge r$. Again, $x \leq s \vee (t \wedge r) \leq z$ and $y \wedge (s \vee (t \wedge r))$ $= (y \wedge s) \vee ((y \wedge r) \wedge (t \wedge r)) = x \vee (x \wedge r)$ implies $s \vee (t \wedge r) \leq s$, i,e. $s = s \vee (t \wedge r)$. Hence $t \wedge r \leq s$, and so $t \wedge r \leq s \wedge r$. This implies $t \wedge r = s \wedge r$.Therefore $s \wedge r$ is the relative pseudocomplement of $y \wedge r$ in $[x \wedge r, z \wedge r]$.□

Theorem 1.6.3. *If* S *is a distributive relatively pseudocomplemented nearlattice, then* S_F *is a distributive relatively pseudocomplemented lattice.*

Proof: By Lemma 1.5.1, S_F is a distributive lattice. Let $[x]$, $[y]$, $[z] \in S_F$ with $[x] \subseteq [y] \subseteq [z]$. Then $[x]=[x \wedge y]$ and $[y] = [y \wedge z]$. Thus, $x \equiv x \wedge y\ (\psi_F)$ and $y \equiv y \wedge z\ (\psi_F)$. This implies $x \wedge f = x \wedge y \wedge f$ and $y \wedge g = y \wedge z \wedge g$ for some $f, g \in F$. Then $x \wedge f \wedge g = x \wedge y \wedge f \wedge g$ and $y \wedge f \wedge g = y \wedge z \wedge f \wedge g$, and so $x \wedge f \wedge g \leq y \wedge f \wedge g \leq z \wedge f \wedge g$, that is $x \wedge h \leq y \wedge h \leq z \wedge h$ where $h = f \wedge g \in F$.

Suppose t is the relative pseudocomplement of $y \wedge h$ in $[x \wedge h, z \wedge h]$. Then $t \wedge y \wedge h = x \wedge h$, and so $[t] \wedge [y \wedge h] = [x \wedge h]$. That is, $[t] \wedge [y] = [x]$ as $y \equiv y \wedge h\ (\psi_F)$ and $x \equiv x \wedge h\ (\psi_F)$. Moreover $[t] \wedge [z] = [t] \wedge [z \wedge h] = [t \wedge z \wedge h] = [t]$ implies $[x] \subseteq [t] \subseteq [z]$. We claim that $[t]$ is the relative pseudocomplement of $[y]$ in $[[x],[z]]$ in S_F.

Suppose $[s] \wedge [y] = [x]$ for some $[s] \in [[x],[z]]$. Then $s \wedge y \equiv x\ (\psi_F)$ and so $s \wedge y \wedge f' = x \wedge f'$ for some $f' \in F$. Again $[s] \subseteq [z]$ implies $s \equiv s \wedge z\ (\psi_F)$, and so $s \wedge g' = s \wedge z \wedge g'$ for some $g' \in F$. Then $s \wedge y \wedge f' \wedge g' = x \wedge f' \wedge g'$ and $s \wedge f' \wedge g' = s \wedge z \wedge f' \wedge g'$. Thus, $s \wedge y \wedge k = x \wedge k$ and $s \wedge k = s \wedge z \wedge k$ where

$k = f' \wedge g' \in F$. These imply

$x \wedge h \wedge k \leq s \wedge h \wedge k \leq z \wedge h \wedge k$

and $(s \wedge h \wedge k) \wedge (y \wedge h \wedge k) = x \wedge h \wedge k$. Then by above lemma, $s \wedge h \wedge k \leq t \wedge k$. Hence

$[s] = [s \wedge h \wedge k] \subseteq [t \wedge k] = [t]$ and so $[t]$ is the relative pseudocomplement of $[y]$ in $[[x],[y]]$. Therefore, S_F is relatively pseudocomplemented. □

CHAPTER TWO

GENERALIZED STONE NEARLATTICES

Introduction: Generalized Stone lattices have been studied by several authors including Katrinak [27] and Cornish [13]. In this chapter we generalize the concept to nearlattices.

A distributive pseudocomplemented lattice L is called a *Stone lattice* if $x^* \vee x^{**} = 1$ for each $x \in L$.

A distributive lattice L with 0 is called a *generalized Stone lattice* if $(x]^* \vee (x]^{**} = L$ for each $x \in L$.

We call a distributive nearlattice S with 0 a *generalized Stone nearlattice* if $(x]^* \vee (x]^{**} = S$, for each $x \in S$.

Normal lattices have been studied extensively by Cornish in [13]. In [44] Noor and Latif extended the idea to nearlattices. According to [44], a distributive nearlattice S with 0 is called normal if its every prime ideal contains a unique minimal prime ideal.

In this chapter we have generalized several results of [13] on generalized Stone nearlattices by introducing the notion of quasi-complemented nearlattices. We have also proved that a distributive nearlattice S with 0 is generalized Stone if and only if it is both normal and sectionally quasi-complemented.

In section 1 of this chapter we have given a characterization of minimal prime ideals of a sectionally pseudocomplemented distributive nearlattice. Then we have shown that every generalized Stone nearlattice is normal. Moreover we show that every sectionally pseudocomplemented distributive nearlattice is generalized Stone if and only if every two minimal prime ideals are comaximal; i.e. normal.

In section 2 we have studied distributive quasi-complemented nearlattices. We have proved that a distributive nearlattice S with 0 is generalized Stone if and only if it is both normal and sectionally quasi-complemented.

1. Minimal prime ideals.

Recall that a nearlattice S with 0 is called sectionally pseudocomplemented if $[0,x]$ is pseudocomplemented for each $x \in S$. For any $y \in [0,x]$, we will denote the relative pseudocomplement of y in $[0, x]$ by y^+. Also, for any $y \in [0, x]$ we denote

$(y]^+ = \{t \in [0, x] \ / \ y \wedge t = 0\}$.

For any $x \in S$, set $D((x]) = \{y \leq x \ / \ y^+ = 0\}$ where y^+ is the relative pseudocomplement of y in $[0, x]$.

A prime ideal P of a nearlattice S is called *minimal* if there does not exists a prime ideal Q such that $Q \subset P$.

The following lemma is an extension of a fundamental result in lattice theory; c.f.[19, Lemma 4 pp. 169]. Though our proof is similar to their proof, we include the proof for the convenience of the reader.

Lamma 2.1.1. *Let S be a nearlattice with 0. Then every prime ideal contains a minimal prime ideal.*

Proof : Let P be a prime ideal of S and let $\mathcal{A}$ be the set of all prime ideals Q contained in P. Then $\mathcal{A}$ is nonvoid, since $P \in \mathcal{A}$. If C is a chain in $\mathcal{A}$ and

$Q = \cap (X \ / \ X \in C)$, then Q is nonvoid, since $0 \in Q$ and Q is an ideal; infact Q is prime. In deed, if $a \wedge b \in Q$ for some a, b $\in$ S, then $a \wedge b \in X$, for all $X \in C$; since X is prime, either $a \in X$ or $b \in X$. Thus either $Q = \cap(X \mid a \in X)$ or $Q = \cap(X \mid b \in X)$, proving that either a or $b \in Q$. Therefore we can apply to $\mathcal{A}$ the dual form of Zorn's Lemma to conclude the existence of a minimal member of $\mathcal{A}$. □

Two ideals I and J of a nearlattice S are called *comaximal* if $I \vee J = S$.

To prove the next theorem we need the following Lemmas which are due to [46]. These results are also generalizations of [10, Lemma 4.3].

Lemma 2.1.2. *Let* S *be a distributive nearlattice with* 0. *Let* $0 \leq x \in S$ *and the interval* $[0,x]$ *is pseudocomplemented. If* y^+ *is the relative pseudocomplement of* y *in* $[0, x]$, *then*

$(y^+] = (y]^* \wedge (x]$ *and* $(y^{++}] = (y]^{**} \wedge (x]$.□

Lemma 2.1.3. *Let* S *be a distributive nearlattice with* 0. *For any* $r \in S$ *and any ideal* I, $((r] \wedge I)^* \wedge (r] = I^* \wedge (r]$.□

Following theorem is a generalization of [19, Lemma 5, pp.169].

Theorem 2.1.4. *Let* S *be a sectionally pseudocomplemented distributive nearlattice and* P *be a prime ideal in* S. *Then the following conditions are equivalent :*

(i) P *is minimal*

(ii) $x \in P$ *implies* $(x]^* \not\subseteq P$

(iii) $x \in P$ *implies* $(x]^{**} \subseteq P$

(iv) $P \cap D((t]) = \Phi$ *for all* $t \in S-P$.

Proof : (i) implies (ii) and (ii) implies (iii) are trivial from the proof of [19, Lemma 5, pp.169].

(iii) implies (iv). Choose any $t \in S-P$. Let $x \in P \cap D((t])$. Then $x^+ = 0$, and so $x^{++} = t$. Also by (iii), $(x]^{**} \subseteq P$. Thus by Lemma2.1.2, $(t] = (x^{++}] = (t] \wedge (x]^{**} \subseteq P$, and so $t \in p$, which gives a contradiction. Therefore $P \cap D((t]) = \Phi$ for all $t \in S-P$.

(iv) implies (i). If P is not minimal, then there exists $Q \subset P$ for some prime ideal Q of S. Let $x \in P-Q$. Since Q is prime, so $(x] \wedge (x]^* = (0] \subseteq Q$ implies that $(x]^* \subseteq Q \subset P$. Thus, $(x] \vee (x]^* \subseteq P$. Choose any $t \in S-P$. Then $(t] \wedge ((x] \vee (x]^*) \subseteq P$ and so $(t \wedge x] \vee ((t] \wedge (x]^*) \subseteq P$. Now by Lemma 2.1.3, $(t] \wedge (x]^* = (t] \wedge (t \wedge x]^*$ and by Lemma 2.1.2, $(t] \wedge (t \wedge x]^* = ((t \wedge x)^+]$, where $(t \wedge x)^+$ is the relative pseudocomplement of $t \wedge x$ in $[0, t]$. This implies $(t \wedge x) \vee (t \wedge x)^{+} \in P$. But $((t \wedge x) \vee (t \wedge x)^{+})^{+} = (t \wedge x)^{+} \wedge (t \wedge x)^{++} = 0$ implies that $(t \wedge x) \vee (t \wedge x)^{+} \in D((t])$. Hence $P \cap D((t]) \neq \Phi$, which is a contradiction.

Therefore, P must be minimal.□

Normal lattices have been studied by several authors including Cornish [13] and Monteiro [35], [36]. In [44] Noor and Latif extended the idea to nearlattices. According to [44], a distributive nearlattice S with 0 is called normal if its every prime ideal contains a unique minimal prime ideal. Equivalently, a distributive nearlattice S with 0 is called normal if each prime filter of S is contained in a unique ultrafilter (maximal and proper) of S.

Following theorem gives a description of normal nearlattice which is due to [44].

Theorem 2.1.5. *For a distributive nearlattice* S *with* 0, *the following conditions are equivalent.*

(i) S *is normal.*

(ii) *Each prime ideal of* S *contains a unique minimal prime ideal.*

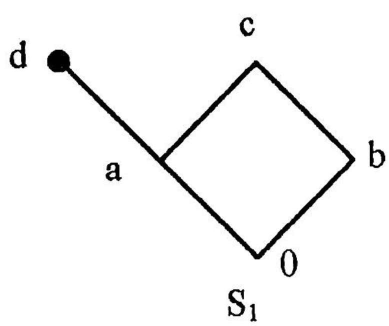
(iii) *Each Prime filter of* S *is contained in a unique ultrafilter of* S.

(iv) *Any two distinct minimal prime ideals are comaximal.*

(v) *For all* $x, y \in S$, $x \wedge y = 0$ *implies* $(x]^* \vee (y]^* = S$.

(vi) $(x \wedge y]^* = (x]^* \vee (y]^*$ *for all* $x, y \in S$. □

Consider the following distributive nearlattices with 0. Observe that in both S_1 and S_2, $(b]$ and $(d]$ are distinct



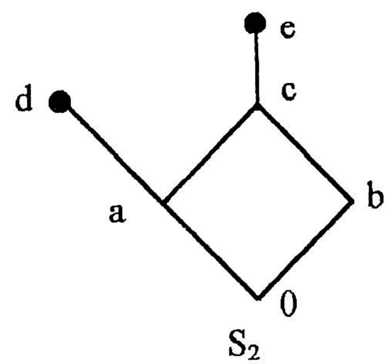


Figure 2.1 **Figure 2.2**

minimal prime ideals. Moreover, $(b]\vee(d]=S_1$ but $(b]\vee(d]\neq S_2$. Therefore, S_1 is normal but S_2 is not. Also observe that in S_2, $\{0,a,b,c,d\}$ is a prime ideal whice contains two prime ideals $(b]$ and $(d]$, and so S_2 is not normal.

Recall that a distributive nearlattice S with 0 is a generalized Stone nearlattice if $(x]^* \vee (x]^{**} = S$ for each $x \in S$.

Katrinak in [27, Lemma 8] has proved the following result for lattices. We generalize it to nearlattices.

Theorem 2.1.6. *A distributive nearlattice* S *with* 0 *is a generalized Stone nearlattice if and only if each interval* $[0,x]$, $0 < x \in S$ *is a Stone lattice.*

Proof : Let S with 0 be a generalized Stone and let $p \in [0,x]$. Then $(p]^* \vee (p]^{**} = S$. So, $x \in (p]^* \vee (p]^{**}$ implies $x = r \vee s$, for some $r \in (p]^*$, $s \in (p]^{**}$. Now $r \in (p]^*$ implies $r \wedge p = 0$, also $0 \leq r \leq x$. Suppose $t \in [0, x]$ such that $t \wedge p = 0$, then $t \in (p]^*$ implies $t \wedge s = 0$. Therefore, $t \wedge x = t \wedge (r \vee s) = (t \wedge r) \vee (t \wedge s) = (t \wedge r) \vee 0 = t \wedge r$ implies $t = t \wedge r$ implies $t \leq r$. So, r is the relative pseudocomplement of p in $[0,x]$,i.e, $r = p^+$.Since $s \in (p]^{**}$ and $r \in (p]^*$, so $s \wedge r = 0$. Let $q \in [0, x]$ such that $q \wedge r = 0$. Then as $x = r \vee s$, so $q \wedge x = (q \wedge r) \vee (q \wedge s)$ implies $q = q \wedge s$ implies $q \leq s$. Hence, s is the relative pseudocomplement of $r = p^+$ in $[0, x]$ i. e, $s = p^{++}$ implies $x = r \vee s = p^+ \vee p^{++}$. Thus $[0, x]$ is a Stone lattice.

Conversely, suppose $[0, x]$, $0 < x \in S$ is a Stone lattice. Let $p \in S$, then $p \wedge x \in [0, p]$. Since $[0, p]$ is a Stone lattice, then $(p \wedge x)^+ \vee (p \wedge x)^{++} = p$, where $(p \wedge x)^+$ is the relative pseudocomplement of$(p \wedge x)$ in $[0,p]$. Therefore $p \in ((p] \cap (p \wedge x]^*) \vee ((p] \cap (p \wedge x]^{**})$. So, we can take $p = r \vee s$, for $r \in (p \wedge x]^*$,$s \in (p \wedge x]^{**}$.Now, $r \in (p \wedge x]^*$ implies $r \wedge p \wedge x = 0$ implies $r \wedge x = 0$ implies $r \in (x]^*$

and $s \in (p \wedge x]^{**}$. Now $p \wedge x \leq x$ implies $(p \wedge x]^{**} \subseteq (x]^{**}$, and so $s \in (x]^{**}$. Therefore $p = r \vee s \in (x]^{*} \vee (x]^{**}$ and so, $S \subseteq (x]^{*} \vee (x]^{**}$. But $(x]^{*} \vee (x]^{**} \subseteq S$ is obvious. Hence $(x]^{*} \vee (x]^{**} = S$ and so S is generalized Stone. □

Following theorem is a generalization of [13, proposition 5.5(b)].

Theorem 2.1.7. *Let* S *be a distributive nearlattice with* 0. *If* S *is generalized Stone, then it is normal.*

Proof: Let P and Q be two minimal prime ideals of S. Then P, Q are unordered. Let $x \in P - Q$. Then $(x] \wedge (x]^{*} = (0] \subseteq Q$ implies $(x]^{*} \subseteq Q$. Since P is minimal, so by Theorem 2.1.4, above $(x]^{**} \subseteq P$. Again, as S is generalized Stone, so $(x]^{*} \vee (x]^{**} = S$. This implies $P \vee Q = S$ and so by Theorem 2.1.5, S is normal.□

Following lemma is due to [46, Lemma 2.8] and so we omit the proof.

Lemma 2.1.8. *If* S_1 *is a subnearlattice of a distributive nearlattice* S *and* P_1 *is a minimal prime ideal in* S_1, *then there exists a minimal prime ideal* P *in* S *such that* $P_1 = S_1 \cap P$. □

Theorem 2.1.9. *A sectionally pseudocomplemented distributive nearlattice* S *is generalized Stone if and only if any two minimal prime ideals are comaximal.*

Proof: Suppose S is generalized Stone. So by Theorem 2.1.7 any two minimal prime ideals are comaximal.

To prove the converse, let P, Q be two minimal prime ideals of S. We need to show that [0,x] is Stone, for each $x \in S$. Let P_1, Q_1 be two minimal prime ideals in [0,x] . Using Lemma 2.1.8. there exist minimal prime ideals P,Q in S such that $P_1 = P \cap [0, x]$, $Q_1 = Q \cap [0, x]$. Therefore $P_1 \vee Q_1 = (P \cap [0,x]) \vee (Q \cap [0,x]) = [P \vee Q] \cap [0, x]$ $= S \cap [0, x] = [0, x]$.

Therefore by [20, Theorem 6. p.115], [0, x] is Stone. So, by Theorem 2.1.6 above, S is generalized Stone.□

Thus we have the following corollary.

Corollary 2.1.10. *A distributive nearlattice S is generalized Stone if and only if it is sectionally pseudocomplemented and normal.*□

Thus the nearlattice S_1 of Figure 2.1 is in fact a generalized Stone nearlattice, as it is both sectionally pseudocomplemented and normal.

2. Quasi-complemented nearlattices.

Quasi-complemented lattices have been studied by several authors including Varlet [56], Speed [52] and Cornish [13]. These lattices are generalizations of pseudocomplemented lattices. Here we generalize these to nearlattices.

A distributive nearlattice S with 0 is called *quasi-complemented* if for each $x \in S$ there exists $x' \in S$ such that $x \wedge x' = 0$ and $((x] \vee (x'])^* = (0]$.

A distributive nearlattice S with 0 is called *sectionally quasi-complemented* if each interval $[0,x]$, $x \in S$ is quasi-complemented. Of course every sectionally pseudocomplemented nearlattice is sectionally quasi-complemented.

Following theorem generalizes [13, Proposition 5.5].

Theorem 2.2.1. *Let* S *be a distributive nearlattice with* 0. *Then* S *is quasi-complemented if and only if it is sectionally quasi-complemented and possesses an element* d *such that* $(d]^* = (0]$.

Proof: Suppose S is quasi-complemented. Then there exists an element d such that $0 \wedge d = 0$ and

$(d]^* = ((0] \vee (d])^* = (0]$.

We now show that an arbitrary interval $[0,x]$ is quasi-complemented . Let $y \in [0,x]$. Then there exists $y' \in S$ such that $y \wedge y' = 0$ and $((y] \vee (y'])^* = (0]$. Put $z = x \wedge y'$. Then $z \wedge y = (x \wedge y') \wedge y = x \wedge (y \wedge y') = 0$ and $z \in [0, x]$. If $w \in [0, x]$ and $(w] \wedge ((y] \vee (z]) = (0]$, then $(w \wedge y] = (0] = (w \wedge z] = (w \wedge x \wedge y'] = (w \wedge y']$. Thus $(w \wedge y] \vee (w \wedge y'] = (0]$, and so $(w] \wedge ((y] \vee (y']) = (0]$. Hence $w=0$, and so $[0, x]$ is quasi-complemented.

Conversely, suppose S is sectionally quasicomplemented and there exists an element $d \in S$ with $(d]^* = (0]$. Let $x \in S$ and consider the interval $[0, d]$. Then $x \wedge d \in [0, d]$. Since S is sectionally quasi-complemented, so there exists an element $x' \in [0, d]$ with $x \wedge d \wedge x' = 0$ and $\{y \in [0, d] \ / \ y \wedge ((x \wedge d) \vee x') = 0\} = (0]$. Now let $z \in ((x] \vee (x'])^*$. Then $z \wedge r = 0$ for all $r \in (x] \vee (x']$. Since $(x \wedge d) \vee x' \in (x] \vee (x']$, so $z \wedge ((x \wedge d) \vee x') = 0$. Thus, $z \wedge d \wedge ((x \wedge d) \vee x') = 0$ and $z \wedge d \in [0, d]$; so $z \wedge d = 0$. This implies $z \in (d]^* = (0]$. Hence $z = 0$ and $x \wedge d \wedge x' = 0$ implies $x \wedge x' = 0$. Therefore S is quasi-complemented.□

Following result is also a generalization of [13, Theorem 5.6]

Theorem 2.2.2. *If* S *is a quasi-complemented and normal nearlattice then it is a generalized Stone nearlattice. The converse of this is true if there exists* $d \in S$ *such that* $(d]^* = (0]$.

Proof : Let $x \in S$. Then there exists $x' \in S$ such that $x \wedge x' = 0$ and $((x] \vee (x'])^* = (x]^* \wedge (x']^* = (0]$. Then by an easy computation we have $(x]^{**} = (x']^*$. Since S is normal, so by Theorem 2.1.5, $(x]^* \vee (x']^* = S$, and so $(x]^* \vee (x]^{**} = S$. Therefore, S is generalized Stone.

Conversely, suppose S is generalized Stone. Then by Theorem 2.1.7, S is normal. Also by Theorem 2.1.6, S is sectionally pseudocomplemented, and so it is sectionally quasi-complemented.

Hence if there exists $d \in S$ with $(d]^* = (0]$, then by Theorem 2.2.1 S is quasi-complemented. □

Following result is a generalization of [13, Theorem 5.7]. Of course Cornish's [13] work was a generalization of Varlet's [56] and Speed's [52] work. That was also an extension of Grätzer and Schmidt's [23] characterization of Stone lattices.

Theorem 2.2.3. *A nearlattice* S *with* 0 *is generalized Stone if and only if it is both normal and sectionally quasi-complemented.*

Proof: Suppose S is both normal and sectionally quasi-complemented. Consider any interval $[0,x]$, $x \in S$. Since S is normal so by [44], $(x]$ is a normal sublattice of S. Let $t \in [0,x]$. Since $[0,x]$ is quasi-complemented, so there exists $t' \in [0,x]$ such that $t \wedge t' = 0$ and $(t \vee t']^{+} = (0]$, where $(t \vee t']^{+}$ is the relative pseudocomplement of $(t \vee t']$ in $[0, x]$. This implies $(t]^{+} \wedge (t']^{+} = (0]$, and so by an easy computation we have $(t']^{+} = (t]^{++}$. Since $(x]$ is normal and $t \wedge t' = 0$, so by Theorem 2.1.5, $(t]^{+} \vee (t']^{+} = (x]$; That is $(t]^{+} \vee (t]^{++} = (x]$. Then $x = p \vee q$ for some $p \in (t]^{+}$ and $q \in (t]^{++}$. Then $p \wedge t = 0$. Also for any $r \in (x]$, if $r \in (t]^{+}$, then $q \wedge r = 0$. Then from $x = p \vee q$ we have $r = x \wedge r = p \wedge r$, and so $r \leq p$. Hence p is the relative pseudocomplement of t in $[0, x]$. Therefore $[0, x]$ is pseudocomplemented and $p = t^{+}$. Similarly $q = t^{++}$ and so $x = t^{+} \vee t^{++}$. This shows that $[0,x]$ is a Stone lattice for each $x \in S$. Therefore by Theorem 2.1.6, S is generalized Stone.

Converse is trivial by Theorem 2.1.6 and 2.1.7.□

Corollary 2.2.4. *A distributive nearlattice* S *with* 0 *is generalized Stone if and only if it is normal and sectionally pseudocomplemented.*□

A nearlattice S with 0 is called *dense* if $(x]^* = (0]$ for each $x \neq 0$ in S. The following theorem is an extension of Theorem 4.1 of Cornish [13].

Theorem 2.2.5. *For a distributive sectionally pseudocomplemented nearlattice* S *, the following hold:*

(i) *If* S *is generalized Stone then* S_F *is Stone for any filter* F *of* S .

(ii) S *is generalized Stone if and only if for each prime filter* F *of* S, S_F *is a dense lattice.*

Proof : (i) Let $\psi_F(x), \psi_F(y) \in S_F$ be such that $\Psi_F(x) \wedge \Psi_F(y) = \overline{0}$. Then, $x \wedge y \equiv 0(\Psi_F)$, which implies that $x \wedge y \wedge f = 0$ for some $f \in F$. Since S is generalized Stone, then by Theorem 2.1.7, S is normal, so $(x]^* \vee (y \wedge f]^* = S$, by Theorem 2.1.5. Hence $(\Psi_F(x)]^* \vee (\Psi_F(y)]^* = (\Psi_F(x)]^* \vee (\Psi_F(y \wedge f)]^*$ $= \Psi_F((x]^* \vee (y \wedge f]^*) = \Psi_F(S) = S_F$. Thus, by Theorem 2.1.5, S_F is normal. Again, since S is sectionally pseudocomplemented, then by Theorem 1.6.1, S_F is pseudocomplemented. Hence by Theorem 2.1.5 and G. Grätzer [20, Theorem 6, pp. 165] S_F is Stone.

(ii) Suppose S is generalized Stone. Let $\Psi_F(x) \neq \overline{0}$ and $\Psi_F(q) \in (\Psi_F(x)]^*$. Then $\Psi_F(q) \wedge \Psi_F(x) = \overline{0}$. Then by Theorem 2.1.7 and Theorem 2.1.5, F is contained in a unique ultrafilter of S. Thus by Theorem 1.5.5, S_F has a unique ultrafilter; and so S_F has a unique minimal prime ideal. But the zero ideal of S_F (as $0 \in S$) is the intersection of all the minimal prime ideals of S_F. Hence, by uniqueness, it is (minimal) prime ideal of S_F. Hence $\Psi_F(q) = \overline{0}$, showing that S_F is dense.

Conversely, let S_F be dense for each prime filter F of S. Suppose $x, y \in S$ are such that $x \wedge y = 0$. Then,

$\Psi_F(x \wedge y) = \Psi_F(0) = \overline{0}$. That is $\Psi_F(x) \wedge \Psi_F(y) = \overline{0}$ which implies that $\Psi_F(x) = \overline{0}$ or $\Psi_F(y) = \overline{0}$ as S_F is dense. Hence, either$(\Psi_F(x)]^* = S_F$ or $(\Psi_F(y)]^* = S_F$. Thus by Theorem 1.5.7, $\Psi_F((x]^* \vee (y]^*) = S_F = \Psi_F(S)$, and so by Theorem 1.5.6 $(x]^* \vee (y]^* = S$. Therefore S is normal. Again, since S is sectionally pseudocomplemented, so by Theorem 2.2.4, S is generalized Stone.□

CHAPTER THREE

RELATIVE ANNIHILATORS IN NEARLATTICES

Introduction : Throughout this chapter we will be concerned with the relative annihilators in nearlattices. For $a, b \in S$, we define $< a, b> = \{x \in S \ / \ a \wedge x \leq b\}$. According to [33], $<a, b>$ is known as an *annihilator of* a *relative to* b or simply a *relative annihilator*. It is very easy to see that inpresence of distributivity $<a, b>$ is an ideal of S. Relative annihilators in lattices have been studied by many authors including Mandelker [33] and Varlet [57]. Also Cornish [13] has used the annihilators in studing relative normal lattices.

In section 1 of this chapter we have studied extensively the relative annihilators in nearlattices.We also include characterizations of modular and distributive nearlattices in terms of relative annihilators.Then we have generalized some of the results of [33] on relative annihilators. We have shown that in a distributive nearlattice S, $<a,b> \vee <b, a> = S$ for all $a,b \in S$ if and only if the filters containing any given prime filter form a chain.

For the background material in Lattice theory see Gratzer [19].Mandelker [33], Varlet [58] and Grätzer and Schmidt [23] have studied relatively Stone lattices. In section

2 we have introduced the notion of relatively Stone nearlattices.

Recall that a pseudocomplemented lattice L is called a Stone lattice if for each $x \in L$, $x^* \vee x^{**} = 1$. We call a distributive nearlattice S a ***relatively Stone nearlattice*** if each closed interval $[x,y]$ with $x < y$ $(x, y \in S)$ is a Stone lattice.

In section 2 we have given several characterizations of relatively Stone nearlattices. We show that for a distributive nearlattice S in which every closed interval is pseudocomplemented is relatively Stone if and only if any two incomparable prime ideals of S are comaximal.

1. Some characterization of relative annihilators in nearlattices.

Recall that a nearlattice S is distribuitve if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $y \vee z$ exists. Since for all $x, y, z \in S$, $(x \wedge y) \vee (x \wedge z)$ always exists by the upper bound property, we give an alternative definition of distributivity of S by the following Lemma.

Lemma 3.1.1 . *A nearlattice* S *is distributive if and only if for all* $t, x, y, z \in S$, $t \wedge ((x \wedge y) \vee (x \wedge z))$
$= (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

Proof : Suppose S is distributive . Then obviously,
$t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

Conversely, suppose S has the given property. Let $a, b, c \in S$ with $b \vee c$ exists. Set $t = b \vee c$. Then
$a \wedge (b \vee c) = a \wedge ((t \wedge b) \vee (t \wedge c))$
$= (a \wedge t \wedge b) \vee (a \wedge t \wedge c) = (a \wedge b) \vee (a \wedge c)$.
Therefore S is distributive .□

Recall that a nearlattice S is modular if for all $x, y, z \in S$ with $z \leq x$ and whenever $y \vee z$ exists then $x \wedge (y \vee z) = (x \wedge y) \vee z$. Like Lemma 3.1.1, we can also easily characterize modular nearlattices by the following result.

Lemma 3.1.2 . *A nearlattice* S *is modular if and only if for all* $t, x, y, z \in S$ *with* $z \leq x$, $x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

Proof : Suppose S is modular. Then obviously, $x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

Conversely, suppose S has the given property. Let $a, b, c \in S$ with $c \leq a$ and whenever $b \vee c$ exists. Set $t = b \vee c$, then $a \wedge (b \vee c) = a \wedge ((t \wedge b) \vee (t \wedge c)) = (a \wedge t \wedge b) \vee (t \wedge c) = (a \wedge b) \vee c$. Therefore S is modular .□

Now we generalize Theorem 1 and Theorem 2 of [33].

Theorem 3.1.3 . *For a nearlattice* S *the following conditions are equivalent* :

(i) S *is distributive.*

(ii) $<a,b>$ *is an ideal for all* $a, b \in S$.

(iii) $<a, b>$ *is an ideal whenever* $b \leq a$.

Proof : Since (i) implies (ii) and (ii) implies (iii) are trivial, we shall prove only (iii) implies (i).

Suppose (iii) holds. Let $t, x, y, z \in S$. Then $(t \wedge x \wedge y) \vee (t \wedge x \wedge z) \leq x$ implies $< x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z) >$ is an ideal. Again $(t \wedge x \wedge y) \leq (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$ implies $t \wedge y \in < x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z) >$.
Similarly, $t \wedge z \in < x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z) >$.
Hence $(t \wedge y) \vee (t \wedge z) \in < x, (t \wedge x \wedge y) \vee (t \wedge x \wedge z) >$.
Thus, $x \wedge ((t \wedge y) \vee (t \wedge z)) \leq (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.
Since the reverse inequality is trivial, so
$x \wedge ((t \wedge y) \vee (t \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.
Therefore by Lemma 3.1.1, S is distributive.□

Theorem 3.1.4. *A nearlattice* S *is modular if and only if whenever* $b \leq a$, *if* $t \wedge x \in (b]$ and $t \wedge y \in < a, b >$ for any $t \in S$, then $(t \wedge x) \vee (t \wedge y) \in <a, b>$.

Proof : Suppose S is modular. Since $t \wedge y \in <a, b>$, so $a \wedge t \wedge y \leq b$. Also $t \wedge x \leq b \leq a$. Thus by modularity of S, $a \wedge ((t \wedge x) \vee (t \wedge y)) = (a \wedge t \wedge y) \vee (t \wedge x) \leq b$, and so $(t \wedge x) \vee (t \wedge y) \in <a, b>$.

Conversely, let the given condition holds, suppose

$t, x, y, z \in S$ with $z \leq x$. Then $(t \wedge z) \vee (t \wedge x \wedge y) \leq x$ and $t \wedge z \in ((t \wedge z) \vee (t \wedge x \wedge y)]$. Also,

$t \wedge x \wedge y \leq (t \wedge z) \vee (t \wedge x \wedge y)$

implies $t \wedge y \in < x, (t \wedge z) \vee (t \wedge x \wedge y) >$. Then by hypothesis $(t \wedge z) \vee (t \wedge y) \in <x, (t \wedge z) \vee (t \wedge x \wedge y)>$. This implies $x \wedge ((t \wedge y) \vee (t \wedge z)) \leq (t \wedge x \wedge y) \vee (t \wedge z)$. Since the reverse inequality is trivial, so by Lemma 3.1.2, S is modular.□

Following result is a generalization of a Lemma of [33] in section 3.

Lemma 3.1.5 . *In any distributive nearlattice* S, *each of the following conditions on a given filter* F *implies the next.*

(i) *For all* a, b $\in$ S, *there exists an element* $x \in F$ *such that* $a \wedge x$ *and* $b \wedge x$ *are comparable.*

(ii) *The filters containing* F *form a chain.*

(iii) F *is prime.*

(iv) F *contains a prime filter.*

Proof : (i) implies (ii). Suppose (i) holds. If (ii) fails then there exist noncomparable filters G and H containing F. Choose elements $a \in G - H$ and $b \in H - G$ .Then by (i) there exists $x \in F$ such that $a \wedge x$ and $b \wedge x$ are comparable. Suppose $a \wedge x \leq b \wedge x$. Since $x \in F - G$, so $a \wedge x \in G$.

Then $a \wedge x \leq b$ implies $b \in G$, which gives a contradiction. Therefore (ii) holds.

(ii) implies (iii). Suppose (ii) holds. Let $a, b \in S$ with $a \vee b$ exists and $a \vee b \in F$. Let $G = F \vee [a)$ and $H = F \vee [b)$. By (ii), either $G \subseteq H$ or $H \subseteq G$. Suppose $G \subseteq H$. Then $a \in H$, and so $a = x \wedge b$ for some $x \in F$. Since $x, a \vee b \in F$, so $x \wedge (a \vee b) \in F$. Thus by distributivity of S, $(x \wedge a) \vee (x \wedge b) = (x \wedge a) \vee a = a \in F$. Therefore F is prime.

(iii) implies (iv) is trivial .□

For a lattice L, the identity $< a, b> \vee <b, a> = L$ for all $a, b \in L$ is well known in Lattice theory. This identity in fact, characterizes relatively Stone and relatively normal lattices ; c.f. [33] and [13].

Theorem 3.1.6 . *For a distributive nearlattice* S *the identity* $< a, b> \vee <b, a> = S$ *for all* $a, b \in S$ *holds if and only if all the conditions of Lemma 3.1.5 are equivalent.*

Proof : Suppose the identity holds . We need only to show that (iv) implies (i) of Lemma 3.1.5. Let $a,b \in S$. Suppose P is a prime filter contained in F. Choose $z \in P$. Since

$< a, b > \vee <b, a> = S$, so $z = x \vee y$ for some $x \in <a,b>$ and $y \in < b, a >$. Since P is prime, either $x \in P$ or $y \in P$. Suppose $x \in P$. Then $x \in F$, and $x \in <a, b>$ implies $a \wedge x \leq b$ and so $a \wedge x \leq b \wedge x$. Therefore (i) holds.

Conversely, suppose all the conditions of the Lemma 3.1.5 are equivalent. Let there exist $a, b \in S$ such that $I = < a, b> \vee <b, a>$ is a proper ideal of S. Then by Theorem 1.2.7, there exists a prime filter P disjoint from I. Then by (iii) implies (i), there exists $x \in P$ such that $a \wedge x$ and $b \wedge x$ are comparable. Suppose $a \wedge x \leq b \wedge x$. Then $a \wedge x \leq b$ implies $x \in <a, b>$ which is a contradiction as $P \cap I = \Phi$. Therefore $< a, b> \vee <b, a> = S$. □

We conclude this section with the following generalization of [33, Theorem 4].

Theorem 3.1.7. *For any distributive nearlattice* S, *the following conditions are equivalent :*

(i) *For all* $a, b \in S$, $< a, b> \vee <b, a> = S$.

(ii) *For any prime filter* P *and for any* $a, b \in S$ *there exists* $x \in P$ *such that* $a \wedge x$ *and* $b \wedge x$ *are comparable.*

(iii) *The filters containing any given prime filter form a chain.*

Proof : (i) implies (ii) easily follows from the proof of first part of Theorem 3.1.6 ; while (ii) implies (iii) holds by Lemma 3.1.5.

(iii) implies (i). Suppose (iii) holds. Let for $a, b \in S$, $I = \langle a, b\rangle \vee \langle b\ a\rangle$ be a proper ideal of S. Then by Stones representation theorem there exists a prime filter P disjoint from I. Let $G = P \vee [a)$ and $H = P \vee [b)$. By (iii) either $G \subseteq H$ or $H \subseteq G$. Suppose $G \subseteq H$. Then $a \in P \vee [b)$ implies $a = x \wedge b$ for some $x \in P$. Then $x \in \langle b, a\rangle$, which is a contradiction as $P \cap I = \Phi$. Therefore $\langle a, b\rangle \vee \langle b\ a\rangle = S$.□

2. Relatively Stone nearlattices.

We start this section with the following characterization of relatively Stone nearlattices, which is a generalization of [33, Theorem 5].

Theorem 3.2.1. *Let* S *be a distributive nearlattice in which every closed interval is pseudocomplemented. Then the following conditions are equivalent :*

(i) S *is relatively Stone.*

(ii) *For all* $x, y \in S$, $< x, y > \vee < y, x > = S$.

Proof : (i) implies (ii). Suppose S is relatively Stone. Let $x, y \in S$. For any $a \in S$ consider $I = [\, x \wedge y \wedge a \, , \, a \,]$ in S .Let + denotes the pseudocomplement in I. Now , $x \wedge y \wedge a = (x \wedge a) \wedge (y \wedge a)$. Since I is Stone , so by [19; Theorem 3, pp.161] $a = (x \wedge y \wedge a)^{+} = ((x\wedge a)\vee(y\wedge a))^{+} = (x \wedge a)^{+} \vee (y \wedge a)^{+}$. Thus $a = r\vee s$ where $r = (x \wedge a)^{+}$, $s = (y \wedge a)^{+}$. Then $x \wedge a \wedge r = y \wedge a \wedge s = x \wedge y \wedge a$. Since $r, s \leq a$, we have $x \wedge y \wedge a = x \wedge r = y \wedge s$. This implies $x \wedge r \leq y$ and $y \wedge s \leq x$ and so $a = r \vee s \in < x, y > \vee < y, x >$. Hence (ii) holds.

(ii) implies (i). Let [a,b] be any closed interval in S and let + denotes pseudocomplement in [a, b]. Let $x \in [a, b]$. By hypothesis $<x^+, x^{++}> \vee <x^{++}, x^+> = S$. Hence by [14, Theorem 1.1] $b = r \vee s$ for some $r \in <x^+, x^{++}>$ and $s \in <x^{++}, x^+>$. Since $a, r, s \leq b$, so by the upper bound property $a \vee r$, $a \vee s$ exist.Now $r \wedge x^+ \leq x^{++}$ and $s \wedge x^{++} \leq x^+$. Thus, $x^+ \wedge (a \vee r) \leq x^{++}$. Moreover $x^+ \wedge (a \vee r) \leq x^+$ is obvious. Hence $x^+ \wedge (a \vee r) \leq x^{++} \wedge x^+ = a$. Since $a \vee r \in [a, b]$, so $a \vee r \leq x^{++}$. Similarly $a \vee s \leq x^+$. Hence $b = (a \vee r) \vee (a \vee s) \leq x^+ \vee x^{++} \leq b$.This implies $x^+ \vee x^{++} = b$ and so $[a, b]$ is a Stone lattice. In other words, S is relatively Stone. □

Definition 3.2.2. *A filter* F *of a nearlattice* S *is called meet irreducible if* $F = G \wedge H$ *implies either* $F = G$ *or* $F = H$, *where* G *and* H *are filters of* S.

Theorem 3.2.3 . *Let* S *be a distributive nearlattice. A filter* F *of* S *is prime if and only if it is meet irreducible.*

Proof : Suppose F is prime and $F = G \wedge H$ for some filters G and H of S. If $G \neq F$. Then there exists $g \in G$ such that $g \notin F$. Suppose $h \in H$ .Then for any $f \in F$, $g \wedge f \in G$ and $h \wedge f \in H$. Hence $(g \wedge f) \vee (h \wedge f) \in G \wedge H = F$. But

$g \wedge f \notin F$ as $g \notin F$. Since F is prime so $h \wedge f \in F$ which implies $h \in F$. This implies $H \subseteq F$. As $F \subseteq H$ is obvious, so $F = H$. Therefore F is meet irreducible.

Conversely, suppose F is meet irreducible. Let $a, b \in S$ such that $a \vee b$ exists and $a \vee b \in F$. Set $G = F \vee [a)$ and $H = F \vee [b)$, clearly, $F \subseteq G \wedge H$. Now, let $x \in G \wedge H$, then $x \geq f_1 \wedge a$, $x \geq f_2 \wedge b$, for some $f_1, f_2 \in F$. Hence, $x \geq f_1 \wedge f_2 \wedge a$, $x \geq f_1 \wedge f_2 \wedge b$. Put $f = f_1 \wedge f_2$, then we get $x \geq f \wedge a$, $x \geq f \wedge b$, which implies that $x \geq (f \wedge a) \vee (f \wedge b)$.

Now $(f \wedge a) \vee (f \wedge b) = f \wedge (a \vee b)$, as S is distributive and $a \vee b$ exists. Therefore, $(f \wedge a) \vee (f \wedge b) \in F$ as $a \vee b \in F$. Hence $x \in F$.

Therefore, $G \wedge H \subseteq F$, and so $G \wedge H = F$. Since F is meet irreducible, so either $G = F$ or $H = F$, that is either $a \in F$ or $b \in F$. Hence F is prime .□

Following theorem generalizes a result of [58].

Theorem 3.2.4 . *In a distributive nearlattice* S, *the following conditions are equivalent.*

(i) *Any proper filter which contains a prime filter is prime.*

(ii) *For any pair of non-comparable prime ideals* P *and* Q,

$P \vee Q = S$.

Proof : (i) implies (ii). Let S be a distributive nearlattice and let P and Q be two non-comparable prime ideals in S such that $P \vee Q \neq S$. Then by Theorem 1.2.7, there exists a prime filter F disjoint from the ideal $P \vee Q$. $S - P$ and $S - Q$ are non-comparable prime filters such that
$(S - P) \wedge (S-Q) = G \supset F$, where G is a filter and by assumption (i), G is prime, which is impossible. Because, then by theorem 3.2.3 ,G is meet-irreducible. Hence for any pair of non-comparable prime ideals P and Q, $P \vee Q = S$.

(ii) implies (i). Let S be a distributive nearlattice and let there exists a prime filter F and a non-prime proper filter G such that $F \subset G$. Thus, G is not meet irreducible. Then there exist filters $A \neq G$ and $B \neq G$ such that $G = A \cap B$. So, we can find two elements a and b such that $a \in A$, $a \notin B$ and $b \in B$, $b \notin A$. Then by Theorem 1.2.7 again, there exists a prime filter A_1 containing A and disjoint from (b] and a prime filter B_1 containing B and disjoint from (a]. A_1 and B_1 contain G and are non-comparable.Thus by assumption (ii), $(S - A_1) \vee (S - B_1) = S$. Which would imply that any element of F is the join of two elements not belonging to F, hence a contradiction as F is prime. This completes the proof.□

Recall that two prime ideals P and Q of a nearlattice S are said to be *comaximal* if $P \vee Q = S$.

Following result is due to [42, Theorem 2.7].

Theorem 3.2.5 . *For any distributive nearlattice* S *the following conditions are equivalent :*

(i) *For all* $a, b \in S$, $< a,b > \vee < b, a > = S$.

(ii) *The filters containing any given filter form a chain.* □

Theorem 3.2.6 . *Let* S *be a distributive nearlattice in which every closed interval is pseudocomplemented. Then the following conditions are equivalent :*

(i) S *is relatively Stone.*

(ii) *The set of all prime ideals contained in a prime ideal is a chain.*

(iii) *Any two incomparable prime ideals are comaximal.*

(iv) *The set of all prime filters of* S *containing a prime filter is a chain.*

(v) *Any proper filter which contains a prime filter is prime.*

(vi) S_F *is a chain for each prime filter* F *of* S.

Proof : (i) implies (ii). Suppose (i) holds. Then by Theorem 3.2.1, $<x, y> \vee <y, x> = S$ for all $x, y \in S$. If (ii) does not hold, then there exist prime ideals P, Q, R with $P \supseteq Q, R$; and Q and R are incomparable. Let $x \in Q - R$ and $y \in R - Q$. Then $< x, y > \subseteq R$ and $< y, x > \subseteq Q$. Thus $S = <x, y> \vee <y, x> \subseteq Q \vee R \subseteq P \neq S$, which is a contradiction. Hence (ii) holds.

(ii) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) are trivial.

(iii) $\Leftrightarrow$ (v) holds by Theorem 3.2.4.

(iv) implies (vi) .Suppose (iv) holds. Then by Corollary 1.5.5, the prime filters of S_F form a chain for any prime filter F of S. But, in a distributive lattice if the set of prime filters form a chain, then the lattice itself is a chain. Therefore S_F is a chain for each prime filter F of S.

(vi) implies (i). Let F be any prime filter of S. By (vi) S_F is a chain, and so for any x, y in S, we have either $\psi_F(x) \leq \psi_F(y)$ or $\psi_F(y) \leq \psi_F(x)$. In either case, $< \psi_F(x) , \psi_F(y) > \vee < \psi_F(y) , \psi_F(x) > = S_F$. i.e. $\psi_F(< x, y > \vee < y , x >) = \psi_F(S)$, and so by the principle of localization, $<x, y> \vee <y, x> = S$. Hence by Theorem 3.2.1, S is relatively Stone.□

Theorem 3.2.7. *If* F *is a filter in a relatively Stone nearlattice* S, *then* S_F *is relatively Stone* .

Proof : Suppose S is relatively Stone. Let $\psi_F(x)$, $\psi_F(y) \in S_F$. Then by Proposition 1.5.7,

$< \psi_F(x) , \psi_F(y) > \vee < \psi_F(y) , \psi_F(x) >$

$= \psi_F < x, y > \vee \psi_F < y, x > = \psi_F [< x, y > \vee < y, x >]$

$= \psi_F (S) = S_F$, as S is relatively Stone.

Hence by Theorem 1.6.3 and Theorem 3.2.1, S_F is relatively Stone.□

We conclude this section with the following examples . Notice that both the nearlattices are relatively pseudocomplemented. In nearlattice of Figure 3.1, notice

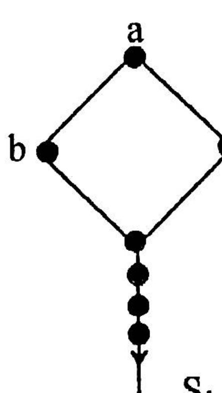


Figure 3.1

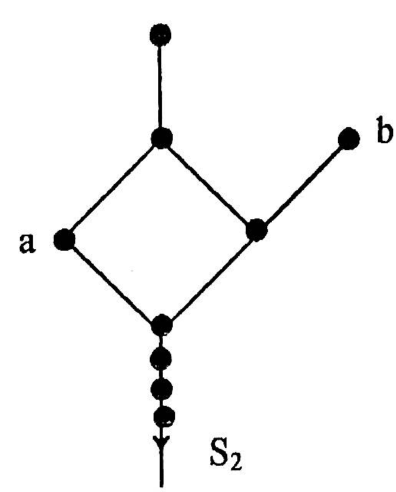


Figure 3.2

that (a], (b] and (c] are only prime ideals. Here both (a] and (b] are incomparable with (c] . Moreover, (a] $\vee$ (c] = (b] $\vee$ (c] $= S_1$. Therefore S_1 is relatively Stone. But for nearlattices of Figure 3.2, observe that (a] and (b] are incomparable prime ideals . But (a] $\vee$ (b] $\neq S_2$. Therefore, S_2 is not relatively Stone .

Also notice that though S_1 is relatively Stone, it is not generaliged Stone as $0 \notin S_1$.

CHAPTER FOUR

NEARLATTICES WHICH ARE SECTIONALLY (RELATIVELY) IN B_N

Introduction : Lee in [31] has determined the lattice of all equational subclasses of the class of all pseudocomplemented distributive lattices. They are given by

$B_{-1} \subset B_0 \subset B_1 \subset \text{----------------} \subset B_n \subset \text{---------------------} \subset B_\omega$

where all the inclusions are proper and B_ω is the class of all pseudocomplemented distributive lattices, B_{-1} consists of all one element algebras, B_0 is the variety of Boolean algebras while B_n, for $1 \leq n < \omega$ consists of all algebras satisfying the equation

$(x_1 \wedge x_2 \wedge \text{----------} \wedge x_n)^*$

$\vee\ (x_1 \wedge \text{---------------} \wedge\ x_{i-1} \wedge\ x_i^* \wedge\ x_{i+1} \wedge\ \text{---------} \wedge\ x_n)^* =1,$

where x^* denotes the pseudocomplements of x. Thus B_1 consists of all Stone algebras.

A nearlattice S is said to be *sectionally* in B_n, $1 \leq n \leq \omega$, if each interval [0, x], $x \in S$ is in B_n.

A distributive nearlattice S is said to be *relatively* in B_n, $-1 \leq n \leq \omega$ if its each interval [a, b], a, b $\in$ S, a < b is in B_n.

Cornish in [8] have studied n - normal lattices. Then Noor in [41] has extended the idea to nearlattices and generalized some results of [8]. By [41], a distributive nearlattice S with 0 is called n- *normal* if every prime ideal contains at most n minimal prime ideals.

Sectionally B_n - lattices and relatively B_n - lattices have been studied by Davey in [16].

In section 1, we have studied sectionally B_n - nearlattices. We have given several characterization to sectionally B_n-nearlattices. We show that a distributive nearlattice is sectionally in B_n if and only if it is n- normal and sectionally pseudocomplemented.

In section 2, we have shown that a distributive nearlattice S is relatively in B_n if and only if any n+1 pairwise incomparable prime ideals are comaximal which is a generalization of some works of Davey [16] and Cornish[8].

1. Nearlattices which are sectionally in B_n.

We start this section with the following lemma which is an extension of 2.3 of Cornish [8] and which will be needed in the proof of our next theorem. Since the proof of the Lemma 4.1.1. follows easily from Cornish's proof, so we omit the proof.

Lemma 4.1.1. *Let* J *be an ideal of a distributive nearlattice* S. *For a given positive integer* $n > 1$, *the following conditions are equivalent*:

(i) *For any* $x_0, x_1,$ --------------, $x_n \in S$, *which are "pair wise in* J" i.e. $x_i \wedge x_j \in J$ *for any* $i \neq j$, *there exists* k *such that* $x_k \in J$.

(ii) J *is the intersection of at most* n *distinct prime ideals.* □

Following theorem is a generalization of a result of Cornish [10, Theorem 4.5].

Theorem 4.1.2. *Let* S *be a sectionally pseudocomplemented distributive nearlattice . For given* n *such that* $1 \leq n < \omega$, *the following conditions are equivalent* :

(i) S *is sectionally in* B_n.

(ii) *For any* $y \in S$, *and for* $x_1, x_2,$ -----------------, $x_n \in (y]$,

$(y] \subseteq ((x_1] \wedge$ ----------$\wedge\ (x_n])^* \vee ((x_1]^* \wedge$-------- $\wedge\ (x_n])^*$

$\vee$ --------------- $\vee\ ((x_1] \wedge$ ------------$\wedge\ (x_n]^*)^*$

(iii) *For any* $x_1,$------------, $x_n \in S$,

$((x_1] \wedge \text{---------------} \wedge (x_n])^* \vee ((x_1]^* \wedge \text{------------} \wedge (x_n])^*$
$\vee \text{-------------------} \vee ((x_1] \wedge \text{--------------} \wedge (x_n]^*)^* = S.$

(iv) *Each prime ideal contains at most* n *minimal prime ideals.*

(v) *For any* $n+1$ *distinct minimal prime ideals*

$P_1, \text{----------} P_{n+1}, P_1 \vee \text{---------} \vee P_{n+1} = S.$

(vi) *For any* $x_0, x_1, \text{----------}, x_n \in S$ *such that* $x_i \wedge x_j = 0$ *for*

$(i \neq j)$, $i = 0, 1, 2, \text{---------} n$; $j = 0, 1, 2, \text{---------} n$;

$(x_0]^* \vee (x_1]^* \vee \text{- - - - - - - - - - - -} \vee (x_n]^* = S.$

Proof : (i) implies (ii). Suppose $2 \leq n$. Let $x_i{}^*$ be the pseudocomplement of x_i in $[0, y]$. By Lemma 2.1.2,

$(x_1] \wedge \text{- - - - - - - - - - - - -} \wedge (x_i]^* \wedge \text{- - - - - - - -} \wedge (x_n].$

$= (x_1] \wedge \text{- - - - - - - - - - - - -} \wedge (x_i]^* \wedge (y] \wedge \text{- - - - - - - -} \wedge (x_n].$

$= (x_1] \wedge \text{- - - - - - - - - - - - -} \wedge (x_i{}^+] \wedge \text{- - - - - - - -} \wedge (x_n].$

$= (x_1 \wedge \text{- - - - - - - - - - - - -} \wedge x_i{}^+ \wedge \text{- - - - - - - -} \wedge x_n].$

Since (i) holds, so

$$(y] = ((x_1 \wedge \text{-------} \wedge x_n)^+ \vee \bigvee_{i=1}^{n} (x_1 \wedge \text{-------} \wedge x_i{}^+ \wedge \text{------} \wedge x_n)^+].$$

$$= ((x_1 \wedge \text{--------} \wedge x_n)^+] \vee \bigvee_{i=1}^{n} ((x_1 \wedge \text{-------} \wedge x_i{}^+ \wedge \text{-----} \wedge x_n)^+].$$

$$= ((x_1 \wedge \text{----} \wedge x_n]^* \wedge (y]) \vee \bigvee_{i=1}^{n} ((x_1 \wedge \text{-----} \wedge x_i{}^+ \wedge \text{-----} \wedge x_n]^*$$

$$\leq ((x_1] \wedge \text{---------} \wedge (x_n])^* \vee \bigvee_{i=1}^{n} ((x_1] \wedge \text{----} \wedge (x_i{}^+] \wedge \text{-----} \wedge (x_n])^*$$

$$= ((x_1] \wedge \text{------} \wedge (x_n])^* \vee \bigvee_{i=1}^{n} ((x_1] \wedge \text{-----} \wedge (x_i]^* \wedge \text{----} \wedge (x_n])^*,$$

by Lemma 2.1.2 and as each $x_i \leq y$. If $n = 1$, then by (i) and using Lemma 2.1.2 we have $(y] = (x_1^{+} \vee x_1^{++}] = (x_1^{+}] \vee (x_1^{++}]$ $= ((x_1]^* \wedge (y]) \vee ((x_1]^{**} \wedge (y]) \subseteq (x_1]^* \vee (x_1]^{**}$.

(ii) Implies (iii) Firstly suppose $2 \leq n$. Let $x_1, \text{-------}, x_n \in S$.

Choose any $r \in S$. Then by (ii)

$(r] \subseteq ((r \wedge x_1] \wedge \text{---------} \wedge (r \wedge x_n])^* \vee \bigvee_{i=1}^{n} ((r \wedge x_1] \wedge \text{--------}$

$\wedge (r \wedge x_i]^* \wedge \text{-----------} \wedge (r \wedge x_n])^*$, and so,

$(r] = (((r \wedge x_1] \wedge \text{----------} \wedge (r \wedge x_n])^* \wedge (r]) \vee \bigvee_{i=1}^{n} (((r \wedge x_1]$

$\wedge \text{----------} \wedge (r \wedge x_i]^* \wedge \text{----------} \wedge (r \wedge x_n])^* \wedge (r])$.

Now, by Lemma 2.1.3,

$((r \wedge x_1] \wedge \text{-------------} \wedge (r \wedge x_n])^* \wedge (r]$

$= ((x_1] \wedge \text{-------------} \wedge (x_n])^* \wedge (r]$

Again, for each $1 \leq i \leq n$, $r \wedge x_i \leq x_i$ implies $(r \wedge x_i]^* \supseteq (x_i]^*$

Thus, $(r \wedge x_1] \wedge \text{-----------} \wedge (r \wedge x_i]^* \wedge \text{-----------} \wedge (r \wedge x_n]$

$\supseteq (r \wedge x_1] \wedge \text{-------------} \wedge (x_i]^* \wedge \text{----------} \wedge (r \wedge x_n]$, ans so,

$((r \wedge x_1] \wedge \text{-----------} \wedge (r \wedge x_i]^* \wedge \text{-----------} \wedge (r \wedge x_n])^* \wedge (r]$

$\subseteq ((r \wedge x_1] \wedge \text{-------------} \wedge (x_i]^* \wedge \text{------------} \wedge (r \wedge x_n])^* \wedge (r]$

$= ((x_1] \wedge \text{------------} \wedge (x_i]^* \wedge \text{------------} \wedge (x_n])^* \wedge (r]$, by using Lemma 2.1.3 again.

Therefore, $(r] \subseteq ((x_1] \wedge \text{--------} \wedge (x_n])^* \vee ((x_1]^* \wedge \text{------} \wedge (x_n])^*$

$\vee \text{---------} \vee ((x_1] \wedge \text{---------} \wedge (x_n]^*)^*$, which implies that

$((x_1] \wedge \text{--------} \wedge (x_n])^* \vee ((x_1]^* \wedge \text{---------} \wedge (x_n])^* \vee \text{-----------}$

$\vee ((x_1] \wedge \text{- - - - - - - -} \wedge (x_n]^*)^* = S$.

If $n = 1$, then for any $r \in S$, we have by (ii) that

$(r] \subseteq (r \wedge x_1]^* \vee (r \wedge x_1]^{**}$.

Thus, $(r] = ((r \wedge x_1]^* \cap (r]) \vee ((r \wedge x_1]^{**} \cap (r])$

$= ((x_1]^* \cap (r]) \vee ((r \wedge x_1]^{**} \cap (r]) \subseteq (x_1]^* \vee (x_1]^{**}$,

by Lemma 2.1.3 and hence $(x_1]^* \vee (x_1]^{**} = S$.

(iii) implies (i) follows exactly from the same proof of [10, Theorem 4.5 (iv) implies (i)].

(iv) implies (v). Suppose(iv)holds,and $P_1, P_2, \text{---------}, P_{n+1}$ are distinct minimal prime ideals. If $P_1 \vee \text{--------} \vee P_{n+1} \neq S$, then by Theorem 1.2.7, there exists a prime ideal P containing $P_1, \text{----------}, P_{n+1}$, which contradicts (iv).

(v) implies (iv). Suppose (v) holds. If (v) does not hold, then there exists a prime ideal P which contains more than n minimal prime ideals. Then by (v) ,P = S, which is impossible.

(iv) implies (vi). Suppose (iv) holds. Then by Corollary 1.5.5, for any prime filter F of S, S_F has at most n ultra filters and so S_F has at most n minimal prime ideals. Since every ideal is the intersection of all of its minimal prime divisors, the zero ideal of S_F is the intersection of at most n minimal (distinct) prime ideals.

Now, let $x_0, x_1, ----------------, x_n \in S$ be such that

$x_i \wedge x_j = 0$ for $i \neq j$, $i = 0, 1, 2, ---------- n$; $j = 0, 1, 2, ------- n$.

Then $\psi_F (x_i) \wedge \psi_F (x_j) = \bar{0}$ (the zero of S_F), for $i \neq j$. Hence by Lemma 4.1.1, there exists k, $0 \leq k \leq n$ such that $\psi_F (x_k) = \bar{0}$. Consequently, $(\psi_F (x_k)]^* = S_F$. Then

$\psi_F ((x_0]^* \vee (x_1]^* \vee ---------- \vee (x_n]^*)$.

$= \psi_F (x_0]^* \vee \psi_F (x_1]^* \vee -------- \vee \psi_F (x_n]^*$

$= (\psi_F (x_0)]^* \vee (\psi_F (x_1)]^* \vee -------- \vee (\psi_F (x_n)]^* = S_F = \psi_F (S)$.

Thus, by Theorem 1.5.6, $(x_0]^* \vee (x_1]^* \vee ----------\vee (x_n]^* = S$.

(vi) implies (iv). Suppose (vi) holds and F is any prime filter of S. If (iv) does not hold then let $F \subseteq Q_0 ,------------, Q_n$, where Q_i are ultrafilters of S. Notice that $Q_i \vee Q_j = S$ for $i \neq j$. Thus for each Q_i, Q_j, $i \neq j$, there exists $x_i \in Q_i$ and $x_j \in Q_j$ such that $x_i \wedge x_j = 0$. Then it is not hard to find elements $y_0, y_1, --------, y_n$ with $y_i \in Q_i$, $y_j \in Q_j$, such that $y_i \wedge y_j = 0$ whenever $i \neq j$. Then by (vi), $(y_0]^* \vee (y_1]^* \vee --------\vee (y_n]^* = S$. Now, if $t \in (y_k]^*$ for some k; $0 \leq k \leq n$, then $t \wedge y_k = 0$. This implies $t \notin Q_k$, otherwise $0 \in Q_k$ as $y_k \in Q_k$. Thus $t \in S - Q_k \subseteq S - F$, and so $(y_k]^* \subseteq S - F$ for each k; $0 \leq k \leq n$. Hence $S=(y_0]^* \vee (y_1]^* \vee$ -------- $\vee (y_n]^* \subseteq S - F$, which is a contradiction. Therefore (iv) holds.

(iii) implies (v). We omit this proof, as it can be proved exactly in a similar way that Cornish has proved (iv) implies (vi) in [10, Theorem 4.5].

(v) implies (i). Suppose (v) holds and $a \in S$. Let Q_1,----------- Q_{n+1} be n+1 distinct minimal prime ideals in [0,a]. By Lemma 2.1.8, there are minimal prime ideals P_i in S such that $Q_i = [0,a] \cap P_i$ for each $1 \leq i \leq n+1$. Since Q_i are distinct , all P_i are also distinct. By (v), $(a] = (a] \wedge (P_1 \vee$ -------- $\vee P_{n+1})$ $= ((a] \wedge P_1) \vee$ ---------- $\vee ((a] \wedge P_{n+1}) = Q_1 \vee$ ----------- $\vee Q_{n+1}$.

Since each interval [0, a] is pseudocomplemented, so $[0, a] \in B_n$ by [31,Theorem 1] and hence S is sectionally in B_n. □

Thus we have the following corollaries :

Corollary 4.1.3. *A nearlattice which is sectionally in* B_n *is* n- *normal.* □

Corollary 4.1.4. *A distributive nearlattice* S *with* 0 *is sectionally in* B_n *if and only if it is normal and sectionally pseudocomplemented.* □

Following theorem extends Theorem 3.5 of Davey [16].

Theorem 4.1.5. *For a distributive sectionally pseudocomplemented nearlattice* S *the following conditions hold:*

(i) *If* S is sectionally in B_n, *then* S_F *is in* B_n *for any filter* F *of* S.

(ii) S *is sectionally in* B_n *if and only if for each prime filter* F *of* S, S_F *has at most n minimal prime ideals.*

Proof : Let $\psi_F(x_0), \psi_F(x_1), \text{----------}, \psi_F(x_n) \in S_F$ be such that $\psi_F(x_i) \wedge \psi_F(x_j) = \bar{0}$ for all $i \neq j$, $i = 0, 1, 2, \text{---------} n$;

$j = 0, 1, 2, \text{---------} n$.

Then $x_i \wedge x_j \equiv 0\ (\psi_F)$ for each i, j $(i \neq j)$. This implies $x_i \wedge x_j \wedge f_{ij} = 0$ for some $f_{ij} \in F$. Set $f = \bigwedge_{i \neq j} f_{ij}$, where $i = 0, 1, 2 \text{---------} n$; $j = 0, 1, 2, \text{---------} n$. Then $x_i \wedge x_j \wedge f = 0$.
Since S is sectionally in B_n, so by Theorem 4.1.2,

$(x_0 \wedge f]^* \vee (x_1 \wedge f]^* \vee \text{----------} \vee (x_n \wedge f]^* = S$.

Hence $(\psi_F(x_0)]^* \vee (\psi_F(x_1)]^* \vee \text{-----------} \vee (\psi_F(x_n)]^*$

$= (\psi_F(x_0 \wedge f)]^* \vee (\psi_F(x_1 \wedge f)]^* \vee \text{---------} \vee (\psi_F(x_n \wedge f)]^*$.

$= \psi_F((x_0 \wedge f]^*) \vee \psi_F((x_1 \wedge f]^*) \vee \text{----------} \vee \psi_F((x_n \wedge f]^*)$

by Theorem 1.5.7.

$= \psi_F[(x_0 \wedge f]^* \vee (x_1 \wedge f]^* \vee \text{-----------} \vee (x_n \wedge f]^*]$.

$= \psi_F(S) = S_F$. Hence S_F is sectionally in B_n by Theorem 4.1.2.

(ii) This is trivial by Theorem 1.5.5.□

2. Nearlattices which are relatively in B_n

We start this section with the following characterization of nearlattices which are relatively in B_n. This will be needed in our next theorem.

Theorem 4.2.1. *Let* S *be a relatively pseudocomplemented distributive nearlattice. Then the following conditions are equivalent :*

(i) S *is relatively in* B_n.

(ii) *For all* $x_0, x_1, \text{---------------}, x_n \in S$,

$< x_1 \wedge x_2 \wedge \text{---------}\wedge x_n, x_0 > \vee < x_0 \wedge x_2 \wedge\text{------------}\wedge x_n, x_1>$

$\vee \text{------------}\vee < x_0 \wedge x_1 \wedge\text{------------}\wedge x_{n-1}, x_n> = S.$

(iii) *For all* $x_0, x_1,\text{---------}, x_n\ z \in S, < x_0 \wedge x_1 \wedge\text{------}\wedge x_n, z >$

$= <x_1 \wedge x_2 \wedge\text{-----------}\wedge x_n, z> \vee < x_0 \wedge x_2 \wedge \text{------------}\wedge x_n, z>$

$\vee\text{-------------}\vee < x_0 \wedge x_1 \wedge\text{----------}\wedge x_{n-1}, z>.$

Proof : (i) implies (ii). Let $a \in S$, consider the interval $I = [x_0 \wedge x_1 \wedge\text{---------------}\wedge x_n \wedge a, a]$ in S. For $0 \leq i < n$, the set of elements $t_i = x_0 \wedge x_1 \wedge\text{-----------}\wedge x_{i-1} \wedge x_{i+1}\wedge\text{-----------}\wedge x_n \wedge a$, are obviously pairwise disjoint in the interval I. Since I is in B_n, so by Theorem 4.1.2. $(t_0]^+ \vee (t_1]^+ \vee \text{----------}\vee (t_n]^+ = I$, where $(t_i]^+ = (t_i]^* \cap I$. So, $a \in (t_0]^+ \vee (t_1]^+ \vee\text{----------}\vee (t_n]^+$. Thus,

$a = p_0 \vee p_1 \vee\text{------------}\vee p_n$, where

$p_0 \wedge t_0 = p_1 \wedge t_1 = \text{--------------} = p_n \wedge t_n = 0$ of I

$= x_0 \wedge x_1 \wedge$-----------------$\wedge\, x_n \wedge a$.

Now, $p_0 \wedge t_0 = x_0 \wedge x_1 \wedge$-----------$\wedge\, x_n \wedge a$ implies $p_0 \wedge t_0 \leq x_0$.

Again, $p_0 \wedge t_0 = p_0 \wedge x_1 \wedge$--------------$\wedge\, x_n \wedge a$

$= p_0 \wedge x_1 \wedge$----------------$\wedge\, x_n$, as $p_o \leq a$.

This implies, $p_0 \wedge x_1 \wedge$---------------$\wedge\, x_n \leq x_0$ and so

$p_0 \in < x_1 \wedge$ ---------$\wedge\, x_n, x_0>$. Similarly,

$p_1 \in <x_0 \wedge x_2\wedge$---------$\wedge\, x_n , x_1>$.

$p_n \in <x_0 \wedge x_1 \wedge$------------- $\wedge\, x_{n-1} , x_n>$. Therefore

$a \in <x_1 \wedge$----------$\wedge\, x_n , x_0> \vee <x_0 \wedge x_2 \wedge$---------------$\wedge\, x_n, x_1>$
$\vee$--------------$\vee <x_0 \wedge x_1 \wedge$----------------$\wedge\, x_{n-1} , x_n>$ and hence

$S=<x_1 \wedge$-----------$\wedge\, x_n ,x_0>\vee <x_0 \wedge x_2 \wedge$---------------$\wedge\, x_n , x_1>$
$\vee$------------$\vee$ $<x_0 \wedge x_1 \wedge$-------------$\wedge\, x_{n-1} , x_n>$

(ii) implies (iii). Suppose $b \in <x_0 \wedge x_1 \wedge$----------$\wedge\, x_n, z>$.

Then by (ii) $b = s_0 \vee s_1 \vee$-----------------$\vee\, s_n$, for some

$s_0 \in \langle x_1 \wedge x_2 \wedge$------------$\wedge\ x_n, x_0\rangle$

$s_1 \in \langle x_0 \wedge x_2 \wedge$------------$\wedge\ x_n , x_1\rangle$

$s_n \in \langle x_0 \wedge x_1 \wedge$------------$\wedge\ x_{n-1} , x_n\rangle$.

Thus, $x_1 \wedge x_2 \wedge$ ------------$\wedge\ x_n \wedge s_0 \leq x_0$

$x_0 \wedge x_2 \wedge$---------------$\wedge\ x_n \wedge s_1 \leq x_1$

$x_0 \wedge x_1 \wedge$ ---------------$\wedge\ x_{n-1} \wedge s_n \leq x_n$.

Then, $x_1 \wedge x_2 \wedge$------------$\wedge\ x_n \wedge s_0 = x_0 \wedge x_1 \wedge$----------$\wedge\ x_n \wedge s_0$.

$\leq x_0 \wedge x_1 \wedge$-------------$\wedge\ x_n \wedge b \leq z$.

Hence, $s_0 \in \langle x_1 \wedge x_2 \wedge$------------------$\wedge\ x_n, z\rangle$. Similarly

$s_1 \in \langle x_0 \wedge x_2 \wedge \text{---------------} \wedge x_n , z\rangle$

$s_n \in \langle x_0 \wedge x_1 \wedge \text{----------------} \wedge x_{n-1} , z\rangle$.

Therefore, $b \in \langle x_1 \wedge x_2 \wedge \text{----------} \wedge x_n, z\rangle \vee$

$\langle x_0 \wedge x_2 \wedge \text{---------} \wedge x_n ,z\rangle \vee \text{----------} \vee \langle x_0 \wedge x_1 \wedge \text{----------} \wedge x_{n-1}, z\rangle$.

So, $\langle x_0 \wedge x_1 \wedge \text{-----------} \wedge x_n , z\rangle$

$\subseteq \langle x_1 \wedge x_2 \wedge \text{--------------} \wedge x_n , z\rangle$

$\vee \langle x_0 \wedge x_2 \wedge \text{----------------} \wedge x_n , z\rangle$

$\vee \text{--------------------} \vee \langle x_0 \wedge x_1 \wedge \text{---------} \wedge x_{n-1} , z\rangle$.

Since the reverse inequality always holds. Therefore

$\langle x_0 \wedge x_1 \wedge \text{-------------} \wedge x_n, z\rangle$

$= \langle x_1 \wedge x_2 \wedge \text{-----------} \wedge x_n , z\rangle \vee \langle x_0 \wedge x_2 \wedge \text{----------} \wedge x_n , z\rangle$

$\vee \text{---------------} \vee \langle x_0 \wedge x_1 \wedge \text{-------------} \wedge x_{n-1} , z\rangle$.

(iii) implies (i). Let $a, b \in S$ with $a < b$. Let

$x_0, x_1, \text{-----------}, x_n \in [a, b]$ such that $x_i \wedge x_j = a$ for all $i \neq j$.

Let $d_0 = x_1 \vee x_2 \vee \text{-------------} \vee x_n$

$d_1 = x_0 \vee x_2 \vee$ --------------$\vee\ x_n$

$d_n = x_0 \vee x_1 \vee$ --------------$\vee\ x_{n-1}$.

Note that $d_0, d_1, d_2,$------------ $,d_n$ exists by the upper bound property of S. Then $a \leq d_i \leq b$ for all i. Now using

$x_i \wedge x_j = a$ for all $i \neq j$. We can easily show by some routine calculations that $x_0 = d_1 \wedge d_2 \wedge$-------------$\wedge\ d_n$

$x_1 = d_0 \wedge d_2 \wedge$------------ $\wedge\ d_n$

$x_n = d_0 \wedge d_1 \wedge$-------------$\wedge\ d_{n-1}$

Then $[a, b] \cap \{ < x_0 , a> \vee < x_1, a> \vee$ --------------$\vee < x_n, a>\}$

$= [a, b] \cap \{ < d_1 \wedge d_2 \wedge$ -------------$\wedge d_n, a>$

$\vee < d_0 \wedge d_2 \wedge$ ----------------$\wedge\ d_n, a> \vee$ ----------------$\vee$

$< d_0 \wedge d_1 \wedge$ ---------------$\wedge\ d_{n-1}, a>\}$.

$= [a, b] \cap \langle d_0 \wedge d_1 \wedge d_2 \wedge \text{---------------} \wedge d_n, a\rangle$, [by (iii)]

$= [a, b] \wedge \langle a, a\rangle = [a, b] \cap S = [a, b].$

Hence by 4.1.2,[a, b] is in B_n. Therefore S is relatively in Bn.□

Following characterization on nearlattices which are relatively in B_n are extensions of some work of Cornish [8] and Davey [16, Theorem 3.4].

Theorem 4.2.2. *For a relatively pseudocomplemented distributive nearlattice* S, *the following conditions are equivalent :*

(i) S *is relatively in* B_n.

(ii) *For any* n+1 *pairwise incomparable prime ideals*

$P_0, P_1, \text{--------------}, P_n$, $P_0 \vee P_1 \vee \text{-----------} \vee P_n = S$.

(iii) *Any prime ideal of* S *contains at most* n *mutually incomparable prime ideals.*

Proof : (i) implies (ii). Suppose S is relatively in B_n. Let $P_0, P_1, \text{-------------}, P_n$ be n+1 pairwise incomparable prime ideals. Then there exists $x_0, x_1, \text{----------------}, x_n \in S$ such that $x_i \in P_j - \bigcup_{\substack{i=1 \\ i\neq j}}^{n} P_i$. Since S is relatively in B_n. Then by Theorem 4.2.2, $\langle x_1 \wedge x_2 \wedge \text{----------} \wedge x_n, x_0\rangle \vee \langle x_0 \wedge x_2 \wedge \text{-----------} \wedge x_n, x_1\rangle \vee \text{----------------} \vee \langle x_0 \wedge x_1 \wedge \text{------------} \wedge x_{n-1}, x_n\rangle = S$.

Let $t_0 \in \langle x_1 \wedge x_2 \wedge \text{-------------} \wedge x_n, x_0\rangle$.

Then $t_0 \wedge x_1 \wedge x_2 \wedge$ --------------$\wedge\ x_n \leq x_0 \in P_0$.

Thus $t_0 \wedge x_1 \wedge x_2 \wedge$ ---------------$\wedge\ x_n \in P_0$. Since P_0 is prime and $x_1 \wedge x_2 \wedge$-----------------$\wedge\ x_n \notin P_0$, so $t_0 \in P_0$. Therefore

$<x_1 \wedge x_2 \wedge$----------------$\wedge\ x_n, x_0> \subseteq P_0$. Similarly,

$<x_0 \wedge x_2 \wedge$----------------$\wedge\ x_n, x_1> \subseteq P_1$

$<x_0 \wedge x_1 \wedge x_3 \wedge$----------------$\wedge\ x_n, x_2> \subseteq P_2$

$< x_0 \wedge x_1 \wedge x_2 \wedge$ ------------$\wedge\ x_{n-1}, x_n> \subseteq P_n$.

Hence, $P_0 \vee P_1 \vee P_2 \vee$---------------$\vee\ P_n = S$.

(ii) implies (i). Let any n+1 pairwise incomparable prime ideals in S are comaximal. Consider an interval [a, b] of S. Let $P_0^{/}, P_1^{/}$,------------------,$P_n^{/}$ be n+1 distinct minimal prime ideals of [a,b]. Then by Lemma 2.1.8, there exists prime ideals

P_0, P_1,---------------, P_n of S such that $P_0^{/} = P_0 \cap [a, b]$,

$P_1^{/} = P_1 \cap [a, b]$. ------------------,$P_n^{/}=P_n \cap [a,b]$ .Since

$P_0^{/}, P_1^{/}$,-------------,$P_n^{/}$ are incomparable, so P_0, P_1,------------,P_n are incomparable. Now by (ii) $P_0 \vee P_1 \vee$----------------$\vee\ P_n = S$.

Hence $P_0' \vee P_1' \vee \text{------------} \vee P_n' = (P_0 \vee P_1 \vee \text{-------------} \vee P_n) \cap [a, b]$ $= S \cap [a, b] = [a, b]$. Therefore $[a, b]$ is in B_n and so S is relatively in B_n.

(ii) = (iii) is trivial. □

Finally, we extend a result of Davey [16, Theorem 3.6]

Theorem 4.2.3. *For a relativley pseudocomplemented distributive nearlattice* S, *if* S *is relatively in* B_n *then* S_F *is relatively in* B_n *for each filter* F *of* S.

Proof : Suppose S is relatively in B_n. Choose

$\psi_F(x_0), \psi_F(x_1), \text{---------------}, \psi_F(x_n) \in S$. Then

$$< \psi_F(x_1) \wedge \psi_F(x_2) \wedge \text{----------------} \wedge \psi_F(x_n), \psi_F(x_0) >$$

$$\vee < \psi_F(x_0) \wedge \psi_F(x_2) \wedge \text{---------------} \wedge \psi_F(x_n), \psi_F(x_1) >$$

$$\vee \text{---------------} \vee < \psi_F(x_0) \wedge \psi_F(x_1) \wedge \text{-------------} \wedge \psi_F(x_{n-1}), \psi_F(x_n) >$$

$$= \psi_F(< x_1 \wedge x_2 \wedge \text{------------} \wedge x_n, x_0 > \vee < x_0 \wedge x_2 \wedge \text{---------} \wedge x_n, x_1 >$$

$$\vee \text{-----------} \vee < x_0 \wedge x_1 \wedge \text{--------------} \wedge x_{n-1}, x_n >) \text{ (by Theorem1.5.7)}$$

$$= \psi_F(S) = S_F \text{ ,by Theorem 4.2.1.}$$

Therefore by Theorem 1.6.3 and Thmeorem 4.2.1 again, S_F is relatively in B_n.□

CHAPTER FIVE

ANNULETS AND α - IDEALS IN A DISTRIBUTIVE NEARLATTICE.

Introduction: *Annulets and α-ideals* in a distributive lattice have been studied extensively by W. H. Cornish in [10]. In a distributive lattice L with 0, set of all ideals of the form $(x]^*$ can be made into a lattice A_0 (L), which is by [10] called the lattice of annulets of L.

A_0 (L) is a sublattice of the Boolean algebra of the annihilator ideals in L. According to Banaschewski [5] the lattice of annulets is no more than the dual of the so called lattice of filters. Subramanian [54, section 4.3, p 20] studied h-ideals with respect to the space of maximal l-ideals in an f-ring. Of course Cornish's α - ideals and his h -ideals were both suggested by the z-ideals of Gilman and Jersion [18, chapter - 2]. On the other hand, Bigard [6] has studied α-ideals in the context of lattice ordered groups. In this chapter we have studied annulets and α-ideals of nearlattices and generalized several results of [10].

By a *"dual nearlattice"* we will mean a join semilattice with the lower bound property. That is, a dual nearlattice S is a join semilattice together with the property that any two elements possessing a common lower bound, have an infimum. So the concept of a dual nearlattice is dual to the concept of a nearlattice.

By [10], for an ideal J in L we define $\alpha(J) = \{(x]^* / x \in J\}$. Also for a filter F in $A_0(L)$, $\alpha^{\leftarrow}(F) = \{x \in L / (x]^* \in F\}$. It is easy to see that $\alpha(J)$ is a filter in $A_0(L)$ and $\alpha^{\leftarrow}(F)$ is an ideal in L. An ideal J in L is called an α - ideal if $\alpha^{\leftarrow}\alpha(J) = J$.

In section 1 we have studied annulets of a distributive nearlattice with 0 and generalized several results of [10]. We have shown that for a distributive nearlattice S with 0, $A_0(S)$ is relatively complemented if and only if S is sectionally quasi-complemented.

Section 2 deals with α-ideals in a distributive nearlattice. We include several generalizations of results in [10]. We have shown that a distributive nearlattice S with 0 is generalized Stone if and only if each prime ideal contains a unique prime α - ideal.

1. Annulets

For a distributive nearlattice S with 0, I (S) the lattice of ideals of S is pseudocomplemented. An ideal J of S is called an *annihilator ideal* if $J = J^{**}$. The pseudocomplement of an ideal J is the annihilator ideal
$J^* = \{ x \in S \,/\, x \wedge j = 0 \text{ for all } j \in J \}$. It is well known by [19, Theorem 4, pp. 58] that the set of annihilator ideals A (S) is a Boolean algebra, where the supremum of J and K in A (S) is given by $J \veebar K = (J^* \cap K^*)^*$. Ideals of the form $(x]^*$ $(x \in S)$ are called the annulets of S. Thus for two annulets $(x]^*$ and $(y]^*$, $(x]^* \veebar (y]^* = ((x]^{**} \cap (y]^{**})^*$ $= ((x \wedge y]^{**})^* = (x \wedge y]^*$. Hence the set of all annulets $A_0(S)$ of S is a join subsemilattice of A (S). But $A_0(S)$ is not necessarily a meet semilattice. But for any $x, y \in S$ if $x \vee y$ exists then $(x]^* \cap (y]^* = (x \vee y]^*$.

Proposition 5.1.1. *Let* S *be a distributive nearlattice with* 0. *Then* $A_0(S)$ *is a dual nearlattice and it is a dual subnearlattice of* A (S). *Moreover* $A_0(S)$ *has the same largest element* $S = (0]^*$ *as* A (S).

Proof : We have already shown that $A_0(S)$ is a join subsemilattice of A (S). Now suppose $(x]^* \supseteq (t]^*$ and $(y]^* \supseteq (t]^*$ for some $x, y, t \in S$. Then $(x]^* \cap (y]^* = ((x]^* \cap (y]^*) \veebar (t]^*$ $= ((x]^* \veebar (t]^*) \cap ((y]^* \veebar (t]^*) = (x \wedge t]^* \cap (y \wedge t]^*$

$= ((x \wedge t) \vee (y \wedge t)]^*$ as $((x \wedge t) \vee (y \wedge t))$ exists by the upper bound property of S. This shows that A_0 (S) has the lower bound property. Hence A_0 (S) is a dual nearlattice and so a dual subnearlattice of A (S). □

Proposition 5.1.2. *Let* S *be a distributive nearlattice with* 0. A_0 (S) *has a smallest element (then of course, it is a lattice) if and only if* S *possesses an element* d *such that* $(d]^* = (0]$.

Proof: If there is an element $d \in S$ with $(d]^* = (0]$ then clearly $(0]$ is the smallest element in A_0 (S).

Conversely, if A_0 (S) has a smallest element $(d]^*$, then for any $x \in S$, $(x]^* = (x]^* \underline{\vee} (d]^* = (x \wedge d]^*$. Thus $x \wedge d = 0$ implies $(x]^* = (0]^* = S$, so that $x = 0$, and hence $(d]^* = (0]$.□

Following result gives a characterization of a normal nearlattice which is a generalization of [11, Proposition 2.2]

Theorem 5.1.3. *A distributive nearlattice* S *with* 0 *is normal if and only if* A_0 (S) *is a join subsemilattice of* I (S).

Proof: By Proposition 5.1.1, A_0 (S) is a join subsemilattice of A (S), and for any $x, y \in S$, $(x]^* \underline{\vee} (y]^* = (x \wedge y]^*$. Now by Theorem 2.1.5. S is normal if and only if $(x \wedge y]^*$ $= (x]^* \vee (y]^*$ for all $x, y \in S$. This proves the theorem.□

A distributive nearlattice S with 0 is called *disjunctive* if for $0 \le a < b$ $(a, b \in S)$ there is an element $x \in S$ such that $a \wedge x = 0$ where $0 < x \le b$. It is easy to check that S is disjunctive if and only if $(a]^* = (b]^*$ implies $a = b$ for any $a, b \in S$. Thus we have the following result.

Proposition 5.1.4. *A disjunctive normal nearlattice* S *is dual isomorphic to* $A_0(S)$.*Hence* S *has a largest element (in that case* S *is a lattice) if and only if there exists* $d \in S$ such that $(d]^* = (0]$.

Proof : If S is normal, then by Theorem 5.1.3, $A_0(S)$ is a join subsemilattice of I (S), and for any $x, y \in S$, $(x \wedge y]^* = (x]^* \vee (y]^*$. Also for any nearlattice S, $(x]^* \cap (y]^* = (x \vee y]^*$ if $x \vee y$ exists in S. Hence the map $x \dashrightarrow (x]^*$ is a dual homomorphism from S onto $A_0(S)$. If S is disjunctive then obviously this map is one-one and so is a dual isomorphism.

Second part is trivial. □

Recall from chapter 2 that a distributive nearlattice S with 0 is quasicomplemented if for each $x \in S$ there is an $x' \in S$ such that $x \wedge x' = 0$ and $(x]^* \cap (x']^* = (0]$. The following result generalizes [11, Proposition 2.4].

Theorem 5.1.5. *A distributive nearlattice* S *with* 0 *is quasi-complemented if and only if* $A_0(S)$ *is a Boolean subalgebra of* A (S).

Proof: Suppose S is quasi-complemented. Then by Theorem 2.2.1, S has an element d such that $(d]^* = (0]$. Then by Proposition 5.1.2, $A_0(S)$ has a smallest element and so it is a sublattice of A (S). Moreover for each $x \in S$ there exists $x' \in S$ such that $x \wedge x' = 0$ and $(x]^* \cap (x']^* = (0]$. Then $(x]^* \veebar (x']^* = (x \wedge x']^* = (0]^* = S$. Therefore $A_0(S)$ is a Boolean subalgebra of A (S).

Conversely, if A_0 (S) is a Boolean subalgebra of A (S), then for any $x \in S$ there exists $y \in S$ such that $(x]^* \cap (y]^* = (0]$ and $(x]^* \veebar (y]^* = S$. But $(x]^* \veebar (y]^* = (x \wedge y]^*$, and so $x \wedge y = 0$. Therefore, S is quasi- complemented .□

Now we generalize [11, Proposition 2.5]. To prove this we need the following lemma. The proof of the Lemma is trivial.

Lemma 5.1.6. *Let* $I = [0, x]$, $0 < x$ *be an interval in a distributive nearlattice* S *with* 0. *For* $a \in I$, $(a]^+ = \{y \in I \,/\, y \wedge a = 0\}$ *is the annihilator of* (a] *with respect to* I. *Then* (i) if $a, b \in I$ *and* $(a]^+ \subseteq (b]^+$ *then* $(a]^* \subseteq (b]^*$

(ii) *if* $w \in S$, $(w]^* \cap I = (w \wedge x]^+$.□

Theorem 5.1.7. *For a distributive nearlattice* S *with* 0, $A_0(S)$ *is relatively complemented if and only if* S *is sectionally quasi-complemented.*

Proof : Suppose A_0 (S) is relatively complemented. Consider the interval I=[0, x] and let a $\in$ I; then $(x]^* \subseteq (a]^* \subseteq (0]^* = S$. Since the interval $[(x]^*, S]$ is complemented in A_0 (S), there exists w $\in$ S such that $(a]^* \cap (w]^* = (x]^*$ and $(a]^* \veebar (w]^* = S$. Then $(a]^* \veebar (w]^* = (a \wedge w]^*$ gives $a \wedge w=0$. Then $a \wedge w \wedge x=0$ and $w \wedge x \in I$. Moreover, intersecting $(a]^* \cap (w]^* = (x]^*$ with (x] and using the Lemma 5.1.6, we have $(a]^+ \cap (w \wedge x]^+ = (0]$. This shows that I is quasi-complemented.

Conversely, suppose S is sectionally quasi-complemented. Since $A_0(S)$ is distributive, it suffices to prove that the interval $[(a]^*, S]$ is complemented for each a $\in$ S. Let $(b]^* \in [(a]^*, S]$. Then $(a]^* \subseteq (b]^* \subseteq S$, so $(b]^* = (a]^* \veebar (b]^* = (a \wedge b]^*$. Now consider the interval I = [0, a] in S. Then $a \wedge b \in I$. Since I is quasi-complemented, there exists $w \in I$ such that $w \wedge a \wedge b = 0$ and $(w]^+ \cap (a \wedge b]^+ = (0] = (a]^+$. This implies $(w \vee (a \wedge b)]^+ = (a]^+$, as $w \vee (a \wedge b)$ exists in S. Then by Lemma 5.1.6, $(a]^* = (w \vee (a \wedge b)]^* = (w]^* \cap (a \wedge b]^*$

$= (w]^* \cap (b]^*$. Also from $w \wedge a \wedge b = 0$ we have $w \wedge b = 0$, hence $(w]^* \veebar (b]^* = S$. Therefore $A_0(S)$ is relatively complemented.□

Since by Theorem 2.2.3, a nearlattice S with 0 is generalized Stone if and only if it is both normal and sectionally quasi-complemented, combining 5.1.7. and 5.1.3. we have the following result :

Theorem 5.1.8. *A nearlattice* S *with* 0 *is a generalized Stone nearlattice if and only if* $A_0(S)$ *is a relatively complemented dual subnearlattice of* I (S). □

2. α - Ideals

In this section we introduce a special class of ideals. We start with the following proposition.

Proposition 5.2.1. *Let* S *be a distributive nearlattice with* 0, *then the following hold:*

(i) *For an ideal* I *in* S, $\alpha(I) = \{(x]^* / x \in I\}$ *is a filter in* $A_0(S)$.

(ii) *For a filter* F *in* $A_0(S)$, $\alpha^{\leftarrow}(F) = \{x \in S / (x]^* \in F\}$ *is an ideal in* S.

(iii) *If* I_1, I_2 *are ideals in* S *then* $I_1 \subseteq I_2$ *implies that* $\alpha(I_1) \subseteq \alpha(I_2)$; *and if* F_1, F_2 *are filters in* $A_0(S)$ *then* $F_1 \subseteq F_2$ *implies that* $\alpha^{\leftarrow}(F_1) \subseteq \alpha^{\leftarrow}(F_2)$.

(iv) *The map* $I \to \alpha^{\leftarrow}\alpha(I)$ $\{= \alpha^{\leftarrow}(\alpha(I))\}$ *is a closure operation on the lattice of ideals,* i.e.

(a) $\alpha^{\leftarrow}\alpha(\alpha^{\leftarrow}\alpha(I)) = \alpha^{\leftarrow}\alpha(I)$,

(b) $I \subseteq \alpha^{\leftarrow}\alpha(I)$,

(c) $I \subseteq J$ *implies that* $\alpha^{\leftarrow}\alpha(I) \subseteq \alpha^{\leftarrow}\alpha(J)$ *for any ideals* $I, J \in S$.

Proof : (i) By Proposition 5.1.1, $A_0(S)$ is a join semilattice with the lower bound property. Let $(x]^*$, $(y]^* \in \alpha(I)$, and $(t]^* \in A_0(S)$, where $x, y \in I$, $t \in S$. Then

$((t]^* \veebar (x]^*) \wedge ((t]^* \veebar (y]^*) = (t \wedge x]^* \wedge (t \wedge y]^*$
$= ((t \wedge x) \vee (t \wedge y)]^* \in \alpha(I)$, as $(t \wedge x) \vee (t \wedge y) \in I$. Also, if $(x]^* \in \alpha(I)$ and $(t]^* \in A_0(S)$ with $(x]^* \subseteq (t]^*$, then $(t]^* = (t]^* \veebar (x]^* = (t \wedge x]^* \in \alpha(I)$. So, $\alpha(I)$ is a filter in $A_0(S)$.

(ii) Let $x, y \in \alpha^{\leftarrow}(F)$ and $t \in S$, then $(x]^*, (y]^* \in F$, and $(t]^* \in A_0(S)$. Since F is a filter of $A_0(S)$, so $((t]^* \veebar (x]^*) \wedge ((t]^* \veebar (y]^*) \in F$ implies that $((t \wedge x) \vee (t \wedge y)]^* \in F$ implies that $(t \wedge x) \vee (t \wedge y) \in \alpha^{\leftarrow}(F)$.

Also, if $x \in \alpha^{\leftarrow}(F)$ and $t \in S$, with $t \leq x$, then $(t]^* \supseteq (x]^*$ and $(x]^* \in F$ implies that $(t]^* \in F$. So, $t \in \alpha^{\leftarrow}(F)$. Hence $\alpha^{\leftarrow}(F)$ is an ideal in S.

(iii) Let $(x]^* \in \alpha(I_1)$, then $x \in I_1 \subseteq I_2$ implies that $(x]^* \in \alpha(I_2)$ implies that $\alpha(I_1) \subseteq \alpha(I_2)$. Let $x \in \alpha^{\leftarrow}(F_1)$, then $(x]^* \in F_1 \subseteq F_2$ implies that $x \in \alpha^{\leftarrow}(F_2)$ implies that $\alpha(F_1) \subseteq \alpha(F_2)$.

(iv) is trivial. □

In a join semilattice S_1 with the lower bound property (i.e. S_1 is a dual nearlattice) a non-empty subset F of S_1 is called a filter if

(i) for any $x, y \in F$, $x \wedge y \in F$ if $x \wedge y$ exists and

(ii) $x \in F$ and $y \geq x$ $(y \in S_1)$ implies that $y \in F$.

Observe that this definition is dual to the definition of an ideal in a nearlattice. Now we give an equivalent definition of a filter in a dual nearlattice which is very easy to prove. This will be needed for further development of this section.□

Theorem 5.2.2. *In a dual nearlattice* S_1 *a non-empty sub set* F *of* S_1 *is a filter if and only if* (i) *for* $f \in F$ *and* $x \geq f$ $(x \in S_1)$ *implies that* $x \in F$, *and* (ii) *for any* $f_1, f_2 \in F$ *and* $x \in S_1$, $(x \vee f_1) \wedge (x \vee f_2) \in F$. □

For an ideal I in a nearlattice S, $\alpha(I) = \{(x]^* / x \in I\}$ is a filter in $A_0(S)$ and conversely for a filter F in $A_0(S)$, $\alpha^{\leftarrow}(F) = \{x \in S / (x]^* \in F\}$ is an ideal in S.

An ideal I of a nearlattice S is called an *α-ideal* if $\alpha^{\leftarrow}\alpha(I) = I$ i.e. α-ideals are simply the closed elements with respect to the closure operation of the Proposition 5.2.1.

Proposition 5.2.3. *The α-ideals of a nearlattice* S *with* 0 *form a complete distributive lattice isomorphic to the lattice of filters, ordered by set inclusion of* $A_0(S)$.

Proof : Let $\{I_i\}$ be any class of α- ideals of S. Then $\alpha^{\leftarrow}\alpha(I_i) = I_i$ for all i. By Proposition 5.2.1 (iv), $\cap I_i \subseteq \alpha^{\leftarrow}\alpha(\cap I_i)$. Again $\alpha^{\leftarrow}\alpha(\cap I_i) \subseteq \alpha^{\leftarrow}\alpha(I_i) = I_i$

for all i implies that $\alpha^{\leftarrow}\alpha(\cap I_i) \subseteq \cap I_i$, and so $\alpha^{\leftarrow}\alpha(\cap I_i) = \cap I_i$. Thus $\cap I_i$ is an α-ideal. Trivially lattice of α- ideals is distributive. Hence α- ideals form a complete distributive lattice. □

For an α- ideal I, $\alpha^{\leftarrow}\alpha(I) = I$. Also, it is easy to see that for any filter F of $A_0(S)$, $\alpha^{\leftarrow}\alpha(F) = F$. Moreover, by Proposition 5.2.1(iii), both α and $\alpha^{\leftarrow}$ are isotone. Hence the lattice of α- ideals of S is isomorphic to the lattice of filters of $A_0(S)$.

Corollary 5.2.4. *Let* S *be a distributive lattice with* 0. *Then the set of prime* α *- ideals of* S *are isomorphic to the set of prime filters of* $A_0(S)$. □

Now we give a characterization of α- ideals of a nearlattice which generalizes [11, Proposition 3.3]

Proposition 5.2.5. *For an ideal* I *in a distributive nearlattice* S *with* 0 *the following conditions are equivalent :*

(i) I *is an* α *-ideal.*

(ii) *for* $x, y \in S$, $(x]^* = (y]^*$ *and* $x \in I$ *implies* $y \in I$.

(iii) $I = \cup_{x \in I} (x]^{**}$ *(where* $\cup$ = *set theoretic union).*

Proof : (i) implies (ii). Suppose I is an α-ideal, then $\alpha^{\leftarrow}\alpha(I) = I$. Let $x, y \in S$, $(x]^* = (y]^*$ and $x \in I$. So, $(x]^* \in \alpha(I)$ implies that $(y]^* \in \alpha(I)$ implies that $y \in \alpha^{\leftarrow}\alpha(I) = I$.

(ii) implies (i). Let I be an ideal of S. $I \subseteq \alpha^{\leftarrow}\alpha(I)$ is always true. Suppose $x \in \alpha^{\leftarrow}\alpha(I)$ then $(x]^* \in \alpha(I)$ implies that $(x]^* = (y]^*$ for some $y \in I$. So, by (ii) $x \in I$ implies that $\alpha^{\leftarrow}\alpha(I) \subseteq I$ implies that $\alpha^{\leftarrow}\alpha(I) = I$.

(ii) implies (iii). Clearly $I \subseteq \cup_{x \in I} (x]^{**}$. If $x \in I$ and $y \in (x]^{**}$ then $(x]^* \subseteq (y]^*$ implies that $(y]^* = (x]^* \veebar (y]^* = (x \wedge y]^*$. Then $x \wedge y \in I$ implies that $y \in I$. Thus $\cup_{x \in I} (x]^{**} \subseteq I$. So $I = \cup_{x \in I} (x]^{**}$.

(iii) implies (ii). If $x, y \in S$, $(x]^* = (y]^*$ and $x \in I$, then $(x]^{**} = (y]^{**}$ implies that $y \in (x]^{**} \subset I$ implies that $y \in I$. This completes the proof. □

Proposition 5.2.6. *In a distributive nearlattice* S *with* 0 *the following conditions are equivalent* :

(i) *each ideal is an* α*-ideal.*

(ii) *each prime ideal is an* α*- ideal.*

(iii) S *is disjunctive.*

Proof : (i) implies (ii). Suppose P is any prime ideal of S, then by (i) P is an α - ideal, that is $\alpha^{\leftarrow}\alpha$ (P) = P. Let I be any ideal of S then we have $I = \cap\ (P / P \supseteq I)$ implies
$\alpha^{\leftarrow}\alpha\ (I) = \alpha^{\leftarrow}\alpha(\cap\ (P / P \supseteq I)) = \cap\ (\alpha^{\leftarrow}\alpha(P)\ /\ P \supseteq I)$
$= \cap\ (P\ /\ P \supseteq I) = I$ implies that $\alpha^{\leftarrow}\alpha\ (I) = I$. So I is an α- ideal.

(ii) implies (i) is trivial.

(i) implies (iii). For any x, y $\in$ S, let $(x]^* = (y]^*$. Since $(x]$ is an α- ideal, so by definition of α- ideal, y $\in$ (x].

Therefore, $y \leq x$. Similarly $x \leq y$, and so x = y. Hence S is disjunctive.

(iii) implies (i). Suppose I is any ideal of S. By 5.2.1, I $\subseteq \alpha^{\leftarrow}\alpha$ (I). For the reverse inclusion, let $x \in \alpha^{\leftarrow}\alpha$ (I). Then by definition $(x]^* \in \alpha$ (I), and so $(x]^* = (y]^*$ for some $y \in I$. This implies x = y, as S in disjunctive. So x $\in$ I, and hence $\alpha^{\leftarrow}\alpha$ (I) = I. Therefore I is an α - ideal of S. □

Proposition 5.2.3 implies that there is an order isomorphism between the prime α- ideals of S and the prime filters of $A_0(S)$. It is not hard to show that each α - ideal is an intersection of prime α- ideals.

Following theorem is a generalization of [11, Theorem 3.6]. It was proved for bounded lattices in [38] and announced in general in [37]; an explicit proof is given in [22, p.276]. We need the following lemma which is due to [11, Lemma 3.5].

Lemma 5.2.7. *A distributive nearlattice* S *with* 0 *is relatively complemented if and only if every prime filter is an ultra filter (Proper and maximal).*

Proof : By [45, Theorem 2.11] S is relatively complemented if and only if its prime ideals are unordered. Thus the result follows. □

Theorem 5.2.8. *Let* S *be a distributive nearlattice with* 0. *Then the following conditions are equivalent :*

(i) S *is sectionally quasi-complemented.*

(ii) *each prime* α - *ideal is a minimal prime ideal.*

(iii) *each* α - *ideal is an intersection of minimal prime ideals.*

Moreover, the above conditions are equivalent to S *being quasi – complemented if and only if there is an element* d $\in$ S *such that* $(d]^* = (0]$.

Proof: (i) implies (ii). Suppose S is sectionally quasi-complemented. Then by Theorem 5.1.7, A_0 (S) is relatively complemented. Hence its every prime filter is an ultra filter. Then by Corollary 5.2.4, each prime α - ideal is a minimal prime ideal.

(ii) implies (iii). It is not hard to show that each ideal of S is an intersection of prime α - ideals. This shows (ii) implies (iii).

(iii) implies (ii) is obvious by the minimality property of prime α - ideals.

(ii) implies (i). Suppose (ii) holds. Then by Corollary 5.2.4, each prime filter of $A_0(S)$ is maximal. Then by Lemma 5.2.7, $A_0(S)$ is relatively complemented, and so by [11, Proposition 2.7], S is sectionally quasi-complemented.□

We conclude the thesis with the following result which is a generalization of [11, Theorem 3.7].

Theorem 5.2.9. *A nearlattice* S *with* 0 *is a generalized Stone nearlattice if and only if each prime ideal contains a unique prime* α *- ideal.*

Proof : Since minimal prime ideals are α- ideals, so by the given condition every prime ideal contains a unique minimal prime ideal. Hence S is normal. Also, by the given condition each prime α- ideal contains a unique prime α- ideal. That is each prime α- ideal contains no other prime α- ideals than itself. Since each minimal prime ideal is also prime α- ideal, so by the condition, each prime α- ideal is itself a minimal prime ideal. Hence by Theorem 5.2.8, S is sectionally quasi- complemented. Therefore, by 2.2.4., S is generalized Stone.

Conversely, if S is generalized Stone then by Theorem 2.1.8, S is normal. So each prime ideal contains a unique minimal prime ideal. Thus the result follows as each minimal prime ideal is a prime α- ideal. □

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