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On Ro and R1 Properties in Fuzzy Topological Spaces

Azam, S. M. Faqruddin Ali

University of Rajshahi

On R_0 and R_1 Properties in Fuzzy Topological Spaces

A

Thesis submitted to the Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh for the degree of Master of philosophy in Mathematics.

By

S. M. Faqruddin Ali Azam

Under the supervision of

Dr. Dewan Muslim Ali

Professor, Department of Mathematics

University of Rajshahi

Rajshahi-6205

Bangladesh

Dedicated
To My Parents

Acknowledgement

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Department of Mathematics

February-2011 (S. M. Faqruddin Ali Azam)

Dr. Dewan Muslim Ali
M. Sc. (Raj.) Ph. D. (Benaras)
Professor
Department of Mathematics

University of Rajshahi
Rajshahi-6205, Bangladesh
Phone-0721-750041-9/4108(dept.)
0721-750528 (Resi.)
Fax- 0088-0721-750528)
E-mail: profdma@ru.ac.bd

No............

Dated............

Certificate

This is to certify that the M. Phil. Thesis entitled "On R_0 and R_1 Properties in Fuzzy Topological Spaces" which is being submitted by S. M. Faqruddin Ali Azam in fulfillment of the requirement for the degree of M. Phil. in Mathematics, University of Rajshahi, Rajshahi, Bangladesh is a record of bona fide research work done by him under my supervision. I believe that the results embodied in the thesis are new and it has not been submitted elsewhere for any degree.

To the best of my knowledge S. M. Faqruddin Ali Azam bears a good moral character and is mentally and physically fit to get the degree. I wish him a bright future and every success in his life.

Supervisor

Muslim 3.2.11

Dr. Dewan Muslim Ali
Professor
Department of Mathematics
University of Rajshahi
Rajshahi-6205
Bangladesh

STATEMENT OF ORIGINALITY

*I declare that the contents in my M. Phil. thesis entitled "**On R_0 andR_1 Properties in Fuzzy Topological Spaces**" are original and accurate to the best of my knowledge. I also certify that the materials contained in my research work have not been previously published or written by any person for a degree or diploma.*

University of Rajshahi
February – 2011

M. Phil. Research Fellow

Faqruddin
03.02.2011

(S. M. Faqruddin Ali Azam)

ABSTRACT

The goal of this thesis is to find out some new R_1-concepts for fuzzy topological spaces. Besides some concepts of fuzzy R_0, R_1, T_0, T_1, T_2 and *regular* topological spaces that are already existing in the literature are recalled. In this work, twelve R_1-axioms of fuzzy R_1-topological spaces are introduced and studied in detail. Interrelations among various R_1 concepts of fuzzy topological spaces are discussed. In analogy with the well known topological properties, a complete answer is given with regard to all possible $(R_1 \wedge T_0 \Leftrightarrow T_2)$ and $(R_1 \Rightarrow R_0)$-type implications for fuzzy topological spaces. It is also shown that, though a *regular* topological space is also a R_1-topological space, this is not true for fuzzy topological spaces.

CONTENTS

INTRODUCTION

The concept of fuzzy sets was first introduced, in 1965, by L. A. Zadeh in his new classical paper [42] as an attempt to mathematically handle those phenomena which are inherently vague, imprecise or fuzzy in nature. He interpreted a fuzzy set on a set X is as a mapping from X into the closed unit interval I = [0, 1]. Various merits and applications as well as some limitations of fuzzy set theory have since been demonstrated by Zadeh and a large number of subsequent workers.

The advent of fuzzy set theory has also led to the development of some new areas of study in mathematics. It has become a concern and a new tool for the mathematicians working in many different areas of mathematics. These have been generally accomplished by replacing subsets, in various existing mathematical structures, by fuzzy sets. In 1968, Chang C. L. [10] did 'fuzzification' of topology by replacing subsets in the definition of fuzzy topology by fuzzy sets. Since then a large body of concepts and results have been growing in this area which has come to be known as "fuzzy topology". In 1971, Goguen [21] defined fuzzy set by replacing the unit interval I by a completely distributive lattice L with an order reversing involution. A further development of L-fuzzy topology was made by Sarkar Mira [33, 34] and Hutton B. [22, 23]. The present state of ongoing research in fuzzy topology can be divided in two separate sections, one of which is exclusively using the unit interval I to describe fuzziness (Chang's fuzzy topologies) and the other using L-fuzzy topologies. In our investigation, we have preferred the concepts of fuzzy topology developed by Chang C. L. [10].

A major deviation in the definition of fuzzy topology was made, in 1976, by Lowen R. [25, 26]. He gave a modified definition of fuzzy topology by including all constant fuzzy sets in a fuzzy topology. Furthermore, in 1977, Lowen R. introduced the notions of initial and final fuzzy topologies, which are two appropriate concepts to generalize the topological ones from the categorical point of view.

In 1980, Pu Pao-Ming and Liu Ying-Ming [31, 32] gave a new definition of fuzzy point. They also introduced the notions of quasi-coincidence and quasi-neighborhoods of fuzzy points. With these new concepts they established the Moore-Smith convergence of fuzzy setting.
In 1974, Wong C. K. [39, 40] extended the notions of product and quotient topologies to fuzzy setting and later many authors including Lowen R. [25, 26], Hotton B. [22, 23], Pu Pao-Ming and Liu Ying-Ming [31, 32], Mashhour et al [27, 28], Christoph F. T. [12] and Erceg M. A. [15] etc.

The concepts of R_0-type and R_1-type axioms for fuzzy topological spaces was first introduced by Hutton B. and Reilly I. [23] in 1980. In 1990, Ali D.M., P Wuyts, A.K. Srivastava [6] introduced some other definitions of fuzzy R_0- axioms. Later Srivastava [35] and Ali D. M. [3, 4] gave some new concepts of R_1-property in fuzzy topology.

The present thesis entitled "*On $\mathcal{R}_0$ and $\mathcal{R}_1$ Properties in Fuzzy Topological Spaces*" is devoted to the study of some R_0 and R_1-propeties for fuzzy topological spaces. The material of this thesis has been divided into five chapters and a brief discussion of this are mentioned below:

The first chapter is incorporated with some basic concepts, definitions and known results on fuzzy sets, fuzzy topological spaces and different mapping on fuzzy topological spaces which are necessary for the subsequent chapters. Results are provided without proof and can be seen in papers referred to.

In chapter two we recall various concepts of fuzzy R_0 properties, fuzzy T_0-properties and fuzzy T_1- properties. We have added some new results of these concepts.

In chapter three, we introduce some new concepts of R_1-axioms for fuzzy topological spaces. We study their interrelations, goodness and initialities. Some other results are also added regarding to this concepts.

In chapter four we recall some existing R_1-properties for fuzzy topological spaces. We study their interrelations and their relations with the R_1-properties introduced in the previous chapter.

In chapter five, the relations between a fuzzy R_1-space and a fuzzy R_0-space are discussed. Besides this, we recall some fuzzy regularity concepts from [4, 5] and it is shown that, though the regularity axiom implies R_1-axiom in 'general topological spaces' this is not true, in general, in 'fuzzy topological spaces'.

CHAPTER -1

Preliminaries

In this chapter we recall some definitions and basic results (which we label as facts) on fuzzy sets and fuzzy topological spaces. This chapter is considered as the base and background for the study of subsequent chapters and we shall keep on referring back to it as and when needed.

1.1 Fuzzy sets:

1.1.1. Definition [42]: Let X be a non-empty set and I the unit closed interval [0, 1]. A fuzzy set is a function u: $X \to I$, $\forall x \in X$; u(x) denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X. Ordinary subsets of X (crisp sets) are also considered as the members of I^X which take the values 0 and 1 only. A crisp set which always takes the value 0 is denoted by 0; similarly a crisp set which always takes the value 1 is denoted by 1.

1.1.2. Definition [4]: Let u: $X \to I$. Then the set $\{x \in X: u(x) > 0\}$ is called the support of u and is denoted by u_0 or supp(u). If $A \subseteq X$, Then by 1_A we denote the characteristics function A. The characteristics function of a singleton set {x} is denoted by 1_x.

1.1.3. Definition [4]: Let u be fuzzy sets in X. Then by u^c, we denote the complement of u which is defined as $u^c(x) = 1 - u(x)$ $\forall x \in X$.

1.1.4. Definition [42]: Let u and v be two fuzzy sets in X. We define

(i) $u = v$ if and only if $u(x) = v(x)$ $\forall x \in X$.

(ii) $u \subseteq v$ if and only if $u(x) \leq v(x)$ $\forall x \in X$.

(iii) $(u \vee v)(x) = \max\{u(x), v(x)\}$, where $x \in X$.

(iv) $(u \wedge v)(x) = \min\{u(x), v(x)\}$, where $x \in X$.

1.1.5. Definition [42]: For a family of fuzzy sets $\{u_i : i \in J\}$ in X. We define

(i) $\bigcup_{i \in J} u_i(x) = \sup_{i \in J}\{u_i(x)\}\ \forall x \in X$.

(ii) $\bigcap_{i \in J} u_i(x) = \inf_{i \in J}\{u_i(x)\}\ \forall x \in X$.

1.1.6. Fact. Let u, v and w are fuzzy sets in X. Then

(i) $u \vee u = u$ and $u \wedge u = u$.

(ii) $u \vee v = v \vee u$ and $u \wedge v = v \wedge u$.

(iii) $(u \vee v) \vee w = u \vee (v \vee w)$ and $(u \wedge v) \wedge w = u \wedge (v \wedge w)$.

(iv) $(u \vee v) \wedge u = u$ and $(u \wedge v) \vee u = u$.

(v) $u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w)$ and $u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w)$.

(vi) $(u^c)^c = u$

(vii) $(u \vee v)^c = u^c \wedge v^c$ and $(u \wedge v)^c = u^c \vee v^c$.

1.1.7. Fact. Complementary law of cantor set doesn't hold for fuzzy set in general. That is, if $u \in I^X$, then $u \vee u^c \neq 1$ and $u \wedge u^c = 0$, in general.

1.1.8. Definitions [31]: A *fuzzy point* x_α in X is a special type of fuzzy set in X with the membership function $x_\alpha(x) = \alpha$ and $x_r(y) = 0$ if $x \neq y$, where $0 < \alpha < 1$ and x, y $\in$ X. The fuzzy point x_α is said to have support x and value α. We also write this as $\alpha 1_x$.

1.1.9. Definitions [31]: Let $\alpha 1_x$ be a fuzzy point in X and u be a fuzzy set in X. Then $\alpha 1_x \in u$ if and only if $\alpha < u(x)$.

1.1.10. Fact.: For all fuzzy points $\alpha 1_x$ and for all fuzzy sets u, v in X, we have

(i) $u \subseteq v$ if and only if $\alpha 1_x \in u \Rightarrow \alpha 1_x \in v$.

(ii) $u = v$ if and only if $\alpha 1_x \in u \Leftrightarrow \alpha 1_x \in v$.

(iii) $\alpha 1_x \in u \vee v$ if and only if $\alpha 1_x \in u$ or $\alpha 1_x \in v$.

(iv) $\alpha 1_x \in u \wedge v$ if and only if $\alpha 1_x \in u$ and $\alpha 1_x \in v$.

More generally,

(v) $\alpha 1_x \in \bigvee_{i=1}^{n} u_i$ if and only if $\alpha 1_x \in u_i$ for some i.

(vi) $\alpha 1_x \in \bigwedge_{i\in J}^{n} u_i$ if and only if $\alpha 1_x \in u_i$ for all $i \in J$.

1.1.11. Fact. [31]: A fuzzy set u in X is the union of all its fuzzy points, i.e. $u = \bigvee_{\alpha 1_x \in u}^{n} \alpha 1_x$.

1.1.12. Definition [10]: Let f: X → Y be a mapping and u be a fuzzy set in X. Then the image f(u) is a fuzzy set in Y which is defined as

$$f(u)(y) = \begin{cases} \sup\{u(x) : f(x) = y\} & \text{if } f^{-1}(y) \neq \varnothing \\ 0 \ \text{ if } f^{-1}(y) \neq \varnothing \end{cases}$$

1.1.13. Definition [10]: Let f: X → Y be a mapping and u be a fuzzy set in X. Then the inverse image $f^{-1}(u)$ is the fuzzy set in X which is defined by $f^{-1}(u)(x) = u(f(x)) \ \forall x \in X$.

1.1.14. Fact. [10]: Let f: X → Y be a mapping. Then

(i) $u_1 \leq u_2 \Rightarrow f(u_1) \leq f(u_2) \ \forall\ u_1, u_2 \in I^X$.

(ii) $u_1 \leq u_2 \Rightarrow f^{-1}(u_1) \leq f^{-1}(u_2) \ \forall\ u_1, u_2 \in I^Y$.

(iii) $u_1 \leq f^{-1}(f(u)) \ \forall\ u \in I^X$.

(iv) $f(f^{-1}(u)) \leq u \ \forall\ u \in I^Y$.

(v) $f^{-1}(u^c) \leq (f^{-1}(u))^c \ \forall\ u \in I^Y$.

(vi) $(f(u))^c \leq f(u^c) \ \forall u \in I^X$

(vii) Let $f : X \to Y$ and $g : Y \to Z$ be two functions and $g \circ f : X \to Z$ be the composition of f and g. Then $(g \circ f)^{-1}(u) = f^{-1}(g^{-1}(u))$ $\forall$ $u \in I^Z$.

1.1.15. Fact [10]: If $f : X \to Y$ is a function, $\{u_i : i \in K\}$ is a family of fuzzy sets in X and $\{v_j : j \in J\}$ is a family of fuzzy sets in Y, then

(i) $f^{-1}\left(\bigvee_{j \in J} v_j\right) = \bigvee_{j \in J} f^{-1}(v_j)$.

(ii) $f^{-1}\left(\bigwedge_{j \in J} v_j\right) = \bigwedge_{j \in J} f^{-1}(v_j)$.

(iii) $f\left(\bigvee_{i \in K} u_i\right) = \bigvee_{i \in K} f(u_i)$.

(iv) $f\left(\bigwedge_{i \in K} u_i\right) = \bigwedge_{i \in K} f(u_i)$.

1.1.16. Fact [10]: If $f : X \to Y$ is a function, $u \in I^X$ and $v \in I^Y$, then the following hold:

(i) If x_α is a fuzzy point in X, then $f(x_\alpha) = [f(x)]_\alpha$ is a fuzzy point in Y.

(ii) If x_α is a fuzzy point in $u \in I^X$, then $f(x_\alpha)$ is a fuzzy point in $f(u) \in I^Y$.

(iii) If $f(x_\alpha)$ is a fuzzy point in $u \in I^Y$, then x_α is a fuzzy point in $f^{-1}(u) \in I^X$.

(iv) If x_α is a fuzzy point in Y, then $f^{-1}(x_\alpha)$ need not to be a fuzzy point in X. However, if f is injective and $x_\alpha \in f(X)$, then $f^{-1}(x_\alpha)$ is a fuzzy point in X and is then defined as $f^{-1}(x_\alpha) = [f^{-1}(x)]_\alpha$.

1.2. Fuzzy topological spaces:

Chang C. L. defined a fuzzy topological space as follows:

1.2.1. Definition [10]: Let X be a set. A class t of fuzzy sets in X is called a fuzzy topology on X if t satisfies the following conditions:

(i) $0, 1 \in t$,

(ii) if $u, v \in t$ then $u \wedge v \in t$

and (iii) if $\{u_i : i \in K\}$ is a family of fuzzy sets in t, then $\vee_{i \in K} u_i \in t$.

The pair (X, t) is then called a fuzzy topological space, in short fts. The members of t are called t-open sets (or open sets) and their complements are called t-closed set (or closed sets).

1.2.2. Definition [26]:

Lowen R. modified the definition of a fuzzy topological space defined by Chang C.L. [24] by adding another condition. In the sense of Lowen R. the definition of a fuzzy topological space is as follows:

Let X be a set and t is a family of fuzzy sets in X. Then t is called a fuzzy topology on X if the following conditions hold:

(i) $0, 1 \in t$,

(ii) if $u, v \in t$ then $u \wedge v \in t$,

(iii) if $\{u_i : i \in K\}$ is a family of fuzzy sets in t, then $\vee_{i \in K} u_i \in t$

and (iv) t contains all constant fuzzy sets in X.

The pair (X, t) is called a fuzzy topology.

We shall use the concept of fuzzy topological space due to Lowen R. unless otherwise stated.

1.2.3. Definition: Let X be a set and d be the class of all fuzzy sets in X. Observe that d satisfies all the axioms of a fuzzy topology. This fuzzy topology is called the discrete fuzzy topology on X and the pair (X, d) is called the discrete fuzzy topological space.

1.2.4. Definition: Let X be a set and i be a fuzzy topology on X consisting of fuzzy sets 0 and 1 alone. Then $i = \{0, 1\}$ is called the indiscrete fuzzy topology on X and the pair (X, i) is called the indiscrete fuzzy topological space.

1.2.5. Definition [4]: Let u be a fuzzy set in an fts (X, t). Then the fuzzy closure $\overline{u}$ and the fuzzy interior u^o of u are defined as follows:

$\overline{u} = \inf\{\lambda : u \leq \lambda \text{ and } \lambda \in t^c\}$.

$u^0 = \sup\{\lambda : \lambda \leq u \text{ and } \lambda \in t\}$.

1.2.6. Fact. For a fuzzy topological space (X, t), the following hold:

(i) $\overline{u} = 1 - u^0$

(ii) $u \in I^X$ is fuzzy open if and only if $u = u^o$.

(iii) u is fuzzy closed if and only if $u = \overline{u}$.

(iv) For any fuzzy set u in X, $u^0 \leq u \leq \overline{u}$.

(v) If $u \leq v$, then $\overline{u} \leq \overline{v}$ and $u^0 \leq v^0$.

(vi) $\overline{\overline{u}} = \overline{u}$ and $(u^o)^o = u^0$.

1.2.7. Definition [20]: Let (X, t) be a fuzzy topological space and $A \subseteq X$. We define the relative fuzzy topology for A by $t_A = \{A \wedge u : u \in t\}$. The pair (A, t_A) is called the fuzzy subspace of (X, t). A fuzzy subspace is called fuzzy open (closed) subspace of (X, t) if the set A is a fuzzy open (closed) set in X.

1.2.8. Fact [20]: Let (A, t_A) be a fuzzy subspace of an fts (X, t) and u be any fuzzy set in (A, t_A). Then

(i) u is t_A-closed if and only if $u = A \wedge v$ for some t-closed fuzzy set v in X.

(ii) the t_A-closure $cl_A(u)$ and the t-closure $\overline{u}$ of u are related by $cl_A(u) = A \wedge \overline{u}$.

1.2.9. Definition [4]: Let (X, t) be an fts. Then the a subfamily B of t is called a *base* for t if and only if each member of t can be expressed as a supremum of members of B; and a subfamily S of t is called a *subbase* for t if the family of all the finite intersection of members of S is a base for t.

1.2.10. Fact [4]: In an fts (X, t), a subfamily B of t is a base for t if and only if for each $\alpha \in I_{0,1}$, $u \in t$ and $x \in X$ with $\alpha < u(x)$, there exists $v \in B$ such that $\alpha < v(x)$ and $v \leq u$.

1.2.11. Fact [4]: Let (X,t) be an fts and λ be a fuzzy set in X. Then for $\alpha \in I_0$ and $x \in X$, $\alpha 1_x \leq \overline{\lambda}$ if and only if for each $u \in t$ with $\alpha + u(x) > 1$, there exists some $y \in X$ such that $u(y) + \lambda(y) > 1$.

1.2.12. Definition [10]: Let (X, t) and (Y, s) be two fuzzy topological spaces and $f:(X,t) \to (Y,s)$ be any function. Then f is called

(i) fuzzy continuous if and only if $f^{-1}(u) \in t$ for each $u \in s$.

(ii) fuzzy open if and only if $f(u) \in s$ for each $u \in t$.

(iii) fuzzy closed if and only if $f(u) \in s^c$ for each $u \in t^c$.

(iv) fuzzy homeomorphism if and only if f is fuzzy bijective, fuzzy continuous and fuzzy open.

(v) fuzzy identification if and only if f is fuzzy continuous, surjective and for each $u \in I^Y$, $f^{-1}(u) \in t$ implies $u \in s$.

1.2.13. Fact [4]: Let $f:(X, t) \to (Y, S)$ be a function. Then the following are equivalent:

(i) f is continuous.

(ii) $f^{-1}(\lambda)$ is t-closed for each s-closed λ.

(iii) $f(\overline{\lambda}) \leq \overline{f(\lambda)}$ for each fuzzy set λ in X.

(iv) $\overline{f^{-1}(u)} \leq f^{-1}(\overline{u})$ for each fuzzy set u in Y.

1.2.14. Definition [4]: Let $\{(X_i, t_i) : i \in K\}$ be a collection of fuzzy topological spaces. Let $X = \prod_{i \in K} X_i$ be their Cartesian product and $p_i : X \to X_i$ be the projection map. Then the fuzzy topology t on X generated by $\{p_i^{-1}(u_i) : i \in K, u_i \in t_i\}$ is called the product fuzzy topology on X and the pair (X, t) is called the product fuzzy topological space. It can be verified that $p_i^{-1}(u_i)$, $i \in K$, as defined above, can be expressed as $\prod_{k \in K} \lambda_k$ where $\lambda_k = u_i$ if $k = i$ and $\lambda_k = X_k$ if $k \neq i$.

The product fuzzy topology t is also called the coarsest fuzzy topology on X

1.2.15. Fact [4]: For a family $\{(X_i, t_i) : i \in K\}$ of fuzzy topological spaces and a fuzzy topology t on $X = \prod_{i \in K} X_i$, the following are equivalent:

(i) t is the product of the fuzzy topologies t_i's.

(ii) t is the smallest fuzzy topology on X which makes each projection $p_i : X \to X_i$ $i \in K$, continuous.

(iii) For each fuzzy topological space (Y, s) and function $f : X \to Y$, $f : (X, t) \to (Y, s)$ is continuous if and only if for all $i \in K$, $p_i \circ f$ is continuous.

1.2.15. Definition [4]: Let $\{f_j : X \to (X_j, t_j); j \in J\}$ be a family of functions from a set X to fuzzy topological spaces $(X_j, t_j), j \in J$. Then the initial fuzzy topology on X induced by the family $\{f_j : j \in J\}$, say t, is the smallest fuzzy topology on X, making each f_j, $j \in J$, continuous. It can be verified that t is generated by the family of fuzzy sets $f_j^{-1}(u_j) : u_j \in t_j$ and $j \in J$. For example, the product fuzzy topology is the initial fuzzy topology induced by the family of projections. Similarly, the subspace topology is also the initial fuzzy topology induced by the inclusion map.

1.2.16. Definition [4]: A fuzzy topological property FP is said to be an initial property if for each family of functions $\{f_j : X \to (X_j, t_j); j \in J\}$, whenever each fts $(X_j, t_j); j \in J$, has FP, then (X, t) also has FP, t being the initial fuzzy topology on X induced by the family $\{f_j : j \in J\}$.

1.2.17. Definition [4]: A real-valued function f on a topological space X is called *lower semi-continuous* (l.s.c.) if and only if for every $\alpha \in \mathbf{R}$, the set $f^{-1}(\alpha, \infty)$ is open.. For a topological space (X, T), the l.s.c. fuzzy topology on X associated with T is denoted by ω(T) and is defined as $\omega(T) = \{u \in I^X : u \text{ is l.s.c.}\}$.

1.2.18. Fact [4]: Let (X, T) be a topological space. Then

(i) $u \in I^X$ is ω(T) closed if and only if for all $\alpha \in I$, $u^{-1}[\alpha, 1]$ is T-closed.

(ii) $A \subseteq X$ is T-open if and only if 1_A is ω(T)-open.

(iii) $A \subseteq X$ is T-closed if and only if 1_A is ω(T)-closed.

(iv) $\overline{u^{-1}(\alpha, 1]} \subseteq (\overline{u})^{-1}[\alpha, 1]$.

(v) $\overline{\alpha 1_A} = \alpha 1_{\overline{A}}$.

(vi) $\{1_U : U \in T\}$ is a subbase for ω(T).

(vii) $\{\alpha 1_U : \alpha \in I_0 \text{ and } U \in T\}$ is a base for ω(T).

1.2.19. Definition [4]: Let P be a property of topological space and FP be its fuzzy topological analogue. Then FP is called a *good extension* of P if and only if the statement "(X, T) has P if and only if (X, ω(T)) has FP" holds good for every topological space (X, T).

CHAPTER-2

Fuzzy R_0 topological spaces

1. Introduction: In this chapter we recall nine R_0-type axioms for fuzzy topological spaces from [6]. We study their interrelations, goodness and initiality. Also a complete answer is given with regard to all possible ($T_1 \Rightarrow R_0$)-type and ($T_0 \wedge R_0 \Leftrightarrow T_1$)-type properties.

2. R_0- properties

We recall from [6], nine definitions of the R_0^k-axioms of a fuzzy topological space used in the sequel:

2.1. Definitions [6]: We define, for fuzzy topological spaces (X, t), R_0-properties as follows:

R_0^1 : For every pair x, y$\in$X, $x \neq y$, $\overline{1_y}(x) = 0 \Rightarrow \overline{1_x}(y) = 0$

R_0^2 : For every pair x, y$\in$X, $x \neq y$, $\left(\forall\alpha \in I_0 : \overline{\alpha 1_x}(y) = \alpha \Leftrightarrow \forall\beta \in I_0 : \overline{\beta 1_y}(x) = \beta\right)$

R_0^3 : $\forall\lambda \in t$, $\forall x \in X$ and $\forall\alpha < \lambda(x)$, $\overline{\alpha 1_x} \leq \lambda$

R_0^4 : $\forall\lambda \in t$, $\forall x \in X$ and $\forall\alpha \leq \lambda(x)$, $\overline{\alpha 1_x} \leq \lambda$

R_0^5 : For every pair x, y$\in$X, $x \neq y$, $\overline{1_x}(y) = 1 \Rightarrow \overline{1_y}(x) = 1$

R_0^6 : For every pair x, y$\in$X, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x)$

R_0^7 : For every pair x, y$\in$X, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$

R_0^8 : For every pair x, y$\in$X, $x \neq y$ and $\forall\alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha \Rightarrow \overline{\alpha 1_y}(x) = \alpha$

R_0^9 : For every pair x, y$\in$X, $x \neq y$ and $\forall\alpha \in I_0$, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$

2.1.1. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^1, i.e., for every pair x, y$\in$X, $x \neq y$, $\overline{1_y}(x) = 0 \Rightarrow \overline{1_x}(y) = 0$

(b) For every pair x, y∈X, x ≠ y, $\overline{1_x}(y) = 0 \Leftrightarrow \overline{1_y}(x) = 0$

(c) For every pair x, y∈X, x ≠ y, if there exists λ∈t, λ(x) = 1, λ(y) = 0, then there exists μ∈t, such that μ(x) = 0 and μ(y) = 1.

Proof:

(a)⇒(b): Suppose (X, t) is R_0^1. Suppose $\overline{1_x}(y) = 0$. Then since (X, t) is R_0^1, so $\overline{1_y}(x) = 0$. On the other hand if $\overline{1_y}(x) = 0$, then by R_0^1, $\overline{1_x}(y) = 0$. Thus we see that $\overline{1_x}(y) = 0 \Leftrightarrow \overline{1_y}(x) = 0$.

(b)⇒(c): Suppose x, y∈X, x ≠ y and there exists λ∈t such that λ(x) = 1 and λ(y) = 0. Put k = 1– λ. Then k∈t^c, k(x) = 0 and k(y) = 1.

We have for every x such that x ≠ y, k(x) = 0. Therefore k = $\overline{1_y}$ and so $\overline{1_y}$ (x) = 0. By (b) $\overline{1_x}(y) = 0$. This implies that there exists a t-closed set m such that m(x) = 1 and m(y) = 0. put μ = 1 – m. Then clearly μ∈t, μ(x) = 0 and μ(y) = 1.

(c)⇒(a): Suppose x, y∈X, x ≠ y and $\overline{1_x}(y) = 0$. This implies that there exists a t-closed set k such that k(y) = 0 and k(x) = 1. Put λ = 1 – k. Then λ is a t-open set such that λ(x) = 0 and λ(y) = 1. By (c) there exists a t-open set μ such that μ(x) = 1 and μ(y) = 0. Put m = 1 – μ . Then m is a t-closed set such that m(y) = 1 and m(x) = 0. Thus there exist a t-closed set m such that m(y) = 1 and m(x) = 0. Therefore, $\overline{1_y}(x) = 0$.

2.1.2. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^2 i.e., for every pair x, y∈X, x ≠ y, $\left(\forall \alpha \in I_0 : \overline{\alpha 1_x}(y) = \alpha \Leftrightarrow \forall \beta \in I_0 : \overline{\beta 1_y}(x) = \beta\right)$

(b) For every x, y∈ X, x ≠ y, if there exists α∈I_0 such that $\overline{\alpha 1_x}(y) < \alpha$, then there exists β∈$I_0$ such that $\overline{\beta 1_y}(x) < \beta$

(c) For every x, y∈ X, x ≠ y, if there exists a t-open set λ such that λ(y) < λ(x), then there exists a t-open set μ such that μ(x) < μ(y).

(d) For every x, y∈ X, x ≠ y, σ(x, y) = 0 ⇔ σ(y, x) = 0.

Where, $\sigma: X \times X \rightarrow I: (x, y) \rightarrow \sup_{\lambda \in t}(\lambda(y) - \lambda(x)) = \sup_{\alpha \in I}(\alpha - \overline{\alpha 1_x}(y))$

Proof:

__(a)⇒(b):__ Suppose x, y ∈X, x ≠ y and there exists $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) < \alpha$ ....(1)

Suppose for every $\beta \in I_0$, $\overline{\beta 1_x}(y) = \beta$. Then by (a) for every $\alpha \in I_0$ $\overline{\alpha 1_x}(y) = \alpha$ which contradicts (1). Therefore there exists $\beta \in I_0$ such that $\overline{\beta 1_y}(x) < \beta$.

__(b)⇒(c):__ Suppose for every x, y∈ X, x ≠ y, there exists a t-open set λ such that λ(y) < λ(x). Let β = λ(y). Then $\overline{\beta 1_y}(x) < \beta$. Hence by (b), there exist $\alpha_0 \in I_0$ such that $\overline{\alpha_0 1_x}(y) < \alpha_0$. This implies that there exists a t-closed set, say η such that $\eta(y) \leq \alpha_0 < \eta(x)$. And so η(y) < η(x). Put μ = 1 − η. Then μ is a t-open set and μ(x) < μ(y)

__(c)⇒(d):__ Suppose, x, y ∈X, x ≠ y and σ(x ,y) = 0. If σ(y ,x) > 0 then there exists λ∈t such that, λ(x) − λ(y) > 0. By (c), there exists μ∈ t such that μ(y) −μ(x) > 0. Therefore, σ(x, y) > 0, a contradiction. Therefore, σ(x ,y) = 0 ⇒ σ(y, x) = 0. Similarly we can show that, σ(y, x) = 0 ⇒ σ(x ,y) = 0

__(d)⇒(a):__ Suppose for every pair x, y∈X. x ≠ y and for every $\alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha$. Then σ(x ,y) = 0. By (d), σ(y, x) = 0. Thus, $\sup_{\beta \in I}(\beta - \overline{\beta 1_y}(x)) = 0$. And so, $\overline{\beta 1_y}(x) = \beta$, for every $\beta \in I_0$, for if there exists a $\beta \in I_0$ such that $\overline{\beta 1_y}(x) < \beta$, then $\beta - \overline{\beta 1_y}(x) > 0$ and so, σ(y, x) ≠ 0, a contradiction.

2.1.3. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^3 i.e., $\forall\lambda\in t$, $\forall x\in X$ and $\forall\alpha<\lambda(x)$, $\overline{\alpha 1_x}\le\lambda$

(b) For every $\lambda\in t$, there exists $M\subset t^c$ such that $\lambda = \mathrm{Sup}\mu,\ \mu\in M$.

Proof:

(a)⇒(b): Let $\lambda\in t$. Put M = $\{\overline{\alpha 1_x} : x\in X, \alpha<\lambda(x)\}$. By R_0^3, for every $\alpha<\lambda(x)$, $\overline{\alpha 1_x}\le\lambda$. Clearly $\lambda = \mathrm{Sup}\mu,\ \mu\in M$.

(b)⇒(a): Let $x\in X$ and λ is a t-open set such that $\alpha<\lambda(x)$. By (b) there exists $M\subset t^c$ such that $\lambda = \mathrm{Sup}\mu,\ \mu\in M$. Thus there exists $\mu\in M$ such that $\alpha<\mu(x)$. That is $\alpha 1_x\le\mu$ and so $\overline{\alpha 1_x}\le\mu\le\lambda$. Thus (X, t) is R_0^3.

2.1.4. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^4 i.e., $\forall\lambda\in t$, $\forall x\in X$ and $\forall\alpha\le\lambda(x)$, $\overline{\alpha 1_x}\le\lambda$

(b) For every $\lambda\in t$ and for every $x\in X$, $\overline{\lambda(x)1_x}\le\lambda$.

(c) For every $\lambda\in t$, $\lambda = \mathrm{Sup}\overline{\lambda(x)1_x},\ x\in X$.

(d) For every pair x, y ∈ X , x ≠ y and for every $\lambda\in t$, there exists $\mu\in t^c$ such that $\mu(x)=\lambda(x)$ and $\mu(y)=\lambda(y)$.

(e) For every pair x, y ∈ X , x ≠ y, the subspace ({x, y}, t|{x, y}) is self duel, i.e. $(\{x,y\},\ t\,|\,\{x,y\}) = (\{x,y\},\ t^c\,|\,\{x,y\})$.

(f) For every pair x, y ∈ X , x ≠ y and for every pair α, $\beta\in I$, $\overline{\alpha 1_x}(y)\le\beta \Rightarrow \overline{(1-\beta)1_y}(x)\le 1-\alpha$.

Proof:

(a)⇒(b): Let $\lambda\in t$ and $x\in X$, Put $\alpha=\lambda(x)$. By R_0^4, $\overline{\alpha 1_x}\le\lambda$. Thus $\overline{\lambda(x)1_x}\le\lambda$.

(b)⇒(c): Suppose $\lambda \in t$. If (b) is satisfied, then for every $x \in X$, $\overline{\lambda(x)1_x} \leq \lambda$. Therefore

$\mathrm{Sup}\overline{\lambda(x)1_x} \leq \lambda$, $x \in X$......................(1)

Now if $y \in X$, we also have $\lambda(y) = \overline{\lambda(y)1_y}(y) \leq \mathrm{Sup}\overline{\lambda(x)1_x}(y)$, $x \in X$. Thus

$\lambda \leq \mathrm{Sup}\overline{\lambda(x)1_x}$ ....(2)

From (1) and (2) $\lambda = \mathrm{Sup}\overline{\lambda(x)1_x}$, $x \in X$.

(c)⇒(d): Let $\lambda \in t$ and $x, y \in X$ such that $x \neq y$. Without loss of generality suppose $\alpha = \lambda(x) \leq \lambda(y) = \beta$. Then $\overline{\beta 1_y}(x) \leq \lambda(x) = \alpha$. Put $\mu_1 = \overline{\beta 1_y}$. Now μ_1 is a t-closed set such that $\mu_1(y) = \lambda(y) = \beta$ and $\mu_1(x) \leq \alpha = \lambda(x)$. Put $\mu = \mu_1 \vee \alpha$. Now $\mu(x) = \alpha = \lambda(x)$, and $\mu(y) = \beta = \lambda(y)$. Thus we see that there exists a $\mu \in t^c$ such that $\mu(x) = \lambda(x)$, and $\mu(y) = \lambda(y)$.

(d) ⇔ (e): Suppose (d) is satisfied. Therefore with the notations of (d) we have $\lambda|\{x, y\} = \mu|\{x, y\}$. Thus $(\{x,y\}, t\,|\,\{x,y\}) = (\{x,y\}, t^c\,|\,\{x,y\})$. On the other hand, suppose (e) is satisfied, i.e. $(\{x,y\}, t\,|\,\{x,y\}) = (\{x,y\}, t^c\,|\,\{x,y\})$, Then for every pair $x, y \in X$, $x \neq y$ and for every $\lambda \in t$, there exists $\mu \in t^c$ such that $\mu(x) = \lambda(x)$ and $\mu(y) = \lambda(y)$.

(d)⇒(f): Suppose $x, y \in X$, $x \neq y$, and $\alpha, \beta \in I$ such that $\overline{\alpha 1_x}(y) \leq \beta$. If $\alpha = \beta$, then there is nothing to prove. If $\beta < \alpha$ there is a $\mu \in t^c$ such that $\mu(x) = \alpha$ and $\mu(y) \leq \beta$. Let $\mu_1 = \mu \vee \beta$. Then $\mu_1(x) = \alpha$ and $\mu_1(y) = \beta$. If (d) is satisfied, there is a $\lambda \in t$ such that $\lambda(x) = \alpha$ and $\lambda(y) = \beta$. Let $\eta = 1 - \lambda$. Then $\eta \in t^c$. Now $\eta(x) = 1 - \alpha$, $\eta(y) = 1 - \beta$. Therefore, $\overline{(1-\beta)1_y}(x) = \mathrm{Inf}\{\eta(x) : \eta \in t^c \text{ and} (1-\beta)1_y \leq \eta\} \leq \eta(x) = 1 - \alpha$. Therefore $\overline{(1-\beta)1_y}(x) \leq 1 - \alpha$.

(f)⇒(a): Suppose (f) is satisfied, $\lambda \in t$ and $\alpha \leq \lambda(x)$. We have to show that $\overline{\alpha 1_x} \leq \lambda$.

Let $y \in X - \{x\}$ and $\lambda(y) = \beta$. If $\beta > \alpha$, then it is clear that $\overline{\alpha 1_x} \leq \lambda$.

Suppose $\beta < \alpha$. Let $\mu = 1 - \lambda$. Then $\mu \in t^c$ such that $\mu(y) = 1 - \beta > 1 - \alpha \geq 1 - \lambda(x) = \mu(x)$.

Thus we have, $\mu(x) < \mu(y)$. Therefore, $\overline{\mu(y)1_y}(x) \leq \mu(x)$.

Applying (f), $\overline{(1-\mu(x))1_x}(y) \leq 1-\mu(y)$.

$$\Rightarrow \overline{\lambda(x)1_x}(y) \leq \lambda(y)$$
$$\Rightarrow \overline{\alpha 1_x}(y) \leq \lambda(y) \ [\text{Since}, \alpha \leq \lambda(x)]$$

Therefore, $\overline{\alpha 1_x} \leq \lambda$. (Proved)

2.1.5. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^5 i.e., for every pair x, y∈X, $x \neq y, \overline{1_x}(y) = 1 \Rightarrow \overline{1_y}(x) = 1$

(b) For every pair x, y ∈ X, $x \neq y$, $\overline{1_x}(y) = 1 \Leftrightarrow \overline{1_y}(x) = 1$

(c) For every pair x, y ∈ X, $x \neq y$, $\overline{1_x}(y) < 1 \Leftrightarrow \overline{1_y}(x) < 1$

(d) For every pair x, y ∈ X, $x \neq y$, if there exists a t-closed set μ such that $\mu(y) < 1 = \mu(x)$, then there exists a t-closed set ν such that $\nu(x) < 1 = \nu(y)$.

Proof:

(a)⇒(b): Trivial.

(b)⇒(c): Suppose, $\overline{1_x}(y) < 1$. We have to show that, $\overline{1_y}(x) < 1$. If $\overline{1_y}(x)$ is not less than 1, then $\overline{1_y}(x) = 1$. Then by (b) $\overline{1_x}(y) = 1$ which is a contradiction. Therefore, $\overline{1_y}(x) < 1$. Thus we see that $\overline{1_x}(y) < 1 \Rightarrow \overline{1_y}(x) < 1$. Similarly we can show that $\overline{1_y}(x) < 1 \Rightarrow \overline{1_x}(y) < 1$.

(c)⇒(d): Suppose there exists a t-closed set μ such that $\mu(y) < 1 = \mu(x)$. Then $\overline{1_x}(y) < 1$. By (c) $\overline{1_y}(x) < 1$. Put $\nu = \overline{1_y}$. Then clearly $\nu(y) = 1$ and $\nu(x) < 1$. Thus we see that there exists a t-closed set, say ν such that $\nu(x) < 1 = \nu(y)$.

(d)⇒(a): Suppose $\overline{1_y}(x) = 1$. We have to show that $\overline{1_x}(y) = 1$.

Suppose $\overline{1_x}(y) < 1$ and $\overline{1_x} = \mu$. Thus μ is a t-closed set such that $\mu(y) < 1 = \mu(x)$. By (d) there exists a t-closed set ν such that $\nu(x) < 1 = \nu(y)$. This implies that $\overline{1_y}(x) < 1$, which is a contradiction. Therefore, $\overline{1_x}(y) = 1$.

Thus we see that, for every pair x, y$\in$X, $x \neq y, \overline{1_x}(y) = 1 \Rightarrow \overline{1_y}(x) = 1$. Thus (a) is satisfied.

2.1.6. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^6 i.e., for every pair x, y$\in$X, $x \neq y, \overline{1_x}(y) = \overline{1_y}(x)$

(b) For every pair x, y $\in$ X, $x \neq y$ and for every $\alpha \in I_1$, $\overline{1_x}(y) \leq \alpha \Rightarrow \overline{1_y}(x) \leq \alpha$

(c) For every pair x, y $\in$ X, $x \neq y$ and for every $\alpha \in I_0$, if there exists a t-open set λ such that $\lambda(y) = 0 < \alpha = \lambda(x)$, then $\exists$ a t-open set μ such that $\mu(x) = 0 < \alpha = \mu(y)$.

Proof:

(a)$\Rightarrow$(b):

Suppose, x, y $\in$ X, $x \neq y$ and $\alpha \in I_1$ such that $\overline{1_x}(y) \leq \alpha$. By (a), $\overline{1_y}(x) = \overline{1_x}(y)$. Therefore, $\overline{1_y}(x) \leq \alpha$.

(b)$\Rightarrow$(c):

Suppose, x, y $\in$ X, $x \neq y$, $\alpha \in I_0$, and $\exists$ a t-open set λ such that $\lambda(y) = 0 < \alpha = \lambda(x)$. Put $\eta = 1 - \lambda$. Then η is a t-closed set such that, $\eta(y) = 1$ and $\eta(x) = 1 - \alpha$. Therefore $\overline{1_y}(x) \leq 1 - \alpha$. Hence by (b) $\overline{1_x}(y) \leq 1 - \alpha$. This implies that there exists a t-closed set ν such that $\nu(x) = 1$ and $\nu(y) = 1 - \alpha$. Put $\mu = 1 - \nu$. Then μ is a t-open set such that $\mu(x) = 0$ and $\mu(y) = \alpha$.

(c)$\Rightarrow$(a):

Suppose, $\overline{1_y}(x) < \overline{1_x}(y)$. Let $\eta = \overline{1_y}$ and $\alpha = \eta(x) \neq 1$. Then η is t-closed set such that $\eta(y) = 1$, $\eta(x) = \alpha < \overline{1_x}(y)$. Let $\lambda = 1 - \eta$. Then λ is a t-open set such that $\lambda(y) = 0$ and $\lambda(x) = 1 - \alpha > 0$. By (c), $\exists$ $\mu \in t$ such that, $\mu(x) = 0$ and $\mu(y) = 1 - \alpha$. Put $\nu = 1 - \mu$. Then ν is a t-closed such that $\nu(x) = 1$ and $\nu(y) = \alpha$. This implies that $\overline{1_x}(y) < \alpha = \overline{1_y}(x)$, a contradiction.

2.1.7. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^7 i.e., for every pair $x, y \in X$, $x \neq y$, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$

(b) $\{\overline{1_x} : x \in X\}$ defines a partition of 1, i.e. there is a partition $\mathcal{A}$ of X such that for every $x \in A \in \mathcal{A}$. $\overline{1_x} = 1_A$.

Proof:

(a)⇒(b):

We have, for every distinct pair x, y $\in X$ $\overline{1_x}(y) = \overline{1_y}(x) \in \{0,1\}$. Therefore, $\overline{1_x}(X) \subset \{0,1\}$, and so there exists, for each $x \in X$, an $A(x) \subset X$ such that $\overline{1_x} = 1_{A(x)}$. Now if $y \in A(x)$, then $\overline{1_x}(y) = 1$. i.e. $1_y \leq 1_{A(x)}$.

It follows that $1_{A(y)} \leq 1_{A(x)}$, so $A(y) \subset A(x)$. Now $\overline{1_x}(y) = \overline{1_y}(x) = 1$. Therefore, $x \in A(y)$, hence $A(x) \subset A(y)$. Therefore $A(x) = A(y)$. Hence $\{A(x): x \in X\}$ is a partition of X.

(b)⇒(a): Given $\{\overline{1_x} : x \in X\}$is a partition of X. This implies that, either $\overline{1_x} = \overline{1_y}$ or $\overline{1_x} \wedge \overline{1_y} = 0$. If $\overline{1_x} = \overline{1_y}$, then clearly $\overline{1_x}(y) = \overline{1_y}(x) = 1$. On the other hand, if $\overline{1_x} \wedge \overline{1_y} = 0$, then $(\overline{1_x} \wedge \overline{1_y})(x) = 0$ and $(\overline{1_x} \wedge \overline{1_y})(y) = 0$. Therefore, $\overline{1_x}(y) = 0 = \overline{1_y}(x)$.

Thus $\overline{1_x}(y) = \overline{1_y}(x) \in \{0, 1\}$

2.1.8. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^8 i.e., for every pair x, y∈X, x ≠ y and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha \Rightarrow \overline{\alpha 1_y}(x) = \alpha$

(b) For every pair, x, y ∈ X, x ≠ y and for every $\alpha \in I_0$, $\overline{\alpha 1_x}(y) < \alpha \Rightarrow \overline{\alpha 1_y}(x) < \alpha$.

Proof:

(a)⇒(b):

Suppose x, y ∈ X, x ≠ y and $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) < \alpha$. Suppose $\overline{\alpha 1_y}(x) = \alpha$. Then by (a), $\overline{\alpha 1_x}(y) = \alpha$, which is a contradiction. Therefore $\overline{\alpha 1_y}(x) < \alpha$.

(b)⇒(a):

Suppose x, y ∈ X and $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) = \alpha$. Suppose $\overline{\alpha 1_y}(x) \neq \alpha$. Therefore, $\overline{\alpha 1_y}(x) < \alpha$. Then by (b), $\overline{\alpha 1_x}(y) < \alpha$, which is a contradiction. Therefore $\overline{\alpha 1_y}(x) = \alpha$.

2.1.9. Lemma [6]: For any fuzzy topological space (X, t), the following are equivalent:

(a) R_0^9, i.e., for every pair x, y∈X, x ≠ y and $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$

(b) For every pair, x, y ∈ X, x ≠ y and for every t-closed set, μ there exists a t-closed set, ν such that ν(x) = μ(y), ν(y) = μ(x).

Proof:

(a)⇒(b):

Let x, y ∈X, x ≠ y and μ is a t-closed set. Let α = μ(x) and β = μ(y). This implies that $\overline{\alpha 1_x}(y) \leq \beta$. Therefore, $\overline{\alpha 1_y}(x) \leq \beta$.

Hence there exists a t-closed set ν such that ν(y) = α and ν(x) = β. Thus μ(x) = ν(y) and μ(y) = ν(x).

(b)⇒(a):

Without loss of any generality suppose, $\overline{\alpha 1_x}(y) < \overline{\alpha 1_y}(x)$ .... (1).

Let $\mu = \overline{\alpha 1_x}$. Then $\alpha = \mu(x)$. Let $\beta = \mu(y)$. Then by (1) $\beta < \overline{\alpha 1_y}$ (x)........................ (2)

By (b) there exists t-closed set ν such that $\nu(x) = \mu(y) = \beta$ and $\nu(y) = \mu(x) = \alpha$.

We have, $\overline{\nu(y)1_y}(x) \leq \nu(x)$

$\Rightarrow \overline{\alpha 1_y}(x) \leq \beta$.

Using (1), $\overline{\alpha 1_x}(y) < \overline{\alpha 1_y}(x) \leq \beta$.

Or $\overline{\alpha 1_x}(y) < \beta$

Or $\mu(y) < \beta$ which is a contradiction.

Therefore, $\overline{\alpha 1_x}(y) < \overline{\alpha 1_y}(x)$ is not true. Similarly we can show that $\overline{\alpha 1_y}(x) < \overline{\alpha 1_x}(y)$ is also not true. Therefore $\overline{\alpha 1_y}(x) = \overline{\alpha 1_x}(y)$.

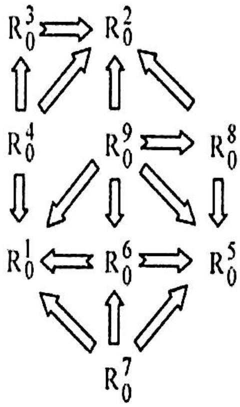
2.2. Remarks: (a) If x, y ∈ X and $\overline{\alpha 1_x}(y) \leq \beta$, there exists for each $\gamma > \beta$ a $\lambda \in t$ such that $\lambda(x) = 1-\alpha, \lambda(y) = 1-\gamma$. If then (X, t) is R_0^3, it follows that $\overline{(1-\gamma)1_y} \leq \lambda$. So, if (X, t) is R_0^3 and $\overline{\alpha 1_x}(y) \leq \beta$, then $\overline{\delta 1_y}(x) \leq 1-\alpha$ for each $\delta < 1-\beta$.

(b) In particular, if (X, t) is R_0^3 and $\overline{1_x}(y) \leq \beta$, then $\overline{\delta 1_y}(x) = 0$ for each $\delta < 1-\beta$.

3. Relations between the R_0-properties

In this section we study the interrelations between the fuzzy R_0-properties.

3.1 Theorem [6]: Between the R_0-porperties, mentioned in the section 2.1, there exist the following implications:



Proof:

1. Suppose (X, t) is R_0^4. Let $\lambda \in t$, $x \in X$ and $\alpha < \lambda(x)$. Then since (X, t) is R_0^4, hence $\overline{\alpha 1_x} \le \lambda$. Therefore (X, t) is R_0^3.

Suppose, there exists $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) = \beta < \alpha$. Take $\beta < \gamma < \alpha$. Let $\lambda = 1 - \overline{\alpha 1_x}$. Then $\lambda(x) = 1 - \alpha$, $\lambda(y) = 1 - \beta > 1 - \gamma$. Since (X, t) is R_0^3, $\overline{(1-\gamma)1_y} \le \lambda$. Now $\overline{(1-\gamma)1_y}(x) \le \lambda(x) = 1 - \alpha < 1 - \gamma$. Thus we see that, if $\overline{\alpha 1_x}(y) < \alpha$.then there exists $\delta \in I_0$ such that $\overline{\delta 1_y}(x) < \delta$. Therefore by lemma-2.2.2, (X, t) is R_0^3.

2. Suppose (X, t) is R_0^4. Then by lemma-2..2.4, for every pair $x, y \in X$, $x \neq y$ and for every pair $\alpha, \beta \in I$, $\alpha \neq \beta$, $\overline{\alpha 1_x}(y) \le \beta \Rightarrow \overline{(1-\beta)1_y}(x) \le 1 - \alpha$. Take $\alpha = 1$ and $\beta = 0$, $\overline{1_x}(y) \le 0 \Rightarrow \overline{1_y}(x) \le 0$. Or $\overline{1_x}(y) = 0 \Rightarrow \overline{1_y}(x) = 0$.

3. Suppose (X, t) is R_0^7. Then clearly, for every $x, y \in X$, $x \neq y$ $\overline{1_x}(y) = \overline{1_y}(x)$. Therefore (X, t) is R_0^6.

Let, $\overline{1_x}(y) = 0$. As (X, t) is R_0^6, $\overline{1_x}(y) = \overline{1_y}(x)$. And so $\overline{1_y}(x) = 0$.Therefore, (X, t) is R_0^1.

4. Suppose (X, t) is R_0^6. Then $\overline{1_x}(y) = \overline{1_y}(x)$. Therefore if $\overline{1_x}(y) = 1$, then $\overline{1_y}(x) = 1$. Therefore (X, t) is R_0^5.

5. Suppose (X, t) is R_0^9. Therefore for every pair x, y$\in$X, x $\neq$ y and $\forall \alpha \in I_0, \overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$. Therefore, if $\overline{\alpha 1_x}(y) = \alpha$ then $\overline{\alpha 1_y}(x) = \alpha$. Hence (X, t) is R_0^8.

Again, suppose $\forall \alpha \in I_0$, $\overline{\alpha 1_x}(y) = \alpha$. Then clearly $\overline{\beta 1_x}(y) = \beta$, $\forall \beta \in I_0$. Since, (X, t) has R_0^8, $\overline{\beta 1_x}(y) = \beta \Rightarrow \overline{\beta 1_y}(x) = \beta$. Therefore, we see that, for every pair x, y$\in$X, x $\neq$ y, $\left(\forall \alpha \in I_0 : \overline{\alpha 1_x}(y) = \alpha \Rightarrow \forall \beta \in I_0 : \overline{\beta 1_y}(x) = \beta\right)$. Similarly we can show that, for every pair x, y$\in$X, x $\neq$ y, $\left(\forall \beta \in I_0 : \overline{\beta 1_y}(x) = \beta \Rightarrow \forall \alpha \in I_0 : \overline{\alpha 1_x}(y) = \alpha\right)$. Thus (X, t) has R_0^2.

6. Suppose (X, t) is R_0^8, x, y $\in$ X and $\overline{1_x}(y) = 1$. Since (X, t) is R_0^8, $\overline{1_y}(x) = 1$. Therefore (X, t) is R_0^5.

7. Suppose (X, t) is R_0^9, then for every pair x, y $\in$ X, x $\neq$ y and for every $\alpha \in I_0$, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$. In particular, if $\alpha = 1$, $\overline{1_x}(y) = \overline{1_y}(x)$. Therefore, (X, t) is R_0^6.

4. Goodness and permanency properties:

In this section we show that all R_0^k $(1 \le k \le 9)$ properties are good extensions of their topological counter parts; all of them are found hereditary, seven of them are initial and therefore productive, and two of them are found not productive and therefore are not initial.

4.1. Theorem [6]: All R_0^k $(1 \le k \le 9)$ are good extensions of the topological R_0-property. That is,

(a) If $(X, \mathcal{T})$ is an R_0-space, then $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_0^k $(1 \le k \le 9)$.

(b) If $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_0^k $(1 \le k \le 9)$ then $(X, \mathcal{T})$ is an R_0-space.

Proof (a): Suppose $(X, \mathcal{T})$ is an R_0-space. Let $\lambda \in \mathcal{W}(\mathcal{T}) = \{u \in I^X : u^{-1}(\alpha, 1] \in \mathcal{T}, \alpha \in I_1\}$, $\lambda(x) = \alpha < \lambda(y) = \beta$. Let $F = \lambda^{-1}(0, \alpha]$, then F is closed in $(X, \mathcal{T})$. We have $y \notin F$. Therefore, $F \cap \overline{\{y\}} = \varnothing$. Also $\overline{\{x\}} \subset F$. Put $\mu = \alpha 1_{\overline{\{x\}}} \vee \beta 1_{\overline{\{y\}}}$. Then μ is closed in $\mathcal{W}(\mathcal{T})$. Now, $\mu(x) = \alpha$ and $\mu(y) = \beta$. Thus $\mu(x) = \lambda(x)$ and $\mu(y) = \lambda(y)$. Therefore $(X, \mathcal{W}(\mathcal{T}))$ is R_0^4. We know $R_0^4 \Rightarrow R_0^3 \Rightarrow R_0^2$ and $R_0^4 \Rightarrow R_0^1$.

Again, $\overline{\alpha 1_x} = \alpha 1_{\overline{\{x\}}}, \overline{\alpha 1_y} = \alpha 1_{\overline{\{y\}}}$. We have $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x) = \alpha$ if and only if $\overline{\{x\}} = \overline{\{y\}}$ and $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x) = 0$ if and only if $\overline{\{x\}} \cap \overline{\{y\}} = \varnothing$. So $(X, \mathcal{W}(\mathcal{T}))$ is R_0^k. We know, $R_0^9 \Rightarrow R_0^8$. Thus, $(X, \mathcal{W}(\mathcal{T}))$ is R_0^k $(1 \le k \le 9)$.

Proof (b):

(1) Suppose $(X, \mathcal{W}(\mathcal{T}))$ is a R_0^1 space and $x \in \overline{\{y\}}$, then $\overline{1_y}(x) = 1_{\overline{\{y\}}}(x) = 1 \ne 0$, and so $1_{\overline{\{x\}}}(y) = \overline{1_x}(y) \ne 0$. Therefore, $y \in \overline{\{x\}}$ which proves that $(X, \mathcal{T})$ is an R_0-space.

(2) Suppose $(X, \mathcal{W}(\mathcal{T}))$ is a R_0^2 space and $x \in \overline{\{y\}}$, then $\overline{\alpha 1_y}(x) = \alpha 1_{\overline{\{y\}}}(x) = \alpha$ for all $\alpha \in I_0$. Therefore $\overline{\beta 1_x}(y) = \beta \, \forall \beta \in I_0$. So in particular $\overline{1_x}(y) = 1_{\overline{\{x\}}}(y) = 1$. Hence $y \in \overline{\{x\}}$ which proves that $(X, \mathcal{T})$ is an R_0-space.

(3) Suppose $(X, \mathcal{W}(\mathcal{T}))$ is a R_0^5 space and $x \in \overline{\{y\}}$, then $\overline{1_y}(x) = 1_{\overline{\{y\}}}(x) = 1$. By R_0^5, $\overline{1_x}(y) = 1_{\overline{\{x\}}}(y) = 1$. Therefore, $y \in \overline{\{x\}}$ which proves that $(X, \mathcal{T})$ is an R_0-space.

Thus we see that, if $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_0^k $(k = 1, 2, 5)$ then $(X, \mathcal{T})$ is an R_0-space. Also we know that, $R_0^4 \Rightarrow R_0^3 \Rightarrow R_0^2$, $R_0^7 \Rightarrow R_0^6 \Rightarrow R_0^1$, $R_0^9 \Rightarrow R_0^8 \Rightarrow R_0^2$.

Therefore, If $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_0^k $(1 \le k \le 9)$ then $(X, \mathcal{T})$ is an R_0-space.

4.2. Theorem [6]: The properties R_0^k, $k \in \{2, 3, 5, 6, 7, 8, 9\}$ are initial, i.e., if $(f_j: X \to (X_j, t_j))$ is a source in fts where all (X_j, t_j) are R_0^k, then the initial fuzzy topology is also R_0^k.

Proof:

(a) Let $\{(X_j, t_j): j \in J\}$ be a family of R_0^2 fuzzy topological spaces, $\{f_j: X \to (X_j, t_j); j \in J\}$ be a family of functions and t be the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \neq y$ and there exists $\lambda \in t$ such that $\lambda(y) < \lambda(x)$. We can find basic t-open sets λ_j, $j \in J$ such that $\lambda = \sup\{\lambda_i: j \in J\}$. Also this λ_j must be expressible as $\lambda_j = \inf\{f_{jk}^{-1}(\lambda_{jk}): 1 \leq k \leq n\}$ where $\lambda_{jk} \in t_{jk}$ and $j_k \in J$. Now we can find some $k(1 \leq k \leq n)$, say k_1 such that $f_{jk}^{-1}(\lambda_{jk_1})(y) < f_{jk_1}^{-1}(\lambda_{jk_1})(x) \Rightarrow \lambda_{jk_1} f_{jk_1}(y) < \lambda_{jk_1} f_{jk_1}(x)$. Since (X_{jk_1}, t_{jk_1}) is R_0^2, there exists $V_{jk_1} \in t_{jk_1}$ such that $V_{jk_1} f_{jk_1}(x) < V_{jik_1} f_{jk_1}(y) \Rightarrow f_{jk_1}^{-1}(V_{jk_1})(x) < f_{jk_1}^{-1}(V_{jk_1})(y)$. Put $V = f_{jk_1}^{-1}(V_{jk_1}) \in t$. Thus, $V(x) < V(y)$. Thus (X, t) is R_0^2.

(b) Let $\{(X_j, t_j): j \in J\}$ be a family of R_0^3 fuzzy topological spaces, $\{f_j: X \to (X_j, t_j); j \in J\}$ be a family of functions and t be the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $\alpha \in I_{0,1}$, $x \in X$ and $u \in t$ with $\alpha 1_x < u$. Since $u \in t$, we can find basic t-open sets u_i, $i \in I$ such that $u = \sup\{u_j: j \in J\}$. Also this u_j must be expressible as $u_j = \inf\{f_{jk}^{-1}(u_{jk}): 1 \leq k \leq n\}$ where $u_{jik} \in t_{jk}$ and $j_k \in I$.. Now we can find some k $(1 \leq k \leq n)$, say k_1 such that $\alpha 1_x < f_{jk_1}^{-1}(u_{jk_1})$. That is, $\alpha < f_{jk_1}^{-1}(u_{jk_1})(x)$ or $\alpha < u_{jk_1}(f_{jk_1}(x))$. Since (X, t_{jk_1}) is R_0^3, $\overline{\alpha 1_{f_{jk_1}(x)}} \leq u_{jk_1}$. Since f is continuous, $f_{jk_1}(\overline{\alpha 1_x}) \leq \overline{\alpha 1_{f_{jk_1}(x)}}$. Thus $f_{jk_1}(\overline{\alpha 1_x}) \leq u_{jk_1} \Rightarrow \overline{\alpha 1_x} \leq f_{jk_1}^{-1}(u_{jk_1})$ But each $f_{jk_1}^{-1}(u_{jk_1}) \leq u$. Therefore, $\overline{\alpha 1_x} \leq u$. Hence (X, t) is R_0^3.

(c) Let $\{(X_j. t_j): j\in J\}$ be a family of R_0^5 fuzzy topological spaces, $\{f_j: X\to(X_j, t_j); j\in J\}$ be a family of functions and t be the initial fuzzy topology on X induced by the family $\{f_j: j\in J\}$. Suppose x, y $\in$X, x $\neq$ y and there exists $\lambda\in t^c$ such that $\lambda(y) < 1 = \lambda(x)$. Put $u = 1 - \lambda$. Then, $u \in t$ such that $u(x) = 0$ and $u(y) > 0$. Since $u\in t$, we can find basic t-open sets u_i such that $u = \sup\{u_j: j\in J\}$. Also each u_j must be expressible as, $u_j = \inf\{f_{jk}^{-1}(u_{jk}): 1 \le k \le n\}$. Since $u(x) = 0$ and $u(y) > 0$, we can find some k ($1 \le k \le n$), say k_1 such that $f_{jk}^{-1}(u_{jk_1})(x) = 0$ and $f_{jk}^{-1}(u_{jk_1})(y) > 0$. This implies that, $u_{jk_1} f_{jk_1}(x) = 0$ and $u_{jk_1} f_{jk_1}(y) > 0$. Since (X_{jk_1}, t_{jk_1}) is R_0^5, there exists $v_{jk_1} \in t_{jk_1}$ such that $v_{jk_1} f_{jk_1}(y) = 0$ and $v_{jk_1} f_{jk_1}(x) > 0$. This implies that $f_{jk_1}^{-1} v_{jk_1}(y) = 0$ and $f_{jk_1}^{-1} v_{jk_1}(x) > 0$. Now let $v = 1 - f_{jk_1}^{-1} v_{jk_1}$. Then $v \in t^c$ such that $v(x) < 1 = v(y)$. This implies that (X, t) is R_0^5.

(d) Let $\{(X_j. t_j): j\in J\}$ be a family of R_0^6 fuzzy topological spaces, $\{f_j: X\to(X_j, t_j); j\in J\}$ be a family of functions and t be the initial fuzzy topology on X induced by the family $\{f_j: j\in J\}$. Suppose x, y $\in$X, x $\neq$ y, $\alpha\in I_0$ and $\lambda\in t$ such that $\lambda(y) = 0 < \alpha = \lambda(x)$. Since $\lambda\in t$, there exists basic t-open sets λ_j such that $\lambda = \sup\{\lambda_j: j\in I\}$. Also each λ_j must be expressible as $\lambda_j = \inf\{f_{jk}^{-1}\lambda_{jk} : 1 \le k \le n\}$. Since $\lambda(y) = 0 < \alpha = \lambda(x)$, we can find some k ($1 \le k \le n$), say k_1 such that $f_{jk_1}^{-1}\lambda_{jk_1}(y) = 0 < \alpha = f_{jk_1}^{-1}\lambda_{jk_1}(x)$. This implies that $\lambda_{jk_1} f_{jk_1}(y) = 0 < \alpha = \lambda_{jk_1} f_{jk_1}(x)$. Since (X_{jk_1}, t_{jk_1}) is R_0^6, there exists $\mu_{jk_1} \in t_{jk_1}$ such that $\mu_{jk_1} f_{jk_1}(x) = 0 < \alpha = \mu_{jk_1} f_{jk_1}(y)$ or $f_{jk_1}^{-1}\mu_{jk_1}(x) = 0 < \alpha = f_{jk_1}^{-1}\mu_{jk_1}(y)$. Now, let $f_{jk_1}^{-1}\mu_{jk_1} = \mu\in t$. Then $\mu(x) = 0 < \alpha = \mu(y)$. Hence (X, t) is R_0^6.

(7) Let $\{(X_j. t_j): j\in J\}$ be a family of R_0^6 fuzzy topological spaces, $\{f_j: X\to(X_j, t_j); j\in J\}$ be a family of functions and t be the initial fuzzy topology on X induced by the family $\{f_j: j\in J\}$. Suppose x, y $\in$X, x $\neq$ y,

4.3. Corollary: Since initiality implies productivity and heredity, the properties R_0^k, $k \in \{2, 3, 5, 6, 7, 8, 9\}$, are productive and hereditary.

4.4. Corollary [6]: All the properties $R_0^k (1 \le k \le 9)$ are hereditary.

Proof: It is enough to show that the properties R_0^1 and R_0^4 are hereditary. Consider a fts (X, t). Let $A \subset X$. Consider the subspace (A, t_A).

We have, $t\text{-cl}(1_x) \cap 1_A = t_A\text{-cl}(1_x)$.

(1). Let $x, y \in A$, $x \neq y$ and $(t_A\text{-cl}(1_x))(y) = 0$. Therefore, $(t\text{-cl}(1_x) \cap 1_A)(y) = 0$. $\Rightarrow (t\text{-cl}(1_x))(y) \wedge 1_A(y) = 0 \Rightarrow (t\text{-cl}(1_x))(y) = 0$, Since, $y \in A$. Now, $x, y \in X$, $x \neq y$, and $(t\text{-cl}(1_x))(y) = 0$. So if (X, t) has R_0^1, then $(t\text{-cl}(1_y))(x) = 0$. Now $(t_A\text{-cl}(1_y))(x) = (t\text{-cl}(1_y) \cap 1_A)(x) = (t\text{-cl}(1_y))(x) \wedge 1_A(x) = 0$. This implies that, (A, t_A) has R_0^1.

(2). Let $x \in A$, $\lambda \in t_A$ such that $\alpha \le \lambda(x)$. There exist $\lambda' \in t$ such that $1_A \cap \lambda' = \lambda$. Since $x \in A$, $\lambda(x) = \lambda'(x)$. Now $\lambda' \in t$ and $\alpha \le \lambda'(x)$. So if So if (X, t) has R_0^4, then $t\text{-cl}(\alpha 1_x) \le \lambda'$. Now, $t_A\text{-cl}(\alpha 1_x) = 1_A \cap (t\text{-cl}(\alpha 1_x)) \le 1_A \cap \lambda' = \lambda$. Therefore, (A, t_A) has R_0^4.

4.5. Theorem [6]. If X is a set, (X', t') a fuzzy topological space having the property $R_0^k (1 \le k \le 9)$, then the reciprocal topology t on X for $f: X \to (X', t')$ also has R_0^k.

Proof: Suppose (X', t') a fuzzy topological space having the property $R_0^k (1 \le k \le 9)$. Suppose, $t = \{ f^{-1}(U) : U \in t' \}$. Now (X, t) is a fuzzy topological space. We have to show that (X, t) has $R_0^k (1 \le k \le 9)$. We have,

$\overline{\alpha 1_x} = f^{-1}\left(\overline{f(\alpha 1_x)}\right) = f^{-1}\left(\overline{\alpha 1_{f(x)}}\right)$ i.e., $\forall y \in X, \overline{\alpha 1_x}(y) = \overline{\alpha 1_{f(x)}}(f(y))$ ....(**).

1. Suppose x, y ∈ X, x ≠ y, $\overline{1_x}(y) = 0$, then $\overline{1_{f(x)}}(f(y)) = 0$ and since (X′, t′) has R_0^1, $\overline{1_{f(y)}}(f(x)) = 0$. Using (**), $f^{-1}(\overline{1_{f(y)}})(x) = 0$, and so $\overline{1_y} = 0$. Therefore, (X, t) has R_0^1.

2. Suppose x, y ∈ X, x ≠ y, $\alpha \in I_0$ and $\overline{\alpha 1_x}(y) = 0$. Then $\overline{\alpha 1_{f(x)}}(f(y)) = 0$. and since (X′, t′) has R_0^2, $\overline{\beta 1_{f(y)}}(f(x)) = 0$, for every $\beta \in I_0$. Using (**), $\overline{\beta 1_y}(x) = 0$ for every $\beta \in I_0$. This implies that (X, t′) has R_0^2.

3. Suppose $x \in X$, $\lambda \in t$ and $\alpha < \lambda(x)$. There is a $\lambda' \in t'$ such that $\lambda = f^{-1}(\lambda') = \lambda' \circ f$. Now, $\alpha < \lambda(x) = \lambda'(f(x))$. Since (X′, t′) has R_0^3, $\overline{\alpha 1_{f(x)}} \le \lambda'$ .Now, Using (**), $\overline{\alpha 1_x} = f^{-1}\left(\overline{\alpha 1_{f(x)}}\right) \le f^{-1}(\lambda') = \lambda$. Therefore (X, t) has R_0^3.

4. Suppose $x \in X$, $\lambda \in t$ and $\alpha \le \lambda(x)$. There is a $\lambda' \in t'$ such that $\lambda = f^{-1}(\lambda') = \lambda' \circ f$. Now, $\alpha \le \lambda(x) = \lambda'(f(x))$. Since (X′, t′) has R_0^4, $\overline{\alpha 1_{f(x)}} \le \lambda'$. Now $\overline{\alpha 1_x} = f^{-1}\left(\overline{\alpha 1_{f(x)}}\right) \le f^{-1}(\lambda') = \lambda$. Therefore (X, t) has R_0^4.

5. Suppose x, y ∈ X, x ≠ y, $\overline{1_x}(y) = 1$. Then $\overline{1_{f(x)}}(f(y)) = 1$ and since (X′, t′) has R_0^5, $\overline{1_{f(y)}}(f(x)) = 1$ and so $\overline{1_y}(x) = 1$. Therefore, (X, t) has R_0^5.

6. Suppose x, y ∈ X, x ≠ y. If (X′, t′) has R_0^6, then $\overline{1_{f(x)}}(f(y)) = \overline{1_{f(y)}}(f(x))$. Therefore, $\overline{1_x}(y) = \overline{1_y}(x)$. Therefore, (X, t) has R_0^6.

7. Suppose x, y ∈ X, x ≠ y. If (X′, t′) has R_0^7, then $\overline{1_{f(x)}}(f(y)) = \overline{1_{f(y)}}(f(x)) \in \{0,1\}$. Therefore, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0,1\}$. Therefore, (X, t) has R_0^7.

8. Suppose x, y $\in$ X, x $\neq$ y, $\alpha \in I_0$. Suppose, $\overline{\alpha 1_x}(y) = \alpha$. Using (**), $\overline{\alpha 1_{f(x)}}(f(y)) = \alpha$ If (X', t') has R_0^8, then $\overline{\alpha 1_{f(y)}}(f(x)) = \alpha$. Using (**), $\overline{\alpha 1_y}(x) = \alpha$. Therefore, (X, t) has R_0^8.

9. Suppose x, y $\in$ X, x $\neq$ y, $\alpha \in I_0$. If (X', t') has R_0^9, then $\overline{\alpha 1_{f(x)}}(f(y)) = \overline{\alpha 1_{f(y)}}(f(x))$. Using (**), $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$. Therefore, (X, t) has R_0^9.

5. Relations among T_0, R_0, T_1

In this section we recall from [41] the definitions and some properties of the T_0- and T_1-separation axioms used in the sequel:

5.1. Definitions: A fuzzy topological space (X, t) is called:

WT_0: if for every x, y $\in$X, x $\neq$ y, $\overline{1_x}(y) \wedge \overline{1_y}(x) < 1$.

T_0''' : if for every x, y $\in$X, x $\neq$ y and for every $\alpha \in I_0$, $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$.

T_0'' : if for every x, y $\in$X, x $\neq$ y and for every α, $\beta \in I_0$, $\overline{\alpha 1_x}(y) < \alpha$ or $\overline{\beta 1_y}(x) < \beta$.

T_0' : if for every x, y $\in$X, x $\neq$ y and for every α, $\beta \in I_0$, $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) < \alpha \wedge \beta$.

T_0 : if for every x, y $\in$X, x $\neq$ y: $\overline{1_x}(y) \wedge \overline{1_y}(x) = 0$.

WT_1: if for every x, y $\in$X, x $\neq$ y: $\overline{1_x}(y) < 1$.

T_1' : if for every x, y $\in$X, x $\neq$ y and for every $\alpha \in I_0$: $\overline{\alpha 1_x}(y) < \alpha$.

T_1 : if for every x $\in$X: $\overline{1_x} = 1$.

5.1.2. Theorem [6, 41]: Between the T_0 and T_1 properties, mentioned in the section 5.1, there exist the following implications:

$$
\begin{array}{ccccccc}
T_1 & \Rightarrow & T_1' & & \Longrightarrow & & WT_1 \\
\Downarrow & & \Downarrow & & & & \Downarrow \\
T_0 & \Rightarrow & T_0' & \Rightarrow & T_0'' \Rightarrow T_0''' & \Rightarrow & WT_0
\end{array}
$$

Proof: Suppose, $x, y \in X$, $x \neq y$, $\alpha, \beta \in I_0$.

Let (X, t) is T_1. Now, $\overline{1_x}(y) \wedge \overline{1_y}(x) = 1_x(y) \wedge 1_y(x) = 0 \wedge 0 = 0$, . Thus we see that, $T_1 \Rightarrow T_0$.

Again, let (X, t) is T_0. Now, $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) \leq \overline{1_x}(y) \wedge \overline{1_y}(x) = 0 < \alpha \wedge \beta$. Thus we see that, $T_0 \Rightarrow T_0'$.

Again, let (X, t) is T_0'. If $\overline{\alpha 1_x}(y) = \alpha$,

Then $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) < \alpha \wedge \beta \Rightarrow \alpha \wedge \overline{\beta 1_y}(x) < \alpha \wedge \beta \Rightarrow \overline{\beta 1_y}(x) < \beta$. Thus we see that, $T_0' \Rightarrow T_0''$.

Again, let (X, t) is T_0''. Then $\overline{\alpha 1_x}(y) < \alpha$ or $\overline{\beta 1_y}(x) < \beta$, for every $\alpha, \beta \in I_0$. In particular, we have, $\overline{\alpha 1_x}(y) < \alpha$ or $\overline{\alpha 1_y}(x) < \alpha$. Therefore $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$. Hence (X, t) is T_0'''.

Again let, (X, t) is T_0''', then for every pair $\alpha, \beta \in I_0$, $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$. Take $\alpha = 1$, then $\overline{1_x}(y) \wedge \overline{1_y}(x) < 1$. Thus we see that, $T_0''' \Rightarrow WT_0$.

Again, let (X, t) is T_1. Now $\overline{\alpha 1_x}(y) \leq \overline{1_x}(y) = 1_x(y) = 0 < \alpha$. Thus we see that, $T_1 \Rightarrow T_1'$.

Again, let (X, t) is T_1'. Then for every $\alpha \in I_0$, $\overline{\alpha 1_x}(y) < \alpha$. Take $\alpha = 1$. Then $\overline{1_x}(y) < 1$. Thus we see that, $T_1' \Rightarrow WT_1$.

Again, let (X, t) is WT_1. Then $\overline{1_x}(y) \wedge \overline{1_y}(x) < 1 \wedge 1 = 1$. Thus we see that, $WT_1 \Rightarrow WT_0$.

5.1.3. Theorem: Between the T_0 and T_1 properties, mentioned in the section 5.1, there exist the following non-implications:

(1) $WT_1 \not\Rightarrow T_1'$

(2) $T_1' \not\Rightarrow T_1$

(3) $T_0 \not\Rightarrow WT_1$

(4) $WT_1 \not\Rightarrow T_0'''$

(5) $T_0' \not\Rightarrow T_0$

(6) $T_0'' \not\Rightarrow T_0'$

(7) $T_0''' \not\Rightarrow T_0''$

(8) $WT_0 \not\Rightarrow T_0'''$

Proof:

(1) $WT_1 \not\Rightarrow T_1'$

Example-1: Consider a fuzzy topological space (X, t), where X = {x, y}, t = {0, u, v, 1} ∪{constants}; u(x) = 0, u(y) = 0.5, v(x) = 0.5 and v(y) = 0. Now, $u'(x) = 1$, $u'(y) = 0.5$, $v'(x) = 0.5$, $v'(y) = 1$.

Thus $\overline{1_x} = u', \overline{1_y} = v', \overline{1_x}(y) = 0.5 < 1$ and $\overline{1_y}(x) = 0.5 < 1$. $\therefore$(X, t) is WT_1.

Again, $\alpha 1_x \subseteq u'$ and $\alpha 1_x \subseteq m$, a constant fuzzy set with value $\geq \alpha$. $\therefore \overline{\alpha 1_x} = m$.

Let $\alpha = 0.5$, we see that $\overline{\alpha 1_x}(y) = 0.5 \nless 0.5$. This implies that (X, t) is not T_1'.

(2) $T_1' \not\Rightarrow T_1$

Example – 2: Let X ={x, y} be a set and m and n be two fuzzy sets in X defined by $m(x) = \alpha = n(y)$, $m(y) = r = n(x)$, where $0 < r < \alpha \leq 1$.

Let t be the fuzzy topology on X where t = {0, u, v, 1}∪{constants}such that u = 1– m, v = 1 – n. Now we see that $\overline{\alpha 1_x} = m$ and $\overline{\alpha 1_y} = n$ and $\overline{\alpha 1_x}(y) = r < \alpha$, $\overline{\alpha 1_y}(y) = r < \alpha$. $\therefore$ (X, t) is T_1'. But we observe that $\overline{1_x} \neq 1_x$. $\therefore$(X, t) is not T_1.

(3) $T_0 \not\Rightarrow WT_1$

Example-3: Consider a fuzzy topological space (X, t) where X = {x, y} and t = {0, u, 1}∪{constants}; u(x) = 0, u(y) = 1. $\therefore u'(x) = 1$, $u'(y) = 0$ and $\overline{1_x} = u'$. Also

$\overline{1_y} = 1$. Now $\overline{1_x}(y) \wedge \overline{1_y}(x) = 0 \wedge 1 = 0$. This implies that (X, t) is T_0. But $\overline{1_y}(x) = 1$, this implies that (X, t) is not WT_1.

(4) $WT_1 \not\Rightarrow T_0^{///}$

Exmple-1 will serve the purpose. Hence we have $WT_1 \not\Rightarrow T_0^{//}, T_0^{/}, T_0, T_1^{/}$ and T_1.

(5) $T_0^{/} \not\Rightarrow T_0$

Example-4: Consider a fuzzy topological space (X, t) where X = {x, y} and $t = \{0, u, v, 1\} \cup \{\text{constants}\}$; u(x) = 0 = v(y), u(y) = 0.8 = v(x).

Let $\alpha = 0.6$, $\beta = 0.7$. Then, $\overline{\alpha 1_x}(y) = 0.2$ and $\overline{\beta 1_y}(x) = 0.2$.

Thus, $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) = 0.2 < \alpha \wedge \beta$. $\therefore$ (X, t) is $T_0^{/}$. But (X, t) is not T_1; since $\overline{1_x}(y) \wedge \overline{1_y}(x) = 0.2 \neq 0$.

(6) $T_0^{//} \not\Rightarrow T_0^{/}$

Example-5: Consider a fuzzy topological space (X, t), where X = {x, y}, $t = \{0, u, 1\} \cup \{\text{constants}\}$; u is defined as u(x) = 0 and u(y) = 0.5. Let $\alpha = 0.6$, then $\overline{\alpha 1_x}(y) = 0.5 < 0.6 \Rightarrow$ (X, t) is $T_0^{//}$. Again, let $\beta = 0.8$. Then $\overline{\beta 1_y}$ = constant fuzzy set with value β. $\therefore \overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) = 0.6 \wedge 0.8 = 0.6 \nless 0.6 \wedge 0.8 = 0.6$. This implies that (X, t) is not $T_0^{/}$.

(7) $T_0^{///} \not\Rightarrow T_0^{//}$

Example-6: Consider a fuzzy topological space (X, t), where X = {x, y}, and $t = \{0, u, v, 1\} \cup \{\text{constants}\}$; u(x) = 0 = v(y), u(y) = 0.5, v(x) = 0.6.

Let α = 0.5. Then $\overline{\alpha 1_x}(y) = 0.5$ and $\overline{\alpha 1_y}(x) = 0.4$, so $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) = 0.4 < 0.5$.

This implies that (X, t) is $T_0^{///}$.

Now, let β = 0.4. Then $\overline{\alpha 1_y}(x) = 0.5 \nless \alpha$ and $\overline{\beta 1_y}(x) = 0.4 \nless \beta$. This implies that (X, t) is not $T_0^{//}$.

(8) $WT_0 \nRightarrow T_0^{///}$

In **Example-1**, we take α = 0.5, then we have $\overline{1_x}(y) \wedge \overline{1_y}(x) = 0.5 \wedge 0.5 = 0.5 < 1$, but $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) = 0.5 \wedge 0.5 = 0.5 \nless 0.5$. $\therefore$(X, t), is WT_0 but not $T_0^{///}$.

5.1.4. Theorem [6]: For fuzzy topological spaces, we have the following:

(a) $WT_1 \Rightarrow R_0^k$ for $k \in \{2, 5\}$

(b) WT_1 does not imply R_0^k for $k \in \{1, 3, 4, 6, 7, 8, 9\}$

(c) $T_1^/ \Rightarrow R_0^k$ for $k \in \{2, 5, 8\}$

(d) $T_1^/$ does not imply R_0^k for $k \in \{1,3, 4, 6, 7, 9\}$

(e) $T_1 \Rightarrow R_0^k$ for all $1 \le k \le 9$.

Proof:

(a) Suppose (X, t) is a fuzzy topological space; x, y $\in$ X, x $\neq$ y and there exists $\alpha \in I_0$ such that $\overline{\alpha 1_x}(y) < \alpha$. If (X, t) is WT_1, we have $\overline{1_x}(y) < 1$. Let β = 1. Thus we see that, there exists a $\beta \in I_0$ such that $\overline{\beta 1_x}(y) < \beta$. Therefore, (X, t) is R_0^2. Again, let $\overline{1_x}(y) < 1$. If (X, t)is WT_1, then $\overline{1_y}(x) < 1$. Thus (X, t) is R_0^5.

(b) On X = I we define t by

$t^c = \{\lambda \in I^X: \frac{1}{2} \le \lambda \le 1\} \cup \{\lambda \in I^X: \lambda$ is non- decreasing and $0 \le \lambda \le \frac{1}{2}\}$. Then it is easily seen,

$$\overline{\alpha 1_x}(y) = \begin{cases} \frac{1}{2} & \text{if } \frac{1}{2} < \alpha \le 1,\ y \ne x \\ 0 & \text{if } 0 < \alpha \le \frac{1}{2},\ y \ge x \\ \alpha & \text{if } y = x \text{ or } 0 < \alpha \le \frac{1}{2},\ y < x \end{cases}$$

(X, t) has WT_1, but not R_0^8. The rest will follow from (d).

(c) This is trivial.

(d) We only have to show that $T_0' \not\Rightarrow R_0^1$ or R_0^3. On X = I we define t by $t^c = \{\mu \in I^X : x < y \Rightarrow \lambda(y) \le 2\lambda(x)$. Then it is easily seen,

$$\overline{\alpha 1_x}(y) = \begin{cases} \frac{1}{2}\alpha & \text{if } y < x \\ \alpha & \text{if } y = x \\ 0 & \text{if } y > x \end{cases}$$

So this space has R_0^8, but it has neither R_0^1 (evident) nor R_0^3, as from $\overline{1_x}(y) = 0$ for y > x, it would follow from Remark 2.2(b) that $\overline{\delta 1_y}(x) = 0$ for all $\delta < 1$. However, it clearly has T_0 and T_1' but not T_1.

(e) We only have to show that T_1 implies R_0^4, R_0^7 and R_0^9. Let (X, t) is T_1. Then $\overline{1_x} = 1_x$ for every $x \in X$. Let $\lambda \in t$, $x \in X$ and $\alpha \le \lambda(x)$. Now $\overline{\alpha 1_x} \le \overline{1_x} = 1_x$. So, $\overline{\alpha 1_x}(y) = 0 \le \lambda(y)$, for every $y \in X$, $y \ne x$. Therefore, $\overline{\alpha 1_x} \le \lambda$. Hence (X, t) is R_0^4. Again, $\overline{1_x}(y) = 1_x(y) = 0 = 1_y(x) = \overline{1_y}(x)$. Hence (X, t) is R_0^7. Again, let $\alpha \in I_0$. $\overline{\alpha 1_x}(y) \le \overline{1_x}(y) = 1_x(y) = 0$ and $\overline{\alpha 1_y}(x) \le \overline{1_y}(x) = 1_y(x) = 0$. Therefore, $\overline{\alpha 1_x}(y) = \overline{\alpha 1_y}(x)$, for every $\alpha \in I_0$. Hence (X, t) is R_0^9

5.1.5. Theorem [6]. For fuzzy topological spaces, we have the following:

(a) $R_0^1 \wedge T_0 \Rightarrow T_1$

(b) $R_0^5 \wedge WT_0 \Rightarrow WT_1$

(c) $R_0^6 \wedge T_0^{/} \Rightarrow T_1^{/}$

(d) $R_0^7 \wedge WT_0 \Rightarrow T_1$

(e) $R_0^8 \wedge T_0^{///} \Rightarrow T_1^{/}$

(f) $R_0^3 \wedge T_0$ does not imply WT_1

(g) $R_0^4 \wedge T_0^{/}$ does not imply WT_1

(h) $R_0^5 \wedge T_0$ does not imply $T_1^{/}$

(i) $R_0^6 \wedge T_0^{//}$ does not imply $T_1^{/}$

(j) $R_0^8 \wedge T_0$ does not imply T_1

(k) $R_0^9 \wedge WT_0$ does not imply $T_1^{/}$

(l) $R_0^9 \wedge T_0^{/}$ does not imply T_1.

Proof:

(a) Let (X, t) be a fuzzy topological space which is both R_0^1 and T_0. Let x, y $\in$X such that x $\neq$ y. By T_0, $\overline{1_x}(y) \wedge \overline{1_y}(x) = 0$. Therefore, either $\overline{1_x}(y) = 0$ or $\overline{1_y}(x) = 0$. Suppose $\overline{1_x}(y) = 0$. By R_0^1, $\overline{1_y}(x) = 0$. On the other hand, if $\overline{1_y}(x) = 0$ then $\overline{1_x}(y) = 0$. Thus we have $\overline{1_x}(y) = 0$ for every y$\in$X such that x $\neq$ y. Therefore, $\overline{1_x} = 1_x$ for every x$\in$X. Hence (X, t) is T_1.

(b) Let (X, t) be a fuzzy topological space which is both R_0^5 and WT_0. By WT_0, $\overline{1_x}(y) \wedge \overline{1_y}(x) < 1$. Therefore, either $\overline{1_x}(y) < 1$ or $\overline{1_y}(x) < 1$. Suppose, $\overline{1_x}(y) < 1$, by R_0^5, $\overline{1_y}(x) < 1$. On the other hand if, $\overline{1_y}(x) < 1$ then by R_0^5, $\overline{1_x}(y) < 1$. Thus we have, $\overline{1_y}(x) < 1$, for every x, y$\in$X such that x $\neq$ y. hence (X, t) is WT_1.

(c) Let (X, t) be a fuzzy topological space which is both R_0^6 and T_0'. Let x, y $\in$X such that x $\neq$ y. By T_0', $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) < \alpha \wedge \beta$ for every pair, α, $\beta \in I_0$ such that $\alpha \neq \beta$. Take $\beta = 1$, then $\overline{\alpha 1_x}(y) \wedge \overline{1_y}(x) < \alpha$. By R_0^6, $\overline{1_x}(y) = \overline{1_y}(x)$. Then $\overline{\alpha 1_x}(y) \wedge \overline{1_x}(y) < \alpha$ or $\overline{\alpha 1_x}(y) < \alpha$. Thus we have, $\overline{\alpha 1_x}(y) < \alpha$ for every $\alpha \in I_0$ and for every pair x , y $\in$X such that x $\neq$ y. Hence, (X, t) is T_1'.

(d) Let (X, t) be a fuzzy topological space which is both R_0^7 and WT_0. Let x, y$\in$X such that x $\neq$ y. By WT_0, $\overline{1_x}(y) \wedge \overline{1_y}(x) < 1$.

Therefore, either $\overline{1_x}(y) < 1$ or $\overline{1_y}(x) < 1$.

By R_0^7, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0,1\}$. Therefore, $\overline{1_x}(y) = \overline{1_y}(x) = 0$. Thus we have, $\overline{1_x}(y) = 0$ for every pair x, y $\in$X such that x $\neq$ y. This implies that, $\overline{1_x} = 1_x$. Hence (X, t) is T_1.

(e) Let (X, t) be a fuzzy topological space which is both R_0^8 and T_0'''. Let x, y$\in$X such that x $\neq$ y and $\alpha \in I_0$. T_0''', $\overline{\alpha 1_x}(y) \wedge \overline{\alpha 1_y}(x) < \alpha$.. Therefore, either $\overline{\alpha 1_x}(y) < \alpha$ or $\overline{\alpha 1_y}(x) < \alpha$. If $\overline{\alpha 1_x}(y) < \alpha$, then by R_0^8, $\overline{\alpha 1_y}(x) < \alpha$ and conversely. Therefore, $\overline{\alpha 1_x}(y) < \alpha$ for every pair x, y $\in$X such that x $\neq$ y and for every $\alpha \in I_0$. Hence (X, t) is T_1'.

(f) We take, X = I and define t by $t^c = \{\mu \in I^X$: if $\exists x$, $\mu(x) = 1$, then $\mu(y) = 1$ for $x \leq y\}$. as $\alpha 1_x \in t^c$ for $\alpha < 1$, each $\nu \in I^X$, and a fortiori each $\lambda \in t$, is a supremum of closed functions, and so (X, t) has R_0^3. However

$$\overline{1_x}(y) = \begin{cases} 0 & \text{if } x > y \\ 1 & \text{if } x \leq y \end{cases}$$

Hence (X, t) has not R_0^1. Moreover it has T_0 but not WT_1.

(g) On X = I we define $t = t_1 \cup t_2 \cup t_3$, where

$$t_1^c = \{\lambda \in I^X: 0 \le \lambda \le \frac{1}{4} \text{ and } \forall x \in X,$$

$$0 \vee (\lambda(0) + (\lambda(0) - \frac{1}{2})x) \le \lambda(x) \le (\lambda(0) + \lambda(0)x) \wedge \frac{1}{4}\},$$

$$t_2^c = \{\lambda \in I^X: \frac{1}{4} \le \lambda \le \frac{3}{4},$$

$$t_3^c = \{\lambda \in I^X: \frac{3}{4} \le \lambda \le 1 \text{ and } \forall x \in X,$$

$$\frac{3}{4} \vee (\lambda(0) + (\lambda(0) - 1)x) \le \lambda(x) \le (\lambda(0) + (\lambda(0) - \frac{1}{2})x) \wedge 1\}.$$

It is only a matter of standard calculations to prove that t is indeed a fuzzy topology and that $t = t^c$. This space has R_0^4 and T_0' but not WT_1.

(h) $R_0^5 \wedge T_0$ does not imply T_1'.

(i) We take a set X with at least two points, elements $a \neq b$ in X and define t by

$$t^c = \{\mu \in I^X: \frac{1}{2} \le \mu \le 1\} \cup \{\mu \in I^X: 0 \le \mu \le \frac{1}{2} \text{ and } \mu(a) = \frac{1}{2} \Rightarrow \mu(b) = \frac{1}{2}\}$$

Then,

$$\overline{\alpha 1_x}(y) = \begin{cases} \frac{1}{2} & \text{if } \frac{1}{2} < \alpha \le 1 \\ 0 & \text{if } 0 \le \alpha < \frac{1}{2} \\ 0 & \text{if } \alpha = \frac{1}{2} \text{ and } x \neq a \text{ or } x = a,\ y \neq b \\ \frac{1}{2} & \text{if } \alpha = \frac{1}{2} \text{ and } x = a,\ y = b \end{cases}$$

This space has T_0' and R_0^6, but not T_1'.

(j) On $X = I$ we define t by

$t^c = \{\mu \in I^X: x < y \Rightarrow \lambda(y) \le 2\lambda(x)$. Then it is easily seen,

$$\overline{\alpha 1_x}(y) = \begin{cases} \frac{1}{2}\alpha & \text{if } y < x \\ \alpha & \text{if } y = x \\ 0 & \text{if } y > x \end{cases}$$

So this space has R_0^8 and T_0 but not T_1

(k) $R_o^9 \wedge WT_0$ doesn't imply T_1'.

(l) On the set X with at least two points we define t by

$$t^c = \{\mu \in I^X : 0 \le \mu \le \frac{1}{2}\} \cup \{\mu \in I^X : \frac{1}{2} \le \mu \le 1\}$$

Then,

$$\overline{\alpha 1_x}(y) = \begin{cases} \frac{1}{2} & \text{if } \frac{1}{2} < \alpha \le 1 \\ 0 & \text{if } 0 \le \alpha \le \frac{1}{2} \end{cases}$$

(X, t) has R_0^9 and T_0' but not T_1.

5.1.6. Theorem [6]. For fuzzy topological spaces, we have the following:

(a) $WT_1 \Leftrightarrow WT_0 \wedge R_0^5$

(b) $T_1' \Leftrightarrow T_0' \wedge R_0^8 \Leftrightarrow T_0'' \wedge R_0^8 \Leftrightarrow T_0''' \wedge R_0^8$

(c) $T_1 \Leftrightarrow T_0 \wedge R_0^k$ for $k \in \{1, 4, 6, 7, 9\}$

Proof:

(a) It follows from the definition that, $WT_1 \Rightarrow R_0^5$. Also we know $WT_1 \Rightarrow WT_0$. Thus $WT_1 \Rightarrow WT_0 \wedge R_0^5$. In theorem. 6.2(b), we have proved that, $WT_0 \wedge R_0^5 \Rightarrow WT_1$. Thus $WT_1 \Leftrightarrow WT_0 \wedge R_0^5$.

(b) In theorem, 6.1.(c), we have proved that $T_1' \Rightarrow R_0^k$, for $k \in \{2, 5, 8\}$. Also $T_1' \Rightarrow T_0'$. Thus $T_1' \Rightarrow T_0' \wedge R_0^8$.

Conversely, let (X, t) be a fuzzy topological space which has T_0' and R_0^8. Let $\overline{\alpha 1_y}(x) = \alpha$, where $x, y \in X$, $x \neq y$ and $\alpha \in I_0$. By R_0^8, $\overline{\alpha 1_x}(y) = \alpha$. Again, by T_0', $\overline{\alpha 1_x}(y) \wedge \overline{\beta 1_y}(x) < \alpha \wedge \beta$ for every pair $x, y \in X$, $x \neq y$ and for every pair α, $\beta \in I_0$. Take $\beta = 1$. Then, $\alpha \wedge \overline{1_y}(x) < \alpha$

6. Topological Properties

6.1. Theorem: Every homeomorphic image R_0^k - fts is also an R_0^k-fts, $(1 \leq k \leq 9)$.

Proof:

1. Let $f: (X, t_1) \to (Y, t_2)$ be a homeomorphism between fts, where (X, t_1) has R_0^1.

Then, $\overline{1_{f(x_1)}}(f(x_2)) = \overline{1_{x_1}}(x_2)$, for every pair, $x_1, x_2 \in X$.

Let $y_1, y_2 \in Y$, $y_1 \neq y_2$ such that $\overline{1_{y_1}}(y_2) = 0$. Let $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Then, $x_1 \neq x_2$. Since $\overline{1_{y_1}}(y_2) = 0$, $\overline{1_{x_1}}(x_2) = 0$. Again, since (X, t_1) has R_0^1, $\overline{1_{x_2}}(x_1) = 0$, and therefore, $\overline{1_{f(x_2)}}(f(x_1)) = \overline{1_{y_2}}(y_1) = 0$. This implies that (Y, t_2) is an R_0^1 fts.

2. Let $f: (X, t_1) \to (Y, t_2)$ be a homeomorphism between fts, where (X, t_1) has R_0^2.

Then, $\overline{\alpha 1_{f(x_1)}}(f(x_2)) = \overline{\alpha 1_{x_1}}(x_2)$, for every pair, $x_1, x_2 \in X$ and for every $\alpha \in I_0$.

Let $y_1, y_2 \in Y$, $y_1 \neq y_2$ and $\alpha \in I_0$ such that $\overline{\alpha 1_{y_1}}(y_2) = \alpha$. Let $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Then, $x_1 \neq x_2$. Since $\overline{\alpha 1_{y_1}}(y_2) = \alpha$, $\overline{\alpha 1_{x_1}}(x_2) = \alpha$. Again, since (X, t_1) has R_0^2, $\overline{\beta 1_{x_1}}(x_2) = \beta$ for every $\beta \in I_0$. Therefore, $\overline{\beta 1_{f(x_1)}}(f(x_2)) = \beta \Rightarrow \overline{\beta 1_{y_1}}(y_2) = \beta$. This implies that (Y, t_2) is an R_0^2 fts.

3. Let f: $(X, t_1) \rightarrow (Y, t_2)$ be a homeomorphism between fts, where (X, t_1) has R_0^3. Then, $\overline{\alpha 1_{f(x)}} = f\left(\overline{\alpha 1_x}\right)$, for every $x \in X$ and for every $\alpha \in I_0$.

Let $y \in Y$, $\lambda \in t_2$ and $\alpha \in I_0$ such that $\alpha < \lambda(y)$. Let $f^{-1}(y) = x$ and $f^{-1}(\lambda) = \mu$. Then, $x \in X$ and $\mu \in t_2$ such that $\alpha < \mu(x)$. Since (X, t_1) is R_0^3, $\overline{\alpha 1_x} \leq \mu$. Now $\overline{\alpha 1_y} = \overline{\alpha 1_{f(x)}} = f\left(\overline{\alpha 1_x}\right) \leq f(u) = \lambda$ This implies that (Y, t_2) is an R_0^3 fts.

4. Let f: $(X, t_1) \rightarrow (Y, t_2)$ be a homeomorphism between fts, where (X, t_1) has R_0^4. Then, $\overline{\alpha 1_{f(x)}} = f\left(\overline{\alpha 1_x}\right)$, for every $x \in X$ and for every $\alpha \in I_0$.

Let $y \in Y$, $\lambda \in t_2$ and $\alpha \in I_0$ such that $\alpha \leq \lambda(y)$. Let $f^{-1}(y) = x$ and $f^{-1}(\lambda) = \mu$. Then, $x \in X$ and $\mu \in t_2$ such that $\alpha \leq \mu(x)$. Since (X, t_1) is R_0^4, $\overline{\alpha 1_x} \leq \mu$. Now $\overline{\alpha 1_y} = \overline{\alpha 1_{f(x)}} = f\left(\overline{\alpha 1_x}\right) \leq f(\mu) = \lambda$ This implies that (Y, t_2) is an R_0^4 fts.

5. Let f: $(X, t_1) \rightarrow (Y, t_2)$ be a homeomorphism between fts, where (X, t_1) is R_0^5. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$, $\mu \in t_2$ such that $\overline{1_{y_1}}(y_2) = 1$. Let $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Since f is a homeomorphism, $\overline{1_{x_1}}(x_2) = \overline{1_{f(x_1)}}(f(x_2)) = \overline{1_{y_1}}(y_2) = 1$. By the R_0^5 property of (X, t_1) we have $\overline{1_{x_2}}(x_1) = 1$. Now, $\overline{1_{y_2}}(y_1) = \overline{1_{f(x_2)}}(f(x_1)) = \overline{1_{x_2}}(x_1) = 1$. This implies that (Y, t_2) is R_0^5.

6. Let f: $(X, t_1) \rightarrow (Y, t_2)$ be a homeomorphism between fts, where (X, t_1) is R_0^6. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$, $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$. Then, $x_1 \neq x_2$. Bt the R_0^6 property of (X, t_1) we have $\overline{1_{x_1}}(x_2) = \overline{1_{x_2}}(x_1)$. Since f is a homeomorphism

$\overline{1_{f(x_1)}}(f(x_2)) = \overline{1_{x_1}}(x_2)$ for every $x_1, x_2 \in X$ which together with $\overline{1_{x_1}}(x_2) = \overline{1_{x_2}}(x_1)$ imply that $\overline{1_{y_1}}(y_2) = \overline{1_{y_2}}(y_1)$. Therefore, (Y, t_2) is R_0^6.

7. Let (X, t_1) and (Y, t_2) be two fuzzy topological spaces, where (X, t_1) is R_0^7, Let f: (X, t_1) → (Y, t_2) be a homeomorphism. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Let $\overline{1_{y_1}}(y_2) \notin \{0, 1\}$. This implies that there exists $\lambda \in t_2^c$ such that $\lambda(y_1) = 1$ but $0 < \lambda(y_2) < 1$. Since f is a homeomorphism we have $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(\lambda) \in t_1^c$ such that $(f^{-1}(\lambda))(f^{-1}(y_1)) = 1$ and $0 < (f^{-1}(\lambda))(f^{-1}(y_2)) < 1$. This implies that $\overline{1_{f^{-1}(y_1)}}(f^{-1}(y_2)) \notin \{0, 1\}$ which is a contradiction since (X, t_1) is R_0^7. Again let $\overline{1_{y_1}}(y_2) \neq \overline{1_{y_2}}(y_1)$. Without any loss of generality we can assume that $0 = \overline{1_{y_1}}(y_2) < \overline{1_{y_2}}(y_1) = 1$. This implies that there exist $\eta, \lambda \in t_2^c$ such that $\eta(y_1) = 1, \eta(y_2) = 0$, $\lambda(y_1) = 0$ and $\eta(y_2) = 1$. Now, since f is a homeomorphism, we have $f^{-1}(\eta), f^{-1}(\lambda) \in t_1^c$ such that $(f^{-1}(\eta))(f^{-1}(y_1)) = 1$, $(f^{-1}(\eta))(f^{-1}(y_2)) = 0$, $(f^{-1}(\lambda))(f^{-1}(y_1)) = 0$ and $(f^{-1}(\lambda))(f^{-1}(y_2)) = 1$. This implies that $\overline{1_{f^{-1}(y_1)}}(f^{-1}(y_2)) = 0$ and $\overline{1_{f^{-1}(y_2)}}(f^{-1}(y_1)) = 1$.
Therefore, $\overline{1_{f^{-1}(y_1)}}(f^{-1}(y_2)) \neq \overline{1_{f^{-1}(y_2)}}(f^{-1}(y_1))$, which is also a contradiction. Therefore, $\overline{1_{y_1}}(y_2) = \overline{1_{y_2}}(y_1) \in \{0, 1\}$, and so, (Y, t_2) is R_0^7.

8. Let f: (X, t_1) → (Y, t_2) be a homeomorphism between fts, where (X, t_1) is R_0^8. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ and $\alpha \in I_0$ such that $\overline{\alpha 1_{y_1}}(y_2) = \alpha$. Again let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. $\overline{\alpha 1_{f(x_1)}}(f(x_2)) = \overline{\alpha 1_{x_1}}(x_2)\ \forall x_1, x_2 \in X$, since f is a homeomorphism,. Now, $\alpha = \overline{\alpha 1_{y_1}}(y_2) = \overline{\alpha 1_{f(x_1)}}(f(x_2)) = \overline{\alpha 1_{x_1}}(x_2)$. By the R_0^8 property of (X, t_1), $\overline{\alpha 1_{x_2}}(x_1) = \alpha$. Now, $\overline{\alpha 1_{y_2}}(y_1) = \overline{\alpha 1_{f(x_2)}}(f(x_1)) = \overline{\alpha 1_{x_2}}(x_1) = \alpha$. Therefore, (Y, t_2) is R_0^8.

9. Let f: (X, t_1) → (Y, t_2) be a homeomorphism between fts, where (X, t_1) is R_0^9. Let $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ and $\alpha \in I_0$ such that. Again let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. By the R_0^9 property of (X, t_1), $\overline{\alpha 1_{x_1}}(x_2) = \overline{\alpha 1_{x_2}}(x_1)$. We have, $\overline{\alpha 1_{f(x_1)}}(f(x_2)) = \overline{\alpha 1_{x_1}}(x_2)\ \forall x_1, x_2 \in X$, since f is a homeomorphism. Now $\overline{\alpha 1_{y_1}}(y_2) = \overline{\alpha 1_{f(x_1)}}(f(x_2)) = \overline{\alpha 1_{x_1}}(x_2)$. Similarly, $\overline{\alpha 1_{y_2}}(y_1) = \overline{\alpha 1_{x_2}}(x_1)$. Therefore, $\overline{\alpha 1_{y_1}}(y_2) = \overline{\alpha 1_{y_2}}(y_1)$. Therefore, (Y, t_2) is R_0^9.

CHAPTER-3

Fuzzy R_1 topological spaces

1. Introduction: In this chapter we introduce twelve R_1-type axioms for fuzzy topological spaces. We study their interrelations, goodness and initiality. A complete answer is given with regard to all possible $(R_1 \wedge T_0 \Rightarrow T_2)$ and $(T_2 \Rightarrow R_1)$-type implications.

2. R_1- properties

In this section we introduce twelve R_1-axioms for fuzzy topological spaces.

2.1. Definitions: We define, for fuzzy topological spaces (X, t), R_1-properties as follows:

R_1^1 :If $\forall$ x, y $\in$ X, x $\neq$ y, $\exists$ w $\in$ t such that either w(x) > $\alpha \in I_{0,1}$, and w(y) = 0 or w(y) > $\alpha \in I_{0,1}$, and w(x) = 0, then $\exists$ μ, $\nu \in$ t such that $\overline{1_x} \le \mu$, $\overline{1_y} \le \nu$ and $\mu \wedge \nu = 0$.

R_1^2 : If $\forall$ x, y $\in$ X, x $\neq$ y, $\exists$ w $\in$ t such that either w(x) > $\alpha \in I_{0,1}$, and w(y) = 0 or w(y) > $\alpha \in I_{0,1}$, and w(x) = 0, then $\exists$ μ, $\nu \in$ t such that $\overline{1_x} \le \mu$, $\overline{1_y} \le \nu$ and $\mu \le 1 - \nu$.

R_1^3 : If $\forall$ x, y $\in$ X, x $\neq$ y, $\exists$ w $\in$ t such that either w(x) > $\alpha \in I_{0,1}$, and w(y) = 0 or w(y) > $\alpha \in I_{0,1}$, and w(x) = 0, then $\exists$ μ, $\nu \in$ t such that $\mu(x) = 1 = \nu(y)$ and $\mu \wedge \nu = 0$.

R_1^4 : If $\forall$ x, y $\in$ X, x $\neq$ y, $\exists$ w $\in$ t such that either w(x) > $\alpha \in I_{0,1}$, and w(y) = 0 or w(y) > $\alpha \in I_{0,1}$, and w(x) = 0, then $\exists$ μ, $\nu \in$ t such that $\mu(x) = 1 = \nu(y)$ and $\mu \le 1 - \nu$.

R_1^5 : If ∀ x, y ∈ X, x ≠ y, ∃ w ∈ t such that either w(x) > α ∈ $I_{0,1}$, and w(y) = 0 or w(y) > α∈ $I_{0,1}$, and w(x) = 0, then ∀ β, δ∈ $I_{0,1}$, ∃ μ, ν ∈ t such that $\mu(x) > \beta$, $\nu(y) > \delta$ and $\mu \wedge \nu = 0$.

R_1^6 : If ∀ x, y ∈ X, x ≠ y, ∃ w ∈ t such that either w(x) > α ∈ $I_{0,1}$, and w(y) = 0 or w(y) > α∈ $I_{0,1}$, and w(x) = 0, then ∃ μ, ν ∈ t such that $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$.

R_1^7 : If ∀ x, y ∈ X, x ≠ y, ∃ w ∈ t such that either w(x) = α ∈ $I_{0,1}$, and w(y) = 0 or w(y) = α∈ $I_{0,1}$, and w(x) = 0, then ∃ μ, ν ∈ t such that $\overline{I_x} \leq \mu$, $\overline{I_y} \leq \nu$ and $\mu \wedge \nu = 0$.

R_1^8 : If ∀ x, y ∈ X, x ≠ y, ∃ w ∈ t such that either w(x) = α ∈ $I_{0,1}$, and w(y) = 0 or w(y) = α∈ $I_{0,1}$, and w(x) = 0, then ∃ μ, ν ∈ t such that $\overline{1_x} \leq \mu$, $\overline{1_y} \leq \nu$ and $\mu \leq 1 - \nu$.

R_1^9 : If ∀ x, y ∈ X, x ≠ y, ∃ w ∈ t such that either w(x) = α ∈ $I_{0,1}$, and w(y) = 0 or w(y) = α∈ $I_{0,1}$, and w(x) = 0, then ∃ μ, ν ∈ t such that $\mu(x) = 1 = \nu(y)$ and $\mu \wedge \nu = 0$.

R_1^{10} : If ∀ x, y ∈ X, x ≠ y, ∃ w ∈ t such that either w(x) = α ∈ $I_{0,1}$, and w(y) = 0 or w(y) = α∈ $I_{0,1}$, and w(x) = 0, then ∃ μ, ν ∈ t such that $\mu(x) = 1 = \nu(y)$ and $\mu \leq 1 - \nu$.

R_1^{11} : If ∀ x, y ∈ X, x ≠ y, ∃ w ∈ t such that either w(x) = α ∈ $I_{0,1}$, and w(y) = 0 or w(y) = α∈ $I_{0,1}$, and w(x) = 0, then ∀ β, δ∈ $I_{0,1}$, ∃ μ, ν ∈ t such that $\mu(x) > \beta$, $\nu(y) > \delta$ and $\mu \wedge \nu = 0$.

R_1^{12} : If $\forall$ x, y $\in$ X, x $\neq$ y, $\exists$ w $\in$ t such that either w(x) = $\alpha \in I_{0,1}$, and w(y) = 0 or w(y) = $\alpha \in I_{0,1}$, and w(x) = 0, then $\exists$ μ, $\nu \in$ t such that $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$.

3. Relations between the R_1^k -properties

3.1. Theorem: The following implications hold among the R_1-properties mentioned in the section 2.1:

$$\begin{array}{ccccc} R_1^1 \Leftrightarrow R_1^3 \Rightarrow R_1^5 & & R_1^7 \Leftrightarrow R_1^9 \Rightarrow R_1^{11} \\ \Downarrow \quad \Downarrow \quad \Downarrow & & \Downarrow \quad \Downarrow \quad \Downarrow \\ R_1^2 \Leftrightarrow R_1^4 \quad R_1^6 & & R_1^8 \Leftrightarrow R_1^{10} \quad R_1^{12} \end{array}$$

Proof:

$\underline{R_1^1 \Rightarrow R_1^3}$: Let (X, t) be an fts which has the property, R_1^1. Suppose that, x, y $\in$ X, $x \neq y$, and $w \in t$ such that $w(x) > \alpha \in I_{0,1}$ and $w(y) = 0$. Then, by the R_1^1- property of (X, t), there exist $u, v \in t$ such that $\overline{1}_x \leq u$, $\overline{1}_y \leq v$ and $u \wedge v = 0$. Clearly, $u(x) = 1 = v(y)$ and $u \wedge v = 0$. Hence, (X, t) has the property R_1^3.

Thus $R_1^1 \Rightarrow R_1^3$. Similarly we can show that $R_1^7 \Rightarrow R_1^9$.

$\underline{R_1^1 \Rightarrow R_1^2}$: Let (X, t) be an fts which has the property, R_1^1. Suppose that, x, y $\in$ X, $x \neq y$, and $w \in t$ such that $w(x) > \alpha \in I_{0,1}$ and w(y) = 0. Then, by the R_1^1 property of (X, t), there exist $u, v \in t$ such that $\overline{1}_x \leq u$, $\overline{1}_y \leq v$ and $u \wedge v = 0$. Clearly, $u \leq 1 - v$. Hence, (X, t) has the property R_1^2.

Thus $R_1^1 \Rightarrow R_1^2$. Similarly we can show that $R_1^7 \Rightarrow R_1^8$.

$R_I^2 \Rightarrow R_I^4$: Let (X, t) be an fts which has the property, R_I^2. Suppose that, $x, y \in X$, $x \neq y$, and $w \in t$ such that $w(x) > \alpha \in I_{0,1}$ and $w(y) = 0$. Then, by the R_I^2 property of (X, t), there exist $u, v \in t$ such that $\overline{1_x} \leq u, \overline{1_y} \leq v$ and $u \leq 1-v$. Clearly, $u(x) = 1 = v(y)$ and $u \leq 1-v$.. Hence, (X, t) has the property R_I^4.

Thus $R_I^2 \Rightarrow R_I^4$. Similarly we can show that $R_I^8 \Rightarrow R_I^{10}$.

$R_1^{10} \Rightarrow R_1^8$: Consider a R_1^{10}-fts (X, t). Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) = \alpha$ and $w(y) = 0$. Then by R_1^{10}, $\exists$ $u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \leq 1 - v$. Let $z \in X$ and $\beta \in I_{0,1}$ such that $\beta 1_z \nleq u$. This implies that $\beta > u(z)$. Now, let $u(z) = \delta \in I_{0,1}$. Then $u(z) = \delta \in I_{0,1}$ and $u(y) = 0$ together imply that $\exists$ $\eta, \lambda \in t$ such that $\eta(y) = 1 = \lambda(z)$ and $\lambda \leq 1 - \eta$. Now $1 - \lambda(y) = 1$. Therefore, $\overline{1_y} \leq 1 - \lambda$. Now, $\overline{1_y}(z) \leq 1 - \lambda(z) = 0$ and so $\beta 1_z \nleq \overline{1_y}$. Therefore, $\overline{1_y} \leq u$, which is a contradiction as $u(y) \neq 1$. Therefore $u(z) = 0$. Now, $\beta \wedge u \in t$ such that $\beta \wedge u(z) = 0$, $\beta \wedge u(x) = \beta$. Therefore $\exists$ $\eta, \lambda \in t$ such that $\eta(x) = 1 = \lambda(z)$ and $\lambda \leq 1 - \eta$. Now, $(1 - \lambda)(x) = 1$. Therefore, $\overline{1_x} \leq 1 - \lambda$. But $\overline{1_x}(z) \leq 1 - \lambda.(z) = 0$. Therefore, $\beta 1_z \nleq \overline{1_x}$. Thus we see that, if $\beta 1_z \nleq u$ then $\beta 1_z \nleq \overline{1_x}$. Hence, $\overline{1_x} \leq u$. Similarly we can show that $\overline{1_y} \leq v$. Therefore, (X, t) is R_1^8. Thus $R_1^{10} \Rightarrow R_1^8$.

Similarly we can show that $R_I^3 \Rightarrow R_I^1$, $R_I^9 \Rightarrow R_I^7$ and $R_1^4 \Rightarrow R_1^2$.

$R_I^3 \Rightarrow R_I^5$: Let (X, t) be an fts which has the property, R_I^3. Suppose that, $x, y \in X$, $x \neq y$, and $w \in t$ such that $w(x) > \alpha \in I_{0,1}$ and $w(y) = 0$. Then, by the R_I^3 property of (X, t), there exist $u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \wedge v = 0$. Clearly, $u(x) > \alpha$, $v(y) > \alpha$ and $u \wedge v = 0$. Hence, (X, t) has the property R_I^5.

Thus $R_I^3 \Rightarrow R_I^5$. Similarly we can show that $R_I^9 \Rightarrow R_I^{11}$.

$R_1^5 \Rightarrow R_1^6$: Let (X, t) be an fts which has the property, R_1^5.. Suppose that, $x, y \in X$, $x \neq y$, and $w \in t$ such that $w(x) > \alpha \in I_{0,1}$ and $w(y) = 0$. Then, by the R_1^5 property of (X, t), there exist $u, v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \wedge v = 0$. Clearly, $u(x)>0$, $v(y)>0$ and $u \wedge v = 0$. Hence, (X, t) has the property R_1^6.

Thus $R_1^5 \Rightarrow R_1^6$. Similarly we can show that $R_1^{11} \Rightarrow R_1^{12}$.

$R_1^3 \Rightarrow R_1^4$: Let (X, t) be an fts which has the property, R_1^3. Suppose that, $x, y \in X$, $x \neq y$, and $w \in t$ such that $w(x) > \alpha \in I_{0,1}$ and $w(y) = 0$. Then, by the R_1^3 property of (X, t), there exist $u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \wedge v = 0$. Clearly, $u \leq 1 - v$. Hence, (X, t) has the property R_1^4.

Thus $R_1^3 \Rightarrow R_1^4$. Similarly we can show that $R_1^9 \Rightarrow R_1^{10}$.

Counter examples:

Example-1: $X = \{x, y\}$ and $t = <\{u, v\} \cup \{\text{constants}\}>$, where $u(x) = 0.6$, $u(y) = 0$, $v(x) = 0.4$ and $v(y) = 0.4$. Then (X, t) is an fts. For $\alpha = 0.6$, (X, t) vacuously satisfies the R_1^1-property. Now, $u(x) = 0.6 = \alpha$ and $u(y) = 0$. But there exist no $u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \wedge v = 0$. Therefore, (X, t) is not R_1^{10}. Thus we see that, $R_1^1 \not\Rightarrow R_1^{10}$.

This example also shows that, $R_1^1 \not\Rightarrow R_1^{12}$.

Thus, $R_1^p \not\Rightarrow R_1^q$ (p = 1, 2, ..., 6 and q = 7, 8, ...,12)

Example-2: $X = \{x, y, z\}$ and $t = <\{u, v\} \cup \{\text{constants}\}>$, where $u(x) = 1$, $u(y) = 0$, $u(z) = 0.4$, $v(x) = 0$, $v(y) = 1$, $v(z) = 0$. For $\alpha = 0.5$, we see that, (X, t) vacuously satisfies the R_1^7-property. But (X, t) is not R_1^4 as $v(y) = 1$ and $v(z) = 0$ and there exist no $\lambda, u \in t$ such that $\lambda(y) = 1 = u(z)$ and $\lambda \wedge u = 0$. Thus we see that, $R_1^7 \not\Rightarrow R_1^4$. In this example, taking $u(z) = 0$, we observe that (X, t) is not R_1^6. Thus $R_1^7 \not\Rightarrow R_1^6$. Hence we have $R_1^p \not\Rightarrow R_1^q$ (p = 7, 8, ..., 12 and q = 1, 2, ...,6)

Example-3 [4]: Let X be an infinite set and for any x, y ∈ X, we define u_{xy}, a fuzzy set in X, as follows:

$u_{xy}(x) = 1$, $u_{xy}(y) = 0$ and $u_{xy}(z) = 0.5\ \forall\ z \in X,\ z \neq x, y$. Now consider the fuzzy topology, t on X generated by $\{u_{xy}: x, y \in X, x \neq y\} \cup$ {constants}. It is clear that, $\overline{1_x} \le u_{xy}$, $\overline{1_y} \le u_{yx}$ and $u_{xy} \le 1 - u_{yx}$. Thus, (X, t) is R_1^2. But (X, t) is not R_1^6 as $u_{xy} \wedge u_{yx}$ can never be zero. Thus, $R_1^2 \not\Rightarrow R_1^6$ and so $R_1^4 \not\Rightarrow R_1^6$.

Thus $R_1^p \not\Rightarrow R_1^q$ ($p = 2, 4$ and $q = 1, 3, 5, 6$)

Example-4: Let $X = \{x, y\}$ and $t = < \{\beta 1_x, \alpha 1_y\} \cup$ {constants}>, where $\beta > \alpha$, $\alpha, \beta \in I_{0,1}$. Then it is clear that (X, t) is R_1^5. But (X, t) is not R_1^4, since there exist no $u, v \in t$ such that $u(x) = 1 = v(y)$ and $u \le 1 - v$. Thus we see that, $R_1^5 \not\Rightarrow R_1^4$.

Thus $R_1^p \not\Rightarrow R_1^q$ ($p = 5, 6$ and $q = 1, 2, 3, 4$).

Example-5: Let $X = \{x, y\}$ and $t = <\left\{\frac{1}{2}1_x, \frac{1}{2}1_y\right\} \cup$ {constants}>. Then (X, t) is an fts and it is R_1^6. But (X, t) is not R_1^5. For, if we take $\beta, \delta \in I_{0,1}$ such that $\beta > 0.5$ and $\delta > 0.5$ there exist no u, v ∈ t such that $u(x) > \beta$, $v(y) > \delta$ and $u \wedge v = 0$. Thus we see that, $R_1^6 \not\Rightarrow R_1^5$. This example also shows that $R_1^{12} \not\Rightarrow R_1^{11}$.

Example-6: Let $X = \{x, y, z\}$ and $t = <\{u, v, w\} \cup$ {constants}>, where $u(x) = 1$, $u(y) = 0$, $u(z) = 0.5$, $v(x) = 0$, $v(y) = 1$, $v(z) = 0.5$, $w(x) = 0.6$, $w(y) = 0$ and $w(z) = 1$. Let $\alpha = 0.6$. Then (X, t) is R_1^8 as $\overline{1_x} \le u, \overline{1_y} \le v$ and $u \le 1 - v$. However (X, t) is not R_1^{12} as $u \wedge v = 0$ doesn't hold. Thus we see that, $R_1^8 \not\Rightarrow R_1^{12}$.

Therefore, $R_1^p \not\Rightarrow R_1^q$ ($p = 8, 10$ and $q = 7, 9, 11, 12$).

Example-7: Let $X = \{x, y\}$. We define fuzzy sets u, v on X as follows:

$u(x) = \alpha$, $u(y) = 0$ and $v(x) = 0$, $v(y) = \alpha$, $\alpha \in I_{0,1}$.

Then (X, t) is R_1^{11}. But it is clear that, (X, t) is not R_1^{10}. Thus $R_1^{11} \not\Rightarrow R_1^{10}$.

Therefore, $R_1^p \not\Rightarrow R_1^q$ ($p = 11, 12$ and $q = 7, 8, 9, 10$).

4. Goodness and permanency properties:

4.1. Theorem: All R_1^k $(1 \le k \le 12)$ are good extensions of the topological R_1-property. That is, $(X, \mathcal{T})$ is an R_1-space, if and only if $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^k $(1 \le k \le 12)$.

Note: By theorem 3.1, we have only to prove the following:

(a) If $(X, \mathcal{T})$ is an R_1-space, then $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^1 and R_1^7.

(b) If $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^k ($k \in \{4, 6, 10, 12\}$), then $(X, \mathcal{T})$ is an R_1-space.

Proof:

(a) Suppose $(X, \mathcal{T})$ is an R_1- topological space. Let $x, y \in X$, $x \neq y$, and $\alpha \in I_{0,1}$, and $w \in t$ such that $w(x) > \alpha$ and $w(y) = 0$. Now $w^{-1}(\alpha, 1] \in \mathcal{W}(\mathcal{T})$ such that $x \in w^{-1}(\alpha, 1]$ and $y \notin w^{-1}(\alpha, 1]$. This implies that $x \notin \overline{\{y\}}$ in $\mathcal{T}$. Hence there exist $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$ and $\mathcal{U} \cup \mathcal{V} = \emptyset$. Since an R_1-topological space is also an R_0- topological space, $\overline{\{x\}} \subseteq \mathcal{U}$ and $\overline{\{y\}} \subseteq \mathcal{V}$. Also we know that, $1_{\overline{\{x\}}} = \overline{1_x}$ and $\overline{1_y} = 1_{\overline{\{y\}}}$. Therefore, $\overline{1_x} \le 1_{\mathcal{U}}$ and $\overline{1_y} \le 1_{\mathcal{V}}$. Moreover, $1_{\mathcal{U}} \wedge 1_{\mathcal{V}} = 0$. Hence $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^1.

Again, suppose $(X, \mathcal{T})$ is an R_1- topological space. Let $x, y \in X$, $x \neq y$, and $\alpha \in I_{0,1}$, and $w \in t$ such that $w(x) = \alpha$ and $w(y) = 0$. Take $\beta \in I_{0,1}$ such that $\alpha > \beta$. Then $w(x) > \beta$. Now $w^{-1}(\beta, 1] \in \mathcal{W}(\mathcal{T})$ such that

$x \in w^{-1}(\beta, 1]$ and $y \notin w^{-1}(\beta, 1]$. This implies that $x \notin \overline{y}$ in $\mathcal{T}$. Hence there exist $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$ and $\mathcal{U} \cup \mathcal{V} = \emptyset$. Since an R_1-topological space is also an R_0- topological space, $\overline{\{x\}} \subseteq \mathcal{U}$ and $\overline{\{y\}} \subseteq \mathcal{V}$. Also we know that, $1_{\overline{\{x\}}} = \overline{1_x}$ and $\overline{1_y} = 1_{\overline{\{y\}}}$. Therefore, $\overline{1_x} \leq 1_{\mathcal{U}}$ and $\overline{1_y} \leq 1_{\mathcal{V}}$. Moreover, $1_{\mathcal{U}} \wedge 1_{\mathcal{V}} = 0$. Hence $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^7.

(b) Suppose $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^4. Let $x, y \in X$ such that $x \notin \overline{\{y\}}$ in $\mathcal{T}$. Then $\exists$ $w \in \mathcal{T}$ such that $x \in w$ and $y \notin w$. Now $1_w \in \mathcal{W}(\mathcal{T})$ such that $1_w(y) = 0$ and $1_w(x) = 1 > \alpha \ \forall\ \alpha \in I_{0,1}$. Therefore $\exists$ $\mu, \nu \in \mathcal{W}(\mathcal{T})$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \leq 1 - \nu$. Take $U = \mu^{-1}\left(\frac{1}{2}, 1\right]$ and $V = \nu^{-1}\left(\frac{1}{2}, 1\right]$. Clearly, $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Therefore, $(X, \mathcal{W}(\mathcal{T}))$ is an R_1-topological space.

Suppose $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^6. Let $x, y \in X$ such that $x \notin \overline{\{y\}}$ in $\mathcal{T}$. Then $\exists$ $w \in \mathcal{T}$ such that $x \in w$ and $y \notin w$. Now $1_w \in \mathcal{W}(\mathcal{T})$ such that $1_w(y) = 0$ and $1_w(x) = 1 > \alpha \ \forall\ \alpha \in I_{0,1}$. Therefore $\exists$ $\mu, \nu \in \mathcal{W}(\mathcal{T})$ such that $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$. Now, $x \in \mu^{-1}(0, 1] \in \mathcal{T}$, $y \in \nu^{-1}(0, 1] \in \mathcal{T}$ such that $\mu^{-1}(0, 1] \cap \nu^{-1}(0, 1] = \emptyset$. Therefore, $(X, \mathcal{W}(\mathcal{T}))$ is an R_1-topological space.

Again, suppose $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^{10}. Let $x, y \in X$ such that $x \notin \overline{\{y\}}$ in $\mathcal{T}$. Then $\exists$ $w \in \mathcal{T}$ such that $x \in w$ and $y \notin w$. Let $\alpha \in I_{0,1}$. Now $\alpha 1_w \in \mathcal{W}(\mathcal{T})$, $\alpha 1_w(x) = \alpha$ and $\alpha 1_w(y) = 0$. Then $\exists$ $\mu, \nu \in \mathcal{W}(\mathcal{T})$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \leq 1 - \nu$ Take, $\mathcal{U} = \mu^{-1}\left(\frac{1}{2}, 1\right]$ $\mathcal{V} = \nu^{-1}\left(\frac{1}{2}, 1\right]$. Then $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ such that $x \in \mathcal{U}$ and $y \in \mathcal{V}$. Moreover $\mathcal{U} \cap \mathcal{V} = \emptyset$. For, if $z \in \mathcal{U} \cap \mathcal{V}$, then $\frac{1}{2} < \mu(z) \leq 1 - \nu(z) < \frac{1}{2}$, a contradiction. Therefore, $(X, \mathcal{W}(\mathcal{T}))$ is an R_1-topological space.

Again suppose $(X, \mathcal{W}(\mathcal{T}))$ satisfies R_1^{12}. Let $x, y \in X$ such that $x \notin \overline{\{y\}}$ in $\mathcal{T}$. Then $\exists$ $w \in \mathcal{T}$ such that $x \in w$ and $y \notin w$. Let $\alpha \in I_{0,1}$. Now $\alpha 1_w \in \mathcal{W}(\mathcal{T})$, $\alpha 1_w(x) = \alpha$ and $\alpha 1_w(y) = 0$. Therefore $\exists$ $\mu, \nu \in \mathcal{W}(\mathcal{T})$ such that $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$. Now, $x \in \mu^{-1}(0, 1] \in \mathcal{T}$, $y \in \nu^{-1}(0, 1] \in \mathcal{T}$ such that $\mu^{-1}(0, 1] \cap \nu^{-1}(0, 1] = \emptyset$. Therefore, $(X, \mathcal{W}(\mathcal{T}))$ is an R_1-topological space.

<u>4.2. Theorem:</u> The properties R_1^k, $(1 \leq k \leq 12)$ are initial, i.e., if $(f_j: X \to (X_j, t_j))$ is a source in fts where all (X_j, t_j) are R_1^k, then the initial fuzzy topology t on X is also R_1^k.

<u>Proof:</u>

(1) Let $\{(X_j, t_j): j \in J\}$ be a family of R_1^1-fts, $\{f_j: X \to (X_j, t_j); j \in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) > \alpha$ and $w(y) = 0$.

Since $w \in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p \in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \leq k \leq n\right\}$. As $w(x) > \alpha$ and $w(y) = 0$, we can find some k $(1 \leq k \leq n)$, say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) > \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) > \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^1, there exists $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\overline{1_{f_{p_{k'}}(x)}} \leq \mu_{p_{k'}}$, $\overline{1_{f_{p_{k'}}(y)}} \leq \nu_{p_{k'}}$ and $\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$. Also since $f_{p_{k'}}$ is continuous, we have $f_{p_{k'}}\left(\overline{1_x}\right) \leq \overline{1_{f_{p_{k'}}(x)}}$. Now put

$\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Then $\mu, \nu \in t$ such that $\overline{1_x} \le \mu$, $\overline{1_y} \le \nu$ and $\mu \wedge \nu = 0$. Hence (X, t) is R_1^1.

(2) Let $\{(X_j, t_j): j\in J\}$ be a family of R_1^2-fts, $\{f_j: X\to(X_j, t_j); j\in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j\in J\}$. Let $x, y \in X$, $x \ne y$, $\alpha\in I_{0,1}$ and $w\in t$ such that $w(x) > \alpha$ and $w(y) = 0$.

Since $w\in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p\in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1\le k\le n\right\}$. As $w(x) > \alpha$ and $w(y) = 0$, we can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) > \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) > \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^2, there exists $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\overline{1_{f_{p_{k'}}(x)}} \le \mu_{p_{k'}}$, $\overline{1_{f_{p_{k'}}(y)}} \le \nu_{p_{k'}}$ and $\mu_{p_{k'}} \le 1 - \nu_{p_{k'}}$. Also since $f_{p_{k'}}$ is continuous, we have $f_{p_{k'}}\left(\overline{1_x}\right) \le \overline{1_{f_{p_{k'}}(x)}}$. Now put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Then $\mu, \nu \in t$ such that $\overline{1_x} \le \mu$, $\overline{1_y} \le \nu$ and $\mu \le 1 - \nu$. Hence (X, t) is R_1^2.

(3) Let $\{(X_j, t_j): j\in J\}$ be a family of R_1^3-fts, $f_j: X\to(X_j, t_j); j\in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j\in J\}$. Let $x, y \in X$, $x \ne y$, $\alpha\in I_{0,1}$ and $w\in t$ such that $w(x) > \alpha$ and $w(y) = 0$.

Since $w\in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p\in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1\le k\le n\right\}$. As $w(x) > \alpha$ and $w(y) = 0$, we can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) > \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) > \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^3, there exist $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\mu_{p_{k'}}(f(x)) = 1 = \nu_{p_{k'}}(f(y))$, $\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$. Put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Since $f_{p_{k'}}$ is continuous, $\mu, \nu \in t$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \wedge \nu = 0$. Hence (X, t) is R_1^3.,

(4) Let $\{(X_j, t_j): j \in J\}$ be a family of R_1^4-fts, $f_j: X \to (X_j, t_j); j \in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) > \alpha$ and $w(y) = 0$.

Since $w \in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p \in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \le k \le n\right\}$. As $w(x) > \alpha$ and $w(y) = 0$, we can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) > \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) > \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^4, there exist $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\mu_{p_{k'}}(f(x)) = 1 = \nu_{p_{k'}}(f(y))$, $\mu_{p_{k'}} \le 1 - \nu_{p_{k'}}$. Put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Since $f_{p_{k'}}$ is continuous, $\mu, \nu \in t$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \le 1 - \nu$. Hence (X, t) is R_1^4.

(5) Let $\{(X_j, t_j): j \in J\}$ be a family of R_1^5-fts, $f_j: X \to (X_j, t_j); j \in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) > \alpha$ and $w(y) = 0$.

Since $w \in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p \in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \le k \le n\right\}$. As $w(x) > \alpha$ and $w(y) = 0$, we can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) > \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) > \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^5, there exist $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\mu_{p_{k'}}(f(x)) > \beta$, $\nu_{p_{k'}}(f(y)) > \delta$ and

$\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$ for every $\beta, \delta \in I_{0,1}$. Put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Now $\mu, \nu \in t$, since $f_{p_{k'}}$ is continuous and $\mu(x) > \beta$, $\nu(y) > \delta$ and $\mu \wedge \nu = 0$. Hence (X, t) is R_1^5.

(6) Let $\{(X_j, t_j): j \in J\}$ be a family of R_1^6-fts, $f_j: X \to (X_j, t_j); j \in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) > \alpha$ and $w(y) = 0$.
Since $w \in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p \in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \leq k \leq n\right\}$. As w(x) > α and w(y) = 0 , we can find some k ($1 \leq k \leq n$), say k′ such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) > \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) > \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^6, there exist $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\mu_{p_{k'}}(f(x)) > 0$, $\nu_{p_{k'}}(f(y)) > 0$ and $\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$. Put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Now $\mu, \nu \in t$, since $f_{p_{k'}}$ is continuous and, $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$. Hence (X, t) is R_1^6.

(7) Let $\{(X_j, t_j): j \in J\}$ be a family of R_1^7-fts, $\{f_j: X \to (X_j, t_j); j \in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) = \alpha$ and $w(y) = 0$.
Since $w \in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p \in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \leq k \leq n\right\}$. As w(x) = α and w(y) = 0 , we can find some k ($1 \leq k \leq n$), say k′ such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) = \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) = \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^7,

there exists $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\overline{1_{f_{p_{k'}}(x)}} \le \mu_{p_{k'}}, \overline{1_{f_{p_{k'}}(y)}} \le \nu_{p_{k'}}$ and $\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$. Also since $f_{p_{k'}}$ is continuous, we have $f_{p_{k'}}\left(\overline{1_x}\right) \le \overline{1_{f_{p_{k'}}(x)}}$. Now put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Then $\mu, \nu \in t$ such that $\overline{1_x} \le \mu$, $\overline{1_y} \le \nu$ and $\mu \wedge \nu = 0$. Hence (X, t) is R_1^7.

(8) Let $\{(X_j, t_j): j\in J\}$ be a family of R_1^8-fts, $\{f_j: X\to(X_j, t_j); j\in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j\in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha\in I_{0,1}$ and $w\in t$ such that $w(x) = \alpha$ and $w(y) = 0$.
Since $w\in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p\in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \le k \le n\right\}$. As $w(x) = \alpha$ and $w(y) = 0$, we can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) = \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$. This implies that $w_{p_{k'}} f_{p_{k'}}(x) > \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^8, there exists $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\overline{1_{f_{p_{k'}}(x)}} \le \mu_{p_{k'}}, \overline{1_{f_{p_{k'}}(y)}} \le \nu_{p_{k'}}$ and $\mu_{p_{k'}} \le 1 - \nu_{p_{k'}}$. Also since $f_{p_{k'}}$ is continuous, we have $f_{p_{k'}}\left(\overline{1_x}\right) \le \overline{1_{f_{p_{k'}}(x)}}$. Now put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Then $\mu, \nu \in t$ such that $\overline{1_x} \le \mu$, $\overline{1_y} \le \nu$ and $\mu \le 1 - \nu$. Hence (X, t) is R_1^8.

(9) Let $\{(X_j, t_j): j\in J\}$ be a family of R_1^9-fts, $f_j: X\to(X_j, t_j); j\in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j\in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha\in I_{0,1}$ and $w\in t$ such that $w(x) = \alpha$ and $w(y) = 0$.
Since $w\in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p\in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \le k \le n\right\}$. As $w(x) = \alpha$ and $w(y) = 0$, we

can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) = \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$.

This implies that $w_{p_{k'}} f_{p_{k'}}(x) = \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^9, there exist $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\mu_{p_{k'}}(f(x)) = 1 = \nu_{p_{k'}}(f(y))$, $\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$. Put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Since $f_{p_{k'}}$ is continuous, $\mu, \nu \in t$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \wedge \nu = 0$. Hence (X, t) is R_1^9.

(10) Let $\{(X_j, t_j): j \in J\}$ be a family of R_1^{10}-fts, $f_j: X \to (X_j, t_j); j \in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \ne y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) = \alpha$ and $w(y) = 0$.
Since $w \in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p \in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \le k \le n\right\}$. As $w(x) = \alpha$ and $w(y) = 0$, we can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) = \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$.

This implies that $w_{p_{k'}} f_{p_{k'}}(x) = \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^{10}, there exist $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\mu_{p_{k'}}(f(x)) = 1 = \nu_{p_{k'}}(f(y))$, $\mu_{p_{k'}} \le 1 - \nu_{p_{k'}}$. Put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Since $f_{p_{k'}}$ is continuous, $\mu, \nu \in t$ such that $\mu(x) = 1 = \nu(y)$ and $\mu \le 1 - \nu$. Hence (X, t) is R_1^{10}.

(11) Let $\{(X_j, t_j): j \in J\}$ be a family of R_1^{11}-fts, $f_j: X \to (X_j, t_j); j \in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \ne y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) = \alpha$ and $w(y) = 0$.
Since $w \in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p \in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \le k \le n\right\}$. As $w(x) = \alpha$ and $w(y) = 0$, we

can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) = \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$.

This implies that $w_{p_{k'}} f_{p_{k'}}(x) = \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^{11}, there exist $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\mu_{p_{k'}}(f(x)) > \beta$, $\nu_{p_{k'}}(f(y)) > \delta$ and $\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$ for every $\beta, \delta \in I_{0,1}$. Put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Now $\mu, \nu \in t$, since $f_{p_{k'}}$ is continuous and $\mu(x) > \beta$, $\nu(y) > \delta$ and $\mu \wedge \nu = 0$. Hence (X, t) is R_1^{11}.

(12) Let $\{(X_j, t_j): j \in J\}$ be a family of R_1^{12}-fts, $f_j: X \to (X_j, t_j); j \in J\}$ a family of functions and t the initial fuzzy topology on X induced by the family $\{f_j: j \in J\}$. Let $x, y \in X$, $x \neq y$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) = \alpha$ and $w(y) = 0$.

Since $w \in t$, there exist basic t-open sets, w_p such that $w = \sup\{w_p: p \in P\}$. Also each w_p must be expressible as $w_p = \inf\left\{f_{p_k}^{-1} w_{p_k} : 1 \le k \le n\right\}$. As $w(x) = \alpha$ and $w(y) = 0$, we can find some k ($1 \le k \le n$), say k' such that $f_{p_{k'}}^{-1} w_{p_{k'}}(x) = \alpha$ and $f_{p_{k'}}^{-1} w_{p_{k'}}(y) = 0$.

This implies that $w_{p_{k'}} f_{p_{k'}}(x) = \alpha$ and $w_{p_{k'}} f_{p_{k'}}(y) = 0$. Since $\left(X_{p_{k'}}, t_{p_{k'}}\right)$ is R_1^{12}, there exist $\mu_{p_{k'}}, \nu_{p_{k'}} \in t_{p_{k'}}$ such that $\mu_{p_{k'}}(f(x)) > 0$, $\nu_{p_{k'}}(f(y)) > 0$ and $\mu_{p_{k'}} \wedge \nu_{p_{k'}} = 0$. Put $\mu = f_{p_{k'}}^{-1}\left(\mu_{p_{k'}}\right)$ and $\nu = f_{p_{k'}}^{-1}\left(\nu_{p_{k'}}\right)$. Now $\mu, \nu \in t$, since $f_{p_{k'}}$ is continuous and, $\mu(x) > 0$, $\nu(y) > 0$ and $\mu \wedge \nu = 0$. Hence (X, t) is R_1^{12}.

4.3. Corollary: Since initiality implies productivity and heredity all the properties R_1^k $(1 \le k \le 12)$ are productive and hereditary.

5. Relationships of the R_1^k-concepts with some fuzzy separation concepts

Recall:

5.1. Definition [4]: A fuzzy topological space (X, t) is called

$FT_0(i)$: iff for every x, y $\in$X, x $\neq$ y, there exists u$\in$t such that either u(x) = 1 and u(y) = 0 or u(x) = 0 and u(y) = 1.

$FT_0(ii)$: iff for every x, y $\in$X, x $\neq$ y, there exists u$\in$t such that either u(x) > 0 and u(y) = 0 or u(x) = 0 and u(y) > 0.

$FT_0(iii)$: iff for every x, y $\in$X, x $\neq$ y, there exists u$\in$t such that either u(x) > u(y) or u(y) > u(x).

The following relations hold between the FT_0-properties:

$$FT_0(i) \Rightarrow FT_0(ii) \Rightarrow FT_0(iii)$$

5.2. Definition [4]: A fuzzy topological space (X, t) is called

$FT_2(i)$: iff for every x, y $\in$X, x $\neq$ y, there exist u, v $\in$ t such that u(x) = 1 = v(y) and $u \wedge v = 0$.

$FT_2(ii)$: iff for every x, y $\in$X, x $\neq$ y, and for every α, $\beta \in I_0$, there exist u, v $\in$ t such that u(x) > α, v(y) > β and $u \wedge v = 0$.

$FT_2(iii)$: iff for every x, y $\in$X, x $\neq$ y, there exist u, v $\in$ t such that u(x) > 0, v(y) > 0 and $u \wedge v = 0$.

FT_2(iv): iff for every x, y $\in$X, x $\neq$ y, there exist u, v $\in$ t such that either u(x) = 1 = v(y) and u $\leq$ 1 – v.

FT_2(v): iff for every x, y $\in$X, x $\neq$ y, and for every α, $\beta \in I_0$, there exist u $\in$ t such that $\alpha \lesssim u(x)$ and $\beta \nleq \bar{u}(y)$

*** By $\alpha \lesssim u(x)$, we mean u(x) = 1 when α = 1 and α < u(x) if $\alpha \neq 1$.

The following relations hold between the FT_2-properties:

$$\begin{array}{l} FT_2(i) \Rightarrow FT_2(ii) \Rightarrow FT_2(iii) \\ \Downarrow \\ FT_2(iv) \Rightarrow FT_2(v) \end{array}$$

5.1. Theorem: For the fuzzy topological spaces, the following are true:

(a) $R_1^1 + FT_0(iii) \not\Rightarrow FT_2(iii), FT_2(v)$, $R_1^7 + FT_0(iii) \not\Rightarrow FT_2(iii), FT_2(v)$

(b) $R_1^6 + FT_0(ii) \Rightarrow FT_2(iii)$, $R_1^{12} + FT_0(ii) \Rightarrow FT_2(iii)$

(c) $R_1^5 + FT_0(ii) \not\Rightarrow FT_2(ii)$, $R_1^{11} + FT_0(ii) \not\Rightarrow FT_2(ii)$

(d) $R_1^1 + FT_0(ii) \Rightarrow FT_2(i)$, $R_1^7 + FT_0(ii) \Rightarrow FT_2(i)$

(e) $R_1^2 + FT_0(ii) \Rightarrow FT_2(iv)$, $R_1^8 + FT_0(ii) \Rightarrow FT_2(iv)$

(f) $R_1^2 + FT_0(i) \not\Rightarrow FT_2(iii)$, $R_1^8 + FT_0(i) \not\Rightarrow FT_2(iii)$

(g) $R_1^6 + FT_0(i) \not\Rightarrow FT_2(ii), FT_2(iv)$, $R_1^{12} + FT_0(i) \not\Rightarrow FT_2(ii), FT_2(iv)$

(h) $R_1^5 + FT_0(i) \not\Rightarrow FT_2(v)$, $R_1^{11} + FT_0(i) \not\Rightarrow FT_2(v)$

Proof (a):

Example-1: Consider a fuzzy topological space, (X, t) where X = {x, y}, w(x) = 0.5, w(y) = 0.4 and t = < {w} ∪{constants}>. Clearly (X, t) is FT_0(iii) and R_1^1 and R_1^7. But (x, t) is neither FT_2(iii) nor FT_2(v).

Proof (b): Let (X, t) be a fuzzy topological space which is both R_1^6 and FT_0(ii). Let x, y ∈X, x ≠ y. By FT_0(ii), there exists w∈t such that either w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0. Definitely either w(x) > α, w(y) = 0 or w(x) = 0, w(y) > α for some α ∈ $I_{0,1}$. Now by R_1^6, there exist u, v ∈ t such that u(x) > 0, u(y) >0 and u ∧ v = 0. Thus (X, t) is FT_2(iii).

Again, let (X, t) be a fuzzy topological space which is both R_1^{12} and FT_0(ii). Let x, y ∈X, x ≠ y. By FT_0(ii), there exists w∈t such that either w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0. Definitely either w(x) = α, w(y) = 0 or w(x) = 0, w(y) = α for some α ∈ $I_{0,1}$. Now by R_1^{12}, there exist u, v ∈ t such that u(x) > 0, u(y) >0 and u ∧ v = 0. Thus (X, t) is FT_2(iii).

Proof (c): Let (X, t) be a fuzzy topological space which is both R_1^5 and FT_0(ii). Let x, y ∈ X and x ≠ y. Since (X, t) is FT_0(ii), there exists w ∈ t such that either w(x) > 0 and w(y) = 0 or w(y) > 0 and w(x) = 0. It is possible to find α ∈ $I_{0,1}$ such that either w(x) > α and w(y) = 0 or w(y) > α and w(x) = 0. Since (X, t) is also R_1^5, there exist u, v ∈ t such that u(x) > β, v(y) > δ ∀ β, δ ∈ $I_{0,1}$. Therefore,(X, t) is FT_2(ii).

Again, let (X, t) be a fuzzy topological space which is both R_1^{11} and FT_0(ii). Let x, y ∈ X and x ≠ y. Since (X, t) is FT_0(ii), there exists w ∈ t such that either w(x) > 0 and w(y) = 0 or w(y) > 0 and w(x) = 0. It is possible to find α ∈ $I_{0,1}$ such that either w(x) = α and w(y) = 0 or w(y) > α and w(x) = 0. Since (X, t) is also R_1^{11}, there exist u, v ∈ t such that u(x) > β, v(y) > δ ∀ β, δ ∈ $I_{0,1}$. Therefore,(X, t) is FT_2(ii).

Proof (d): Let (X, t) be a fuzzy topological space which is both R_1^1 and FT_0(ii). Let x, y $\in$X, x $\neq$ y. By FT_0(ii), there exists w $\in$ t such that either w(x)>0, w(y) = 0. Clearly either w(x) > α, w(y) = 0 or w(x) = 0 and w(y) >α, for some $\alpha\in I_{0,1}$. By R_1^3, there exist u, v $\in$ t such that u(x) = 1 = v(y) and u $\wedge$ v = 0. Thus (X, t) is FT_2(i).

Again, let (X, t) be a fuzzy topological space which is both R_1^7 and FT_0(ii). Let x, y $\in$X, x $\neq$ y. By FT_0(ii), there exists w $\in$ t such that either w(x)>0, w(y) = 0. Clearly either w(x) = α, w(y) = 0 or w(x) = 0 and w(y) = α, for some $\alpha\in I_{0,1}$. By R_1^7, there exist u, v $\in$ t such that u(x) = 1 = v(y) and u $\wedge$ v = 0. Thus (X, t) is FT_2(i).

Proof (e): Let (X, t) be a fuzzy topological space which is both R_1^2 and FT_0(ii). Let x, y $\in$X, x $\neq$ y. By FT_0(ii), there exists w$\in$t such that either w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0. Definitely, for some $\alpha\in I_{0,1}$, either w(x) > α, w(y) = 0 or w(x) = 0, w(y) > α. By R_1^2, there exist u, v $\in$ t such that u(x) = 1 = v(y) and u $\leq$ 1 – v. Therefore, (X, t) is FT_2(iv).

Again, let (X, t) be a fuzzy topological space which is both R_1^8 and FT_0(ii). Let x, y $\in$X, x $\neq$ y. By FT_0(ii), there exists w$\in$t such that either w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0. Definitely, for some $\alpha\in I_{0,1}$, either w(x) = α, w(y) = 0 or w(x) = 0, w(y) = α. By R_1^8, there exist u, v $\in$ t such that u(x) = 1 = v(y) and u $\leq$ 1 – v. Therefore, (X, t) is FT_2(iv).

Proof (f):

Example-2 [4]: Let X be an infinite set.. For any x, y $\in$ X, x $\neq$ y, let u_{xy} be a fuzzy set in X such that u_{xy} (x) = 1, u_{xy} (y) = 0 and u_{xy} (z) = 0.5 where z$\in$X such that x $\neq$ z, z $\neq$ y. Now consider the fuzzy topology t on X generated by {u_{xy}: x, y $\in$X, x $\neq$ y } $\cup$ {constants}. Then the fts, (X, t) is FT_0(i). However, it is clear that, $\overline{1_x} \leq u_{xy}$, $\overline{1_y} \leq u_{yx}$

and $u_{xy} \leq 1 - u_{yx}$. Thus (X, t) is R_1^2 and R_1^8. But (X, t) is not FT_2(iii) as the intersection of two non-trivial fuzzy sets cannot be zero.

Proof (g):

Example-3: Let, X = {x, y}, u and v are two fuzzy sets on X defined as follows: u(x) = 1, u(y) = 0, v(x) = 0, v(y) = 0.5. Let t be the fuzzy topology on X generated by the sets {u, v} ∪ {constants}. Then it is clear that, (X, t) is FT_0(i), R_1^6 and R_1^{12}. However, (X, t) is not FT_2(ii) for, if we take $\alpha = \beta = 0.7$, we see that, there exist no u, v ∈ t such that $u(x) > \alpha$, $v(y) > \beta$ and $u \wedge v = 0$. Again (X, t) is not FT_2(iv) as there exist no u, v ∈ t such that u(x) = 1 = v(y).

Proof (h):

Example-4: Let X = I and t be the fuzzy topology on X generated by $B = B_1 \cup B_2 \cup B_3 \cup B_4$. Where, $B_1 = \{1_x : x \in I_{0,1}\}$,

$B_2 = \{u_m : m \in \mathbf{N}\}$,

Where u_m is a fuzzy set in X defined by $u_m = 1_{\left[0, \frac{1}{m+1}\right]}$,

$B_3 = \{v_{n,F} : n \in \mathbf{N}$ and F is a finite crisp subset of X },

Where $v_{n,F}$ is a fuzzy set in X defined by $v_{n,F} = \left(\frac{n}{n+1}\right) 1_{\left[\frac{1}{n+1}, 1\right] - F}$

And B_4 = {constants}.

Now, (X, t) is R_1^5, R_1^{11} and FT_0(i) but not FT_2(v). (c.f. [On certain separation and connectedness concepts in fuzzy topology-By D.M. Ali])

5.2. Theorem: For the fuzzy topological spaces the following are true.

(a) $FT_2(iii) \Rightarrow R_1^6$

(b) $FT_2(iii) \nRightarrow R_1^5$

(c) $FT_2(ii) \Rightarrow R_1^5$

(d) $FT_2(ii) \not\Rightarrow R_1^2$

(e) $FT_2(i) \Rightarrow R_1^1$

(f) $FT_2(iv) \not\Rightarrow R_1^6$

(g) $FT_2(iv) \Rightarrow R_1^4$

Proof (a): Trivial.

Proof (b):

Example-5: Consider a fuzzy topological space (X, t) where X ={x , y}, u(x) = 0.5, u(y) = 0, v(x) = 0, v(y) = 0.5, w(x) = 0.6, w(y) = 0 and t = <{u, v, w}∪{constants}>. It can be checked that (X, t) is $FT_2(iii)$ but not R_1^5.

Proof (c): Trivial.

Proof (d): In example-4, (X ,t) is $FT_2(ii)$ but it is not R_1^2.

Proof (e): Trivial.

Proof (f): In example-2, (X ,t) is $FT_2(iv)$ but it is not R_1^6.

CHAPTER-4

Some remarks on fuzzy R_1 topological spaces.

1. Introduction: In this chapter we recall eighteen axioms of fuzzy R_1-type axioms from [3]. We study their interrelations, goodness and initiality. The relations between these axioms with the axioms studied in chapter three are also discussed. It is shown that, the reciprocal pre-image and homeomorphic image of a fuzzy R_1-topological space is also a fuzzy R_1-topological space.

1.1 FR_1 Properties [3]:

In this section we recall some definitions of fuzzy R_1-topological spaces from [3].

Definitions:

$\underline{FR_1\text{(i)}}$: An fts (X, t) is called FR_1(i) iff for all distinct x, y ∈ X, if there exists $w \in t$ with $w(x) \neq w(y)$, then there exist u, v ∈ t with $\overline{1_x} \leq u, \overline{1_y} \leq v$ and $u \wedge v = 0$.

$\underline{FR_1\text{(ii)}}$: An fts (X, t) is called FR_1(ii) iff for all distinct x, y ∈ X, if there exists $w \in t$ with $w(x) \neq w(y)$, then there exist u, v ∈ t with $\overline{1_x} \leq u, \overline{1_y} \leq v$ and $u \leq 1 - v$.

$\underline{FR_1\text{(iii)}}$: An fts (X, t) is called FR_1(iii) iff for all distinct x, y ∈ X, if there exists $w \in t$ with $w(x) \neq w(y)$, then there exist u, v ∈ t with $u(x) = 1 = v(y)$ and $u \wedge v = 0$

$\underline{FR_1\text{(iv)}}$: An fts (X, t) is called FR_1(iv) iff for all distinct x, y ∈ X, if there exists $w \in t$ with $w(x) \neq w(y)$, then there exist u, v ∈ t with $u(x) = 1 = v(y)$ and $u \leq 1 - v$

$\underline{FR_1(v)}$: An fts (X, t) is called $FR_1(v)$ iff for all distinct x, y $\in$ X, if there exists w$\in$t with $w(x) \neq w(y)$, then for all α, $\beta \in I_{0,1}$, there exist u, v $\in$ t with $u(x) > \alpha$, $v(y) > \beta$ and $u \wedge v = 0$.

$\underline{FR_1(vi)}$: An fts (X, t) is called $FR_1(vi)$ iff for all distinct x, y $\in$ X, if there exists w$\in$t with $w(x) \neq w(y)$, then there exist u, v $\in$ t with $u(x) > 0$, $v(y) > 0$ and $u \wedge v = 0$,

$\underline{FR_1(vii)}$: An fts (X, t) is called $FR_1(vii)$ iff for all distinct x, y $\in$ X, if there exists w$\in$t with w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0, then there exist u, v $\in$ t with $\overline{1_x} \leq u, \overline{1_y} \leq v$ and $u \wedge v = 0$.

$\underline{FR_1(viii)}$: An fts (X, t) is called $FR_1(viii)$ iff for all distinct x, y $\in$ X, if there exists w$\in$t with w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0, then there exist u, v $\in$ t with $\overline{1_x} \leq u, \overline{1_y} \leq v$ and $u \leq 1 - v$.

$\underline{FR_1(ix)}$: An fts (X, t) is called $FR_1(ix)$ iff for all distinct x, y $\in$ X, if there exists w$\in$t with w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0, then there exist u, v $\in$ t with $u(x) = 1 = v(y)$ and $u \wedge v = 0$

$\underline{FR_1(x)}$: An fts (X, t) is called $FR_1(x)$ iff for all distinct x, y $\in$ X, if there exists w$\in$t with w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0, then there exist u, v $\in$ t with $u(x) = 1 = v(y)$ and $u \leq 1 - v$

$\underline{FR_1(xi)}$: An fts (X, t) is called $FR_1(xi)$ iff for all distinct x, y $\in$ X, if there exists w$\in$t with w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0, then for all α, $\beta \in I_{0,1}$, there exist u, v $\in$ t with $u(x) > \alpha$, $v(y) > \beta$ and $u \wedge v = 0$.

<u>FR_1 (xii):</u> An fts (X, t) is called FR_1(xii) iff for all distinct x, y ∈ X, if there exists w∈t with w(x) > 0, w(y) = 0 or w(x) = 0, w(y) > 0, then there exist u, v ∈ t with $u(x) > 0, v(y) > 0$ and $u \wedge v = 0$,

<u>FR_1 (xiii):</u> An fts (X, t) is called FR_1(xiii) iff for all distinct x, y ∈ X, if there exists w∈t with w(x) = 1, w(y) = 0 or w(x) = 0, w(y) = 1, then there exist u, v ∈ t with $\overline{1_x} \le u, \overline{1_y} \le v$ and $u \wedge v = 0$.

<u>FR_1 (xiv):</u> An fts (X, t) is called FR_1(xiv) iff for all distinct x, y ∈ X, if there exists w∈t with w(x) = 1, w(y) = 0 or w(x) = 0, w(y) = 1, then there exist u, v ∈ t with $\overline{1_x} \le u, \overline{1_y} \le v$ and $u \le 1 - v$.

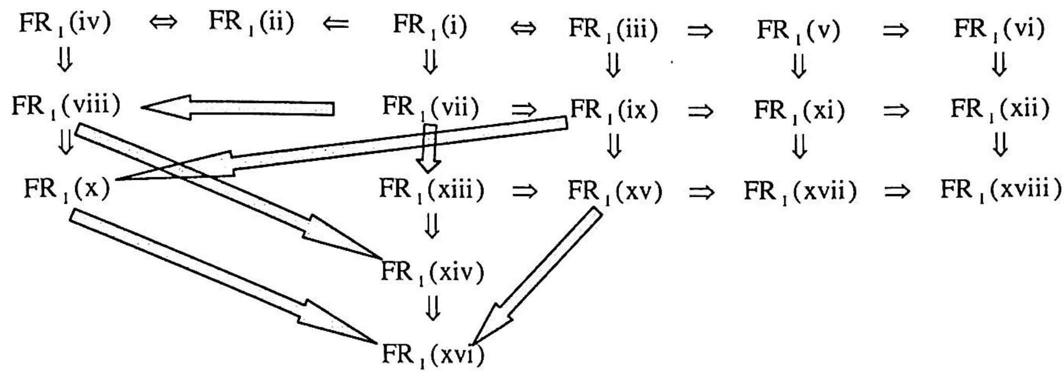
<u>FR_1 (xv):</u> An fts (X, t) is called FR_1(xv) iff for all distinct x, y ∈ X, if there exists w∈t with w(x) = 1, w(y) = 0 or w(x) = 0, w(y) = 1, then there exist u, v ∈ t with $u(x) = 1 = v(y)$ and $u \wedge v = 0$

<u>FR_1 (xvi):</u> An fts (X, t) is called FR_1(xvi) iff for all distinct x, y ∈ X, if there exists w∈t with w(x) = 1, w(y) = 0 or w(x) = 0, w(y) = 1, then there exist u, v ∈ t with $u(x) = 1 = v(y)$ and $u \le 1 - v$

<u>FR_1 (xvii):</u> An fts (X, t) is called FR_1(xvii) iff for all distinct x, y ∈ X, if there exists w∈t with w(x) = 1, w(y) = 0 or w(x) = 0, w(y) = 1, then for all α, β∈$I_{0,1}$, there exist u, v ∈ t with $u(x) > \alpha$, $v(y) > \beta$ and $u \wedge v = 0$.

<u>FR_1 (xviii):</u> An fts (X, t) is called FR_1(xviii) iff for all distinct x, y ∈ X, if there exists w∈t with w(x) = 1, w(y) = 0 or w(x) = 0, w(y) = 1, then there exist u, v ∈ t with $u(x) > 0, v(y) > 0$ and $u \wedge v = 0$.

1.1. Theorem [3]: The following implications hold between the FR_1-propertits of an fts.



1.2. Theorem [3]:

All FR_1-properties mentioned in the section 1.1 are good extension of their topological counter parts, i.e. A topological space (X, T) is R_1 if and only if $(X, \omega(T))$ is $FR_1(p)$ (p = i, ii, ..., xviii).

1.3. Theorem [3]: $FR_1(p)$ (p = i, ii, ..., xii) are initial, and therefore productive and hereditary.

2. Relations between $FR_1(p)$ fuzzy topological space and R_1^k-fuzzy topological space:

In this section we study the relations between the FR(p) fuzzy topological space mentioned in the section 1.1 and the R_1^k- fuzzy topological space discussed in the chapter three.

2.1. Theorem: The following implications hold between $FR_1(p)$ and R_1^q properties ($p = i, ii, ..., xviii$ and $q = 1, 2, ..., 12$):

(1). $FR_1(vii) \Rightarrow R_1^1 \Rightarrow FR_1(xiii)$

(2). $FR_1(viii) \Rightarrow R_1^2 \Rightarrow FR_1(xiv)$

(3). $FR_1(ix) \Rightarrow R_1^3 \Rightarrow FR_1(xv)$

(4). $FR_1(x) \Rightarrow R_1^4 \Rightarrow FR_1(xvi)$

(5). $FR_1(xi) \Rightarrow R_1^5 \Rightarrow FR_1(xvii)$

(6). $FR_1(xii) \Rightarrow R_1^6 \Rightarrow FR_1(xviii)$

(7). $FR_1(vii) \Rightarrow R_1^7 \Rightarrow FR_1(xiii)$

(8). $FR_1(viii) \Rightarrow R_1^8 \Rightarrow FR_1(xiv)$

(9). $FR_1(ix) \Rightarrow R_1^9 \Rightarrow FR_1(v)$

(10). $FR_1(x) \Rightarrow R_1^{10} \Rightarrow FR_1(xvi)$

(11). $FR_1(xi) \Rightarrow R_1^{11} \Rightarrow FR_1(xvii)$

(12). $FR_1(xii) \Rightarrow R_1^{12} \Rightarrow FR_1(xviii)$

Proof:

$FR_1(vii) \Rightarrow R_1^1$:

Let (X, t) be an fts which has the property $FR_1(vii)$. Suppose that, $x, y \in X$, $\alpha \in I_{0,1}$ and $w \in t$ such that $w(x) > \alpha$ and $w(y) = 0$. Then clearly $w(x) > 0$ and $w(y) = 0$. Therefore, by $FR_1(vii)$ property of (X, t), there exist $u, v \in t$ such that $\overline{1_x} \leq u, \overline{1_y} \leq u$ and $u \wedge v = 0$. Therefore, (x, t) has the property R_1^1.

$R_1^1 \Rightarrow FR_1(xiii)$:

Again let (X, t) has the property R_1^1. Let $x, y \in X$, and $w \in t$ such that $w(x) = 1$ and $w(y) = 0$. Then clearly $w(x) > \alpha$ and $w(y) = 0$, $\alpha \in I_{0,1}$. Therefore, by the R_1^1-property of (X, t), there exist u, v ∈ t such that $\overline{1_x} \leq u, \overline{1_y} \leq u$ and $u \wedge v = 0$ and therefore (X, t) is FR_1(xiii).

<u>$FR_1(vii) \Rightarrow R_1^7$:</u>

Let (X, t) be an fts which has the property FR_1(vii). Suppose that, x, y ∈ X, $\alpha \in I_{0,1}$ and w∈t such that w(x) = α and w(y) = 0. Then clearly w(x) > 0 and w(y) = 0. Therefore, by FR_1(vii) property of (X, t), there exist u, v ∈ t such that $\overline{1_x} \leq u, \overline{1_y} \leq u$ and $u \wedge v = 0$. Therefore, (x, t) has the property R_1^7.

<u>$R_1^7 \Rightarrow FR_1(xiii)$:</u>

Again let (X, t) has the property R_1^7. Let x, y ∈ X and w∈t such that w(x) = 1 and w(y) = 0. Let $\alpha \in I_{0,1}$. Define $w' = w \wedge \alpha$. Clearly $w' \in t$ such that $w'(x) = \alpha$ and $w'(y) = 0$. Therefore, by the R_1^7-property of (X, t), there exist u, v ∈ t such that $\overline{1_x} \leq u, \overline{1_y} \leq u$ and $u \wedge v = 0$ and therefore (X, t) is FR_1(xiii).

All other proofs are similar.

<u>Counter examples:</u>

Example-1: X = {x, y} and t = <{u, v}∪{constants}>, where u(x) = 0.6, u(y) = 0. For α = 0.6, (X, t) vacuously satisfies the R_1^1-property. Now, u(x) > 0 and u(y) = 0. But there exist no u, v ∈ t such that u(x) > 0, v(y) >0 and $u \wedge v = 0$. Therefore, (X, t) is not FR_1(xii).

Therefore, $R_1^1 \not\Rightarrow FR_1(xii)$. This example also shows that, $R_1^1 \not\Rightarrow FR_1(x)$.

Example-2: Consider a fuzzy topological space, (X, t) where X = {x, y}, t = < {w} ∪ {constants}>; w is defined as w(x) = 0.5 and w(y) = 0. For α = 0.4, (X, t) vacuously satisfies the property R_1^7. But, (X, t) is neither FR_1(x) nor FR_1(xii).

Example-3: Consider a fuzzy topological space, (X, t) where X = {x, y}, t = < {w} ∪ {constants}>; w is defined as w(x) = 0.5 and w(y) = 0. Vacuously, (X, t) satisfies the property, FR_1(xiii). We see that:

- (X, t) doesn't satisfy the property, R_1^4. For, take α = 0.4. Then w(x) > α and w(y) = 0, but there don't exist u, v ∈ t such that u(x) = 1 = v(y) and u ∧ v = 0.

- (X, t) doesn't satisfy the property, R_1^6. For, take α = 0.4. Then w(x) > α and w(y) = 0, but there don't exist u, v ∈ t such that u(x) > 0, v(y) > 0 and u ∧ v = 0.

- (X, t) doesn't satisfy the property, R_1^{10}. For, take α = 0.5. Then w(x) = α and w(y) = 0, but there don't exist u, v ∈ t such that u(x) = 1 = v(y) and u ∧ v = 0.

- (X, t) doesn't satisfy the property, R_1^{12}. For, take α = 0.5. Then w(x) = α and w(y) = 0, but there don't exist u, v ∈ t such that u(x) > 0, v(y) > 0 and u ∧ v = 0.

3. Reciprocal pre-image and homeomorphic image of fuzzy R_1 topological spaces

3.1 Definition: Suppose X be a set and (X′, t′) be an fts. Consider a function, f: X → (X′, t′). Let t = {f^{-1}(u): u∈t′}. Then t is a fuzzy topology on X. We call t, the reciprocal topology on X.

3.2. Theorem. If X is a set, (X', t') a fuzzy topological space having the property $R_1^k (1 \le k \le 12)$, then the reciprocal topology t on X for f: $X \to (X', t')$ also has R_1^k.

Proof:

(1) Let (X', t') be an R_1^1 fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y ∈ X, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) > \alpha$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) > \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, v ∈ t′ such that $\overline{1_{x'}} \le u, \overline{1_{y'}} \le v$ and $u \wedge v = 0$. We have, $f(\overline{1_z}) \le \overline{1_{f(z)}}$ for every $z \in X$ since f is continuous.

Thus, $f(\overline{1_x}) \le \overline{1_{f(x)}} = \overline{1_{x'}} \le u$ and $f(\overline{1_y}) \le \overline{1_{f(y)}} = \overline{1_{y'}} \le v$. Thus, $\overline{1_x} \le f^{-1}(u)$, $\overline{1_y} \le f^{-1}(v)$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an R_1^1 fts.

(2) Let (X', t') be an R_1^2 fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y ∈ X, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) > \alpha$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) > \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, v ∈ t′ such that $\overline{1_{x'}} \le u, \overline{1_{y'}} \le v$ and $u \le 1 - v$. We have, $f(\overline{1_z}) \le \overline{1_{f(z)}}$ for every $z \in X$ since f is continuous.

Thus, $f(\overline{1_x}) \le \overline{1_{f(x)}} = \overline{1_{x'}} \le u$ and $f(\overline{1_y}) \le \overline{1_{f(y)}} = \overline{1_{y'}} \le v$. Thus, $\overline{1_x} \le f^{-1}(u)$, $\overline{1_y} \le f^{-1}(v)$. Moreover, $f^{-1}(u) \le 1 - f^{-1}(v)$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an R_1^2 fts.

(3) Let (X', t') be an R_1^3 fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y ∈ X, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) > \alpha$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) > \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, v ∈ t′ such that $u(x') = v(y') = 1$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) =$

$u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an R_1^3 fts.

(4)Let (X', t') be an R_1^4 fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let x, $y \in X$, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) > \alpha$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) > \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, $v \in t'$ such that $u(x') = v(y') = 1$ and $u \leq 1 - v$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \leq 1 - f^{-1}(v)$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an R_1^4 fts.

(5) Let (X', t') be an R_1^5 fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let x, $y \in X$, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) > \alpha$ and $w(y) = 0$. Choose $\beta, \delta \in I_{0,1}$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) > \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, $v \in t'$ such that $u(x') > \alpha$, $v(y') > \beta$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > \alpha$. Similarly, we can show that $f^{-1}v(y) > \beta$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an R_1^5 fts.

(6) Let (X', t') be an R_1^6 fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let x, $y \in X$, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) > \alpha$ and $w(y) = 0$. Choose $\beta, \delta \in I_{0,1}$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) > \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, $v \in t'$ such that $u(x') > 0$, $v(y') > 0$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > 0$. Similarly, we can show that $f^{-1}v(y) > 0$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an R_1^6 fts.

(7) Let (X', t') be an R_1^7 fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y ∈ X, w∈t, α∈$I_{0,1}$ such that w(x) = α and w(y) = 0. Let f(x) = x′ and f(y) = y′. As w∈t, there exists w′∈t′ such that w = $f^{-1}(w')$.

Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, v ∈ t′ such that $\overline{1_{x'}} \le u, \overline{1_{y'}} \le v$ and $u \wedge v = 0$. We have, $f(\overline{1_z}) \le \overline{1_{f(z)}}$ for every z∈X since f is continuous. Thus, $f(\overline{1_x}) \le \overline{1_{f(x)}} = \overline{1_{x'}} \le u$ and $f(\overline{1_y}) \le \overline{1_{f(y)}} = \overline{1_{y'}} \le v$. Thus, $\overline{1_x} \le f^{-1}(u)$, $\overline{1_y} \le f^{-1}(v)$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an R_1^7 fts.

(8) Let (X', t') be an R_1^8 fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y ∈ X, w∈t, α∈$I_{0,1}$ such that w(x) = α and w(y) = 0. Let f(x) = x′ and f(y) = y′. As w∈t, there exists w′∈t′ such that w = $f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, v ∈ t′ such that $\overline{1_{x'}} \le u, \overline{1_{y'}} \le v$ and $u \le 1 - v$. We have, $f(\overline{1_z}) \le \overline{1_{f(z)}}$ for every z∈X since f is continuous.

Thus, $f(\overline{1_x}) \le \overline{1_{f(x)}} = \overline{1_{x'}} \le u$ and $f(\overline{1_y}) \le \overline{1_{f(y)}} = \overline{1_{y'}} \le v$. Thus, $\overline{1_x} \le f^{-1}(u)$, $\overline{1_y} \le f^{-1}(v)$. Moreover, $f^{-1}(u) \le 1 - f^{-1}(v)$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an R_1^8 fts.

(9) Let (X', t') be an R_1^9 fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y ∈ X, w∈t, α∈$I_{0,1}$ such that w(x) = α and w(y) = 0. Let f(x) = x′ and f(y) = y′. As w∈t, there exists w′∈t′ such that w = $f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists u, v ∈ t′ such that $u(x') = v(y') = 1$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an R_1^9 fts.

(10) Consider an R_1^{10} fts, (X', t') and let t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) = \alpha$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists $u, v \in t'$ such that $u(x') = v(y') = 1$ and $u \leq 1 - v$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \leq 1 - f^{-1}(v)$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an R_1^{10} fts.

(11) Consider an R_1^{11} fts, (X', t') and let t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) = \alpha$ and $w(y) = 0$. Choose $\beta, \delta \in I_{0,1}$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists $u, v \in t'$ such that $u(x') > \alpha$, $v(y') > \beta$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > \alpha$. Similarly, we can show that $f^{-1}v(y) > \beta$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an R_1^{11} fts.

(12) Let (X', t') be an R_1^{12} fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$, $\alpha \in I_{0,1}$ such that $w(x) = \alpha$ and $w(y) = 0$. Choose $\beta, \delta \in I_{0,1}$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = \alpha$. Similarly, $w'(y') = 0$. Therefore, there exists $u, v \in t'$ such that $u(x') > 0$, $v(y') > 0$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > 0$. Similarly, we can show that $f^{-1}v(y) > 0$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an R_1^{12} fts.

3.3. Theorem. If X is a set, (X', t') a fuzzy topological space having the property $FR_1(k)$ $(i \leq k \leq xviii)$, then the reciprocal topology t on X for $f: X \to (X', t')$ also has $FR_1(k)$.

Proof:

1. Let (X', t') be an FR_1(i)-fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y $\in$ X, w$\in$t such that $w(x) \neq w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As w$\in$t, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') \neq w'(y')$, and so, there exists u, v $\in$ t' such that $\overline{1_{x'}} \leq u, \overline{1_{y'}} \leq v$ and $u \wedge v = 0$. We have, $f(\overline{1_z}) \leq \overline{1_{f(z)}}$, for every z$\in$X, since f is continuous. Now, $f(\overline{1_x}) \leq \overline{1_{f(x)}} = \overline{1_{x'}} \leq u$ and $f(\overline{1_y}) \leq \overline{1_{f(y)}} = 1_{y'} \leq v$. Thus, $\overline{1_x} \leq f^{-1}(u)$, $\overline{1_y} \leq f^{-1}(v)$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an FR_1(i)-fts.

2. Let (X', t') be an FR_1(ii)-fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y $\in$ X, w$\in$t such that $w(x) \neq w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As w$\in$t, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') \neq w'(y')$, and so, there exists u, v $\in$ t' such that $\overline{1_{x'}} \leq u, \overline{1_{y'}} \leq v$ and $u \leq 1 - v$. We have, $f(\overline{1_z}) \leq \overline{1_{f(z)}}$ for every z$\in$X since f is continuous.

Thus, $f(\overline{1_x}) \leq \overline{1_{f(x)}} = \overline{1_{x'}} \leq u$ and $f(\overline{1_y}) \leq \overline{1_{f(y)}} = 1_{y'} \leq v$. Thus, $\overline{1_x} \leq f^{-1}(u)$, $\overline{1_y} \leq f^{-1}(v)$. Moreover, $f^{-1}(u) \leq 1 - f^{-1}(v)$. Clearly, $f^{-1}(u)$, $f^{-1}(v) \in t$. Hence (X, t) is an FR_1(ii)-fts.

3. Let (X', t') be an FR_1(iii)-fts, t be the reciprocal topology on X for f: $X \to (X', t')$. Let x, y $\in$ X, w$\in$t such that $w(x) \neq w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As w$\in$t, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') \neq w'(y')$, and so, there exists u, v $\in$ t' such that $u(x') = v(y') = 1$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly,

$f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(iii)- fts.

4. Let (X', t') be an FR_1(iv)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let x, y ∈ X, $w \in t$ such that $w(x) \neq w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') \neq w'(y')$, and so, there exists u, v ∈ t' such that $u(x') = v(y') = 1$ and $u \leq 1 - v$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \leq 1 - f^{-1}(v)$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(iv)- fts.

5. Let (X', t') be an FR_1(v)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let x, y ∈ X, $w \in t$ such that $w(x) \neq w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') \neq w'(y')$, and so, there exists u, v ∈ t' such that $u(x') > \alpha$, $v(y') > \beta$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > \alpha$. Similarly, we can show that $f^{-1}v(y) > \beta$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(v)-fts.

6. Let (X', t') be an FR_1(vi)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let x, y ∈ X, $w \in t$ such that $w(x) \neq w(y)$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') \neq w'(y')$, and so, there exists u, v ∈ t' such that $u(x') > 0$, $v(y') > 0$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > 0$. Similarly, we can show that $f^{-1}v(y) > 0$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(vi)-fts.

7. Let (X', t') be an FR_1(vii)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') > 0$ and $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $\overline{1_{x'}} \leq u, \overline{1_{y'}} \leq v$ and $u \wedge v = 0$. We have, $f(\overline{1_z}) \leq \overline{1_{f(z)}}$, for every $z \in X$, since f is continuous. Now, $f(\overline{1_x}) \leq \overline{1_{f(x)}} = \overline{1_{x'}} \leq u$ and $f(\overline{1_y}) \leq \overline{1_{f(y)}} = \overline{1_{y'}} \leq v$. Thus, $\overline{1_x} \leq f^{-1}(u)$, $\overline{1_y} \leq f^{-1}(v)$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(vii)-fts.

8. Let (X', t') be an FR_1(viii)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') > 0$ and $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $\overline{1_{x'}} \leq u, \overline{1_{y'}} \leq v$ and $u \leq 1 - v$. We have, $f(\overline{1_z}) \leq \overline{1_{f(z)}}$ for every $z \in X$ since f is continuous. Thus, $f(\overline{1_x}) \leq \overline{1_{f(x)}} = \overline{1_{x'}} \leq u$ and $f(\overline{1_y}) \leq \overline{1_{f(y)}} = \overline{1_{y'}} \leq v$. Thus, $\overline{1_x} \leq f^{-1}(u)$, $\overline{1_y} \leq f^{-1}(v)$. Moreover, $f^{-1}(u) \leq 1 - f^{-1}(v)$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(viii)-fts.

9. Let (X', t') be an FR_1(ix)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') > 0$ and $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $u(x') = v(y') = 1$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(xi)- fts.

10. Let (X', t') be an $FR_1(x)$-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') > 0$ and $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $u(x') = v(y') = 1$ and $u \le 1 - v$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \le 1 - f^{-1}(v)$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an $FR_1(x)$- fts.

11. Let (X', t') be an $FR_1(xi)$-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') > 0$ and $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $u(x') > \alpha$, $v(y') > \beta$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > \alpha$. Similarly, we can show that $f^{-1}v(y) > \beta$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an $FR_1(xi)$-fts.

12. Let (X', t') be an $FR_1(xii)$-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x)$. Similarly, $w'(y') = w(y)$. Therefore $w'(x') > 0$ and $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $u(x') > 0$, $v(y') > 0$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > 0$. Similarly, we can show that $f^{-1}v(y) > 0$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an $FR_1(xii)$-fts.

13. Let (X', t') be an $FR_1(xiii)$-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As

$w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = 1$. Similarly, $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $\overline{1_{x'}} \le u, \overline{1_{y'}} \le v$ and $u \wedge v = 0$. We have, $f(\overline{1_z}) \le \overline{1_{f(z)}}$, for every $z \in X$, since f is continuous. Now, $f(\overline{1_x}) \le \overline{1_{f(x)}} = \overline{1_{x'}} \le u$ and $f(\overline{1_y}) \le \overline{1_{f(y)}} = 1_{y'} \le v$. Thus, $\overline{1_x} \le f^{-1}(u)$, $\overline{1_y} \le f^{-1}(v)$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(xiii)-fts.

14. Let (X', t') be an FR_1(xiv)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = 1$. Similarly, $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $\overline{1_{x'}} \le u, \overline{1_{y'}} \le v$ and $u \le 1 - v$. We have, $f(\overline{1_z}) \le \overline{1_{f(z)}}$ for every $z \in X$ since f is continuous. Thus, $f(\overline{1_x}) \le \overline{1_{f(x)}} = \overline{1_{x'}} \le u$ and $f(\overline{1_y}) \le \overline{1_{f(y)}} = 1_{y'} \le v$. Thus, $\overline{1_x} \le f^{-1}(u)$, $\overline{1_y} \le f^{-1}(v)$. Moreover, $f^{-1}(u) \le 1 - f^{-1}(v)$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(xiv)-fts.

15. Let (X', t') be an FR_1(xv)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = 1$. Similarly, $w'(y') = 0$, and so, there exists $u, v \in t'$ such that $u(x') = v(y') = 1$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(xv)-fts.

16. Let (X', t') be an FR_1(xvi)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let $x, y \in X$, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = 1$. Similarly, $w'(y') = 0$, and so, there exists u, v $\in t'$ such that $u(x') = v(y') = 1$ and $u \le 1 - v$. Now, $f^{-1}u(x) = uf(x) = u(x') = 1$. Similarly, $f^{-1}v(y) = 1$. Moreover, $f^{-1}(u) \le 1 - f^{-1}(v)$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(xvi)-fts.

17. Let (X', t') be an FR_1(xvii)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let x, y $\in$ X, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = 1$. Similarly, $w'(y') = 0$, and so, there exists u, v $\in t'$ such that $u(x') > \alpha$, $v(y') > \beta$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > \alpha$. Similarly, we can show that $f^{-1}v(y) > \beta$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(xvii)-fts.

18. Let (X', t') be an FR_1(xviii)-fts, t be the reciprocal topology on X for $f: X \to (X', t')$. Let x, y $\in$ X, $w \in t$ such that $w(x) > 0$ and $w(y) = 0$. Let $f(x) = x'$ and $f(y) = y'$. As $w \in t$, there exists $w' \in t'$ such that $w = f^{-1}(w')$. Now, $w'(x') = w'(f(x)) = (f^{-1}(w'))(x) = w(x) = 1$. Similarly, $w'(y') = 0$, and so, there exists u, v $\in t'$ such that $u(x') > 0$, $v(y') > 0$ and $u \wedge v = 0$. Now, $f^{-1}u(x) = uf(x) = u(x') > 0$. Similarly, we can show that $f^{-1}v(y) > 0$. Moreover, $f^{-1}(u) \wedge f^{-1}(v) = 0$. Clearly, $f^{-1}(u), f^{-1}(v) \in t$. Hence (X, t) is an FR_1(xviii)-fts.

3.4. Theorem: Every homeomorphic image of R_1^k-fts is also an R_1^k- fts ($1 \le k \le 12$).

<u>**Proof:**</u>

1. Let (X, t) be an R_1^1-fts and let $f: (X, t) \to (Y, s)$ be a homeomorphism between fts. Suppose $y_1, y_2 \in Y$, $\alpha \in I_{0,1}$ and $w_2 \in s$ such that $w_2(y_1) = \alpha$ and $w_2(y_2) = 0$.

Now $f^{-1}(y_1), f^{-1}(y_2) \in X$ and $f^{-1}(w_2) \in t$ such that $(f^{-1}(w_2))(f^{-1}(y_1)) = \alpha$ and $(f^{-1}(w_2))(f^{-1}(y_2)) = 0$. Since (X, t) is R_1^1, there exist u, v ∈ t such that $\overline{1_{f^{-1}(y_1)}} \le u$, $\overline{1_{f^{-1}(y_2)}} \le v$ and $u \wedge v = 0$. Since f is a homeomorphism, $\overline{1_{f^{-1}(y)}} = f^{-1}(\overline{1_y})\ \forall\ y \in Y$. Now f(u), f(v) ∈ S such that $\overline{1_{y_1}} \le f(u)$, $\overline{1_{y_2}} \le f(v)$ and $f(u) \wedge f(v) = 0$. Therefore, (Y, s) is R_1^1.

All other proofs are similar.

3.5. Theorem: Every homeomorphic image of FR(k)-fts is also an FR(k)-fts (i≤ k≤xviii).

The proofs are similar.

CHAPTER-5

Relations between fuzzy R_0, R_1 and regularity concepts

Introduction: In this chapter, a complete answer is given with regard to all possible $(R_1 \Rightarrow R_0)$-type implications for fuzzy topological spaces, where the R_0 and R_1-axioms, considered in the previous chapters are taken into account. Besides, we recall five definitions of fuzzy regular axioms from [1, 4], and it is shown that, though the regularity axiom implies R_1 axiom in 'general topological spaces', this in not true in 'fuzzy topological spaces', in general.

1. Relations between fuzzy R_0 and R_1-axioms

1.1. Theorem: The following relations hold between the fuzzy R_1-axioms and fuzzy R_0-axioms

(a) $FR_1(xvi) \Rightarrow R_0^1$, and so $FR_1(k) \Rightarrow R_0^1$, where $k \in \{i-iv, vii-x, xiii-xvi\}$.

Proof: Let (X, t) be an $FR_1(xvi)$-fts. Let x, y $\in$ X, x $\neq$ y and $\lambda \in$ t such that $\lambda(x) = 1$, $\lambda(y) = 0$. Since (X, t) is an $FR_1(xvi)$-fts, there exist u, v $\in$ t such that u(x) = 1 = v(y) and $u \leq 1 - v$. Clearly, v(x) = 0. Hence (X, t) is R_0^1.

(b) $FR_1(xiii) \not\Rightarrow R_0^5$, and so $FR_1(k) \not\Rightarrow R_0^m$, where $k \in \{xiii, xiv, ..., xviii\}$ and $m \in \{5, 6, ..., 9\}$.

Proof:

Example-1: Consider a fuzzy topological space (X, t), where X ={x, y}, u(x) = 0.5, u(y) = 0 and t = < {u} $\cup$ {constants} >. Clearly, (X, t) is $FR_1(xiii)$ but it is not R_0^5. For $\overline{1_x}(y) = 1$ but $\overline{1_y}(x) < 1$.

(c) $FR_1(v) \Rightarrow R_0^8$, and so $FR_1(k) \Rightarrow R_0^m$ where $k \in \{i, iii, v\}$ and $m \in \{2, 8\}$.

Proof: Let (X, t) be an $FR_1(v)$-fts. Let x, y ∈ X, x ≠ y, α ∈ I_0 such that $\overline{\alpha 1_x}(y) < \alpha$. This implies that there exists m ∈ t^c such that m(x) = α and m(y) < α. Let w = 1 – m ∈ t. Then w(x) ≠ w(y). Since (X, t) is an $FR_1(v)$-fts, there exist u, v ∈ t such that $u(x) > \alpha_1$, $v(y) > \alpha_2$ and $u \wedge v = 0$ for every $\alpha_1, \alpha_2 \in I_{0,1}$. Choose α_1, α_2 such that $\alpha = \alpha_2$ and $\alpha_1 > 1 - \alpha$. Now $\alpha 1_y < v \le 1 - u$. Therefore, $\overline{\alpha 1_y} \le \overline{1 - u} = 1 - u$ and so $\overline{\alpha 1_y}(x) \le 1 - u(x) < 1 - \alpha_1 < \alpha$. Therefore, (X, t) is R_0^8.

(d) $FR_1(vi) \Rightarrow R_0^2$, and so $FR_1(k) \Rightarrow R_0^2$ where $k \in \{i, iii, v, vi\}$.

Proof: Let (X, t) be an $FR_1(vi)$-fts. Let x, y ∈ X, x ≠ y and w ∈ t such that w(x) > w(y). Then, by $FR_1(vi)$, there exist u, v ∈ t such that u(x) > 0, v(y) > 0 and u ∧ v = 0. Clearly, v(y) > v(x). Therefore, (X, t) is R_0^2.

(e) $FR_1(vi) \not\Rightarrow R_0^8$, and so $FR_1(vi) \not\Rightarrow R_0^m$, where $m \in \{8, 9\}$.

Proof:

Example-2: Consider an fts (X, t) where X = {x, y}, t = < $\{u_1, u_2, u_3, u_4\} \cup \{\text{constants}\}$>, $u_1(x) = u_1(y) = u_2(x) = 0.6$, $u_2(y) = 0.7$, $u_3(x) = u_4(y) = 0$, $u_3(y) = 0.8$ and $u_4(y) = 0.4$. It can be checked that (X, t) is $FR_1(vi)$. Let $m_k = 1 - u_k$, $k = 1, 2, 3, 4$. Now $m_1(x) = 0.4 = m_2(x)$, $m_3(x) = 1, m_4(x) = 0.6$, $m_1(y) = 0.4, m_2(y) = 0.3$, $m_3(y) = 0.2$ and $m_4(y) = 1$. Take α = 0.4. Then $\overline{\alpha 1_x}(y) = 0.2 < \alpha$. But $\overline{\alpha 1_y}(x) = 0.4 = \alpha$. Therefore, (X, t) is not R_0^8.

(f) $FR_1(vi) \not\Rightarrow R_0^3$, and so $FR_1(vi) \not\Rightarrow R_0^m$, where $m \in \{3, 4\}$.

Proof:

Example-3: Consider a fuzzy topological space (X, t) where X = {x, y}, u(x) = 0.6, u(y) = 0 = v(x) and v(y) = 0.4. Clearly, (X, t) is $FR_1(vi)$. Let α = 0.5. Now α < u(x).

It can be checked that $\overline{\alpha 1_x}(y) = \alpha > u(y)$. Therefore, $\overline{\alpha 1_x} \nleq u$. Therefore, (X, t) is not R_0^3.

(g) $FR_1(iv) \Rightarrow R_0^4$, and so $FR_1(k) \Rightarrow R_0^m$ where $k \in \{i - iv\}$ and $m \in \{2,3,4\}$.

Proof: Let (X, t) be an $FR_1(iv)$-fts. Let x∈X, λ∈t and α∈I_1 such that α∈ λ(x).

Suppose $\overline{\alpha 1_x} \nleq \lambda$. This implies that there exist y∈X, x ≠ y such that $\overline{\alpha 1_x}(y) > \lambda(y)$. Thus λ(x) ≠ λ(y). Hence there exist p, q ∈ t such that p(x) = 1 = q(y) and p ≤ 1 – q. Put m = 1 – p and n = 1 – q. Now m, n ∈ t^c such that m(x) = 0 = n(y) and m(y) = 1 = n(x). Therefore, $\overline{\alpha 1_x}(y) \le n(y) = 0$, which is a contradiction. Therefore, $\overline{\alpha 1_x} < \lambda$. Hence (X, t) is R_0^4.

(h) $R_1^1 \nRightarrow R_0^2$, and so $R_1^k \nRightarrow R_0^m$, where $k \in \{1,2,3,4,5,6\}$ and $m \in \{2,3,4,8,9\}$.

Proof:

Example-4: Consider a fuzzy topological space (X, t) where X = {x, y}, u(x) = 0.1, u(y) = 0.2 and t = < {u} ∪ {constants} >. Vacuously, (X, t) is R_1^1, but (X, t) is not R_0^2. For, we have u(x) < u(y), but there exists no λ ∈ t such that λ(y) < λ(x).

(i) $R_1^4 \Rightarrow R_0^1$, and so $R_1^k \Rightarrow R_0^1$, where $k \in \{1,2,3,4\}$.

Proof (i): Let (X, t) be an R_1^4-fts. Let λ∈t, x, y ∈ X, x ≠ y such that λ(x) = 1 and λ(y) = 0. Consider, α, β ∈ $I_{0,1}$ such that α < β. Let w = β ∧ λ ∈ t. Now, w(x) > α and w(y) = 0. Since (X, t) is an R_1^4-fts, there exists u, v ∈ t such that u(x) = 1 = v(y) and u ≤ 1 – v. Clearly v(x) = 0. Thus (X, t) is R_0^1.

(j) $R_1^6 \Rightarrow R_0^5$, and so $R_1^k \Rightarrow R_0^5$, where $k \in \{1,3,5,6\}$.

Proof (j): Let (X, t) be an R_1^6-fts. Let u∈ t^c such that u (y) < 1 = u(x). Take w = 1 – u ∈ t. Then w(x) = 0 and w(y) = 1. Therefore, w(y) > α for every α ∈ $I_{0,1}$. Since (X, t) is an R_1^6-fts, there exists p, q ∈ t such that p(x) > 0, q(y) > 0 and p ∧ q = 0. Now p(y) = 0 = q(x). Take λ = 1 – p. Then λ(x) < 1 = λ(y). Therefore, (X, t) is an R_0^5-fts.

(k) $R_1^5 \not\Rightarrow R_0^1$, and so $R_1^k \not\Rightarrow R_0^m$, where $k \in \{5,6\}$ and $m \in \{1,4,6,7,9\}$.

Proof:

Example-5 [2]: Let X = I and t be the fuzzy topology on X generated by B = $B_1 \cup B_2 \cup B_3 \cup B_4$. Where, $B_1 = \{1_x : x \in I_{0,1}\}$,

$B_2 = \{u_m : m \in \mathbf{N}\}$,

Where u_m is a fuzzy set in X defined by $u_m = 1_{\left[0, \frac{1}{m+1}\right]}$.

$B_3 = \{v_{n,F} : n \in \mathbf{N}$ and F is a finite crisp subset of X $\}$,

Where $v_{n,F}$ is a fuzzy set in X defined by $v_{n,F} = \left(\frac{n}{n+1}\right) 1_{\left[\frac{1}{n+1}, 1\right] - F}$

And $B_4 = \{\text{constants}\}$.

It can be checked that (X, t) is R_1^5 but not R_0^1. (c.f. [On certain separation and connectedness concepts in fuzzy topology-By D.M. Ali])

(l) $R_1^4 \Rightarrow R_0^7$, and so $R_1^k \Rightarrow R_0^m$, where $k \in \{1,2,3,4\}$ and $m \in \{1,5,6,7\}$.

Proof (l): Suppose (X, t) is an R_1^4-fts. Let x, y ∈ X, x ≠ y such that $\overline{1_y}(x) \notin \{0,1\}$. This implies that there exists m ∈ t^c such that m(y) = 1 and 0 < m(x) < 1. Put w = 1 – m ∈ t. Now w(x) > 0 and w(y) = 0. Then, w(x) > α for some α ∈ $I_{0,1}$. Since (X, t) is an R_1^4-fts, there exist u, v ∈ t such that u(x) = 1 = v(y) and u ∧ v = 0. Put n = 1 – u ∈ t^c. Now n(x) = 0 and n(y) = 1. Therefore, $\overline{1_y} \le n$ and so $\overline{1_y}(x) = 0$, which is a contradiction. Again let $\overline{1_y}(x) \ne \overline{1_x}(y)$. Without any loss of generality, let $0 = \overline{1_x}(y) < \overline{1_y}(x) = 1$. This implies that there exist λ_1, λ_2 ∈ t^c such that $\lambda_1(x) = \lambda_2(x) = \lambda_2(y) = 1$ and $\lambda_1(y) = 0$. Take w = 1 – λ_1. Now w ∈ t such that w(x) = 0 and w(y) =

1. Clearly, w(y) > α for every $\alpha \in I_{0,1}$. Since (X, t) is an R_1^4-fts, there exist p, q ∈ t such that p(x) = 1 = q(y) and p ∧ q = 0. Put $n_1 = 1 - p$ and $n_2 = 1 - q$. Now, $n_1, n_2 \in t^c$ such that $n_1(x) = 0 = n_2(y)$ and $n_1(y) = 1 = n_2(x)$. Clearly, $\overline{1_y} \leq n_1$ and so $\overline{1_y}(x) = 0$, which is also a contradiction. Therefore, $\overline{1_x}(y) = \overline{1_y}(x) \in \{0,1\}$. Therefore (X, t) is an R_0^7-fts.

(m) $R_0^m \not\Rightarrow FR_1(k)$, where $k \in \{i - xviii\}$ and $m \in \{1-9\}$.

Proof:

Example-6: Let X be an infinite set. For x, y ∈X, we define $U_{xy} \in I^X$ as follows:

$$U_{xy}(z) = \begin{cases} 0 \text{ if } z \in \{x, y\} \\ 1 \text{ if } z \notin \{x, y\} \end{cases}$$

Let t be the fuzzy topology generated by $\{U_{xy}: x, y \in X\} \cup \{\text{constants}\}$. It can be checked that if x ≠ y, then $\overline{1_x}(y) = 0$. Therefore, (X, t) is R_0^4, R_0^7 and R_0^9. But (X, t) is neither $FR_1(xvi)$ nor $FR_1(xviii)$ as there exists no u, v ∈ t such that u ≤ 1 – v.

2. Fuzzy regular axioms.

4.1 Definition: A fuzzy topological space (X, t) is called

(a) FR(i) if and only if $\alpha \in I_0$, $\lambda \in t^c$, x∈X and λ≤ 1 – λ(x) imply that there exist u, v ∈ t such that α ≤ u(x), λ ≤ v and u ≤ 1 – v.

(b) FR(ii) if and only if $\alpha \in I_0$, $\lambda \in t^c$, x ∈ X and $\alpha \lesssim 1 - \lambda(x)$ imply that there exist u, v ∈ t such that $\alpha \lesssim u(x)$, λ ≤ v and u ≤ 1 – v.

(c) FR(iii) if and only if each u ∈t is a supremum of $u_j, j \in J$, where ∀ j, $u_j \in t$ and $\overline{u_j} \leq u$.

(d) FR(iv) if and only if $\lambda \in t^c$, $x \in X$ and $\lambda(x) = 0$ imply that there exist u, $v \in t$ such that $u(x) = 1$, $\lambda \le v$ and $u \le 1 - v$.

(e) FR(v) if and only if $\lambda \in t^c$, $x \in X$ and $1 - \lambda(x) > 0$ imply that there exist u, $v \in t$ such that $u(x) > 0$, $\lambda \le v$ and $u \le 1 - v$.

1. Note: Let $x \in X$ and λ be a fuzzy set in X. Then for $\alpha \in I_0$, "$\alpha \lesssim \lambda(x)$" means $\alpha < \lambda(x)$ if $\alpha \neq 1$ and $\lambda(x) = 1$ if $\alpha = 1$.
2. Note: The following implications exist among FR(i), FR(ii), …., FR(v).

$$
\begin{array}{l}
FR(i) \Rightarrow FR(ii) \Rightarrow FR(iii) \Rightarrow FR(v) \\
\qquad\qquad \Downarrow \\
\qquad\quad FR(iv)
\end{array}
$$

Example-4: Let $X = \{x, y, z\}$. For $x, y \in X$, $x \neq y$, we define U_{xy} as follows:

$U_{xy}(x) = 1, U_{xy}(y) = 0$ and $U_{xy}(z) = 0.5$.

Let t be the fuzzy topology of X generated by $\{U_{xy} : x, y \in X, x \neq y\}$. Then (X, t) is easily seen to be FR(i). But (X, t) does not satisfy any of R_1^4, R_1^6, R_1^{10}, R_1^{12}, FR_1(xvi) and FR_1(xviii). Therefore, $FR(j) \not\Rightarrow R_1^k$ and $FR(j) \not\Rightarrow FR_1(m)$, j = i, ii, ….,v; k = 1, 2, …..,12 and m = i, ii, ……, xviii.

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