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Use of Fox Derivatives for Solution of Group Theoretic Problem

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USE OF FOX DERIVATIVES FOR SOLUTION OF GROUP THEORETIC PROBLEMS

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by

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To

my parents

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Certified that the thesis entittled "Use of Fox Derivatives for Solution of Group Theoretic Problems" submitted by Quazi Selina Sultana in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, University of Rajshahi, Rajshahi, has been completed under my supervision. I believe that this research work is an original one and that it has not been submitted elsewhere for any degree.

Subrata Majumdar
(Subrata Majumdar)

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STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

Quazi Selina Sultana.

(Quazi Selina Sultana)

SYNOPSIS

The thesis is an exposition of use of Fox derivatives for solution of topological and group theoretic problems. Our own contribution is in the latter area. Developed by Fox, free differential calculus emerged as a powerful tool for the solution of topological problems. In addition to reviewing application of this tool in various mathematical situations, (i) we have applied it to provide new proofs of a few very important results in combinatorial group theory, (ii) solved the conjugacy problem for torsion-free polycyclic-by-finite groups and torsion-free groups with single defining relations and (iii) determined homology and cohomology of a few classes of groups.

In the first chapter the free differential calculus of Fox has been introduced and its application to classification of topological spaces, has been briefly described. In this context, Lens spaces, knots, links and braids have been defined and use of Alexander polynomials in their classification described A few useful terms in combinatorial group theory have been defined and a number of fundamental results stated.

The second chapter describes Birman's work on the relationship of the Jacobians with automorphisms of free groups. This has been used to determine the automorphism group of a free group of rank 2.

The third chapter provides a new proof of Freiheitssatz, a fundamental theorem in the theory of single relator groups. This was first proved by Magnus who used it to solve the word problem for such groups. Many proofs (some of them geometric) of his theorem and its generalisations have so far been given by eminent mathematicians, viz. Lyndon, Schupp, Burns, etc. We proved this theorem using Fox derivatives following Majumdar's ideas.

In the fourth chapter a new proof of Lyndon's famous Identity Theorem has been given. We have also determined the roots of words of the form $x_1^{n_1}......x_r^{n_r}$ thereby generalising a result of Steinberg. The Identity theorem has been used by Lyndon and Huebschmann to compute cohomology of important classes of groups. Determination of roots is algorithmic in nature and in a sense complementary to solution of word problems, the nature of the latter being existential.

Chapter five deals with the solution of conjugacy problem for torsion-free polycyclic-by-finite groups and torsion-free groups with a single relation.

In Chapter six, we have constructed free resolutions for a few classes of groups using Majumdar-Akhter techenique of extending Lyndon's partial resolution. The classes of groups considered by us are finite the dihedral groups, the fundamental group of union of n Tori $T_1,.......,T_n$ such that $T_j \cap T_{j+1}$ is a single point, the quasi-cyclic group, the unpermuted braid group and the 3-dimensional Heisenberg group H_3. The homology and in most cases cohomology also have been calculated from these resolutions.

CONTENTS

CHAPTER - 1

ALGEBRAIC AND TOPOLOGICAL BACKGROUND OF FOX DERIVATIVES

1.1 Introduction

In this chapter necessary terminology has been introduced, useful ideas described and important results stated in several areas, viz., Free Differential Calculus, Topolgy, Knot Theory, Combinatorial Group Theory and The Theory of Riemann Surfaces. S ome of those have been diectly applied in our work while the others are needed to describe the broad perspective of our problems and the method of their solution.

1.2. Free Differential Calculus and Fox Derivatives

The free differential calculus was developed by R. H. Fox ([32], [33], [34] in connection with his analysis of the ideas regarding Alexander's polynomial. His objective was to solve the topological classification of the 3-dimensiomal Lens spaces by a scheme involving a generalisation of Alexander polynomial. This he succeeded by showing that Reideimster's combinatorial classification [94] of the Lens spaces is also the topological classification. The free differential calculus has been being used in the study of knots ([3], [4], [8], [16], [37], [38], [87], [90], [92], [94],[100]) and also in connection with group theoretic problems including determination of homology and cohomology of groups ever since its development (see for example, Fox [32], [33], ..., [37], Lyndon [68], Gruenberg [50], Majumdar ([72], [73], [74], [75], [76], [77], [78], [79]) Birman [14], Balanchfield [17], Majumdar - Akhter ([80], [81], [82]).

The derivatives of the free diffrential calculus will be called *Fox derivatives*. Here we shall briefly describe these works and define the necessary algebraic and topological concepts. For definitions, figures and notation we shall mostly follow Moran [91], Crowell and Fox [25], W.S. Massey [87], Hocking and Young [58], Griffiths [43], Hilton and Wylie [57].

We start with some importnt definitions and results about free groups and presentation of groups. We follow Magnus, Karras, and Solitar [85] and Lyndon and Schupp [71].

Free Groups

Let X be a subset of a free group F, then F is a *free group with basis* X if every function φ from X into a group H can be extented uniquely onto a homomnorphim $\bar{\varphi}$ from F into H. All bases for a given free group F have same cardinal, the rank of F.

A group P is *projective* provided the following holds : if G and H are any group and γ is a homomorphism from G onto H and π a homomorphism from P into H, then there is a homomorphism φ from P into G such that $\gamma\varphi = \pi$.

A homomorphism ρ from a group G onto a subgroup S of G is called a *retraction* and S is called a *retract* of G if $\rho^2 = \rho$.

Theorem 1.1 ([71], p. 2).

The projectives groups are precisely retracts of free groups.

Corollary 1.2 ([71], p.2).

The projective groups are precisely the free groups.

We now give an explicite construction of free groups and thereby establish the existence of free groups. Let X be a nonempty set and Y be a set disjoint from X and in 1-1 correspondence with X. If y corresponds with x, we write $y = x^{-1}$ and $x = y^{-1}$ and called x, y are inverse of each other. We also write $Y = X^{-1}, X = Y^{-1}$. Let S be $X \cup Y = X \cup X^{-1}$. Now the elements of S are called *letters*.

A finite sequence $(a_1, a_2, \cdots a_n)$, $n \geq 0$ of letters is called a *word* and is denoted by w. We also write a_i for (a_i) and $a_1 a_2 \cdots a_n$ for$(a_1 a_2 \cdots a_n)$. If $n = 0$ we write $w = 1$. It is the empty word. Let W be the set of all words. Define a multiplication in W as follows:

If $w_1 = a_1 \cdots a_m$ and $w_2 = b_1 \cdots b_n$, then $w_1 w_2 = a_1 \cdots a_m b_1 \cdots b_n$. W, then becomes a semigroup with the identity 1. If $w = a_1 \cdots a_m$, we define $w^{-1} = a_m^{-1} \cdots a_1^{-1}$. w^{-1} is called the *inverse of* w. Two words w_1 and w_2 are called *equivalent* and write $w_1 \sim w_2$ if w_2 is obtained from w_1 by inserting or deleting a finite number of words of the form aa^{-1}. Clearly $\sim$ is an equivalence relation on w which preserves multiplication and inversions. The quotient semi-group $F = \frac{W}{\sim}$ is clearly a group. F is a free group with basis $\bar{X} = \{\bar{x} \mid x \in X\}$, where $\bar{x}$ denotes the equivalence class of x under $\sim$ in W. We shall often identify a word with its equivalence class. Thus we shall often write x for $\bar{x}$.

Each element of F other than the identity has a unique presentation as a *reduced word* $w = s_1 s_2 \cdots s_n$ in which two successive letters $s_j s_{j+1}$ form an inverse pair $x_i x_i^{-1}$ or $x_i^{-1} x_i$. n is called the *length* of w and is denoted by $\mid w \mid$.

A reduced word w is called *cyclically reduced* if $s_i \neq s_1^{-1}$ and if there is no cancellation in forming the product $z = u_1 \cdots u_n$, we write $z \equiv u_1 \cdots u_n$. A subset T of F is called *symmetrised* if all all elements of T are cyclically reduced and, for each $t \in T$, all cyclically reduced conjugates of both t and t^{-1} also belong to T.

Theorem 1.3 ([71] p.16).

Every subgropup of a free group is free.

Theorem 1.4 ([71] p.2).

A group generated by a set of its n elements (n finite or infinite)is a quotient of factor group of rank n.

Thus, every group G can be written as $G = \frac{F}{R}$, where F is a free group and R is a normal subgroup of F. Let G be a group and S be a non-empty subset of G. The normal closure $N_c[S]$ of S is the intersection of all those normal subgroup of G which contains S. The normal closure $N_c[S]$ consists of all finite products of the conjugates of the elements of S and the inverse in G. If $S = \{x_1, x_2, \cdots, x_n\}$, $N_c[S]$ is also called a *normal closure of* $x_1, x_2, \cdots, x_n$.

If F is generated by a finite of elements , say $x_1, x_2, \cdots, x_m$ and R is the normal closure of finite number of elements, say $r_1, r_2, \cdots, r_n$ of F, then G is said to be *finitely prensented.*

We often express the above presentation of G either by

$$< x_1, \cdots, x_m \mid r_1, \cdots, r_n >$$

or, by

$$\text{generatos :} \qquad x_1, \cdots, x_m$$

$$\text{relation :} \qquad r_1, \cdots, r_n \; .$$

It is cleaer that in the above situation an element $w \in F$ belong to R if and only if $w = (w_1^{-1} r_{i_1}^{e_i} w_1) \cdots (w_u^{-1} r_{i_u}^{e_u} w_u)$, for some non negative integer u, where $w_k \in F$, $e_k = \pm 1$ if $u = 0$, $w = 1$.

An element of R too is called a *relator.* The fundamental groups of topological spaces are often obtained in terms of presentations with the help of van Kampen's Theorem [] in algebric topology. Conversely any finitely presented group is a fundamental group of 4-manifold (see Massey [80] p.143). The fundamental group of compact 2-manifold is a group with single-relator. Hence this class of groups has been subjected to vigorus study by various mathematicians.

For a group $G = \frac{F}{R}$ with a presentation, M Dehn [26] posed three fundamental dicision problems.

I. **Word problem**

To decide in a finite number of steps , whether two elements w_1 and w_2 represent the same elements in G. When it is possible G is said to have *solvable word problem.*

II. **Conjugacy problem**

To decide a finite number of steps, if w_1 and w_2 represents the conjugate element in G. G has the *solvable conjugacy problem* when the answere is in the affirmative.

III. **Isomorphism problem.**

To decide in a finite number of steps whether two groups with given finite presentations are isomorphic.

Dehn [26] solved all this presentatins for cannonical presentations of fundamental groups of 2-manifolds. For a group with a single defining relation the word problem was solved by Magnus [83]. There are finitely presented groups with unsolvable word problem. (see Britton [18], G.Higman [56], Rotman [100]). Also, there are groups with solveable word problem but unsolvable conjugacy problem (see Fridman [40], Collins [23], Miller [90]). Lyndon [68] used Fox derivatives for computation of the cohomology of groups of the single defining relation. His work yielded a partial free resolution for finitely presentated groups. This solution has been extended to a full resolution and has been used to compute homology and cohomology for many classes of groups by Majumdar and Akhter [80]-[82].

1.3. Derivation in a free group ring.

Let F be a free group with a free set of generators X. An element of F is an equivalence u of words is represented by a unique reduced word $\prod_{x=1}^{n} x_i^{e_i}, e_i = \pm 1,\ e_i + e_{i+1} \neq 1$. n is called the *length of* u. The identity element 1 is represented by the empty word and is of length 0. The inverse u^{-1} of u is represented by the word $\prod_{i=n}^{1} x_i^{-e_i}$. The group $\mathbb{Z}F$ is the integral group ring of the free group F i.e., the set of all formal sums $\sum_{u\in F} a_u u,\ a_u \in \mathbb{Z}$, with $a_u \neq 0$ only for a finite number of u's, together with addition and multiplication defined by $\sum a_u u + \sum b_u u = \sum (a_u + b_u)u$;
$(\sum a_u u)(\sum b_u u) = \sum c_u u$, where $c_u = \sum_{u=vw} a_u b_w$.
It is clear that $\mathbb{Z}F$ is a group ring. The map $\varepsilon : \mathbb{Z}F \to \mathbb{Z}$ given by $\varepsilon(\sum a_u u) = \sum a_u$ is a ring homomorhism and is called *augmentation homomorphism.* The kernel of ε is denoted by $\mathfrak{F}$ and is called the *fundamental ideal* of $\mathbb{Z}F$.

In general for any group G, the integral group ring $\mathbb{Z}G$ is defined in exactly the same way as $\mathbb{Z}F$. The fundamental ideal of $\mathbb{Z}G$ too is defined in the case of $\mathbb{Z}F$. $\pi : \mathbb{Z}F \to \mathbb{Z}G$

is defined the ring homomorphism obtained by linearly extending the canonical group homomorphism $F \to G$. Kerπ, is the kernel of π, is denoted by $\Re$.

We note that the following result will be extensively used in our work.

Theorem 1.5 ([22])

If F is a free group with basis X, the $\mathfrak{F}$ is a right (left) free $\mathbb{Z}G$ - module on $\{x-1 \mid x \in X\}$, and conversely.

Theorem 1.6 ([44])

If F is a free group and R is a normal subgroup of F with a basis Y, then $\Re$ is a free right(left) $\mathbb{Z}F$-module on $\{y-1 \mid y \in Y\}$, and conversely.

Theorem 1.7 (Gruenberg [48]).

Let F be a free group. If M_1 and M_2 are ideal of $\mathbb{Z}F$ which are free right (left) $\mathbb{Z}F$ -module on $\{y-1 \mid y \in Y\}$, $\{z-1 \mid z \in Z\}$, then M_1M_2 is a right (left) free $\mathbb{Z}F$-module on $\{(y-1)(z-1) \mid y \in Y,\, z \in Z,\, Y, Z \subseteq F\}$.

Theorem 1.8

Let F be a free group, and let $G = \frac{F}{R}$. If M is an ideal of $\mathbb{Z}F$ which is a right (left) free $\mathbb{Z}F$-module on $\{y-1 \mid y \in Y \subseteq F\}$, then $\frac{M}{M\Re}(\frac{M}{\Re M})$ is right (left) $\mathbb{Z}F$ - module on $\{(y-1) \in Y\}$.

A left (resp. right) derivation in a group ring $\mathbb{Z}G$ is a mapping $D : \mathbb{Z}G \to \mathbb{Z}G$ such that
$D(\varphi + \psi) = D\varphi + D\psi$
$D_l(\varphi\psi) = D(\varphi)\varepsilon(\psi) + D(\psi)$, (resp.$D(\varphi) + \varepsilon(\varphi)D(\psi)$)
Thus, for $g, h \in G$, $D(gh) = D(g) + gD(h)$ (resp. $D(h) + D(g)h$)

We have the following chacterisation of derivativation in a free group ring. For description of the works of various mathematicians we define both the left and right derivatives although Fox's [38] paper defines only the left derivative.

Theorem 1.9 ([38], p.550).

For a free group F with a basis X, there corresponds a left (resp. right) derivation

$$D : \mathbb{Z}F \to \mathbb{Z}F$$

given by

$$D(\varphi) = (\frac{\partial\varphi}{\partial x})_L((\frac{\partial\varphi}{\partial x})_R),$$

which has the property that

$$(\frac{\partial x_i}{\partial x_j})_L = \delta_{ij}(\frac{\partial x_i}{\partial x_j})_R.$$

Furthermore, there is one and only one derivation D such that $D(x_i) = w(x)$ and is given by

$$D_L(\varphi) = \sum \frac{\partial \varphi}{\partial x} w(x).$$

We also have the following fundamental result :

Theorem 1.10 (Moran [91], p.146)

Let $\varphi \in \mathbb{Z}F$, then,

$$\varphi - \varepsilon(\varphi) = \sum_x (\frac{\partial \varphi}{\partial x})_L (x-1) = \sum_x (x-1)(\frac{\partial \varphi}{\partial x})_R. \tag{1.1}$$

(1.1) is often called the *fundamental formula.*

If $\varphi \in \Im$, then

$$\varphi = \sum_x (\frac{\partial \varphi}{\partial x})_L (x-1) = \sum_x (x-1)(\frac{\partial \varphi}{\partial x})_R. \tag{1.2}$$

In particular, if $w = \prod_{i=1}^{n} x_i^{e_i}$ is an element of F, then

$$\left(\frac{\partial w}{\partial x}\right)_L = \sum_{i=1}^{n} e_i \prod_{j=1}^{i-1} x_j^{e_j} x_i^{\frac{(e_i-1)}{2}} \tag{1.3}$$

and

$$\left(\frac{\partial w}{\partial x}\right)_R = \sum_{i=1}^{n} e_i x_i^{\frac{e_i-1}{2}} \prod_{j=i+1}^{n} x_j^{e_j}. \tag{1.4}$$

For $\varphi \in \mathbb{Z}F$, $(\frac{\partial \varphi}{\partial x})_L$ and $(\frac{\partial \varphi}{\partial x})_R$ are called the left (resp. right) *Fox derivative* of φ.

The higher order derivatives of φ are defined inductively

$$\frac{\partial^n \varphi}{\partial x_{j_n} \partial x_{j_{n-1}} \cdots \partial x_{J_1}} = \frac{\partial}{\partial x_{j_n}} \left(\frac{\partial^{n-1} \varphi}{\partial x_{j_{n-1}} \cdots \partial x_{j_1}} \right), (n \geq 2).$$

By applying the fundamental formula (1.1) we obtain the tailor series with remainder.

Theorem 1.11 ([38], p. 553)

Let F be a free group with basis X,for each $\varphi \in \mathbb{Z}F$,

$$\begin{aligned}\varphi(x) &= \varphi(1) + \sum_{j_1} (D_{j_1}\varphi(1))(x_{j_1}) + \sum_{j_2,j_1} D_{j_2 j_1}\varphi(1)(x_{j_2}-1)(x_{j_1}-1) + \cdots \\ &+ \sum_{j_{n-1},\cdots,j_1} \varphi(1)(x_{j_{n-1}-1})\cdots(x_{j_1}-1) \\ &+ \sum_{j_n,\cdots,j_1} D_{j_n\cdots j_1}\varphi(x)(x_{j_n}-1)\cdots(x_{j_1}-1). \end{aligned} \quad (1.5)$$

One also obtain a formal "Tailor series" expansion

$$\varphi(x) = \varphi(1) + \sum D_j\varphi(1)(x_j - 1) + \sum_{j,k} D_{jk}\varphi(1)(x_j-1)(x_k-1) + \cdots . \quad (1.6)$$

In (1.5) and (1.6) $\varphi(1)$ standsfor $\varepsilon(\varphi), D_j\varphi(1)$ for $\varepsilon(D_j(\varphi))$, and so on.
In particular the formal expansion of

$$\varphi(x) = x_j$$

and

$$\varphi(x) = x_j^{-1}$$

are

$$x_j = 1 + (x_j - 1)$$

$$x_j^{-1} = 1 - (x_j-1) + (x_j-1)^2 - (x_j-1)^3 + \cdots .$$

These expansions are identical with the corresponding expansion (1.1) of Magnus if one writes a for x_j and s for $x_j - 1$.

Structure of $\mathbb{Z}F$

The free differential calculus (Fox derivatives) have been applied by Fox [38] to through much light on the structure of the free group ring $\mathbb{Z}F$. We enlist below some of these results. Here we have used left derivatives.

(A) $\varphi(x)$ in $\mathbb{Z}F$ belong to $\mathfrak{F}^n$ if and only if all of its derivatives of order $0, 1, \cdots, n-1$ vanish at n=1. The length $l(\varphi(x))$ of a non-zero $\varphi = a_1u_1 + \cdots a_mu_m$ in $\mathbb{Z}F$ is defined as $max\{l(u_1), \cdots, l(u_m)\}$. Assuming that $u_i \neq u_j$ for $i \neq j$ and that $a_1, \cdots a_m \neq 0$. Also $l(0) = 0$.

(B) If $\varphi(X) \in \mathfrak{F}^n$ and $\varphi(x) \neq 0$, then $l(\varphi(x)) \geq \frac{n}{2}$.

(C) Uniqueness theorem for power series expansion.

If $\varphi(x)$ and $\psi(x)$ in $\mathbb{Z}F$ are such that

$$\varphi(1) = \psi(1),\ D_j(\varphi(1)) = D_j(\psi(1)),\ D_{ij}(\varphi(1))\text{etc},$$

then

$$\varphi(x) = \psi(x).$$

(D) Corollary

$$\cap \mathfrak{F}^n = 0.$$

(E) Let $\mathfrak{R}$ be an ideal of $\mathbb{Z}F$ and let $\varphi(x) \in \mathbb{Z}F$, then $\varphi(x) \in \mathfrak{R}\mathfrak{F}^n$ if and only if all its derivatives of order n belong to $\mathfrak{R}$ (in which case its derivatives of order i belong to $\mathfrak{R}\mathfrak{F}^{n-1}$, $i = 1, \cdots, n$).

(F) The ideal $\mathfrak{F}^n$ determines the n-th lower central group F_n.

(G) The ideal $\mathfrak{G}^n$ determines the n-th lower central group G_n.

(H) $\mathbb{Z}F$ has no zero divisor.

(I) The ideal $\mathfrak{R}$, $\mathfrak{F}$ determines the commutator subgroup $[R, R]$ of R.

(J) In order that an element $w - 1$ of $\mathbb{Z}F$ belong to $\mathfrak{R}$, it is necessary and sufficient that there exist an element $r \in R$ such that $v - 1 \in \mathfrak{R}\mathfrak{F}$.

J. S. Birman also applied Fox derivative for study of free groups. We state her result below. She used the left derivatives (1.4).

Inverse Function Theorem 1.12 (Birman [14], p. 635)

Let $\{y_1, \cdots, y_k\}$ be a set of $k \leq n$ elements of F_n. Let J_{kn} denote the $k \times n$ matrix $\| \frac{\partial y_i}{\partial x_j} \|$

(i) *If $k = n$ a necessary and sufficient condition for $\{y_1, \cdots, y_n\}$ to be a generating set of F_n is that J_{nn} have a right inverse.*

(ii) *If $k < n$, a necessary condition for $\{y_1, \cdots, y_k\}$ extend to a generating set $\{y_1, \cdots, y_n\}$ is that J_{nn} have a right inverse.*

Corollary 1.13 ([14], p.636)

Any set of n elements which generates a free group of rank n are a set of free geneators.

Corollary 1.14 ([14], p.637)

Let J^{α}_{nn} denote the image of J_{nn} under the abalianising homomorphism α acting on $\mathbb{Z}F_n$. Then $\{y_1, \cdots, y_n\}$ is a basis for F_n only if $detJ^{\alpha}_{nn}$ is a unit in $\mathbb{Z}F_n$.

Alexander Polynomial.

Let G be a group with a finite presentation

$$\text{generators} \quad x_1, \cdots, x_m;$$

$$\text{relarors} \quad r_1, \cdots, r_n. \tag{1.7}$$

Then $G = \frac{F}{R}$, where f is a free group generated by $x_1, \cdots, x_m$ and R is the normal closure of $\{r_1, \cdots, r_n\}$. Let $\pi : \mathbb{Z}F \to \mathbb{Z}G$ be the ring homomorphism induced by the canonical homomorphism $F \to G$ and let $\alpha : \mathbb{Z}G \to \mathbb{Z}(\frac{G}{G'})$ be the ring homomorphism induced by the canonical homomorphism $G \to \frac{G}{G'}$. Then the $m \times n$ matrix $(\alpha\pi)(\frac{\partial r_i}{\partial x_j})$ with the centre is an abelian group $\mathbb{Z}(\frac{G}{G'})$ is called the *Alexander matrix* associate with the presentation (1.7).

We shall now introduce an equivalence the marices which will very important in conception with the Alexander matrices.

An Equivalence of Matrices.

Let A_1, A_2 be matrices with entries from a fixed commutative ring with 1. Then A_1 is said to be equivalent to A_2 if and only if A_2 is obtained from A_1 by a finite number of elementary operation of the following form or their inverses:

(a) Permute the order of the rows;

(b) Permute the order of the colunms;

(c) Add to any row, any linear combination of the remaining rows;

(d) Add to any column, any linear combination of the remaining rows;

(e) Insert a row of zeroes;

(f) Insert a new border of the form

$$A \longrightarrow \begin{bmatrix} & & & & 0 \\ & & & & 0 \\ & & A & & \cdot \\ & & & & \cdot \\ & & & & \cdot \\ & & & & 0 \\ * & \cdot & \cdot & * & u \end{bmatrix}.$$

.

Then the above equivalent is a genuine equivalence relaion.

Theorem 1.15 (Uniqeness of Alexander Matrix) ([91], p.163)

If a group has two finite presentation then their Alexander matrices are euivalent.

Elementary Ideal of Alexander Matrix.

Let A be a Alexander matrix which has entries from a commutative ring R with unit element 1. If A is an $m \times n$ matrix, then we define the *elementary ideal* $E(A)$ in R to be the ideal generated in R by all determinants of every $(n-1) \times (n-1)$ submatrix of A.

Theorem 1.16 ([91], p.173) (Invariance of Elementary Ideals)

Suppose that A_1 and A_2 are equivalent matrices. Then $E(A_1) = E(A_2)$.

1.4. Knots, Links, Braids

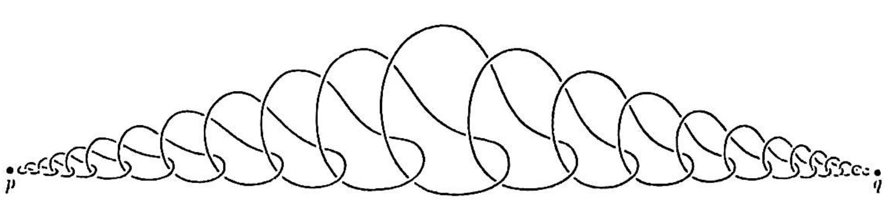
A *knot* K is the image in $\mathbb{R}^3$ of a continuous mapping $f : S^1 \rightarrow \mathbb{R}^3$ of the unit circle S^1, which is one to one. A mathematical knot is an obstraction of a common knot of threads or strings we came across in our daily life with the extra requirment that the two ends are tied together.

A knot K is said to be *equivalent* to a knot K' if and only if there exist a homomorphism φ of $\mathbb{R}^3$ onto $\mathbb{R}^3$ such that $\varphi(K) = K'$.

A homomorphism φ of $\mathbb{R}^3$ onto $\mathbb{R}^3$ is said to be *orientation preserving (reversing)* if and only if it maps the right handed corck screw onto a right(left) handed corck screw.

A knot K is *tame* if and only if it is homeomorphic to the union of a finite number of straight lines segment in $\mathbb{R}$.

A knot which is not tame is calld *wild.* Artin and Fox gave an interesting and surprising example of a wild arc which is given by the following figure.



A simple arc in E^3 whose complement is not simply connected.

Figure - 1.1.

A knot K is said to be *oriented equivalent* to a knot K' if and only if there exists an orientation preserving homeomorphism φ of $\mathbb{R}^3$ onto $\mathbb{R}^3$ such that $\varphi(K) = K'$.

The knot K is said to be *string isotopic* to the knot K' if and only if there exist a continuous mapping $F : S^1 \times I \to \mathbb{R}^3$ such that $F(S^1, 0) = K$, $F(S^1, 1) = K'$ and for each fixed value of t(with $0 \leq t \leq 1$) the mapping $S \to F(s,t)$ with $s \in S^1$ is one to one giving rise to the knot $\{F(s,t),\ s \in S^1\}$ which is polygonal.

Theorem 1.17 ([91], p.67) If the knots K and K' are string isotopic. Then K and K' are oriented equivalent knots.

A simple closed curve is said to be trivial if it is equivalent to the plane circle in $\mathbb{R}^3$ with the equation $x_1^2 + x_2^2 = 1,\ x_3 = 0$.

A knot K is said to be *unknotted* if and only if it is equivalent to the trivial knot

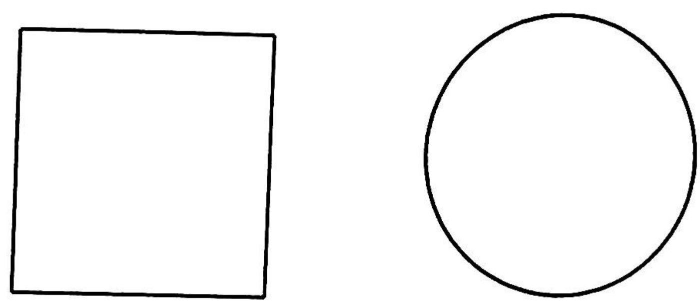


Figure - 1.2.

otherwise it is said to be *knotted.*

Examples of some knots:

(i) The left handed trefoil knot;

(ii) The right handed trefoil knot;

(iii) The granny knot;

(iv) The square knot;

(v) The figure eight knot.

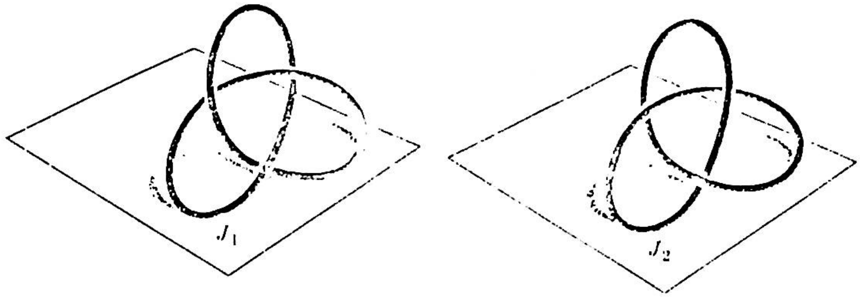


Figure - 1.3. (Right and left-and trefoil knots)

Knot Group

The *group of a knot* K is the fundamental group $\pi(\mathbb{R}^3 - K.)$

Theorem 1.18 ([87], p.139) The group of the trivial knot in $\mathbb{R}^3$ is infinite cyclic.

In fact the group of a knot K is infinite cyclic if and only if K is a trivial knot. (see [95], [100])

Let T be a totrus obtained by identifying the opposite edges of the unite square $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1\ 0 \leq y \leq 1\}$. Let L be a line through the origin in $\mathbb{R}^2$ with slop $\frac{m}{n}$, where $1 < m < n$, m and n are relatively prime integers. Under the above identification map $p : \mathbb{R}^2 \to T$, L reduces a simple closed curve K on T and spirals round Tm times while going round it n times the other way. K is a knot in $\mathbb{R}^3$ and is called a *torus knot of type (m,n).*

Theorem 1.19 ([87], p.140) *The group of a torus knot* (m, n) *has a presentation:*

Generators : $x,\ y;$

Relation : $x^m y^n.$

Remark

The chief problem of knot theory is to classify knots. Two equivalent knot have isomorphic knot group. However oriented equivalance classes are ot fully characterised by knot groups, *since there are nonequivalent knots having isomorphic knot groups. The right and left trefoil knots have isomorphic knot groups with generators* x_1, x_2, x_3, *relations* $x_2x_1 = x_3x_2$; $x_3x_2 = x_1x_3$. *(see [91], [99])*

Schoenflies [103] proved the following result:
Let $\mathcal{C}$ be a simple closed curve in the plane $\mathbb{R}^2$ and let h be a homeomorphism of $\mathcal{C}$ onto the unit circle S^1 in $\mathbb{R}^2$, then h may be extended to a homeomorphism h^{-1} of $\mathbb{R}^2$ onto itself.
This means that there are non-trival knot in a plane. The existence of non-trivial knots in $\mathbb{R}^3$ constitutes an obstruction in generalising results from $\mathbb{R}^2$ to $\mathbb{R}^3$.

For example the above result of Schenflies cannot be generalised by replacing $\mathbb{R}^2$ to $\mathbb{R}^3$. Infact if S is simple closed surface in $\mathbb{R}^3$ and h a homeomorphism of S onto the unit sphere S^2 in $\mathbb{R}^3$. The question whether h can be extended to a homeomorphism $\bar{h}$ of $\mathbb{R}^3$ onto itself , is to answered by Alexander [2] in the following way. He showed the answere is affirmative in the spacial cae when S is a finite polytop in $\mathbb{R}^3$ but negative in the general case. He gave a famous example, the Alexander horned sphere, for which the anwere is negative. The following figure shows the Alexander horned sphere;

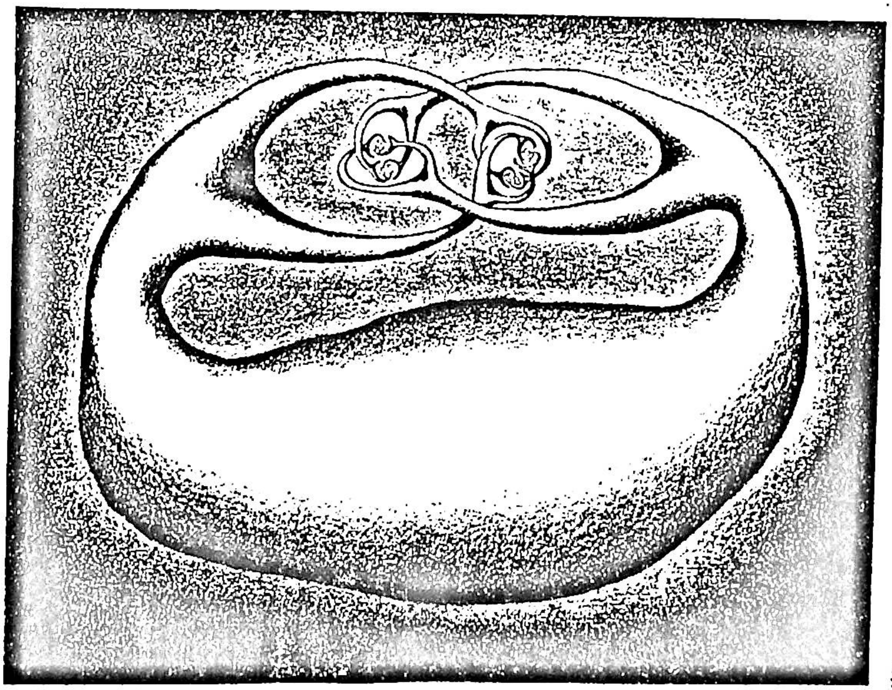


Figure -1.4.(Alexander horned sphere)

A *link* in $\mathbb{R}^3$ is the union of finite number of disjoint polygonal knots in $\mathbb{R}^3$.

*Equivalence, oriented equivalence,string isotopy*for liks are defined as in the case of kots.

A link L is string isotpic to links L_1 and L_2 such that L_1 and L_2 lie inside the disjoint balls in $\mathbb{R}^3$, then L is said to be *splittable.*

The trivial link with of two components

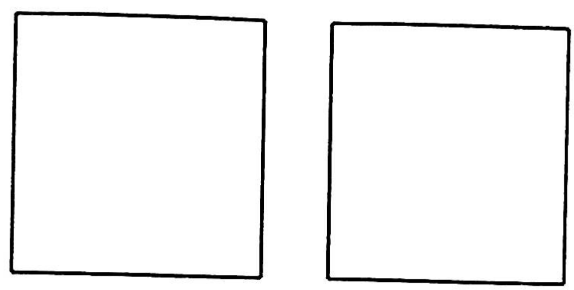


Figure - 1.5.

is splittable, but

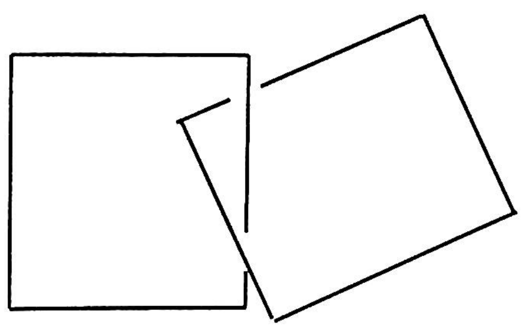


Figure - 1.6.

is unsplitted.

The *group of a link* is the fundamental group $\pi(\mathbb{R}^3 - L)$.

Theorem 1.20 ([91], p.128)

If L and L′ are links such that their group are not isomorphic then L and L′ are not equivalent.

Knopts links are related with Riemann surfaces through the following result.

There is a close relationship between the study of Riemann surfaces and that of algebraic curves: any irreducible plane algebraic curve admits a holomorphic parametric representation and the domain of dfinition of this representation is a compact Riemann surface. The relationship is given by the following two theorems. Before stating this theorem we recall the definition of Riemann surface.

Definition 1.21

A *Riemann surface* is a connected Housdorff topological space S together with an open covering $\{U_\alpha\}$ of S and the family of mappings $Z_\alpha : U_\alpha \to \mathbb{C}$ such that

(i) each $Z_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism of $U_\alpha$onto an open subset $Z_\alpha(U_\alpha)$ of $\mathbb{C}$;

(ii) if $U_\alpha \cap U_\beta \neq \phi$, then the function $Z_\beta \circ Z_\alpha^{-1} : Z_\alpha(U_\alpha \cap U_\beta) \to Z_\beta(U_\alpha \cap U_\beta)$ is biholomorphic , i.e., the function is itself and its inverse are both holomorphic. (U_α, Z_α) is called a *local holomorphic co-ordinate* and $\{U_\alpha\}$ is called a *holomorphic co-ordinate covering.*

The extended complex numbers $\Sigma = \mathbb{C} \cup \{\infty\}$ (one point compactification of complex numbers) is an example of a Riemann surface with the covering $\{U_0, U_\alpha\}$, $U_0 = \Sigma - \{\infty\} = \mathbb{C}$, $U_1 = \Sigma - \{0\}$ and the mappings $Z_0 : U_0 \to \mathbb{C}$ and $Z_1 : U_1 \to \mathbb{C}$ are given by $Z_0(z) = z$ and

$$Z_1(z) = \begin{cases} 0, & z = \infty, \\ \frac{1}{z}, & z \neq \infty. \end{cases}$$

Theorem 1.22 (Normalization Theorem) ([43], p.5)

For any irreducible algebric curve$C \subset P^2\mathbb{C}$, there exist a compact Riemann surface $\bar{C}$ and holomorphic mapping $\sigma : \hat{C} \to P^2C$ such that $\sigma(\bar{C}) = C$, and σ is injective in the inverse image of the set of smooth points of C.

The holomorphic mapping σ is called the *normalization* of C. Also any compact Riemann surface can be represented by an algebric curve:

Theorem 1.23

Any compact Riemann surface $\bar{C}$ can be obtained through the normalization of a certain plane algebraic curve C with atmost ordinary double points. i.e., there exist a holomorphic mapping $\sigma : \bar{C} \to P^2C$ such that $\sigma(\bar{C})$ is an algebric curve possesing atmost ordinary double points.

Their connection with 3- dimensional manifols is given in Alexander work:

Theorem 1.24 (Alexander [2])

Every 3-dimensional closed manifold may be generated by rotation about an axis of a Riemann surface with a fixed number of simple brance point ever crosse the axis or merges into other.

n-**braid or braid or n strings** is defined by the following :

(i) n points $P_1, P_2, \cdots P_n$ in $\mathbb{R}^3$ which have the same z-cordinate, $z = a$, say, and whose x-cordinate strictly increases as one goes from P_i to P_{i+1} along the line segment P_iP_{i+1}, for each i;

(ii) n points $Q_1, Q_2, \cdots, Q_n$ in $\mathbb{R}^3$ which have the same z-co-ordinate $z = b$, say, and whose x -co-ordinate strictly increases as one goes from Q_i to Q_{i+1} along the line segment Q_iQ_{i+1}, for each i;

(iii) for every i there is a finite polygonal path joining $P - i$ to $Q_{i\mu}$, where μ is a permutation of 1, 2, $\cdots$, n so that as one travels along this path from P_i to $Q_{i\mu}$ the z-co-ordinate strictly decreases;

(iv) $a > b$ and no two distinct path intersect.

In an n-braid , the path joining P_i to $Q_{i\mu}$ is called the *i-th string*, where $1 \leq i \leq n$.

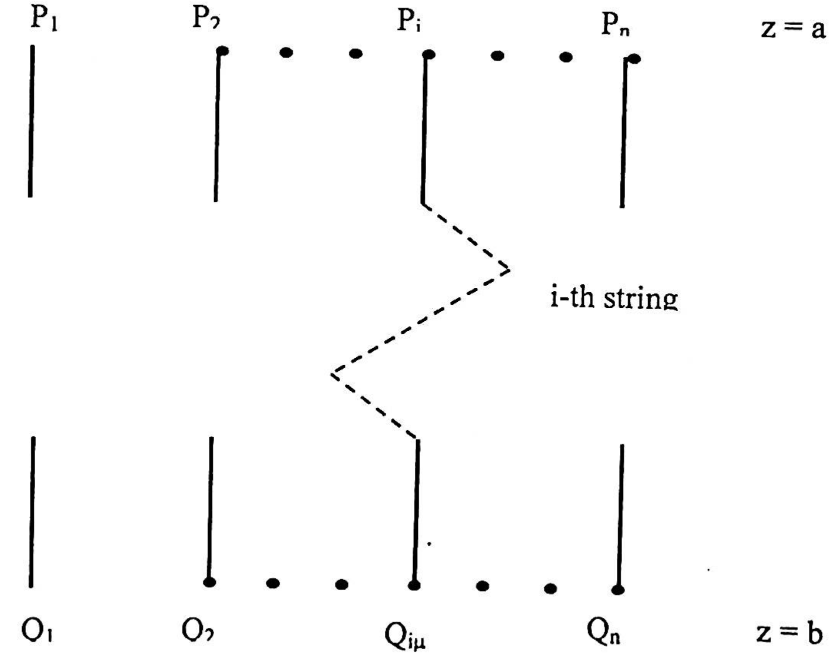


Figure - 1.7.

Two n-braids are said to be *equal* or *string isotopic* if and only if there is a continuous deformation of one braid onto the other n-braid which satisfy the above condition (i) - (iv) throughout the deformation and the distance between two vertices is never less than a fixed number $\delta > 0$.

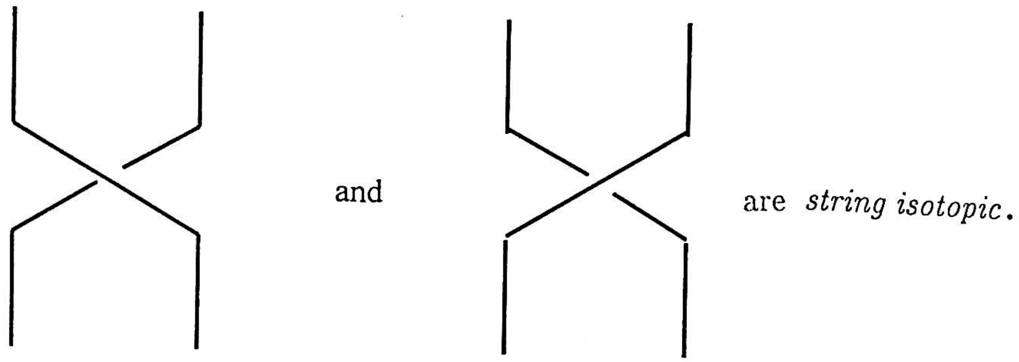


Figure - 1.8 .

We shall assume that all braids are given in the form that their projection onto xz-plane satisfy the following conditions:

(i) A vertex does not project onto a double point;

(ii) The only mutiple points of the projection are double point.

For a positive integer n, the set B_n of all n-braids can be turned into group taking string isoptopy as equivalence relation and the following operation as the product: If σ and σ' are n-braids, then their product $\sigma\sigma'$ is obtained by first constructing an n-braid σ'' which is string isotopic to σ' so that the initial points of the string of σ'' coincide with the end points of the string of σ and the placing σ'' under σ.

The product of the 3-braids

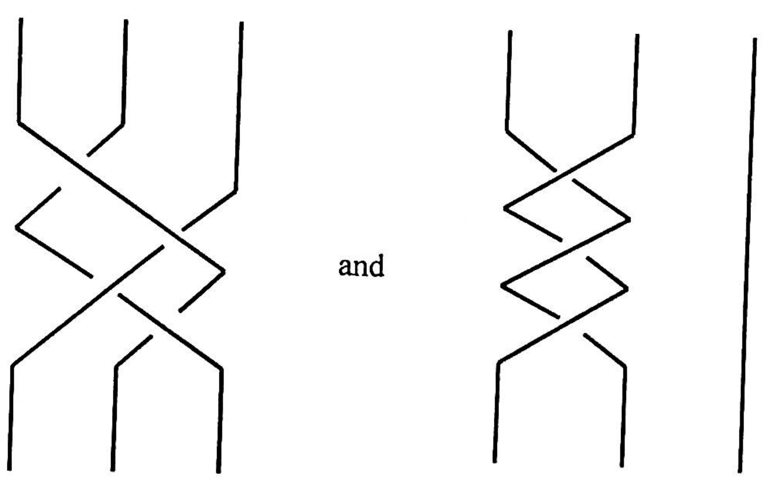


Figure - 1.9 .

is

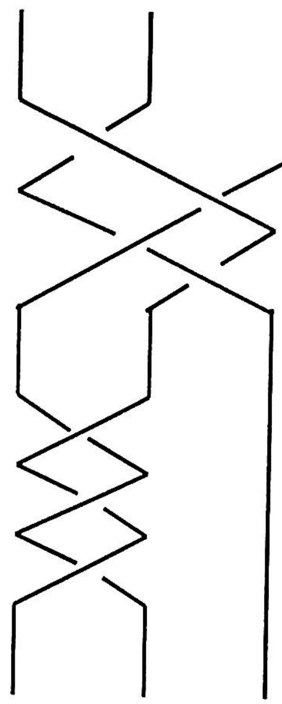


Figure - 1.10.

The n-braid

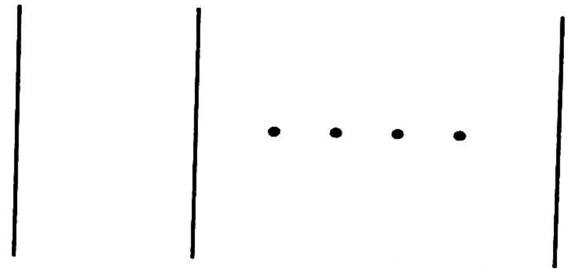
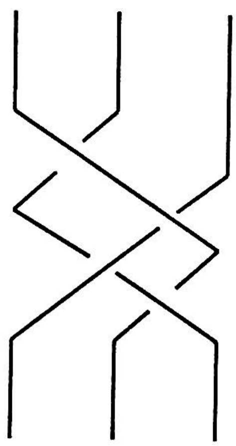


Figure 1.11.

is the identity element.

The invese of a braid σ is obtained in the following way.

Reflect σ in a line $z = a$, where a is a real number such that σ lies in the region $z < a$ of $\mathbb{R}^3$. The inverse of σ =



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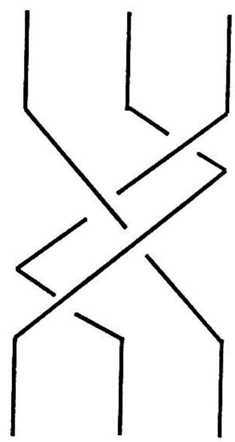

Figure - 1.12.

Theorem 1.25 ([91] p.92)

The group of n-braids is defined by
Generators: $\sigma_i\sigma_j = \sigma_j\sigma_i$, $\mid i - j \mid \geq 2$ *and*
$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $1 \leq i \leq n-1$

Link corresponding to a Braid

If σ is an n-braid with yhe i-th string joining P_i to Q_i, then the corresponding link $L(\sigma)$ is obtained from σ by identifying P_i to Q_i.

Theorem 1.26 (Artin and Birmann) ([91], p.119)([15])

Suppose that σ is an n-braid. The group $G(L(\sigma))$ of the link $L(\sigma)$ has a presentation of the form :

Generators : $x_1, \cdots, x_n$

Relations : $x = \bar{\sigma}(x_1), \cdots, x_n = \bar{\sigma}(x_n)$, *where $\bar{\sigma}$ denotes the automorphism of the fre group $F = < x_1, \cdots, x_n >$ determined by σ. Conversely the group of every link is given in this manner.*

Consequence: ([91], p.124)

If σ is an n-braid, then
$\sigma(L(\sigma)) \cong < x_1, \cdots, x_n; x_1 = \sigma(x_1), \cdots, x_i = \sigma(x_i) \cdots, x_n = \sigma(x_n) >$ for every i with $1 \leq i \leq n$.

Theorem 1.27 ([91], p.137)

Let the link $L(\sigma)$ has components and let $G(L(\sigma))$ be the group of a link σ, then

$$\left(\frac{G}{G'}\right) \cong \left[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_c^{\pm 1}\right].$$

1.5 Alexander Polynomial of a Knot

The elementary ideal of a knot group is perticularly easy to calculate. For suppose that the link associated with an n-braid σ is a knot $K(\sigma)$. Then by the theorem of Artin and Birman and its consequence, we have
$G(K(\sigma)) \cong < x_1, \cdots, x_n : \bar{\sigma}(x_1).x_1^{-1}, \cdots \bar{\sigma}(x_{n-1})x_{n-1}^{-1} >$.

Hence the Alexander matrix of this group is the $(n-1) \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \left(\frac{\partial(\bar{\sigma}(x_i).x_i^{-1})}{\partial x_j}\right)_{x_1 = \cdots x_n = t}$$

for $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

It follows that the Alexander matrix of $G(K(\sigma))$ is equivalent to the matrix

$$\left[\left(\frac{\partial(\bar{\sigma}(x_i).x_i^{-1})}{\partial x_j}\right)_{x_1 = \cdots = x_n = t} \quad 0_{n,1}\right].$$

Thus the elementary ideal of $G(K(\sigma))$ is the principal ideal generated by the element

$$det\left[\left(\frac{\partial(\bar{\sigma}(x_i).x_i^{-1})}{\partial x_j}\right) \begin{matrix} x_1 = \cdots = x_n = t \\ i, j \leq n-1 \end{matrix}\right]$$

in $\mathbb{Z}[t, t^{-1}]$.

A generator of this ideal is of the form $f(t)$ where $f(t)$ is a polynomial in t with integer coefficient and nopnzero positive constant term when $f(t) \neq 0$. Such a polynomial is called the *Alexander polynomial of the knot* $K(\sigma)$. the Alexander polynomial of knots are equivalent if and only if they generate the same ideal in the ring $\mathbb{Z}[t, t^{-1}]$, i.e., if and only if one is a multiple of the other by an invertible element in $\mathbb{Z}[t, t^{-1}]$.

It follows from the invariance of elementary ideals that the Alexander polynomial of equivalent knots are equivalent. Hence if two knots have nonequivalent Alexander polynomials, then they cannot be equivalent.

Alexander Polynomial of a Link

Let σ be an n-braid and $L(\sigma)$ be the corresponding link with more than one component. Then by the theorem of Artin and Birman and consequence

$$G(L(\sigma)) \cong < x_1, \cdots, x_n : \bar{(}x_1) = x_1, \cdots, \bar{\sigma}(x_n) = x_n >$$

Suppose that under the group homomorphism

$$\alpha\pi : F < x_1, \cdots x_n > \rightarrow \frac{G(L(\sigma)}{G'(L(\sigma))},$$

we have by theorem of Artin and Birman , that

$$\alpha\pi(x_j) = t_{\delta(j)} \text{ for } 1 \leq j \leq n,$$

where $1 \leq \delta(j) \leq c > 1$ with c being the number of components of the link $L(\sigma)$. Then the Alexander matrix of the group $G(L(\sigma))$ is the $(n-1) \times n$ matrix $A = (a_{ij})$,where

$$a_{ij} = \left(\frac{\partial(\bar{\sigma}(x_i).x_i^{-1})}{\partial x_j}\right)_{x_1 = \cdots x_n = t}$$

and $x_j = t_{\delta(j)}$. In order to calculate the elementary ideal $E(A)$ one has to evaluate the determinant Δ_J of the submatrix of A, which is obtained from A by deleting the the j-th column of A, for $j = 1, 2, \cdots, n$.

The elementary ideal $E(A)$ of A is the ideal generated in $[{t_1}^{pm}, \cdots, t_c^{\pm}]$ by the elements $\Delta_1, \Delta_2, \cdots, \Delta_n$.
If

$$(-1)^j \frac{\Delta_j}{t_{\delta(j)^{-1}}} = \Delta,$$

then $E(A)$ is the ideal generated in $[t_1^{\pm}, \cdots, t_c^{\pm}]$ by the elements

$$\Delta.(t_1 - 1), \Delta.(t_2 - 1), \cdots, \Delta(t_c - 1).$$

The element Δ of $[t_1^{pm}, \cdots, t_c^{\pm}]$ is called the *Alexander Polynomial of the link* $L(\sigma)$ with more than one component.

1.6. We conclude thi chapter by giving a brief account of Lens spaces referred in the first paragraph in connection with Fox's work related to this spaves.

Lens spaces (Siefart and Threlfall [106], Hilton and Wylie [57])

We follow Hilton and Wylie [53] in the following description of Lens spaces.
Let p, q, $p > q$ be two co-prime non-integers. the *Lens spcase* $L(p,q)$ is a 3-dimensional manifold dfined as follows. consider a close rgion in $\mathbb{R}^2$ bounded by regular p-sided polygon a_0^0, a_1^0, $\cdots$, a_{p-1}^0. We shall regard the subscript in a_i^0 as being an element in $\mathbb{Z}_p$, so that a_i^0 is defined for all. Join this region two points of $\mathbb{R}^3$, a_0^2 and a_q^2, one on each side of $\mathbb{R}^2$, to form a solid double pyramid P on the given polygonal base. The frontier of P is covered by triangles $a_i^0 a_{i+1}^0 a_j^2$ $j = 0\, or\, q$; $L(p,q)$ is obtained from P by identifying certain points on its frontier. There is a unique linear order preserving homeomorphism between $a_i^0 a_{i+1}^0 a_0^2$ and $a_{i+q}^0 a_{i+q+1}^0 a_q^2$; two points related undersuch a homeomorphism $0 \leq i \leq p-1$, are to be identified. The sloping faces of a pyramid are matches with those of the other after twist of $\frac{2q\pi}{P}$. It is clear that if $q \equiv q'(modp)$, then $L(p,q)$ is homeomorphic to $L(p,q')$.

Theorem 1.28 ([57] p.224)

A necessary condition that $L(p,q)$ is homeomorphic to $L(p,q')$ is that qq' be a quadratic residue modulo p.

J. H. C. Whitehead [113] has proved that this necessary condition is also sufficient for $L(p,\, q)$ and $L(p,\, q')$ to be of the same homotopy type. The homotopy classification of Lens spaces is therefore complete. $L(7,\, 1)$ and $L(7,\, 2)$ provide example of two non-homeomorphic 3-dimensional manifolds that are of the same homotopy type.

CHAPTER - 2

AUTOMORPHISMS OF FREE GROUPS

2.1 Introduction

The present chapter deals with automorphisms of free groups. The central result in this context is that of Birman [15] in which a necessary and suffiecint condition has been given for an endomorphism of a free group to be an automorohism. Topping's [110] result have been briefly described. We have determined the automorphism group of a free group of rank 2 and have obtained here a necessary and sufficient condition for an endomorphsm of a finitely presented group to be an automorphism. Fox derivative have been extensively used.

2.2 Characterisation of an Automorphism of a Free Group of a Finite Rank.

J. S. Birman [14], has characterised the automorphisms of a free group with a finite basis with the help of Fox derivatives. Jacobian matrices play a fundamental role in her characterisation. We start with her fundamental result.

Theorem 2.1

Let F be a free group with basis $\{x_1, \cdots, x_n\}$.If $u_1, \cdots, u_n$ are elements of F,then the endomorphism of F is defined by $x_i \rightarrow u_i$ is an automorphoism if and only if the Fox Jacobian $(\frac{\partial u_i}{\partial x_j})$ is a unit, i.e., if $\left(\frac{\partial u_i}{\partial x_j}\right)$ with the entries in group ring $\mathbb{Z}F$, F' has a left inverse.

This is a consequence of Birman's Inverse Function Theorem and its consequence. Topping [110] had established independently certain cases of this result. It has been established thatIf φ is a free metabelian group with $x_1, \cdots, x_n$ and $u_1, \cdots, u_n$ are elements of φ, then the map defined by $x_i \rightarrow u_i$ is an automorphism if and only if the Fox Jacobian evaluated now over the integral group ring of $\frac{\varphi}{[\varphi,\varphi]}$, has determinant $\pm g$ for some $g \in \frac{\varphi}{[\varphi,\varphi]}$.

For any basis x (not necessarily finite) and x in X, let α_x be the endomorphism carying x into x^{-1} and leaving $X - \{x\}$ fixed. For any x, y in X, where $x \neq y$, let β_{xy} be the endopmrphism carrying x into xy and leaving $X - \{x\}$ fixed. In both cases it is easy to see that the image of X is an another basis, whence α_x and β_{xy} are automorphisms. For finite X we prove this using Birman's Theorem:

Theorem 2.2

Let X be a set with a finite basis of a free group F. Then each α_x and β_{xy} are automorphisms of F.

Proof

Let $X = \{x_1, \cdots, x_n\}$ and let F be free on X. We have to show that α_x and β_{xy} are automorphisms.By Birman [14] it suffices to show that $\left|\frac{\partial(\alpha_x(x_i))}{\partial x_j}\right|$ and $\left|\frac{\partial(\beta_{xy}(x_i))}{\partial x_j}\right|$ are units in $\psi(\mathbb{Z}F)$.

We have for every $x \in X$

$$\left(\frac{\partial(\alpha_x(x_i))}{\partial x_j}\right) = \begin{pmatrix} \frac{\partial x_1^{-1}}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & \frac{\partial x_2^{-1}}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial x_n} & \frac{\partial x_2}{\partial x_n} & \cdots & \frac{\partial x_n^{-1}}{\partial x_n} \end{pmatrix}$$

$$= \begin{pmatrix} -x_1^{-1} & 0 & \cdots & 0 \\ 0 & -x_2^{-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -x_n^{-1} \end{pmatrix}.$$

which has an inverse

$$\begin{pmatrix} -x_1 & 0 & \cdots & 0 \\ 0 & -x_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -x_n \end{pmatrix}.$$

Again

$$\left(\frac{\partial(\beta_{x_k y_l}(x_i))}{\partial x_j}\right) = \begin{pmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \cdots & \frac{\partial x_k x_l}{\partial x_1} & \cdots & \frac{\partial x_l}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & \frac{\partial x_2}{\partial x_2} & \cdots & \frac{\partial x_k x_l}{\partial x_2} & \cdots & \frac{\partial x_l}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial x_k} & \frac{\partial x_2}{\partial x_k} & \cdots & \frac{\partial x_k x_l}{\partial x_k} & \cdots & \frac{\partial x_l}{\partial x_k} & \cdots & \frac{\partial x_n}{\partial x_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial x_l} & \frac{\partial x_2}{\partial x_l} & \cdots & \frac{\partial x_k x_l}{\partial x_l} & \cdots & \frac{\partial x_l}{\partial x_l} & \cdots & \frac{\partial x_n}{\partial x_l} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_1}{\partial x_n} & \frac{\partial x_2}{\partial x_n} & \cdots & \frac{\partial x_k x_l}{\partial x_n} & \cdots & \frac{\partial x_l}{\partial x_n} & \cdots & \frac{\partial x_n}{\partial x_n} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & x_l & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}.$$

Which has a left inverse

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & x_l^{-1} & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -x_l^{-1} & \cdots & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}.$$

Therefore α_x and β_{xy} are automorphisms in F for all x in F.

Theorem 2.3

Let $X = \{x_1, \cdots, x_n\}$ be a finite set and let F be free on X. Then α_{x_i}, $\beta_{x_i x_j}$ generate $AutF$.

The proof is outlined in [71], p.22.

This Theorem is a particular case of the following general result.

Theorem 2.4 ([71] Prop. 4.1, p.23)

Let F be free with basis X, and let $Aut_f(F)$ be the subgroup of $Aut(F)$ generated by the elementary Nielson transformation α_x and β_{xy}. Then $Aut_f(F)$ is dense in $Aut(F)$ in the sense that if $u_1, \cdots, u_n$ are elements of F and $\alpha \in Aut(F)$ there exist $\beta \in Aut_f(F)$ such that $u_1\alpha, \cdots, u_n\alpha = u_n\beta$. In perticular if F has finite rank, then $Aut_f(F) = Aut(F)$.

We shall now determine $AutF_2$ by disovering relations satisfying α_x, α_y, β_{xy}, and β_{yx}.

We write $\alpha_x(x,y)$ for $(\alpha_x(x), \alpha_x(y))$, $\beta_{xy}(x,y)$ for $(\beta_{xy}(x), \beta_{xy}(y))$, etc.
Then

$$\left.\begin{aligned} \alpha_x(x,\, y) &= (x^{-1},\, y) \\ \alpha_y(x,\, y) &= (x,\, y^{-1}) \\ \beta_{xy}(x,\, y) &= (xy,\, y) \\ \beta_{yx}(x,\, y) &= (x,\, yx) \end{aligned}\right\}. \qquad (A)$$

Using (A),we have

$$\alpha_x^2(x,y) = \alpha_(x^{-1},y) = (x,y),$$

so that

$$\alpha_x^2 = 1\,.$$

And

$$\alpha_y^2(x,y) = \alpha_(x,y^{-1}) = (x,y).$$

Hence

$$\alpha_y^2 = 1\,.$$

$$\therefore\ \alpha_x^2 = \alpha_y^2 = 1\,. \qquad (2.1)$$

$$\alpha_x\beta_{xy}(x,y) = \alpha(xy,y) = (x^{-1}y,y),$$

and

$$\beta_{xy}\alpha_x(x,y) = \beta_{x\,y}(x^{-1},y) = (y^{-1}x^{-1},y).$$

Thus,

$$\alpha_x\beta_{xy} \neq \beta_{xy}\alpha_x\,. \tag{2.2}$$

$$(\alpha_x\beta_{yx})^2(x,y) = \alpha_x\beta_{yx}\alpha_x\beta_{yx}(x,y) = \alpha_x\beta_{yx}\alpha_x(x,yx)$$
$$= \alpha_x\beta_{yx}(x^{-1},yx^{-1}) = \alpha_x(x^{-1},y) = (x,y)\,.$$

Hence

$$(\alpha_x\beta_{yx})^2 = 1\,. \tag{2.3}$$

$$\begin{aligned}(\alpha_y\beta_{xy})^2(x,y) &= \alpha_y\beta_{xy}\alpha_y(xy,y)\\ &= \alpha_y\beta_{xy}(xy^{-1},y^{-1})\\ &= \alpha_y(xyy^{-1},y^{-1}) = \alpha_y(x,y^{-1}) = (x,y),\end{aligned}$$

so that

$$(\alpha_y\beta_{xy})^2 = 1\,. \tag{2.4}$$

$$\alpha_y\beta_{yx}(x,y) = \alpha_y(x,yx) = (x,y^{-1}x),$$

$$\beta_{yx}\alpha_y(x,y) = \beta_{yx}(x,y^{-1}) = (x,x^{-1}y^{-1})),$$

$$\therefore\ \alpha_y\beta_{yx} \neq \beta_{yx}\alpha_y\,. \tag{2.5}$$

$$\alpha_x\beta_{yx}(x,y) = \alpha_x(x,yx) = (x^{-1},yx^{-1}),$$

$$\beta_{yx}\alpha(x,y) = \beta_{yx}(x^{-1},y) = (x^{-1},yx)\,.$$

$$\therefore\ \alpha_x\beta_{yx}(x,y) \neq \beta_{yx}\alpha(x,y)\,. \tag{2.6}$$

$$\alpha_y\beta_{xy}(x,y) = \alpha_y(xy,y) = (xy^{-1},y^{-1})$$

and

$$\beta_{xy}\alpha_y(x,\, y) \;=\; \beta_{xy}(x,\, y^{-1}) \;=\; (xy,\, y^{-1})\cdot$$

$$\therefore\ \alpha_y\beta_{xy} \neq \beta_{xy}\alpha_y. \qquad (2.7)$$

Therefore

$$AutF_2 = <\ \alpha_x,\, \alpha_y,\, \beta_{xy},\, \beta_{yx},\ |\ \alpha_x^2 = \alpha_y^2 = (\alpha_x\beta_{yx})^2 = (\alpha_y\beta_{xy})^2 = 1\ >\cdot$$

Let

$$\alpha_x = z_1,\, \alpha_y = z_2,\, \beta_{xy} = z_3,\, \beta_{yx} = z_4,$$

then

$$AutF_2 = <\ z_1,\, z_2,\, z_3,\, z_4\ |\ z_1^2 = z_2^2 = (z_1z_4)^2 = (z_2z_3)^2\ >\cdot$$

We may obtain from Birman's Theorem a necessary and sufficient condition for an automorphism of a finitely pesented group to be an automorphism. This is expressed in the following theorem.

Theorem 2.5

Let $G = \frac{F}{R}$, where F is a free group with basis $\{x_1, \cdots, x_m\}$ and R is the normal closure of $\{r_1, \cdots, r_n\}$. Let π be the ring homomorphism induced by the canonical homomorphism $F \to G$.An endomorphism f of G is an automorphism if $\| \frac{\partial f(x_i)}{\partial x_j} \|$ has a unit inverse and $\pi(\bar{f}(r_k')) = 1$,where $\bar{f} : F \to F$ is the homomorphism which induces f, $r_k' = r_k(\bar{f}(x_1), \cdots, \bar{f}(x_m))$.

Proof

If $\| \frac{\partial f(x_i)}{\partial x_j} \|$ has a left inverse, by Birman's Theorem,then f is an automorphism of F and so has an inverse. Let r_k' be $r_k(\bar{f}(x_1), \cdots, \bar{f}(x_m))$. Then G is also given by

$$G = <\ \bar{f}(x_1), \cdots, \bar{f}(x_m)\ |\ r_1', \cdots r_n'\ > .$$

Hence if for each k, $\pi\bar{f}^{-1}(r_k') = 1$, *i.e.*, $\bar{f}^{-1}(r_k') \in R$, then $\bar{f}^{-1}$ induces an endomophism g and g must be the inverse of f. Thus f is an automorphiosm of G.

CHAPTER-3

THE FREIHEITSSATZ

3.1 Introduction

In this chapter we shall give a new proof of a very important result in the theory of a groups with a single defininig relation. This is the famous Freiheitssatz proved by Magnus [83] to establish the solvability of word problem for this class of groups. We have briefly outlined an account of works by other mathematicians about this theorem, its generalizations and other proofs. Our method of proof is based on the use of Fox derivatives.We have used Majumdar ideas.

3.2 Groups with a Single Defininng Relations

Groups with a single defining relations have been studied extensively by many mathematicians. The importance of this class of groups lies mainly in the fact that the fundamental groups of compact connected 2-manifolds, i.e., surfaces, belong to this class. We recall that such surfaces are (i) spheres or connected sums of a finite number of tori (if orientable) and (ii) connected sum of either a projective plane or a Klein bottle and connected sum of orientable surfaces(otherwise).

The fundamental groups of the surfaces are obtained from their canonical representations by plane figures.

Thus the fundamental groups of some of these are given below:

(a) The connected sum of n tori

$$< a_1, b_1, \cdots, a_n, b_n | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} >,$$

(b) The connected sum of n projective planes

$$< a_1, \cdots, a_n | a_1^2 \cdots a_n^2 > \cdot$$

Dehn ([26], [27])showed the solvablity of the word problem, the conjugacy problem and the isomorphism problem

for fundamental groups of 2-manifolds. Magnus proved that the word problem for groups with a single defining relation is solvable. The fundamental tool used by Magnus in this proof of this result was a theorem (namely, the Freiheitssatz) which is very significant and important by itself. This theorem and the technique of proof have been very useful in proving other resuls for single-relator groups.

The Freiheitssatz thus occupies a central place in the theory of groups with a single defining relation.

We state the Freiheitssatz below.

Theorem 3.1 (The Freiheitssatz)[85] ,[71] and [83])

Let X be a non-empty set and let r be a cyclically reduced word in the elements of X such that r involves $x \in X$. Then the subgroup of $G =< x : r >$ which is generated by them; in other words, every non-trivial relation of G must involves x.

Sharper result have been obtained in the "Spelling Theorem" of Newman [93] for the case where the defining relation is a proper power. Gurevich [51] has strengthened Newmann's results, and Schupp [104] has proved a theorem which strengthens both the Freiheitssatz for one relation in general and Newmann's results in the torsion case. Gurevich[51] announced this same result for the case where the relation is a proper power. He later informed Schupp [105] that his methods of proof would also yield Schupp's result even when the relation is not a proper power. Lyndon [69] proved a theorem which extends Freiheitssatz from free groups that are free products of groups isomorphic to subgroups of the additive group of real numbers, and at the same time, Lyndon's own Identity Theorem [68]. Lyndon obtained his result in an attempt to translate Magnus method of proof of the Freiheitssatz into the language of combinatonial geometry, using a result of Van Kampen[111]. Schupp's proof too is geometric and makes use of Lyndon's "maximum modulus " approach to Freiheitssatz. Majumdar [79] used Fox derivatives in this context.

We shall now present our own proof of the Freiheitssatz (Th. 3.1) using Fox's free partial derivatives. In this chapter all the derivatives used will be left derivatives. we shall write $\frac{\partial w}{\partial x}$ for $(\frac{\partial w}{\partial x})_L$.

3.3 Proof of the Freiheitssatz

(A). Let $w = s^{-1}r^e s$. Then, w does not involve x only if the non-trivial occurences of x in r^e are cancelled by those in both s and s^{-1}. But this implies that r^e and hence r is of the form $t^{-1}wt$. This contradicts that r is cyclically reduced. Hence w involves x.

(B) Next let $n \geq 2$. We prove that w involves x by showing that $\frac{dw}{dx} \neq 0$. If possible let $\frac{dw}{dx} = 0$.

The case when at least one s_i involves x.

Since $w = (s_1^{-1}r_1^{e_1}s_1)\cdots(s_n^{-1}r_n^{e_n}s_n)$, it is required to prove that $\frac{\partial w}{\partial x} = 0$ leads to a contradictions. Without loss of generality we may assume s_1 involves x. Hence s_1^{-1} involves x, then $\frac{\partial s_1^{-1}}{\partial x} \neq 0$. This impies that $\frac{\partial s_1^{-1}}{\partial x}$ is a non-trivial sum of terms of the form s_1' for some subword s_1' of s_1^{-1}. All of these terms cannot cancel with one another; for, otherwise $\frac{\partial s_1^{-1}}{\partial x}$ will be 0. Let s_1' be a surviving term in $\frac{\partial s_1^{-1}}{\partial x}$ (i.e., one which does not cancel with any other term in $\frac{\partial s_1^{-1}}{\partial x}$.)

Since $\frac{\partial w}{\partial x} = 0$, s_1' cancels with some summand in the term $s_1^{-1}e_1 r^{\frac{e_1-1}{2}}\frac{\partial r}{\partial x}$ or the term $s_1^{-1}r_1^e\frac{\partial s_1}{\partial x}$.

Case I

Suppose s_1' cancels with some term in $s_1^{-1}r^{e_1}\frac{\partial s_1}{\partial x}$, say, $s_1^{-1}r^{e_1}s_1''$ or ($s_1^{-1}r^{e_1}s_1''x^{-1}$), s_1'', being subword of s_1 . Then , $s_1' = s_1^{-1}r^{e_1}s_1''$ or $s_1'' = s_1^{-1}r^{e_1}s_1''x^{-1}$. Since s_1^{-1} does not cancel even partially on the right with r^{e_1}, and s_1, and hence $s_1''x^{-1}$ and s_1'', do not cancel even partially on the left with r^{e_1}, $s_1/$ and hence s_1^{-1} contains r^{e_1} as a factor- a contradiction.

Case II

Suppose s_1^{-1} cancels with some term in

$$s_1^{-1}e_1r^{\frac{e_1-1}{2}}\frac{\partial r}{\partial x},$$

say, $s_1^{-1}e_1r^{\frac{e_1-1}{2}}r'$ for some summand r' of r (i.e. $r = r'r''$).

(1) Let $e_1 = -1$. Then $s_1' = s_1^{-1}r^{-1}r'$. If $r = r'$ then $s_1' = s_1^{-1}$. So the definition of Fox derivatives yields,

$$s_1^{-1} = s_1''x^{-1},\ r^{-1} = (r_1''x^{-1})^{-1} = xr_1''^{-1}$$

for some s_1'' and r_1''. This is a contradiction to condition (A). So $r \neq r'$, and thus, $r^{-1}r' \neq 1$. So, $r^{-1}r'$ or $(r^{-1}r'x^{-1})$ is a proper subword of r^{-1}. So $s_1^{-1} = s_1\prime(r^{-1}r')^{-1}$ and $r^{-1} = (r^{-1}r')r'^{-1}$. This contradicts the condition (A).

(2) Let $e_1 = 1$. Then $s_1' = s_1^{-1}r'$, for some subword r' of r, where $r = r'r''$. If $s_1^{-1} = s_1's_1''$, then $1 = s_1''r'$, i.e., $s_1'' = r'^{-1}$. Hence $s_1^{-1} = s_1'r'^{-1}$.

If $r' = 1$, $s_1^{-1} = s_1'$. But in this case $r^{-1} = xr''$ (first term in the derivative $\frac{\partial r^{-1}}{\partial x}$), $s_1^{-1} = s_1' = s_1^{-1}x^{-1}$ (a term in the derivative $\frac{\partial s_1^{-1}}{\partial x}$), for some s_1. We have a contradiction to the condition (A). Hence s_1' can not cancel with any term in $s_1^{-1}e_1r^{\frac{e_1-1}{2}}\frac{\partial r}{\partial x}$. Therefore, s_1' does not cancel with any term in

$$\frac{\partial}{\partial x}(s_1^{-1}r_1^e s_1).$$

If $-s_1^{-1}s_1'$(or $s_1^{-1}s_1'x^{-1}$) cancels with a term not contained in $\frac{\partial}{\partial x}(s_1^{-1}r_1x^{-1})$, then

$$s_1^{-1}s_1'(\text{or } s_1^{-1}s_1'x^{-1}) = (s_1^{-1}r_1x^{-1}) \times \text{some other terms}.$$

. Again, this implies that s' and hence s_1 contains r^{-1} as a factor, a contradiction. Thus we may conclude that no s_i involves x.

(B) The case when no s_i involves x.

(α) First let $n = 2$, then $w = (s_1^{-1}r^{e_1}s_1)(s_2^{-1}r^{e_2}s_2)$, and let $s_1,\ s_2$ be free of x. If possible let $\frac{\partial w}{\partial x} = 0$.

Case I ($e_1 = e_2$)

Suppose $e_1 = e_2$. Hence $r^{e_1} = r^{e_2} = r_1r_2r_3$, say, where $r_1,\ r_3$ do not involve x and r_2 involves x and begins and ends with x or x^{-1}.

(1) Suppose r_2 starts with x.

Then $s_1^{-1}r_1$ (the left most term in $\frac{\partial w}{\partial x}$) can not cancel with any summand in $\frac{\partial}{\partial x}(s_1^{-1}r^{e_1}s_1)$. It must cancel with some summand in $(s_1^{-1}r^{e_1}s_1)\frac{\partial w}{\partial x}(s_2^{-1}r^{e_2}s_2)$, (since $e_1 = e_2$), say

$$(s_1^{-1}r^{e_1}s_1)(s_2^{-1}r_1r_2'x^{-1}),$$

where

$$r_2 = r_2'x^{-1}r_2'' \qquad (3.1),$$

say, since $s_1^{-1}r_1$ and other term must have opposite signs.

Then $$s_1^{-1}r_1 = (s_1^{-1}r_1r_2r_3s_1)(s_2^{-1}r_1r_2'x^{-1})$$

$$\Rightarrow\ r_2r_3s_1s_2^{-1}r_1r_2'x^{-1} = 1 \qquad (3.2)$$

$$\Rightarrow\ w = (s_1^{-1}r_1r_2r_3s_1)(s_2^{-1}r_1r_2'x^{-1}r_2''r_3s_2)$$

$$= s_1^{-1}r_1r_2''r_3s_2). \quad \text{by (3.2)} \qquad (3.3)$$

r'' involves x, otherwise, r_3 will be replaced by $r_2''r_3$. Also s_1, r_1, r_3, s_2 does not involve x. (3.4)

(a) If $r_2'' \neq 1$, the right hand side involves x by (3.3) and so, $\frac{\partial w}{\partial x} \neq 0$, a contradiction.

(b) If $r_2'' = 1$, then $r_2' x^{-1} = r_2$ by (3.1) and so, from (3.2), we have

$r_2(r_3 s_1 s_2^{-1} r_1) r_2 = 1$, so that

$r_3 s_1 s_2^{-1} r_1 = r_2^{-2}$.

The left hand side is free of x and so, the right hand side is free of x. (3.5)

But $\frac{\partial}{\partial x}(r_2^{-2}) = \frac{\partial r_2}{\partial x}(r_2+1)r_2^{-2} \neq 0$, since $r_2 + 1 \neq 0$ and $\frac{\partial r_2}{\partial x} \neq 0$, since r_2 involves x and since has no zero divisors. Hence r_2^{-2} involves x, a contradiction to (3.5).

[Case I $e_1 = e_2$]

(2)**Suppose r_2 starts with x^{-1}**

In this case arguing as before we see that $s^{-1} r_1 x^{-1}$(the left most term with oppsite sign) cancels with

$$(s_1^{-1} r^{e_1} s_1)(s_2^{-1} r_1 r_2'),$$

a summand of

$$(s_1^{-1} r^{e_1} s_1) \frac{\partial}{\partial x}(s_2^{-1} r^{e_1} s_1)$$

(since $e_1 = e_2$), for some r_2' with

$$r_2 = r_2' x r_2'' . \qquad (3.6)$$

Therefore,

$$s_1^{-1} r_1 x^{-1} = (s_1^{-1} r_1 r_2 r_3 s_1)(s_2^{-1} r_1 r_2').$$

Thus

$$x^{-1} = (r_2 r_3 s_1)(s_2^{-1} r_1 r_2').$$

i.e.,

$$r_2 r_3 s_1 s_2^{-1} r_1 r_2' x = 1 \ .$$

and so,from (3.6)

$$r_2 r_3 s_1 s_2^{-1} r_2 r_2''^{-1} = 1 \tag{3.7}$$

From(3.2) we see that

$$r_3 s_1 s_2^{-1} = r_2^{-1} r_2'' r_2^{-1}. \tag{3.8}$$

(*c*) If $r_2'' = 1$, then from (3.8), we have

$$r_3 s_1 s_2^{-1} = r_2^{-2}. \tag{3.9}$$

The left hand side is free of x. But $\frac{\partial}{\partial x}(r_2^{-2}) = -\frac{\partial r_2}{\partial x}(r_2 + 1)r_2^{-2} \neq 0$, since $r_2 + 1 \neq 0$ and hence $\frac{\partial r_2}{\partial x} \neq 0$ (since r_2 involves x). Hence r_2^{-2} involves x, a contradiction (from (3.9)).

(*d*) If $r_2'' \neq 1$, then the right hand side involves x since the existing x's in two r_2^{-1}'s can not be cancelled by the x's in r_2'', r_2'' being subword of r_2. Since the left hand side is free of x, we have a contradiction.

Case II $(e_1 = -e_2)$

Suppose $e_1 = -e_2$, then $r^{e_1} = r_1 r_2 r_3 = r^{-e_2}$ as before.

Therefore,

$$w = (s_1^{-1}r_1r_2r_3s_1)(s_2^{-1}r_3^{-1}r_2^{-1}r_1^{-1}s_1). \qquad (3.10)$$

(1) **Suppose** r_2 **starts with** x, then $s_1^{-1}r_1$ cancels with

$$(s_1^{-1}r_1r_2r_3s_1)(s_2^{-1}r_3^{-1}r_2'x^{-1}),$$

a summand of

$$(s_1^{-1}r_1r_2r_3s_1)\frac{\partial}{\partial x}(s_2^{-1}r_3^{-1}r_2^{-1}r_1^{-1}x^{-1}),$$

for some r_1'', r_2'' .

where

$$r_2^{-1} = r_2'x^{-1}r_2''. \qquad (3.11).$$

So $s_1^{-1}r_1 = (s_1^{-1}r_1r_2r_3s_1)(s_2^{-1}r_3^{-1}r_2'x^{-1})$.

$$\Rightarrow \quad r_2r_3s_1s_2^{-1}r_3^{-1}r_2'x^{-1} = 1 \qquad (3.12)$$

$$\Rightarrow \quad r_2r_3s_1s_2^{-1}r_3^{-1}r_2^{-1}r_2''^{-1} = 1$$

$$\Rightarrow \quad w = s_1^{-1}r_1r_2''r_1^{-1}s_2 \qquad (3.13),$$

using(3.12).

(e) If $r_2'' = 1$, then from (B''), $r_2^{-1} = r_2'x^{-1}$, and (3.11) and (3.12) yields $s_1s_2^{-1} = r_2^{-1}r_3^{-1}r_3r_2 = 1$.

(*f*) If $r_2'' \neq 1$, then from (D''), the right hand side involves x and then $\frac{\partial w}{\partial x} \neq 0$, a contradiction.

(2)Suppose r_2 starts with x^{-1}.

Then $s_1^{-1}r_1x^{-1}$ cancels with

$$(s_1^{-1}r_1r_2r_3s_1)(s_2^{-1}r_3^{-1}r_2'),$$

a summand of

$$(s_1^{-1}r_1r_2r_3s_1)\frac{\partial}{\partial x}(s_2^{-1}r_3^{-1}r_2^{-1}r_1^{-1}s_1),$$

where

$$r_2^{-1} = r_2'xr_2'' \tag{3.14}$$

for some r_2', r_2'' cancelling terms must have opposite sign.

$$\begin{aligned} \text{So} \qquad & s_1^{-1}r_1x^{-1} = (s_1^{-1}r_1r_2r_3s_1)(s_2^{-1}r_3^{-1}r_2') \\ \Rightarrow\ & x^{-1} = r_2r_3s_1s_2^{-1}r_3^{-1}r_2' \\ \Rightarrow\ & r_2r_3s_1s_2^{-1}r_3^{-1}r_2'x = 1 \qquad (3.15) \\ \Rightarrow\ & r_2r_3s_1s_2^{-1}r_3^{-1}r_2^{-1}r_2'' = 1. \text{ using } (3.14) \end{aligned}$$

$$\begin{aligned} \text{Then } w &= (s_1^{-1}r_1r_2r_3s_1)(s_2^{-1}r_3^{-1}r_2^{-1}r_1^{-1}s_2) \\ &= s_1^{-1}r_1r_2''r_1^{-1}s_2). \qquad (3.16) \end{aligned}$$

(g) If $r_2'' = 1$, then from (C''') $w = 1$, contradiction since w is a non-trivial relations.

(h) If $r_2'' \neq 1$. As before r_2'' involves x (otherwise we may replace r_1^{-1} by $r_2''r_1^{-1}$ i.e., replace r_1 by $r_1r_2''^{-1}$). So w involves x.

Therefore, $\frac{\partial w}{\partial x} \neq 0$, a contradiction.

Thus when the number of conjugate factors $(s_i^{-1}r^{e_1}s_i)$ in w is 1 or 2, w involves x.

Now we prove the theorem by induction on n. Suppose $n \geq 2$ and this is true for n. We prove it to be true for $n+1$.

$$\text{Let} \qquad w = \prod_{i=1}^{n+1}(s_i^{-1}r^{e_i}s_i)$$
$$= (s_1^{-1}r^{e_1}s_1)\prod_{k=2}^{n}(s_k^{-1}r^{e_k}s_k)(s_{n+1}^{-1}r^{e_{n+1}}s_{n+1}).$$

Case I ($e_1 = e_{n+1}$)

Suppose $e_1 = e_{n+1}$, then $r^{e_1} = e^{e_{n+1}} = r_1r_2r_3$, where r_1, r_3 are free of x and r_2 involves x and starts with x or x^{-1}.

(1) Suppose r_2 starts with x.

Then the first term in $\frac{\partial w}{\partial x} = s_1^{-1}r_1$. By induction it can not cancel with any term in

$$\frac{\partial}{\partial x}(\prod_{j=2}^{n}s_j^{-1}r^{e_j}s_j).$$

It must cancel with

$$\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)s_{n+1}^{-1}r_1r_2'x^{-1},$$

a summand of

$$\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)\frac{\partial}{\partial x}(s_{n+1}^{-1}r^{e_1}s_{n+1})$$

(since $e_1 = e_{n+1}$), where

$$r_2 = r_2'x^{-1}r_2'' \qquad (3.17),$$

since $s_1^{-1}r_1$ and other terms must have opposite signs.

Therefore

$$s_1^{-1}r_1 = (s_1^{-1}r_1r_2r_3s_1)\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_1r_2'x^{-1})$$

$$\Rightarrow \; r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_1r_2'x^{-1}) = 1 \qquad (3.18)$$

$$\Rightarrow w = s_1^{-1}r_1r_2''s_{n+1}^{-1} \qquad (3.19)$$

using (3.17) and (3.18)

(*i*) If $r_2'' = 1$, then from (3.19)

$$w = s_1^{-1} r_1 r_3 s_{n+1},$$

which contradicts the definition of w as

$$w = \prod_{i=1}^{n+1} (s_i^{-1} r^{e_i} s_i),$$

since r_1, r_3 is a proper subword of r^{e_i}. Also if $r_1 = r_3$, then $w = 1$, i.e., w is trivial, a contradiction.

(*j*) If $r_2'' \neq 1$, then from ((3.19)), the right hand side involves x and then $\frac{\partial w}{\partial x} \neq 0$, a contradiction.

(2) Suppose r_2 starts with x^{-1}.

Then the first term in $\frac{\partial w}{\partial x} = -s_1^{-1} r_1^{-1} x^{-1}$. By induction it can not cancel with any term in

$$\frac{\partial}{\partial x}\left(\prod_{j=2}^{n} s_j^{-1} r^{e_j} s_j\right).$$

It must cancel with

$$\prod_{j=2}^{n} \left(s_j^{-1} r^{e_j} s_j\right) s_{n+1}^{-1} r_1 r_2',$$

a summand of

$$\prod_{j=2}^{n} \left(s_j^{-1} r^{e_j} s_j\right) \frac{\partial}{\partial x} (s_{n+1}^{-1} r^{e_1} s_{n+1})$$

(since $e_1 = e_{n+1}$), where

$$r_2 = r_2' x r_2'' \qquad (3.20),$$

for some r_2' and r_2''.

Therefore $$s_1^{-1}r_1x^{-1} = (s_1^{-1}r^{e_1}s_1)\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_1r_2')$$

$$= s_1^{-1}r_1r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_1r_2').$$

Hence $$x^{-1} = r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_1r_2')$$

$$\Rightarrow \quad r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_1r_2'x = 1$$

$$\Rightarrow \quad r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_1r_2'r_2''^{-1} = 1 \text{ using (3.20)} \qquad (3.21)$$

$$\Rightarrow \quad w = s_1^{-1}r_1r_2''^{-1}r_3s_{n+1} \qquad \text{(3.22) from (1) and (3.21)}$$

(k) If $r_2'' = 1$, then from (3.22)

$$w = s_1^{-1}r_1r_3s_{n+1}. \qquad (3.23)$$

Since s_1^{-1} does not cancel even partially with r_1 on the right (as $r^{e_1} = r_1r_2r_3$) and s_{n+1} does not cancel partially with r_3 on the left (as $r^{e_{n+1}} = r_1r_2r_3$).

(X) If r_1 does not cancel with r_3, then, since each of s_1^{-1}, r_1, r_3 of s_{n+1} contains atmost only a proper subword r or r^{-1} shows that w is a product of atmost two conjugate of r or r^{-1}. From case (2) in (A) of our proof, w involves x, contradiction.

(Y) Since atmost a proper subword of r or r^{-1} can occur as a subword in each of s_1^{-1}, r_1, r_3 and s_{n+1}, $w = s_1^{-1}r_1, r_3s_{n+1}$ can be non-trivial product of atmost two conjugates of r or r^{-1}. Hence by Case I, (2) in (B), w involves x, a contradiction.

We have a contradiction to the fact that w is of the form $\prod_{i=1}^{n+1}(s_i^{-1}r^{e_i}s_i)$ by definition, since r_1r_3 is a proper subword of r^{e_1}. Also if $r_1 = r_3^{-1}$, then $w = 1$, i.e., w is trivial, a contradiction.

(l) If $r_2'' \neq 1$, then the right hand side of (3.22) involves x i.e., $\frac{\partial w}{\partial x} \neq 0$, a contradiction.

Case II ($e_1 = -e_{n+1}$)

Suppose $e_1 = -e_{n+1}$, then $r^{e_1} = r_1r_2r_3 = r^{-e_{n+1}}$.

(1) Suppose r_2 starts with x.

By induction it can not cancel with any term in

$$\frac{\partial}{\partial x}(\prod_{j=2}^{n} s_j^{-1}r^{e_j}s_j).$$

It must cancel with

$$\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)s_{n+1}^{-1}r_3^{-1}r_2'x^{-1},$$

a summand of

$$\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)\frac{\partial}{\partial x}(s_{n+1}^{-1}r^{-e_1}s_{n+1}),$$

where

$$r_2^{-1} = r_2'x^{-1}r_2'' \tag{3.24}$$

by considering signs.

Therefore $$s_1^{-1}r_1 = (s_1^{-1}r_1r_2r_3s_1)\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_3^{-1}r_2'x^{-1})$$

$$\Rightarrow \quad r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_3^{-1}r_2'x^{-1}) = 1$$

$$\Rightarrow \quad r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_3^{-1}r_2^{-1}r_2''^{-1}) = 1\,. \tag{3.25}$$

(m) If $r_2'' = 1$, then from (3.25)

$$s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)s_{n+1}^{-1} = 1$$

$$\Rightarrow \quad \prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j) = s_1^{-1}s_{n+1}\,.$$

The left hand side involves x but the right hand side does not involve x, a contradiction.

(n) If $r_2'' \neq 1$, then from (B_3), and (1)

$$w = s_1^{-1}r_1r_2''r_1^{-1}s_{n+1},$$

since r_2'' involves x $\frac{\partial w}{\partial x} \neq 0$, a contradiction.

(2) Suppose r_2 starts with x^{-1}.

Then by induction $-s_1^{-1}r_1^{-1}x^{-1}$ can not cancel with any term in

$$\frac{\partial}{\partial x}(\prod_{j=2}^{n} s_j^{-1}r^{e_j}s_j).$$

It must cancel with any term in

$$\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)s_{n+1}^{-1}r_3^{-1}r_2',$$

a summand of

$$\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)\frac{\partial}{\partial x}(s_{n+1}^{-1}r^{-e_1}s_1),$$

where

$$r_2^{-1} = r_2' x r_2'' \qquad (3.26).$$

$$\begin{aligned} \text{Therefore} \quad s_1^{-1}r_1x^{-1} &= (s_1^{-1}r_1r_2r_3s_1)\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_3^{-1}r_2') \\ &\Rightarrow x^{-1} = r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_3^{-1}r_2') \\ &\Rightarrow r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_3^{-1}r_2'x) = 1 \\ &\Rightarrow r_2r_3s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)(s_{n+1}^{-1}r_3^{-1}r_2^{-1}r_2''^{-1}) = 1. \quad (3.27) \end{aligned}$$

using (3.26)

(o) If $r_2'' = 1$, then from (3.27)

$$s_1\prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j)s_{n+1}^{-1} = 1$$

$$\Rightarrow \prod_{j=2}^{n}(s_j^{-1}r^{e_j}s_j) = s_1^{-1}s_{n+1}.$$

The left hand side involves x but the right hand side does not involve x which is a contradiction.

(p) If $r_2'' \neq 1$, then from (3.27), and (1) we get

$w = s_1^{-1} r_1 r_2'' r_1^{-1} s_1$. Then since r_2'' involves x, $\frac{\partial w}{\partial x} \neq 0$, a contradiction.

Thus we always have a contradiction to the assumption on the s_i's.

Therefore, $\frac{\partial w}{\partial x} \neq 0$ i.e., w involves x.

CHAPTER - 4

PROOF OF THE IDENTITY THEOREM AND DETERMINATION OF ROOTS

4.1 Introduction

In this chapter we have given a new proof of the famous Identity Theorem due to Lyndon [68]. We have used Fox derivatives to prove the theorem for torsion free single-relator groups. Our proof of the general theorem uses two results due to Lyndon. We have also determined the root of the word $w = x_1^{p_1} \cdots x_k^{p_k}$, and there by have generalised certain cases of Steinberg's work [109].

4.2 Lyndon [68] proved an important theorem called the Identity Theorem in connection with his complete determination of the cohomology of groups with a single defining relation.This theorem has also been used by Huebschmann [61] for determination of cohomology of small cancellation groups.

The Identity Theorem in its general form is stated below.

We need a few definitions.
A word w in a free group on generators x_i is said to be *reduced* if it does not contain adjacent symble $x_i^{e_i}$ and $x_i^{-w_i}$ and it is said to be *cyclically reduced* if its first and last symbles are not $x_i^{e_i}$ and $x_i^{-e_i}$, $e_i = \pm 1$.

Identity Theorem 4.1 ([68], p.658)

Let F be the free group on generators $x_1, \cdots, x_n$ (and possibly other generators y_i);let $r_1, \cdots r_n$ be cyclically reduced words in F such that for each t, x_t and x_{t+s} are the first and last (in order of subscript) of the x_i that occure in r_t. Let each $r_t = w_i^{ft}$, for ft maximal and R be the smallest normal subgroup of Fcontaining $r_1, \cdots, r_n$. If

$$\prod_{i=1}^{m} s_i^{-1} r_{t_i}^{e_i} s_i = 1(s_i \in F;\ e_i = \pm,\ t_i = 1, \cdots, n),$$

then the indices $1, \cdots, m$ can be grouped into pairs (i, j) such that $t_i = t_j$, $e_i = -e_j$, and, for certain integers c_i, $s_i \equiv s_j q_{t_i}^{e_i} moduloR$.

In case of a single relation r, the condition that r be cyclically reduced is supperflous and we obtain :

The Simple Identity Theorem 4.2

If $r = r'^q$, for q is maximal, is a word in the free group, and R is the normal closure of r, then

$$\prod s_I^{-1} r^{e_i} s_i = 1$$

implies that the indices can be grouped into pairs (i,j) such $e_i = -e_j$,, and, for certain integer c_i, $s_i \equiv s_j q^{c_i} (modR)$.

Theorem 4.3 ([68], p.660)

If r is a power of a generator of the free group F, then r satisfies the Simple Identity Theorem.

Theorem 4.4 ([68], p.659)

Let r and $r_1, \cdots, r_n$ be given as in the Identity Theorem and in the Simple Identity Theorem. If the Simple Identity Theorem holds for each of $r_1, \cdots, r_n$ separately, then the Identity Theorem holds for them all together.

The proof of Theorwm 4.3 uses fox drivatives.
Lyndon also proved the following corollary of the Identity Theorem.

Theorem 4.5 ([68], p.659)

The Identity Theorem is equivalent to the theorem obtained by replacing the condition that $\prod(s_i^{-1} r_i^{e_i} s_i) = 1$ by the condition that this product lies in the commutator subgroup $[R, R]$.

4.3 We shall now describe our own proof of the Identity Theorem with the help of Fox derivatives. We start with the following form of the Simple Identity Theorem:

Theorem 4.6

Let $G = \frac{F}{R}$ be a torsion free group with a single defining relation, Where F is a free group with the basis X and R ia the normal closure of r. If

$$w = \prod_{i=1}^{n} (s_i^{-1} r^{e_i} s_i), \; (s_i \in F, \, e_i = \pm 1)$$

is an element of $R' = [R, R]$, then the indices i's can be grouped into pairs (j, k) such that $e_j = -e_k$ and $s_j \equiv s_k (mod R)$.

We shall use the following result in the proof.

Theorem 4.7 ([50])

If F is a free group with basis X and R is a normal subgroup of F with basis Y, Then $\frac{r}{r'}$ is a left $\mathbb{Z}G$-module isomorphic to $\frac{\mathfrak{R}}{\mathfrak{R}\mathfrak{F}}$, where $\mathfrak{F}$ and $\mathfrak{R}$ are defined as in Chapter ...p..., and $\frac{R}{R'} \to \frac{\mathfrak{R}}{\mathfrak{R}\mathfrak{F}}$ given by $f(rR') = (r-1) + \mathfrak{R}\mathfrak{F}$ is a $\mathbb{Z}G$-isomorphism.

Proof of Theorem 4.6

Let $w = \prod_{i=1}^{n}(s_i^{-1}r^{e_i}s_i) \in R'$, then by the **Theorem 4.1** $w - 1 \in \mathfrak{F}\mathfrak{R}$. Since

$$w - 1 = \sum_{x \in X} \frac{\partial w}{\partial x}(x-1)',$$

Theorem 4.7 implies that, for each $x \in X$, $\frac{\partial w}{\partial x} \in \mathfrak{R}$.
Now

$$\begin{aligned}
\frac{\partial w}{\partial x} &= [-s_1^{-1}\frac{\partial s_1}{\partial x} + s_1^{-1}e_1 r^{\frac{e_1-1}{2}}\frac{\partial r}{\partial x} + s_1^{-1}r^{e_1}\frac{\partial s_1}{\partial x}] \\
&+ \sum_{j=2}^{n}\prod_{k=1}^{j-1}(s_k^{-1}r^{e_k}s_k)[-s_j^{-1}\frac{\partial s_j}{\partial x} + s_j^{-1}e_j r^{\frac{e_j-1}{2}}\frac{\partial r}{\partial x} + s_j^{-1}r^{e_j}\frac{\partial s_j}{\partial x}] \\
&= [s_1^{-1}(r^{e_1-1})\frac{\partial s_1}{\partial x} + s_1^{-1}e_1 r^{\frac{e_1-1}{2}}\frac{\partial r}{\partial x}] \\
&+ \sum_{j=2}^{n}\prod_{k=1}^{j-1}(s_k^{-1}r^{e_k}s_k)[-s_j^{-1}\frac{\partial s_j}{\partial x} + s_j^{-1}e_j r^{\frac{e_j-1}{2}}\frac{\partial r}{\partial x} + s_j^{-1}r^{e_j}\frac{\partial s_j}{\partial x}]. \qquad (4.1)
\end{aligned}$$

For each $\varphi \in \mathbb{Z}F$, we denote $\pi(\varphi)$ by $\bar{\varphi}$ where, $\pi :: \mathbb{Z}F \to \mathbb{Z}G$ is as before, the ring homomorphism induced by the canonical homomorphism $F \to G$, since

$$\frac{\partial w}{\partial x} \in \mathfrak{R}, \ \frac{\partial \bar{w}}{\partial x} = 0.$$

Hence from (4.6) we obtain

$$(\sum_{i=1}^{n} e_i \bar{s}_i^{-1})\frac{\partial \bar{r}}{\partial x} = 0 \qquad (4.2)$$

in $\mathbb{Z}G$.

Now $\frac{\bar{\partial r}}{\partial x} \neq 0$; for otherwise, $\frac{\partial w}{\partial x} \in \Re$, for each x, and so , $r - 1 \in \Re\mathfrak{F}$. So by Theorem 4.7, $r \in R'$. So, $\frac{R}{R'} = 0$. Since $\frac{R}{R'}$ is a free abelian group with basis $\{(y-1) \mid y \in Y\}$, we have a contradiction to the definition of R. By the Theorem of Brown [19] $\mathbb{Q}G$ and hence $\mathbb{Z}G$ has no zero divisors. Therefore from (4.7) we obtain

$$(\sum_{i=1}^{n} e_i \bar{s}_i^{-1}) = 0. \tag{4.3}$$

Thus the indices are groped into pairs (j,k)such that $e_j = -e_k$ and $\bar{s}_j \equiv \bar{s}_k$ i.e., $s_j \equiv s_k (mod R)$.

We now consider a group with a single defining relation which is not torsion free.

Theorem 4.8

If r is a power of a word w in F, say $r = w^q, (q > 1)$, then r satisfies the Simple Identity Theorem.

Proof

The theorem follows from the Theorem 4.3 and Theorem 4.5, since w^q is a finite product of powers of x's.

Theorem 4.4, shows that The (general) Identity Theorem is a consequence of Theorem 4.6 and 4.7.

4.4 Root of a Word

Let F be a free group with a basis $\{x_1, \cdots, x_n\}$. Let $r \in F$. An element r in F is called a *root of w(in F)* if w is contained in the normal closure of r.
There is another definition of a root [71] which we donot consider here.
The word problem for a single-relator groups solved by Magnus [83] is the algorithmic problem of determining whether r is a root of w. However the problem of determining all roots of r of a given word w is difficult and has been solved only in simple cases.

If w is a word in two generators x_1 and x_2. It is useful to find all roots of $x^k y^l$ in connection with the characterisation of one relator group with non-trivial centre [88]. For, if

$$G = < x_1, x_2 \mid r = [x_1^k, x_2^l] \text{ or } r = x_1^k x_2^l >,$$

then, x_1^k is in the centre of G. Moreover if a root $r'(x_1, x_2)$ of w is not a root of any word $\bar{r}(x_1, x_2)$ which represents a given element z in the cenre of G, then the group

$$k = < x_1, x_2 \mid r' >$$

will have a non-trivial centre. It has been conjectured that any one relator group with non-trivial centre is one of the groups K.
Meskin proved the following theorem.

Theorem 4.9 ([109] p.1)

Let ζ be the class of non abelian group G with centre $C(G)$. If $G \in \zeta$ and GG' is free abelian of rank 2, then G has a presentation

$$G = < x_1, x_2 \mid r = r(x_1, x_2) >$$

and an integer $k > 0$ exists such that $r(x_1, x_2)$ is a root of $[x_1^k, x_2^l]$. Furthermore, for any such presentation if k is minimal, then $C(G) = < x_k >$. Moreover, Steinberg [109]

determined all the roots of $x_1^k x_2^l$.

Theorem 4.10 ([109] p.2)

For k and l both prime, the only cyclically reduced roots of $x_1^k x_2^l$ other than $x_1^k x_2^l$ itself are $P(x_1, x_2) \neq 1$, where $P(x_1, x_2)$ is a primitive in the free group on x_1 and x_2, $P(x_1, x_2)$ is unique upto conjugation and inversion. If $k \neq l$, then $P(x_1, x_2)$ has exponent sum k on x_1 and l on x_2, if $k = l$, then $P(x_1, x_2) = x_1 x_2$.
Its proof is long and uses three lemmas. Nielsons transformations play a significant role in their proof.

We shall determine roots of $x_1^{p_1}, \cdots, x_k^{p_k}$ in the free group F with basis $\{x_1, \cdots, x_k, \cdots, \}$, where $p_1, \cdots, p_k$ are prime. We thereby generalise by the theorem of Steinberg for the case when k and l are primes. Our result is stated in the following theorem.

Theorem 4.11

Let F be a free group with a basis $X = \{x_1, \cdots, x_k, \cdots, \}$, and $w = x_1^{p_1}, \cdots, x_k^{p_k}$, where $p_1, \cdots, p_k$ are primes . Then the root of w are w and primitives in F.

Proof

Let r be a root of w. Then

$$w = \prod_{i=1}^{u} (s_i^{-1} r^{e_i} s_i). \qquad (4.4)$$

Then

$$\begin{aligned}
\frac{\partial w}{\partial x} &= [-s_1^{-1}\frac{\partial s_1}{\partial x} + s_1^{-1}e_1 r^{\frac{e_1-1}{2}}\frac{\partial r}{\partial x} + s_1^{-1}r^{e_1}\frac{\partial s_1}{\partial x}] \\
&\quad + \sum_{j=2}^{u}\prod_{k=1}^{j-1}(s_k^{-1}r^{e_k}s_k)[-s_j^{-1}\frac{\partial s_j}{\partial x} + s_j^{-1}e_j r^{\frac{e_j-1}{2}}\frac{\partial r}{\partial x} + s_j^{-1}r^{e_j}\frac{\partial s_j}{\partial x}] \\
&= [s_1^{-1}(r^{e_1-1})\frac{\partial s_1}{\partial x} + s_1^{-1}e_1 r^{\frac{e_1-1}{2}}\frac{\partial r}{\partial x}] \\
&\quad + \sum_{j=2}^{u}\prod_{k=1}^{j-1}(s_k^{-1}r^{e_k}s_k)[-s_j^{-1}\frac{\partial s_j}{\partial x} + s_j^{-1}e_j r^{\frac{e_j-1}{2}}\frac{\partial r}{\partial x} + s_j^{-1}r^{e_j}\frac{\partial s_j}{\partial x}].
\end{aligned}$$

$$\therefore\ \frac{\partial \bar{w}}{\partial x} = \sum_{i=1}^{n} e_i\bar{s}_i^{-1})\frac{\bar{\partial r}}{\partial x}.$$

Now

$$\left.\begin{aligned}
\frac{\partial w}{\partial x_1} &= x_1^{p_1-1}+\cdots+x_1+1 \\
\frac{\partial w}{\partial x_2} &= x_1^{p_1}[x_2^{p_2-1}+\cdots+x_2+1] \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{\partial w}{\partial x_k} &= x_1^{p_1}\cdots x_{k-1}^{p_{k-1}}[x_k^{p_k-1}+\cdots+x_k+1] \\
\frac{\partial w}{\partial x_l} &= 0, l>k
\end{aligned}\right\}. \tag{4.5}$$

From (4.5) $\frac{\partial r}{\partial x_l} = 0, l > k$, by the Freiheitssatz. Thus

$$\left.\begin{aligned}
x_1^{p_1-1}+\cdots+x_1+1 &= \sum_{i=1}^{u} e_i\bar{s}_i^{-1})\frac{\bar{\partial r}}{\partial x_1} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
x_1^{p_1}\cdots x_{\alpha-1}^{p_\alpha-1}[x_\alpha^{p_\alpha-1}+\cdots+x_\alpha+1] &= \sum_{i=1}^{u} e_i\bar{s}_i^{-1})\frac{\bar{\partial r}}{\partial x_\alpha} 2\le\alpha\le k
\end{aligned}\right\}. \tag{4.6}$$

$x_\beta^{\beta_1-1}+\cdots+x_\beta+1$ is irreducible in $\mathbb{Z}G$, (4.6) implies that either

$$\sum_{i=1}^{n} e_i\bar{s}_i^{-1}),$$

or $\frac{\partial \bar{r}}{\partial x_\alpha}$ is a unit in $\mathbb{Z}G$, (4.6) shows that

$$\sum_{i=1}^{n} e_i \bar{s}_i^{-1}) = 1,$$

and

$$\frac{\partial \bar{r}}{\partial x_1} = x_1^{p_1 - 1} + \cdots + x_1 + 1,$$

and so,

$$\frac{\partial \bar{r}}{\partial x_\alpha} = x_1^{p_1} \cdots x_{\alpha-1}^{p_\alpha - 1}[x_\alpha^{\alpha_1 - 1} + \cdots + x_\alpha + 1], \;\; 2 \leq \alpha \leq k;$$

so that for $1 \leq \beta \leq k$,

$$\frac{\partial \bar{r}}{\partial x_\beta} = \frac{\partial \bar{w}}{\partial x_\beta}.$$

The nature of derivatives imply that $r = w$. On the other hand, $\sum_{i=1}^{n} e_i \bar{s}_i^{-1})$ is a unit , then $\frac{\partial \bar{r}}{\partial x_\alpha}$ is a unit for each α. In this case $\frac{\partial r}{\partial x} \neq 0$ in $\mathbb{Z}F$, and so, r is a primitive in $\mathbb{Z}F$.

CHAPTER - 5

ON THE CONJUGACY PROBLEM FOR A CLASS OF GROUPS

5.1 Introduction

In this chapter we have considered the conjugacy problem for groups. We started with a brief historical survey of works in this area. Later we have obtained a set of suffitient conditions for solvablity of cojugacy problem for finitely pesented groups and applied our result to prove that torsion-free single-relator groups and torsion- free polycyclic-by- finite groups have solvable conjugacy problm.

5.2 We recall that if G is a group with a presentation $G = < X; Y >$, G is said to have solvable conjugacy problem if for any two elements $g_1, g_2 \in G$ it is possible to determine in a finite umber of steps whether g_2 is a conjugate of g_1. Dehn [26] posed the problem for in general and solved it for the fundamental group of closed orietable two dimensional manifolds. Dehn's method for geometric used regular tesselations of the hyperbolic plane. Later works in this context used combinatorial group theory independent of any geometric arguments [27].

Magnus proved the solvablity of the word problem for single relator groups using the Freiheitssatz. The geometric character of Dehn's argument was restored in the form of elementary combinatorial geometry with the emergence of of the small cancellation theory as a unified and powerful theory. The basic idea of this geometric approach is as follows. Let a G group has presentation $G = < X, Y >$. Let F be a free group with basis X and R be the normal closure of Y in F. Let a word w in F be given by $w = c_1 \cdots c_n$. where $c_i = s_i^{-1} r_i^{e_i} s_i$, $(e_i = \pm 1)$. With such a product is associated a map in the Euclidian plane which contains every necessary information about the product $w = c_1 \cdots c_n$. This maps acts as adequate tool for studying membership in the normal subgroup R of F and hence for studying equality in the group G. van Kampen [111], (1933) discovered the diagrams but did not make much use of them. Lyndon [70] and Weinbaum [112] discovered these diagrams indepedently and used them for a geometric study of a small cancellation theory. The latter was applied to settle the word problem and the conjugacy problem for groups in various situations. In his study the word problem for fundamental groups of orientable two dimensional maifolds if they showed that if a freely reduced on trivial word w is equality in the fundamental group, then w contains more than half of some cyclic permutation of the defining relation and its inverse. He used it to obtain a algorithm, called Dehn's algorithm, for the word problem.

Let S be a symmetrized subset of a free group F. Let r_1 and r_2 be distinct element of S with $r_1 \equiv bc_1$ and $r_2 \equiv bc_2$, then b is called the *piece relative to the set S*. Since b is cancelled in the product $r_1^{-1}r_2$, and S is symmetrized, a piece is simply a subword of an element of S which can be cancelled by the multiplication of two non-inverse element of S. Let Y be a symmetrized subset of a free group F. Then the hypotheses of small cancellation asserts that pieces are relatively small parts of elements of S.

In small cancellation theory there are some fundamental metric conditions $C'(\lambda), C(p), T(q)$, very much useful in connection with word problems.

These are:
Condition $C'(\lambda)$: if $y \in Y$ and $y \equiv bc$, where b is a piece, then $|b| < \lambda|y|$.
Condition $C(p)$: element of y is a product of power of p pieces.
Condition $T(q)$: Let $3 \leq h \leq q$. Suppose $y_1, \cdots y_h$ are element of Y with no succesive elements are r_i, r_{i+1} in inverse pair. Then at least one of the products $r_1r_2, \cdots, r_{h-1}r_h, r_hr_1$ is reduced without cancellation.

A group G with presentation $G = < X; Y >$ is called a small cancellation group if Y is symmetrized and satisfies at least one of the condition $C'(\lambda)$, $C(p)$ or $C(q)$.

Small cancellation Theory established that Dehn's Algorithm is valid for Y satisfying either the metric hypotheses $C'(\frac{1}{6})$ or $C(\frac{1}{4})$ and $T(4)$. Greendlinger [44] proved a stronger result, called Greendlinger's Lemma(see [44], [49], [71] Thm. 4.9 and Thm 4.6 p. 250).

By establishing Area Theorem ([71], p.260) for maps associated with the presentation and applying it cleverly Lyndon [71] solved the word problem for groups $G = < X; Y >$ (Y is symmetrized) in Y satisfies $C(6)$, or $C(4)$ and $T(4)$, or $C(3)$ and $T(6)$. Schupp [105] proved that the Layer Theorem ([71], p.264) for maps and applied it to show that $G = < X; Y >$ (Y symmetrized) has a solvable conjugacy problem if $C(6)$ or $C(4)$ and $T(4)$, or $C(3)$ and $T(6)$ holds.

Later weaker small cacellation hypotheses were proposed and applied for solution of the word problem and the conjugcay problem by Juhasz ([63], [64], [65], Rips [102] and Mallick [86]). The weaker condition introduced by Juhasz as denoted by $w(4)$, while the condition introduced by Mallick [86] were denoted by $S_1(T_4)$ and $S_2(T_4)$. The woprd problem and the conjugacy problem were shown to be solvable for these groups.

Majumdar ([73],[78]) used Fox derivatives for solution of the word problems and conjugacy problems for certain classes of groups.

5.3 We shall now describe our own work on the solvablity of the conjugacy problem for groups. Our techique reliesheavilyon use of Fox derivatives. We follow Majumdar's ideas. Here derivatives means left derivatives.

Let F be a free group with basis X and R, the normal closure of $r_1, r_2, \cdots, r_n$ in F.

Then if $\mathfrak{F}$ and $\mathfrak{R}$ denote respectively the kernels of the ring homomorphisms $\mathbb{Z}F \to \mathbb{Z}$ and $\mathbb{Z}F \to \mathbb{Z}G$ given by $x_i \to 1$ and $x_i \to x_i\mathfrak{R}$, then $\mathfrak{F}$ is freely generated as a left $\mathbb{Z}F$ by all $x - 1$, $x \in X$. Also $\frac{\mathfrak{R}}{\mathfrak{R}\mathfrak{F}}$ is generated as a left $\mathbb{Z}G-$ module by $(r_j - 1) + \mathfrak{R}\mathfrak{F}$, where $j = 1, 2, \cdots, n$. See Gruenberg ([49], [50]).

Before we proceed to our first and fundamental result about the conjugacy problem, we note the following.

There are certain classes of groups G which are finitely presented and are such that the consistency of any finite system of linear equations over $\mathbb{Z}G$ can be decided and if it is consistent, the solutions can be obtained in a finite number of steps. Free groups, torsion-free single-relator groups and torsion-free groups are examples of such groups. This follows from Artin [6], Since the work of J.Lewin and T. Lewin [67], P.Hall [54] Farkas and Snider [30], Formanek [31] shows that $\mathbb{Q}G$ and hence $\mathbb{Z}G$, can be embeded in a skew field. We

thus prove our result.

Theorem 5.1

Let G be a group such that , the following condition hold:

(i) G is finitely presented,

(ii) G has solvable word problem,

(iii) given a finite system of linear equations over $\mathbb{Z}G$, there exist an algorithm to decide in a finite number of steps whether the system has solution and an algorithm to find the solution if it is cosistent.

Then G has a solvable conjugacy problem.

Proof

Let G be a group satisfying the condition of the theorem, let G be given by $G = < x_1, x_2, \cdots, x_m; r_1, r_2, \cdots, r_n >$.
Let F be a free group generated by $x_1, x_2, \cdots, x_m$ and let R be the normal closure of $r_1, r_2, \cdots, r_n$.

Let g_1 and g_2 be two elements of G and let $g_1 = w_1R$ and $g_2 = w_2R$, where $w_1, w_2 \in F$.

G has a solvable conjugacy problem if it is possible to determine in a finite number of steps whether there exists $g \in G$ such that $g_2 = g^{-1}g_1g$,
i.e., there exist $w \in F$ such that

$$\begin{aligned} & w_2 \equiv w^{-1}w_1w \,(mod R) \\ \Longleftrightarrow\ & w_2w^{-1}w_1w \in R \\ \Longleftrightarrow\ & (w_2w^{-1}w_1w - 1) + \mathfrak{R}\mathfrak{F} = \frac{\mathfrak{R}}{\mathfrak{R}\mathfrak{F}}, \end{aligned}$$

since $rR' \leftrightarrow (r-1+\Re\mathfrak{F})$ is an isomorphism $\frac{R}{R'} \cong \frac{\Re}{\Re\mathfrak{F}}$ of abelian groups (see Gruenberg [49]).

$\iff$ there exist $\varphi_1, \varphi_2, \cdots, \varphi_n \in \mathbb{Z}F$ such that

$$(w_2 w^{-1} w_1 w - 1) + \Re\mathfrak{F} = \sum_{j=1}^{n} \varphi_j (r_j - 1) + \Re\mathfrak{F},$$

by the description in the begining of this section.

$\iff$ there exist $\varphi_1, \varphi_2, \cdots, \varphi_n \in \mathbb{Z}F$ such that $(w_2 w^{-1} w_1 w - 1) - \sum_{j=1}^{n} \varphi_j (r_j - 1) \in \Re\mathfrak{F}$

$\iff$ for each $i = 1, 2, \cdots, m$, there exist $\varphi_1, \varphi_2, \cdots, \varphi_n \in \mathbb{Z}F$, such that

$$\frac{\partial}{\partial x_i}((w_2 w^{-1} w_1 w) - \sum_{j=1}^{n} \varphi_j \frac{\partial r_j}{\partial x_i} \in \Re,$$

since $\mathfrak{F}$ is free on $\{x_i - 1\}$. The latter condition holds if and only if

$$\frac{\partial w_2}{\partial x_i} - w_2 w^{-1} \frac{\partial w}{\partial x_i} - w_2 w^{-1} w_1^{-1} \frac{\partial w_1}{\partial x_i} + w_2 w^{-1} w_1^{-1} \frac{\partial w}{\partial x_i} - \sum_{j=1}^{n} \varphi_j \frac{\partial r_j}{\partial x_i} \in \Re \,.$$

i.e.,

$$\frac{\partial \bar{w}_2}{\partial x_i} - w_2 w^{-1} \frac{\partial \bar{w}}{\partial x_i} - w_2 w^{-1} w_1^{-1} \frac{\partial \bar{w}_1}{\partial x_i} + w_2 w^{-1} w_1^{-1} \frac{\partial \bar{W}}{\partial x_i} - \sum_{j=1}^{n} \nu_j \frac{\partial \bar{r}_j}{\partial x_i} = 0, \qquad (5.1)$$

where $g_2 = \pi(w_2), g_1 = \pi(w_1), g = \pi(w), \nu_j = \pi(\varphi_j),\ j = 1, 2, \cdots, n$.
(5.1) is equivalent to

$$g_1 g g_2^{-1} \frac{\partial \bar{w}_2}{\partial x_i} - g_1 \frac{\partial \bar{w}}{\partial x_i} - \frac{\partial \bar{w}_1}{\partial x_i} + \frac{\partial \bar{w}}{\partial x_i} - g_1 g g_2^{-1} \nu_j \frac{\partial \bar{r}_j}{\partial x_i} = 0 \,. \qquad (5.2)$$

We write (5.2) as

$$g' \frac{\partial \bar{w}_2}{\partial x_i} + (1 - g_1) \frac{\partial \bar{w}_1}{\partial x_i} - \frac{\partial \bar{w}_1}{\partial x_i} - g' \nu_j \frac{\partial \bar{r}_j}{\partial x_i} = 0, \qquad (5.3)$$

where $g' = g_1 g g_2^{-1}$, so that $g = g_1^{-1} g' g_2$.
which is the same as

$$g' \frac{\partial \bar{w}_2}{\partial x_i} + (1 - g_1) \frac{\partial \bar{w}}{\partial x_i} - \frac{\partial \bar{w}_1}{\partial x_i} - \lambda'_j \frac{\partial \bar{r}_j}{\partial x_i} = 0 \,, \lambda' = g' \nu_j \,. \qquad (5.4)$$

Now (5.4) is the system of m linear equations over $\mathbb{Z}G$ in m+n+1 unknowns

$$\frac{\partial \bar{w}}{\partial x_i}, \quad i = 1, 2, \cdots, m, \ \lambda'_j,\ j = 1, 2, \cdots, n \text{ and } g'.$$

By the condition (iii) it is possible to decide in a finite number of steps whether (5.4) is consistent .

If (5.4) is consistent , then by the same condition it is possible to find the value of $\frac{\partial \bar{w}}{\partial x_i}$, g', and λ'_j satisfying (5.4) since the coefficients of the equations are unambiguous by the solvablity of the word problem. Hence it is possible to find in a finite number of steps the values of $\frac{\partial w}{\partial x_i}$, g, λ_j satisfying (5.2). Also it is possible to verify whether the relation

$$g - 1 = \sum_{i=1}^{m} \frac{\partial w}{\partial x_i}(h_i - 1),$$

where

$$g = \pi(w), h_i = \pi(x_i), w - 1 = \sum_{i=1}^{m} \frac{\partial w}{\partial x_i}(x_i - 1)$$

are satisfied. If it is found out that the above conditions are satisfied then g_2 is a conjgate of g_1. The conjugacy problem is thus solvable.

Corollary 5.2

Let G be a torsion-free group with a single relation. Then the conjugacy problem for G is solvable.

Proof

G satisfies (i) and (ii) of Theorem by Magnus [83] . It has been proved by J.Lewin and T.Lewin[67] that $\mathbb{Q}G$ and hence $\mathbb{Z}G$, can be embeded in a skew field S. Given a system of linear equations over S, there is a finite algorithm (see Artin [6]), for deciding whether it has a solution, and to find the solution when it exist. Hence it is also possible to decide in a finite number of steps whether the solution, if it is exists belong to $\mathbb{Z}G$, the Theorem 5.1 implies that G has a solvabe conjugacy problem.

Corollary 5.3

Let G be a polycyclic-by-finite group. Then the conjugacy problem for G is solvable.

Proof

It follows from ([30],p.192]) that $\mathbb{Q}G$, has no zero devisors. Also, by [54], $\mathbb{Z}G$ satisfies a.c.c on right ideals. Hence $\mathbb{Z}G$ is semiprime with a.c.c, by Theorem 29 of [28], can be embeded in a skew field. The other arguments are similar to those in the proof of Corollary 5.2.

CHAPTER - 6

DETERMINATION OF HOMOLOGY AND COHOMOLOGY OF A FEW CLASSES OF GROUPS

6.1 Introduction

In this final chapter we have used Fox derivatives to obtain free resolutions for a few classes of groups and used this derivatives to determine their homology and cohomology. The classes of groups considered here consist of the dihedral group,the fundamental group of the union of tori $T_1, \cdots, T_n$ $(n \geq 2)$ such that $T_j \cap T_{j+1}$ intersect at a single point, the unpermuted braid group, the quasi-cyclic group and the 3- dimensional Heisenberg group.

6.2 Lyndon was the first to use Fox deivatives in conection with dtermination of cohomology of groups. Using his Identity Thorem he completely determined the cohomology of groups with a single defining relation. In the process he indirectly indicated the construction of a pertial (4-term) free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ [68][49][71]for a finitely presented group G.

The Fox derivative used in this chapter are right derivatives.

Lyndon's Pertial (4-term) Resolutions

Let $G = \frac{F}{R}$, where F is a free group with basis $X = \{x_1, \cdots, x_m\}$ and R, a normal closure of $Y = \{r_1, \cdots, r_n\}$, so that G may be written as $G = < x_1, \cdots, x_m \mid r_1, \cdots, r_n >$. As before let $\pi : \mathbb{Z}F \rightarrow \mathbb{Z}G$ be the ring homomorphism induced by the canonical homomorphism $F \rightarrow G$. Let $\pi(x_i) = h_i$, $i = 1, \cdots, m$. Then the following is a pertial (4-term) free resoluton of $\mathbb{Z}$:

$$\cdots\cdots\cdots Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0, \qquad (6.1)$$

$$Y_0 \text{is a right} \mathbb{Z}G\text{-module free on} \alpha_1, \cdots, \alpha_m,$$

$$Y_1 \text{is a right} \mathbb{Z}G\text{-module free on} \beta_1, \cdots, \beta_n,$$

and the ϵ, d_0 and d_1 are $\mathbb{Z}G$-homomorphism given by

$$\begin{aligned}
\epsilon(g) &= 1, \\
d_0(\alpha_i) &= (h_i - 1),\ i = 1, \cdots, m, \\
d_1(\beta_j) &= \sum_{i=1}^{m} \alpha_i \frac{\partial r_j}{\partial x_i}\ .
\end{aligned}$$

6.3 The Lydon's 4-terms resolutions was extended to a full resolution by Majumdar ([73], [74]) for a number of important classes of groups. Who use this to complete the corresponding homology and the cohomology of group. The general method of extending Lyndon's partial resolution to a full resolutions for a finitely presented groups was achieved by Majumdar and Akhter ([80-82]). The construction of the resolution is straight-forward and consists of solution of systems of linear equations over the corresponding integral group ring. It is immediately applicable for computation the homology and the cohomology of the group concerned. They applied the technique for determination of the homology and the cohomology of many important classes of groups.

6.4 We shall apply the above method of extension of Lyndon's partial resolution for the following very useful classes of groups:

(i) The fundamental group a union of n tori ;

(ii) the dihedral group;

(iii) the unpermutated braid group;

(iv) the Heisenberg group H_3 and

(v) the quasi-cyclic group.

In our construction of the free resolutions we have repeatedly used two results due to Majumdar ([66], Prop.1,2.). We describe these below.

Let G be a group and H is a subgroup of G. Let $\{g_i \mid j \in J\}$ be a right transversal of H in G, for some indexing set J.Then $\mathbb{Z}G$ is a left $\mathbb{Z}H$-module free on $\{g_i \mid j \in J\}$, and hence each element γ of $\mathbb{Z}G$ can be writtn uniquely in the form

$$\gamma = \sum_{j \in J} \sum h \in H N_{hj} h_{gj}, \tag{6.2}$$

where, for each j and h, $N_{hj} \in \mathbb{Z}$,only a finite number of them being nonzero.

We introduce the symble $s((H)$ as follows

$$s(H) = \sum h, \text{ if} H \text{ is finite,}$$
$$= 0, \text{ otherwise.}$$

For $\gamma \in \mathbb{Z}G$, let $\mathbb{Z}G.\gamma$ and $\gamma\mathbb{Z}G$ denote respectively the left ad right ideals of $\mathbb{Z}G$ generated by γ For any subset S of $\mathbb{Z}G$, let $Ann_L S$ and $Ann_R S$denote respectively the left and right annihilatos of S.

Let $\bar{H}$ denotes the set of generators of H.

Lemma 6.1

If H is finite, $Ann_r\{s(H)\} = \sum(h-1).\mathbb{Z}G$.

Lemma 6.2

$$\bigcap_{h \in H} Ann_R\{h-1\} = s(H).\mathbb{Z}G.$$

6.4.1 Dihedral group D_{2m}

The dihedral group $G = D_{2m}$ is the symmeric group of an n sided regular plane polygon and prestation :

generators : h_1, h_2;

relations : $h_1^2 = h_2^2 = (h_1h_2)^2 = 1.$ (6.4.1)

Here h_1 represents a relation by $\frac{2\pi}{M}$ about the axis through the centre of the polygon and perpendicular to the plane of the polygon and h_2 is a reflexion in one of the lines of symmetries of the polygon.

The dihedral group D_{2m} can also be viewed differently. It belong to the family of Coxeter groups.
The Coxeter group has a presentation [89]

generators : $h_1, \cdots, h_n$

relations : $h_i^2 = (h_ih_j)^{m_{ij}} = 1, i < j$

ranges over $\{1, \cdots, n\} \times \{1, \cdots, n\}$, $m_{ij} \geq 2$.

For n=2 , this reduces to the dihedral group D_{2m} with generators h'_1, h'_2,where $h'_1 = h_1h_2$ and $h_2 = h_2$.

Recently Yang [97] has shown that the dihedral group D_{2p}, where p is an odd prime can be realised as the automorphism group of the compact Riemann surface.

Free resolution for D_{2m}

$G = D_{2m}$ is given by $G = \frac{F}{R}$,where F is a fre group generated by x_1, x_2 and R is the normal closure of r_1, r_2, r_3 in F, where $r_1 = x_1^m$, $r_2 = x_2^2$ and $\gamma_3 = (x_1x_2)^2$.
Then the Fox derivatives of r_1, r_2 and r_3 are :

$$\frac{\partial r_1}{\partial x_1} = x_1^{m-1} + \cdots + 1; \qquad \frac{\partial r_1}{\partial x_2} = 0;$$

$$\frac{\partial r_2}{\partial x_1} = 0; \qquad \frac{\partial r_2}{\partial x_2} = x_2 + 1;$$

$$\frac{\partial r_3}{\partial x_1} = x_2x_1x_2 + x_3; \qquad \frac{\partial r_3}{\partial x_2} = x_1x + 2 + 1 .$$

Writing $\pi(x_i) = h_i$, $i = 1, 2$, we have

$$\pi(\frac{\partial r_1}{\partial x_1}) = h_1^{m-1} + \cdots + 1; \qquad \pi(\frac{\partial r_1}{\partial x_2}) = 0$$

$$\pi(\frac{\partial r_2}{\partial x_1}) = 0; \qquad \pi(\frac{\partial r_2}{\partial x_2}) = h_2 + 1;$$

$$\pi(\frac{\partial r_3}{\partial x_1}) = h_2(h_2h_2 + 1); \qquad \pi(\frac{\partial r_3}{\partial x_2}) = h_1h_2 + 1.$$

We then have

Theorem 6.3

The following 6-term is a free resolution of $\mathbb{Z}$:

$$Y_3 \xrightarrow{d_3} Y_2 \xrightarrow{d_2} Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where Y_0, Y_1, Y_2 and Y_3 are right $\mathbb{Z}G-$*modules* freely generated by $\{\alpha_1, \alpha_2\}$, $\{\beta_1, \beta_2, \beta_3\}$, $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ and $\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ and d_0, d_1, d_2, d_3 are $\mathbb{Z}G$ – *homomorphisms* defined by

$$
\begin{aligned}
d_0(\alpha_1) &= \alpha_1(h_1 - 1), \\
d_0(\alpha_2) &= \alpha_2(h_2 - 1), \\
d_1(\beta_1) &= \alpha_1(h_1^{m-1} + \cdots + 1), \\
d_1(\beta_2) &= \alpha_2(h_2 + 1), \\
d_1(\beta_3) &= \alpha h_2(h_1 h_2 + 1) + \alpha_2 h_2(h_1 h_2 + 1), \\
d_2(\gamma_1) &= \beta_1(h_1 - 1), \\
d_2(\gamma_2) &= \beta_2(h_2 - 1), \\
d_2(\gamma_3) &= -\beta_3(h_1 h_2 - 1), \\
d_2(\gamma_4) &= \beta_1(h_2 + 1) + \beta_2(h_1^{m-1} + \cdots + 1) - \beta_3(h_1^{m-1} + \cdots + 1),
\end{aligned}
$$

$$
\begin{aligned}
d_3(\delta_1) &= \gamma_1(h_1^{m-1} + \cdots + 1), \\
d_3(\delta_2) &= \gamma_2(h_2 + 1), \\
d_3(\delta_3) &= \gamma_3(h_1 h_2 + 1), \\
d_3(\delta_4) &= -\gamma_1[(h_1^{m-1} + \cdots + 1) + h_2 h_1 - 1] + \gamma_4(h_1 - 1), \\
d_3(\delta_5) &= -\gamma_2(h_1^{m-1} + \cdots + 1) - \gamma_3(h_1^{m-1} + \cdots + 1) + \gamma_4(h_2 - 1).
\end{aligned}
$$

Proof

Exactness at Y_1 from the right

$$d_1d_2(\gamma_1) = d_1(\beta_1(h_1-1)) = d_1(\beta_1)(h_1-1) = \alpha_1(h_1^{m-1}+\cdots+1)(h_1-1) = 0.$$

$$d_1d_2(\gamma_2) = d_1(\beta_2(h_2-1)) = d_1(\beta_2)(h_2-1) = \alpha_2(h_2+1)(h_2-1) = 0.$$

$$\begin{aligned} d_1d_2(\gamma_3) &= d_1(-\beta_3(h_1h_2-1)) = -d_1(\beta_3)(h_1h_2-1) \\ &= -\alpha_1h_2(h_1h_2+1)(h_1h_2-1) - \alpha_2h_2(h_1h_2+1)(h_1h_2-1) \\ &= 0. \end{aligned}$$

$$\begin{aligned} d_1d_2(\gamma_4) &= d[\beta_1(h_2+1) + \beta_2(h_1^{m-1}+\cdots+1) - \beta_3(h_1^{m-1}+\cdots+1)] \\ &= \alpha_1(h_1^{m-1}+\cdots+1)(h_2+1) + \alpha_2(h_2+1)(h_1^{m-1}+\cdots+1) \\ &\quad -\alpha_1h_2(h_1h_2+1)(h_1^{m-1}+\cdots+1) - \alpha_2h_2(h_1h_2+1)(h_1^{m-1}+\cdots+1) \\ &= \alpha_1(h_1^{m-1}+\cdots+1)(h_2+1) + \alpha_2(h_2+1)(h_1^{m-1}+\cdots+1) \\ &\quad -\alpha_1(h_1^{m-1}+h_2)(h_1^{m-1}+\cdots+1) - \alpha_2(h_1^{m-1}+h_2)(h_1^{m-1}+\cdots+1) \\ &= \alpha_1(h_1^{m-1}+\cdots+1) + \alpha_2(h_1^{m-1}+\cdots+1) - \alpha_1h_1^{m-1}(h_1^{m-1}+\cdots+1) \\ &\quad -\alpha_2h_1^{m-1}(h_1^{m-1}+\cdots+1) \\ &= \alpha_1(h_1^{m-1}+\cdots+1) + \alpha_2(h_1^{m-1}+\cdots+1) - \alpha_1(h_1^{m-1}+\cdots+1) \\ &\quad -\alpha_2(h_1^{m-1}+\cdots+1) \\ &= 0. \end{aligned}$$

$$\therefore \ker d_1 \supseteq \operatorname{Im} d_2 \,.$$

Conversely, let $d_1(\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3) = 0$, for some $\gamma_i \in \mathbb{Z}G$, $i = 1,2,3$.
Then $\alpha_1(h_1^{m-1}+\cdots+1)\gamma_1 + \alpha_2(h_2+1)\gamma_2 + [\alpha_1h_2(h_1h_2+1) + \alpha_2h_2(h_1h_2+1)]\gamma_3 = 0$.
Since Y_0 is free on α_1, α_2, we have

$$\left.\begin{aligned}(h_1^{m-1}+\cdots+1)\gamma_1+h_2(h_1h_2+1)\gamma_3=0\\(h_2+1)\gamma_2+(h_1h_2+1)\gamma_3=0\end{aligned}\right\}.$$

or,

$$\left.\begin{aligned}(h_1^{m-1}+\cdots+1)\gamma_1+h_2(h_1h_2+1)\gamma_3= 0 \qquad (i)\\(h_2+1)\gamma_2+h_2(h_1h_2+1)\gamma_3=0 \qquad (ii)\end{aligned}\right\}. \tag{6.4.1}$$

From (i) and (ii)

$$(h_1^{m-1}+\cdots+1)\gamma_1=(h_2+1)\gamma_2=-h_2(h_1h_2+1)\gamma_3=\gamma \text{ (say)}. \tag{6.4.2}$$

$$\therefore \left.\begin{aligned}(h_1-1)\gamma_1=0\\(h_2-1)\gamma_2=0\end{aligned}\right\} \implies \quad gr=\gamma \,\forall\, g\in G.$$

$$\implies \quad \gamma=\sum_{g\in G} g\gamma'. \tag{6.4.3}$$

From (6.4.2) and (6.4.3),we have

$$(h_1^{m-1}+\cdots+1)\gamma_1 = \sum_{g\in G} g\gamma' = (h_1^{m-1}+\cdots+1)(h_2+1)\gamma'.$$

$$\therefore\ (h_1^{m-1}+\cdots+1)[\gamma_1-(h_2+1)\gamma'] = 0$$

$$\therefore\ \gamma_1-(h_2+1)\gamma' = (h_1-1)\gamma_1', \text{for some } \gamma_1'\in\mathbb{Z}G,$$

by Majumdar's Lemma

$$\therefore\ \gamma_1 = (h_1-1\gamma_1'+(h_2+1)\gamma'. \tag{6.4.4}$$

From (6.4.2)

$$(h_2+1)\gamma_2 = \gamma = \sum_{g\in G} g\gamma' = (h_2+1)(h_1^{m-1}+\cdots+1)\gamma',$$

for some $\gamma'\in\mathbb{Z}G$.

$$\therefore\ (h_2+1)[\gamma_2-(h_1^{m-1}+\cdots+1)\gamma'] = 0\,.$$

$$\therefore\ \gamma_2 = (h_2-1)\gamma_2'+(h_1^{m-1}+\cdots+1)\,. \tag{6.4.5}$$

Also from (6.4.2) and (6.4.3)

$$-h_2(h_1h_2+1)\gamma_3 = \gamma = \sum_{g\in G} g\gamma' = (h_1h_2+1)(h_1^{m-1}+\cdots+1)\gamma'\,.$$

Multiplying by h_2,we have

$$-(h_1h_2+1)\gamma_3 = (h_1h_2+1)(h_1^{m-1}+\cdots+1)\gamma'$$

$$\Longrightarrow \quad \gamma_3 = -(h_1^{m-1}+\cdots+1)\gamma' - (h_1h_2-1)\gamma_3' \,. \tag{6.4.6}$$

Hence

$$\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3$$

$$= \beta_1(h_1-1)\gamma_1' + \beta_1(h_2+1)\gamma_4 + \beta_2(h_2-1)\gamma_2' + \beta_2(h_1^{m-1}+\cdots+1)\gamma_4$$

$$-\beta_3(h_1^{m-1}+\cdots+1)\gamma_4 - \beta_3(h_1h_2-1)\gamma_3'$$

$$= \beta_1(h_1-1)\gamma_1' + \beta_2(h_2-1)\gamma_2' - \beta_3(h_1h_2-1)\gamma_3' + [\beta_1(h_2+1)$$

$$+\beta_2(h_1^{m-1}+\cdots+1) - \beta_3(h_1^{m-1}+\cdots+1)]\gamma_4$$

$$= d_2(\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3+\beta_4\gamma_4) \,.$$

$$\therefore\ ker d_1 \subseteq Im d_2 \,.$$

Exactness at Y_2 from the right

$$\begin{aligned}
d_2d_3(\delta_1) &= d_2(\gamma_1(h_1^{m-1}+\cdots+1)) = d_2(\gamma)(h_1^{m-1}+\cdots+1) \\
&= \beta_1(h_1-1)(h_1^{m-1}+\cdots+1) = 0 \,. \\
d_2d_3(\delta_2) &= d_2(\gamma_2(h_2+1)) = \beta_2(h_2-1)(h_2+1) = 0 \,. \\
d_2d_3(\delta_3) &= d_2(\gamma_3(h_1h_2+1)) = -\beta_3(h_1h_2-1)(h_1h_2+1) = 0 \,. \\
d_2d_3(\delta_4) &= d_2[-\gamma_1\{(h_1^{m-1}+\cdots+1)h_2 + (h_1h_2-1)\} + \gamma_4(h_1-1)] \\
&= [-\beta_1[(h_1-1)(h_1^{m-1}+\cdots+1)h_2 + (h_1h_2+1)] + \beta_1(h_2+1) \\
&\quad +\beta_2(h_1^{m-1}+\cdots+1) - \beta_3(h_1^{m-1}+\cdots+1)](h_1-1) = 0 \,. \\
d_2d_3(\delta_5) &= d_2[-\gamma_2(h_1^{m-1}+\cdots+1) - \gamma_3(h_1^{m-1}+\cdots+1) + \gamma_4(h_2-1)] \\
&= -\beta_2(h_2-1)(h_1^{m-1}+\cdots+1) - \gamma_3(h_1^{m-1}+\cdots+1) + \gamma_4(h_2-1) = 0 \,.
\end{aligned}$$

$$\therefore\ \ker d_2 \supseteq \operatorname{Im} d_3 .$$

Conversely, suppose $(\gamma_1\eta_1 + \gamma_2\eta_1 + \gamma_3\eta_1 + \gamma_4\eta_1) \in ker d_2$, for some $\eta_i \in \mathbb{Z}G$,

$$\therefore\ d_2(\gamma_1\eta_1 + \gamma_2\eta_2 + \gamma_3\eta_3 + \gamma_4\eta - 4) = 0 .$$

$$\therefore\ \beta_1(h_1 - 1)\eta_1 + \beta_2(h_2 - 1)\eta_2 - \beta_3(h_1h_2 - 1)\eta_3 + [\beta_1(h_2 - 1)$$

$$+\beta_2(h_1^{m-1} + \cdots + 1) - \beta_3(h_1^{m-1} + \cdots + 1)]\eta_4 = 0 .$$

$$\therefore\ \beta_1[(h_1 - 1)\eta_1 + (h_2 - 1)\eta_4] + \beta_2[(h_2 - 1)\eta_2 + (h_1^{m-1} + \cdots + 1)]\eta_4$$

$$-\beta_3[(h_1h_2 - 1)\eta_3 + (h_1^{m-1} + \cdots + 1)]\eta_4 = 0 .$$

Since Y_2 is free on β_1, β_2 and β_3, we have

$$\left.\begin{aligned} (h_1 - 1)\eta_1 + (h_2 - 1)\eta_4 &= 0 \qquad (iii) \\ (h_2 - 1)\eta_2 + (h_1^{m-1} + \cdots + 1)\eta_4 &= 0 \qquad (iv) \\ (h_1h_2 - 1)\eta_3 + (h_1^{m-1} + \cdots + 1)\eta_4 &= 0 \qquad (v) \end{aligned}\right\} . \tag{6.4.7}$$

From (iv), we have

$$(h_2 + 1)(h_1^{m-1} + \cdots + 1)\eta_4 = 0$$

$$\implies \sum g\eta_4 = 0$$

$$\implies \eta_4 = (h_1 - 1\eta_4' + (h_2 - 1)\eta_4'' . \tag{6.4.8}$$

From (iii) and (6.4.7), we have

$$(h_1 - 1)\eta_1 + (h_2 + 1)(h_1 - 1)\eta_4' = 0$$

$$\Rightarrow\ (h_1 - 1)\eta_1 + (h_2h_1 + h_1 - h_2 - 1)\eta_4' = 0$$

$$\Rightarrow\ (h_1 - 1)\eta_1 + (h_1^{m-1}h_2 - 1)\eta_4' + h_1\eta_4' - h_2\eta_4' = 0$$

$$\Rightarrow\ (h_1 - 1)\eta_1 + (h_1^{m-1} - 1)(h_2 + 1)\eta_4' + h_1\eta_4' - H_1^{m-1}\eta_4' = 0$$

$$\Rightarrow\ (h_1 - 1)\eta_1 - (h_1 - 1)(h_1^{m-2} + \cdots + 1)(h_2 + 1)\eta_4'$$

$$-(h_1 - 1)(h_1^{m-3} + \cdots + 1)h_1\eta_4' = 0 .$$

$$\Rightarrow \quad \eta_1' = (h_1^{m-1}+\cdots+1)\eta_1' + (h_1^{m-3}+\cdots+1)h_1\eta_4'$$

$$-(h_1^{m-2}+\cdots+1)(h_2+1)\eta_4'$$

$$= (h_1^{m-1}+\cdots+1)\eta_1' - (h_1^{m-2}+\cdots+1)h_2\eta_4' - \eta_4' \,.$$

$$\therefore \quad \eta_1 = (h_1^{m-1}+\cdots+1)[\eta_1' - h_2\eta_4'] + h_1^{m-1}h_2\eta_1' - \eta_4' \,. \tag{6.4.9}$$

From (iv)and (6.4.8), we have

$$(h_2-1)\eta_2 + (h_1^{m-1}+\cdots+1)(h_2-1)\eta_4'' = 0$$

$$\Rightarrow \quad (h_2-1)\eta_2 + (h_2-1)(h_1^{m-1}+\cdots+1)\eta_4'' = 0\,.$$

$$\therefore \quad \eta_2 = (h_2+1)\eta_2' - (h_1^{m-1}+\cdots+1)\eta_4'' \,. \tag{6.4.10}$$

From (v)and (6.4.8), we have

$$-(h_1h_2-1)\eta_3 - (h_1^{m-1}+\cdots+1)(h_2-1)\eta_4'' = 0$$

$$\Rightarrow \quad (h_1h_2-1)\eta_3 + (h_1^{m-1}+\cdots+1)(h_1h_2-1)\eta_4'' = 0\,.$$

$$\therefore \quad \eta_3 = (h_1h_2+1)\eta_3' - (h_1^{m-1}+\cdots+1)\eta_4'' \,. \tag{6.4.11}$$

$$\therefore \quad \gamma_1\eta_1 + \gamma_2\eta_2 + \gamma_3\eta_3 + \gamma_4\eta_4$$

$$= \gamma_1[(h_1^{m-1}+\cdots+1)[\eta_1' - h_2\eta_4'] + h_1^{m-1}h_2\eta_1' - \eta_4']$$

$$+\gamma_2[(h_2+1)\eta_2' - (h_1^{m-1}+\cdots+1)\eta_4''] + \gamma_3[(h_1h_2+1)\eta_3'$$

$$-(h_1^{m-1}+\cdots+1)\eta_4''] + \gamma_4[(h_1-1)\eta_4' + (h_2-1)\eta_4'']$$

$$= \gamma_1(h_1^{m-1}+\cdots+1)\eta_1' + \gamma_2(h_2+1)\eta_2' + \gamma_3(h_1h_2+1)\eta_3'$$

$$+[-\gamma_1\{(h_1^{m-1}+\cdots+1)h_2 + (h_1h_2-1)\} + \gamma_4(h-1-1)]\eta_4'$$

$$+[-\gamma_2(h_1^{m-1}+\cdots+1) - \gamma_3(h_1^{m-1}+\cdots+1) + \gamma_4(h_2-1)]\eta_4'']$$

$$= d_3(\gamma\delta_1\eta - 1 + \gamma_2\eta_2 + \gamma_3\eta_3 + \gamma_4\eta_4 + \gamma_5\eta_5)\,.$$

$$\therefore\ \ker d_2 \subseteq \mathrm{Im} d_3.$$

Homology groups of G

Let A be a $\mathbb{Z}G-$ *module.* Then the homology groups $H_n(G, A)$ is the homology of the complex

$$Y_3 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_3\otimes 1} Y_2 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_2\otimes 1} Y_1 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_1\otimes 1} Y_0 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_0\otimes 1} \mathbb{Z}G \bigotimes_{\mathbb{Z}G} A \longrightarrow 0$$

or, equivalently by

$$A^5 \xrightarrow{\bar{d}_3} A^4 \xrightarrow{d_2} A^3 \xrightarrow{\bar{d}_1} A^2 \xrightarrow{\bar{d}_0} A \longrightarrow 0,$$

where $\bar{d}_0$, $\bar{d}_1$, $\bar{d}_2$ and $\bar{d}_3$ are given by

$$\bar{d}_0(a_1, a_2) = (h_1-1)a_1 + (h_2-1)a_2\,,$$

$$\bar{d}_1(a_1, a_2, a_3) = ((h_1^{m-1}+\cdots+1)a_1 + h_2(h_1h_2+1)a_3, ((h_2+1)a_2$$
$$+h_2(h_1h_2+1)a_3))\,,$$

$$\bar{d}_2(a_1, a_2, a_3, a_4) = ((h_1-1)a_1 + (h_2+1)a_4, (h_2-1)a_2 + (h_1^{m-1}$$
$$+\cdots+1)a_4, -(h_1h_2-1)a_3 - (h_1^{m-1}+\cdots+1)a_4)\,,$$

$$\bar{d}_3(a_1, a_2, a_3, a_4, a_5) = ((h_1^{m-1}+\cdots+1)a_1 - h_2(h_1^{m-1}+\cdots+1)a_4, (h_2+1)a_2$$
$$-(h_1^{m-1}+\cdots+1)a_5, (h_1h_2+1)a_3 - (h_1^{m-1}$$
$$+\cdots+1)a_5, (h_1-1)a_4 + (h_2-1)a_4)\,.$$

If A is trivial, then

$$\begin{aligned} d_0(a_1, a_2) &= 0. \\ d_1(a_1, a_2, a_3) &= (ma_1 + 2a_3, 2a_2 + 2a_3). \\ d_2(a_1, a_2, a_3, a_4) &= (2a_4, ma_4, -ma_4). \\ d_3(a_1, a_2, a_3, a_4, a_5) &= (ma_1 - ma_4, 2a_2 - ma_5, 2a_3 - ma_5, 0). \end{aligned}$$

Let $A = \mathbb{Z}$,then we have

$$\begin{aligned} H_0(G, \mathbb{Z}) &= \frac{\mathbb{Z}}{\mathrm{Im}\bar{d}_0} = \frac{\mathbb{Z}}{0} \cong \mathbb{Z}. \\ H_1(G, \mathbb{Z}) &= \frac{\ker\bar{d}_0}{\mathrm{Im}\bar{d}_1} = \frac{\{(a_1, a_2)|a_1, a_2 \in \mathbb{Z}\}}{\{(ma_1 + 2a_3, 2a_2 + 2a_3)|a_1, a_2, a_3 \in \mathbb{Z}\}} \\ &= \frac{<(1, 0)> \oplus <(0, 1)>}{<(m, 0)a_1 + (0, 2)a_2 x + (2, 2)a_3, >} \\ &= \frac{<(1, 0)> \oplus <(0, 1)>}{<m(1, 0)> + <2(0, 1)> + <2(1, 0) + 2(0, 1)>} \\ &= <x, y|\, mx = 2y = 2(x+y) = 0> \\ &= <x, y|\, mx = 2x = 2y = 0> \\ &= \begin{cases} <x, y|\, x = 0, 2y = 0> & \text{if } m \text{ is odd} \\ <x, y\,|2x = 2y = 0> & \text{if } m \text{ is even} \end{cases} \\ &= \begin{cases} <y|\, 2y = 0> & \text{if } m \text{ is odd} \\ <x, y\,|2x = 2y = 0> & \text{if } m \text{ is even} \end{cases} \end{aligned}$$

$$= \begin{cases} \mathbb{Z}_2, & \text{if } m \text{ is odd} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } m \text{ is even .} \end{cases}$$

$$\begin{aligned} H_2(G,\mathbb{Z}) &= \frac{ker\bar{d}_1}{Im\bar{d}_2} = \frac{\{(a_1,\ a_2,\ a_3)\,|\, ma_1 + 2a_3 \ =\ 2a_2 + 2a_3\ =\ 0\}}{\{(2a_4,\ ma_4,\ -ma_4)\,|\, a_4 \in \mathbb{Z}\}} \\ &= \frac{\{(2,\ m,\ -m)a\,|\, a_\in \mathbb{Z}\}}{\{(2,\ m,\ -m)\,|\, a \in \mathbb{Z}\}} \\ &= \frac{<\ (2,\ m,\ -m)\ >}{<\ (2,\ m,\ -m)\ >} \\ &= 0. \end{aligned}$$

$$\begin{aligned} H_3(G,\mathbb{Z}) &= \frac{ker\bar{d}_2}{Im\bar{d}_3} \\ &= \frac{\{(a_1,\ a_2,\ a_3,\ a_4)\,|\, 2a_4\ =\ ma_4\ =\ -ma_4\ =\ 0\ =\ 0\}}{\{(ma_1 - ma_4,\ 2a_2 - ma_5,\ 2a_2 - ma_5,\ 0)\,|\, a_1,\ a_2a_4,\ a_5 \in \mathbb{Z}\}} \\ &= \frac{\{(a_1,\ a_2,\ a_3,\ 0\}}{\{(ma_1 - ma_4,\ 2a_2 - ma_5,\ 2a_2 - ma_5,\ 0)\}}\ . \end{aligned}$$

$$H_3(G,\mathbb{Z}) = \frac{<(1,0,0,0)> \oplus <(0,1,0,0)> \oplus <(0,0,1,0)}{<m(1,0,0,0)> + <2(0,1,0,0)>)> + <2(0,0,1,0)> + <-m(1,0,0,0,0)> - <m(0,1,0,0) + m(0,0,1,0)>}$$

$$= \ <\ x,\ y,\ z\,|\, mx\ =\ 2y\ =\ 2z\ =\ -mx\ =\ -m(y+z)\ =\ 0\ >$$

$$= \begin{cases} <\ x,\ y,\ z\,|\, mx\ =\ 2y\ =\ 2z\ =\ 0\ > & \text{if } m \text{ is even} \\ <\ x,\ y,\ z\,|\, mx\ =\ 2y\ =\ 0\ > & \text{if } m \text{ is odd} \end{cases}$$

$$= \begin{cases} \mathbb{Z}_m \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, & \text{if } m \text{ is even} \\ \mathbb{Z}_m \oplus \mathbb{Z}_2, & \text{if } m \text{ is odd .} \end{cases}$$

Detemination of cohomology groups of $\mathbb{Z}$

Let A be a right $\mathbb{Z}G-module$. Then the cohomology groups $H^n(G,A)$ is the cohomology of the complex

$$\cdots\cdots \longleftarrow \mathrm{Hom}_{\mathbb{Z}G}(Y_3, A) \xleftarrow{d_3^*} \mathrm{Hom}_{\mathbb{Z}G}(Y_2, A) \xleftarrow{d_2^*} \mathrm{Hom}_{\mathbb{Z}G}(Y_1, A) \xleftarrow{d_1^*} \mathrm{Hom}_{\mathbb{Z}G}(Y_0, A) \xleftarrow{d_0^*} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \longleftarrow 0\ ,$$

or equivalently,

$$\cdots\cdots \longleftarrow A^5 \overset{d_3^{**}}{\longleftarrow} A^4 \overset{d_2^{**}}{\longleftarrow} \mathrm{Hom}A^3 \overset{d_1^{**}}{\longleftarrow} A^2 \overset{d_0^{**}}{\longleftarrow} A \longleftarrow 0,$$

where d_0^{**}, d_1^{**}, d_2^{**}, d_3^{**}, $\cdots$ are given by

$$\begin{aligned}
d_0^{**}(a) &= (a(h_1-1),\, a(h_2-1)),\\
d_1^{**}(a_1,\, a_2) &= (a_1(h_1^{m-1}+\cdots+1),\, a_2(h_2+1), a_1h_2(h_1h_2+1)+a_2h_2(h_1h_2+1)),\\
d_2^{**}(a_1,\, a_2,\, a_3) &= (a_1(h_1-1),\, a_2(h_2-1),\, a_3(h_1h_2-1),\, a_1(h_2+1)\\
&\quad +a_2(h_1^{m-1}+\cdots+1)-a_3(h_1^{m-1}+\cdots+1)),\\
d_2^{**}(a_1,\, a_2,\, a_3) &= (a_1(h_1^{m-1}+\cdots+1),\, a_2(h_2+1),\, a_3(h_1h_2+1))\\
&\quad a_1[-h_2(h_1^{m-1}+\cdots+1)+(h_1h_2-1)]+a_4(h_1-1),\\
&\quad -a_2(h_1^{m-1}+\cdots+1)-a_3(h_1^{m-1}+\cdots+1)+a_4(h_2-1)).
\end{aligned}$$

If A is trivial, then

$$\begin{aligned}
d_0^{**}(a) &= (0,\, 0).\\
d_1^{**}(a_1,\, a_2) &= (ma_1,\, 2a_2,\, 2a_1+2a_2).\\
d_2^{**}(a_1,\, a_2,\, a_3) &= (0,\, 0,\, 0,\, 2a_1+ma_2-ma_3).\\
d_3^{**}(a_1,\, a_2,\, a_3,\, a_4) &= (ma_1,\, 2a_2,\, 2a_3,\, ma_1,\, -ma_2-ma_3).
\end{aligned}$$

Let $A = \mathbb{Z}$, then we have

$$\begin{aligned}
H^0(G,\, \mathbb{Z}) &= \frac{\ker d_0^{**}}{0} = \frac{\mathbb{Z}}{0} \cong \mathbb{Z}.\\
H^1(G,\, \mathbb{Z}) &= \frac{\ker d_1^{**}}{\mathrm{Im} d_0^{**}}\\
&= \frac{\{(a_1,\, a_2)\,|\, ma_1 = 2a_2 = 2a_1+2a_2 = 0,\ a_1, a_2 \in \mathbb{Z}\}}{\{(0,\, 0)\}}\\
&= 0.
\end{aligned}$$

$$H^2(G, \mathbb{Z}) = \frac{\ker d_2^{**}}{\mathrm{Im} d_1^{**}}$$

$$= \frac{\{(a_1,\ a_2,\ a_3)\,|\,2a_1 + ma_2 - ma_3 = 0,\ a_1, a_2, a_3 \in \mathbb{Z}\}}{\{(ma_1,\ 2a_2,\ 2a_1 + 2a_2)\}}.$$

Let m be even, say $m = 2m'$. Then

$$H^2(G, \mathbb{Z}) = \frac{\{(a_1,\ a_2,\ a_3)\,|\,2a_1 + 2m'a_2 - 2m'a_3 = 0,\ a_1, a_2, a_3 \in \mathbb{Z}\}}{\{(ma_1,\ 2a_2,\ 2a_1 + 2a_2)\}}$$

$$= \frac{\{(a_1,\ a_2,\ a_3)\,|\,a_1 + m'a_2 - m'a_3 = 0\}}{\{(ma_1,\ 2a_2,\ 2a_1 + 2a_2)\}}$$

$$= \frac{\{(m'(a_3 - a_2),\ a_2,\ a_3\}}{\{(2m'a_1,\ 2a_2,\ 2a_1 + 2a_2)\}}$$

$$\cong \mathbb{Z}_2 .$$

Let $m$be odd, say, $m = 2m' + 1$.

$$\therefore H^2(G, \mathbb{Z}) = \frac{\{(a_1, a_2, a_3)\ |\ 2a_1 + (2m'+1)(a_2 - a_3) = 0,\ a_1, a_2, a_3 \in \mathbb{Z}\}}{\{((2m'+1)a_1,\ 2a_2,\ 2a_1 + 2a_2)\}}$$

$$= \frac{\{((2m'+1)a_1', a_2, a_2 + 2a_1')\ |\ a_1, a_2, a_3 \in \mathbb{Z}\}}{\{(2m'+1)a_1,\ 2a_2,\ 2a_1 + 2a_2\}}$$

$$= \frac{\{a_1'((2m'+1), 0, 0) + a_2(0, 1, 1)\}}{\{a_1'((2m'+1), 0, 0) + a_2(0, 2, 2)\}}$$

$$= \frac{< ((2m'+1), 0, 0) > + < (0, 1, 1) >}{< ((2m'+1), 0, 0) > + < (0, 2, 2) >}$$

$$= < (0, 1, 1) + Imd_1^{**} >$$

$$= \mathbb{Z}_2 \cdot$$

$$
\begin{aligned}
H^3(G,\mathbb{Z}) &= \frac{ker d_3^{**}}{Im d_2^{**}} \\
&= \frac{\{(a_1,\ a_2,\ a_3,\ a_4)\,|\,2ma_1 = 2a_2 = 2a_3 = ma_1 = -ma_2 - ma_3 \ = \ 0 \ a_1, a_2, a_3 \in \mathbb{Z}\}}{\{(0,\ 0,\ 0,\ 2a_1 + ma_2 - ma_3)\}} \\
&= \frac{\{(0,\ 0,\ 0,\ a_4)\,|\,a_4 \in \mathbb{Z}\}}{\{(0,\ 0,\ 0,\ 2a_1 + ma_2 - ma_3)\}} \\
&= \begin{cases} \frac{\{(0,0,0,a_4)\,|\,a_4 \in \mathbb{Z}\}}{\{(0,0,0,a)\,|\,a \in \mathbb{Z}\}} & \text{if } m \text{ is odd} \\ \frac{\{(0,0,0,a_4)\,|\,a_4 \in \mathbb{Z}\}}{\{(0,0,0,2)\,|\,a \in \mathbb{Z}\}} & \text{if } m \text{ is even} \end{cases} \\
&= \begin{cases} \frac{<(0,0,0,1)>}{<(0,0,0,1)>} \cong 0 \\ \frac{<(0,0,0,1)>}{<(0,0,0,2)>} \cong \mathbb{Z}_2 \ . \end{cases}
\end{aligned}
$$

6.5. The Fundamental Group of the Union of Tori $T_1, \cdots, T_n$ $(n \geq 2)$ such that T_j and $T_{j+1}(i \leq j \leq n-1)$ intersect at a Single Point.

6.5.1 Introduction

Let $T_i, \cdots, T_n$ be n two dimensional tori such that each T_j and $T_{j+1}(i \leq j \leq n-1)$ intersect at a single point. Let $X = \cup T_i$, $i = 1, 2, \cdots, n$. In this section we constructed a free resolution of $\pi(X)$ and from there determined its homology and cohomology.

By Seifert- van Kampen Theorem (Massey [87],Th.3.1,p.122)
$\prod(X) \cong \prod(T_i) * \cdots * \prod(T_n)$, where $*$ denotes the free product. Hence $\prod(X)$ is given by the presention

$$\text{generators}: \quad x_1, x_1, \cdots, x_{2n-1}, x_{2n};$$

$$\text{relations}: \quad x_1^{-1}x_1^{-2}x_1x_2,\ x_3^{-1}x_4^{-1}x_3x_4, \cdots, x_{2n-1}^{-1}x_{2n}^{-1}x_{2n-1}x_{2n}.$$

6.5.2 Free Resolution for $\mathbb{Z}$

We write G for $\prod(X)$. Then $G = \frac{F}{R}$, where F is a free group generated by $x_1, x_1, \cdots, x_{2n-1}, x_{2n}$ and R is the normal closure of $r_1, \cdots, r_n$.
$r_1 = x_1^{-1}x_1^{-2}x_1x_2$, $r_2 = x_3^{-1}x_4^{-1}x_3x_4, \cdots, r_n = x_{2n-1}^{-1}x_{2n}^{-1}x_{2n-1}x_{2n}$

Then the Fox derivatves of $r_1, r_2, \cdots, r_n$ are

$$\frac{\partial r_1}{\partial x_1} = -r_1 + x_2; \qquad \frac{\partial r_i}{\partial x_1} = 0, \ i \neq 1,$$

$$\frac{\partial r_1}{\partial x_2} = -x_2^{-1}x_1x_2 + 1; \qquad \frac{\partial r_i}{\partial x_2} = 0, \ i \neq 1,$$

$$\frac{\partial r_2}{\partial x_3} = -r_2 + x_4; \qquad \frac{\partial r_i}{\partial x_3} = 0, \ i \neq 2,$$

$$\frac{\partial r_2}{\partial x_4} = -x_4^{-1}x_3x_4 + 1; \qquad \frac{\partial r_i}{\partial x_4} = 0, \ i \neq 2,$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{\partial r_n}{\partial x_{2n-1}} = -r_n + x_{2n}; \qquad \frac{\partial r_i}{\partial x_{2n-1}} = 0, \ i \neq n,$$

$$\frac{\partial r_n}{\partial x_{2n}} = -x_{2n-1}^{-1}x_{2n-1}x_{2n} + 1; \qquad \frac{\partial r_i}{\partial x_{2n}} = 0, \ i \neq n,$$

Writing $\pi(x_i) = h_i, i = 1, 2,$we have

$$\frac{\partial r_1}{\partial x_1} = -1 + h_2; \qquad \frac{\partial r_i}{\partial x_1} = 0, \ i \neq 1,$$

$$\frac{\partial r_1}{\partial x_2} = -h_2^{-1}h_1h_2 + 1; \qquad \frac{\partial r_i}{\partial x_2} = 0, \ i \neq 1.$$

$$\frac{\partial r_2}{\partial x_3} = -1 + h_4; \qquad \frac{\partial r_i}{\partial x_3} = 0, \ i \neq 2.$$

$$\frac{\partial r_2}{\partial x_4} = -h_4^{-1}h_3h_4 + 1; \qquad \frac{\partial r_i}{\partial x_4} = 0, \ i \neq 2.$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{\partial r_n}{\partial x_{2n-1}} = -1 + h_{2n}; \qquad \frac{\partial r_i}{\partial x_{2n-1}} = 0, \ i \neq n.$$

$$\frac{\partial r_n}{\partial x_{2n}} = -h_{2n}^{-1}h_{2n-1}h_{2n} + 1; \qquad \frac{\partial r_i}{\partial x_{2n}} = 0, \ i \neq n.$$

We then have

Theorem 6.5.1

The following is a free resolution of $\mathbb{Z}$;

$$0 \longrightarrow Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where Y_0, Y_1 are right $FG-modules$ freely generated by $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n-1}, \alpha_{2n}\}$ and $\{\beta_1, \beta_2, \cdots, \beta_n\}$ and ε, d_0 and d_1 are defined by

$$\varepsilon(g) = 1, \qquad \forall g \in G.$$

$$d_0(\alpha_i) = h_i - 1, \qquad i = 1, 2, \cdots, 2n;$$

$$d_1(\beta_j) = \sum_{i=1}^{2n} \alpha_i \prod\left(\frac{\partial r_j}{\partial x_i}\right), \qquad j = 1, 2, \cdots, n;$$

$$\begin{aligned} d_1(\beta_j) &= \sum_{i=1}^{2n} \alpha_i \prod\left(\frac{\partial r_j}{\partial x_i}\right); \\ &= \alpha_1(h_{j+1} - 1) + \alpha_2(1 - h_j), \qquad j = 1, 2, \cdots, n. \end{aligned}$$

Proof

Exactness at Y_1 from the right

Suppose

$$\beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_n\gamma_n \in \ker d_1.$$

Then

$$d_1(\beta_1\gamma_1 + \beta_2\gamma_2 + \cdots + \beta_n\gamma_n) = 0.$$

$$\therefore\ [\alpha_1(h_2 - 1) + (1 - h_1)]\gamma_1 + [\alpha_3(h_4 - 1) + (1 - h_3)]\gamma_2$$

$$+ \cdots + [\alpha_{2n-1}(h_{2n} - 1) + \alpha_{2n}(1 - h_{2n-1})]\gamma_n = 0.$$

Since Y_0 is free on $\alpha_1, \alpha_2, \cdots, \alpha_{2n}$, we have

$$\left.\begin{aligned} (h_2 - 1)\gamma_1 &= 0 \quad (i) \\ (1 - h_1)\gamma_1 &= 0 \quad (ii) \end{aligned}\right\} \qquad (6.5.1)$$

$$\left.\begin{aligned} (h_4 - 1)\gamma_1 &= 0 \quad (iii) \\ (1 - h_3)\gamma_1 &= 0 \quad (iv) \end{aligned}\right\} \qquad (6.5.2)$$

....

....

$$\left.\begin{aligned} (h_{2n} - 1)\gamma_1 &= 0 \quad (v) \\ (1 - h_{2n-1})\gamma_1 &= 0 \quad (vi) \end{aligned}\right\}. \qquad (6.5.3)$$

We write Equation (6.5.1), (6.5.2)and (6.5.3) as

$$\left.\begin{aligned} (h_2 - 1)\gamma_1 &= 0 \quad (vii) \\ (h_1 - 1)\gamma_1 &= 0 \quad (viii) \end{aligned}\right\} \qquad (6.5.4)$$

$$\left.\begin{aligned} (h_4 - 1)\gamma_1 &= 0 \quad (ix) \\ (h_3 - 1)\gamma_1 &= 0 \quad (x) \end{aligned}\right\} \qquad (6.5.5)$$

....

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left.\begin{array}{lll}(h_{2n}-1)\gamma_1 &= 0 & (xi) \\ (h_{2n-1}-1)\gamma_1 = 0 & & (xii)\end{array}\right\}. \qquad (6.5.6)$$

$\therefore \gamma_1 = \gamma_2 = \cdots = \gamma_n = 0$, since $h_1, h_2, \cdots, h_{2n}$ do not have finite orders.

Homology groups of G

Let A be a $\mathbb{Z}G-$ *module.* Then the homology groups $H_n(G, A)$ is the homology of the complex

$$0 \longrightarrow Y_1 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_1 \otimes 1} Y_0 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_0 \otimes 1} \mathbb{Z}G \bigotimes_{\mathbb{Z}G} A \longrightarrow 0$$

or, equivalently by

$$0 \longrightarrow A^n \xrightarrow{\bar{d}_1} A^{2n} \xrightarrow{\bar{d}_0} A \longrightarrow 0,$$

where $A^i = A \oplus \cdots \oplus A(i\text{-copies})$.

Since $Y_i \cong \mathbb{Z}G \oplus \mathbb{Z}G \oplus \cdots \oplus \mathbb{Z}G$ and $\bar{d}_0$, $\bar{d}_1$,are given by

$$\begin{aligned}
\bar{d}_0(a_1, a_2, \cdots, a_{2n}) &= (h_1-1)a_1 + (h_2-1)a_2 + \cdots + (h_{2n}-1)a_{2n} \\
\bar{d}_1(a_1, a_2, \cdots, a_n) &= \left(\sum_{j=1}^{n} \Pi\left(\frac{\partial r_j}{\partial x_1}\right) a_1, \sum_{j=1}^{n} \Pi\left(\frac{\partial r_j}{\partial x_2}\right) a_2, \cdots, \sum_{j=1}^{n} \Pi\left(\frac{\partial r_j}{\partial x_{2n}}\right) a_{2n}\right) \\
&= ((h_2-1)a_1, (h_1-1)a_1, (h_4-1)a_2, (h_3-1)a_2, \cdots, \\
&\quad (h_{2n}-1)a_n, (h_{2n-1}-1)a_n).
\end{aligned}$$

If A is trivial, then

$$\begin{aligned}
\bar{d}_0(a_1, a_2, \cdots a_{2n}) &= 0 \\
\bar{d}_1(a_1, a_2, \cdots, a_n) &= (0, 0, \cdots, 0)_{2n\text{-copies}}.
\end{aligned}$$

If $A = \mathbb{Z}$, then

$$H_0(G,\mathbb{Z}) = \frac{\mathbb{Z}}{0} = \mathbb{Z}.$$

$$H_1(G,\mathbb{Z}) = \frac{\ker d_0}{\operatorname{Im} d_1} = \frac{\{(a_1, a_2, \cdots, a_{2n}) \mid a_i \in \mathbb{Z}\}}{\{(0,0,\cdots,0)\}}$$

$$\cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z})_{2n\text{-copies}}$$

$$H_2(G,\mathbb{Z}) = \frac{\ker d_1}{0} = \frac{\{(a_1, a_2, \cdots, a_n) \mid a_i \in \mathbb{Z}\}}{\{(0,0,\cdots,0)\}}$$

$$\cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z})_{n\text{-copies}}$$

$$H_n(G,\mathbb{Z}) = 0,\ n > 2\,.$$

Determination of cohomology groups of $\mathbb{Z}$.

Let A be a right $\mathbb{Z}G-$*module*. Then the cohomology groups $H^n(G,A)$ is the cohomology of the complex

$$0 \longleftarrow \operatorname{Hom}(Y_1, A) \overset{d_1^*}{\longleftarrow} \operatorname{Hom}(Y_0, A) \overset{d_0^*}{\longleftarrow} \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \longleftarrow 0$$

or equivalently,

$$0 \longleftarrow A^n \overset{d_1^{**}}{\longleftarrow} A^{2n} \overset{d_0^{**}}{\longleftarrow} A \longleftarrow 0,$$

where d_0^{**}, d_1^{**} are given by

$$d_0^{**}(a_1,\ a_2) = (a(h_1-1),\ a(h_2-1),\ \cdots,\ a(h_{2n}-1))$$

$$d_1^{**}(a_1,\ a_2,\ \cdots,\ a_{2n}) = ([a_1(h_2-1)+a_2(h_1-1)], [a_3(h_4-1)$$
$$+a_4(h_3-1)], [a_{2n-1}(h_{2n}-1)+a_{2n}(h_{2n-1}-1)])\,.$$

If A is trivial , then

$$d_0^{**}((a) = (0,\ 0,\ \cdots,\ 0)(2n-\text{copies})$$

$$d_1^{**}(a_1,\ a_2,\ \cdots,\ a_{2n}) = (0,\ 0,\ \cdots,\ 0)(n-\text{copies})\,.$$

Let $A = \mathbb{Z}$, then

$$H^0(G, \mathbb{Z}) = \frac{A}{0} \cong \mathbb{Z}$$

$$H^1(G, \mathbb{Z}) = \frac{\ker d_1^{**}}{\mathrm{Im} d_0^{**}} = \frac{\{(a_1, a_2, \cdots, a_{2n}) \mid a_i \in \mathbb{Z}\}}{\{(0, 0, \cdots, 0)\}}$$

$$\cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z})(2n - \text{copies})$$

$$H^2(G, \mathbb{Z}) = \frac{A^n}{\mathrm{Im} d_1^{**}} = \frac{\{(a_1, a_2, \cdots, a_n) \mid a_i \in \mathbb{Z}\}}{\{(0, 0, \cdots, 0)\}}$$

$$\cong (\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z})(n - \text{copies}).$$

6.6. Unpermuted Braid Group

6.6.1 We now consider the unpermuted braid group [92] which has presentation $G = < a, b, c \mid [b,\ c] = [a,\ c] = 1 >$ and, determined its homology and cohomology.

6.6.2 Free Resolution for $\mathbb{Z}$

Let $G = \frac{F}{R}$,where F is free group generated by x_1, x_2, x_3 and R is the normal subgroup generated by r_1, r_2 , $r_1 = [x_1, x_2]$ and $r_2 = [x_2, x_3]$.

Then the Fox derivatives of r_1, r_2, r_3 are:

$$\frac{\partial r_1}{\partial x_1} = -r - 1 + x_2; \qquad \frac{\partial r_1}{\partial x_2} = -x_2^{-1}x_1x_2 + 1;$$

$$\frac{\partial r_1}{\partial x_3} = 0; \qquad \frac{\partial r_2}{\partial x_1} = 0;$$

$$\frac{\partial r_2}{\partial x_2} = -r_2 + x_3; \qquad \frac{\partial r_2}{\partial x_3} = -x_3^{-1}x_2x_3 + 1.$$

Writing $\pi(x_i) = h_i\,, i = 1, 2$,we have

$$\frac{\partial r_1}{\partial x_1} = -1 + h_2; \qquad \frac{\partial r_1}{\partial x_2} = -h_2^{-1}h_1h_2 + 1;$$

$$\frac{\partial r_1}{\partial x_3} = 0; \qquad \frac{\partial r_2}{\partial x_1} = 0;$$

$$\frac{\partial r_2}{\partial x_2} = -1 + h_3; \qquad \frac{\partial r_2}{\partial x_3} = -h_3^{-1}h_2h_3 + 1.$$

We then have

Theorem 6.6.1

The following is a free resolution of $\mathbb{Z}$:

$$0 \longrightarrow Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where Y_0, Y_1 are right $\mathbb{Z}G-$ *modules* freely generated by $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2\}$ and

ε, d_0 and d_1 are defined by

$$\begin{aligned}
\varepsilon(g) &= 1, \quad \forall g \in G; \\
d_0(\alpha_i) &= h_i - 1 \quad i = 1,\, 2,\, 3; \\
d_1(\beta_1) &= \alpha_1(h_1 - 1) + \alpha_2(1 - h_1)\,; \\
d_1(\beta_2) &= \alpha_2(h_3 - 1) + \alpha_3(1 - h_2)\,.
\end{aligned}$$

Proof

Exactness at Y_1

Suppose

$$\beta_1\gamma_1 + \beta_2\gamma_2 \in \ker d_1,$$

Then

$$d_1(\beta_1\gamma_1 + \beta_2\gamma_2) = 0\,.$$

$$\therefore \quad [\alpha_1(h_2 - 1) + \alpha_2(1 - h_1)]\gamma_1 + [\alpha_2(h_3 - 1) + \alpha_3(1 - h_2)]\gamma_2 = 0\,.$$

Since Y_0 is free on $\alpha_1, \alpha_2\alpha_3$,we have

$$\left.\begin{aligned}
(h_2 - 1)\gamma_1 &= 0 \qquad (i) \\
(1 - h_1)\gamma_1 + (h_3 - 1)\gamma_2 &= 0 \qquad (ii) \\
(1 - h_2)\gamma_2 &= 0 \qquad (iii)
\end{aligned}\right\}. \qquad (6.7.1)$$

(i) and $(iii) \Longrightarrow \gamma_1 = \gamma_2 = 0$, since h_2 has infinite order.

Homology groups of G

Let A be a $\mathbb{Z}G - module$.Then the homology groups $H_n(G, A)$ is the homology of the complex

$$0 \longrightarrow Y_1 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_1 \otimes 1} Y_0 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_0 \otimes 1} \mathbb{Z}G \bigotimes_{\mathbb{Z}G} A \longrightarrow 0$$

or, equivalently by

$$0 \longrightarrow A^2 \xrightarrow{\bar{d}_1} A^3 \xrightarrow{\bar{d}_0} A \longrightarrow 0,$$

where and $\bar{d}_0$, $\bar{d}_1$, are given by

$$\bar{d}_0(a_1, a_2, a_3) = (h_1 - 1)a_1 + (h_2 - 1)a_2 + (h_3 - 1)a_3 \;;$$

$$\bar{d}_1(a_1, a_2) = = ((h_2 - 1)a_1, (1 - h_1)a_1 + (h_3 - 1)a_2, (1 - h_3)a_2).$$

If A is trivial, then

$$\bar{d}_0(a_1, a_2, a_3) = 0.$$
$$\bar{d}_1(a_1, a_2) = (0, 0, 0).$$

If $A = \mathbb{Z}$, then

$$H_0(G, \mathbb{Z}) = \frac{\mathbb{Z}}{0} = \mathbb{Z}.$$

$$H_1(G, \mathbb{Z}) = \frac{\ker d_0}{\operatorname{Im} d_1} = \frac{\{(a_1, a_2, a_3) \mid a_i \in \mathbb{Z}\}}{\{(0, 0, 0)\}}$$
$$\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

$$H_2(G, \mathbb{Z}) = \frac{\ker d_1}{0} = \frac{\{(a_1, a_2) \mid a_i \in \mathbb{Z}\}}{\{(0, 0)\}}$$
$$\cong \mathbb{Z} \oplus \mathbb{Z}.$$

Determination of cohomology groups of $\mathbb{Z}$.

Let A be a right $\mathbb{Z}G-$*module*. Then the cohomology groups $H^n(G, A)$ is the cohomology of the complex

$$0 \longleftarrow \operatorname{Hom}(Y_1, A) \overset{d_1^*}{\longleftarrow} \operatorname{Hom}(Y_0, A) \overset{d_0^*}{\longleftarrow} \operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \longleftarrow 0$$

or equivalently,

$$0 \longleftarrow A^2 \overset{d_1^{**}}{\longleftarrow} A^3 \overset{d_0^{**}}{\longleftarrow} A \longleftarrow 0,$$

where d_0^{**}, d_1^{**} are given by

$$d_0^{**}(a) = (a(h_1 - 1), a(h_2 - 1), a(h_3 - 1));$$

$$d_1^{**}(a_1, a_2, a_3) = ([(a_1(h_2 - 1) + a_2(1 - h_1)], [a_2(h_3 - 1) + a_3(1 - h_2)]).$$

If A is trivial, then

$$d_0^{**}(a) = (0, 0, 0);$$

$$d_1^{**}(a_1, a_2, a_3) = (0, 0).$$

Let $A = \mathbb{Z}$, then

$$H^0(G, \mathbb{Z}) = \frac{\mathbb{Z}}{0} \cong \mathbb{Z}.$$

$$\begin{aligned} H^1(G, \mathbb{Z}) &= \frac{\ker d_1^{**}}{\operatorname{Im} d_0^{**}} = \frac{\{(a_1, a_2, a_3) \mid a_i \in \mathbb{Z}\}}{\{(0, 0, 0)\}} \\ &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \oplus \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} H^2(G, \mathbb{Z}) &= \frac{\mathbb{Z}^2}{\operatorname{Im} d_1^{**}} = \frac{\{(a_1, a_2) \mid a_i \in \mathbb{Z}\}}{\{(0, 0)\}} \\ &\cong \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

6.7 Quasi-cyclic Group

6.7.1 Introduction

In this section we shall construct a free $\mathbb{Z}G$-resolution for the quasi-cyclic group $G = \mathbb{Z}(p^\infty)$ and determined its homology. The quasi-cyclic group [41] (Prufer group)[60] is an ifinitely generatd abelian group given by

$$\text{generatos}: \quad x_1, x_2, x_3, \cdots$$

$$\text{relations}: \quad x_1^p = 1,\ x_2^p = x_1,\ x_3^p = x_2, \cdots$$

It is dual to an infinite cyclic group. All of its subgroups except itself are finite cyclic while all of its factor group except 1 are isomorphic to the whole group.

6.7.2. We first prove two results about zero-divisors of a group ring. We shall need these for the for the construction of the free resoluton.

Lemma 6.7.1 *Let G be a group and h an element of G of order p^n, $n > 1$. Let $\gamma \in \mathbb{Z}G$ be such that*

$$(1 + h + \cdots + h^{p-1})\gamma = 0, \tag{6.7.1}$$

then

$$\gamma = (1 + h^p + h^{2p} + \cdots + h^{p^n - p})(1 - h)\gamma' \tag{6.7.2}$$

$$= (1 + h^p + \cdots + h^{(p-1)p})(1 + h^{p^2} + \cdots + h^{(p-1)p^2}) \cdots$$

$$\cdots (1 + h^{p^{n-1}} + \cdots + h^{(p-1)p^{n-1}}(1 - h)) \tag{6.7.3}$$

for some $\gamma' \in \mathbb{Z}G$.

Proof

It is easily seen that the right hand sides of (6.7.2) and (6.7.3) are equal.

We prove (6.7.2).

γ may be written as

$$\gamma = \sum_\alpha \left(\sum_{i=0}^{p^n - 1} n_i h^i \right) g_\alpha,$$

where $\{g_\alpha\}$ is a right transversal of the cyclic subgroup H of G generated by h. $\mathbb{Z}G$ is free. Left $\mathbb{Z}H$-module generated by $\{g_\alpha\}$.

Hence we have from (6.7.1)

$$\left(1+h+\cdots+h^{p-1}\right)\left(\sum_{i=0}^{p^n-1} n_i h^i\right) g_\alpha = 0, \tag{6.7.4}$$

$$\therefore\ n_0+n_1+\cdots+n_{p-1}=0 \quad \text{and} \quad n_i = n_{i+p}, \quad 0 \le i \le p_n - p,$$

where the sum $i+p$ is considered modulo p^n.

It therefore follows that

$$\sum_{i=0}^{p^n-1} n_i h^i = (1+h^p+h^{2p}+\cdots+h^{p^n-p})(1-h)\bar{\gamma}_\alpha, \quad \text{for some} \quad \bar{\gamma}_\alpha \in \mathbb{Z}H\,.$$

So, $\gamma = (1+h^p+h^{2p}+\cdots+h^{p^n-p})(1-h)\gamma'$, where $\gamma' = \sum_\alpha \bar{\gamma}_\alpha g_\alpha \in \mathbb{Z}G$.

Lemma 6.7.2 *Let G be a group and h an element of G of order p^n, $n>1$. Let $\gamma \in \mathbb{Z}G$ be such that*

$$(1+h^p+\cdots+h^{(p-1)p})(1+h^{p^2}+\cdots+h^{(p-1)p^2})\cdots(1+h^{p^{n-1}}+\cdots+h^{(p-1)p^{n-1}})(1-h)\gamma = 0, \tag{6.7.5}$$

then

$$\gamma = (1+h+\cdots+h^{p-1})\gamma' \quad \textit{for some} \quad \gamma' \in \mathbb{Z}G.$$

Proof

From Proposition 2 of Majumdar [72]

$$(1+h^p+\cdots+h^{(p-1)p})(1+h^{p^2}+\cdots+h^{(p-1)p^2})\cdots(1+h^{p^{n-1}}+\cdots+h^{(p-1)p^{n-1}})\gamma$$

$$= (1+h+\cdots+h^{p^n-1})\bar{\gamma} \quad \text{for some} \quad \bar{\gamma}\in\mathbb{Z}G$$

$$= (1+h+\cdots+h^{p-1})(1+h^p+\cdots+h^{(p-1)p})\cdots(1+h^{p^{n-1}}+\cdots+h^{(p-1)p^{n-1}})\bar{\gamma}.$$

$$\therefore\ (1+h^p+\cdots+h^{(p-1)p})\cdots(1+h^{p^{n-1}}+\cdots+h^{(p-1)p^{n-1}})\left[\gamma-(1+h+\cdots+h^{p-1})\bar{\gamma}\right]=0$$

$$\text{i.e.,} \quad (1+h^p+\cdots+h^{p^n-p})\left[\gamma-(1+h+\cdots+h^{p-1})\bar{\gamma}\right]=0.$$

Hence by Proposition 1 of Majumdar [66],

$$\begin{aligned}\gamma-(1+h+\cdots+h^{p-1})\bar{\gamma} &= (1-h^p)\bar{\bar{\gamma}}, \quad \text{for some} \quad \bar{\bar{\gamma}}\in\mathbb{Z}G.\\ &= (1+h+\cdots+h^{p-1})(1-h^p)\bar{\bar{\gamma}}\,.\end{aligned}$$

$$\therefore \gamma = (1+h+\cdots+h^{p-1})[(1-h)\bar{\bar{\gamma}} - \bar{\gamma}]$$

$$= (1+h+\cdots+h^{p-1})\gamma', \quad \text{where} \quad \gamma' = ((1-h)\bar{\bar{\gamma}} - \bar{\gamma}) \in \mathbb{Z}G.$$

Hence the lemma.

6.7.3 The quasi-cyclic group (Prufer group) $G = \mathbb{Z}(p^\infty)$ has presentation $G = \frac{F}{R}$, where F is the free group generated by $\{x_i, i \in \mathbb{N}\}$ and R is the normal clousere of $\{r_j, j \in \mathbb{N}\}$, $\{r_1 = x_1^p, r_{n+1} = x_n^{-1}x_{n+1}, n \in \mathbb{N}\}$ (see Fuchs [41], Huebschmann [59]). It is clear that G is abelian and is the union of an infinite ascending sequence of cyclic groups $C_p \subset C_{p^2} \subset C_{p^3} \subset \cdots$.

Though G is not finitely presented, the Lyndon's partial resolution can still be constructed for G, since each r_j contains a finite number of x_i's. Here

$$\begin{array}{lll}
\frac{\partial r_1}{\partial x_1} = x_1^{p-1} + \cdots + 1, & & \frac{\partial r_1}{\partial x_j} = 0, \quad i \neq 1 \\
\frac{\partial r_2}{\partial x_1} = -r_2, & \frac{\partial r_2}{\partial x_2} = x_2^{p-1} + \cdots + 1, & \frac{\partial r_2}{\partial x_i} = 0, \quad i \neq 1, 2 \\
\cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \\
\frac{\partial r_{i+1}}{\partial x_i} = -r_{i+1}, & \frac{\partial r_{i+1}}{\partial x_{i+1}} = x_{i+1}^{p-1} + \cdots + 1, & \frac{\partial r_{i+1}}{\partial x_j} = 0, \quad j \neq i, i+1 \\
\cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots
\end{array}$$

Using $\pi(x_i) = h_i$, $i \in \mathbb{N}$, we have

$$\begin{array}{lll}
\frac{\partial r_1}{\partial x_1} = h_1^{p-1} + \cdots + 1, & & \frac{\partial r_1}{\partial x_j} = 0, \quad i \neq 1 \\
\frac{\partial r_2}{\partial x_1} = -1, & \frac{\partial r_2}{\partial x_2} = h_2^{p-1} + \cdots + 1, & \frac{\partial r_2}{\partial x_i} = 0, \quad i \neq 1, 2 \\
\cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \\
\frac{\partial r_{i+1}}{\partial x_i} = -1, & \frac{\partial r_{i+1}}{\partial x_{i+1}} = h_{i+1}^{p-1} + \cdots + 1, & \frac{\partial r_{i+1}}{\partial x_j} = 0, \quad j \neq i, i+1 \\
\cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots
\end{array} \tag{6.7.6}$$

Theorem 6.7.3

The following is a free $\mathbb{Z}G$-resolution for $\mathbb{Z}$:

$$\cdots \xrightarrow{d_2} Y \xrightarrow{d_1} Y \xrightarrow{d_2} Y \xrightarrow{d_1} Y \xrightarrow{d_0} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0, \tag{6.7.7}$$

where Y is a right $\mathbb{Z}G$-module free on $\{\alpha_i, i \in \mathbb{N}\}$ and ϵ, d_0, d_1, d_2 are $\mathbb{Z}G$- homomorphisms given by

$$\epsilon(g) = 1, \quad \text{for each } g \in G,$$

$$d_0(\alpha_i) = h_i - 1, \quad i \in \mathbb{N},$$

$$
\begin{aligned}
d_1(\alpha_1) &= \alpha_1(h_1^{p-1} + \cdots + 1), \\
d_1(\alpha_{i+1}) &= -\alpha_i + \alpha_{i+1}\left(h_{i+1}^{p-1} + \cdots + 1\right), \quad i \in \mathbb{N}, \\
d_2(\alpha_1) &= \alpha_1(h_1 - 1), \\
d_2(\alpha_{i+1}) &= \left[\alpha_1 + \alpha_2(h_1^{p-1} + \cdots + 1)\right] + \cdots \\
&\quad + \left[\alpha_{i+1}(h_i^{p-1} + \cdots + 1)(h_{i-1}^{p-1} + \cdots + 1) \cdots (h_1^{p-1} + \cdots + 1)\right](h_{i+1} - 1), \quad i \in \mathbb{N}.
\end{aligned}
$$

Proof

We first observe that

$$
d_1(\alpha_j) = \sum_{i \in \mathbb{N}} \alpha_i \Pi(\frac{\partial r_j}{\partial x_i}), \quad j \in \mathbb{N},
$$

with a finite number of nonzero summands on the right side of each equality sign.

By (6.7.1)and (6.7.7) is exact at $\mathbb{Z}$, $\mathbb{Z}G$ and the first Y from the right.

We, therefore, have only to verify the exactness at the second and the third Y from the right.

Exactness at the second Y from the right

$$
\begin{aligned}
d_1 d_2(\alpha_1) &= d_1(\alpha_1(h_1 - 1)) = d_1(\alpha_1)(h_1 - 1) = (h_1^{p-1} + \cdots + 1)(h_1 - 1) = h_1^p - 1 = 0, \\
d_1 d_2(\alpha_{i+1}) &= d_1\left[[\alpha_1 + \alpha_i(h_1^{p-1} + \cdots + 1) + \cdots\right. \\
&\quad \left. + \alpha_{i+1}(h_i^{p-1} + \cdots + 1) \cdots (h_1^{p-1} + \cdots + 1)(h_{i+1} - 1)]\right] \\
&= \left[d_1(\alpha_1) + d_1(\alpha_2)(h_1^{p-1} + \cdots + 1) + \cdots\right. \\
&\quad \left. + d_1(\alpha_{i+1})(h_i^{p-1} + \cdots + 1) \cdots (h_1^{p-1} + \cdots + 1)\right](h_{i+1} - 1) \\
&= (h_1^{p-1} + \cdots + 1)(h_{i+1} - 1) \\
&\quad + \left[-\alpha_1 + \alpha_2(h_2^{p-1} + \cdots + 1)\right](h_1^{p-1} + \cdots + 1)(h_{i+1} - 1) + \cdots \\
&\quad + \left[-\alpha_i + \alpha_{i+1}(h_{i+1}^{p-1} + \cdots + 1)\right](h_i^{p-1} + \cdots + 1) \cdots (h_1^{p-1} + \cdots + 1)(h_{i+1} - 1) \\
&= 0.
\end{aligned}
$$

Therefore, Ker $d_1 \supseteq$ Im d_2.
Let $x = \sum_{i \in \mathbb{N}} \alpha_i \gamma_i \in$Ker d_1. Since x is a finite sum, there exists $n \in \mathbb{N}$ such that $\gamma_{n+k+1} = 0$, for all $k \in \mathbb{N}$.

$$\therefore \quad x = \alpha_1\gamma_1 + \cdots + \alpha_{n+1}\gamma_{n+1}.$$

$$\therefore \quad d_1(\alpha_1\gamma_1 + \cdots + \alpha_{n+1}\gamma_{n+1}) = 0.$$

$$\therefore \quad \alpha_1(h_1^{p-1} + \cdots + 1)\gamma_1 + \cdots + \left[-\alpha_n + \alpha_{n+1}(h_{n+1}^{p-1} + \cdots + 1)\right]\gamma_{n+1} = 0.$$

$$\therefore \begin{cases} (h_1^{p-1} + \cdots + 1)\gamma_1 - \gamma_2 = 0 \\ (h_2^{p-1} + \cdots + 1)\gamma_2 - \gamma_3 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (h_n^{p-1} + \cdots + 1)\gamma_n - \gamma_{n+1} = 0 \\ (h_{n+1}^{p-1} + \cdots + 1)\gamma_{n+1} = 0\,. \end{cases} \tag{6.7.8}$$

Using the result of the Lemma 6.7.1 and solving the system of equations (6.7.8) we obtain

$$\begin{cases} \gamma_{n+1} = (h_n^{p-1} + \cdots + 1)\cdots(h_1^{p-1} + \cdots + 1)(h_{n+1}^{p-1} - 1)\gamma'_{n+1} \\ \gamma_n \quad = (h_{n-1}^{p-1} + \cdots + 1)\cdots(h_1^{p-1} + \cdots + 1)\left[(h_n - 1)\gamma'_n + (h_{n+1} - 1)\gamma'_{n+1}\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma_2 \quad = (h_1 + \cdots + 1)\left[(h_1 - 1)\gamma'_2 + (h_3 - 1)\gamma'_3\right] \\ \gamma_1 \quad = (h_1 - 1)\gamma'_1 + (h_2 - 1)\gamma'_2. \end{cases} \tag{6.7.9}$$

From (6.7.9), we see that $x = d_2(\alpha_1\gamma_1 + \cdots + \alpha_{n+1}\gamma'_{n+1})$ so that $x \in$Im d_2,

$$\therefore \text{Ker}\, d_1 \subseteq \text{Im}\, d_2\,.$$

Exactness at the third Y from the right

$$\begin{aligned}
d_2d_1(\alpha_1) &= d_2(\alpha_1(h_1+\cdots+1)) = \alpha_1(h_1-1)(h_1^{p-1}+\cdots+1)=0.\\
d_2d_1(\alpha_2) &= d_2[-\alpha_1+\alpha_2(h_2^{p-1}+\cdots+1)]\\
&= -d_2(\alpha_1)+d_2(\alpha_2)(h_2^{p-1}+\cdots+1)\\
&= -\alpha_1(h_1-1)+[\alpha_1+\alpha_2(h_1^{p-1}+\cdots+1)](h_2-1)(h_2^{p-1}+\cdots+1)\\
&= -\alpha_1(h_1-1)+\alpha_1(h_2^p-1)+\alpha_2(h_1^{p-1}+\cdots+1)(h_2^p-1)\\
&= -\alpha_1(h_1-1)+\alpha_1(h_1-1)+\alpha_2(h_1^{p-1}+\cdots+1)(h_1-1)=\alpha_2(h_1^p-1)\\
&= 0.
\end{aligned}$$

$$\therefore \operatorname{Ker} d_2 \supseteq \operatorname{Im} d_1 .$$

Suppose $x \in$ Ker d_2. Then, as before,

$$d_2(\alpha_1\gamma_1+\alpha_2\gamma_2+\cdots+\alpha_{n+1}\gamma_{n+1})=0,$$

$$\begin{aligned}
\therefore \quad & \alpha_1(h_1-1)\gamma_1+[\alpha_1+\alpha_2(h_1^{p-1}+\cdots+1)](h_2-1)\gamma_2+\cdots\\
&+[\alpha_1+\alpha_2(h_1^{p-1}+\cdots+1)+\cdots\\
&+\alpha_{n+1}(h_n^{p-1}+\cdots+1)\cdots(h_1^{p-1}+\cdots+1)](h_{n+1}-1)\gamma_{n+1}=0 . \qquad (6.7.10)
\end{aligned}$$

$$\therefore \; (h_n^{p-1}+\cdots+1)\cdots(h_1^{p-1}+\cdots+1)(h_{n+1}-1)\gamma_{n+1}=0 .$$

By Lemma (6.7.2) we have

$$\gamma_{n+1}=(h_{n+1}^{p-1}+\cdots+1)\gamma_{n+1}'.$$

From (6.7.10)

$$(h_{n-1}^{p-1} + \cdots + 1)\cdots(h_1^{p-1} + \cdots + 1)(h_n - 1)\gamma_n$$

$$+(h_{n-1}^{p-1} + \cdots + 1)\cdots(h_1^{p-1} + \cdots + 1)(h_{n+1} - 1)\gamma_{n+1} = 0.$$

$$\therefore \quad (h_{n-1}^{p-1} + \cdots + 1)\cdots(h_1^{p-1} + \cdots + 1)(h_n - 1)\gamma_n$$

$$+(h_{n-1}^{p-1} + \cdots + 1)\cdots(h_1^{p-1} + \cdots + 1)(h_{n+1} - 1)(h_{n+1}^{p-1} + \cdots + 1)\gamma'_{n+1} = 0.$$

$$\therefore \ (h_{n-1}^{p-1} + \cdots + 1)\cdots(h_1^{p-1} + \cdots + 1)(h_n - 1)[\gamma_n + \gamma'_{n+1}] = 0.$$

$$\therefore \begin{cases} \gamma_n = (h_n^{p-1} + \cdots + 1)\gamma'_n - \gamma'_{n+1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \gamma_2 = (h_2^{p-1} + \cdots + 1)\gamma'_2 - \gamma'_3 \\ \gamma_1 = (h_1^{p-1} + \cdots + 1)\gamma'_1 - \gamma'_2. \end{cases} \tag{6.7.11}$$

From (6.7.11) we see that $x = d_1(\alpha_1\gamma'_1 + \cdots + \alpha_{n+1}\gamma'_{n+1})$.

$$\therefore \ \mathrm{Ker}\, d_1 \subseteq \mathrm{Im}\, d_2.$$

The proof is thus complete.

6.7.4 Homology of G

Let A be a left $\mathbb{Z}G$-module. Then the homology groups $H_n(G, A)$ are given by the homology of the complex

$$\cdots\cdots\cdots \longrightarrow Y \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_1\otimes 1} Y \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_2\otimes 1} Y \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_1\otimes 1} Y \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_0\otimes 1} \mathbb{Z}G \bigotimes_{\mathbb{Z}G} A \longrightarrow 0$$

or, equivalently by

$$\cdots\cdots\cdots \xrightarrow{d_2} A^\omega \xrightarrow{d_1} A^\omega \xrightarrow{d_2} A^\omega \xrightarrow{d_1} A^\omega \xrightarrow{d_0} A \longrightarrow 0,$$

where ω is an ordinal number of $\mathbb{N}$ and $\bar{d}_0$, $\bar{d}_1$, $\bar{d}_2$ are induced by d_0, d_1, d_2 and are given by

$$\bar{d}_0(a_1, a_2, \cdots, a_n, \cdots) = (h_1 - 1)a_1 + \cdots + (h_n - 1)a_n + \cdots,$$

$$\bar{d}_1(a_1, a_2, \cdots, a_n, \cdots) = ((h_1^{p-1} + \cdots + 1)a_1 - a_2, (h_2^{p-1} + \cdots + 1)a_2 - a_3, \cdots, (h_n^{p-1} + \cdots + 1)a_n - a_{n+1}, \cdots).$$

$$\bar{d}_2(a_1, a_2, \cdots, a_n, \cdots) = ((h_1 - 1)a_1 + (h_2 - 1)a_2, (h_1^{p-1} + \cdots + 1)(h_2 - 1)a_2 + (h_3 - 1)a_3, \cdots, (h_n - 1)a_n, \cdots).$$

Here each $(a_1, a_2, \cdots)$ has only a finite number of non-zero entries.

$$\therefore \quad H_0(G, A) = \frac{A}{\mathrm{Im}\bar{d}_0} = \frac{A}{\{(h_1 - 1)a_1 + (h_2 - 1)a_2 + \cdots\}} = A_G,$$

$$H_1(G, A) = \frac{\mathrm{Ker}\bar{d}_0}{\mathrm{Im}\bar{d}_1} = \frac{\{(a_1, a_2, \cdots) \mid (h_1 - 1)a_1 = (h_2 - 1)a_2 = \cdots = 0\}}{\{(h_1^{p-1} + \cdots + 1)a_1 - a_2, (h_2^{p-1} + \cdots + 1)a_2 - a_3, \cdots\}},$$

$$H_{2n}(G, A) = \frac{\mathrm{Ker}\bar{d}_1}{\mathrm{Im}\bar{d}_2} = \frac{\{(a_1, \cdots) \mid (h_1^{p-1} + \cdots + 1)a_1 - a_2 = (h_2^{p-1} + \cdots + 1)a_2 - a_3 = \cdots = 0\}}{\{((h_1 - 1)a_1 + (h_2 - 1)a_2, (h_1^{p-1} + \cdots + 1)(h_2 - 1)a_2 + (h_3 - 1)a_3, \cdots)\}},$$

$$H_{2n+1}(G, A) = \frac{\{(a_1, a_2, \cdots) \mid (h_1 - 1)a_1 + (h_2 - 1)a_2 = (h_1^{p-1} + \cdots + 1)(h_2 - 1) + (h_3 - 1)a_3 = \cdots = 0\}}{\{((h_1^{p-1} + \cdots + 1)a_1 - a_2, (h_2^{p-1} + \cdots + 1)a_2 - a_3, \cdots)\}}.$$

Integral Homology

If $A = \mathbb{Z}$, considered as a trivial $\mathbb{Z}G$-module, then

$$\bar{d}_0(a_1, a_2, a_3, \cdots) = 0,$$

$$\bar{d}_1(a_1, a_2, a_3, \cdots) = (pa_1 - a_2, pa_2 - a_3, pa_3 - a_4, \cdots),$$

$$\bar{d}_2(a_1, a_2, a_3, \cdots) = (0, 0, 0, \cdots).$$

$$\therefore \quad H_0(G, \mathbb{Z}) = \frac{\mathbb{Z}}{\mathrm{Im}\bar{d}_0} = \frac{\mathbb{Z}}{0} = \mathbb{Z}.$$

$$H_1(G, \mathbb{Z}) = \frac{\mathrm{Ker}\bar{d}_0}{\mathrm{Im}\bar{d}_1} = \frac{\{(a_1, a_2, \cdots) \mid a_i \in \mathbb{Z})\}}{\{(pa_1 - a_2, pa_2 - a_3, \cdots)\}}.$$

$$H_1(G,\mathbb{Z}) = \frac{\{a_1(1,0,0,\cdots)+a_2(0,1,0,\cdots)+\cdots\}}{\{a_1(p,0,0,\cdots)+a_2[(0,1,0,\cdots)-(1,0,0,\cdots)]+a_3[(0,0,p,\cdots)-(0,1,0,\cdots)+\cdots\}},$$

$$\begin{aligned} H_1(G,\mathbb{Z}) &= \frac{< y_1,\, y_2,\, y_3,\, \cdots >}{< py_1,\, py_2 - y_1,\, py_3 - y_2,\, \cdots >} \\ &= < y_1,\, y_2,\, y_3,\, \cdots \mid py_1 = 0,\, py_2 = y_1,\, py_3 = y_2,\, \cdots > \\ &\cong G. \end{aligned}$$

where $y_k = (0,\, 0,\, \cdots,\, 0,\, 1,\, 0,\, \cdots)$ with 1 at the kth place.

For each $n \geq 1$,

$$\begin{aligned} H_{2n}(G,\mathbb{Z}) = \frac{\mathrm{Ker}\bar{d}_1}{\mathrm{Im}\bar{d}_2} &= \frac{\{(a_1,\, a_2,\, a_3,\, \cdots) \mid a_i \in \mathbb{Z}\,\text{and}\, pa_1 - a_2 = pa_2 - a_3 = \cdots = 0\}}{\{(0,\, 0,\, 0,\, \cdots)\}} \\ &= \frac{\{a_1(1,\, p,\, p^2,\, \cdots) \mid a_1 \in \mathbb{Z}\}}{\{(0,\, 0,\, 0,\, \cdots)\}} \\ &= \frac{\{(0,\, 0,\, 0,\, \cdots)\}}{\{(0,\, 0,\, 0,\, \cdots)\}} \\ &\cong 0, \qquad \text{since each sequence in the numerator must have} \\ &\qquad\qquad \text{finite number of non-zero entries.} \end{aligned}$$

$$H_{2n+1}(G,\mathbb{Z}) = \frac{\mathrm{Ker}\bar{d}_2}{\mathrm{Im}\bar{d}_1} = \frac{\{(a_1,\, a_2,\, a_3,\, \cdots) \mid a_i \in \mathbb{Z}\}}{\{(pa_1 - a_2,\, pa_2 - a_3,\, \cdots)\}} \cong G.$$

6.8 The 3-Dimensional Heisenberg Group H_3

6.8.1 Introduction

Here we shall construction a full free resolution of $\mathbb{Z}$ for the integral group ring of the 3-dimesional Heisenberg group, using the technique mentioned earlier. We compute the integral homology and cohomology from the resolution obtained.

6.8.2 H_3, the 3-dimensional Heisenberg group has a presetation

$$H_3 = < x, y, z \,|\, [x, z], = [y, z] = 1, [x, y] = z > \text{, (Burillo [114], p.2).}$$

It is a member of widely studied important class of Lie groups called the Heisenberg group. It is nilpotent of class 2. Huebschmann [61] used his sophisticated perturbation theory techenique to determine the cohomology of the generalised Heiseberg group given by

$$G = < x, y, z \,|\, [x, z] = [y, z] = 1, [x, y] = z^k > .$$

$$G = H_3, \text{ if } k = 1.$$

The (2n+1)Heiseberg group $H_{2n+1}$is the group of upper triangular $(n + 2) \times (n + 2)$ matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y^T \\ 0 & 0 & 1 \end{pmatrix},$$

where $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, Y_n)$, I is the $n \times n$ unit mtrix. Thus H_3 is the group of all upper triangular matrices :

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}, (x, y, z \in \mathbb{R}).$$

The Heisenberg group H_3 has a cubic Dehn function ([114], p.1) the latter being a best possible choice for isopermetric function. Isopermetric inequalities have been used fruitfully in the study of hyperbolic groups and automatic groups (Gromov [45], Epstien [29]). The cubic nature of Dehn function for H_3 shows that it is neither hyperbolic nor automatic, since these have respectively a linear Dehn function and a quardratic Dehn function. Thurston proved these facts by combinatorial methods. He also shows that H_3 is not combable.

6.8.3 Before constructing our free resolution for H_3, we give a few definitions, state a few known results and prove a number of results that will be needed for construction and its proof.

Lemma 6.8.1

H_3 is torsion-free.

Proof

$$H_3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

Now

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}^n$$

$$= \begin{pmatrix} 1 & nx & nz + nxy \\ 0 & 1 & ny \\ 0 & 0 & 1 \end{pmatrix}.$$

So,

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}^n$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

if and only if $n = 0$.

Hence H_3 is torsion-free.

The following definitios are due to Higman [55].

Definition 6.8.2

A group G is said to be *indexed* if it can be mapped homomorphically onto a nonzero subgroupof $\mathbb{Z}$.

Definition 6.8.3

A group G is said to be *indicable throughout* if every subgroup $H\,(\neq 1)$ of G can be indexed.

We state two theorems due to Higman :

Theorem 6.8.4 ([55], p.242)

If G is indicable throughout and R is a ring with 1, and has no zero devisors, then RG has no zeo devisors.

Theorem 6.8.5 ([55], p.243)

If G is indicable throughout and R is a ring with 1 and has no zero devisors, then the units of RG are trivial.

We shall use these results to prove :

Theorm 6.8.6

H_3 is indicable throughout.

Proof

We write G for H_3 and let H be a subgroup of G and let $H \neq 1$.

Case I

First suppose that $H \subseteq G'$. By the definition of G, $G' \subseteq Z(G)$. This implies that G' is abelian. G' is the normal subgroup generated by the commutator $[h_1, h_2]$, h_1, h_2 are the images of x_1, x_2 in G. So a tripical element g' of G' is $\prod(g_i^{-1}([h_1, h_2]^{e_i})$, $e_i \ \pm 1$.

$$\therefore \ g' = \prod_{i=1}^{n} [h_1, h_2]^{e_i},$$

since $[h_1,\ h_2]$ is a commutator of h_1 and h_2. Since G' is infinite cyclic, and since $H \neq 1$, H too is infinite cyclic. So H can be indexed.

Case II

Suppose $H \not\subseteq G'$. Then $HG' \neq G'$, and so, $\frac{HG'}{G'}$ is a subgroup of $\frac{G'}{G'}$, and is not the identity subgroup. Hence $\frac{HG'}{G'}$ is free abelian. Then there is a homomorphism $f : \frac{HG'}{G'} \longrightarrow \mathbb{Z}$ such that Im$f \neq \{0\}$. If φ is the canonical homomorphism $\varphi : H \longrightarrow \frac{HG'}{G'}$, then φ is onto and Im$\bar{f} \neq \{0\}$, where $\bar{f} : H \longrightarrow \mathbb{Z}$ is the composite $\bar{f} = f\varphi$.
Thus H can be indexed.
Hence G is indicable throughout.

As a consequence of Theorem 6.8.4 ad Theorem 6.8.5, Theorem 6.8.6 implies the following:

Corollary 6.8.7

$\mathbb{Z}G$ has no zero devisors.

Corollary 6.8.8

The units of $\mathbb{Z}G$ are trivial.

Free resolution of $\mathbb{Z}$

Let $G = H_3 = \frac{F}{R}$, where F is the free group generated by x_1, x_2 and R is the normal closure of r_1, r_2.where $r_1 = [x_1, [x_1, x_2]]$ and $r_2 = [x_2, [x_1, x_2]]$.

i.e.,

$$r_1 = x_1^{-1}x_2^{-1}x_1^{-1}x_2x_1x_1x_1^{-1}x_2^{-1}x_1x_2$$

and

$$r_2 = x_2^{-1}x_2^{-1}x_1^{-1}x_2x_1x_2x_1^{-1}x_2^{-1}x_1x_2$$

Then the Fox derivatives of r_1, r_2 are:

$$\frac{\partial r_1}{\partial x_1} = -r_1 - x_2x_1r_1 + x_2^{-1}x_1x_2 + x_2;$$

$$\frac{\partial r_1}{\partial x_2} = -x_1r_1 + x_1x_2^{-1}x_1x_2 - x_2^{-1}x_1x_1 + 1;$$

$$\frac{\partial r_2}{\partial x_1} = -x_2^2r_2 + x_2x_1^{-1}x_2^{-1}x_1x_2 - x_1^{-1}x_2^{-1}x_1x_2 + x_2;$$

$$\frac{\partial r_2}{\partial x_2} = -r_2 - x_2r_2 + x_2^{-1}x_1x_2^2r_2 + x_1^{-1}x_2^{-1}x - 1x_2 - x_2^{-1}x - 1x_2 + 1.$$

Writing $\pi(x_i) = h_i\,, i = 1, 2$,we have

$$\frac{\partial r_1}{\partial x_1} = -1 - h_2h_1 + h_2^{-1}h_1h_2 + h_2;$$

$$\frac{\partial r_1}{\partial x_2} = -h_1 + h_1h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 + 1;$$

$$\frac{\partial r_2}{\partial x_1} = -h_2^2 + h_2h_1^{-1}h_2^{-1}h_1h_2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2;$$

$$\frac{\partial r_2}{\partial x_2} = -1 - h_2 + h_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 + 1.$$

To construct the free resolution of $\mathbb{Z}$ we proceed as follows. In Lyndon partial free resolution of $\mathbb{Z}$, let

$$\beta_1\gamma_1 + \beta_2\gamma_2 \in \ker d_1,$$

where

$$\gamma_1\gamma_2 \in \mathbb{Z}G.$$

Then

$$d_1(\beta_1\gamma_1 + \beta_2\gamma_2) = 0.$$

$\therefore$ $[\alpha_1(h_2 - 1 - h_2h_1 - h_1^{-1}h_1h_2) + \alpha_2(1 - h_1 - h_2^{-1}h_1h_2 + h_1h_2^{-1}h_1h_2)]\gamma_1 + [\alpha_1(-h_2^2 - h_1^{-1}h_2^{-1}h - 1h_2^2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2) + \alpha_2(h_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 - h_2)]\gamma_2 = 0$

or,

$\alpha_1[(h_2 - 1 - h_2h_1 - h_1^{-1}h_1h_2) + (-h_2^2 - h_1^{-1}h_2^{-1}h - 1h_2^2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2)] + \alpha_2[(1 - h_1 - h_2^{-1}h_1h_2 + h_1h_2^{-1}h_1h_2)]\gamma_1 + (H_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}h_1h_2 - h_2^{-1}h_1h_2 - h_2)]\gamma_2] = 0.$

Since Y_0 is free on α_1, α_2, we have

$$\left.\begin{array}{lll} [(h_2 - 1 - h_2h_1 - h_2^{-1}h_1h_2)]\gamma_1 & & \\ +[(h_2^2 - h_1^{-1}h_2^{-1}h_1h_2^2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2)]\gamma_2 & = 0 & (i) \\ (1 - h_1 - h_2^{-1}h_1h_2 + h_1h_2^{-1}h_1h_2)\gamma_1 & & \\ +(H_2^{-1}h_1h_2^2 + h_1^{-1}h_2^{-1}H_1h_2 - h_2^{-1}h_1h_2 - h_2)\gamma_2 & = 0 & (ii) \end{array}\right\}. \tag{6.8.1}$$

We write Equation (6.8.1)as

$$\left.\begin{array}{ll} a\gamma_1 + b\gamma_2 = 0 & (i) \\ c\gamma_1 + d\gamma_2 = 0 & (ii) \end{array}\right\}, \tag{I}$$

where

$$a = h_2 - 1 - h_2h_1 - h_2^{-1}h_1h_2 \text{ etc.}$$

Solving the equation (I) in $\mathbb{Q}$, we have $\gamma_1 = \gamma_1'$, where γ_1' is an arbitray element of $\mathbb{Z}G$, and $\gamma_2 = -b^{-1}a\gamma_1' = -d^{-1}c\gamma_1'$, so that $-b^{-1}a = -d^{-1}c$. γ_1' being the arbitrary element of $\mathbb{Z}G$, $-b^{-1}a = -d^{-1}c \in \mathbb{Z}G$. Since $b^{-1}a$ has an inverse $a^{-1}b$, $b^{-1} = g_0$, for some $g_0 \in G$, by corollary (6.8.8).

Define Y_2 as the right $\mathbb{Z}G-$ module freely generated by δ and define

$$d_2 : Y_2 \longrightarrow Y_1$$

by

$$d_2(\delta) = \beta_1 - \beta_2 g_0.$$

Then,

$$\begin{aligned} d_2(\delta\gamma_1) = (\beta_1 - \beta_2 g_0)\gamma_1 &= \beta_1\gamma_1 - \beta_2 g_0\gamma_1 \\ &= \beta_1\gamma_1 + \beta_2\gamma_2 . \end{aligned}$$

Hence

$$\beta_1\gamma_1 + \beta_2\gamma_2 \in \mathrm{Im} d_2.$$

Also,

$$\begin{aligned}(d_1 d_2)(\delta) &= d_1(\beta - 1 - \beta_2 g_0) \\ &= (\alpha_1 a\gamma_1 + \alpha_2 a\gamma_2) - (\alpha_1 b g_0\gamma_1 + \alpha_2 d g_0\gamma_2) \\ &= \alpha_1(a-a)\gamma_1 + \alpha_2(c-c)\gamma_2 \\ &= 0.\end{aligned}$$

$$\therefore\ ker d_2 \supseteq Im d_1 .$$

Now, let $\delta \in ker d_2$, then

$$\alpha_1\gamma - \alpha_2 g_0\gamma = 0$$

$$\Longrightarrow \gamma = 0 .$$

Thus we have the following free resolution of $\mathbb{Z}$:

$$0 \longrightarrow Y_2 \xrightarrow{d_2} Y_1 \xrightarrow{d_1} Y_0 \xrightarrow{d_0} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where Y_0, Y_1, Y_2 are right $\mathbb{Z}G - modules$ freely generated by $\{\alpha_1, \alpha_2\}, \{\beta_1, \beta_2\}$, $\{\delta\}$ and ε, d_0, d_1, d_2 are defined by

$$\begin{aligned}\varepsilon(g) &= 1, \forall g \in G; \\ d_0(\alpha - 1) &= h_1 - 1 ; \\ d_0(\alpha_2) &= h_2 - 1 ; \\ d_1(\beta_1) &= \alpha_1(h_2 - 1 - h_2 h_1 - h_1^{-1} h_1 h_2) + \alpha_2(1 - h_1 - h_2^{-1} h_1 h_2 + h_1 h_2^{-1} h_1 h_2) ; \\ d_1(\beta_2) &= \alpha_1(-h_2^2 - h_1^{-1} h_2^{-1} h - 1 h_2^2 - h_1^{-1} h_2^{-1} h_1 h_2 + h_2) \\ &\quad + \alpha_2(h_2^{-1} h_1 h_2^2 + h_1^{-1} h_2^{-1} h_1 h_2 - h_2^{-1} h_1 h_2 - h_2). \\ d_2(\delta) &= \beta_1 - \beta_2 g_0 .\end{aligned}$$

Homology groups of G

Let A be a $\mathbb{Z}G$ – *module*. Then the homology groups $H_n(G, A)$ is the homology of the complex

$$0 \longrightarrow Y_2 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_2\otimes 1} Y_1 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_1\otimes 1} Y_0 \bigotimes_{\mathbb{Z}G} A \xrightarrow{d_0\otimes 1} \mathbb{Z}G \bigotimes_{\mathbb{Z}G} A \longrightarrow 0$$

or, equivalently by

$$0 \longrightarrow A \xrightarrow{\bar{d}_2} A^2 \xrightarrow{\bar{d}_1} A^2 \xrightarrow{\bar{d}_0} A \longrightarrow 0,$$

where, $\bar{d}_0, \bar{d}_1$, d_2, is given by

$$\begin{aligned}
\bar{d}_0(a_1,\, a_2) &= (h_1 - 1)a_1 + (h_2 - 1)a_2\ ; \\
\bar{d}_1(a_1,\, a_2) &= ([(h_2 - 1 - h_2h_1 - h_2^{-1}h_1h_2)]a_1,\ [(h_2^2 \\
&\quad -h_1^{-1}h_2^{-1}h_1h_2^2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2)]a_2)\ ; \\
\bar{d}_2(a) &= (a,\, -g_0a).
\end{aligned}$$

If A is trivial, then

$$\begin{aligned}
\bar{d}_0(a_1, a_2) &= 0\ ; \\
\bar{d}_1(a_1, a_2) &= (0, 0)\ ; \\
\bar{d}_2(a) &= (a, -a).
\end{aligned}$$

If $A = \mathbb{Z}$, then

$$\begin{aligned}
H_0(G,\, \mathbb{Z}) &= \frac{A}{0} = \mathbb{Z}. \\
H_1(G,\, \mathbb{Z}) &= \frac{\ker\bar{d}_0}{\mathrm{Im}\bar{d}_1} = \frac{\{(a_1,\, a_2) \mid a_i \in \mathbb{Z}\}}{\{(0,\, 0)\}} \\
&\cong \mathbb{Z} \oplus \mathbb{Z}.
\end{aligned}$$

$$\begin{aligned} H_2(G, \mathbb{Z}) &= \frac{\ker \bar{d}_1}{\mathrm{Im}\bar{d}_2} = \frac{\{(a_1, a_2) \mid a_i \in \mathbb{Z}\}}{\{(a, -a)\}} \\ &= \frac{\{(a, -a) + (0, a_2)\}}{\{(a, -a) \mid a \in \mathbb{Z}\})} \\ &\cong \frac{< (1, -1) > \oplus < (0, 1) >}{< (1, -1) >} \\ &\cong < (0, 1) > \\ &\cong \mathbb{Z}. \end{aligned}$$

$$H_3(G, \mathbb{Z}) = \frac{\ker \bar{d}_2}{0} = 0.$$

Detemination of cohomology groups of $\mathbb{Z}$

Let A be a right $\mathbb{Z}G-module$. Then the cohomology groups $H^n(G, A)$ is the cohomology of the complex

$$0 \longleftarrow \mathrm{Hom}(Y_2, A) \xleftarrow{d_2^*} \mathrm{Hom}(Y_1, A) \xleftarrow{d_1^*} \mathrm{Hom}(Y_0, A) \xleftarrow{d_0^*} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \longleftarrow 0$$

or equivalently,

$$0 \longleftarrow A \xleftarrow{d_2^{**}} A^2 \xleftarrow{d_1^{**}} A^2 \xleftarrow{d_0^{**}} A \longleftarrow 0,$$

where d_0^{**}, d_1^{**}, d_2^{**} are given by

$$\begin{aligned} d_0^{**}(a) &= (a(h_1 - 1), a(h_2 - 1)), \\ d_1^{**}(a_1, a_2) &= ([a_1(h_2 - 1 - h_2h_1 - h_2^{-1}h_1h_2)], [a_2(h_2^2 \\ &\quad -h_1^{-1}h_2^{-1}h_1h_2^2 - h_1^{-1}h_2^{-1}h_1h_2 + h_2)]), \\ d_2^{**}(a_1, a_2) &= a_1 - g_0a_1. \end{aligned}$$

If A is trivial , then

$$\begin{aligned} d_0^{**}(a_1, a_2) &= (0, 0)), \\ d_1^{**}(a_1, a_2) &= (0, 0), \\ d_2^{**}(a_1, a_2) &= 0. \end{aligned}$$

Let $A = \mathbb{Z}$, then

$$H^0(G, \mathbb{Z}) = \frac{\mathbb{Z}}{0} \cong \mathbb{Z}.$$

$$\begin{aligned} H^1(G, \mathbb{Z}) &= \frac{kerd-1^{**}}{Imd_0^{**}} = \frac{\{(a_1, a_2) \mid a_i \in \mathbb{Z}\}}{\{(0,0)\}} \\ &\cong \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} H^2(G, \mathbb{Z}) &= \frac{d_2^{**}}{Imd_1^{**}} = \frac{\{(a_1, a_2) \mid a_i \in \mathbb{Z}\}}{\{(0,0)\}} \\ &\cong \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

$$\begin{aligned} H^3(H, \mathbb{Z}) &= \frac{\{a \mid a \in \mathbb{Z}\}}{\{0\}} \\ &\cong \mathbb{Z}. \end{aligned}$$

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