2008

Damped Forced Vibration of Some Quasi-Linear Differential Systems

Dey, Pinakee

University of Rajshahi

DAMPED FORCED VIBRATION OF SOME QUASI-LINEAR DIFFERENTIAL SYSTEMS

Ph.D. Thesis

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

SUBMITTED BY

PINAKEE DEY

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF RAJSHAHI
RAJSHAHI-6205, BANGLADESH
November-2008

DAMPED FORCED VIBRATION OF SOME QUASI-LINEAR DIFFERENTIAL SYSTEMS

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy In Mathematics

SUBMITTED BY

PINAKEE DEY

Roll No. 122, Registration No.0053

Session: July 2004

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF RAJSHAHI
RAJSHAHI-6205, BANGLADESH
November-2008

DECLARATION

The thesis entitled "Damped Forced Vibration of Some Quasi-Linear Differential Systems" is written by me and has been submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh. Here I confirm that this research work is an original one and it has not been submitted elsewhere for any degree.

Pinakee Dey

Pinakee Dey
(Candidate)
Roll No: 122, Session: July2004, Reg No:0053
Department of Mathematics
University of Rajshahi, Rajshahi-6205, Bangladesh.

Date: 20.11.08

DEDICATED

TO

MY BELOVED MOTHER

CERTIFICATE

This is to certify that the research work entitled "**Damped Forced Vibration Of Some Quasi-Linear Differential Systems**" presented in this dissertation is based on the study carried out by **Pinakee Dey**, Roll No. 122, Registration No. 0053, Session-July 2004 in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh, has been completed under our supervision. We believe that this research work is an original one and it has not been submitted elsewhere for any degree.

Prof. Dr. M. Abdus Sattar
Supervisor
Department of Mathematics
University of Rajshahi, Rajshahi, Bangladesh.

Prof. Dr. M. Zulfikar Ali
Co-Supervisor
Department of Mathematics
University of Comilla, Comilla, Bangladesh.

Prof. Dr. M. Shamsul Alam
Co-Supervisor
Department of Mathematics
Rajshahi University of Engineering and Technology, Rajshahi, Bangladesh.

Date: 20.11.08

Acknowledgement

All the praises and glory is to the Almighty, the Lord of the Universe. The Sustainer of all creatures, who, with His mercy and blessings, enabled me to finish this study.

I would like to express my sincere admiration, appreciation and gratitude to my Supervisor, Professor Dr. M A. Sattar, Co-supervisors, Professor Dr. M. Zulfikar Ali and Professor Dr. M. Shamsul Alam for their guidance, invaluable suggestions and farsighted advice in implementing the research work.

I am also highly indebted to Professor Dr. Subrata Majunder, Rajshahi University, Professor Dr. M. Majedur Rahman, Professor Dr. Bellal Hossain, Assistant Prof. Kamales Chandra Roy, Abul Kalam Azad, Rajshahi RUET, Assistant Prof. Sharif Uddin, Khulna University, Dr. M. Ali Akbar, Department of Applied Mathematics, Rajshahi University, for their various advice and help during my research work.

I also express my heart felt sense of gratitude, profound regards Mr. Dulal Dey, Professor Anil Chandra Basak, Principal Professor Khalilur Rahaman, Dr. Asit Roy, Dr. Sibajee Dey, Dr. Paddini Roy for their kind co-operation and valuable suggestions to make this study a success.

I am thankful to the University of Rajshahi and in particular to the Department of Mathematics for extending all facilities and co-operation during the course of my Ph. D. program.

Finally, I express my deep sense of appreciation to my beloved wife Sabita Dey for her constant encouragement and continuous support in various ways during the whole period of study. Thanks are due to my affectionate daughters Arundhati and Arunima for their sacrifice during my busy days is research.

Abstract

There are many approaches for approximating solutions of nonlinear vibrating problems. The most common methods for constructing approximate analytical solutions to the nonlinear vibrating problems are the perturbation methods. These methods are developed to find only periodic vibrations of the nonlinear differential systems. In order to investigate the transients of nonlinear vibrations, Krylov and Bogoliubov introduced a perturbation method to discuss the transients in the second order autonomous systems with small nonlinearities. The method is well known as an "asymptotic averaging method" in the theory of nonlinear vibrations. Then the method was amplified and justified by Bogoliubov and Mitropolskii. These methods were applied to autonomous systems. Later, Arya and Bojadziev, Bojadziev and Hung, and Shamsul extended the Krylov-Bogoliubov-Mitropolskii (KBM) method to some time dependent nonlinear differential systems. In this dissertation, we extend the work of KBM and investigate some other time dependent non-linear differential systems.

Firstly, a second order time dependent nonlinear differential system is considered. Then a new perturbation technique is developed to find an asymptotic solution of nonlinear vibrations in presence of a slowly decaying external force. We then find an asymptotic solution of a time dependent nonlinear differential system with slowly varying coefficients using the KBM method. Later, we find the perturbation solutions of damped forced vibrations using the modified KBM method, in which the coefficients change slowly varying with time. Further, this technique is used to obtain the second approximate solution of second order forced vibrations. Finally, this technique is used to obtain the higher approximate solution of an n-th order damped forced vibrating problem in the resonance case, and the stability of the stationary regime of vibrations has also been investigated. The methods are illustrated by several examples.

Table of Contents

Declaration		(*i*)
Dedication		(*ii*)
Certificate		(*iii*)
Acknowledgment		(*iv*)
Abstract		(*v*)
Table of Contents		(*vii*)
List of Table and Figures		(*ix*)
Introduction		1-3
Chapter 1	**The Survey and the Proposal**	**4-24**
1.1	The Survey	4
1.2	The Proposal	24
Chapter 2	**An Asymptotic Method for Second Order Time Dependent Non-Linear Over-Damped Systems**	**25-33**
2.1	Introduction	25
2.2	The Method	26
2.3	Example	29
2.4	Results and Discussions	31
2.5	Conclusion	33
Chapter 3	**An Asymptotic Method for Second Order Time Dependent Non-Linear Systems with Varying Coefficients**	**34-48**
3.1	Introduction	34
3.2	The Method	35
3.3	Example	37
3.4	Results and Discussions	44
3.5	Conclusion	45

Chapter 4 **Perturbation Theory for Second Order Damped Forced Vibrations with Slowly Varying Coefficients** **49-63**

4.1 Introduction 49

4.2 The Method 50

4.3 Example 53

4.4 Results and Discussions 58

4.5 Conclusion 59

Chapter 5 **Second Approximate Solution of Second Order Time Dependent Weakly Nonlinear Differential Systems** **64-75**

5.1 Introduction 64

5.2 The Method 65

5.3 Example 68

5.4 Results and Discussions 72

5.5 Conclusion 73

Chapter 6 **Higher Approximate Solution of *n*-th Order Weakly Non-Linear Non-Autonomous Differential Systems with Damping** **76-124**

6.1 Introduction 76

6.2 The Method 77

6.3 Example 82

6.4 Results and Discussions 115

6.5 Conclusion 117

References **125-136**

List of Table and List of Figures

List	Page no:
Table.2.1	(31)
Fig. 3.1	(46)
Fig. 3.2	(47)
Fig. 3.3	(48)
Fig. 4.1	(60)
Fig. 4.2	(61)
Fig. 4.3	(62)
Fig. 4.4	(63)
Fig. 5.1	(74)
Fig. 5.2	(75)
Fig. 6.1	(118)
Fig. 6.2	(119)
Fig. 6.3	(120)
Fig. 6.4	(121)
Fig. 6.5	(122)
Fig. 6.6	(123)
Fig. 6.7	(124)

Introduction

Vibrations are ubiquitous in all fields of fundamental and applied sciences. A number of physical, mechanical, chemical, biological, biochemical and some economic laws and relations appear mathematically in the form of differential equations which are ordinary or partial, linear or nonlinear, autonomous or non-autonomous. The modeling of the involved physical phenomena leads very often to ordinary differential equations that, in most of the cases, are nonlinear. Solving nonlinear ordinary differential equations is thus of great importance for gaining insights into the real world. Methods of solutions of linear differential equations are comparatively easy and well established. On the contrary, the techniques of solutions of nonlinear differential equations are less available and in general, linear approximations are frequently used. The method of small oscillations is a well-known example of the linearization of problems, which are essentially nonlinear. With the discovery of numerous phenomena of self-excitation of circuits containing nonlinear conductors of electricity, such as electron tubes, gaseous discharge, etc., and in many cases of nonlinear mechanical vibrations of special types, the method of small oscillations becomes inadequate for their analytical treatment. There exists an important difference between the phenomena which oscillate in steady state and the phenomena governed by linear differential systems with constant coefficients, e.g., oscillations of a pendulum with small amplitudes, in that the amplitude of the ultimate stable oscillation seems to be entirely independent of the initial conditions, whereas in oscillations governed by linear differential systems with constant coefficients, it depends upon the initial conditions.

Van der Pol first paid attention to the new (self-excitation) oscillation and found that their existence is inherent in the nonlinearity of the differential systems characterizing the process. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential systems in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential systems there exist some methods. Among the methods, the method of perturbations, *i. e.*, asymptotic expansions in terms of a small parameter, are foremost. According to these techniques, the solutions are presented by the first two terms to avoid rapidly growing algebraic complexity. Although these perturbation expansions may be divergent, they can be more useful for qualitative and quantitative representations than the expansions that are uniformly convergent.

Perturbation methods are one of the fundamental tools used by all applied mathematicians and theoretical physicists and widely used in science to obtain approximate solutions based on known exact solutions to nearby problems. Such asymptotic techniques can also be used to provide initial guesses for numerical approximations, and they can now be generated through smart use of symbolic computation. An example of this occurs in boundary layer problem where the regions of rapid change of quantities are fluid velocity, temperature or concentration. This method is most effectively used to analyze problems in solid and fluid mechanics, control theory, celestial mechanics, optics, shock waves, nonlinear vibrations, nonlinear wave propagations, and reaction-diffusion systems arising in several physical and biological contexts.

In this dissertation, we shall discuss nonlinear vibrating problems that can be described by the dynamical vibrations of second and nth order time dependent nonlinear differential systems with small nonlinearities by use of the modified Krylov-Bogoliubov-Mitropolskii (KBM) method. An important approach to study such nonlinear oscillatory problems is the small parameter expansion. Two widely spread methods are mainly used; one is averaging, particularly the KBM method and the other is the method of variation of parameters. According to the KBM technique the solution starts with the solution of linear equation, termed as generating solution, assuming that, in the nonlinear case, the amplitude and phase of the solution of the linear differential equation are time-dependent functions rather than constants. This method introduces an additional condition on the first derivative of the generating solution for determining the solution of a second order equation. Originally, the method was developed by Krylov-Bogoliubov to obtain the periodic solutions of second order nonlinear differential systems. Now, the method is used to obtain oscillatory, damped oscillatory and non-oscillatory solutions of second, third etc. order nonlinear differential systems by imposing some restrictions to make the solutions uniformly valid.

Most of the authors, found the solutions of autonomous nonlinear differential systems. Only a diminutive number of authors investigated damped forced nonlinear vibrating problems. In this dissertation, some second order and an n-th order time dependent nonlinear vibrating problems have been studied and their solutions are investigated.

The results may be useful to researchers working in the field of nonlinear mechanics, mathematical physics, control theory, population dynamics, etc.

Chapter 1

The Survey and the Proposal

1.1 The Survey

The study of nonlinear vibrating problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most physical phenomena in our real world are essentially nonlinear and are described by nonlinear equations. In the mathematical formulations, many physical problems often result in differential equations that are nonlinear. However, in many cases it is possible to replace a nonlinear differential equation with a related linear differential equation that approximates the actual equation closely enough to give useful results. Often such linearization is not possible or feasible; when it is not, the original nonlinear equation itself must be tackled.

During the last several decades a number of Russian scientists, like, Mandelstam and Papalexi [47], Andronov [6], Krylov and Bogoliubov [37], Bogoliubov and Mitropolskii [12] worked jointly and have investigated nonlinear problems. Among them, Krylov and Bogoliubov are certainly to be found most active.

Krylov and Bogoliubov considered primarily equations of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(t, x, \frac{dx}{dt}, \varepsilon) \tag{1.1}$$

where ε is a small positive quantity and f is a power series in ε, whose coefficients are polynomials in $x, \frac{dx}{dt}, \sin t, \cos t$. In fact, generally f contains neither ε nor t. Similar equations are well known in astronomy and have been the object of systematic investigations by Lindstedt [44,45], Gylden [33], Liapounoff [42] and, above all by Poincare [77]. In general sense, it seems that, Krylov and Bogoliubov applied the same methods. However, the applications in which they viewed are quite different, being mainly in engineering, technology or physics, notably electrical circuit theory. The method has also been used in plasma physics, theory of oscillations and control theory.

In the treatment of nonlinear oscillations, by perturbation methods, Lindstedt [44,45], Gylden [33], Liapounoff [42], Poincare [77], discussed only periodic solutions, transients were not considered. Krylov and Bogoliubov (KB) first discussed transient response. The method of KB starts with the solution of the linear equation, assuming that, in the nonlinear case, the amplitude and phase in the solution of the linear equation are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the solution.

Extensive uses have been made and some important works are done by Stoker [118], McLachlan [48], Minorsky [51], Nayfeh [60,61], Bellman *et al* [11].

Most probably, Poisson initiated to find approximate solutions of nonlinear differential equations around 1830 and the technique was formally introduced by Liouville [46]. Duffing [29] investigated many significant results concerning the periodic solutions of the equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + x = -\varepsilon x^3 \tag{1.2}$$

Some different nonlinear phenomena occur when the amplitude of the dependent variable of a dynamical system is less than or greater than unity, the damping is negative when the amplitude is less than unity and the damping is positive when the amplitude of the dependent variable is greater than unity. The governing equation, having these phenomena is given by

$$\frac{d^2x}{dt^2} - \varepsilon(1-x^2)\frac{dx}{dt} + x = 0 \qquad (1.3)$$

This equation is known as Van der Pol [120] equation. This equation has a very extensive field of application in connection with self-excited oscillations in electron-tube circuits.

Since, in general, f contains neither ε nor t, the equation (1.1) therefore takes the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f\left(x, \frac{dx}{dt}\right) \qquad (1.4)$$

The method of KB is very similar to that of Van der Pol and related to it. Van der Pol applied the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\frac{d^2x}{dt^2} + \omega^2 x = 0$, on the other hand KB applied the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus in the KB method the varied constants are a and φ, while in the Van der Pol's method the constants are a and b. The method of KB seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation.

If $\varepsilon = 0$, then the equation (1.4) reduces to linear equation and its solution is

$$x = a\cos(\omega t + \varphi) \qquad (1.5)$$

where a and φ are arbitrary constants to be determined from given initial conditions.

If $\varepsilon \neq 0$, but is sufficiently small, then KB assumed that the solution is still given by (1.5) with the derivative of the form

$$\frac{dx}{dt} = -a\omega\sin(\omega t + \varphi) \tag{1.6}$$

where a and φ are functions of t, rather than being constants. Thus the solution of the equation (1.4) is of the form

$$x = a(t)\cos(\omega t + \varphi(t)) \tag{1.7}$$

and the derivative of the solution is chosen to be of the form

$$\frac{dx}{dt} = -a(t)\omega\sin(\omega t + \varphi(t)) \tag{1.8}$$

Differentiating the assumed solution (1.7) with respect to t, gives

$$\frac{dx}{dt} = \frac{da}{dt}\cos\psi - a\,\omega\sin\psi - a\frac{d\varphi}{dt}\sin\psi, \qquad \psi = \omega t + \varphi \tag{1.9}$$

Therefore,

$$\frac{dx}{dt}\cos\psi - a\frac{d\varphi}{dt}\sin\psi = 0 \tag{1.10}$$

by using (1.6).

Again differentiating (1.8) with respect to t, gives

$$\frac{d^2x}{dt^2} = -\frac{da}{dt}\omega\,\sin\psi - a\,\omega^2\cos\psi - a\,\omega\frac{d\varphi}{dt}\cos\psi \tag{1.11}$$

Substituting (1.11) into the equation (1.4) and using equations (1.7)-(1.8), gives

$$\frac{da}{dt}\omega\sin\psi + a\omega\frac{d\varphi}{dt}\cos\psi = -\varepsilon\, f(a\cos\psi, -a\omega\sin\psi) \tag{1.12}$$

Solving (1.10) and (1.12) for $\frac{da}{dt}$ and $\frac{d\varphi}{dt}$ yields

$$\frac{da}{dt} = -\varepsilon f(a\cos\psi, -a\omega\sin\psi)\sin\psi/\omega$$
$$\frac{d\varphi}{dt} = -\varepsilon f(a\cos\psi, -a\omega\sin\psi)\cos\psi/a\omega \tag{1.13}$$

Thus instead of the single differential equation (1.4) of the second order in the unknown x, we obtain two differential equations of the first order in the unknowns a and φ. Since $\frac{da}{dt}$ and $\frac{d\varphi}{dt}$ are proportional to the small parameter ε; a and φ are slowly varying functions of the time t with the period $T = 2\pi/\omega$ and, as a first approximation, they are constants.

Expanding $f(a\cos\psi, -a\omega\sin\psi)\sin\psi$ and $f(a\cos\psi, -a\omega\sin\psi)\cos\psi$ in Fourier series in the total phase ψ, the first approximate solution of (1.4), by averaging (1.13) over one period is

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi d\psi$$
$$\left\langle \frac{d\varphi}{dt} \right\rangle = -\frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi d\psi \tag{1.14}$$

where a and φ are independent of time under the integrals.

KB called their method asymptotic in the sense that $\varepsilon \to 0$. An asymptotic series itself is not convergent, but for a fixed number of terms the approximate solution tends to the exact solution as ε tends to zero. It is noted that the term asymptotic is frequently used in the theory of oscillation, also in sense that $\varepsilon \to \infty$. But in this case the mathematical method is quite different.

Later, this technique has been amplified and justified mathematically by Bogoliubov and Mitropolskii [12], and extended to non-stationary vibrations by Mitropolskii [53]. They assumed the solution of the nonlinear differential equation (1.4) in the form

$$x = a\cos\psi + \varepsilon\, u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \ldots + \varepsilon^n u_n(a,\psi) + O\left(\varepsilon^{n+1}\right) \qquad (1.15)$$

where u_k, $k = 1, 2, \ldots, n$ are periodic functions of ψ with a period 2π, and the quantities a and ψ are functions of time t, defined by

$$\begin{aligned} \frac{da}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^n A_n(a) + O\left(\varepsilon^{n+1}\right) \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots + \varepsilon^n B_n(a) + O\left(\varepsilon^{n+1}\right) \end{aligned} \qquad (1.16)$$

The functions u_k, A_k and B_k, $k = 1,2,\ldots,n$ are to be chosen in such a way that the equation (1.15), after replacing a and ψ by the functions defined in equation (1.16), is a solution of the equation (1.4). Since there are no restrictions in choosing the functions A_k and B_k, that generate the arbitrariness in the definitions of the functions u_k [13]. To remove this arbitrariness, the following additional conditions are imposed

$$\begin{aligned} &\int_0^{2\pi} u_k(a,\psi)\cos\psi\, d\psi = 0, \\ &\int_0^{2\pi} u_k(a,\psi)\sin\psi\, d\psi = 0, \end{aligned} \qquad (1.17)$$

Differentiating (1.15) two times with respect to t, utilizing relations (1.16), substituting (1.15) and the derivatives $\frac{dx}{dt}, \frac{d^2x}{dt^2}$, in the original equation (1.4), and equating the coefficients of ε^k, $k = 1,2,\ldots,n$ results a recursive system

$$\omega^2\left(\frac{\partial^2 u_k}{\partial \psi^2} + u_k\right) = f^{(k-1)}(a,\psi) + 2\omega(a B_k \cos\psi + A_k \sin\psi), \qquad (1.18)$$

where

$$f^0(a,\psi) = f(a\cos\psi, \quad -\omega a \sin\psi),$$

$$
\begin{aligned}
& f^{(1)}(a,\psi) = u_1 f_x(a\cos\psi, \quad -\omega a\sin\psi) \\
& + \left(A_1 \cos\psi - aB_1 \sin\psi + \omega \frac{\partial u_1}{\partial \psi} \right) \\
& \times f_{\dot{x}}(a\cos\psi, \quad -\omega a\sin\psi) + \left(a B_1{}^2 - A_1 \frac{dA_1}{da} \right)\cos\psi \\
& + \left(2A_1 B_1 - aA_1 \frac{dB_1}{da} \right)\sin\psi - 2\omega \left(A_1 \frac{\partial^2 u_1}{\partial a \partial \psi} + B_1 \frac{\partial^2 u_1}{\partial \psi^2} \right).
\end{aligned}
\tag{1.19}
$$

It is obvious that f^{k-1} is a periodic function of the variable ψ with period 2π, which depends also on the amplitude a. Therefore, f^{k-1} as well as u_k can be expanded in a Fourier series as

$$
\begin{aligned}
& f^{(k-1)}(a,\psi) = g_0{}^{(k-1)}(a) + \sum_{n=1}^{\infty} g_n{}^{(k-1)}(a)\cos n\psi + h_n{}^{(k-1)}(a)\sin n\psi \\
& u_k(a,\psi) = v_0{}^{(k-1)}(a) + \sum_{n=1}^{\infty} v_n{}^{(k-1)}(a)\cos n\psi + w_n{}^{(k-1)}(a)\sin n\psi,
\end{aligned}
\tag{1.20}
$$

where

$$
\begin{aligned}
& g_0{}^{(k-1)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\ \cos\psi, \quad -\omega a\sin\psi)\,d\psi, \\
& g_n{}^{(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\ \cos\psi, \quad -\omega a\sin\psi)\cos n\psi\,d\psi, \\
& h_n{}^{(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\ \cos\psi, \quad -\omega a\sin\psi)\sin n\psi\,d\psi, \qquad n \ge 1
\end{aligned}
\tag{1.21}
$$

Here $v_1{}^{(k-1)} = w_1{}^{(k-1)} = 0$ for all values of k, since both integrals of (1.17) vanish.

Substituting these values into the equation (1.18), it becomes

$$
\begin{aligned}
& \omega^2 v_0{}^{(k-1)}(a) + \sum_{n=1}^{\infty} \omega^2(1-n^2)\left[v_n{}^{(k-1)}(a)\cos n\psi + w_n{}^{(k-1)}(a)\sin n\psi\right] \\
& = g_0{}^{(k-1)}(a) + \left(g_1{}^{(k-1)}(a) + 2\omega a B_k\right)\cos\psi + \left(h_1{}^{(k-1)}(a) + 2\omega B\right)\sin\psi \\
& + \sum_{n=2}^{\infty}\left[g_n{}^{(k-1)}(a)\cos n\psi + h_n{}^{(k-1)}(a)\sin n\psi\right]
\end{aligned}
\tag{1.22}
$$

Now equating the coefficients of harmonics of the same order, we get

$$g_1^{(k-1)}(a) + 2\omega a B_k = 0, \qquad h_1^{(k-1)}(a) + 2\omega A_k = 0,$$
$$v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2}, \quad v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \tag{1.23}$$
$$w_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \geq 1$$

These are the sufficient conditions to obtain the desired order of approximation. For the first order approximation, we have

$$A_1 = -\frac{h_1^{(1)}(a)}{2\omega} = -\frac{1}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, \; -\omega a\sin\psi)\sin\psi\, d\psi,$$
$$B_1 = -\frac{g_1^{(1)}(a)}{2\omega a} = -\frac{1}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, \; -\omega a\sin\psi)\cos\psi d\psi. \tag{1.24}$$

Therefore, the variational equations in (1.16) become

$$\frac{da}{dt} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, \; -\omega a\sin\psi)\sin\psi\, d\psi,$$
$$\frac{d\psi}{dt} = \omega - \frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, \; -\omega a\sin\psi)\cos\psi\, d\psi. \tag{1.25}$$

The equations of (1.25) are similar to the equations in (1.14). Thus the first order solution obtained by Bogoliubov and Mitropolskii [12] is identical with the original solution obtained by KB [37]. In the second case, higher order solution can be found easily. The correction term u_1 is obtained from (1.23) as

$$u_1 = \frac{g_0^{(1)}(a)}{\omega^2} + \sum_{n=2}^{\infty}\frac{g_n^{(1)}(a)\cos n\psi + h_n^{(1)}(a)\cos n\psi}{\omega^2(1-n^2)}. \tag{1.26}$$

The solution (1.15) together with u_1 is known as the first order improved solution in which a and ψ are the solutions of the equation (1.25). If the values of the functions A_1 and B_1 are substituted from (1.24) in the second relation of (1.19), the function $f^{(1)}$, and in the similar way, the unknown functions A_2, B_2 and u_2 can be found. Thus the determination of the higher order approximation is complete.

Volosov [121,122], Museenkov [58] and Zebreiko [124] also obtained higher order approximations.

The KB method has been extended by Kruskal [36] to solve the fully nonlinear differential equation

$$\frac{d^2x}{dt^2} = F(x, \frac{dx}{dt}, \varepsilon) \tag{1.27}$$

The solution of this fully nonlinear equation is based on recurrent relations and is given in the form of power series of the small parameter ε.

Cap [27] has investigated some nonlinear systems of the form

$$\frac{d^2x}{dt^2} + \omega^2 f(x) = \varepsilon F(x, \frac{dx}{dt}) \tag{1.28}$$

He solved this equation by using elliptical functions in the sense of Krylov and Bogoliubov.

Struble [119] developed a technique for treating weakly nonlinear oscillatory systems such as those governed by

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \varepsilon f(x, \frac{dx}{dt}, t) \tag{1.29}$$

He expressed the asymptotic solution of this equation for small ε in the form

$$x = a\cos(\omega_0 t - \theta) + \sum_{n=1}^{N} \varepsilon^n x_n(t) + O(\varepsilon^{Y+1}) \tag{1.30}$$

where a and θ are slowly varying functions of time.

Later, the method of Krylov-Bogoliubov-Mitropolskii (KBM) has been extended by Popov [78] to damped nonlinear systems

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = \varepsilon\, f(x, \frac{dx}{dt}) \tag{1.31}$$

where $-2k\dfrac{dx}{dt}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the method [78] in the physical systems, involving damping force, Mendelson [49] and Bojadziev [23] rediscovered Popov's results. In the case of damped nonlinear systems the first equation of (1.16) has been replaced by

$$\frac{da}{dt} = -ka + \varepsilon\, A_1(a) + \varepsilon^2\, A_2(a) + \ldots + \varepsilon^n\, A_n(a) + O\left(\varepsilon^{n+1}\right) \tag{1.16a}$$

Murty, Deekshatulu and Krishna [56] found a hyperbolic type asymptotic solution of an over-damped system represented by the nonlinear differential equation (1.31) in the sense of KBM method; *i. e.*, in this case $k > \omega$. They used hyperbolic function, $\cosh\varphi$ or $\sinh\varphi$ instead of the harmonic function, $\cos\varphi$, which is used in [12,37,49,78]. In the case of oscillatory or damped oscillatory process $\cos\varphi$ may be used arbitrarily for all kinds of initial conditions. But in the case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions [24,56,57]. Murty and Deekshatulu [55] found another asymptotic solution of the over-damped system represented by the equation (1.31), by the method of variation of parameters. Shamsul [107] extended the KBM method to find solutions of over-damped nonlinear systems, when one root becomes much smaller than the other root. Murty [57] has presented a unified KBM method for solving the nonlinear systems represented by the equation (1.31). Bojadziev and Edwards [24] investigated the solutions of oscillatory and non-oscillatory systems represented by (1.31), when k and ω are slowly varying functions of time t. Arya and Bojadziev [7,8] examined damped oscillatory systems and time-dependent oscillating systems with slowly varying parameters and delay. Shamsul, Alam and Shanta [94] extended the Krylov-Bogoliubov-Mitropolskii method to certain non-oscillatory nonlinear systems with varying coefficients. Later, Shamsul [109] unified the KBM method for solving an n-th order nonlinear differential equation with varying coefficients. Sattar [83] has developed an asymptotic method to solve a critically damped nonlinear system represented by (1.31). He has found the asymptotic solution of the system (1.31) in the form

$$x = a(1+\psi) + \varepsilon u_1(a,\psi) + \ldots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}) \tag{1.32}$$

where a is defined by the equation (1.16a) and ψ is defined by

$$\frac{d\psi}{dt} = 1 + \varepsilon C_1(a) + \ldots + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}) \tag{1.16b}$$

Shamsul [91] has developed an asymptotic method for second-order over-damped and critically damped nonlinear systems. Shamsul [102,110] has also extended the KBM method for certain non-oscillatory nonlinear systems when the eigenvalues of the unperturbed equation are real and non-positive. Shamsul [93] has presented a new perturbation method based on the work of Krylov-Bogoliubov-Mitropolskii to find approximate solutions of nonlinear systems with large damping. Later, he extended the method to n-th order nonlinear differential systems [99]. Shamsul, Hossain and Shanta [97] investigated perturbation solution of a second order time-dependent nonlinear system based on the modified Krylov-Bogoliubov-Mitropolskii method.

Making use of the KBM method, Bojadziev [14] has investigated nonlinear damped oscillatory systems with small time lag. Bojadziev [20,21], Bojadziev and Chan [22] applied the KBM method to problems of population dynamics. Bojadziev [23] used the

KBM method to investigate nonlinear biological and biochemical systems. Lin and Khan [43] have also used the KBM method to some biological problems. Proskurjakov [79], Bojadziev, Lardner and Arya [15] have investigated periodic solutions of nonlinear systems by KBM and Poincare method, and compared the two solutions. Bojadziev and Lardner [16,17] have investigated monofrequent oscillations in mechanical systems including the case of internal resonance, governed by hyperbolic differential equation with small nonlinearities. Bojadziev and Lardner [18] have also investigated hyperbolic differential equations with large time delay. Freedman, Rao and Lakshami [30] used the KBM method to study stability, persistence and extinction in a prey-predator system with discrete and continuous time delay. Freedman and Ruan [31] used the KBM method in three-species food chain models with group defense.

Murty [57] has presented unified KBM method for solving the differential equation (1.31) by using their previous solution [55] as a general solution for the un-damped, damped and over-damped cases, which is the basis of the unified theory. Murty [57] assumed a solution of (1.31) according to the asymptotic method in the form

$$x(t,\varepsilon)=\frac{a}{2}e^{\psi}-\frac{a}{2}e^{-\psi}+\varepsilon\omega_1(a,\psi)+\ldots \tag{1.33}$$

where a and ψ satisfy the first order differential equations

$$\begin{aligned}\frac{da}{dt}&=-ka+\varepsilon A_1(a)+\varepsilon^2 A_2(a)+\ldots\\ \frac{d\psi}{dt}&=\omega_1+\varepsilon B_1(a)+\varepsilon^2 B_2(a)+\ldots\end{aligned} \tag{1.34}$$

with $\lambda_1-\lambda_2=2\omega_1$ and ω_1 is unknown function of a and ψ, where λ_1 and λ_2 are the eigenvalues of the corresponding linear equation of (1.31). In his paper, Murty [57] restricts himself to only the first approximation.

When the eigenvalues of the corresponding linear equation are real, ψ being a real quantity and the first two terms on the right sides of equation (1.15) can be combined as

$$x(t,\varepsilon) = a\sinh\psi + \varepsilon\omega_1(a,\psi) + \dots \tag{1.35}$$

which corresponds to over-damped solution of (1.33). When the eigenvalues of the corresponding linear equation are complex conjugate (*i.e.* for un-damped and under-damped cases) instead of real, putting $a = -ia$, $\psi = i\psi$, $\cosh i\psi = \cos\psi$ and $\sinh i\psi = -i\sin\psi$, the solution in equation (1.31) becomes

$$x(t,\varepsilon) = a\sin\psi + \varepsilon\omega_1(a,\psi) + \dots \tag{1.36}$$

which corresponds to the periodic and under-damped solution of (1.31). Murty's [57] technique is a generalization of the KBM method. Many authors extended this technique in various oscillatory and non-oscillatory systems. Bojadziev and Edwards [24] investigated nonlinear damped oscillatory and non-oscillatory systems with varying coefficients following Murty's [57] unified method.

Most probably, Osiniskii [63], first extended the KBM method to a third order nonlinear differential equation

$$\frac{d^3x}{dt^3} + k_1\frac{d^2x}{dt^2} + k_2\frac{dx}{dt} + k_3x = \varepsilon\, f(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}) \tag{1.37}$$

where ε is a small positive parameter and f is a nonlinear function. Osiniskii assumed the asymptotic solution in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \dots + \varepsilon^n u_n(a,b,\psi) + O(\varepsilon^{n+1}), \tag{1.38}$$

where u_k, $k = 1,2,\dots,n$ are periodic function of ψ with period 2π and, a, b and ψ are functions of time t, given by

$$\frac{da}{dt} = -\lambda a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1})$$
$$\frac{db}{dt} = -\mu b + \varepsilon B_1(b) + \varepsilon^2 B_2(b) + \ldots + \varepsilon^n B_n(b) + O(\varepsilon^{n+1}) \quad (1.39)$$
$$\frac{d\psi}{dt} = \omega + \varepsilon C_1(b) + \varepsilon^2 C_2(b) + \ldots + \varepsilon^n C_n(b) + O(\varepsilon^{n+1})$$

where $-\lambda$, $-\mu \pm i\omega$ are the eigenvalues of the equation (1.37) when $\varepsilon = 0$.

Shamsul and Sattar [87] extended Murty's [57] unified technique for obtaining the transient response of third order nonlinear systems. Recently, Shamsul [98] has presented a unified formula to obtain a general solution of an *n*-th order differential equation with constant coefficients. He considers a weakly nonlinear system as

$$\frac{d^{(n)}x}{dt^{(n)}} + k_1 \frac{d^{(n-1)}x}{dt^{(n-1)}} \ldots + k_n x = \varepsilon f(x, \frac{dx}{dt}, \ldots) \quad (1.40)$$

where over-dot denotes differentiation with respect to t, k_j, $j = 1, 2, \ldots, n$ are constants. Shamsul [98] seeks a solution of (1.40) in the form

$$x(\varepsilon, t) = \sum_{j=1}^{n} a_j(t) e^{\lambda_j t} + \varepsilon w_1(a_1, a_2, \ldots, a_n, t) + \ldots \quad (1.41)$$

where λ_j, $j = 1, 2, \ldots, n$ are the eigenvalues of the corresponding linear equation of (1.40) and each a_j satisfies a first order differential equation

$$\frac{da_j}{dt} = \varepsilon A_1(a_1, a_2, \ldots, a_n, t) + \ldots \quad (1.42)$$

In most treatment of the perturbation techniques an approximate solution is determined in terms of amplitude and phase variables. But the solution (1.40) starts with some new variables $a_1, a_2, \ldots, a_n$, such a choice of variables is important to tackle various nonlinear problems with an easier approach. This technique greatly speeds up the KBM method to determine the asymptotic solution.

Shamsul [109] extended his previous solution [98] to an n-th order differential equation with slowly varying coefficients. Shamsul [112] presented a modified and compact form of KBM unified method for obtaining the transient response of n-th order differential equation with small nonlinearities.

Osiniskii [64] has also extended the KBM method to a third order nonlinear partial differential equation with internal friction and relaxation. Mulholland [54] studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [39] investigated nonlinear damped oscillations governed by a third order partial differential equation. They introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Rauch [80] has studied oscillations of a third order nonlinear autonomous system. Sattar [84] has extended the KBM asymptotic method for three-dimensional over-damped nonlinear systems. Shamsul and Sattar [86] developed a method to solve third order critically damped nonlinear systems. Shamsul [100] redeveloped the method presented in [86] to find approximate solutions of critically damped nonlinear systems in the presence of different damping forces. Later, he unified the KBM method for solving critically damped nonlinear systems [115]. Shamsul and Sattar [92] studied time dependent third order oscillating systems with damping based on an extension of the asymptotic method of Krylov-Bogoliubov-Mitropolskii. Shamsul [103,107], Shamsul, Hossain and Ali Akbar [114] has developed a simple method to obtain the time response of second order over-damped nonlinear systems together with slowly varying coefficients under some special conditions. Later, Shamsul [101], Shamsul and Hossain [108] has extended the method [103,107] to obtain the time response of n-th order ($n \geq 2$), over-damped systems.

Shamsul [104] also developed a method for obtaining non-oscillatory solution of third order nonlinear systems. Shamsul and Sattar [87] presented a unified KBM method for solving third order nonlinear systems. Shamsul [98] has also presented a unified Krylov-Bogoliubov-Mitropolskii method, which is not the formal form of the original KBM method, for solving n-th order nonlinear systems. The solution contains some unusual variables. Yet this solution is very important. Shamsul [112] has also presented a modified and compact form of Krylov-Bogoliubov-Mitropolskii unified method for solving an n-th order nonlinear differential equation. The formula presented in [112] is compact, systematic and practical, and easier than that of [98]. Shamsul Alam, M. M. Abul Kalam Azad and M.A. Hoque [117] presented a general Struble's technique for solving an n-th order weakly non-linear differential system with damping.

Raymond and Cabak [81] examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Lewis [41,42] investigated stability for an autonomous second-order two-degree-of-freedom system and for a control surface with structural nonlinearities in surface flow. Andrianov and Awrejcewicz [4], Awrejcewicz and Andrianov [9] presented some new trends of asymptotic techniques in application to nonlinear dynamical systems in terms of summation and interpolation methods. In this dissertation, we shall not discuss this technique.

Hung and wu [34] obtained an exact solution of a differential system in terms of Bessel's functions, where the coefficients varying with time in an exponential order. Roy and Shamsul [82] found an asymptotic solution of a differential system in which the coefficient changes in an exponential order of slowly varying time.

O'Malley [65] found an asymptotic solution of a semiconductor device problem involving reverse bias. O'Malley [66,67,69,70] presented singular perturbation method for ordinary differential equations with matching and used this singular perturbation method to stiff differential equations. He [68] also presented exponential asymptotic for boundary layer resonance and dynamic meta-stability.

Bojadziev [19] found a mono frequent damped solution of an n-dimensional $n = 2,3,\cdots$ time dependent differential system with strong damping effects, small time delay and slowly varying coefficients. Bojadziev illustrated his method [19] by a second order equation, of the form

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + 2\beta\frac{dx}{dt}(t - \varepsilon\Delta) - cx = \varepsilon(1 - x^2)\frac{dx}{dt} + \varepsilon E \sin \nu t \tag{1.43}$$

Arya and Bojadziev [8] studied a second order time dependent differential eqution with damping, slowly varying coefficients and small time delay in which a non-periodic external force $\varepsilon E e^{-pt} \sin \nu t$ acted. Bojadziev [25], and Bojadziev and Hung [26] used the method of KBM to investigate a 3-dimensional time dependent differential system. Bojadziev extended the result in (1.43) to a time dependent system of the type

$$\frac{dx}{dt} = Ax + \varepsilon F(\theta, x), \qquad \theta = \nu t \tag{1.44}$$

where ε is a small positive parameter, ν is the frequency of the external acting force, $x = (x^{(1)}, x^{(2)}, x^{(3)})^T$ is a vector, $F(\theta, x) = (F^{(1)}(\theta, x), \; F^{(2)}(\theta, x), F^{(3)}(\theta, x))^T$ is a real vector function, 2π periodic in θ, with sufficient number of derivatives with respect to all the arguments in a domain and $F(\theta, 0) = 0$. Bojadziev assumed the asymptotic solution in the form

$$x(t, \varepsilon) = \varphi a + b[\phi e^{i\alpha} + \phi^* e^{-i\alpha}] + \varepsilon u(a, b, \theta, \alpha) + \varepsilon^2 \ldots \tag{1.45}$$

where $u = (u^{(1)}, u^{(2)}, u^{(3)})^T$ is an unknown 2π periodic function in θ and $\alpha = (p/q)\theta + \psi$, p and α are integers. The scalar variables a, b and ψ are functions of t to be determined by the differential equations

$$\frac{da}{dt} = -\xi a + \varepsilon A(a,b,\psi) + \varepsilon^2 \ldots,$$
$$\frac{db}{dt} = -\zeta b + \varepsilon B(a,b,\psi) + \varepsilon^2 \ldots, \qquad (1.46)$$
$$\frac{d\psi}{dt} = \omega - (p/q)\nu + \varepsilon C(a,b,\psi) + \varepsilon^2 \cdots,$$

where $-\xi$, $-\zeta \pm i\omega$ are the eigenvalues of the equation (1.44), when $\varepsilon = 0$.

Shamsul, Hossain and Shanta [97] found an approximate solution of a time dependent nonlinear system in which a strong linear damping force acts. Shamsul [113] developed a general formula based on the extended Krylov-Bogoliubov-Mitropolskii method, for obtaining asymptotic solution of an n-th order time dependent quasi-linear differential equation with damping. Nguyen Van Dinh [62] investigated the stationary oscillation from a variant of the asymptotic procedure in a special case of the type

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(x, \frac{dx}{dt}, \varphi), \qquad \omega = \varphi t \qquad (1.47)$$

where x is an oscillatory variable of the form

$$x = a\cos\psi + \varepsilon u_1(a,\theta,\psi) + \varepsilon^2 u_2(a,\theta,\psi) + \ldots \qquad (1.48)$$

with

$$\frac{da}{dt} = \varepsilon A_1(a,\theta) + \varepsilon^2 A_2(a,\theta) + \ldots$$
$$\frac{d\theta}{dt} = \varepsilon B_1(a,\theta) + \varepsilon^2 B_2(a,\theta) + \ldots \qquad (1.49)$$

$$\psi = \varphi - \theta = \omega t - \theta$$

Bojadziev [25], Bojadziev and Hung [26] used at least two trial solutions to investigate the time dependent differential systems; one is for resonant case and the other is for the non-resonant case. But Shamsul [113] used only one set of variational equations, arbitrarily for both resonant and non-resonant cases.

Shamsul [113] investigated an n-th order time dependent differential equation

$$\frac{d^{(n)}x}{dt^{(n)}}+k_1\frac{d^{(n-1)}x}{dt^{(n-1)}}\ldots+k_nx=\varepsilon f(\nu t,x,\frac{dx}{dt},\ldots) \tag{1.50}$$

where $x^{(i)}$, $i=1,2,\ldots,n-1,n$ represent the i-th derivative, ε is a small parameter, k_j, $j=1,2,\ldots,n$ are constants, f is a nonlinear function and ν is the frequency of the external acting force. Shamsul [94] seeks an asymptotic solution of (1.49) in the form

$$x(\varepsilon,t)=\sum_{j=1}^{n}a_j(t)e^{\lambda_j t}+\varepsilon u_1(a_1,a_2,\ldots a_n)+\ldots+\varepsilon^m u_m(a_1,a_2,\ldots a_n) \tag{1.51}$$

where λ_j, $j = 1,2,\ldots,n$ are the eigenvalues of the unperturbed equation and each a_j satisfy a first order differential equation

$$\frac{da_j}{dt}=\lambda_j a_j+\varepsilon A_j(a_1,a_2,\ldots,a_n,t)+\ldots+\varepsilon^m p_j(a_1,a_2,\ldots,a_n,t) \tag{1.52}$$

For $\varepsilon=0$, Eq. (1.51) with Eq. (1.52) gives the solution of the unperturbed equation

$$x(t,0)=\sum_{j=1}^{n}a_{j,0}e^{\lambda_j t}, \tag{1.53}$$

where $a_{j,0}$, $j=1,2,\ldots,n$ are arbitrary constants. The proposed solution (1.51) is not chosen in a formal form of KBM method, but it can be easily brought to the formal form (1.50)-(1.53) by suitable variable transformations $a_{2l-1}(t)=1/2b_1(t)e^{i\varphi_1(t)}$ and $a_{2l}(t)=1/2b_1(t)e^{-i\varphi_1(t)}$, where $b_l(t)$ and $\varphi_l(t)$, $l=1,2,\ldots,n/2$ are amplitude and phase variables. It can be readily shown that solution (1.51) takes the form

$$\begin{aligned}x(\varepsilon,t)=&\sum_{l=1}^{n/2}1/2b_l(t)(e^{i\varphi_1(t)}+e^{-i\varphi_1(t)})+\varepsilon u_1(b_1,b_2,\ldots,b_{n/2},\varphi_1,\varphi_2,\ldots,\varphi_{n/2})\\&+\ldots+\varepsilon^m u_m(\ldots)\end{aligned} \tag{1.54}$$

and $b_l(t)$ and $\varphi_l(t)$ satisfy the equations

$$\begin{aligned}\frac{db_l}{dt} &= -\mu_l\, b_l + \varepsilon A_l(b_1, b_2, \ldots, b_{n/2}, \varphi_1, t) + \ldots + \varepsilon^n P_n(b_1, b_2, \ldots, b_{n/2}, \varphi_1, t) \\ \frac{d\varphi_l}{dt} &= \omega_l b_l + \varepsilon B_1(b_1, b_2, \ldots, b_{n/2}, \varphi_1, t) + \ldots + \varepsilon^n Q_n(b_1, b_2, \ldots, b_{n/2}, \varphi_1, t)\end{aligned} \qquad (1.55)$$

where $\lambda_{2l-1} = -\mu_l \pm i\omega_l$ are the eigenvalues of the equation (1.50) when $\varepsilon = 0$.

Pinakee Dey *et al* [71] found an asymptotic solution of a second order over-damped nonlinear non-autonomous differential system in presence of a slowly decaying external force. The authors [72] have developed an asymptotic method for time dependent nonlinear differential systems with varying coefficients, in which the coefficients change slowly and periodically with time. Further, the authors [74] have used the KBM method to find perturbation solutions of damped forced vibrations, in which coefficients change slowly with time. The authors [76] have found the second approximate solution of second order forced vibrations. Finally, the authors [75] have found the higher approximate solution of an *n*-th order damped forced vibrating problem in the resonance case, and investigated the stability of the stationary regime of vibrations.

2.2 The Proposal

We propose the perturbation systems governed by second and *n*th order nonlinear non-autonomous differential equations

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2x=\varepsilon\, f(\nu t,x,\frac{dx}{dt}) \qquad (1.56)$$

and

$$\frac{d^{(n)}x}{dt^{(n)}}+k_1\frac{d^{(n-1)}x}{dt^{(n-1)}}\cdots+k_nx=\varepsilon\, f(\nu t,x,\frac{dx}{dt},\ldots), \qquad (1.57)$$

where ε is a small positive parameter and f is a given nonlinear function.

In **Chapter 2,** a new asymptotic solution is introduced for second order time dependent over-damped nonlinear systems. An asymptotic method for second order time dependent nonlinear differential systems with varying coefficients is developed in **Chapter 3**. A perturbation solution of damped forced vibrations with slowly varying coefficients is presented in **Chapter 4**. In **Chapter 5,** a second approximate solution of second order time dependent weakly nonlinear systems has been studied. Finally, higher approximate solution of *n*-th order weakly nonlinear non-autonomous differential systems with damping has been examined in **Chapter 6.**

Chapter 2

An Asymptotic Method for Second Order Time Dependent Nonlinear Over Damped Systems

2.1 Introduction

For more than half a century there have been many analytical techniques developed for solving oscillations of nonlinear systems. It has been a research subject of intensive focus because many engineering oscillatory systems are very often governed by a system of nonlinear differential equations. Generally, these equations can be linearized by imposing certain restrictions and then they are solved in simple approaches. In vibrating processes many problems are solved by linearizing such differential equations when the amplitude of oscillations is small. The nonlinearity of the governing equations increases with the increasing of amplitude. When the amplitudes are not small enough, the linear solutions are not sufficient to describe the vibration. In such cases, the Krylov-Bogoliubov- Mitropolskii (KBM) [37,12,53] perturbation method is one of the most convenient and widely used technique to obtain asymptotic solutions of weakly nonlinear systems. The method was originally developed by Krylov and Bogoliubov [37] for obtaining periodic solution of a second order nonlinear differential equation. The method was amplified and justified by Bogoliubov and Mitropolskii [12]. Popov [78] extended the method to a damped oscillatory process in which a strong linear damping force acts. Murty *et al* [55] developed a method of variation of parameters to obtain the time response of a second order nonlinear over-damped system with a small nonlinearity based on the work of Krylov-Bogoliubov-Mitropolskii. Murty [57] has presented a unified

KBM method for solving second order nonlinear systems. Bojadziev [19] found a mono-frequent damped solution of an n-dimensional time dependent differential system with strong damping and small time delay. Murty, Dekshatulu and Krisna [56] extended the method to over-damped nonlinear systems. Sattar [84] has studied a third order over-damped nonlinear system. Shamsul and Sattar [86] developed a method to solve third order critically damped nonlinear equations. Shamsul and Sattar [87] has presented a unified KBM method for solving third order nonlinear systems. Shamsul [98,99] has presented a unified KBM method for solving an nth order nonlinear differential equation.

Shamsul [91,101,117] investigated over-damped nonlinear systems and found approximate solutions of *Duffing*'s equation when one root of the unperturbed equation was respectively double or triple of the other. Moreover, Shamsul [96-110] investigated some over damped system when the roots are approximately equal. Recently, Shamsul [107] has presented an approximate solution when one root becomes much smaller than the other. But Murty, Dekshatulu and Krisna [56] and Shamsul [91,101,117,98,115] limited their investigations to autonomous systems. The aim of this paper is to further extend the result in [107] to a similar nonlinear non-autonomous system in which a slowly decaying external force acts.

2.2 The Method

Consider a second order nonlinear non-autonomous differential equation with a slowly decaying external force, $Ee^{-\nu t}$,

$$\ddot{x} + 2k_1\dot{x} + k_2 x = \varepsilon f(x,\dot{x}) + \varepsilon E e^{-\nu t}, \tag{2.1}$$

where the over-dots denote differentiation with respect to t, k_1 and k_2 are positive constants, ε is a small parameter, f is the given nonlinear function and E, ν are constants, $\nu > 0$. When $k_1 > \sqrt{k_2}$ the characteristic roots of the linear equation of (2.1) are real and unequal say λ_1, λ_2 and $\lambda_2 < \lambda_1 < 0$. So that (2.1) represents an over-damped system. Therefore, the solution of the unperturbed equation of (2.1) becomes

$$x(t,0) = a_0 e^{\lambda_1 t} + b_0 e^{\lambda_2 t}, \tag{2.2}$$

where a_0 and b_0 are arbitrary constants. We investigate the above nonlinear system when the linear damping force $k_1 \dot{x}$, is large or very large, *i.e.*, $k_1 >> k_2$. We choose an approximate solution of (2.1) in the form of the asymptotic expansion

$$x(t,\varepsilon) = a(t)e^{\lambda_1 t} + b(t)e^{\lambda_2 t} + \varepsilon u_1(a,b,t) + \varepsilon^2 u_2(a,b,t) + \varepsilon^3 \ldots, \tag{2.3}$$

where a and b satisfy the differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,b,t) + \varepsilon^2 A_2(a,b,t) + \varepsilon^3 \ldots, \\ \dot{b} &= \varepsilon B_1(a,b,t) + \varepsilon^2 B_2(a,b,t) + \varepsilon^3 \ldots, \end{aligned} \tag{2.4}$$

Here solution (2.3) together with (2.4) is not considered in a usual form of the classical KBM method. But this solution was early introduced by Murty and Deekshatulu [55] to investigate an over-damped case of equation (2.1). Now it is being used to investigate various oscillatory and non-oscillatory problems (see [91,101,117,98,115] for details).

Confining only to the first few terms, 1,2,…,*m*, in the series expansions of (2.3) and (2.4), we evaluate the functions $u_1, u_2, \ldots,$ and $A_1, A_2, \ldots, B_1, B_2, \ldots,$ such that $a(t)$ and $b(t)$ appearing in (2.3) and (2.4) satisfy the given differential equation (2.1) with an accuracy of ε^{m+1}. Theoretically, the solution can be obtained up to the accuracy of any approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is, in general, confined to a lower order, usually the first [115]. In order to determine these unknown functions it is assumed that the functions $u_1, u_2, \ldots$ do not contain secular-type term te^{-t}(see [91,101,117,107] for details).

Differentiating $x(t,\varepsilon)$ twice with respect to t, substituting the derivatives, $\dot{x}, \ddot{x}$ and $x(t,\varepsilon)$ in the original equation (2.1) and equating the coefficient of ε, we obtain

$$\begin{aligned} & e^{\lambda_1 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)A_1+e^{\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_1+\lambda_2\right)A_2+\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1 \\ & =f^{(0)}(a,b,t)+Ee^{-vt} \end{aligned} \tag{2.5}$$

where $f^{(0)}=f(x_0,\dot{x}_0)$ and $x_0=a(t)e^{\lambda_1 t}+b(t)e^{\lambda_2 t}$.

In general $f^{(0)}$ can be expanded in Taylor series as

$$f^{(0)}=\sum_{r_1=0,r_2=0} F_{r_1,r_2}(a,b)e^{(r_1\lambda_1+r_2\lambda_2)t} \tag{2.6}$$

It was early imposed by Krylov and Bogoliubov [37] that u_1 does not contain secular terms (*e.g.*, $t\cos t$ and $t\sin t$) for obtaining the periodic solution of (2.1) in which $k_1=0$. Popov [78] extended this method to an under-damped case in which $\sqrt{k_2}>k_1>0$. Murty, Deekshatulu and Krisna [56] extended the same method to the over-damped case. *i.e.*, for $k_1>\sqrt{k_2}$. But Murty, Deekshatulu and Krisna's [56] solution gives incorrect result when one root is multiple of the other (see [91,101,117] for details), or one root becomes much smaller than the other [107]. In these situation Shamsul [91,101,117,107] has determined some special type over-damped solutions, subject to the condition that u_1 excludes the term $e^{(r_1\lambda_1+r_2\lambda_2)t}$ of $f^{(0)}$ where $(r_1\lambda_1+r_2\lambda_2)>k(r_1+r_2)$. This assumption assures that u_1 does not contain secular type term te^{-t}(see [91,101,117,107). Under this assumption we are able to find the unknown function u_1 and A_1, B_1 which complete the determination of the first approximate solution of (2.1).

2.3 Example

Let us consider a *Duffing* equation with a slowly decaying external force,

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon x^3 + \varepsilon E e^{-\nu t}, \tag{2.7}$$

Here over dots denote differentiation with respect to t, k and ω are positive constants, E, ν are constants and $\nu > 0$. Here the damping force $2k\dot{x}$ is large, *i.e*, $k >> 1$. When $\varepsilon = 0$, the characteristic equation of (2.1) has two roots λ_1 and λ_2, $\lambda_1 + \lambda_2 = 2k$ and $\lambda_1\lambda_2 = \omega^2$. We obtain $\lambda_1 < k < \omega^2$ and $\lambda_2 > k > \omega^2$, we may consider that $|\lambda_2| >> |\lambda_1|$, $\lambda_1 \cong -\nu$.

The function $f^{(0)}$ for (2.7) becomes

$$f^{(0)} = -\varepsilon(a_1^3 e^{3\lambda_1 t} + 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + a_2^3 e^{3\lambda_2 t}) + \varepsilon E e^{-\nu t} \tag{2.8}$$

According to the restriction imposed by Shamsul [91,101,117] (already denoted in sec 2.2) u_1 excludes the terms $e^{3\lambda_1 t}, e^{(2\lambda_1+\lambda_2)t}$ and $Ee^{-\nu t}$. We substitute $f^{(0)}$ in (2.5) and separate it into two parts as

$$\begin{aligned} &e^{\lambda_1 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)A_1 + e^{\lambda_2 t}\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right)A_2 \\ &= -\varepsilon a_1^3 e^{3\lambda_1 t} - \varepsilon 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + \varepsilon E e^{-\nu t} \end{aligned} \tag{2.9}$$

and

$$\left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)u_1 = -(3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + a_2^3 e^{3\lambda_2 t}). \tag{2.10}$$

The particular solution of (2.10) is

$$u_1 = c_1 a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + c_2 a_2^3 e^{3\lambda_2 t}, \tag{2.11}$$

where $c_1 = \dfrac{-3}{2\lambda_2(\lambda_1+\lambda_2)}$, $c_2 = \dfrac{-1}{2\lambda_2(3\lambda_2-\lambda_1)}$

Now, we have to determine two functions A_1 and A_2 from a single equation (2.10).

In this paper, we have already considered that the damping force is large or very large and one root becomes much smaller than the other, so that $\lambda_2 + 2\lambda_1 \approx \lambda_2$. Therefore, we can equate the coefficient of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ from both sides of (2.9) (according to Shamsul [107]) and obtain the following equations

$$\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right) A_1 = -a_1^3 e^{2\lambda_1 t} + E e^{(-\nu-\lambda_1)t} \tag{2.12}$$

$$\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right) A_2 = -3a_1^2 a_2 e^{2\lambda_1 t} \tag{2.13}$$

The particular solutions of (2.12)-(2.13) are

$$A_1 = n_1 a_1^3 e^{2\lambda_1 t} + E n_2,$$

and

$$A_2 = l_1 a_1^2 a_2 e^{2\lambda_1 t} \tag{2.14}$$

where $n_1 = \dfrac{-1}{3\lambda_1 - \lambda_2}$, $n_2 = \dfrac{1}{\lambda_1 - \lambda_2}$, $l_1 = \dfrac{-3}{\lambda_1 + \lambda_2}$

Therefore, the first approximate solution of (2.8) is

$$x(t,\varepsilon) = a_1(t) e^{\lambda_1 t} + a_2(t) e^{\lambda_2 t} + \varepsilon u_1 \tag{2.15}$$

where a_1 and a_2 are solutions of

$$\dot{a}_1 = \varepsilon(n_1 a_1^3 e^{2\lambda_1 t} + E n_2), \quad \dot{a}_2 = \varepsilon l_1 a_1^2 a_2 e^{2\lambda_1 t} \tag{2.16}$$

and u_1 is given by (2.11).

2.4 Results and Discussions

An asymptotic solution of a second-order damped nonlinear non-autonomous system is obtained based on the KBM method in which a slowly decaying external force acts. In order to test the accuracy of the approximate solution obtained by a perturbation method, we compare the approximate solution to the numerical solution. With regard to such a comparison concerning the presented KBM method of this paper, we refer to the works of Murty, Dekshatulu and Krishna [56] and Shamsul [91,101,117,107]. In this paper, we have compared the perturbation solution (2.15) to those obtained by the Runge-Kutta (Fourth order) method for $\lambda_1 = -.05, \lambda_2 = -4, a_1 = 1, a_2 = 0, \varepsilon = 0.1, E = 3$ with initial conditions $x(0) = 1.0,\ \dot{x}(0) = .000025$ and all the results are presented in Table 2.1.

Table 2.1

t	x	x_{nu}	$E\%$
0.0	1.000000	1.000000	0.0000
0.5	0.999527	0.999795	0.0268
1.0	0.998187	0.998872	0.0685
1.5	0.996054	0.997138	0.1087
2.0	0.9932	0.994644	0.1451
2.5	0.989689	0.991456	0.1782
3.0	0.985581	0.987634	0.2078
4.0	0.975791	0.978322	0.2587

5.0	0.96422	0.96712	0.2998
7.0	0.937006	0.940387	0.3595
10.	0.889547	0.893214	0.4105
20.	0.714881	0.718018	0.4368
30.	0.552333	0.554546	0.3990
40.	0.415312	0.416776	0.3512
50.	0.305002	0.305937	0.3056
60.	0.219255	0.219839	0.2656
70.	0.154635	0.154996	0.2329
80.	0.107261	0.107482	0.2056
90.	0.073350	0.073484	0.1823
100.	0.049561	0.049642	0.1631

From the **Table 2.1**, it is clear that percentage errors are much smaller than 1% and thus (2.15) shows a good coincidence with the numerical solution. In general, the equation (2.16) has no exact solution. Usually, a numerical procedure is used to solve it. In this paper, we have used the *Runge-Kutta* (Fourth order) method. Numerically, it is advantageous to solve the transformed equation (2.16) instead of the original equation (2.7) because a large step size can be used in the integration (see [61] for details).

2.5 Conclusion

An asymptotic solution has been obtained for the second order nonlinear non-autonomous differential system characterized by non-oscillatory process. The method is a generalization of KBM method [37,12] and can be used to obtain the desired solution for certain non-periodic external forces. The solution shows a good coincidence with the numerical solution.

Chapter 3

An Asymptotic Method for Second Order Time Dependent Nonlinear Systems with Varying Coefficients

3.1 Introduction

Most of the well-known perturbation methods (*e.g.*, Poincare method [77], WKB method [123,35,10], Multi time-scale method [32,60] or Krylov-Bogoliubov-Mitropolskii (KBM) method [37,12,53]) were developed to find periodic solutions of nonlinear differential systems with constant and slowly varying coefficients. Among the above methods the KBM method is convenient and widely used. Krylov and Bogoliubov [37] originally developed a perturbation method to obtain an approximate solution of a second order nonlinear differential system described by

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}) \tag{3.1}$$

where the over dots denote the differentiation with respect to t, ω_0 is a positive constant and ε is a small parameter. Then the method was amplified and justified by Bogoliubov and Mitropolskii [12]. Mitropolskii [53] has extended the method to nonlinear differential system with slowly varying coefficients as

$$\ddot{x} + \omega_0^2(\tau).x = -\varepsilon f(x, \dot{x}, \tau), \qquad \tau = \varepsilon t \tag{3.2}$$

Following the extended Krylov-Bogoliubov-Mitropolskii (KBM) method [37,12,53]), Bojadziev and Edwards [24] studied some damped oscillatory and purely non-oscillatory systems with slowly varying coefficients, modeled by

$$\ddot{x} + c(\tau).\dot{x} + \omega^2(\tau).x = -\varepsilon f(x, \dot{x}, \tau) \tag{3.3}$$

where $c(\tau)$ and $\omega(\tau)$ are positive. Murty [57] has presented a unified KBM method for both under-damped and over-damped systems with constant coefficients. Shamsul [108] has presented a unified formula to obtain a general solution of an n-th order ordinary differential equation with constant and slowly varying coefficients. Hung and Wu [34] obtained an exact solution of a differential system in terms of Bessel's functions where the coefficients varying with time in an exponential order. Recently, Roy and Shamsul [82] found an asymptotic solution of a differential system in which the coefficient changes in an exponential order of slowly varying time. The aim of this article is to extend the work of Roy and Shamsul [82] to similar nonlinear problems in which the coefficients change slowly and periodically with time. Such problems arise in different branches of engineering, *e.g.*, rotor with slowly and periodically changing mass.

3.2 The Method

Let us consider the nonlinear differential system

$$\ddot{x} + (k_1^2 + k_2 \sin\tau)x = -\varepsilon f(x, \tau), \qquad \tau = \varepsilon t \tag{3.4}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, k_1, k_2 are constants, $k_2 = O(\varepsilon)$, and f is a given nonlinear function. We assume that $\omega^2(\tau) = (k_1^2 + k_2 \sin\tau)$, where $\omega(\tau)$ is known as frequency.

For $\varepsilon = 0$ and $\tau = \tau_0 =$ constant, we find that $\lambda_1(\tau_0) = i\omega(\tau_0)$, $\lambda_2(\tau_0) = -i\omega(\tau_0)$ are two eigen values of the unperturbed equation of (3.4) and has the solution

$$x(t,0) = a_{1,0}e^{\lambda_1(\tau_0)t} + a_{2,0}e^{\lambda_2(\tau_0)t} \tag{3.5}$$

When $\varepsilon \neq 0$, we seek a solution in accordance with the KBM method, of the form

$$x(t,\varepsilon) = a_1(t,\tau) + a_2(t,\tau) + \varepsilon u_1(a_1,a_2,\tau) + \varepsilon^2 \ldots \tag{3.6}$$

where a_1 and a_2 satisfy the equations

$$\begin{aligned} \dot{a}_1 &= \lambda_1(\tau)a_1 + \varepsilon A_1(a_1,a_2,\tau) + \varepsilon^2 \ldots \\ \dot{a}_2 &= \lambda_2(\tau)a_2 + \varepsilon A_2(a_1,a_2,\tau) + \varepsilon^2 \ldots \end{aligned} \tag{3.7}$$

Confining our attention to the first few terms $1, 2, \ldots, m$ in the series expansions of (3.6) and (3.7), we evaluate functions $u_1, \ldots, A_1, A_2 \ldots,$ such that a_1 and a_2 appearing in (3.6) and (3.7) satisfy (3.4) with an accuracy of ε^{m+1}. In order to determine these unknown functions it was early assumed by Murty [57] and Shamsul [109] that the functions $u_1, \ldots$ exclude all fundamental terms, since these are included in the series expansion (3.6) at order ε^0.

Now differentiating (3.6) twice with respect to t, substituting for the derivatives $\ddot{x}$ and x in (3.4), utilizing relation (3.7) and comparing the coefficient of ε, we obtain

$$\begin{aligned} &\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)A_1 + \lambda_1' a_1 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)A_2 + \lambda_2' a_2 \\ &+ \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)u_1 = -f^{(0)}(a_1,a_2,\tau) \end{aligned} \tag{3.8}$$

where $\lambda_1' = \dfrac{d\lambda_1}{d\tau}$, $\lambda_2' = \dfrac{d\lambda_2}{d\tau}$, $f^{(0)} = f(x_0,\tau)$ and $x_0 = a_1 + a_2$.

It is assumed that $f^{(0)}$ can be expanded in Taylor's series [57,109] as

$$f^{(0)} = \sum_{r_1,r_2=0}^{\infty} F_{r_1,r_2}(\tau) a_1^{r_1} a_2^{r_2} \tag{3.9}$$

We have assumed that u_1 does not contain fundamental terms and for this reason the solution will be free from secular terms, namely $t\cos t$, $t\sin t$ and te^{-t} (see Shamsul [109]). To obtain this solution (4.4), it has been proposed in Shamsul [109] that $u_1,\ldots,$ exclude the terms $a_1^{r_1}a_2^{r_2}$ of $f^{(0)}$, where $r_1 - r_2 = \pm 1$. This restriction guarantees that the solution always excludes *secular*-type terms or first harmonic terms and the KBM solution becomes uniformly valid KBM solution [37,12,53]. Moreover, we assume that A_1 and A_2 respectively contain terms a_1 and a_2. We have already mentioned that equation (3.4) is not a standard form of KBM method. We shall be able to transform (3.6) to the exact formal KBM [37,12,53] solution by substituting $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$. Herein a and φ are respectively amplitude and phase variables (see [26,113]).

3.3 Example

3.3.1 A nonlinear problem in absence of an external force

We consider a second order nonlinear system with constant and slowly varying coefficient

$$\ddot{x} + (k_1^2 + k_2 \sin\tau)x = -\varepsilon x^3 \tag{3.10}$$

Here over dots denote differentiation with respect to t, k_1, k_2 are constants, $k_2 = O(\varepsilon)$, $x_0 = a_1 + a_2$ and the function $f^{(0)}$ becomes

$$f^{(0)} = -(a_1^3 + 3a_1^2a_2 + 3a_1a_2^2 + a_2^3) \tag{3.11}$$

Following the assumption (discussed in Section 3.2) u_1 excludes the terms $3a_1^2a_2$ and $3a_1a_2^2$

We substitute (3.11) in (3.8) and separate it into two parts as

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_1 + \lambda_1' a_1 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 = -(3a_1^2 a_2 + 3a_1 a_2^2) \tag{3.12}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) u_1 = -(a_1^3 + a_2^3) \tag{3.13}$$

The particular solution of (3.13) is

$$u_1 = -\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{3.14}$$

Now we have to solve (3.12) for two functions A_1 and A_2. According to the unified KBM method A_1 contains the term $3a_1^2 a_2$ and A_2 contains the term $3a_1 a_2^2$ (Shamsul [109]) to obtain the following equations

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_1 + \lambda_1' a_1 = -3a_1^2 a_2 \tag{3.15}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 = -3a_1 a_2^2 \tag{3.16}$$

The particular solutions of (3.15) and (3.16) are

$$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1}$$

and

$$A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} \tag{3.17}$$

Substituting the functional values of A_1 and A_2 from (3.17) into (3.7) and rearranging, we obtain

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1}\right)$$

and (3.18)

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(\frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2}\right)$$

Under the transformations, $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$ together with $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$ and the use of $A_1 e^{-i\varphi} + A_2 e^{i\varphi} = \widetilde{A}_1$ and $-i(A_1 e^{-i\varphi} - A_2 e^{i\varphi}) = a\widetilde{B}_1$ (where $\widetilde{A}_1$ and $\widetilde{B}_1$ are usual notations) equations (3.18) reduce to

$$\dot{a} = \varepsilon \widetilde{A}_1(a) + \varepsilon^2 \ldots \text{and } \dot{\varphi} = \omega + \varepsilon \widetilde{B}_1(a) + \varepsilon^2 \ldots \tag{3.19}$$

We shall obtain the variational equations of a and φ in the real form (a and φ are know as amplitude and phase) which transform (3.18) to

$$\dot{a} = -\frac{\varepsilon a \omega'}{2\omega} \tag{3.20}$$

and

$$\dot{\varphi} = \omega + \frac{3\varepsilon a^2}{8\omega}, \tag{3.21}$$

where $\omega = \sqrt{k_1^2 + k_2 \sin\tau}$

The variational equations (3.20) and (3.21) are a form of the KBM solution. The variational equations for amplitude and phase usually appear in a set of first order differential equations and are solved by the numerical technique (see Shamsul [109]).

Therefore, the first approximate solution of the equation (3.10) is

$$x(t,\varepsilon) = a\cos\varphi + \varepsilon u_1 \tag{3.22}$$

where a and φ are the solutions of the equations (3.20) and (3.21) respectively.

3.3.2 Let us consider another form of the nonlinear differential problem (3.10)

$$\ddot{x} + k_1^2 x = -k_2 \sin\tau\, x - \varepsilon x^3 = -\varepsilon\, k \sin\tau\, x - \varepsilon x^3\,, \tag{3.23}$$

where $k_2 = \varepsilon\, k$ and $k_1^2 = \omega^2$. Here,

$$f^{(0)} = -(\alpha_1^3 + 3\alpha_1^2\alpha_2 + 3\alpha_1\alpha_2^2 + \alpha_2^3) - k\sin\tau(\alpha_1 + \alpha_2)\ . \tag{3.24}$$

In our assumption, u_1 excludes the terms $3\alpha_1^2\alpha_2, 3\alpha_1\alpha_2^2$ and $k\sin\tau(\alpha_1+\alpha_2)$. The equations of u_1, A_1 and A_2 become (discussed in Section 3.2)

$$\left(\lambda_1\alpha_1\frac{\partial}{\partial\alpha_1} + \lambda_2\alpha_2\frac{\partial}{\partial\alpha_2} - \lambda_1\right)\left(\lambda_1\alpha_1\frac{\partial}{\partial\alpha_1} + \lambda_2\alpha_2\frac{\partial}{\partial\alpha_2} - \lambda_2\right)u_1 = -(\alpha_1^3 + \alpha_2^3) \tag{3.25}$$

and

$$\begin{aligned}\left(\lambda_1\alpha_1\frac{\partial}{\partial\alpha_1} + \lambda_2\alpha_2\frac{\partial}{\partial\alpha_2} - \lambda_2\right)A_1 &= -3\alpha_1^2\alpha_2 - k\alpha_1\sin\tau,\\ \left(\lambda_1\alpha_1\frac{\partial}{\partial\alpha_1} + \lambda_2\alpha_2\frac{\partial}{\partial\alpha_2} - \lambda_1\right)A_2 &= -3\alpha_1\alpha_2^2 - k\alpha_2\sin\tau\end{aligned} \tag{3.26}$$

Solutions of Eqs. (3.25)-(3.26) are

$$u_1 = -\frac{\alpha_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{\alpha_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)}\,, \tag{3.27}$$

and

$$\begin{aligned}A_1 &= -\frac{3\alpha_1^2\alpha_2}{2\lambda_1} - \frac{k\alpha_1\sin\tau}{\lambda_1 - \lambda_2}\\ A_2 &= -\frac{3\alpha_1\alpha_2^2}{2\lambda_2} + \frac{k\alpha_2\sin\tau}{\lambda_1 - \lambda_2}\end{aligned} \tag{3.28}$$

Substituting the functional values of A_1 and A_2 from (3.28) into (3.7) and rearranging, we obtain

and
$$\dot{\alpha}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{3\alpha_1^2\alpha_2}{2\lambda_1} - \frac{k\alpha_1 \sin\tau}{\lambda_1 - \lambda_2}\right)$$
$$\dot{\alpha}_2 = \lambda_2 a_2 + \varepsilon\left(-\frac{3\alpha_1\alpha_2^2}{2\lambda_2} + \frac{k\alpha_2 \sin\tau}{\lambda_1 - \lambda_2}\right) \qquad (3.29)$$

Under the transformations $\alpha_1 = \alpha e^{i\varphi}/2$ and $\alpha_2 = \alpha e^{-i\varphi}/2$, and substitutions $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, we shall obtain the variational equations of α and φ in the real form (α and φ are know as amplitude and phase) which transform (3.28) to

and
$$\dot{\alpha} = 0$$
$$\dot{\varphi} = \omega + \frac{3\varepsilon\alpha^2}{8\omega} + \frac{\varepsilon k \sin\tau}{2\omega}, \qquad (3.30)$$

where $\omega^2 = k_1^2$.

Therefore, the first approximate solution of the equation (3.23) is

$$x(t,\varepsilon) = a\cos\varphi + \varepsilon u_1 \qquad (3.31)$$

where a and φ are the solutions of the equation (3.30).

3.4 Non-linear System with an External Force

The method is used to similar nonlinear differential system with an external force $E\sin\nu t$,

$$\ddot{x} + (k_1^2 + k_2 \sin\tau)x = -\varepsilon f(x,\tau) + \varepsilon E \sin\nu t, \qquad \tau = \varepsilon t \qquad (3.32)$$

where ν is the frequency of the external force.

3.4.1 Let us consider a second order nonlinear differential system with an external force

$$\ddot{x} + (k_1^2 + k_2 \sin\tau)x = -\varepsilon x^3 + \varepsilon E \sin \nu t \tag{3.33}$$

Here over dots denote differentiation with respect to t; k_1, k_2 are constants, $k_2 = O(\varepsilon)$, $x_0 = a_1 + a_2$ and the function

$$f^{(0)} = -(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3) + \frac{E}{2i}\left(e^{i\nu t} - e^{-i\nu t}\right) \tag{3.34}$$

Under the restriction (discussed in Section 3.2) u_1 excludes the terms $3a_1^2 a_2$ .and $3a_1 a_2^2$. Moreover, in our assumption, u_1 excludes $\varepsilon E\left(e^{i\nu t} - e^{-i\nu t}\right)/(2i)$. We substitute (3.34) in (3.8) and separate it into two parts as

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)A_1 + \lambda_1' a_1 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)A_2 + \lambda_2' a_2$$
$$= -(3a_1^2 a_2 + 3a_1 a_2^2) + \frac{\varepsilon E}{2i}(e^{i\nu t} - e^{-i\nu t}) \tag{3.35}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)u_1 = -(a_1^3 + a_2^3) \tag{3.36}$$

The particular solution of (3.35) is

$$u_1 = -\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{3.37}$$

Now we have to solve (3.34) for two functions A_1 and A_2. According to the unified KBM method A_1 contains the terms $3a_1^2 a_2$ and $Ee^{i\nu t}/(2i)$ and A_2 contains the terms $3a_1 a_2^2$ and $Ee^{-i\nu t}/(2i)$ (see [109]) and thus we obtain the following equations

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_1 + \lambda_1' a_1 = -3a_1^2 a_2 + \frac{E}{2i} e^{i\nu t} \tag{3.38}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 = -3a_1 a_2^2 - \frac{E}{2i} e^{-i\nu t} \tag{3.39}$$

The particular solutions of (3.37) and (3.38) are

$$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1} - \frac{Ee^{i\nu t}}{2(\nu + \omega)}$$

and (3.40)

$$A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} - \frac{Ee^{-i\nu t}}{2(\nu + \omega)}$$

Substituting the functional values of A_1 and A_2 from (3.39) into (3.7) and rearranging, we obtain (See sub-section 3.3.1)

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1} - \frac{Ee^{i\nu t}}{2(\nu + \omega)}\right)$$

and (3.41)

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(\frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} - \frac{Ee^{-i\nu t}}{2(\nu + \omega)}\right).$$

The variational equations of a and φ in the real form (a and φ are know as amplitude and phase) transform (3.40) to

$$\dot{a} = -\frac{\varepsilon a \omega'}{2\omega} - \frac{\varepsilon E \cos(\varphi - \nu t)}{\nu + \omega}$$

and (3.42)

$$\dot{\varphi} = \omega + \frac{3\varepsilon a^2}{8\omega} + \frac{\varepsilon E \sin(\varphi - \nu t)}{a(\nu + \omega)},$$

where $\omega = \sqrt{k_1^2 + k_2 \sin\tau}$

Equations (3.42) are similar to that obtained by the KBM method (see [26,113]).

Therefore, the first approximate solution of the equation (3.33) is

$$x(t,\varepsilon) = a\cos\varphi + \varepsilon u_1, \tag{3.43}$$

where a and φ are the solutions of the equation (3.42).

3.5 Results and Discussions

An approximate solution of second-order time dependent nonlinear differential systems with constant and varying coefficients has been obtained based on the KBM [37,12,53] method. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexities for the derivation of the function, the solution is, in general, confined to a lower order, usually the first. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one can easily compare the approximate solution to the numerical solution (considered to be exact). Due to such a comparison concerning the presented KBM method of this paper, we refer to the works of Murty [57], and Shamsul [109, 82,94] where asymptotic solutions have been compared to the corresponding numerical solutions. In this article we have also compared the perturbation solutions (3.22), (3.31) and (3.43) of *Duffing's* equation (3.10), (3.23) and (3.33) to the numerical solutions obtained by Runge-Kutta (Fourth-order) procedure.

First of all, $x(t,\varepsilon)$ has been computed by perturbation solution (3.22) with initial conditions $[x(0)=1,\ \dot{x}(0)=0]$ or $a=1.00000$, $\varphi=-.001434$ for $\varepsilon=.05$. The corresponding numerical solution has been also computed by the Fourth order Runge-Kutta method. All the results are shown in Fig.3.1. From Fig.3.1 it is clear that the asymptotic solution (3.22) shows a good agreement with the numerical solution of equation (3.10).

We have found the approximate solution of the same problem utilizing the classical KBM method [37,12] (see Sub-section 3.3.2) with initial conditions $[x(0)=1, \dot{x}(0)=0]$ or $\alpha=1.$, $\varphi=0$ and $\varepsilon=.05$ and presented in Fig.3.2. From the graph it is clear that the perturbation solution (3.31) does not agree with the numerical solution after a short time interval. Thus the extended KBM method presented here is better than the classical KBM method.

In Section 3.4.1, a perturbation solution (3.43) has been derived when an external force acts and the solution has been presented in Fig.3.3 for $\varepsilon=.05$, $\nu=1.1$, $E=.5$ with initial conditions$[x(0)=1, \dot{x}(0)=0]$ or $a=1.00534$, $\varphi=.103118$. This solution also shows a good coincidence with the numerical solution.

3.6. Conclusion

A new asymptotic method for a second order nonlinear deferential system with slowly varying coefficients has been found. This method is a generalization of the KBM method. This improved method gives better results than the previous KBM method. The solution for different initial conditions shows good coincidence with the corresponding numerical solution.

Fig 3.1

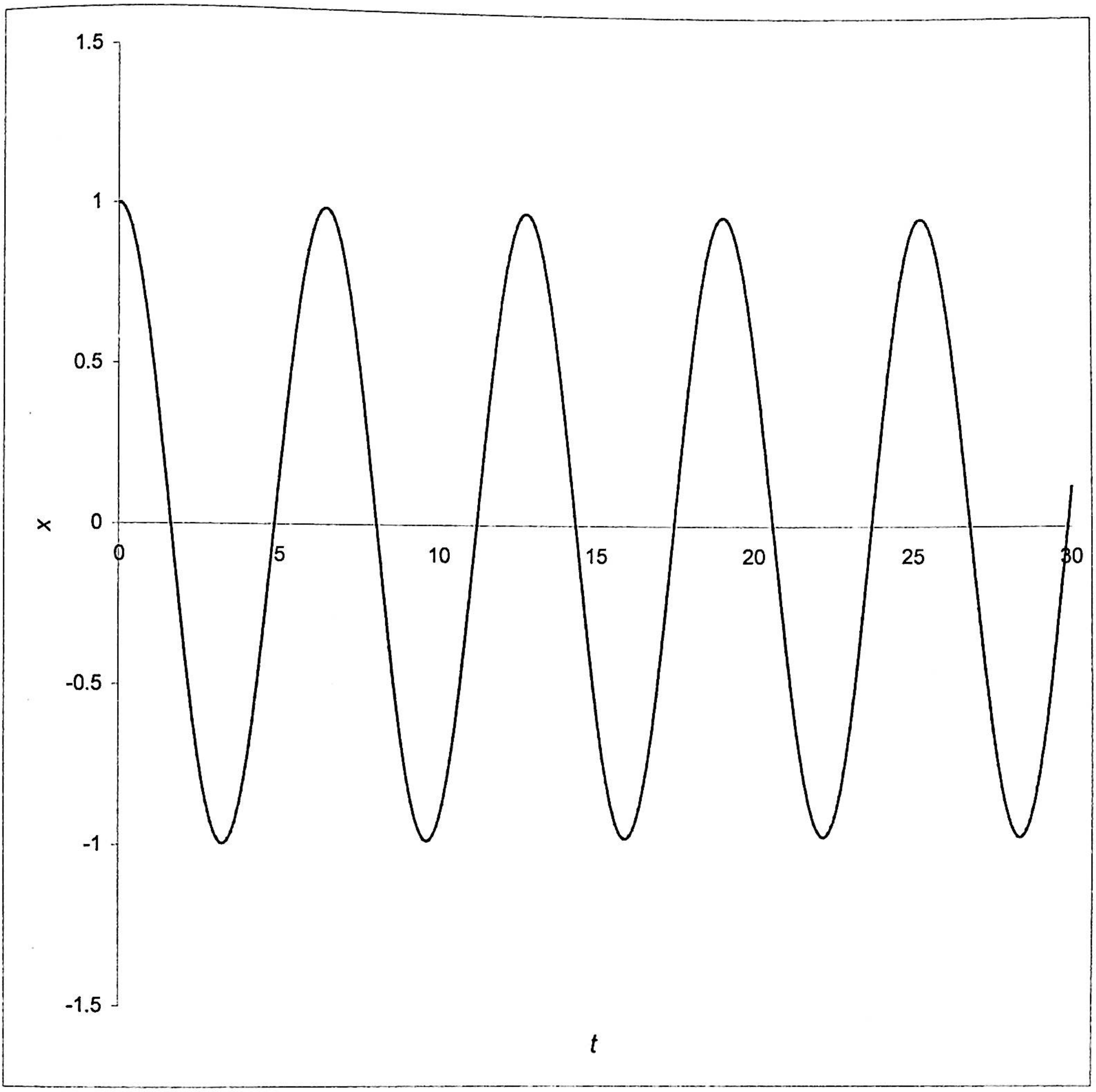

Fig 3.1: Approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when $\nu = 1.1$ together with initial conditions $a = 1$, $\varphi = -.001434$ $[x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ and $e = .05$

Fig 3.2

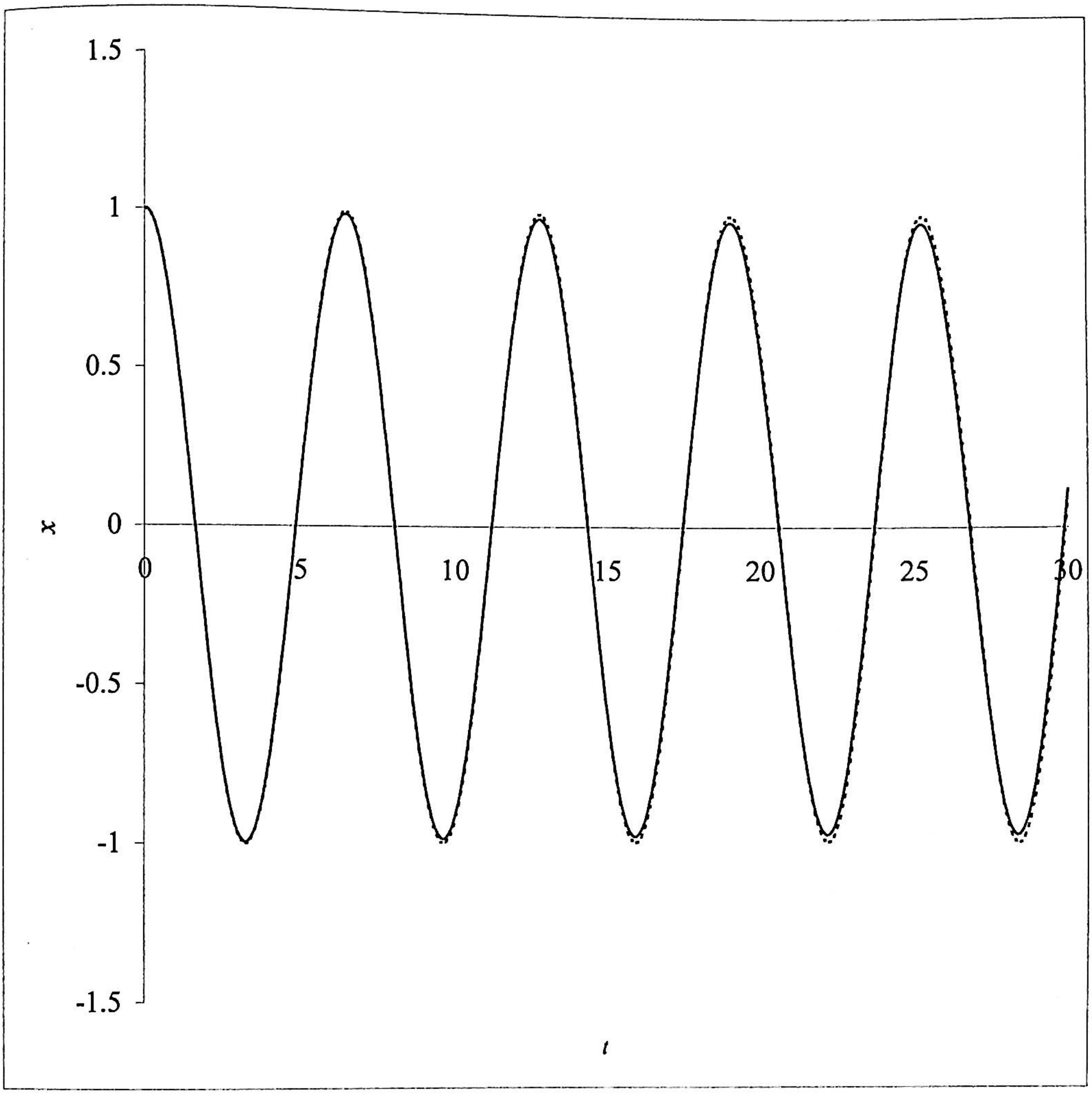

Fig.3.2: Approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when $\nu = 1.1$ together with initial conditions $a = 1, \varphi = .0$ $[x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ and $e = .05$

Fig 3.3

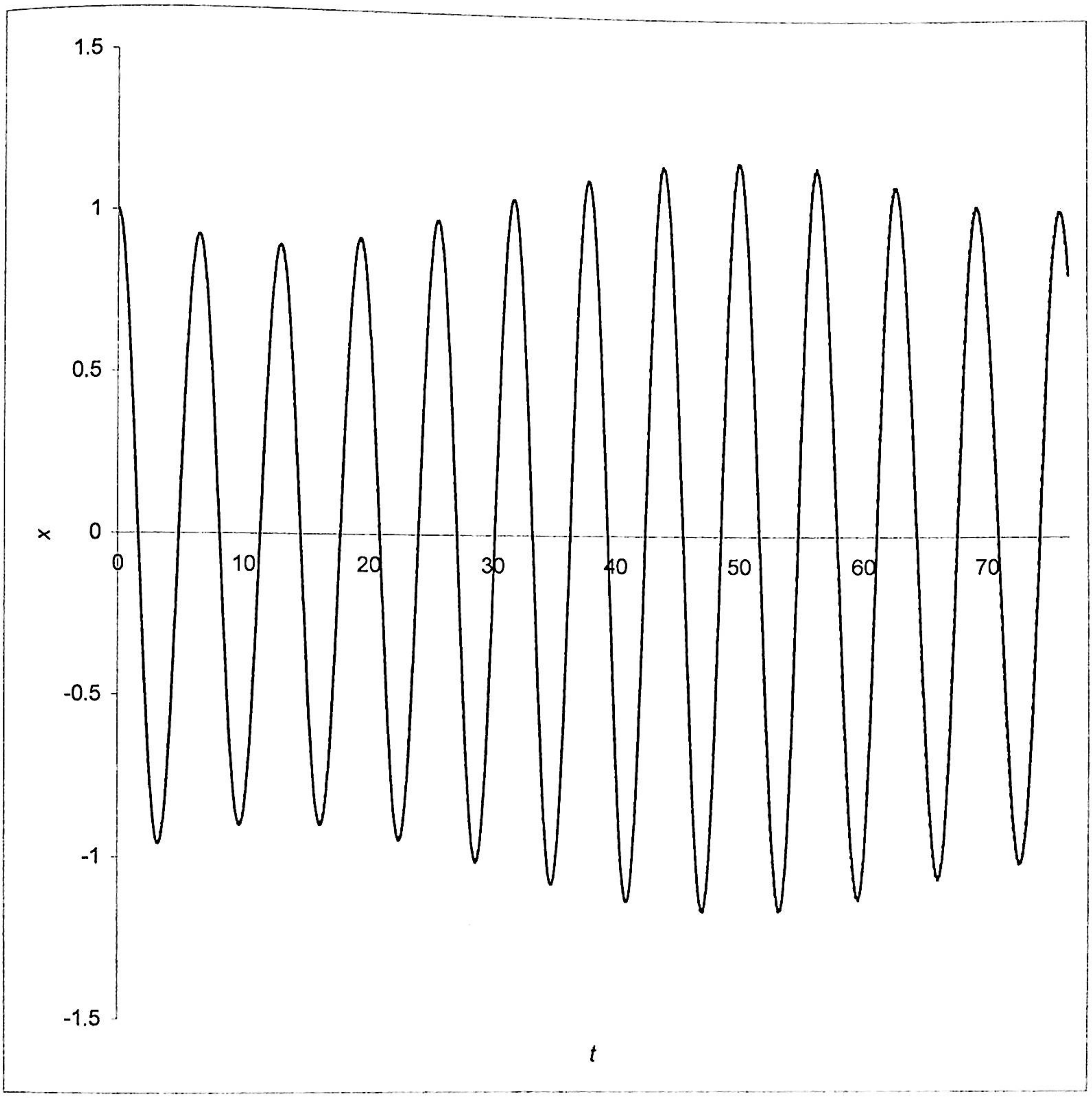

Fig.3.3: Approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when $\nu = 1.1$ together with initial conditions $a = 1.005340$, $\varphi = .103118$ [$x(0) = 1.00000$, $\dot{x}(0) = 0.00000$] and $e = .05$ $E = .5$

Chapter 4

Perturbation Theory for Second Order Time Dependent Damped Forced Vibrations with Slowly Varying Coefficients

4.1 Introduction

An important approach to study nonlinear vibrating processes is the small parameter expansion on which the perturbation theory is based. One widely spread method of this theory, mainly used in literature, is the averaging asymptotic method of Krylov-Bogoliubov-Mitropolskii (KBM) [37,12,53]. Krylov-Bogoliubov-Mitropolskii (KBM) [37,12,53]) developed an asymptotic method to find periodic solutions of nonlinear differential systems with constant and slowly varying coefficients. Krylov and Bogoliubov [37] originally developed a perturbation method to obtain an approximate solution of a second order nonlinear differential system. Then the method was amplified and justified by Bogoliubov and Mitropolskii [12]. Mitropolskii [53] has first used asymptotic method to investigate non-stationary solution of the second order nonlinear differential system with slowly varying coefficients. Following the extended Krylov-Bogoliubov-Mitropolskii (KBM) method, Bojadziev and Edwards [24] studied some damped oscillatory and purely non-oscillatory systems with slowly varying coefficients. Feshchenko, Shkil and Nikolenko [32] have used an asymptotic method to linear differential equations with slowly varying coefficients. Arya and Bojadziev [8] have studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Arya and Bojadziev [7] have also studied a system of second order nonlinear hyperbolic differential equation with slowly varying coefficients. Murty [57] has presented a unified KBM method for both under-damped and over-damped systems with constant coefficients. Hung and Wu [34] obtained an exact solution of a differential system in terms of Bessel's functions, where the coefficients varying with time in an exponential order. Recently, Shamsul [109] has presented a unified formula to obtain a general solution of an n-th order ordinary differential equation with constant and slowly varying coefficients. But Murty [57] and Shamsul [109] limited their investigations to autonomous systems. The aim of this article is to find the approximate solution of second order time dependent non-linear vibrating problems with damping, external forces with slowly varying coefficients.

4.2 The Method

Let us consider the nonlinear differential system

$$\ddot{x} + 2k(\tau)\dot{x} + \omega^2(\tau)x = -\varepsilon f(x, \dot{x}, \tau), \quad \tau = \varepsilon t, \tag{4.1}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, $\tau = \varepsilon t$ is the slowly varying time, $k(\tau) \geq 0$, f is a given nonlinear function and $\omega(\tau)$ is the frequency. The coefficients in Eq. (4.1) are slowly varying in the sense that their time derivatives are proportional to ε [53].

Setting $\varepsilon = 0$ and $\tau = \tau_0$=constant, in Eq.(4.1), we obtain the unperturbed solution of the equation. Let Eq.(4.1) has two eigenvalues $\lambda_j(\tau_0)$, $j = 1,2$, where $\lambda_j(\tau_0)$ are constant, but when $\varepsilon \neq 0$, $\lambda_j(\tau)$ slowly vary with time. The unperturbed solution of Eq. (5.1) becomes

$$x(t,0)=\sum_{j=0}^{2}a_{j,0}e^{\lambda_j(\tau_0)t}\ . \tag{4.2}$$

When $\varepsilon \neq 0$, we seek a solution, in accordance with the KBM method, of the form

$$x(t,\varepsilon)=\sum_{j=1}^{2}a_{j,0}(t,\tau)+\varepsilon u_1(a_1,a_2,\tau)+\varepsilon^2 u_2(a_1,a_2,\tau)+\ldots, \tag{4.3}$$

where $a_{j,0}$, $j=1,2$ satisfy the differential equations

$$\dot{a}_j=\lambda_j(\tau)a_j+\varepsilon A_j(a_1,a_2,\tau)+\varepsilon^2\ldots, \tag{4.4}$$

Usually one retains only the first few terms, $1,2,\ldots,m$ in the series expansions of (4.3) and (4.4), we evaluate the functions $u_1,\ldots,A_1,A_2\ldots$, such that a_1 and a_2 appearing in (4.3) and (4.4) satisfy the given differential equation (4.1) with an accuracy of ε^{m+1} [98]. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexities for the derivation of the function, the solution is, in general, confined to a lower order, usually the first. In order to determine these unknown functions, it was assumed that the functions $u_1,u_2,\ldots$ do not contain the fundamental terms [57,98,109], which are included in the series expansion (4.3) of order ε^0.

According to the KBM technique, solution equations (4.3) is differentiated two times with respect to *t*, substituting for the derivatives $\ddot{x}$ and x in the original equation (4.1) and equating the coefficient of ε, we obtain

$$(\Omega-\lambda_2)A_1+\lambda_1'a_1+(\Omega-\lambda_1)A_2+\lambda_2'a_2+(\Omega-\lambda_1)(\Omega-\lambda_2)u_1=-f^{(0)}(a_1,a_2,\tau), \tag{4.5}$$

where $\Omega\equiv\lambda_1 a_1\dfrac{\partial}{\partial a_1}+\lambda_2 a_2\dfrac{\partial}{\partial a_2}$, $\lambda_1'=\dfrac{d\lambda_1}{d\tau}$, $\lambda_2'=\dfrac{d\lambda_2}{d\tau}$, $f^{(0)}=f(x_0,\dot{x}_0,\tau)$

and $x_0 = a_1 + a_2$.

Followed by the Shamsul[98] assumption u_1 does not contain fundamental terms and for this reason the solution will be free from secular terms, namely $t\cos t$, $t\sin t$ and te^{-t}.

In general, the function $f^{(0)}$ can be expanded in a Taylor series as

$$f^{(0)} = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2}(a_1^{r_1}, a_2^{r_2}) \tag{4.6}$$

To obtain this solution (4.4), it has been proposed in [98,109] that u_1, u_2 exclude the terms $a_1^{r_1} a_2^{r_2}$ of $f^{(0)}$, where $r_1 - r_2 = \pm 1$. This restriction guarantees that the solution always excludes *secular*-type terms or the first harmonics (see [109] for details). According to our assumption, u_1 does not contain the fundamental terms, therefore equation (4.5) can be separated into three individual equations for unknown functions u_1, A_1 and A_2 (see [98] for details). Substituting the functional values of $f^{(0)}$ and equating the coefficients of $e^{\lambda_j t}$, $j = 1,2$, we obtain

$$(\Omega - \lambda_2)A_1 + \lambda_1' a_1 = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2}(a_1^{r_1}, a_2^{r_2}), \qquad \text{if } r_1 = r_2 + 1 \tag{4.7}$$

$$(\Omega - \lambda_1)A_2 + \lambda_2' a_2 = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2}(a_1^{r_1}, a_2^{r_2}), \quad \text{if } r_2 = r_1 + 1 \tag{4.8}$$

and

$$(\Omega - \lambda_1)(\Omega - \lambda_2)u_1 = \sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2}(a_1^{r_1}, a_2^{r_2}), \tag{4.9}$$

where $\sum_{r_1=0,r_2=0}^{\infty,\infty} F_{r_1,r_2}(a_1^{r_1}, a_2^{r_2})$ exclude those terms for $r_1 = r_2 \pm 1$.

Thus the particular solutions of (4.7)-(4.9) give three unknown functions A_1, A_2 and u_1. It is noted that equation (4.1) is not a formal form of KBM method. We shall be able to transform (4.3) to the exact form of the KBM [37,12,53] solution by substituting $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$. Herein, a and φ are respectively amplitude and phase variables (see [26,113]). The particular solution of (4.7), (4.8) and (4.9) gives the unknown functions u_1, A_1 and A_2. This completes the determination of the solution of a second order non-linear problem (4.1). The method can be carried out to higher order in a similar way.

4.3 Example

4.3.1 We consider a second order nonlinear system with constant and slowly varying coefficients

$$\ddot{x} + 2k(\tau)\dot{x} + \omega^2(\tau)x = -\varepsilon x^3 \tag{4.10}$$

Here over dots denote differentiation with respect to t. In this case $x_0 = a_1 + a_2$ and the function $f^{(0)}$ becomes

$$f^{(0)} = -(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3) \tag{4.11}$$

Following the assumption (discussed in Section 4.2) u_1 excludes the terms $3a_1^2 a_2$ and $3a_1 a_2^2$.

We substitute (4.11) in (4.5) and separate it into two parts as

$$(\Omega - \lambda_2)A_1 + \lambda_1' a_1 + (\Omega - \lambda_1)A_2 + \lambda_2' a_2 = -(3a_1^2 a_2 + 3a_1 a_2^2) \tag{4.12}$$

and

$$(\Omega - \lambda_1)(\Omega - \lambda_2)u_1 = -(a_1^3 + a_2^3) \tag{4.13}$$

The particular solution of (5.13) is

$$u_1 = -\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{4.14}$$

Now we have to solve (4.12) for two functions A_1 and A_2. According with the unified KBM method, A_1 contains the term $3a_1^2 a_2$ and A_2 contains the term $3a_1 a_2^2$ (Shamsul [98,109]) and thus we obtain the following equations

$$(\Omega - \lambda_2)A_1 + \lambda_1' a_1 = -3a_1^2 a_2 \tag{4.15}$$

and

$$(\Omega - \lambda_1)A_2 + \lambda_2' a_2 = -3a_1 a_2^2 \tag{4.16}$$

The particular solutions of (4.15) and (4.16) are

$$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1} \tag{4.17}$$

and

$$A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} \tag{4.18}$$

Substituting the functional values of A_1, A_2 from (4.17) and (4.18) into (4.4) and rearranging, we obtain

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1}\right) \tag{4.19}$$

and

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(\frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2}\right) \tag{4.20}$$

Under the transformations, $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$ together with $\lambda_1 = -k + i\omega$, $\lambda_2 = -k - i\omega$ and the replacement $A_1 e^{-i\varphi} + A_2 e^{i\varphi} = \widetilde{A}_1$ and $-i(A_1 e^{-i\varphi} - A_2 e^{i\varphi}) = a\widetilde{B}_1$ (where $\widetilde{A}_1$ and $\widetilde{B}_1$ are usual notations), equations (4.17-4.20) reduce to

$$\dot{a} = \varepsilon \widetilde{A}_1(a) + \varepsilon^2 \ldots$$

and (4.21)

$$\dot{\varphi} = \omega + \varepsilon \widetilde{B}_1(a) + \varepsilon^2 \ldots$$

We shall obtain the variational equations of a and φ in the real form (a and φ are know as amplitude and phase respectively) which transform (4.21) to

$$\dot{a} = -ka - \frac{\varepsilon a \omega'}{2\omega} + \frac{3\varepsilon a^3 k}{8(k^2 + \omega^2)} \tag{4.22}$$

and

$$\dot{\varphi} = \omega - \frac{\varepsilon k'}{2\omega} + \frac{3\varepsilon a^2 \omega}{8(k^2 + \omega^2)} \tag{4.23}$$

The variational equations (4.22) and (4.23) are in the form of the KBM solution. The variational equations for amplitude and phase are usually appeared in a set of first order differential equations and solved by the numerical technique (see Shamsul [98,109]).

Thus the first approximate solution of the equation (4.10) is

$$x(t, \varepsilon) = a\cos\varphi + \varepsilon u_1 \tag{4.24}$$

where a and φ are the solutions of the equations (4.22) and (4.23) respectively.

4.3.2. Let us consider a second order nonlinear differential system with an external force

$$\ddot{x} + 2k(\tau)\dot{x} + \omega^2(\tau)x = -\varepsilon x^3 + \varepsilon E \cos \nu t, \qquad \tau = \varepsilon t \tag{4.25}$$

where over dots denote differentiation with respect to t, ν is the frequency of the external force, $x_0 = a_1 + a_2$ and the function

$$f^{(0)} = -(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3) + \frac{E}{2}\left(e^{i\nu t} + e^{-i\nu t}\right). \tag{4.26}$$

Under the restrictions (discussed in Section 4.2) u_1 excludes the terms $3a_1^2 a_2$, $3a_1 a_2^2$ and $\varepsilon E\left(e^{i\nu t} + e^{-i\nu t}\right)/2$. We substitute (4.26) in (4.5) and separate it into two parts as

$$(\Omega - \lambda_2)A_1 + \lambda_1' a_1 + (\Omega - \lambda_1)A_2 + \lambda_2' a_2 = -(3a_1^2 a_2 + 3a_1 a_2^2) + \frac{E}{2}(e^{i\nu t} + e^{-i\nu t}) \tag{4.27}$$

and

$$(\Omega - \lambda_1)(\Omega - \lambda_2)u_1 = -(a_1^3 + a_2^3) \tag{4.28}$$

The particular solution of (4.28) is

$$u_1 = -\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{4.29}$$

Now we have to solve (4.27) for two functions A_1 and A_2. According with the unified KBM method, A_1 contains the terms $3a_1^2 a_2$, $Ee^{i\nu t}/2$ and A_2 contains the terms $3a_1 a_2^2$, $Ee^{-i\nu t}/2$ (see [98,109]) which lead to the following equations

$$(\Omega - \lambda_2)A_1 + \lambda_1' a_1 = -3a_1^2 a_2 + \frac{E}{2}e^{i\nu t}, \tag{4.30}$$

and

$$\left(\Omega - \lambda_1\right) A_2 + \lambda_2' a_2 = -3 a_1 a_2^2 + \frac{E}{2} e^{-ivt} \tag{4.31}$$

The particular solutions of (4.30) and (4.31) are

$$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3 a_1^2 a_2}{2\lambda_1} + \frac{E e^{ivt}}{2(i\nu - \lambda_2)} \tag{4.32}$$

and

$$A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3 a_1 a_2^2}{2\lambda_2} + \frac{E e^{-ivt}}{-2(i\nu + \lambda_1)} \tag{4.33}$$

Substituting the functional values of A_1 and A_2 into (4.5) and rearranging, we obtain (see Section 4.2)

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3 a_1^2 a_2}{2\lambda_1} + \frac{E e^{ivt}}{2(i\nu - \lambda_2)} \right) \tag{4.34}$$

and

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(\frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3 a_1 a_2^2}{2\lambda_2} - \frac{E e^{-ivt}}{2(i\nu + \lambda_1)} \right). \tag{4.35}$$

The variational equations of a and φ in the real form (a and φ are known as amplitude and phase respectively), transform (4.34) and (4.35) to

$$\dot{a} = -ka - \frac{\varepsilon a \omega'}{2\omega} + \frac{3 \varepsilon a^3 k}{8(k^2 + \omega^2)} + \frac{\varepsilon E\{k \cos(\varphi - \nu t) - (\nu + \omega)\sin(\varphi - \nu t)\}}{k^2 + (\nu + \omega)^2} \tag{4.36}$$

and

$$\dot{\varphi} = \omega - \frac{\varepsilon k'}{2\omega} + \frac{3 \varepsilon a^2 \omega}{8(k^2 + \omega^2)} - \frac{\varepsilon E\{(\nu + \omega)\cos(\varphi - \nu t) + k \sin(\varphi - \nu t)\}}{a\{k^2 + (\nu + \omega)^2\}} \tag{4.37}$$

The variational equations (4.34) and (4.35) are in the form of the KBM solution. The variational equations for amplitude and phase are usually appeared in a set of first order differential equations and solved by a numerical technique (see Shamsul [98,109]).

Thus the first approximate solution of the equation (4.25) is

$$x(t,\varepsilon) = a e^{-k(\tau)} \cos\varphi + \varepsilon u_1, \tag{4.38}$$

where a and φ are the solutions of the equations (4.34) and (4.35) respectively. Equation (4.38) is similar to that obtained by the KBM method (see [98,109]).

4.4 Results and Discussions

An analytical method has been developed to obtain an approximate solution of a second-order time dependent nonlinear differential system with damping, external forced and slowly varying coefficients based on the unified KBM [37,12,53] method. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this article, we refer to the works of Murty [57], and Shamsul [98,109,113,114]. In this article, we may compare the approximate solutions (4.24) and (4.38) of *Duffing's* equations (4.10) and (4.25) to the numerical solutions obtained by Runge-Kutta (fourth-order) procedure.

First of all, x is calculated by (4.10) with initial conditions $[x(0)=1, \dot{x}(0)=0]$ or $a=1.001621$, $\varphi=-.056901$ for $\varepsilon=.1$, $\omega=\omega_0\sqrt{\cos\tau}$, $k=.01\cos\tau$. Then the corresponding numerical solution is also computed by Runge-Kutta method. For $\varepsilon=1$ and strong damping force $k=.1\cos\tau$, x is calculated by (4.10) with initial conditions $[x(0)=1, \dot{x}(0)=0]$ or

$a = 1.000021$, $\varphi = -.058901$. All the results are shown respectively in Fig. 4.1 and Fig.4.2. From Fig.4.1, it is clear that the perturbation results all most coincide to the numerical results. In Fig. 4.2, the perturbation results slightly deviate from the numerical results.

In Sub section 4.3.2, a perturbation solution (4.38) has been derived and the solution has been presented in Fig.4.3 for $\varepsilon = .1, \nu = 1.1$, $E = .5$, $\omega = \omega_0\sqrt{\cos\tau}$, $k = .01\cos\tau$ with initial conditions $[x(0) = 1,\ \dot{x}(0) = 0]$, or, $a = 1.0$, $\varphi = 0.0$. This solution also shows a good agreement with the numerical solution. Further $x(t,\varepsilon)$ is calculated by (4.38) with initial conditions $[x(0) = 1,\ \dot{x}(0) = 0]$ or $a = 1.,\ \varphi = .0$ when $\varepsilon = .1$, $\nu = 1.$, $E = 1.$, $\omega = \omega_0\sqrt{\cos\tau}$, $k = .1\cos\tau$ and all the results are shown in Fig. 4.4. In Fig. 4.4, results show a steady-state solution.

4.5. Conclusion

A perturbation solution of a second order time dependent nonlinear deferential system with slowly varying coefficients is investigated by the modified KBM perturbation technique. The solution for different initial conditions shows good coincidence with the corresponding numerical solution.

Fig 4.1

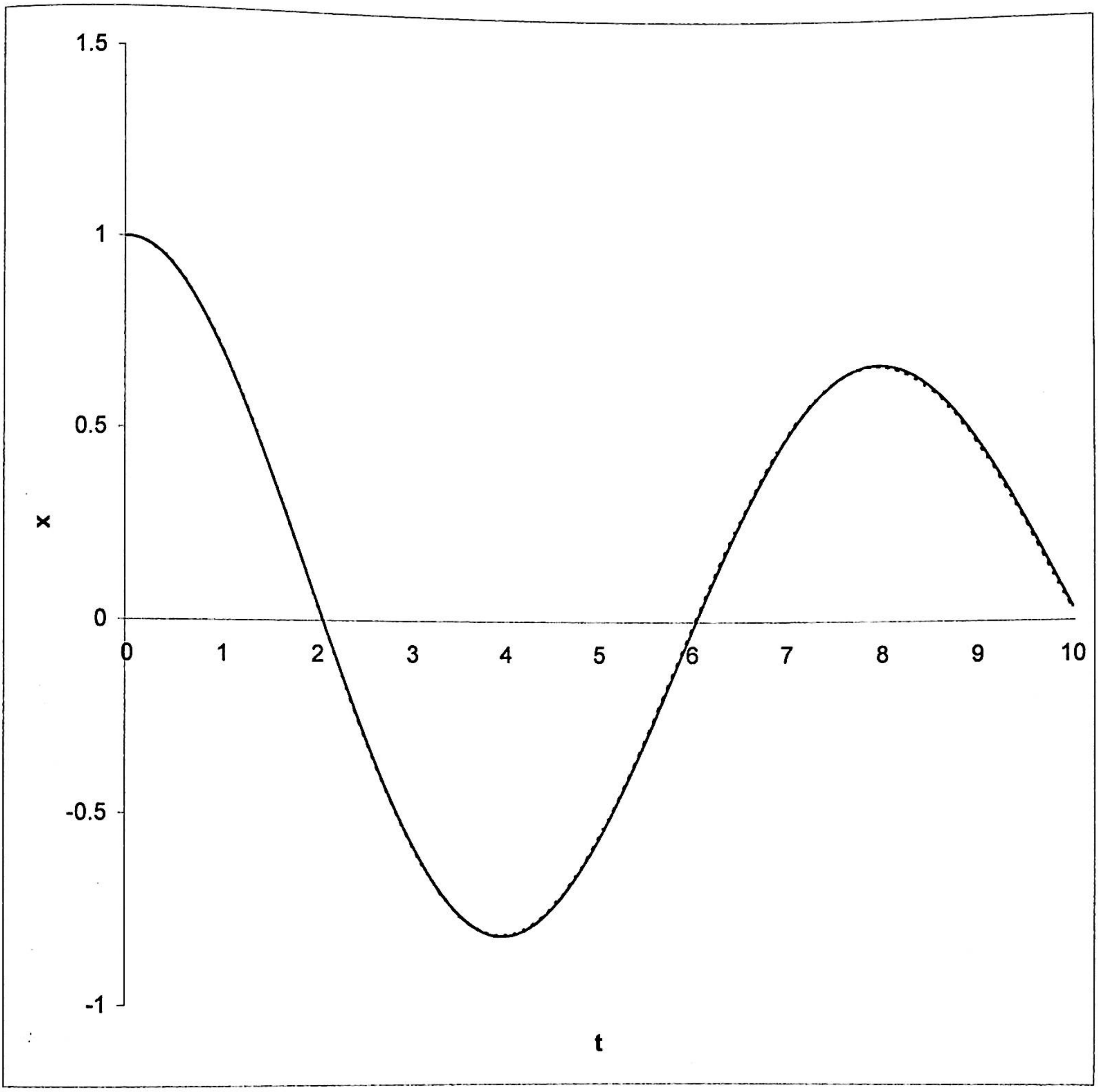

Fig 4.1: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.001621,\ \varphi = -.056901$ $[x(0) = 1.0,\ \dot{x}(0) = 0.0]$ for $e = .1,\ \omega_0 = 1,\ h = .05$.

Fig 4.2

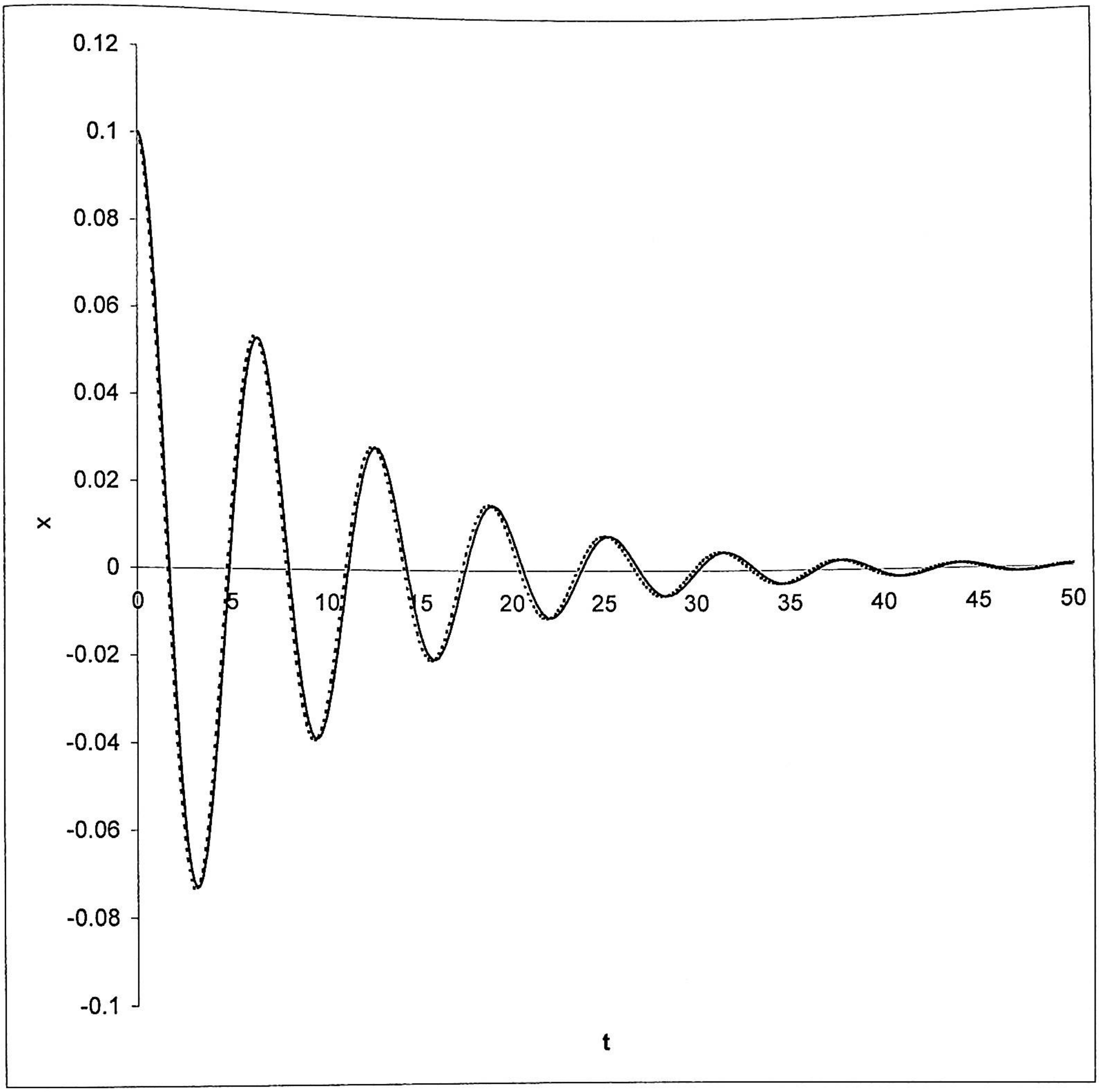

Fig 4.2: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.000021$, $\varphi = -.058901$ [$x(0) = 1.00000$, $\dot{x}(0) = 0.00000$] for $e = 1.$, $\omega_0 = 1.$, $h = .05$.

Fig 4.3

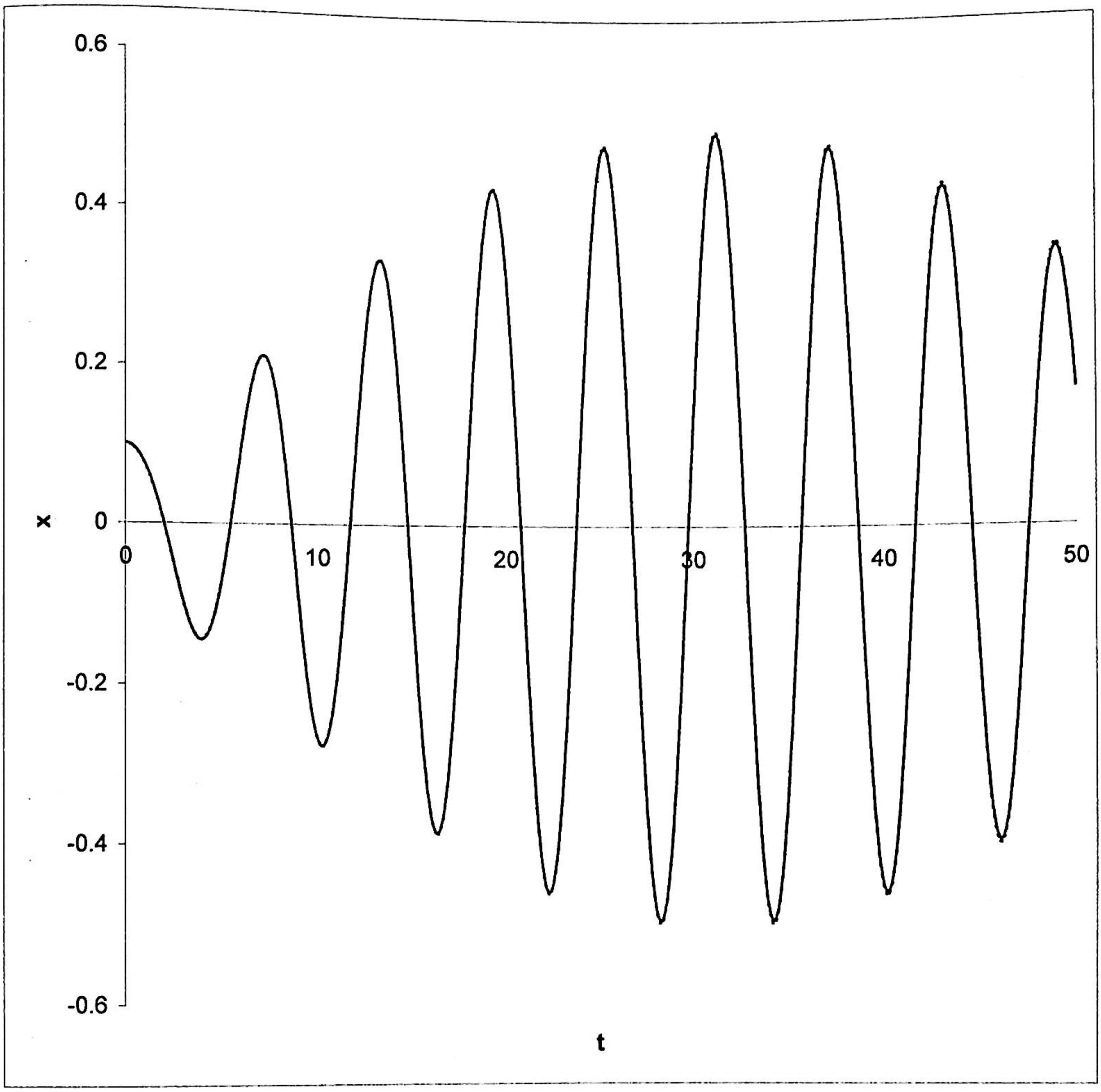

Fig.4.3: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.00000,\ \varphi = 0.00000$ $[x(0) = 1.00000,\ \dot{x}(0) = 0.00000]$ for $e = .1,\ \nu = 1.1,\ E = .5,\ \omega_0 = 1.,\ h = .05$.

Fig 4.4

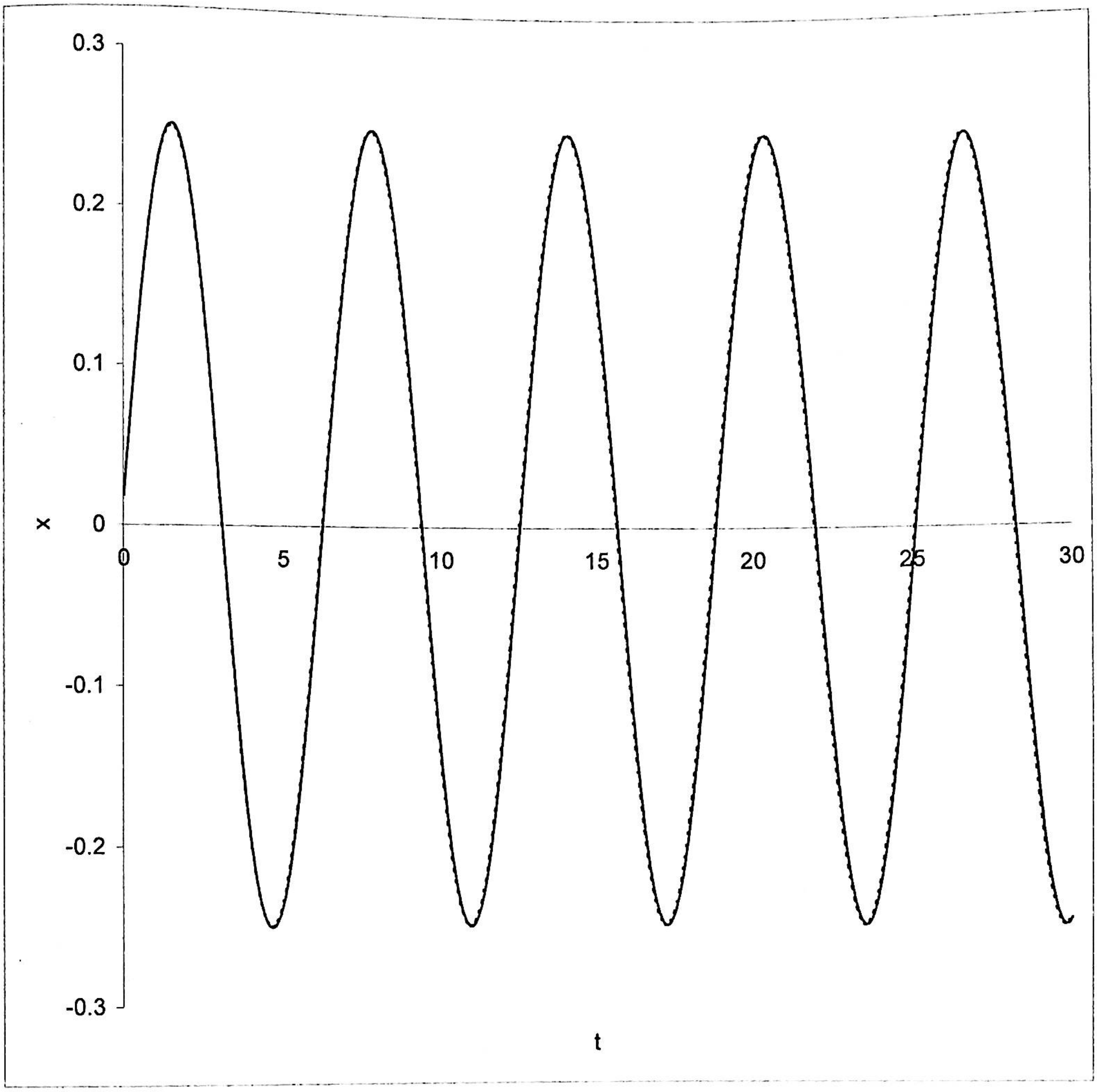

Fig.4.4: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.00000$, $\varphi = 0.00000$ [$x(0) = 1.00000$, $\dot{x}(0) = 0.00000$] for $e = .1$, $\nu = 1.$, $E = 1.$, $\omega_0 = 1.$, $h = .05$.

Chapter 5

Second Approximate Solution of Second Order Time Dependent Weakly Nonlinear Systems

5.1. Introduction

Many of the problems facing today by physicists, engineers and applied mathematicians involve difficulties, such as nonlinear governing equations, variable coefficients, and nonlinear boundary condition at complex known or unknown boundaries, which preclude their solutions exactly. Mathematical modal of such processes commonly result in differential equations. Nonlinear oscillating processes in nature are of great importance. In the last several decades there has been an increased interest in oscillating processes. Great achievements in science have to be attributed to the theory of periodic oscillations. In this connection among many branches of science, astronomy has played a significant role. Consequently, solutions are approximated using numerical techniques, analytic techniques and combinations of both. Among the approximation methods used to study nonlinear systems with a small nonlinearity, Krylov-Bogoliubov-Mitropolskii (KBM) [37,12,53] method is particularly convenient and is the widely used technique to obtain the approximate solutions. Originally the method, developed for systems with periodic solutions, was later extended by Popov [78] and Meldelson [49] for damped nonlinear oscillations. Followed by Popov's [78] technique, Murty *et al.* [56] extended the method to over-damped non-linear systems. They investigated second and fourth order differential equations when all the eigenvalues of the respective linear equation become real and unequal. Murty [57] has developed a unified KBM method for solving second order nonlinear systems which cover the undamped, damped and over-damped cases. Sattar [84] has studied third order over-damped nonlinear systems. Bojadziev [19] found a mono-frequent damped solution of an n-dimensional, $n = 2, 3, \cdots$ time-dependent differential system with strong damping effects, small time-delay and slowly varying coefficients. Arya and Bojadziev [8] studied a second order time dependent differential equation with damping, slowly varying coefficients and small time delay in which a non-periodic external force acted. Shamsul and Sattar [87] have presented a unified KBM method for solving third order nonlinear systems. Shamsul [98] has presented a unified method for solving an n-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes. But the above authors (Murty [57], Sattar [86,87] and Shamsul [94, 98,109,116]) found first approximate solutions of autonomous systems. The aim of the present article is to find second approximate solution of second order time dependent weakly nonlinear vibrating problems with an external force.

5.2 Method

Let us consider the second order time dependent weakly nonlinear differential systems

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}, \nu t), \tag{5.1}$$

where the over-dots denote differentiation with respect to t, ω is a positive constant, ε is a small parameter, f is the given nonlinear function and ν is the frequency of the external forces. When $\varepsilon = 0$, let us consider that the characteristic roots of the linear equation of (5.1) are real and unequal say λ_1, λ_2.

Therefore, the solution of the unperturbed equation of (5.1) become

$$x(t,0) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}, \tag{5.2}$$

where a_1 and a_2 are arbitrary constants. We have investigated the above nonlinear system when the natural frequency of the system and the frequency of the external forces are almost same. We have chosen an approximate solution (see also [98]) of (5.1) in the form of the asymptotic expansion

$$x(t,\varepsilon) = a_1(t)e^{\lambda_1 t} + a_2(t)e^{\lambda_2 t} + \varepsilon u_1(a_1,a_2,t) + \varepsilon^2 u_2(a_1,a_2,t) + \varepsilon^3 \ldots, \tag{5.3}$$

where a_1 and a_2 satisfy the differential equations

$$\dot{a}_1 = \varepsilon A_1(a_1,a_2,t) + \varepsilon^2 B_1(a_1,a_2,t) + \varepsilon^3 \ldots \tag{5.4}$$

and

$$\dot{a}_2 = \varepsilon A_2(a_1,a_2,t) + \varepsilon^2 B_2(a_1,a_2,t) + \varepsilon^3 \ldots. \tag{5.5}$$

Confining only to the first few terms, 1, 2,...,m in the series expansions of (5.3) and (5.4), we evaluate the functions $u_1, u_2, \ldots,$ and $A_1, A_2, \ldots, B_1, B_2, \ldots,$ such that $a_1(t)$ and $a_2(t)$ appearing in (5.3) and (5.4) satisfy the given differential equation (5.1) with an accuracy of ε^{m+1}. In order to determine these unknown functions it is assumed that the functions $u_1, u_2, \ldots$ do not contain secular-type term te^{-t} (see [98,113,115,116] for details).

Differentiating $x(t,\varepsilon)$ twice with respect to t, substituting the derivatives $\dot{x}, \ddot{x}$ and $x(t,\varepsilon)$ in the original equation (5.1) yields

$$\begin{aligned} &e^{\lambda_1 t}\left(\frac{d}{dt} + \lambda_1 - \lambda_2\right)(\varepsilon A_1 + \varepsilon^2 B_1 + \cdots) + e^{\lambda_2 t}\left(\frac{d}{dt} + \lambda_2 - \lambda_1\right)(\varepsilon A_2 + \varepsilon^2 B_2 + \ldots) \\ &+ \left(\frac{d}{dt} - \lambda_1\right)\left(\frac{d}{dt} - \lambda_2\right)(\varepsilon u_1 + \varepsilon^2 u_2 + \ldots) = \varepsilon\, f(..) \end{aligned} \tag{5.6}$$

and equating the coefficient of ε, ε^2, we obtain

$$\begin{aligned} &e^{\lambda_1 t}\left(\frac{d}{dt} + \lambda_1 - \lambda_2\right)(\varepsilon A_1) + e^{\lambda_2 t}\left(\frac{d}{dt} + \lambda_2 - \lambda_1\right)(\varepsilon A_2) \\ &+ \left(\frac{d}{dt} - \lambda_1\right)\left(\frac{d}{dt} - \lambda_2\right)(\varepsilon u_1) = \varepsilon\, f^{(0)}(a_1,\, a_2,\, \nu t) \end{aligned}$$

and

$$e^{\lambda_1 t}\left(\frac{d}{dt}+\lambda_1-\lambda_2\right)(\varepsilon^2 B_1)+e^{\lambda_2 t}\left(\frac{d}{dt}+\lambda_2-\lambda_1\right)(\varepsilon^2 B_2)$$
$$+\left(\frac{d}{dt}-\lambda_1\right)\left(\frac{d}{dt}-\lambda_2\right)(\varepsilon^2 u_2)=\varepsilon\, f^{(1)}(a_1,\, a_2,\, \nu t) \tag{5.7}$$

where $f^{(0)}=f(x_0,\dot{x}_0,\nu t)$ and

$$f^{(1)}=-3(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t})^2u_1-\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)(A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t})$$
$$-\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)u_1-\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1 \tag{5.8}$$

$$x_0=a_1(t)e^{\lambda_1 t}+a_2(t)e^{\lambda_2 t}.$$

The related function to solution equation (5.3) where determine utilizing formulae equation (5.6) under the restrictions that $u_1, u_2,..$exclude terms $a_1^{r_1}a_2^{r_2}e^{(r_1\lambda_1+r_2\lambda_2)t}$ of f where $r_1-r_2=\pm 1$. This restriction guarantees that the solution always excludes *secular*-type terms or the first harmonics terms, otherwise a sizeable error would occur (see [98,115] for details). Herein it is noted that equation (5.3) is not chosen in a formal form of the KBM method. To get the formal solution, a simple variable transformation, namely, $a_1=ae^{i\varphi}/2$ and $a_1=ae^{-i\varphi}/2$ (a and φ are respectively amplitude and phase variable), is used. It is interesting to note that under the said variable transformation equation (5.8) can be transformed to a formal form *i.e.*, in terms of amplitude and phase (Shamsul [98,115,116]). Under this assumption, we shall able to find the unknown functions u_1, u_2 and A_1, A_2, B_1, B_2 which complete the determination of the second approximate solution of a second order non-linear vibrating problem (5.1).

5.3 Example

Let us consider the *Duffing* equation with external forces

$$\ddot{x} + \omega_0^2 x = -\varepsilon x^3 + \varepsilon E \cos \nu t, \tag{5.9}$$

where ν is the frequency of the external forces. When $\varepsilon = 0$, equation (5.9) has two eigenvalues $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$.

Thus for equation (5.9), we obtain

$$\begin{aligned} f^{(0)} = -\{ & a_1^3 e^{3\lambda_1 t} + 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + a_2^3 e^{3\lambda_2 t} \\ & -\frac{E}{2}(e^{i\nu t} + e^{-i\nu t}) + 3\varepsilon\left(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}\right)^2 u_1 + \cdots \}. \end{aligned} \tag{5.10}$$

Therefore, equation (5.6) becomes

$$\begin{aligned} & e^{\lambda_1 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)(\varepsilon A_1 + \varepsilon^2 B_1 + \cdots) + e^{\lambda_2 t}\left(\frac{\partial}{\partial t} - \lambda_1 + \lambda_2\right)(\varepsilon A_2 + \varepsilon^2 B_2 + \cdots) \\ & + \left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)(\varepsilon u_1 + \varepsilon^2 u_2 + \cdots) \\ & = -\varepsilon \{ a_1^3 e^{3\lambda_1 t} + 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + a_2^3 e^{3\lambda_2 t} \\ & -\frac{E}{2}(e^{i\nu t} + e^{-i\nu t}) + 3\,\varepsilon\left(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}\right)^2 u_1 + \cdots \}. \end{aligned} \tag{5.11}$$

Following the assumption (discussed in Section 5.2) u_1 excludes the terms $3\varepsilon a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t}$, $3\varepsilon a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t}$ and $\varepsilon E\left(e^{i\nu t} - e^{-i\nu t}\right)/(2)$. Thus, for equation (5.11), we obtain

$$e^{\lambda_1 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)A_1 = -3\, a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + Ee^{i\nu t}/2 \tag{5.12}$$

$$e^{\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_1\right)A_2 = -3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + Ee^{-i\nu t}/2 \tag{5.13}$$

$$\left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)u_1 = -\left(a_1^3 e^{3\lambda_1 t} + a_2^3 e^{3\lambda_2 t}\right) \tag{5.14}$$

and

$$e^{\lambda_1 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)B_1+e^{\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_1+\lambda_2\right)B_2+\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_2$$

$$=-3(a_1^2e^{2\lambda_1 t}+2a_1a_2e^{(\lambda_1+\lambda_2)t}+a_{21}^2e^{2\lambda_2 t})u_1-\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)\left(A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t}\right) \quad (5.15)$$

$$-\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)u_1-\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1$$

Solving equations (5.12)-(5.14), we obtain

$$A_1=\frac{-3a_1^2a_2\,e^{(\lambda_1+\lambda_2)t}}{2\lambda_1}+\frac{Ee^{(i\nu-\lambda_1)\,t}}{2(i\nu-\lambda_2)} \quad (5.16)$$

$$A_2=\frac{-3a_1\,a_2^2\,e^{(\lambda_1+\lambda_2)t}}{2\lambda_2}+\frac{-Ee^{-(i\nu+\lambda_2)\,t}}{2(i\nu+\lambda_1)} \quad (5.17)$$

and

$$u_1=\frac{-a_1^3\,e^{3\lambda_1 t}}{2\lambda_1(3\lambda_1-\lambda_2)}+\frac{-a_2^3\,e^{3\lambda_2 t}}{2\lambda_2(3\lambda_2-\lambda_1)} \quad (5.18)$$

Substituting the values of A_1, A_2 and u_1 in the right hand side of (5.15), we obtain

$$e^{\lambda_1 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)B_1+e^{\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_1+\lambda_2\right)B_2+\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_2=$$

$$\left\{\frac{3a_1^5e^{5\lambda_1 t}}{2\lambda_1(3\lambda_1-\lambda_2)}+\frac{3a_2^5e^{5\lambda_2 t}}{2\lambda_2(3\lambda_2-\lambda_1)}+\frac{3a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)}+\frac{3a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{\lambda_1(3\lambda_1-\lambda_2)}\right.$$

$$+\frac{3a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)}+\frac{3a_1\,a_2^4e^{(\lambda_1+4\lambda_2)t}}{\lambda_2(3\lambda_2-\lambda_1)}-\frac{9a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_1^2}-\frac{9a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{4\lambda_1\lambda_2}$$

$$-\frac{9a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{4\lambda_1\lambda_2}-\frac{9a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2^2}+\frac{3Ea_1a_2e^{(i\nu+\lambda_1+\lambda_2)t}}{2\lambda_1(i\nu-\lambda_2)}+\frac{3Ea_2^2e^{(i\nu+2\lambda_2)t}}{4\lambda_2(i\nu-\lambda_2)}$$

$$-\frac{3Ea_1^2e^{(-i\nu+2\lambda_1)t}}{4\lambda_1(i\nu+\lambda_1)}-\frac{3Ea_1a_2e^{(-i\nu+\lambda_1+\lambda_2)t}}{2\lambda_2(i\nu+\lambda_1)}-\frac{9a_1^4a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2(3\lambda_1-\lambda_2)}$$

$$+\frac{3Ea_1^2(i\nu+\lambda_1)e^{(i\nu+2\lambda_1)t}}{4\lambda_1(i\nu-\lambda_2)(3\lambda_1-\lambda_2)}+-\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{\lambda_2(3\lambda_2-\lambda_1)}-\frac{3Ea_2^2(-i\nu+2\lambda_2-\lambda_1)e^{(-i\nu+2\lambda_2)t}}{4\lambda_2(i\nu+\lambda_1)(3\lambda_2-\lambda_1)}$$

$$+\frac{9a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2}+\frac{3Ea_1^2e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)}-\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)}-\frac{3Ea_2^2e^{(iv+2\lambda_2)t}}{2(iv+\lambda_1)(3\lambda_2-\lambda_1)}\Bigg\} \quad (5.19)$$

Since u_2 dose not contain the first harmonic terms, so the Eq.(5.19) can be separated for B_1, B_2 in the following way:

$$e^{\lambda_1 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)B_1=\Bigg\{\frac{3a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)}-\frac{9a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_1^2}-\frac{9a_2^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{4\lambda_1\lambda_2}$$
$$+\frac{3Ea_1a_2e^{(iv+\lambda_1+\lambda_2)t}}{2\lambda_1(iv-\lambda_2)}-\frac{3Ea_1^2e^{(-iv+2\lambda_1)t}}{4\lambda_1(iv+\lambda_1)}\Bigg\} \quad (5.20)$$

$$e^{\lambda_2 t}\left(\frac{\partial}{\partial t}-\lambda_1+\lambda_2\right)B_2=\Bigg\{\frac{3a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)}-\frac{9a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2^2}-\frac{9a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{4\lambda_1\lambda_2}$$
$$-\frac{3Ea_1a_2e^{(-iv+\lambda_1+\lambda_2)t}}{2\lambda_2(iv+\lambda_1)}+\frac{3Ea_2^2e^{(iv+2\lambda_2)t}}{4\lambda_2(iv-\lambda_2)}\Bigg\} \quad (5.21)$$

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_2=\Bigg\{\frac{3a_1^5e^{5\lambda_1 t}}{2\lambda_1(3\lambda_1-\lambda_2)}+\frac{3a_2^5e^{5\lambda_2 t}}{2\lambda_2(3\lambda_2-\lambda_1)}+\frac{3a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{\lambda_1(3\lambda_1-\lambda_2)}$$
$$+\frac{3a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{\lambda_2(3\lambda_2-\lambda_1)}-\frac{9a_1^4a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2(3\lambda_1-\lambda_2)}+\frac{3Ea_1^2(iv+\lambda_1)e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)(3\lambda_1-\lambda_2)}$$
$$-\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{\lambda_2(3\lambda_2-\lambda_1)}-\frac{3Ea_2^2(-iv+2\lambda_2-\lambda_1)e^{(-iv+2\lambda_2)t}}{4\lambda_2(iv+\lambda_1)(3\lambda_2-\lambda_1)}-\frac{9a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2}$$
$$+\frac{3Ea_1^2e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)}-\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)}-\frac{3Ea_2^2e^{(iv+2\lambda_2)t}}{2(iv+\lambda_1)(3\lambda_2-\lambda_1)}\Bigg\} \quad (5.22)$$

Solving equations (5.20) and (5.21), we obtain

$$B_1=\Bigg\{\frac{3a_1^3a_2^2e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)}-\frac{9a_1^3a_2^2e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1^2(3\lambda_1+\lambda_2)}-\frac{9a_2^3a_2^2e^{2(\lambda_1+\lambda_2)t}}{4\lambda_1\lambda_2(3\lambda_1+\lambda_2)}$$
$$+\frac{3Ea_1a_2e^{(iv+\lambda_2)t}}{2\lambda_1(iv-\lambda_2)(iv+\lambda_1)}-\frac{3Ea_1^2e^{(-iv+\lambda_1)t}}{4\lambda_1(iv+\lambda_1)(-iv+2\lambda_1-\lambda_2)}\Bigg\} \quad (5.23)$$

$$B_2=\Bigg\{\frac{3a_1^2a_2^3e^{2(\lambda_1+\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)(\lambda_1+3\lambda_2)}-\frac{9a_1^2a_2^3e^{2(\lambda_1+\lambda_2)t}}{2\lambda_2^2(\lambda_1+3\lambda_2)}-\frac{9a_1^2a_2^3e^{2(\lambda_1+\lambda_2)t}}{4\lambda_1\lambda_2(\lambda_1+3\lambda_2)}$$
$$-\frac{3Ea_1a_2e^{(-iv+\lambda_1)t}}{2\lambda_2(iv+\lambda_1)(-iv+\lambda_2)}+\frac{3Ea_2^2e^{(iv+\lambda_2)t}}{4\lambda_2(iv-\lambda_2)(iv-\lambda_1+2\lambda_2)}\Bigg\} \quad (5.24)$$

$$
\begin{aligned}
u_2 &= \frac{3a_1^5 e^{5\lambda_1 t}}{8\lambda_1^2(3\lambda_1-\lambda_2)(5\lambda_1-\lambda_2)} + \frac{3a_2^5 e^{5\lambda_2 t}}{8\lambda_2^2(3\lambda_2-\lambda_1)(5\lambda_2-\lambda_1)} \\
&+ \frac{3a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)} + \frac{3a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t}}{4\lambda_2^2(3\lambda_2-\lambda_1)(\lambda_1+3\lambda_2)} \\
&- \frac{9a_1^4 a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{16\lambda_1^3(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)} + \frac{3Ea_1^2(iv+\lambda_1)e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)(3\lambda_1-\lambda_2)(iv+\lambda_1)(iv+2\lambda_1-\lambda_2)} \\
&- \frac{9a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t}}{4\lambda_2^2(3\lambda_2-\lambda_1)(\lambda_1+3\lambda_2)} - \frac{3Ea_2^2(-iv+2\lambda_2-\lambda_1)e^{(-iv+2\lambda_2)t}}{4\lambda_2(iv+\lambda_1)(3\lambda_2-\lambda_1)(-iv+2\lambda_2-\lambda_1)(-iv+\lambda_2)} \\
&- \frac{9a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t}}{16\lambda_1^3(3\lambda_1+\lambda_2)} + \frac{3Ea_1^2 e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)(iv+\lambda_1)(iv+2\lambda_1-\lambda_2)} - \frac{9a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t}}{8\lambda_2^2(3\lambda_2-\lambda_1)(3\lambda_2+\lambda_1)} \\
&- \frac{3Ea_2^2 e^{(iv+2\lambda_2)t}}{2(iv+\lambda_1)(3\lambda_2-\lambda_1)(iv+2\lambda_2-\lambda_1)(iv+\lambda_2)}
\end{aligned}
\tag{5.25}
$$

Putting the values of A_1, A_2, B_1, B_2 in equations (5.4) and (5.5), we obtain

$$
\begin{aligned}
\dot{a}_1 &= \frac{-3\varepsilon a_1^2 a_2\, e^{(\lambda_1+\lambda_2)t}}{2\lambda_1} + \frac{\varepsilon E e^{(iv-\lambda_1)\,t}}{2(iv-\lambda_2)} \\
&+ \frac{3\varepsilon^2 a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)} - \frac{9\varepsilon^2 a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1^2(3\lambda_1+\lambda_2)} - \frac{9\varepsilon^2 a_2^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{4\lambda_1\lambda_2(3\lambda_1+\lambda_2)} \\
&+ \frac{3\varepsilon^2 E a_1 a_2 e^{(iv+\lambda_2)t}}{2\lambda_1(iv-\lambda_2)(iv+\lambda_1)} - \frac{3\varepsilon^2 E a_1^2 e^{(-iv+\lambda_1)t}}{4\lambda_1(iv+\lambda_1)(-iv+2\lambda_1-\lambda_2)}
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
\dot{a}_2 &= \frac{-3\varepsilon a_1\, a_2^2\, e^{(\lambda_1+\lambda_2)t}}{2\lambda_2} + \frac{-E\varepsilon e^{-(iv+\lambda_2)\,t}}{2(iv+\lambda_1)} \\
&+ \frac{3\varepsilon^2 a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)} - \frac{9\varepsilon^2 a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1^2(3\lambda_1+\lambda_2)} - \frac{9\varepsilon^2 a_2^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{4\lambda_1\lambda_2(3\lambda_1+\lambda_2)} \\
&- \frac{3\varepsilon^2 E a_1 a_2 e^{(-iv+\lambda_1)t}}{2\lambda_2(iv+\lambda_1)(-iv+\lambda_2)} + \frac{3\varepsilon^2 E a_2^2 e^{(iv+\lambda_2)t}}{4\lambda_2(iv-\lambda_2)(iv-\lambda_1+2\lambda_2)}
\end{aligned}
\tag{5.27}
$$

Now, using the variables $a_1 = ae^{i\varphi}/2$, $a_2 = ae^{-i\varphi}/2$ and the eigenvalues $\lambda_1 = i\omega$, $\lambda_2 = -i\omega$ and simplifying, we obtain the variational equations for a and φ in the real form (a and φ are known as amplitude and phase). Therefore, the equations (5.26) and (5.27) transform to

$$\dot{a} = \varepsilon\, Ep_1 \sin(\omega t - \nu t + \varphi) + \varepsilon^2 Ep_2 a^2 \sin(\omega t - \nu t + \varphi) + \varepsilon^2 Ep_3 a^2 \sin(\omega t - \nu t + \varphi) \tag{5.28}$$

and

$$\dot{\varphi} = \varepsilon\, q_1 a^2 + \varepsilon\, Ep_1 \cos(\omega t - \nu t + \varphi)/a + \varepsilon^2 Eq_2 a^4 + \varepsilon^2 Ep_2 a \cos(\omega t - \nu t + \varphi) - \varepsilon^2 Ep_3 a \cos(\omega t - \nu t + \varphi). \tag{5.29}$$

where

$$p_1 = -1/(\omega + \nu),\ p_2 = 3/4\omega(\omega + \nu)^2$$
$$p_3 = 3(3\omega^2 + 2\omega\nu - \nu^2)/8\omega(\omega + \nu)^2(3\omega - \nu)^2$$
$$q_1 = 3/8\omega),\ q_2 = -30/512\omega^3$$

The form of the variational equations (5.28) and (5.29) are same as the form of the KBM solution. The variational equations for the amplitude and phase are usually appeared in a set of first order differential equations and solved by a numerical technique.

Thus the second approximate solution of the equation (5.9) is

$$x(t,\,\varepsilon) = a\,\cos(\omega t + \varphi) + \varepsilon\, u_1 + \varepsilon^2 u_2 + \cdots, \tag{5.30}$$

where a, φ are the solutions of the equations (5.28) and (5.29) and u_1, u_2 are given by (5.18) and (5.25).

5.4 Results and Discussions

An asymptotic method, based on the theory of extended KBM method, has been developed to obtain second approximate solution of a second order non-autonomous nonlinear vibrating problem with small non-linearity. It is laborious task to find second approximate solution of nonlinear vibrating problems by the classical method, but it can be easily solved by this technique.

It is usually compare the perturbation solution to the numerical solution to test the accuracy of an approximate solution. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexities for the derivation of the formulae, the solution in general confined to a low order, usually the first [98]. In this article the solution has been derived to second approximation. With regard to such a comparison concerning the presented the KBM method of this article, we refer to the works of Murty *et al* [57], and Shamsul [98,115]. In this article, we have also compared the approximate solutions of *Duffing's* equation (5.9) to those obtained by Range-Kutta (fourth-order) procedure.

First, x has been calculated by (5.30) with first approximation together with initial conditions $[x(0)=1,\ \dot{x}(0)=0]$ or $a=1.00000$, $\varphi=0.0000$ or $\varepsilon=.25$, $E=1$, $\nu=1$, $\omega=1$. Then corresponding numerical solutions is computed by Runge-Kutta method. All the results are shown in Fig.1. From Fig.1 the asymptotic solutions (5.30) vary with the numerical solution of equation (5.9).

We have again computed x by perturbation method with second approximation together with initial condition $[x(0)=1.007812,\ \dot{x}(0)=0]$ or $a=1.00000$, $\varphi=0.0000$ for $\varepsilon=.25, E=1,\ \nu=1, \omega=1$. The corresponding numerical solution has been computed in this case and all the result are plotted in Fig. 2, we find that the approximate solution show a good agreement with the numerical solution.

5.4 Conclusion.

In this present paper a technique is presented by the extended KBM method for obtaining the second approximate solution of a second order non-autonomous nonlinear vibrating problem with small nonlinearities. In this case perturbation method facilitates the numerical method. The results obtained by this method agree with those obtained by the numerical method.

Fig.5.1

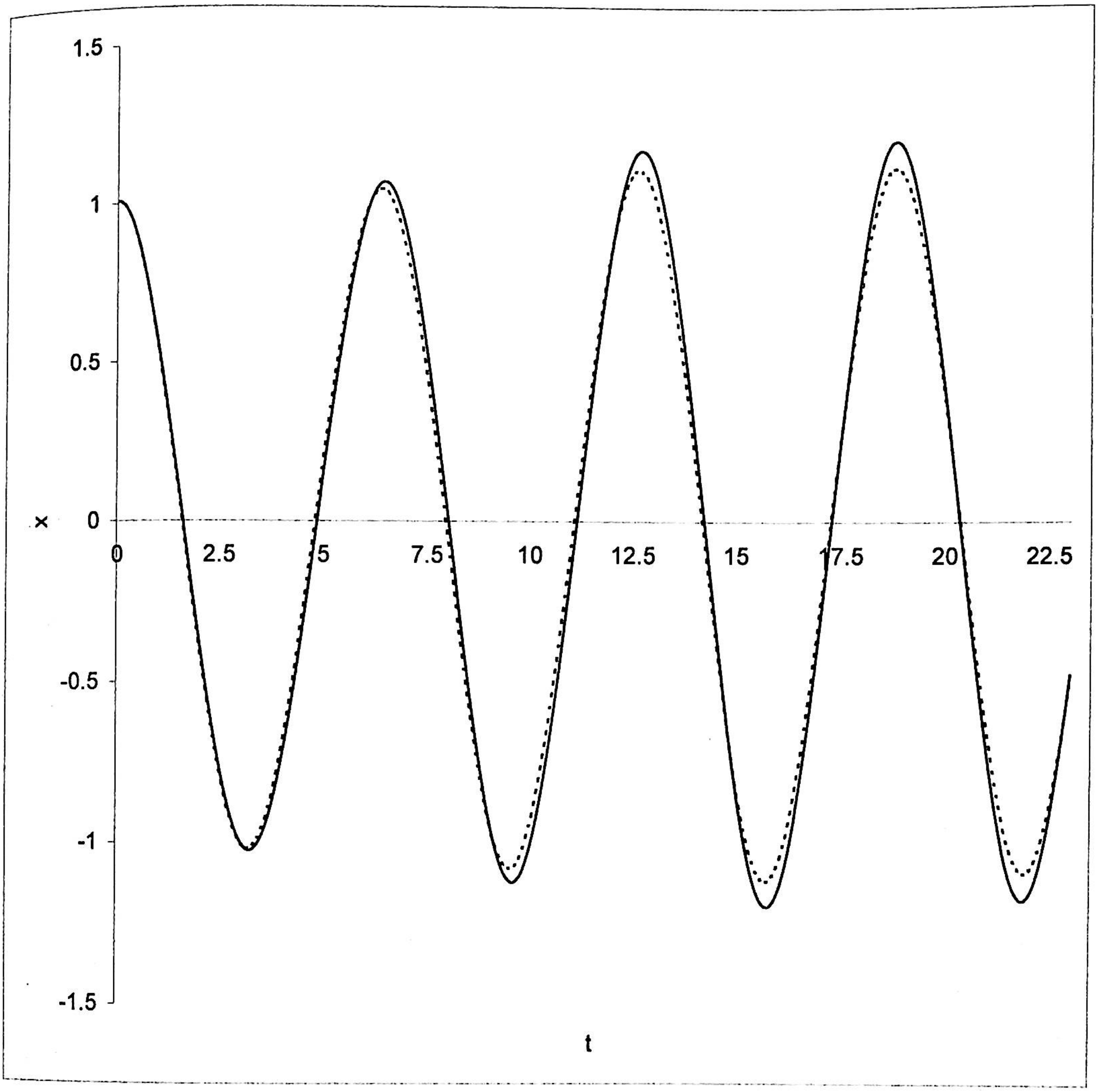

Fig. 5.1: First approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when $\omega = \nu = 1$ together with initial conditions $a = 1.$, $\varphi = 0$ [$x(0) = 1.0000$, $\dot{x}(0) = 0.00000$] and $\varepsilon = 0.25$, $E = 1$.

Fig. 5.2

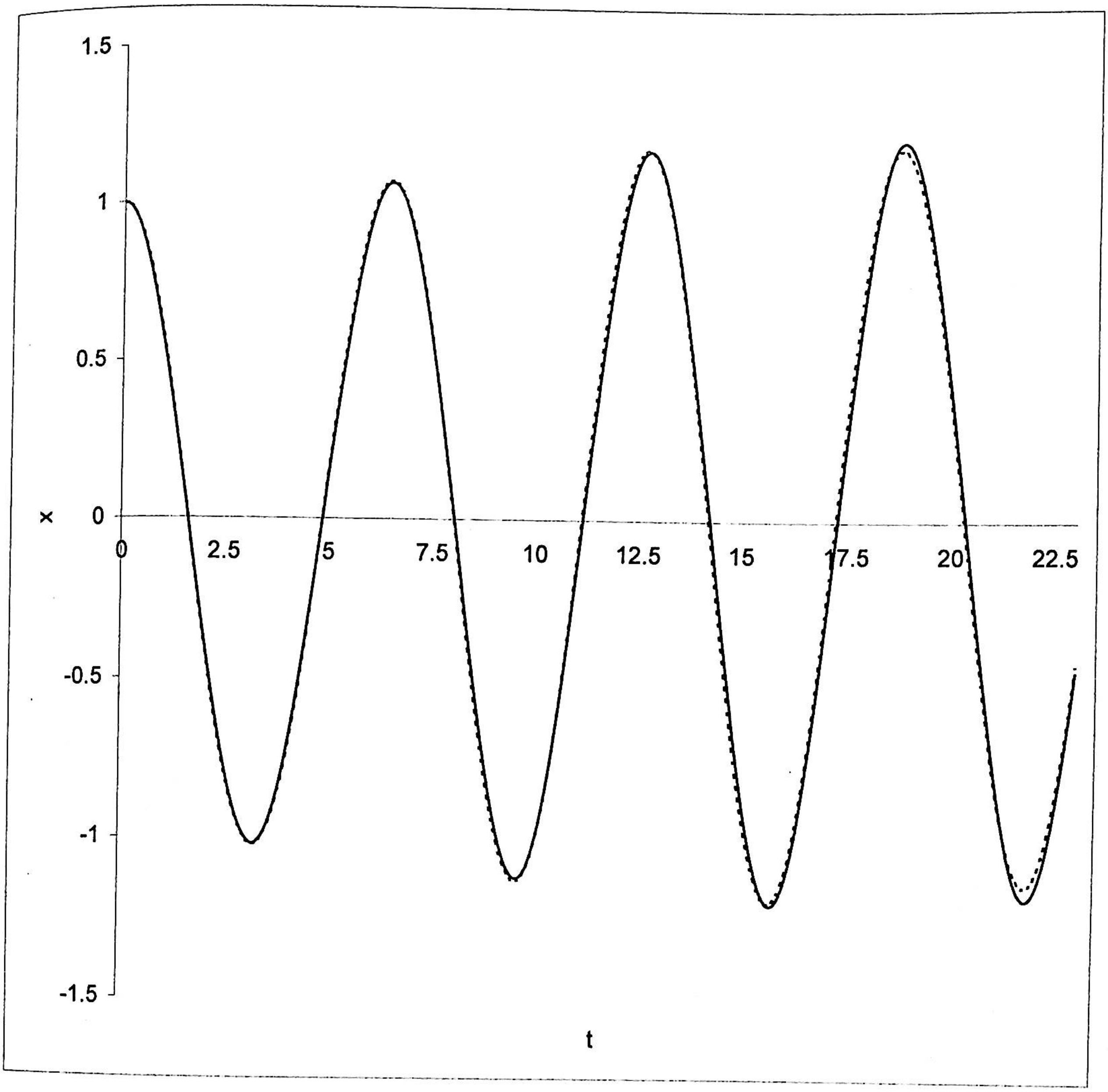

Fig. 5.2: Second approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when $\omega = \nu = 1$ together with initial conditions $a = 1.$, $\varphi = 0$ $[x(0) = 1.007812,\ \dot{x}(0) = 0.00000]$ and $\varepsilon = 0.25,\ E = 1$.

Chapter 6

Higher Approximate Solution of *n*-th Order Weakly Nonlinear Non-autonomous Differential Systems with Damping

6.1. Introduction

Many analytical approaches have been developed for approximating periodic solutions of the nonlinear systems. The most common and widely studied approximate methods for nonlinear differential systems are the perturbation methods, whereby the solution is analytically expanded in power series of a small parameter. The Struble's method [119], Krylov-Bogoliubov-Mitropolskii (KBM) method [37,12,53], multiple time–scales method [61] were originally formulated to find periodic solution of a second order weakly non-linear differential system

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \quad \varepsilon \leq 1 \tag{6.1}$$

Several authors extended these methods to investigate similar non-linear problems with a strong linear damping effect, $-2k\dot{x}$, $k = O(1)$, modeled by the following equation

$$\ddot{x} + +2k\dot{x} + \omega^2 x = -\varepsilon f(x, \dot{x}) \tag{6.2}$$

Popov [78] was familiar among them. He extended the Krylov-Bogoliubov-Mitropolskii (KBM) method and investigated the under-damped case of Eq.(6.2). Then Mendelson [49] rediscovered Popov's [78] results. Followed by Popov's [78] technique, Murty *et al.* [56] extended the method to over-damped non-linear systems. They investigated second and fourth order differential equations when all the eigenvalues of the respective linear equation were real and unequal. Murty [57] has developed a unified KBM method for solving second order nonlinear systems, which cover the undamped, damped, and over-damped cases. Bojadziev [19] found a mono-frequent damped solution of an n-dimensional, $n = 2, 3, \cdots$ time-dependent differential system with strong damping effects, small time-delay and slowly varying coefficients. However, Bojadziev illustrated his method [19] by a second order equation, namely,

$$\ddot{x} + 2b\dot{x} + 2\beta\,\dot{x}(t - \varepsilon\Delta) + cx = \varepsilon\,(1 - x^2)\dot{x} + \varepsilon\,E\sin\nu\,t \tag{6.3}$$

Arya and Bojadziev [8] studied a second order time dependent differential equation with damping, slowly varying coefficients and small time delay in which a non-periodic external force acted. Mulholland [54], Osiniski [64], Bojadziev [25], Bojadziev and Hung [26], Sattar [16], Shamsul and Sattar [17] investigated some third order quasi-linear differential systems. Shamsul [98] has generalized Murty's [57] technique for solving an n-th, $n = 2,3...,$ order autonomous non-linear differential equation. Shamsul [109] has extended the unified method [98] to similar differential systems with slowly varying coefficients. Recently, Shamsul [113] has examined an n-th, $n = 2,3...,$ order time dependent quasi-linear differential system. But none of the above authors investigated higher approximate solution with external forces. The aim of this article is to find the higher approximate solution of n-th, $n = 2,3...,$ order non-autonomous non-linear deferential systems with damping.

6.2 The Method

Consider an n-th order time dependent weakly nonlinear differential system

$$x^{(n)} + k_1 x^{(n-1)} + \cdots + k_n x = \varepsilon\, f(x, \dot{x}, \ddot{x}, \cdots, \nu t) \tag{6.4}$$

where $x^{(j)}$, $j \geq 3$ represents a j-th derivative of x with respect to t, over-dot is used for first, second,... derivatives, ε is a small parameter, k_j, $j = 1, 2, \cdots, n$ are constants, f is a nonlinear function and ν is the frequency of the external acting forces.

When $\varepsilon = 0$, Eq. (6.4) has n eigenvalues, say λ_j, $j = 1, 2, \cdots, n$. If all the eigenvalues are distinct, the unperturbed solution becomes

$$x(t,0) = \sum_{j=1}^{n} a_{j,0} e^{\lambda_j t}, \tag{6.5}$$

where $a_{j,0}$, $j = 1, 2, \cdots, n$ are arbitrary constants.

When $\varepsilon \neq 0$, we seek a solution of the nonlinear differential equation (6.4) of the form [98]

$$x(t,\varepsilon) = \sum_{j=1}^{n} a_{j,0} e^{\lambda_j t} + \varepsilon u_1(a_1, a_2, \cdots, a_n, t) + \varepsilon^2 u_2(a_1, a_2, \cdots, a_n, t) + \varepsilon^3 \ldots, \tag{6.6}$$

where $a_{j,0}$, $j = 1, 2 \cdots, n$ are satisfy the differential equations

$$\dot{a}_j(t) = \varepsilon A_j(a_1, a_2, \cdots, a_n, t) + \varepsilon^2 B_j(a_1, a_2, \cdots, a_n, t) + \varepsilon^3 \ldots, \tag{6.7}$$

Confining only to the first few terms, $1, 2, \ldots, m$ in the series expansions of (6.6) and (6.7), we evaluate the functions $u_1, u_2, \ldots$ and A_j, B_j, $j = 1, 2, \ldots, n$, such that $a_j(t)$ appearing in (6.6) and (6.7) satisfy the given differential equation (6.4) with an accuracy of ε^{m+1}. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexities for the derivation of the function, the solution is in general confined to a lower order, usually the first. In order to determine these unknown functions it is assumed that the functions $u_1, u_2, \ldots$ do not contain fundamental terms, which are included in the series expansion (6.6) of order ε^0 (see [37,12,53,57,98,109]).

One can readily rewrite the Eq. (6.4) as

$$\prod_{k=1}^{n} \left(\frac{d}{dt} - \lambda_k \right) x = \varepsilon f(x, \dot{x}, \cdots, \nu t). \tag{6.8}$$

Substituting the value of x from Eq. (6.6) into the left side of Eq. (6.8), it leads to

$$\prod_{k=1}^{n}\left(\frac{d}{dt}-\lambda_k\right)\left(\sum_{j=1}^{n}a_j(t)e^{\lambda_j t}+\varepsilon u_1+\varepsilon^2 u_2+\cdots\right)=\varepsilon f(..)\,, \tag{6.9}$$

or,

$$\begin{aligned}&\sum_{j=1}^{n}\left(\prod_{k=1}^{n}\left(\frac{d}{dt}-\lambda_k\right)(a_j e^{\lambda_j t})\right)+\varepsilon\prod_{k=1}^{n}\left(\frac{d}{dt}-\lambda_k\right)u_1\\&+\varepsilon^2\prod_{k=1}^{n}\left(\frac{d}{dt}-\lambda_k\right)u_2+\cdots=\varepsilon f(..)\end{aligned} \tag{6.10}$$

On the other hand, Eq. (6.7) can be written as

$$\left(\frac{d}{dt}-\lambda_j\right)(a_j e^{\lambda_j t})=\varepsilon A_j e^{\lambda_j t}+\varepsilon^2 B_j e^{\lambda_j t}+\varepsilon^3\cdots. \tag{6.11}$$

With help of Eq. (6.11), Eq. (6.10) can be rewritten as

$$\begin{aligned}&\sum_{j=1}^{n}\left(\prod_{k=1,\,k\neq j}^{n}\left(\frac{d}{dt}-\lambda_k\right)(\varepsilon A_j e^{\lambda_j t}+\varepsilon^2 B_j e^{\lambda_j t}+\cdots)\right)\\&+\varepsilon\prod_{k=1}^{n}\left(\frac{d}{dt}-\lambda_k\right)u_1+\varepsilon^2\prod_{k=1}^{n}\left(\frac{d}{dt}-\lambda_k\right)u_2+\cdots=\varepsilon f(..).\end{aligned} \tag{6.12}$$

Here all the functions A_j, B_j, u_j, $j=1,2,\cdots$ depend on $a_1,a_2,\cdots,a_n$ and t. To determine the derivatives of these functions, the following notations can be used:

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\sum_{j=1}^{n}\frac{\partial}{\partial a_j}\dot{a}_j=\frac{\partial}{\partial t}+\varepsilon\sum_{j=1}^{n}A_j\frac{\partial}{\partial a_j}+\varepsilon^2\cdots=D+\varepsilon\Phi+\varepsilon^2\cdots, \tag{6.13}$$

where $D\equiv\frac{\partial}{\partial t}$, $\Phi\equiv\sum_{j=1}^{n}A_j\frac{\partial}{\partial a_j}$. Replacing the notation $\frac{d}{dt}$ of Eq.(6.12) by $D+\varepsilon\Phi+\varepsilon^2\cdots$ and then equating the coefficients of ε, ε^2 from both sides, the following formulae are found:

$$\begin{aligned}&\prod_{j=1}^{n}(D-\lambda_j)u_1+\sum_{j=1}^{n}\left(\prod_{k=1,\,k\neq j}^{n}(D-\lambda_k)\right)(A_j e^{\lambda_j t})=f^{(0)}(..),\\&\prod_{j=1}^{n}(D-\lambda_j)u_2+\sum_{j=1}^{n}\left(\prod_{k=1,\,k\neq j}^{n}(D-\lambda_k)\right)(B_j e^{\lambda_j t})=f^{(1)}(..),\end{aligned} \tag{6.14}$$

where

$$
\begin{aligned}
f^{(0)} &= f(\Sigma_{j=1}^{n} a_j e^{\lambda_j t}, \Sigma_{j=1}^{n} \lambda_j a_j e^{\lambda_j t}, \cdots), \qquad n \ge 2, \\
f^{(1)} &= u_1 f_x(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}, \lambda_1 a_1 e^{\lambda_1 t} + \lambda_2 a_2 e^{\lambda_2 t}) + (A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + Du_1) \\
&\quad \times f_{\dot{x}}(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}, \lambda_1 a_1 e^{\lambda_1 t} + \lambda_2 a_2 e^{\lambda_2 t}) - [\Phi(D + c_1) + D\Phi]u_1 \\
&\quad - \Phi(A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}), \quad n = 2, \\
f^{(1)} &= u_1 f_x(\Sigma_{j=1}^{n} a_j e^{\lambda_j t}, \Sigma_{j=1}^{n} \lambda_j a_j e^{\lambda_j t}, \cdots) + (\Sigma_{j=1}^{n} A_j e^{\lambda_j t} + Du_1) \\
&\quad \times f_{\dot{x}}(\Sigma_{j=1}^{n} a_j e^{\lambda_j t}, \Sigma_{j=1}^{n} \lambda_j a_j e^{\lambda_j t}, \cdots) + \cdots - [\Phi(D^{n-1} + c_1 D^{n-2} + c_2 D^{n-3} + \cdots) \\
&\quad + D\Phi(D^{n-2} + c_1 D^{n-3} + \cdots) + D^2\Phi(D^{n-3} + \cdots) + \cdots]u_1 \\
&\quad - \sum_{l'=1, l' \ne l}^{n} [\Phi(D^{n-3} + c_1^{(l)} D^{n-4} + c_2^{(l)} D^{n-5} + \cdots) + D\Phi(D^{n-4} + c_1^{(l)} D^{n-5} + \cdots) \\
&\quad + D^2\Phi(D^{n-5} + \cdots) + \cdots] \\
&\quad \times [(D - \lambda_{2l})(e^{\lambda_{2l-1} t} A_{2l-1}) + (D - \lambda_{2l-1})(e^{\lambda_{2l} t} A_{2l})], \qquad n > 2,
\end{aligned}
\tag{6.15}
$$

and $c_1^{(l)}, c_2^{(l)}, \cdots, c_{n-2}^{(l)}$ are the coefficients of the algebraic equation

$$
\prod_{l'=1, l' \ne l}^{n/2} (\lambda - \lambda_{2l-1})(\lambda - \lambda_{2l}) = 0, \tag{6.16}
$$

or

$$
\left(\prod_{l'=1, l' \ne l}^{(n-1)/2} (\lambda - \lambda_{2l-1})(\lambda - \lambda_{2l}) \right) \times (\lambda - \lambda_n) = 0 \tag{6.17}
$$

respectively for n is even or odd. The formula Eq. (6.14) is an essential part of KBM method in which the unknown functions u_1, u_2, A_j, B_j, $j = 1, 2, \cdots$ are determined. Clearly, the structures of the two members of Eq. (6.14) are same. So the second equation of Eq. (6.14) is solved in a way similar to that of the first [98,109,113,114,]. In general, D and Φ are not commutative. We can show that D and Φ are commutative when the real parts of the eigenvalues, λ_j, $j = 1, 2, \cdots, n$ vanish (see Section 4). In this case, Eq. (6.15) takes the simplest form and $f^{(1)}$ can be evaluated quickly. However, the unknown functions u_1, u_2, A_j, B_j, $j = 1, 2, \cdots$ can be determined from Eq. (6.14) in terms of $a_1, a_2, \cdots, a_n$, t, by imposing the restriction that u_1, u_2 exclude the first harmonic terms (see [98,109] for details). But Eq. (6.14) will be transformed to amplitude and phase according to [109], in which the second approximation (in a usual form) can be found directly. For an even or an odd value of n, Eq. (6.14) can be written respectively as

$$\sum_{l=1}^{n/2}\left(\prod_{k=1,\,k\neq 2l-1,\,2l}^{n}(D-\lambda_k)[(D-\lambda_{2l})(A_{2l-1}e^{\lambda_{2l-1}t})+(D-\lambda_{2l-1})(A_{2l}e^{\lambda_{2l}t})]\right)$$
$$+\prod_{j=1}^{n}(D-\lambda_j)u_1=f^{(0)}(..),$$
$$\sum_{l=1}^{n/2}\left(\prod_{k=1,\,k\neq 2l-1,\,2l}^{n}(D-\lambda_k)[(D-\lambda_{2l})(B_{2l-1}e^{\lambda_{2l-1}t})+(D-\lambda_{2l-1})(B_{2l}e^{\lambda_{2l}t})]\right)$$
$$+\prod_{j=1}^{n}(D-\lambda_j)u_1=f^{(1)}(..), \tag{6.18}$$

or,

$$\sum_{l=1}^{(n-1)/2}\left(\prod_{k=1,\,k\neq 2l-1,\,2l}^{n}(D-\lambda_k)[(D-\lambda_{2l})(A_{2l-1}e^{\lambda_{2l-1}t})+(D-\lambda_{2l-1})(A_{2l}e^{\lambda_{2l}t})]\right)$$
$$+\prod_{j=1}^{n-1}(D-\lambda_j)(A_ne^{\lambda_n t})+\prod_{j=1}^{n}(D-\lambda_j)u_1=f^{(0)}(a_1,a_2,\cdots,a_n,t),$$
$$\sum_{l=1}^{(n-1)/2}\left(\prod_{k=1,\,k\neq 2l-1,\,2l}^{n}(D-\lambda_k)[(D-\lambda_{2l})(B_{2l-1}e^{\lambda_{2l-1}t})+(D-\lambda_{2l-1})(B_{2l}e^{\lambda_{2l}t})]\right)$$
$$+\prod_{j=1}^{n-1}(D-\lambda_j)(B_ne^{\lambda_n t})+\prod_{j=1}^{n}(D-\lambda_j)u_1=f^{(1)}(a_1,a_2,\cdots,a_n,t). \tag{6.19}$$

For the straightforward solution $u_1,u_2,\ldots$do not contain *secular terms,* when damping force is large. Yet it was restricted (by Popov [78]) that $u_1,u_2,\ldots$would be free from the first harmonics, otherwise a sizeable error would occur. We notice that the equation (6.6) is not in the form of the KBM method. But by a simple variable transformation namely $a_{2l-1}=\alpha_l e^{i\varphi_l}/2$ and $a_{2l}=\alpha_l e^{-i\varphi_l}/2$, $l=1,2,\cdots,n/2$ or$(n-1)/2$ (α_l and φ_l are respectively amplitude and phase variables), all these equations and functions can be transformed into the KBM form (see [98,109,113,114]). Under this assumption, we shall be able to find the unknown functions $u_1,u_2,\ldots$ and $A_1,A_2,\ldots,B_1,B_2,\ldots,$ which complete the determination of the second approximate solution of the n-th order non-linear problem (6.4).

6.3 Example

6.3.1. Second-order equation

Let us consider the *Duffing* equation with external forces

$$\ddot{x}+2k\dot{x}+\omega_0^2 x=-\varepsilon x^3+\varepsilon E\cos\nu t \tag{6.20}$$

where k is the linear damping coefficient and ν is the frequency of the external forces. When $\varepsilon=0$, equation (6.20) has two eigenvalues $\lambda_1=-k+i\omega$, $\lambda_2=-k-i\omega$, and $\omega^2=\omega_0^2-k^2$, where $k<\omega_0$.

Thus for equation (6.20), we obtain

$$\begin{aligned}f^{(0)}=&-\{a_1^3e^{3\lambda_1 t}+3a_1^2a_2e^{(2\lambda_1+\lambda_2)t}+3a_1a_2^2e^{(\lambda_1+2\lambda_2)t}+a_2^3e^{3\lambda_2 t}\\&-\frac{E}{2}(e^{i\nu t}+e^{-i\nu t})+3\varepsilon\left(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}\right)^2u_1+\cdots\}.\end{aligned} \tag{6.21}$$

Therefore, equation (6.12) becomes

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\lambda_2\right)(\varepsilon A_1e^{\lambda_1 t}+\varepsilon^2B_1e^{\lambda_1 t}+\cdots)+\left(\frac{\partial}{\partial t}-\lambda_1\right)(\varepsilon A_2e^{\lambda_2 t}+\varepsilon^2B_2e^{\lambda_2 t}+\cdots)\\&+\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)(\varepsilon u_1+\varepsilon^2u_2+\cdots)\\&=-\varepsilon\left\{a_1^3e^{3\lambda_1 t}+3a_1^2a_2e^{(2\lambda_1+\lambda_2)t}+3a_1a_2^2e^{(\lambda_1+2\lambda_2)t}+a_2^3e^{3\lambda_2 t}\right.\\&\left.-\frac{E}{2}(e^{i\nu t}+e^{-i\nu t})+3\varepsilon\left(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}\right)^2u_1+\cdots\right\}.\end{aligned} \tag{6.22}$$

Following the assumption (discussed in section 6.2) u_1 excludes the terms $3\varepsilon a_1^2a_2e^{(2\lambda_1+\lambda_2)t}$ and $3\varepsilon a_1a_2^2e^{(\lambda_1+2\lambda_2)t}$. Moreover, u_1 also excludes the term $\varepsilon E\left(e^{i\nu t}-e^{-i\nu t}\right)/(2)$. Thus, for equation (6.22), we obtain

$$\left(\frac{\partial}{\partial t}-\lambda_2\right)(A_1e^{\lambda_1 t})=-3a_1^2a_2e^{(2\lambda_1+\lambda_2)t}+\frac{E}{2}e^{i\nu t} \tag{6.23}$$

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)(A_2\,e^{\lambda_2 t})=-3a_1\,a_2^2e^{(\lambda_1+2\lambda_2)t}+\frac{E}{2}e^{-i\nu t} \tag{6.24}$$

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1=-\left(a_1^3e^{3\lambda_1 t}+a_2^3e^{3\lambda_2 t}\right) \tag{6.25}$$

and

$$\left(\frac{\partial}{\partial t}-\lambda_2\right)(B_1\,e^{\lambda_1 t})+\left(\frac{\partial}{\partial t}-\lambda_1\right)(B_2\,e^{\lambda_2 t})+\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_2$$
$$=-3(a_1^2e^{2\lambda_1 t}+2a_1a_2e^{(\lambda_1+\lambda_2)t}+a_{21}^2e^{2\lambda_2 t})u_1-\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)\left(A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t}\right) \tag{6.26}$$
$$-\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)u_1-\left(A_1\frac{\partial}{\partial a_1}+A_2\frac{\partial}{\partial a_2}\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1$$

Solving the equations (4.23)-(4.25), we obtain

$$A_1=\frac{-3a_1^2a_2\,e^{(\lambda_1+\lambda_2)t}}{2\lambda_1}+\frac{Ee^{(i\nu-\lambda_1)t}}{2(i\nu-\lambda_2)} \tag{6.27}$$

$$A_2=\frac{-3a_1\,a_2^2\,e^{(\lambda_1+\lambda_2)t}}{2\lambda_2}+\frac{-Ee^{-(i\nu+\lambda_2)t}}{2(i\nu+\lambda_1)} \tag{6.28}$$

and

$$u_1=\frac{-a_1^3\,e^{3\lambda_1 t}}{2\lambda_1(3\lambda_1-\lambda_2)}+\frac{-a_2^3\,e^{3\lambda_2 t}}{2\lambda_2(3\lambda_2-\lambda_1)} \tag{6.29}$$

Substituting the value of A_1, A_2 and u_1 in the right hand side of (6.26), we obtain

$$\left(\frac{\partial}{\partial t}-\lambda_2\right)(B_1\,e^{\lambda_1 t})+\left(\frac{\partial}{\partial t}-\lambda_1\right)(B_2\,e^{\lambda_2 t})+\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_2=$$

$$+3\left(\frac{a_1^5e^{5\lambda_1 t}}{2\lambda_1(3\lambda_1-\lambda_2)}+\frac{a_2^5e^{5\lambda_2 t}}{2\lambda_2(3\lambda_2-\lambda_1)}+\frac{a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)}\right)$$

$$+3\left(\frac{a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{\lambda_1(3\lambda_1-\lambda_2)}+\frac{a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)}+\frac{a_1\,a_2^4e^{(\lambda_1+4\lambda_2)t}}{\lambda_2(3\lambda_2-\lambda_1)}\right)$$

$$+\left(-\frac{9a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_1^2}-\frac{9a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{4\lambda_1\lambda_2}-\frac{9a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{4\lambda_1\lambda_2}-\frac{9a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2^2}\right)$$

$$+\left(\frac{3Ea_1a_2e^{(iv+\lambda_1+\lambda_2)t}}{2\lambda_1(iv-\lambda_2)}+\frac{3Ea_2^2e^{(iv+2\lambda_2)t}}{4\lambda_2(iv-\lambda_2)}-\frac{3Ea_1^2e^{(-iv+2\lambda_1)t}}{4\lambda_1(iv+\lambda_1)}-\frac{3Ea_1a_2e^{(-iv+\lambda_1+\lambda_2)t}}{2\lambda_2(iv+\lambda_1)}\right)$$

$$+\left(-\frac{9a_1^4a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2(3\lambda_1-\lambda_2)}+\frac{3Ea_1^2(iv+\lambda_1)e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)(3\lambda_1-\lambda_2)}\right)$$
$$+\left(-\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{\lambda_2(3\lambda_2-\lambda_1)}-\frac{3Ea_2^2(-iv+2\lambda_2-\lambda_1)e^{(-iv+2\lambda_2)t}}{4\lambda_2(iv+\lambda_1)(3\lambda_2-\lambda_1)}\right)$$

$$+\left(-\frac{9a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2}+\frac{3Ea_1^2e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)}\right)$$
$$+\left(-\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)}-\frac{3Ea_2^2e^{(iv+2\lambda_2)t}}{2(iv+\lambda_1)(3\lambda_2-\lambda_1)}\right) \tag{6.30}$$

Since u_2 dose not contain the first harmonic terms, so the Eq.(6.20) can be separated for B_1, B_2 in the following way:

$$\left(\frac{\partial}{\partial t}-\lambda_2\right)(B_1e^{\lambda_1 t})=\frac{3a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)}-\frac{9a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_1^2}-\frac{9a_2^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{4\lambda_1\lambda_2}$$
$$+\frac{3Ea_1a_2e^{(iv+\lambda_1+\lambda_2)t}}{2\lambda_1(iv-\lambda_2)}-\frac{3Ea_1^2e^{(-iv+2\lambda_1)t}}{4\lambda_1(iv+\lambda_1)} \tag{6.31}$$

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)(B_2e^{\lambda_2 t})=\frac{3a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)}-\frac{9a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2^2}-\frac{9a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{4\lambda_1\lambda_2}$$
$$-\frac{3Ea_1a_2e^{(-iv+\lambda_1+\lambda_2)t}}{2\lambda_2(iv+\lambda_1)}+\frac{3Ea_2^2e^{(iv+2\lambda_2)t}}{4\lambda_2(iv-\lambda_2)} \tag{6.32}$$

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_2=\frac{3a_1^5e^{5\lambda_1 t}}{2\lambda_1(3\lambda_1-\lambda_2)}+\frac{3a_2^5e^{5\lambda_2 t}}{2\lambda_2(3\lambda_2-\lambda_1)}+\frac{3a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{\lambda_1(3\lambda_1-\lambda_2)}$$
$$+\frac{3a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{\lambda_2(3\lambda_2-\lambda_1)}-\frac{9a_1^4a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2(3\lambda_1-\lambda_2)}+\frac{3Ea_1^2(iv+\lambda_1)e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)(3\lambda_1-\lambda_2)}$$
$$-\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{\lambda_2(3\lambda_2-\lambda_1)}-\frac{3Ea_2^2(-iv+2\lambda_2-\lambda_1)e^{(-iv+2\lambda_2)t}}{4\lambda_2(iv+\lambda_1)(3\lambda_2-\lambda_1)}-\frac{9a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2}$$
$$+\frac{3Ea_1^2e^{(iv+2\lambda_1)t}}{4\lambda_1(iv-\lambda_2)}-\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)}-\frac{3Ea_2^2e^{(iv+2\lambda_2)t}}{2(iv+\lambda_1)(3\lambda_2-\lambda_1)} \tag{6.33}$$

Solving Eq. (6.31) and (6.32), we obtain

$$B_1 = \frac{3a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)} - \frac{9a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1^2(3\lambda_1+\lambda_2)} - \frac{9a_2^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{4\lambda_1\lambda_2(3\lambda_1+\lambda_2)} + \frac{3Ea_1a_2e^{(i\nu+\lambda_2)t}}{2\lambda_1(i\nu-\lambda_2)(i\nu+\lambda_1)} - \frac{3Ea_1^2e^{(-i\nu+\lambda_1)t}}{4\lambda_1(i\nu+\lambda_1)(-i\nu+2\lambda_1-\lambda_2)} \quad (6.34)$$

$$B_2 = \frac{3a_1^2 a_2^3 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)(\lambda_1+3\lambda_2)} - \frac{9a_1^2 a_2^3 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_2^2(\lambda_1+3\lambda_2)} - \frac{9a_1^2 a_2^3 e^{2(\lambda_1+\lambda_2)t}}{4\lambda_1\lambda_2(\lambda_1+3\lambda_2)} - \frac{3Ea_1a_2e^{(-i\nu+\lambda_1)t}}{2\lambda_2(i\nu+\lambda_1)(-i\nu+\lambda_2)} + \frac{3Ea_2^2e^{(i\nu+\lambda_2)t}}{4\lambda_2(i\nu-\lambda_2)(i\nu-\lambda_1+2\lambda_2)} \quad (6.35)$$

$$\begin{aligned} u_2 = {} & \frac{3a_1^5 e^{5\lambda_1 t}}{8\lambda_1^2(3\lambda_1-\lambda_2)(5\lambda_1-\lambda_2)} + \frac{3a_2^5 e^{5\lambda_2 t}}{8\lambda_2^2(3\lambda_2-\lambda_1)(5\lambda_2-\lambda_1)} \\ & + \frac{3a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1^2(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)} + \frac{3a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t}}{4\lambda_2^2(3\lambda_2-\lambda_1)(\lambda_1+3\lambda_2)} \\ & - \frac{9a_1^4 a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{16\lambda_1^3(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)} + \frac{3Ea_1^2(i\nu+\lambda_1)e^{(i\nu+2\lambda_1)t}}{4\lambda_1(i\nu-\lambda_2)(3\lambda_1-\lambda_2)(i\nu+\lambda_1)(i\nu+2\lambda_1-\lambda_2)} \\ & - \frac{9a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t}}{4\lambda_2^2(3\lambda_2-\lambda_1)(\lambda_1+3\lambda_2)} - \frac{3Ea_2^2(-i\nu+2\lambda_2-\lambda_1)e^{(-i\nu+2\lambda_2)t}}{4\lambda_2(i\nu+\lambda_1)(3\lambda_2-\lambda_1)(-i\nu+2\lambda_2-\lambda_1)(-i\nu+\lambda_2)} \\ & - \frac{9a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t}}{16\lambda_1^3(3\lambda_1+\lambda_2)} + \frac{3Ea_1^2 e^{(i\nu+2\lambda_1)t}}{4\lambda_1(i\nu-\lambda_2)(i\nu+\lambda_1)(i\nu+2\lambda_1-\lambda_2)} - \frac{9a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t}}{8\lambda_2^2(3\lambda_2-\lambda_1)(3\lambda_2+\lambda_1)} \\ & - \frac{3Ea_2^2 e^{(i\nu+2\lambda_2)t}}{2(i\nu+\lambda_1)(3\lambda_2-\lambda_1)(i\nu+2\lambda_2-\lambda_1)(i\nu+\lambda_2)} \end{aligned} \quad (6.36)$$

Putting the values of A_1, A_2, B_1, B_2 in Eq.(6.7), we obtain

$$\begin{aligned} \dot{a}_1 = {} & \varepsilon\left(\frac{-3a_1^2 a_2 e^{(\lambda_1+\lambda_2)t}}{2\lambda_1} + \frac{Ee^{(i\nu-\lambda_1)t}}{2(i\nu-\lambda_2)}\right) \\ & + \varepsilon^2\left(\frac{3a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)} - \frac{9a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1^2(3\lambda_1+\lambda_2)} - \frac{9a_2^3 a_2^2 e^{2(\lambda_1+\lambda_2)t}}{4\lambda_1\lambda_2(3\lambda_1+\lambda_2)}\right) \\ & + \varepsilon^2\left(+\frac{3Ea_1a_2e^{(i\nu+\lambda_2)t}}{2\lambda_1(i\nu-\lambda_2)(i\nu+\lambda_1)} - \frac{3Ea_1^2e^{(-i\nu+\lambda_1)t}}{4\lambda_1(i\nu+\lambda_1)(-i\nu+2\lambda_1-\lambda_2)}\right) \end{aligned} \quad (6.37)$$

$$
\begin{aligned}
\dot{a}_2 &= \varepsilon\left(\frac{-3a_1 a_2^2 e^{(\lambda_1+\lambda_2)t}}{2\lambda_2}+\frac{-Ee^{-(i\nu+\lambda_2)t}}{2(i\nu+\lambda_1)}\right)\\
&+\varepsilon^2\left(\frac{3a_1^3a_2^2e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)}-\frac{9a_1^3a_2^2e^{2(\lambda_1+\lambda_2)t}}{2\lambda_1^2(3\lambda_1+\lambda_2)}-\frac{9a_2^3a_2^2e^{2(\lambda_1+\lambda_2)t}}{4\lambda_1\lambda_2(3\lambda_1+\lambda_2)}\right)\\
&+\varepsilon^2\left(-\frac{3Ea_1a_2e^{(-i\nu+\lambda_1)t}}{2\lambda_2(i\nu+\lambda_1)(-i\nu+\lambda_2)}+\frac{3Ea_2^2e^{(i\nu+\lambda_2)t}}{4\lambda_2(i\nu-\lambda_2)(i\nu-\lambda_1+2\lambda_2)}\right)
\end{aligned}
\qquad (6.38)
$$

Now, using the variables $a_1 = ae^{i\varphi}/2$, $a_2 = ae^{-i\varphi}/2$ and the eigenvalues $\lambda_1 = -k+i\omega$, $\lambda_2 = -k-i\omega$ and simplifying, we obtain the variational equations for a and φ in the real form (a and φ are known as amplitude and phase). Therefore, the equations (6.37) and (3.38) transform to

$$
\begin{aligned}
\dot{a} &= \varepsilon c_1 a^3 e^{-2kt} + \varepsilon Ec_2 e^{kt}\cos(\omega t-\nu t+\varphi)+\varepsilon Ec_3e^{kt}\sin(\omega t-\nu t+\varphi)\\
&+\varepsilon^2 Ec_4a^5e^{-4kt}+\varepsilon^2Ec_5a^2e^{-kt}\cos(\omega t-\nu t+\varphi)+\varepsilon^2Ec_6a^2e^{-kt}\sin(\omega t-\nu t+\varphi)\\
&+\varepsilon^2Ec_7a^2e^{-kt}\cos(\omega t-\nu t+\varphi)+\varepsilon^2Ec_8a^2e^{-kt}\sin(\omega t-\nu t+\varphi)
\end{aligned}
\qquad (6.39)
$$

and

$$
\begin{aligned}
\dot{\varphi} &= \varepsilon d_1a^2e^{-2kt}+\varepsilon Ec_3e^{kt}\cos(\omega t-\nu t+\varphi)/a-\varepsilon Ec_2e^{kt}\sin(\omega t-\nu t+\varphi)/a\\
&+\varepsilon^2Ed_2a^4e^{-4kt}+\varepsilon^2Ec_6ae^{-kt}\cos(\omega t-\nu t+\varphi)-\varepsilon^2Ec_5ae^{-kt}\sin(\omega t-\nu t+\varphi)\\
&-\varepsilon^2Ec_8ae^{-kt}\cos(\omega t-\nu t+\varphi)+\varepsilon^2Ec_7ae^{-kt}\sin(\omega t-\nu t+\varphi),
\end{aligned}
\qquad (6.40)
$$

where

$$
\begin{aligned}
&c_1 = 3k/8(k^2+\omega^2),\ c_2 = k/\{k^2+(\omega+\nu)^2\},\ c_3 = -(\omega+\nu)/\{k^2+(\omega+\nu)^2\},\\
&c_4 = -3k(16k^4-36k^2\omega^2-65\omega^4)/128(k^2+\omega^2)^2(4k^2+\omega^2)(k^2+4\omega^2),\\
&c_5 = 3k/4(k^2+\omega^2)\{k^2+(\omega+\nu)^2\},\ c_6 = 3\omega/4(k^2+\omega^2)\{k^2+(\omega+\nu)^2\},\\
&c_7 = 3k(k^2+\nu^2-7\omega^2-2\omega\nu)/8(k^2+\omega^2)\{k^2+(\omega+\nu)^2\}\{k^2+(3\omega-\nu)^2\},\\
&c_8 = 3\omega(3\omega^2+2\omega\nu-5k^2-\nu^2)/8(k^2+\omega^2)\{k^2+(\omega+\nu)^2\}\{k^2+(3\omega-\nu)^2\}\\
&d_1 = 3\omega/8(k^2+\omega^2),\\
&d_2 = -3\omega(26k^6+124k^2\omega^2-10\omega^4)/128(k^2+\omega^2)^2(4k^2+\omega^2)(k^2+4\omega^2).
\end{aligned}
$$

The form of the variational equations (6.39) and (6.40) are same as the form of the KBM solution. The variational equations for amplitude and phase are usually appeared in a set of first order differential equations and solved by a numerical technique.

Therefore, the second approximate solution of the equation (6.20) is

$$x(t, \varepsilon) = a\, e^{-k\,t} \cos(\omega t + \varphi) + \varepsilon\, u_1 + \varepsilon^2 u_2 + \cdots, \tag{6.41}$$

where a and φ are the solutions of the equations (6.39) and (6.40) and u_1, u_2 are given by (6.29) and (6.36).

For the damped forced vibration, the stationary oscillation has a great importance. Therefore, to investigate the stationary regime of vibration or to examine the stability of the stationary regime of oscillations, we have to eliminate the time from amplitude and phase. To do this, we have substituted $b = ae^{-kt}$ and $\psi = \omega t - \nu t + \varphi$. This leads the equations (6.39) and (6.40) to

$$\begin{aligned}\dot{b} = &-k\,b + \varepsilon\, c_1 b^3 + \varepsilon\, E\, c_2 \cos\psi + \varepsilon\, E\, c_3 \sin\psi + \varepsilon^2 E\, c_4 b^5 + \varepsilon^2 E\, c_5 b^2 \cos\psi \\ &+ \varepsilon^2 E\, c_6 b^2 \sin\psi + \varepsilon^2 E\, c_7 b^2 \cos\psi + \varepsilon^2 E\, c_8 b^2 \sin\psi\end{aligned} \tag{6.42}$$

and

$$\begin{aligned}\dot{\psi} = &(\omega - \nu) + \varepsilon\, d_1 b^3 + \varepsilon\, b^{-1} E\, c_3 \cos\psi - \varepsilon\, b^{-1} E\, c_2 \sin\psi - \varepsilon^2 E\, d_2 b^4 \\ &+ \varepsilon^2 E\, c_6 b \cos\psi - \varepsilon^2 E\, c_5 b \sin\psi - \varepsilon^2 E\, c_8 b \cos\psi + \varepsilon^2 E\, c_7 b \sin\psi.\end{aligned} \tag{6.43}$$

For steady state solution, setting $\dot{b} = \dot{\psi} = 0$, and neglecting the terms c_7 and c_8, equations (6.42) and (6.43) become

$$k\,b - \varepsilon\, c_1 b^3 - \varepsilon^2 c_4 b^5 = E(\varepsilon\, c_2 + \varepsilon^2 c_3\, b^2)\cos\psi + E(\varepsilon\, c_3 + \varepsilon^2 c_6\, b^2)\sin\psi \tag{6.44}$$

and

$$-(\omega - \nu)\,b - \varepsilon\, d_1 b^3 - \varepsilon^2 d_2 b^5 = E(\varepsilon\, c_3 + \varepsilon^2 c_6\, b^2)\cos\psi - E(\varepsilon\, c_2 + \varepsilon^2 c_5\, b^2)\sin\psi \tag{6.45}$$

In the case of the stationary regime, eliminating ψ from equations (6.44) and (6.45) gives the equation for the resonance curve (see [19]). Therefore, we obtain

$$\begin{aligned}&\varepsilon^4 b^{10}(c_4^2 + d_2^2) + 2\varepsilon^3 b^8 (c_1 c_4 + d_1 d_2) + \varepsilon^2 b^6 \{c_1^2 - 2k\, c_4 + d_1^2 + 2(\omega - \nu)\, d_2\} \\ &+ b^4\{-2\varepsilon\, c_1 k + 2\varepsilon\, d_1 (\omega - \nu) - E^2 \varepsilon^4 c_5^2 + E^2 \varepsilon^4 c_6^2) + b^2\{k^2 + (\omega - \nu)^2 \\ &- 2E^2\, \varepsilon^3\, c_2\, c_5 - 2E^2\, \varepsilon^3 c_3\, c_6\} - E^2 \varepsilon^2 (c_2^2 + c_3^2) = 0\end{aligned} \tag{6.46}$$

which relates the amplitude of the response b, to the frequency ν, of the forcing term and the natural frequency of the system.

6.3.2 Third-order equation

Let us consider a third order nonlinear differential system with an external force

$$\dddot{x} + k_1\ddot{x} + k_2\dot{x} + k_3 x = -\varepsilon x^3 + \varepsilon E\cos\nu t, \tag{6.47}$$

where k_1, k_2, k_3 are constants and ν is the frequency of the external forces. When $\varepsilon = 0$, equation (6.47) has three eigenvalues $\lambda_1 = -\xi$, $\lambda_2 = -k + i\omega$, $\lambda_3 = -k - i\omega$ and $k_1 = \xi + 2k$, $k_2 = 2k\xi + k^2 + \omega^2$, $k_3 = \xi(k^2 + \omega^2)$.

Thus for equation (6.47), we obtain

$$\begin{aligned} f^{(0)} = -\{ & a_1^3 e^{3\lambda_1 t} + 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + a_2^3 e^{3\lambda_2 t} \\ & + 3a_1^2 a_3 e^{(2\lambda_1+\lambda_3)t} + 3a_2^2 a_3 e^{(2\lambda_2+\lambda_3)t} + 3a_1 a_3^2 e^{(\lambda_1+2\lambda_3)t} \\ & + 3a_2 a_3^2 e^{(\lambda_2+2\lambda_3)t} + a_3^3 e^{3\lambda_3 t} + 6a_1 a_2 a_3 e^{(\lambda_1+\lambda_2+\lambda_3)t} \\ & - \frac{E}{2}(e^{i\nu t} + e^{-i\nu t}) + 3\varepsilon\left(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t}\right)^2 u_1 + \cdots \} \end{aligned} \tag{6.48}$$

Therefore, equation (6.8) becomes

$$\begin{aligned} & (D-\lambda_2)(D-\lambda_3)(\varepsilon A_1 e^{\lambda_1 t} + \varepsilon^2 B_1 e^{\lambda_1 t} + \cdots) \\ & + (D-\lambda_1)(D-\lambda_3)(\varepsilon A_2 e^{\lambda_2 t} + \varepsilon^2 B_2 e^{\lambda_2 t} + \cdots) \\ & + (D-\lambda_1)(D-\lambda_2)(\varepsilon A_3 e^{\lambda_3 t} + \varepsilon^2 B_3 e^{\lambda_3 t} + \cdots) \\ & + (D-\lambda_1)(D-\lambda_2)(D-\lambda_3)(\varepsilon u_1 + \varepsilon^2 u_2 + \cdots \\ & = -\{a_1^3 e^{3\lambda_1 t} + 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + a_2^3 e^{3\lambda_2 t} \\ & + 3a_1^2 a_3 e^{(2\lambda_1+\lambda_3)t} + 3a_2^2 a_3 e^{(2\lambda_2+\lambda_3)t} + 3a_1 a_3^2 e^{(\lambda_1+2\lambda_3)t} + 3a_2 a_3^2 e^{(\lambda_2+2\lambda_3)t} + a_3^3 e^{3\lambda_3 t} \\ & + 6a_1 a_2 a_3 e^{(\lambda_1+\lambda_2+\lambda_3)t} - \frac{E}{2}(e^{i\nu t} + e^{-i\nu t}) + 3\varepsilon\left(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t}\right)^2 u_1 + \cdots\} \end{aligned} \tag{6.49}$$

Following the assumption (discussed in Section 6.2) u_1 excludes the terms $a_1^3 e^{3\lambda_1 t}$, $6\varepsilon a_1 a_2 a_3 e^{(\lambda_1+\lambda_2+\lambda_3)t}$, $3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t}$, $3\varepsilon a_2^2 a_3 e^{(2\lambda_2+\lambda_3)t}$, $3a_1^2 a_3 e^{(2\lambda_1+\lambda_3)t}$ and $3\varepsilon a_2 a_3^2 e^{(\lambda_2+2\lambda_3)t}$. Moreover u_1 also

excludes the term $\varepsilon E\left(e^{i\nu t} + e^{-i\nu t}\right)/(2)$. Thus, for equation (6.47), we obtain

$$(D-\lambda_2)(D-\lambda_3)(A_1 e^{\lambda_1 t}) = -a_1^3 e^{3\lambda_1 t} - 6a_1 a_2 a_3 e^{(\lambda_1+\lambda_2+\lambda_3)t} \tag{6.50}$$

$$(D-\lambda_1)(D-\lambda_3)(A_2 e^{\lambda_2 t}) = -3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} - 3a_2^2 a_3 e^{(2\lambda_2+\lambda_3)t} + \frac{E}{2} e^{i\nu t} \tag{6.51}$$

$$(D-\lambda_1)(D-\lambda_2)(A_3 e^{\lambda_3 t}) = -3a_1^2 a_3 e^{(2\lambda_1+\lambda_3)t} - 3a_2 a_3^2 e^{(\lambda_2+2\lambda_3)t} + \frac{E}{2} e^{-i\nu t} \tag{6.52}$$

$$\begin{aligned}&(D-\lambda_1)(D-\lambda_2)(D-\lambda_3)u_1\\ &= -\{3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + 3a_1^2 a_3 e^{(2\lambda_1+\lambda_3)t} + 3a_1 a_3^2 e^{(\lambda_1+2\lambda_3)t} + a_2^3 e^{3\lambda_2 t} + a_3^3 e^{3\lambda_3 t}\}\end{aligned} \tag{6.53}$$

and

$$\begin{aligned}&(D-\lambda_2)(D-\lambda_3)(B_1 e^{\lambda_1 t}) + (D-\lambda_1)(D-\lambda_3)(B_2 e^{\lambda_2 t})\\ &+(D-\lambda_1)(D-\lambda_2)(B_3 e^{\lambda_3 t}) + (D-\lambda_1)(D-\lambda_2)(D-\lambda_3)u_2\\ &= -3(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t})^2 u_1 - \Phi\{(D-\lambda_2)A_1 e^{\lambda_1 t} + (D-\lambda_3)(A_2 e^{\lambda_2 t})\\ &+(D-\lambda_1)(A_3 e^{\lambda_3 t})\} - (D-\lambda_3)\Phi A_1 e^{\lambda_1 t} - (D-\lambda_1)\Phi A_2 e^{\lambda_2 t} - (D-\lambda_2)\Phi A_3 e^{\lambda_3 t}\\ &-\{(D^2 + C_1 D + C_2)\Phi + (D+C_1)\Phi D + \Phi D^2\}u_1\end{aligned} \tag{6.54}$$

To obtain the first approximate solution it can be considered that a_1, a_2 and a_3 are constants (see also [56,57] for details) in the right hand sides of equations (6.50)-(6.52). Thus the particular solutions of equations (6.50)-(6.52) respectively become

$$A_1 = -\frac{a_1^3 e^{2\lambda_1 t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)} - \frac{6a_1 a_2 a_3 e^{(\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)}$$

$$A_1 = l_1 a_1^3 e^{2\lambda_1 t} + l_2 a_1 a_2 a_3 e^{(\lambda_2+\lambda_3)t} \tag{6.55}$$

$$A_2 = -\frac{3a_1^2 a_2 e^{2\lambda_1 t}}{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)} - \frac{3a_2^2 a_3 e^{(\lambda_2+\lambda_3)t}}{2\lambda_2(2\lambda_2-\lambda_1+\lambda_3)} + \frac{E\, e^{(i\nu-\lambda_2)t}}{2(i\nu-\lambda_1)(i\nu-\lambda_3)}$$

$$A_2 = m_1 a_1^2 a_2 e^{2\lambda_1 t} + m_2 a_2^2 a_3 e^{(\lambda_2+\lambda_3)t} + E m_3 e^{(i\nu-\lambda_2)} \tag{6.56}$$

$$A_3 = -\frac{3a_1^2 a_3 e^{2\lambda_1 t}}{(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)} - \frac{3a_2 a_3^2 e^{(\lambda_2+\lambda_3)t}}{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)} + \frac{E\, e^{-(i\nu+\lambda_3)t}}{2(i\nu+\lambda_1)(i\nu+\lambda_2)}$$

$$A_3 = n_1 a_1^2 a_3 e^{2\lambda_1 t} + n_2 a_2 a_3^2 e^{(\lambda_2+\lambda_3)t} + E n_3 e^{-(i\nu+\lambda_3)} \tag{6.57}$$

and

$$u_1 = \frac{-3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)} + \frac{-3a_1 a_3^2 e^{(\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$
$$+\frac{-a_2^3 e^{3\lambda_2 t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)} + \frac{-a_3^3 e^{3\lambda_3 t}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_3)}$$

or,

$$u_1 = c_1 a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + c_2 a_1 a_3^2 e^{(\lambda_1+2\lambda_3)t} + c_3 a_2^3 e^{3\lambda_2 t} + c_4 a_3^3 e^{3\lambda_3 t} \tag{6.58}$$

where

$$l_1 = -\frac{1}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}, \quad l_2 = -\frac{6}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)}$$
$$m_1 = -\frac{3}{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)}, \quad m_2 = -\frac{3}{2\lambda_2(2\lambda_2-\lambda_1+\lambda_3)},$$
$$m_3 = \frac{1}{2(i\nu-\lambda_1)(i\nu-\lambda_3)}, \quad n_1 = -\frac{3}{(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)},$$
$$n_2 = -\frac{3}{2\lambda_3(\lambda_2+2\lambda_3-\lambda_1)}, \quad n_3 = \frac{1}{2(i\nu+\lambda_1)(i\nu+\lambda_2)}$$
$$c_1 = \frac{-3}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}, \quad c_2 = \frac{-3}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$
$$c_3 = \frac{-1}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}, \quad c_4 = \frac{-1}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_3)}$$

Substituting the values of A_1, A_2, A_3 and u_1 in the right hand side of (6.54), we obtain

$$(D-\lambda_2)(D-\lambda_3)(B_1 e^{\lambda_1 t}) + (D-\lambda_1)(D-\lambda_3)(B_2 e^{\lambda_2 t})$$
$$+(D-\lambda_1)(D-\lambda_2)(B_3 e^{\lambda_3 t}) + (D-\lambda_1)(D-\lambda_2)(D-\lambda_3)u_2 =$$

$$-3\{c_1 a_1^3 a_2^2 e^{(3\lambda_1+2\lambda_2)t} + c_2 a_1^3 a_3^2 e^{(3\lambda_1+2\lambda_3)t} + c_3 a_1^2 a_2^3 e^{(2\lambda_1+3\lambda_2)t} + c_4 a_1^2 a_3^3 e^{(2\lambda_1+3\lambda_3)t}$$
$$+c_1 a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t} + c_2 a_1 a_2^2 a_3^2 e^{(\lambda_1+2\lambda_2+2\lambda_3)t} + c_3 a_2^5 e^{5\lambda_2 t} + c_4 a_2^2 a_3^3 e^{(2\lambda_2+3\lambda_2)t} +$$

$$+c_1a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}+c_2a_1a_3^4e^{(\lambda_1+4\lambda_3)t}+c_3a_2^3a_3^2e^{(3\lambda_2+2\lambda_3)t}+c_4a_3^5e^{(5\lambda_3)t}+$$
$$2c_1a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}+2c_2a_1^2a_2a_3^2e^{(2\lambda_1+\lambda_2+2\lambda_3)t}+2c_3a_1a_2^4e^{(\lambda_1+4\lambda_2)t}+2c_4a_1a_2a_3^3e^{(\lambda_1+\lambda_2+3\lambda_3)t}$$

$$+2c_1a_1a_2^3a_3e^{(\lambda_1+3\lambda_2+\lambda_3)t}+2c_2a_1a_2a_3^3e^{(\lambda_1+\lambda_2+3\lambda_3)t}+2c_3a_2^4a_3e^{(4\lambda_2+\lambda_3)t}+2c_4a_2a_3^4e^{(\lambda_2+4\lambda_3)t}+$$
$$2c_1a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}+2c_2a_1^2a_3^3e^{(2\lambda_1+3\lambda_3)t}+2c_3a_1a_2^3a_3e^{(\lambda_1+3\lambda_2+\lambda_3)t}+2c_4a_1a_3^4e^{(\lambda_1+4\lambda_3)t}\Big\}+$$

$$\Bigg\{\frac{-3a_1^5e^{5\lambda_1t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)^2}-\frac{6a_1^3a_2a_3e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)}-\frac{6a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)}$$
$$-\frac{6a_1^4a_3e^{(4\lambda_1+\lambda_3)t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)}-\frac{18a_1^3a_2a_3e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)}$$

$$-\frac{36a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)}-\frac{36a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)}-\frac{36a_1^2a_2a_3^2e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)}$$
$$-\frac{18a_1^3a_2a_3e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)}-\frac{9a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{(\lambda_1+\lambda_2)^2(2\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{18a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)}-\frac{9a_1^2a_2a_3^2e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)}$$

$$-\frac{18a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)}-\frac{9a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)}-\frac{9a_2^3a_3^2e^{(3\lambda_2+2\lambda_3)t}}{\lambda_2(2\lambda_2+\lambda_3-\lambda_1)^2}$$
$$-\frac{9a_2^2a_3^3e^{(2\lambda_2+3\lambda_3)t}}{4\lambda_2\lambda_3(2\lambda_2+\lambda_3-\lambda_1)}+\frac{3Ea_1a_3e^{(iv+\lambda_1+\lambda_3)t}}{(\lambda_1+\lambda_2)(iv-\lambda_1)(iv-\lambda_3)}+\frac{3Ea_1^2e^{(iv+2\lambda_1)t}}{2(\lambda_1+\lambda_2)(iv-\lambda_1)(iv-\lambda_3)}+$$

$$\frac{3Ea_2a_3e^{(iv+\lambda_2+\lambda_3)t}}{2(2\lambda_2+\lambda_3-\lambda_1)(iv-\lambda_1)(iv-\lambda_3)}+\frac{3Ea_3^2e^{(iv+2\lambda_3)t}}{4\lambda_3(iv-\lambda_1)(iv-\lambda_3)}\Bigg\}+\Bigg\{\frac{3Ea_1a_2e^{(-iv+\lambda_1+\lambda_3)t}}{(\lambda_1+\lambda_2)(iv+\lambda_1)(iv+\lambda_2)}$$
$$+\frac{3Ea_2^2e^{(-iv+2\lambda_2)t}}{2(2\lambda_2+\lambda_3-\lambda_1)(iv+\lambda_1)(iv+\lambda_2)}+\frac{3Ea_1^2e^{(-iv+2\lambda_1)t}}{2(iv+\lambda_1)(iv+\lambda_2)(2\lambda_1+\lambda_3-\lambda_2)}$$
$$+\frac{3Ea_2a_3e^{(-iv+\lambda_2+\lambda_3)t}}{2\lambda_3(iv+\lambda_1)(iv+\lambda_2)}-\frac{18a_1^3a_2a_3e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)(2\lambda_1+\lambda_3-\lambda_2)}-\frac{9a_1^4a_3e^{(4\lambda_{12}+\lambda_3)t}}{2(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)^2}$$
$$-\frac{9a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)}-\frac{18a_1^2a_2a_3^2e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)}-$$
$$\frac{9a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}}{\lambda_3(\lambda_1+\lambda_2)(2\lambda_3-\lambda_1+\lambda_2)}-\frac{9a_2^3a_3^2e^{(3\lambda_2+2\lambda_3)t}}{2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)}-\frac{9a_2^2a_3^3e^{(2\lambda_2+3\lambda_3)t}}{2\lambda_3^2(2\lambda_3-\lambda_1+\lambda_2)}$$

$$
\begin{aligned}
&-\frac{9a_1^2a_2a_3^2e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_1-\lambda_2+\lambda_3)}\Bigg\}+\Bigg\{-\frac{3a_1^5(5\lambda_1-\lambda_3)e^{(5\lambda_1)t}}{(3\lambda_1-\lambda_2)^2(3\lambda_1-\lambda_3)^2}-\\
&\frac{36a_1^3a_2a_3(3\lambda_1+\lambda_2)e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)}-\frac{36a_1a_2^2a_3^2(\lambda_1+2\lambda_2+\lambda_3)e^{(\lambda_1+2\lambda_2+2\lambda_3)t}}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)^2}\\
&-\frac{18a_1^3a_2a_3(3\lambda_1+\lambda_2)e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_3)}-\frac{18a_1^3a_2a_3(3\lambda_1+\lambda_2)e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)(2\lambda_1-\lambda_2+_3)(\lambda_1+\lambda_3)^2}\\
&-\frac{18a_1a_2^2a_3^2(\lambda_1+2\lambda_2+\lambda_3)(\lambda_2+\lambda_3)e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{2\lambda_2\lambda_3(\lambda_1+\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)(\lambda_1+\lambda_3)}\\
&+\frac{3Ea_1a_3(-iv+\lambda_1+\lambda_2-\lambda_3)e^{(-iv+\lambda_1+\lambda_3)t}}{2(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(iv-\lambda_1)(iv-\lambda_3)}+\frac{3Ea_1a_2e^{(-iv+\lambda_1+\lambda_3)t}}{2(iv+\lambda_1)(iv+\lambda_2)(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)}\Bigg\}\\
&+\Bigg\{\frac{-6a_1^4a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)}\\
&-\frac{36a_1^2a_2^2a_3(\lambda_1+2\lambda_2+\lambda_3)e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)}-\frac{9a_1^4a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{(2\lambda_1+\lambda_2-\lambda_3)^2(\lambda_1+\lambda_2)^2}\\
&\frac{-9a_1^2a_2^2a_3(\lambda_1+2\lambda_2+2\lambda_3)e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{\lambda_2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2)(2\lambda_2-\lambda_1+\lambda_3)}-\\
&\frac{9a_1^2a_2^2a_3(\lambda_1+2\lambda_2+\lambda_3)e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{2\lambda_2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2)(2\lambda_2-\lambda_1+\lambda_3)}\\
&-\frac{9a_2^3a_3^2(3\lambda_2+\lambda_3-\lambda_1)e^{(3\lambda_2+2\lambda_3)t}}{\lambda_2^2(2\lambda_2-\lambda_1+\lambda_3)^2}-\frac{9a_2^3a_3^2(2\lambda_2+\lambda_3+\lambda_1)e^{(3\lambda_2+2\lambda_3)t}}{2\lambda_2(2\lambda_2-\lambda_1+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)(\lambda_1+\lambda_3)}\\
&-\frac{9a_2^3a_3^2(3\lambda_2+2\lambda_3-\lambda_1)e^{(3\lambda_2+2\lambda_3)t}}{4\lambda_2\lambda_3(2\lambda_3-\lambda_1+\lambda_2)(2\lambda_2-\lambda_1+\lambda_3)}+\frac{3Ea_1^2e^{(iv+2\lambda_1)t}(iv+\lambda_1)}{2(iv-\lambda_1)(iv-\lambda_3)(2\lambda_1-\lambda_3+\lambda_2)(\lambda_1+\lambda_2)}\\
&+\frac{3Ea_2^2e^{(-iv+2\lambda_2)t}(-iv+2\lambda_2-\lambda_1)}{4\lambda_2(2\lambda_2+\lambda_3-\lambda_1)(iv+\lambda_1)(iv+\lambda_2)}+\frac{3Ea_2a_3e^{(iv+\lambda_2+\lambda_3)t}(iv+\lambda_2+\lambda_3-\lambda_1)}{2\lambda_2(2\lambda_2+\lambda_3-\lambda_1)(iv-\lambda_1)(iv-\lambda_3)}\Bigg\}\\
&+\Bigg\{\frac{-6a_1^4a_3(4\lambda_1+\lambda_3-\lambda_2)e^{(4\lambda_1+\lambda_3)t}}{(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)}\\
&-\frac{36a_1^2a_2a_3^2(2\lambda_1+2\lambda_3)e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{(2\lambda_1-\lambda_2+\lambda_3)(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2}-\frac{9a_1^4a_3(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_3)t}}{(2\lambda_1-\lambda_2+\lambda_3)^2(\lambda_1+\lambda_3)^2}
\end{aligned}
\qquad (6.59)
$$

$$-\frac{9a_1^2a_2a_3^2(2\lambda_1+2\lambda_3)e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{2\lambda_3(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2)(\lambda_2-\lambda_1+2\lambda_3)}-$$

$$-\frac{9a_1^2a_2a_3^2(2\lambda_1+2\lambda_3)e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{2\lambda_3(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_3)(\lambda_2-\lambda_1+2\lambda_3)}-$$

$$\frac{9a_1^2a_2a_3^2(2\lambda_1+2\lambda_3)e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{\lambda_3(2\lambda_1-\lambda_2+\lambda_3)(\lambda_1+\lambda_3)(\lambda_2-\lambda_1+2\lambda_3)}$$

$$-\frac{9a_2^2a_3^3(\lambda_2+3\lambda_3)e^{(2\lambda_2+3\lambda_3)t}}{4\lambda_2\lambda_3(2\lambda_2-\lambda_1+\lambda_3)(\lambda_2-\lambda_1+2\lambda_3)}-\frac{9a_2^3a_3^2(3\lambda_3+\lambda_2)e^{(3\lambda_2+2\lambda_3)t}}{2\lambda_2^2(\lambda_2-\lambda_1+2\lambda_3)^2}$$

$$+\frac{3Ea_1^2e^{(-iv+2\lambda_1)t}(-iv+2\lambda_1-\lambda_2)}{2(iv+\lambda_1)(iv+\lambda_2)(2\lambda_1-\lambda_2+\lambda_3)(\lambda_1+\lambda_3)}$$

$$\left.+\frac{3Ea_3^2e^{(-iv+2\lambda_3)t}(iv+2\lambda_3-\lambda_2)}{4\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(iv-\lambda_1)(iv-\lambda_3)}+\frac{3Ea_2a_3e^{(-iv+\lambda_2+\lambda_3)t}(-iv+\lambda_3)}{2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(iv+\lambda_1)(iv+\lambda_2)}\right\}$$

$$-\frac{3a_1^3a_2^2\{(3\lambda_1+2\lambda_2)^2-\sum\lambda_1(3\lambda_1+2\lambda_2)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{3a_1^3a_3^2\{(3\lambda_1+2\lambda_3)^2-\sum\lambda_1(3\lambda_1+2\lambda_3)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$-\frac{18a_1a_2^3a_3\{(\lambda_1+3\lambda_2+\lambda_3)^2-\sum\lambda_1(\lambda_1+3\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$-\frac{18a_1a_2a_3^3\{(\lambda_1+\lambda_2+3\lambda_3)^2-\sum\lambda_1(\lambda_1+\lambda_2+3\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$-\frac{3a_1^3a_2^2\{(3\lambda_1+2\lambda_2)^2-\sum\lambda_1(3\lambda_1+2\lambda_2)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{9a_1^2a_2^3\{(2\lambda_1+3\lambda_2)^2-\sum\lambda_1(2\lambda_1+3\lambda_2)+\sum\lambda_1\lambda_2\}e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(3\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)}$$

$$-\frac{18a_1a_2^3a_3\{(\lambda_1+3\lambda_2+\lambda_3)^2-\sum\lambda_1(\lambda_1+3\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{4\lambda_2^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1-\lambda_3)}$$

$$-\frac{9a_2^4a_3\{(4\lambda_2+\lambda_3)^2-\sum\lambda_1(4\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(2\lambda_2-\lambda_1+\lambda_3)(3\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)}$$

$$+\frac{3Ea_1a_2\{(iv+\lambda_1+\lambda_2)^2-\sum\lambda_1(iv+\lambda_1+\lambda_2)+\sum\lambda_1\lambda_2\}e^{(iv+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(iv-\lambda_1)(iv-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$+\frac{3Ea_2^2\{(iv+2\lambda_2)^2-\sum\lambda_1(iv+2\lambda_2)+\sum\lambda_1\lambda_2\}e^{(iv+2\lambda_2)t}}{4\lambda_2(3\lambda_2-\lambda_1)(iv-\lambda_1)(iv-\lambda_3)(3\lambda_2-\lambda_3)}$$

$$-\frac{18a_1^3a_3^2\{(3\lambda_1+2\lambda_3)^2-\sum\lambda_1(3\lambda_1+2\lambda_3)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(2\lambda_1-\lambda_2+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$-\frac{3a_1^2a_3^3\{(2\lambda_1+3\lambda_3)^2-\sum\lambda_1(2\lambda_1+3\lambda_3)+\sum\lambda_1\lambda_2\}e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{18a_1a_2a_3^3\{(\lambda_1+\lambda_2+3\lambda_3)^2-\sum\lambda_1(\lambda_1+\lambda_2+3\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)^2}$$

$$-\frac{9a_2a_3^4\{(\lambda_1+4\lambda_3)^2-\sum\lambda_1(\lambda_1+4\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_2+4\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$+\frac{6Ea_1a_3\{(-iv+\lambda_1+\lambda_3)^2-\sum\lambda_1(-iv+\lambda_1+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(-iv+\lambda_1+\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)(iv+\lambda_1)(iv+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$+\frac{3Ea_3^2\{(-iv+2\lambda_3)^2-\sum\lambda_1(-iv+2\lambda_3)+\sum\lambda_1\lambda_2\}e^{(-iv+2\lambda_3)t}}{4\lambda_3(3\lambda_3-\lambda_1)(iv+\lambda_1)(iv+\lambda_3)(3\lambda_3-\lambda_2)}$$

$$-\frac{3a_1^3a_2^2\{(3\lambda_1+2\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+2\lambda_2)\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}$$

$$-\frac{3a_1^3a_3^2\{(3\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}$$

$$-\frac{18a_1^3a_2^2\{(3\lambda_1+2\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+2\lambda_2)\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{18a_1a_2^2a_3\{(\lambda_1+3\lambda_2+\lambda_3)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+\lambda_2)\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)}$$

$$-\frac{18Ea_1a_2\{(iv+\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+\lambda_2)\}e^{(iv+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(iv-\lambda_1)(iv-\lambda_3)}$$

$$-\frac{3a_1^2a_2^3\{(2\lambda_1+3\lambda_2)3\lambda_2+\sum\lambda_1 3\lambda_2\}e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$-\frac{9a_2^4a_3\{(4\lambda_2+\lambda_3)3\lambda_2+\sum\lambda_1 3\lambda_2\}e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(2\lambda_2-\lambda_1+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$-\frac{3Ea_2^2\{(iv+2\lambda_2)3\lambda_2+\sum\lambda_1 3\lambda_2\}e^{(iv+2\lambda_2)t}}{4\lambda_2(iv-\lambda_1)(iv-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$-\frac{18a_1^3a_3^2\{(3\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(\lambda_1+2\lambda_3-\lambda_2)(2\lambda_1-\lambda_2+\lambda_3)}$$

$$-\frac{18a_1a_2a_3^3\{(\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)^2}$$

$$-\frac{6Ea_1a_3\{(-i\nu+\lambda_1+\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(-i\nu+\lambda_1+\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(i\nu+\lambda_1)(i\nu+\lambda_2)}$$

$$-\frac{3a_1^2a_3^3\{(3\lambda_3)(2\lambda_1+3\lambda_3)+\sum\lambda_1 3\lambda_3\}e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{9a_2a_3^4\{(3\lambda_3)(\lambda_2+4\lambda_3)+\sum\lambda_1 3\lambda_3\}e^{(\lambda_2+4\lambda_3)t}}{\lambda_3^2(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{3Ea_3^2\{(3\lambda_3)(-i\nu+2\lambda_3)+\sum\lambda_1 3\lambda_3\}e^{(-i\nu+2\lambda_2)t}}{2\lambda_3(i\nu+\lambda_1)(i\nu+\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{3a_1^3a_2^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}$$

$$-\frac{3a_1^3a_3^2(\lambda_1+2\lambda_3)^2e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+\lambda_2)^2}$$

$$-\frac{18a_1^3a_2^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)}$$

$$-\frac{9a_1^2a_2^3(3\lambda_2)^2e^{(2\lambda_1+\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{4\lambda_2^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)}$$

$$-\frac{9a_2^4a_3(3\lambda_2)^2e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(-\lambda_1+2\lambda_2+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$+\frac{3Ea_2^2a_3(3\lambda_2)^2e^{-(i\nu+2\lambda_2)t}}{4\lambda_2(i\nu-\lambda_1)(i\nu-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$+\frac{6Ea_1a_2(\lambda_1+2\lambda_2)^2e^{(i\nu+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(i\nu-\lambda_3)(i\nu-\lambda_1)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{18a_1^3a_3^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$-\frac{9a_1^2a_3^3(3\lambda_3)^2e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{18a_1a_2a_3^3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(-\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$
$$-\frac{9a_2a_3^4(3\lambda_3)^2e^{(\lambda_2+4\lambda_3)t}}{4\lambda_3^2(\lambda_2+2\lambda_3-\lambda_1)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$
$$+\frac{6Ea_1a_3(\lambda_1+3\lambda_3)^2e^{(-iv+\lambda_1+\lambda_3)t}}{4\lambda_3(iv+\lambda_1)(iv+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)}$$
$$+\frac{3Ea_3^2(3\lambda_3)^2e^{(-iv+2\lambda_3)t}}{4\lambda_3(iv+\lambda_1)(iv+\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

Since u_2 dose not contain the first harmonic terms, so the Eq.(6.59) can be separated for B_1 and B_2 in the following way:

$$(D-\lambda_2)(D-\lambda_3)(B_1e^{\lambda_1t})=-3\left\{c_2a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)}+c_1a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)}\right\}+$$
$$\left\{-\frac{3a_1^5e^{5\lambda_1t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)^2}-\frac{6a_1^3a_2a_3e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)}-\frac{18a_1^3a_2a_3e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)}\right.$$
$$-\frac{36a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)}-\frac{18a_1^3a_2a_3e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)}-\frac{18a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)}$$
$$-\frac{18a_1^3a_2a_3e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)(2\lambda_1+\lambda_3-\lambda_2)}-\frac{9a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}}{\lambda_3(\lambda_1+\lambda_2)(2\lambda_3-\lambda_1+\lambda_2)} \qquad (6.60)$$

$$\left.+\frac{3Ea_1a_3e^{(iv+\lambda_1+\lambda_3)t}}{(\lambda_1+\lambda_2)(iv-\lambda_1)(iv-\lambda_3)}+\frac{3Ea_1a_2e^{(-iv+\lambda_1+\lambda_3)t}}{(\lambda_1+\lambda_2)(iv+\lambda_1)(iv+\lambda_2)}\right\}+$$
$$\left\{-\frac{3a_1^5(5\lambda_1-\lambda_3)e^{(5\lambda_1)t}}{(3\lambda_1-\lambda_2)^2(3\lambda_1-\lambda_3)^2}-\frac{36a_1^3a_2a_3(3\lambda_1+\lambda_2)e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)}\right.$$
$$-\frac{36a_1a_2^2a_3^2(\lambda_1+2\lambda_2+\lambda_3)e^{(\lambda_1+2\lambda_2+2\lambda_3)t}}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)^2}-\frac{18a_1^3a_2a_3(3\lambda_1+\lambda_2)e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_3)}$$
$$\left.-\frac{18a_1^3a_2a_3(3\lambda_1+\lambda_2)e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)(2\lambda_1-\lambda_2+_3)(\lambda_1+\lambda_3)^2}-\frac{18a_1a_2^2a_3^2(\lambda_1+2\lambda_2+\lambda_3)(\lambda_2+\lambda_3)e^{(3\lambda_1+\lambda_2+\lambda_3)t}}{2\lambda_2\lambda_3(\lambda_1+\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)(\lambda_1+\lambda_3)}\right\}$$

$$(D-\lambda_1)(D-\lambda_3)(B_2e^{\lambda_2t})=-3c_3a_2^3a_3^2e^{(3\lambda_2+2\lambda_3)}-\frac{9a_2^3a_3^2e^{(3\lambda_2+2\lambda_3)t}}{\lambda_2(2\lambda_2+\lambda_3-\lambda_1)^2}$$
$$-\frac{9a_2^3a_3^2e^{(3\lambda_2+2\lambda_3)t}}{2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)}-\frac{9a_2^3a_3^2(3\lambda_3+\lambda_2)e^{(3\lambda_2+2\lambda_3)t}}{2\lambda_2^2(\lambda_2-\lambda_1+2\lambda_3)^2}$$

$$-\frac{9a_2^3a_3^2(3\lambda_2+2\lambda_3-\lambda_1)e^{(3\lambda_2+2\lambda_3)t}}{4\lambda_2\lambda_3(2\lambda_3-\lambda_1+\lambda_2)(2\lambda_2-\lambda_1+\lambda_3)}\Bigg\}+\Bigg\{-6c_1a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)} \tag{6.61}$$

$$-\frac{36a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)}-\frac{18a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)}$$

$$-\frac{9a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)}-\frac{9a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)}$$

$$-\frac{36a_1^2a_2^2a_3(\lambda_1+2\lambda_2+\lambda_3)e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)}-\frac{27a_1^2a_2^2a_3(\lambda_1+2\lambda_2+\lambda_3)e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{2\lambda_2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2)(2\lambda_2-\lambda_1+\lambda_3)}-$$

$$-\frac{9a_1^2a_2^2a_3(\lambda_1+2\lambda_2+\lambda_3)e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{2\lambda_2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)}\Bigg\}+\Bigg\{-\frac{6a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)}$$

$$-\frac{9a_1^4a_2e^{(4\lambda_1+\lambda_2)t}}{(\lambda_1+\lambda_2)^2(2\lambda_1+2\lambda_2-\lambda_3)}-\frac{6a_1^4a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)}$$

$$-\frac{9a_1^4a_2(3\lambda_1+\lambda_2)e^{(4\lambda_1+\lambda_2)t}}{(2\lambda_1+\lambda_2-\lambda_3)^2(\lambda_1+\lambda_2)^2}\Bigg\}+\Bigg\{\frac{3Ea_1^2e^{(i\nu+2\lambda_1)t}}{2(\lambda_1+\lambda_2)(i\nu-\lambda_1)(i\nu-\lambda_3)}$$

$$+\frac{3Ea_1^2e^{(i\nu+2\lambda_1)t}(i\nu+\lambda_1)}{2(i\nu-\lambda_1)(i\nu-\lambda_3)(2\lambda_1-\lambda_3+\lambda_2)(\lambda_1+\lambda_2)}\Bigg\}+\Bigg\{\frac{3Ea_2^2e^{(-i\nu+2\lambda_2)t}}{2(2\lambda_2+\lambda_3-\lambda_1)(i\nu+\lambda_1)(i\nu+\lambda_2)}$$

$$+\frac{3Ea_2^2e^{(-i\nu+2\lambda_2)t}(-i\nu+2\lambda_2-\lambda_1)}{4\lambda_2(2\lambda_2+\lambda_3-\lambda_1)(i\nu+\lambda_1)(i\nu+\lambda_2)}\Bigg\}+\Bigg\{\frac{3Ea_2a_3e^{(i\nu+\lambda_2+\lambda_3)t}}{(2\lambda_2+\lambda_3-\lambda_1)(i\nu-\lambda_1)(i\nu-\lambda_3)}$$

$$+\frac{3Ea_2a_3e^{(i\nu+\lambda_2+\lambda_3)t}(i\nu+\lambda_2+\lambda_3-\lambda_1)}{2\lambda_2(2\lambda_2+\lambda_3-\lambda_1)(i\nu-\lambda_1)(i\nu-\lambda_3)}\Bigg\}$$

$$(D-\lambda_1)(D-\lambda_2)(B_3\,e^{\lambda_3 t})=-3c_4a_2^2a_3^3e^{(2\lambda_2+3\lambda_3)}-\frac{9a_2^2a_3^3e^{(2\lambda_2+3\lambda_3)t}}{4\lambda_2\lambda_3(2\lambda_2+\lambda_3-\lambda_1)}$$

$$-\frac{9a_2^2a_3^3e^{(2\lambda_2+3\lambda_3)t}}{2\lambda_3^2(2\lambda_3+\lambda_2-\lambda_1)}-\frac{9a_2^2a_3^3(3\lambda_3+\lambda_2)e^{(2\lambda_2+3\lambda_3)t}}{4\lambda_2\lambda_3(\lambda_2-\lambda_1+2\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)}$$

$$-\frac{9a_2^2a_3^3(\lambda_2+3\lambda_3)e^{(2\lambda_2+3\lambda_3)t}}{2\lambda_3^2(2\lambda_3-\lambda_1+\lambda_2)^2}\Bigg\}+\Big\{-6c_2a_1^2a_2a_3^2e(^{2\lambda_1+\lambda_2+2\lambda_3)}$$

$$-\frac{36a_1^2a_2a_3^2e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)}-\frac{9a_1^2a_2a_3^2e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{2\lambda_3(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2)} \qquad (6.62)$$

$$-\frac{9a_1^2a_2a_3^2e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{\lambda_3(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)}-\frac{9a_1^2a_2^2a_3e^{(2\lambda_1+2\lambda_2+\lambda_3)t}}{2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_1+\lambda_3-\lambda_2)}$$

$$-\frac{36a_1^2a_2a_3^2(2\lambda_1+2\lambda_3)e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{(2\lambda_1-\lambda_2+\lambda_3)(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2}-\frac{9a_1^2a_2a_{33}^2(2\lambda_1+2\lambda_3)e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{2\lambda_3(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+\lambda_2)(2\lambda_3+\lambda_2-\lambda_1)}-$$

$$\left.\frac{27a_1^2a_2a_3^2(2\lambda_1+2\lambda_3)e^{(2\lambda_1+\lambda_2+2\lambda_3)t}}{2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)}\right\}+\left\{-\frac{6a_1^4a_3e^{(4\lambda_1+\lambda_3)t}}{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)}\right.$$

$$-\frac{9a_1^4a_3e^{(4\lambda_1+\lambda_3)t}}{(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)^2}-\frac{6a_1^4a_3(4\lambda_1+\lambda_3-\lambda_2)e^{(4\lambda_1+\lambda_3)t}}{(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)}$$

$$\left.-\frac{9a_1^4a_2(4\lambda_1-\lambda_2+\lambda_3)e^{(4\lambda_1+\lambda_3)t}}{(2\lambda_1-\lambda_2+\lambda_3)^2(\lambda_1+\lambda_3)^2}\right\}+\left\{\frac{3Ea_1^2e^{(-iv+2\lambda_1)t}}{2(2\lambda_1+\lambda_3-\lambda_2)(iv+\lambda_1)(iv+\lambda_2)}\right.$$

$$\left.+\frac{3Ea_1^2e^{(iv+2\lambda_1)t}(-iv+2\lambda_1-\lambda_2)}{2(iv+\lambda_1)(iv+\lambda_2)(2\lambda_1+\lambda_3-\lambda_2)(\lambda_1+\lambda_3)}\right\}+\left\{\frac{3Ea_2^3e^{(iv+2\lambda_3)t}}{4(iv-\lambda_1)(iv-\lambda_3)}\right.$$

$$\left.+\frac{3Ea_3^2e^{(-iv+2\lambda_3)t}(iv+2\lambda_3-\lambda_2)}{4\lambda_3(\lambda_2+2\lambda_3-\lambda_1)(iv-\lambda_1)(iv-\lambda_3)}\right\}+\left\{\frac{3Ea_2a_3e^{-(iv+\lambda_2+\lambda_3)t}}{2\lambda_3(iv+\lambda_1)(iv+\lambda_2)}\right.$$

$$\left.+\frac{3Ea_2a_3e^{(-iv+\lambda_2+\lambda_3)t}(-iv+\lambda_3)}{2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(iv+\lambda_1)(iv+\lambda_2)}\right\}$$

$$(D-\lambda_1)(D-\lambda_2)(D-\lambda_3)u_2=$$

$$\frac{9a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}+\frac{9a_1^3a_3^2e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$+\frac{3a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}+\frac{3a_1^2a_3^3e^{(3\lambda_1+2\lambda_2)}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$+\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}+\frac{3a_2^5e^{5\lambda_2}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$+\frac{9a_1a_3^4e^{(\lambda_1+4\lambda_3)}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}+\frac{3a_3^5e^{5\lambda_3}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$+\frac{6a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}+\frac{6a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$+\frac{6a_1a_2a_3^3e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}+\frac{18a_1a_2^3a_3e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$+\frac{18a_1a_2a_3^3e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}+\frac{6a_2^4a_3e^{(4\lambda_2+\lambda_3)t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$+\frac{6a_2a_3^4e^{(\lambda_2+4\lambda_3)t}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_3)}+\frac{18a_1^2a_3^3a_3e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$+\frac{6a_1a_2^3a_3e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}+\frac{6a_1a_3^4e^{(\lambda_1+4\lambda_3)t}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{3a_1^3a_2^2\left\{(3\lambda_1+2\lambda_2)^2-\sum\lambda_1(3\lambda_1+2\lambda_2)+\sum\lambda_1\lambda_2\right\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{3a_1^3a_3^2\left\{(3\lambda_1+2\lambda_3)^2-\sum\lambda_1(3\lambda_1+2\lambda_3)+\sum\lambda_1\lambda_2\right\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$-\frac{18a_1a_2^3a_3\left\{(\lambda_1+3\lambda_2+\lambda_3)^2-\sum\lambda_1(\lambda_1+3\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\right\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$-\frac{18a_1a_2a_3^3\left\{(\lambda_1+\lambda_2+3\lambda_3)^2-\sum\lambda_1(\lambda_1+\lambda_2+3\lambda_3)+\sum\lambda_1\lambda_2\right\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)}$$

$$-\frac{3a_1^3a_2^2\left\{(3\lambda_1+2\lambda_2)^2-\sum\lambda_1(3\lambda_1+2\lambda_2)+\sum\lambda_1\lambda_2\right\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{9a_1^2a_2^3\left\{(2\lambda_1+3\lambda_2)^2-\sum\lambda_1(2\lambda_1+3\lambda_2)+\sum\lambda_1\lambda_2\right\}e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(3\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)}$$

$$-\frac{18a_1a_2^3a_3\left\{(\lambda_1+3\lambda_2+\lambda_3)^2-\sum\lambda_1(\lambda_1+3\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\right\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{4\lambda_2^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1-\lambda_3)}$$

$$-\frac{9a_2^4a_3\left\{(4\lambda_2+\lambda_3)^2-\sum\lambda_1(4\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\right\}e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(2\lambda_2-\lambda_1+\lambda_3)(3\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)}$$

$$+\frac{3Ea_1a_2\left\{(iv+\lambda_1+\lambda_2)^2-\sum\lambda_1(iv+\lambda_1+\lambda_2)+\sum\lambda_1\lambda_2\right\}e^{(iv+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(iv-\lambda_1)(iv-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$+\frac{3Ea_2^2\left\{(iv+2\lambda_2)^2-\sum\lambda_1(iv+2\lambda_2)+\sum\lambda_1\lambda_2\right\}e^{(iv+2\lambda_2)t}}{4\lambda_2(3\lambda_2-\lambda_1)(iv-\lambda_1)(iv-\lambda_3)(3\lambda_2-\lambda_3)}$$

$$
\begin{aligned}
&-\frac{18a_1^3a_3^2\{(3\lambda_1+2\lambda_3)^2-\sum\lambda_1(3\lambda_1+2\lambda_3)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(2\lambda_1-\lambda_2+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}\\
&-\frac{3a_1^2a_3^3\{(2\lambda_1+3\lambda_3)^2-\sum\lambda_1(2\lambda_1+3\lambda_3)+\sum\lambda_1\lambda_2\}e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}\\
&-\frac{18a_1a_2a_3^3\{(\lambda_1+\lambda_2+3\lambda_3)^2-\sum\lambda_1(\lambda_1+\lambda_2+3\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)^2}\\
&-\frac{9a_2a_3^4\{(\lambda_1+4\lambda_3)^2-\sum\lambda_1(\lambda_1+4\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_2+4\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}\\
&+\frac{6Ea_1a_3\{(-iv+\lambda_1+\lambda_3)^2-\sum\lambda_1(-iv+\lambda_1+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(-iv+\lambda_1+\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)(iv+\lambda_1)(iv+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)}\\
&+\frac{3Ea_3^2\{(-iv+2\lambda_3)^2-\sum\lambda_1(-iv+2\lambda_3)+\sum\lambda_1\lambda_2\}e^{(-iv+2\lambda_3)t}}{4\lambda_3(3\lambda_3-\lambda_1)(iv+\lambda_1)(iv+\lambda_3)(3\lambda_3-\lambda_2)}\\
&-\frac{3a_1^3a_2^2\{(3\lambda_1+2\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+2\lambda_2)\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}\\
&-\frac{3a_1^3a_3^2\{(3\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}\\
&-\frac{18a_1^3a_2^2\{(3\lambda_1+2\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+2\lambda_2)\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)}\\
&-\frac{18a_1a_2^2a_3\{(\lambda_1+3\lambda_2+\lambda_3)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+\lambda_2)\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)}\\
&-\frac{18Ea_1a_2\{(iv+\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+\lambda_2)\}e^{(iv+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(iv-\lambda_1)(iv-\lambda_3)}\\
&-\frac{3a_1^2a_2^3\{(2\lambda_1+3\lambda_2)3\lambda_2+\sum\lambda_1 3\lambda_2\}e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}\\
&-\frac{9a_2^4a_3\{(4\lambda_2+\lambda_3)3\lambda_2+\sum\lambda_1 3\lambda_2\}e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(2\lambda_2-\lambda_1+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}\\
&-\frac{3Ea_2^2\{(iv+2\lambda_2)3\lambda_2+\sum\lambda_1 3\lambda_2\}e^{(iv+2\lambda_2)t}}{4\lambda_2(iv-\lambda_1)(iv-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}
\end{aligned}
\qquad (6.63)
$$

$$-\frac{18a_1^3a_3^2\{(3\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(\lambda_1+2\lambda_3-\lambda_2)(2\lambda_1-\lambda_2+\lambda_3)}$$

$$-\frac{18a_1a_2a_3^3\{(\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)^2}$$

$$-\frac{6Ea_1a_3\{(-iv+\lambda_1+\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(-iv+\lambda_1+\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(iv+\lambda_1)(iv+\lambda_2)}$$

$$-\frac{3a_1^2a_3^3\{(3\lambda_3)(2\lambda_1+3\lambda_3)+\sum\lambda_1 3\lambda_3\}e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{9a_2a_3^4\{(3\lambda_3)(\lambda_2+4\lambda_3)+\sum\lambda_1 3\lambda_3\}e^{(\lambda_2+4\lambda_3)t}}{\lambda_3^2(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{3Ea_3^2\{(3\lambda_3)(-iv+2\lambda_3)+\sum\lambda_1 3\lambda_3\}e^{(-iv+2\lambda_2)t}}{2\lambda_3(iv+\lambda_1)(iv+\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{3a_1^3a_2^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}$$

$$-\frac{3a_1^3a_3^2(\lambda_1+2\lambda_3)^2e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+\lambda_2)^2}$$

$$-\frac{18a_1^3a_2^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)}$$

$$-\frac{9a_1^2a_2^3(3\lambda_2)^2e^{(2\lambda_1+\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{4\lambda_2^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)}$$

$$-\frac{9a_2^4a_3(3\lambda_2)^2e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(-\lambda_1+2\lambda_2+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$+\frac{3Ea_2^2a_3(3\lambda_2)^2e^{-(iv+2\lambda_2)t}}{4\lambda_2(iv-\lambda_1)(iv-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$+\frac{6Ea_1a_2(\lambda_1+2\lambda_2)^2e^{(iv+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(iv-\lambda_3)(iv-\lambda_1)(\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{18a_1^3a_3^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)}$$

$$-\frac{9a_1^2a_3^3(3\lambda_3)^2e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$-\frac{18a_1a_2a_3^3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(-\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)}$$
$$-\frac{9a_2a_3^4(3\lambda_3)^2e^{(\lambda_2+4\lambda_3)t}}{4\lambda_3^2(\lambda_2+2\lambda_3-\lambda_1)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$
$$+\frac{6Ea_1a_3(\lambda_1+3\lambda_3)^2e^{(-i\nu+\lambda_1+\lambda_3)t}}{4\lambda_3(i\nu+\lambda_1)(i\nu+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)}$$
$$+\frac{3Ea_3^2(3\lambda_3)^2e^{(-i\nu+2\lambda_3)t}}{4\lambda_3(i\nu+\lambda_1)(i\nu+\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)}$$

$$B_1=a_1^5e^{4\lambda_1t}\{-3(8\lambda_1-\lambda_2-\lambda_3)\ \}/\{(5\lambda_1-\lambda_2)(5\lambda_1-\lambda_3)(3\lambda_1-\lambda_2)^2(3\lambda_1-\lambda_3)^2\}$$
$$+a_1^3a_2a_3e^{(2\lambda_1+\lambda_2+\lambda_3)t}\{-6(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)-18(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)$$
$$(3\lambda_1-\lambda_2(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)-18(\lambda_1+\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)$$
$$(2\lambda_1-\lambda_2+\lambda_3)$$
$$-18(\lambda_1+\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)-36(\lambda_1+\lambda_2)(3\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)$$
$$(2\lambda_1-\lambda_2+\lambda_3)-18(3\lambda_1+\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)-18(3\lambda_1+\lambda_2)$$
$$(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)\}/\{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)$$
$$(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_1+\lambda_2)(3\lambda_1+\lambda_3)\}$$

$$+a_1a_2^2u_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}\{-9\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)$$
$$-9\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)-72\lambda_2\lambda_3(\lambda_1+\lambda_3)$$
$$(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2 \qquad (6.64)$$
$$(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)-18\lambda_2(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2(\lambda_1+2\lambda_2-\lambda_3)$$
$$(2\lambda_2-\lambda_1+\lambda_3)-72\lambda_2\lambda_3(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)$$
$$-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2+\lambda_1-\lambda_3)$$
$$-18(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(2\lambda_2+\lambda_1-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)\}$$
$$/\{2\lambda_2\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)$$
$$(2\lambda_3+\lambda_2+\lambda_1)(2\lambda_2+\lambda_1-\lambda_3)\}$$
$$+Ea_1a_3e^{(i\nu+\lambda_3)t}3/\{(\lambda_1+\lambda_2)(i\nu-\lambda_1)(i\nu-\lambda_3)(i\nu+\lambda_1+\lambda_3-\lambda_2)(i\nu+\lambda_1)\}$$
$$+Ea_1a_2e^{(-i\nu+\lambda_2)t}3/\{(\lambda_1+\lambda_2)(i\nu+\lambda_1)(i\nu+\lambda_2)(-i\nu+\lambda_1-\lambda_3+\lambda_2)(-i\nu+\lambda_1)\}$$

$$B_2 = a_2^3 a_3^2 e^{(2\lambda_2+2\lambda_3)t}\{6\lambda_2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)^2-36\lambda_2\lambda_3$$
$$(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)-18\lambda_2^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)^2$$
$$-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)-9\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)$$
$$(3\lambda_2+2\lambda_3-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)\}/\{4\lambda_2^2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)$$
$$(2\lambda_2-\lambda_1+\lambda_3)^2(3\lambda_2-\lambda_1+2\lambda_3)(3\lambda_2+\lambda_3)\}$$

$$+a_1^2 a_2^2 a_3 e^{(2\lambda_1+\lambda_2+\lambda_3)t}18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)$$
$$-72\lambda_2\lambda_3(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)-72\lambda_2\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)$$
$$(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)-9\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)$$
$$(2\lambda_1-\lambda_2+\lambda_3)-18\lambda_2\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)$$
$$-72(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2+\lambda_1+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)$$
$$(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2+\lambda_1+\lambda_3)-9(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2+\lambda_1+\lambda_3)$$
$$-9\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)\}/\{2\lambda_2\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)$$
$$(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)(2\lambda_2+\lambda_1+\lambda_3)(2\lambda_2+2\lambda_1)\}$$

$$+a_1^4 a_2 e^{4\lambda_1 t}\{-9(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)-6(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)$$
$$(3\lambda_1+\lambda_2)-9(3\lambda_1+\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)\}/\{(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)^2$$
$$(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(3\lambda_1+\lambda_2)(4\lambda_1+\lambda_2-\lambda_3)\}$$

$$+Ea_1^2 e^{(i\nu+2\lambda_1-\lambda_2)t}(i\nu+3\lambda_1+\lambda_2-\lambda_3)/\{2(2\lambda_1+\lambda_2-\lambda_3)(i\nu+\lambda_1)(i\nu+2\lambda_1-\lambda_3)$$
$$(\lambda_1+\lambda_2)(i\nu-\lambda_1)(i\nu-\lambda_3)\}$$
$$+Ea_2^2 e^{(-i\nu+\lambda_2)t}\{3(4\lambda_2-\lambda_1-i\nu)\}\{4\lambda_2(i\nu+\lambda_1)(i\nu+\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)$$
$$(-i\nu+2\lambda_2-\lambda_1)(-i\nu+2\lambda_2-\lambda_3)\}$$
$$+Ea_2 a_3 e^{(i\nu+\lambda_3)t}\{(2+\lambda_2-\lambda_1+i\nu+\lambda_3)\}/\{2\lambda_3(i\nu-\lambda_1)(i\nu-\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)$$
$$(i\nu-\lambda_1+\lambda_2+\lambda_3)(i\nu+\lambda_2)\}$$

$$B_3 = a_2^2 a_3^3 e^{(2\lambda_2+2\lambda_3)t}\{6\lambda_2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)^2(2\lambda_2-\lambda_1+\lambda_3)-9\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)$$
$$(2\lambda_3+\lambda_2-\lambda_1)^2-18\lambda_2(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)$$
$$-9\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(2\lambda_3+\lambda_2-\lambda_1)-18\lambda_2(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(3\lambda_3+\lambda_2)$$
$$(2\lambda_2-\lambda_1+\lambda_3)\}/\{4(3\lambda_3-\lambda_2)\lambda_2\lambda_3^2(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(3\lambda_3+\lambda_2)(3\lambda_3+2\lambda_2-\lambda_1)$$
$$(2\lambda_3+\lambda_2-\lambda_1)^2(2\lambda_2-\lambda_1+\lambda_3)\}$$

$$
\begin{aligned}
&+a_1^2a_2a_3^2e^{(2\lambda_2+\lambda_2+2\lambda_3)t}\{18(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)\\
&-72\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)(2\lambda_3+\lambda_2-\lambda_1)-9(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)\\
&(2\lambda_3-\lambda_2+\lambda_1)(2\lambda_3+\lambda_2-\lambda_1)-18(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)\\
&(2\lambda_3+\lambda_2-\lambda_1)\\
&-9(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)-144)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)\\
&(2\lambda_3+\lambda_2-\lambda_1)-18\lambda_3(\lambda_1+\lambda_3)^2(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)\\
&(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)-36(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)\}/\\
&\{2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)\\
&(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_3+\lambda_2+\lambda_1)(2\lambda_3+2\lambda_2)\}\\
&+a_1^4a_3e^{4\lambda_1t}\{-9(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_1+\lambda_3)-6(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)\\
&(4\lambda_1-\lambda_2+\lambda_3)-9(4\lambda_1-\lambda_2+\lambda_3)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)\}/\{(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)\\
&(\lambda_1+\lambda_3)^2(2\lambda_1-\lambda_2+\lambda_3)^2(3\lambda_1+\lambda_3)(4\lambda_1+\lambda_3-\lambda_2)\}\\
&+Ea_1^2e^{(-i\nu+2\lambda_1-\lambda_3)}t\{3(\lambda_1+\lambda_3)+3(-i\nu+2\lambda_1-\lambda_2)\}/\{2(i\nu+\lambda_1)(i\nu+\lambda_2)\\
&(2\lambda_1+\lambda_3-\lambda_2)(-i\nu+\lambda_1)(-i\nu+2\lambda_1-\lambda_2)(\lambda_1+\lambda_3)\}\\
&+Ea_3^2e^{(i\nu_1+\lambda_3)t}\{3(2\lambda_3+\lambda_2-\lambda_1)+3(i\nu+2\lambda_3-\lambda_2)\}/\{4\lambda_3(i\nu-\lambda_1)(i\nu-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)\\
&(i\nu+2\lambda_3-\lambda_1)(i\nu+2\lambda_3-\lambda_2)\}\\
&+Ea_2a_3e^{(-i\nu+\lambda_2)t}\{3(2\lambda_3+\lambda_2-\lambda_1)+3(-i\nu+\lambda_3)\}/\{2\lambda_3(i\nu+\lambda_1)(i\nu+\lambda_2)(2\lambda_3+\lambda_2-\lambda_1)\\
&(-i\nu-\lambda_1+\lambda_2+\lambda_3)(-i\nu+\lambda_3)\}
\end{aligned}
\tag{6.66}
$$

$$
\begin{aligned}
u_2 &= \frac{9a_1^3a_2^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+2\lambda_2)(3\lambda_1+\lambda_2)(3\lambda_1+2\lambda_2-\lambda_3)}\\
&+\frac{9a_1^3a_3^2e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(2\lambda_1+2\lambda_3)(3\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1+\lambda_3)}\\
&+\frac{3a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(\lambda_1+3\lambda_2)(2\lambda_1+2\lambda_2)(2\lambda_1+3\lambda_2-\lambda_3)}\\
&+\frac{3a_1^2a_3^3e^{(3\lambda_1+2\lambda_2)}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_1+3\lambda_3)(2\lambda_1+3\lambda_3-\lambda_2)(2\lambda_1+2\lambda_3)}\\
&+\frac{9a_1a_2^4e^{(\lambda_1+4\lambda_2)}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(4\lambda_2)(\lambda_1+3\lambda_2)(\lambda_1+4\lambda_2-\lambda_3)}\\
&+\frac{3a_2^5e^{5\lambda_2}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(-\lambda_1+5\lambda_2)(4\lambda_2)(5\lambda_1-\lambda_3)}\\
&+\frac{9a_1a_3^4e^{(\lambda_1+4\lambda_3)}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(4\lambda_3)(\lambda_1+4\lambda_3-\lambda_2)(\lambda_1+3\lambda_3)}\\
&+\frac{3a_3^5e^{5\lambda_3}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(5\lambda_3-\lambda_1)(5\lambda_3-\lambda_2)(4\lambda_3)}
\end{aligned}
$$

$$+\frac{6a_1^2a_2^3e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+3\lambda_2)(2\lambda_1+2\lambda_2)(2\lambda_1+3\lambda_2-\lambda_3)}$$

$$+\frac{6a_1a_2^4e^{(\lambda_1+4\lambda_2)t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(4\lambda_2)(\lambda_1+3\lambda_2)(\lambda_1+4\lambda_2-\lambda_3)}$$

$$+\frac{6a_1a_2a_3^3e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_2+3\lambda_3)(\lambda_1+3\lambda_3)(\lambda_1+\lambda_2+2\lambda_3)}$$

$$+\frac{18a_1a_2^3a_3e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+2\lambda_2)}$$

$$+\frac{18a_1a_2a_3^3e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(\lambda_2+3\lambda_3)(\lambda_1+3\lambda_3)(\lambda_1+\lambda_2+2\lambda_3)}$$

$$+\frac{6a_2^4a_3e^{(4\lambda_2+\lambda_3)t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(4\lambda_2+\lambda_3-\lambda_1)(3\lambda_2+\lambda_3)(4\lambda_2)}$$

$$+\frac{6a_2a_3^4e^{(\lambda_2+4\lambda_3)t}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_3)(\lambda_2+4\lambda_3-\lambda_1)(4\lambda_3)(\lambda_2+3\lambda_3)}$$

$$+\frac{18a_1^2a_3^3a_3e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+3\lambda_3)(2\lambda_1+3\lambda_3-\lambda_2)(2\lambda_1+2\lambda_3)}$$

$$+\frac{6a_1a_2^3a_3e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(3\lambda_2+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(\lambda_1+2\lambda_2)}$$

$$+\frac{6a_1a_3^4e^{(\lambda_1+4\lambda_3)t}}{2\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(4\lambda_3)(\lambda_1+4\lambda_3-\lambda_2)(\lambda_1+\lambda_2+4\lambda_3)}$$

$$-\frac{3a_1^3a_2^2\{(3\lambda_1+2\lambda_2)^2-\sum\lambda_1(3\lambda_1+2\lambda_2)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2+2\lambda_2)(3\lambda_1+\lambda_2)(3\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{3a_1^3a_3^2\{(3\lambda_1+2\lambda_3)^2-\sum\lambda_1(3\lambda_1+2\lambda_3)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(2\lambda_1+2\lambda_3)(3\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1+\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3\{(\lambda_1+3\lambda_2+\lambda_3)^2-\sum\lambda_1(\lambda_1+3\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_2+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(\lambda_1+3\lambda_2)}$$

$$-\frac{18a_1a_2a_3^3\{(\lambda_1+\lambda_2+3\lambda_3)^2-\sum\lambda_1(\lambda_1+\lambda_2+3\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)(\lambda_2+3\lambda_3)(\lambda_1+3\lambda_3)(\lambda_1+\lambda_2+2\lambda_3)}$$

$$-\frac{3a_1^3a_2^2\{(3\lambda_1+2\lambda_2)^2-\sum\lambda_1(3\lambda_1+2\lambda_2)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+2\lambda_2)(3\lambda_1+\lambda_2)(3\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{9a_1^2a_2^3\{(2\lambda_1+3\lambda_2)^2-\sum\lambda_1(2\lambda_1+3\lambda_2)+\sum\lambda_1\lambda_2\}e^{(2\lambda_1+3\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(3\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)(\lambda_1+3\lambda_2)(2\lambda_2+2\lambda_2)(2\lambda_1+3\lambda_2-\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3\{(\lambda_1+3\lambda_2+\lambda_3)^2-\sum\lambda_1(\lambda_1+3\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{4\lambda_2^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1-\lambda_3)(3\lambda_2+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(\lambda_1+3\lambda_2)}$$

$$-\frac{9a_2^4a_3\{(4\lambda_2+\lambda_3)^2-\sum\lambda_1(4\lambda_2+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(2\lambda_2-\lambda_1+\lambda_3)(3\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)(4\lambda_2+\lambda_3-\lambda_1)(3\lambda_2+\lambda_3)(4\lambda_2)}$$

$$+\frac{3Ea_1a_2\{(iv+\lambda_1+\lambda_2)^2-\sum\lambda_1(iv+\lambda_1+\lambda_2)+\sum\lambda_1\lambda_2\}e^{(iv+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(iv-\lambda_1)(iv-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(iv+\lambda_2)(iv+\lambda_1)(iv+\lambda_1+\lambda_2-\lambda_3)}$$

$$+\frac{3Ea_2^2\{(iv+2\lambda_2)^2-\sum\lambda_1(iv+2\lambda_2)+\sum\lambda_1\lambda_2\}e^{(iv+2\lambda_2)t}}{4\lambda_2(3\lambda_2-\lambda_1)(iv-\lambda_1)(iv-\lambda_3)(3\lambda_2-\lambda_3)(iv+2\lambda_2-\lambda_1)(iv+\lambda_2)(iv+2\lambda_2-\lambda_3)}$$

$$-\frac{18a_1^3a_3^2\{(3\lambda_1+2\lambda_3)^2-\sum\lambda_1(3\lambda_1+2\lambda_3)+\sum\lambda_1\lambda_2\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(2\lambda_1-\lambda_2+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(2\lambda_1+2\lambda_3)(3\lambda_1-\lambda_2+2\lambda_3)(3\lambda_1+\lambda_3)}$$

$$-\frac{3a_1^2a_3^3\{(2\lambda_1+3\lambda_3)^2-\sum\lambda_1(2\lambda_1+3\lambda_3)+\sum\lambda_1\lambda_2\}e^{(2\lambda_1+3\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_1+3\lambda_3)(2\lambda_1-\lambda_2+3\lambda_3)(\lambda_3+\lambda_3)}$$

$$-\frac{18a_1a_2a_3^3\{(\lambda_1+\lambda_2+3\lambda_3)^2-\sum\lambda_1(\lambda_1+\lambda_2+3\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)^2(\lambda_2+3\lambda_3)(\lambda_1+3\lambda_3)(\lambda_1+\lambda_2+2\lambda_2)}$$

$$-\frac{9a_2a_3^4\{(\lambda_1+4\lambda_3)^2-\sum\lambda_1(\lambda_1+4\lambda_3)+\sum\lambda_1\lambda_2\}e^{(\lambda_2+4\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_2+4\lambda_3-\lambda_1)(4\lambda_3)(\lambda_2+3\lambda_3)}$$

$$+\frac{6Ea_1a_3\{(-iv+\lambda_1+\lambda_3)^2-\sum\lambda_1(-iv+\lambda_1+\lambda_3)+\sum\lambda_1\lambda_2\}e^{(-iv+\lambda_1+\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)(iv+\lambda_1)(iv+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)(-iv+\lambda_3)(-iv-\lambda_2+\lambda_1+\lambda_3)(-iv+\lambda_1)}$$

$$+\frac{3Ea_3^2\{(-iv+2\lambda_3)^2-\sum\lambda_1(-iv+2\lambda_3)+\sum\lambda_1\lambda_2\}e^{(-iv+2\lambda_3)t}}{4\lambda_3(3\lambda_3-\lambda_1)(iv+\lambda_1)(iv+\lambda_3)(3\lambda_3-\lambda_2)(-iv-\lambda_1+2\lambda_3)(-iv-\lambda_2+2\lambda_3)(-iv+\lambda_3)}$$

$$-\frac{3a_1^3a_2^2\{(3\lambda_1+2\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+2\lambda_2)\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+2\lambda_2)(3\lambda_1+\lambda_2)(3\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{3a_1^3a_3^2\{(3\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\}e^{(3\lambda_1+2\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+2\lambda_3)(3\lambda_1-\lambda_2+2\lambda_3)(3\lambda_1+\lambda_3)}$$

$$-\frac{18a_1^3a_2^2\{(3\lambda_1+2\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+2\lambda_2)\}e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+2\lambda_2)(3\lambda_1+\lambda_2)(3\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3\{(\lambda_1+3\lambda_2+\lambda_3)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+\lambda_2)\}e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(\lambda_1+3\lambda_2)}$$

$$-\frac{18Ea_1a_2\{(iv+\lambda_1+\lambda_2)(\lambda_1+2\lambda_2)+\sum\lambda_1(\lambda_1+\lambda_2)\}e^{(iv+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(iv-\lambda_1)(iv-\lambda_3)(iv+\lambda_2)(iv+\lambda_1)(iv+\lambda_1+\lambda_2-\lambda_3)}$$

$$-\frac{3a_1^2a_2^3\{(2\lambda_1+3\lambda_2)3\lambda_2+\sum\lambda_1 3\lambda_2\}e^{(2\lambda_1+3\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(\lambda_1+3\lambda_2)(2\lambda_1+3\lambda_2-\lambda_3)}$$

$$-\frac{9a_2^4a_3\left\{(4\lambda_2+\lambda_3)3\lambda_2+\sum\lambda_1 3\lambda_2\right\}e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(2\lambda_2-\lambda_1+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(4\lambda_2+\lambda_3-\lambda_1)(3\lambda_2+\lambda_3)(4\lambda_2)}$$

$$-\frac{3Ea_2^2\left\{(i\nu+2\lambda_2)3\lambda_2+\sum\lambda_1 3\lambda_2\right\}e^{(i\nu+2\lambda_2)t}}{4\lambda_2(i\nu-\lambda_1)(i\nu-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(i\nu+2\lambda_2-\lambda_1)(i\nu+\lambda_2)(i\nu+2\lambda_2-\lambda_3)}$$

$$-\frac{18a_1^3a_3^2\left\{(3\lambda_1+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\right\}e^{(3\lambda_1+2\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)^3(\lambda_1+2\lambda_3-\lambda_2)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1+\lambda_3)}$$

$$-\frac{18a_1a_2a_3^3\left\{(\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\right\}e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)^2(\lambda_2+3\lambda_3)(\lambda_1+3\lambda_3)(\lambda_1+\lambda_2+2\lambda_3)}$$

$$-\frac{6Ea_1a_3\left\{(-i\nu+\lambda_1+\lambda_3)(\lambda_1+2\lambda_3)+\sum\lambda_1(\lambda_1+2\lambda_3)\right\}e^{(-i\nu+\lambda_1+\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(i\nu^2-\lambda_1^2)(i\nu+\lambda_2)(-i\nu+\lambda_3)(-i\nu-\lambda_2+\lambda_1+\lambda_3)}$$

$$-\frac{3a_1^2a_3^3\left\{(3\lambda_3)(2\lambda_1+3\lambda_3)+\sum\lambda_1 3\lambda_3\right\}e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_1+3\lambda_3)(2\lambda_1-\lambda_2+3\lambda_3)(2\lambda_1+2\lambda_3)}$$

$$-\frac{9a_2a_3^4\left\{(3\lambda_3)(\lambda_2+4\lambda_3)+\sum\lambda_1 3\lambda_3\right\}e^{(\lambda_2+4\lambda_3)t}}{\lambda_3^2(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_2+4\lambda_3-\lambda_1)(4\lambda_3)(\lambda_2+3\lambda_3)}$$

$$-\frac{3Ea_3^2\left\{(3\lambda_3)(-i\nu+2\lambda_3)+\sum\lambda_1 3\lambda_3\right\}e^{(-i\nu+2\lambda_2)t}}{2\lambda_3(i\nu+\lambda_1)(i\nu+\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(-i\nu+2\lambda_3-\lambda_1)(-i\nu-\lambda_2+2\lambda_3)(-i\nu+\lambda_3)}$$

$$-\frac{3a_1^3a_2^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(3\lambda_1+\lambda_2)(3\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{3a_1^3a_3^2(\lambda_1+2\lambda_3)^2e^{(3\lambda_1+2\lambda_3)t}}{4\lambda_3(\lambda_1+\lambda_3)^2(\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_2+2\lambda_3)(3\lambda_1+\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{2\lambda_2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+\lambda_2)^2 2\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)} \tag{6.67}$$

$$-\frac{18a_1a_2a_3^3(\lambda_1+2\lambda_3)^2e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)^2(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+\lambda_2)(\lambda_2+3\lambda_3)(\lambda_1+3\lambda_3)(\lambda_1+\lambda_2+2\lambda_3)}$$

$$-\frac{18a_1^3a_2^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)^3(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_1+\lambda_2)(3\lambda_1+2\lambda_2-\lambda_3)}$$

$$-\frac{9a_1^2a_2^3(3\lambda_2)^2e^{(2\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)^2(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(\lambda_1+3\lambda_2)(2\lambda_1+3\lambda_2-\lambda_3)}$$

$$-\frac{18a_1a_2^3a_3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+3\lambda_2+\lambda_3)t}}{4\lambda_2^2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)(2\lambda_2+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(\lambda_1+3\lambda_2)}$$

$$-\frac{9a_2^4a_3(3\lambda_2)^2e^{(4\lambda_2+\lambda_3)t}}{4\lambda_2^2(-\lambda_1+2\lambda_2+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(4\lambda_2+\lambda_3-\lambda_1)(3\lambda_2+\lambda_3)(4\lambda_2)}$$

$$+\frac{3Ea_2^2a_3(3\lambda_2)^2e^{-(iv+2\lambda_2)t}}{4\lambda_2(iv-\lambda_1)(iv-\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(-iv-2\lambda_2-\lambda_1)(-iv+\lambda_2)(-iv+2\lambda_2-\lambda_3)}$$

$$+\frac{6Ea_1a_2(\lambda_1+2\lambda_2)^2e^{(iv+\lambda_1+\lambda_2)t}}{4\lambda_2(\lambda_1+\lambda_2)(iv-\lambda_3)(iv-\lambda_1)(\lambda_1+2\lambda_2-\lambda_3)(iv+\lambda_2)(iv+\lambda_1)(iv+\lambda_1+\lambda_2-\lambda_3)}$$

$$-\frac{18a_1^3a_3^2(\lambda_1+2\lambda_2)^2e^{(3\lambda_1+2\lambda_2)t}}{2\lambda_2(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(2\lambda_1+2\lambda_3)(3\lambda_1+2\lambda_3-\lambda_2)(3\lambda_1+2\lambda_3)}$$

$$-\frac{9a_1^2a_3^3(3\lambda_3)^2e^{(2\lambda_1+3\lambda_3)t}}{2\lambda_3(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_1+3\lambda_3)(2\lambda_1+3\lambda_3-\lambda_2)(2\lambda_1+2\lambda_3)}$$

$$-\frac{18a_1a_2a_3^3(\lambda_1+2\lambda_2)^2e^{(\lambda_1+\lambda_2+3\lambda_3)t}}{4\lambda_3^2(-\lambda_1+\lambda_2+2\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(\lambda_2+3\lambda_3)(\lambda_1+3\lambda_3)(\lambda_1+\lambda_2+3\lambda_3)}$$

$$-\frac{9a_2a_3^4(3\lambda_3)^2e^{(\lambda_2+4\lambda_3)t}}{16\lambda_3^3(\lambda_2+2\lambda_3-\lambda_1)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_2+4\lambda_3-\lambda_1)(\lambda_2+3\lambda_3)}$$

$$+\frac{6Ea_1a_3(\lambda_1+3\lambda_3)^2e^{(-iv+\lambda_1+\lambda_3)t}}{4\lambda_3(iv+\lambda_1)(iv+\lambda_2)(\lambda_1+2\lambda_3-\lambda_2)(-iv+\lambda_3)(-iv-\lambda_2+\lambda_1+\lambda_3)(-iv+\lambda_1)}$$

$$+\frac{3Ea_3^2(3\lambda_3)^2e^{(-iv+2\lambda_3)t}}{4\lambda_3(iv+\lambda_1)(iv+\lambda_2)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(-iv+2\lambda_3-\lambda_1)(-iv-\lambda_2+2\lambda_3)(-iv+\lambda_3)}$$

Putting the values of $A_1, A_2, A_3, B_1, B_2\, B_3$ in Eq.(6.7), we obtain

$$\begin{aligned}
&\dot{a}_1=\varepsilon a_1^3e^{2\lambda_1 t}/\{(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)\}-\varepsilon a_1a_2a_3e^{(\lambda_2+\lambda_3)t}(\lambda_1+\lambda_3)(\lambda_1+\lambda_2)\\
&+\varepsilon^2a_1^5e^{4\lambda_1 t}\{-3(8\lambda_1-\lambda_2-\lambda_3)\}/\{(5\lambda_1-\lambda_2)(5\lambda_1-\lambda_3)(3\lambda_1-\lambda_2)^2(3\lambda_1-\lambda_3)^2\}\\
&+\varepsilon^2a_1^3a_2a_3e^{(2\lambda_1+\lambda_2+\lambda_3)t}\{-6(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)\\
&-18(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(3\lambda_1-\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)-18(\lambda_1+\lambda_3)\\
&(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)\\
&-18(\lambda_1+\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)-36(\lambda_1+\lambda_2)(3\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)\\
&(2\lambda_1-\lambda_2+\lambda_3)-18(3\lambda_1+\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)-18(3\lambda_1+\lambda_2)\\
&(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)\}/\{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)\\
&(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(3\lambda_1+\lambda_2)(3\lambda_1+\lambda_3)\}\\
&+\varepsilon^2a_1a_2^2a_3^2e^{(\lambda_1+2\lambda_2+2\lambda_3)t}-9\lambda_2(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)\\
&-9\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)-72\lambda_2\lambda_3(\lambda_1+\lambda_3)\\
&(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2\\
&(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)
\end{aligned} \tag{6.68}$$

$$-18\lambda_2(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)-72\lambda_2\lambda_3(\lambda_1+2\lambda_2-\lambda_3)$$
$$(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)$$
$$(2\lambda_2+\lambda_1-\lambda_3)-18(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)(2\lambda_2+\lambda_1-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)\}/$$
$$\{2\lambda_2\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)(2\lambda_3+\lambda_2+\lambda_1)$$
$$(2\lambda_2+\lambda_1-\lambda_3)\}$$
$$+\varepsilon^2 Ea_1a_3e^{(i\nu+\lambda_3)t}3/\{(\lambda_1+\lambda_2)(i\nu-\lambda_1)(i\nu-\lambda_3)(i\nu+\lambda_1+\lambda_3-\lambda_2)(i\nu+\lambda_1)\}$$
$$+\varepsilon^2 Ea_1a_2e^{(-i\nu+\lambda_2)t}3/\{(\lambda_1+\lambda_2)(i\nu+\lambda_1)(i\nu+\lambda_2)(-i\nu+\lambda_1-\lambda_3+\lambda_2)(-i\nu+\lambda_1)\}$$

$$\dot{a}_2=\varepsilon(-3)a_1^2a_2e^{(2\lambda_1)t}/\{(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)\}+\varepsilon(-3)a_2^2e^{(\lambda_2+\lambda_3)t}/\{(2\lambda_2)(2\lambda_2-\lambda_1+\lambda_3)\}$$
$$+\varepsilon Ee^{(i\nu-\lambda_2)t}/\{2(i\nu-\lambda_1)(i\nu-\lambda_3)\}$$
$$+\varepsilon^2 a_2^3a_3^2e^{(2\lambda_2+2\lambda_3)t}\{6\lambda_2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)^2-36\lambda_2\lambda_3(\lambda_1+\lambda_2)$$
$$(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)-18\lambda_2^2(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)^2$$
$$-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)-9\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)$$
$$(3\lambda_2+2\lambda_3-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)\}/\{4\lambda_2^2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)$$
$$(2\lambda_2-\lambda_1+\lambda_3)^2(3\lambda_2-\lambda_1+2\lambda_3)(3\lambda_2+\lambda_3)\}$$
$$+\varepsilon^2 a_1^2a_2^2a_3e^{(2\lambda_1+\lambda_2+\lambda_3)t}18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)$$
$$-72\lambda_2\lambda_3(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)-72\lambda_2\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)$$
$$(2\lambda_1-\lambda_2+\lambda_3)-9\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)$$
$$-18\lambda_2\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)$$
$$-72(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2+\lambda_1+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)-18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)$$
$$(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2+\lambda_1+\lambda_3)-9(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)(2\lambda_2+\lambda_1+\lambda_3)$$
$$-9\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+2\lambda_2-\lambda_3)\}/\{2\lambda_2\lambda_3(\lambda_1+\lambda_2)^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2-\lambda_3)$$
$$(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_2-\lambda_1+\lambda_3)(2\lambda_2+\lambda_1+\lambda_3)(2\lambda_2+2\lambda_1)\}$$
$$+\varepsilon^2 a_1^4a_2e^{4\lambda_1t}\{-9(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)-6(\lambda_1+\lambda_2)$$
$$(2\lambda_1+\lambda_2-\lambda_3)(3\lambda_1+\lambda_2)-9(3\lambda_1+\lambda_2)(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)\}/\{(\lambda_1+\lambda_2)^2 \qquad (6.69)$$
$$(\lambda_1+\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)^2(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(3\lambda_1+\lambda_2)(4\lambda_1+\lambda_2-\lambda_3)\}$$

$$+\varepsilon^2 Ea_1^2e^{(i\nu+2\lambda_1-\lambda_2)t}(i\nu+3\lambda_1+\lambda_2-\lambda_3)/\{2(2\lambda_1+\lambda_2-\lambda_3)(i\nu+\lambda_1)$$
$$(i\nu+2\lambda_1-\lambda_3)(\lambda_1+\lambda_2)(i\nu-\lambda_1)(i\nu-\lambda_3)\}$$
$$+\varepsilon^2 Ea_2^2e^{(-i\nu+\lambda_2)t}\{3(4\lambda_2-\lambda_1-i\nu)\}\{4\lambda_2(i\nu+\lambda_1)(i\nu+\lambda_2)(2\lambda_2+\lambda_3-\lambda_1)$$
$$(-i\nu+2\lambda_2-\lambda_1)(-i\nu+2\lambda_2-\lambda_3)\}$$
$$+\varepsilon^2 Ea_2a_3e^{(i\nu+\lambda_3)t}\{(2+\lambda_2-\lambda_1+i\nu+\lambda_3)\}/\{2\lambda_3(i\nu-\lambda_1)(i\nu-\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)$$
$$(i\nu-\lambda_1+\lambda_2+\lambda_3)(i\nu+\lambda_2)\}$$

$$\dot{a}_3 = \varepsilon(-3)a_1^2 a_3 e^{(2\lambda_1)t} / \{(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)\} + \varepsilon(-3)a_3^2 e^{(\lambda_2+\lambda_3)t} /$$
$$\{(2\lambda_3)(2\lambda_3-\lambda_1+\lambda_2)\} + \varepsilon E e^{(-iv+\lambda_3)t} / \{2(iv+\lambda_1)(iv+\lambda_3)\}$$
$$+\varepsilon^2 a_2^2 a_3^3 e^{(2\lambda_2+2\lambda_3)t}\{6\lambda_2\lambda_3(2\lambda_3+\lambda_2-\lambda_1)^2(2\lambda_2-\lambda_1+\lambda_3) - 9\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)$$
$$(2\lambda_3+\lambda_2-\lambda_1)^2 - 18\lambda_2(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_2-\lambda_1+\lambda_3)$$
$$-9\lambda_3(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(2\lambda_3+\lambda_2-\lambda_1) - 18\lambda_2(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(3\lambda_3+\lambda_2)$$
$$(2\lambda_2-\lambda_1+\lambda_3)\}/\{4(3\lambda_3-\lambda_2)\lambda_2\lambda_3^2(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(3\lambda_3+\lambda_2)(3\lambda_3+2\lambda_2-\lambda_1)$$
$$(2\lambda_3+\lambda_2-\lambda_1)^2(2\lambda_2-\lambda_1+\lambda_3)\}$$
$$+\varepsilon^2 a_1^2 a_2 a_3^2 e^{(2\lambda_2+\lambda_2+2\lambda_3)t}\{18(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_3+\lambda_2-\lambda_1)$$
$$-72\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)(2\lambda_3+\lambda_2-\lambda_1) - 9(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)$$
$$(2\lambda_3-\lambda_2+\lambda_1)(2\lambda_3+\lambda_2-\lambda_1) - 18(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)$$
$$(2\lambda_3+\lambda_2-\lambda_1)$$
$$-9(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1) - 144)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)$$
$$(2\lambda_3+\lambda_2-\lambda_1) - 18\lambda_3(\lambda_1+\lambda_3)^2(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_3-\lambda_2+\lambda_1) - 18\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)$$
$$(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1) - 36(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)\}/$$
$$\{2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(2\lambda_3-\lambda_2+\lambda_1)$$
$$(2\lambda_3+\lambda_2-\lambda_1)(2\lambda_3+\lambda_2+\lambda_1)(2\lambda_3+2\lambda_2)\} \tag{6.70}$$
$$+\varepsilon^2 a_1^4 a_3 e^{4\lambda_1 t}\{-9(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_1+\lambda_3) - 6(\lambda_1+\lambda_3)(2\lambda_1-\lambda_2+\lambda_3)(4\lambda_1-\lambda_2+\lambda_3)$$
$$-9(4\lambda_1-\lambda_2+\lambda_3)(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)\}/\{(3\lambda_3-\lambda_1)(3\lambda_3-\lambda_2)(\lambda_1+\lambda_3)^2(2\lambda_1-\lambda_2+\lambda_3)^2$$
$$(3\lambda_1+\lambda_3)(4\lambda_1+\lambda_3-\lambda_2)\}$$
$$+\varepsilon^2 E a_1^2 e^{(-iv+2\lambda_1-\lambda_3)}t\{3(\lambda_1+\lambda_3)+3(-iv+2\lambda_1-\lambda_2)\}/\{2(iv+\lambda_1)(iv+\lambda_2)(2\lambda_1+\lambda_3-\lambda_2)$$
$$(-iv+\lambda_1)(-iv+2\lambda_1-\lambda_2)(\lambda_1+\lambda_3)\}$$
$$+\varepsilon^2 E a_3^2 e^{(iv1+\lambda_3)t}\{3(2\lambda_3+\lambda_2-\lambda_1)+3(iv+2\lambda_3-\lambda_2)\}/\{4\lambda_3(iv-\lambda_1)(iv-\lambda_3)$$
$$(2\lambda_3+\lambda_2-\lambda_1)(iv+2\lambda_3-\lambda_1)(iv+2\lambda_3-\lambda_2)\}$$
$$+\varepsilon^2 E a_2 a_3 e^{(-iv+\lambda_2)t}\{3(2\lambda_3+\lambda_2-\lambda_1)+3(-iv+\lambda_3)\}/\{2\lambda_3(iv+\lambda_1)(iv+\lambda_2)$$
$$(2\lambda_3+\lambda_2-\lambda_1)(-iv-\lambda_1+\lambda_2+\lambda_3)(-iv+\lambda_3)\}$$

Now, substituting the variables $a_1 = a,\ a_2 = be^{i\varphi}/2,\ \ a_2 = be^{-i\varphi}/2$ and the eigenvalues by $\lambda_1 = -\xi,\ \lambda_2 = -k+i\omega,\ \lambda_3 = -k-i\omega$ and simplifying, we obtain the variational equations for b and φ in the real form (b and φ are known as amplitude and phase respectively), which transform the equations (6.69) and (6.70) to

$$\dot{a} = \varepsilon P_1 a^5 e^{-4kt} + \varepsilon P_2 ab^2 e^{-2kt}/4 + \varepsilon P_3 a^5 e^{-4\xi t} + \varepsilon P_4 a^3 b^2 e^{-2kt} e^{-2\xi t}/4 + \varepsilon P_5 ab^4 e^{-4kt}$$
$$+\varepsilon^2 Eab\epsilon^{-kt}\{K_1\cos(\omega t - vt + \varphi) + K_2\sin(\omega t - vt + \varphi)\}/2 \tag{6.71}$$

$$\dot{b} = \varepsilon L_1 a^2 b e^{-2\xi t}/2 + \varepsilon L_2 b^3 e^{-2kt}/4 + 2\varepsilon E e^{kt}\{L_3\cos(\omega t - \nu t + \varphi)$$
$$+ M_3 \sin(\omega t - \nu t + \varphi)\} + \varepsilon^2 a^4 b L_4 e^{-4\xi t} + \varepsilon^2 a^2 b^3 L_5 e^{-2\xi t} e^{-2kt}/4 + \varepsilon^2 b^5 L_6 e^{-4kt}/16$$
$$+ \varepsilon^2 2Ea^2 e^{-2\xi t} e^{kt}\{L_7\cos(\omega t - \nu t + \varphi) + M_7 \sin(\omega t - \nu t + \varphi)\} \tag{6.72}$$
$$+ \varepsilon^2 Eb^2 e^{-kt}\{L_8\cos(\omega t - \nu t + \varphi) - M_8 \sin(\omega t - \nu t + \varphi)\}/2$$
$$+ \varepsilon^2 Eb^2 e^{-kt}\{L_9\cos(\omega t - \nu t + \varphi) + M_9 \sin(\omega t - \nu t + \varphi)\}/2$$

$$\dot{\varphi} = \varepsilon M_1 a^2 e^{-2\xi t}/2 + \varepsilon L_2 b^3 e^{-2kt}/4 + 2\varepsilon E e^{kt}\{L_3\cos(\omega t - \nu t + \varphi)$$
$$+ M_3 \sin(\omega t - \nu t + \varphi)\} + \varepsilon^2 a^4 b M_4 e^{-4\xi t} + \varepsilon^2 a^2 b^2 M_5 e^{-2\xi t} e^{-2kt}/4 + \varepsilon^2 b^4 M_6 e^{-4kt}/16$$
$$+ \varepsilon^2 2Ea^2 e^{-2\xi t} e^{kt}\{M_7\cos(\omega t - \nu t + \varphi) - L_7 \sin(\omega t - \nu t + \varphi)\}/b \tag{6.73}$$
$$+ \varepsilon^2 Ebe^{-kt}\{M_8\cos(\omega t - \nu t + \varphi) + L_8 \sin(\omega t - \nu t + \varphi)\}/2$$
$$+ \varepsilon^2 Ebe^{-kt}\{M_9\cos(\omega t - \nu t + \varphi) - L_9 \sin(\omega t - \nu t + \varphi)\}/2$$

where

$$L_1 = \xi(\xi + k) - \omega^2,\; M_2 = \xi\omega + \omega(\xi + k),$$
$$L_2 = -3(3k^2 + k\xi - \omega^2)/2(k^2 + \omega^2)\{(3k+\xi)^2 + \omega^2\},$$
$$M_2 = -3(4k\omega + \omega\xi)/2(k^2 + \omega^2)\{(3k+\xi)^2 + \omega^2\},$$
$$L_3 = -(k\xi + \nu^2 + \nu\omega)/2(\xi^2 + \nu^2)\{k^2 + (\omega+\nu)^2\},$$
$$M_3 = -(k\nu - \xi\nu - \omega\xi)/2(\xi^2 + \nu^2)\{k^2 + (\omega+\nu)^2\},$$
$$L_4 = -6\{(8k^3 + 4k^2\xi + 8k\omega^2 - k\nu^2 + 2k^2\xi + k\xi^2 + 6\omega^2\xi$$
$$- 2\omega\nu\xi)(3k^3\xi + 7k^2\omega\nu + 3k^2\nu^2 + k^2\xi^2 - 2k\omega\nu\xi + k\nu^2\xi - 5k\omega^2\xi - \omega^3\nu$$
$$- \omega^2\nu^2 - \omega\nu\xi^2 - \omega^2\xi^2) + (16k^2\omega - 8k^2\nu + 16k\omega\xi - 6k\nu^2\xi + 3\omega\xi^2 - \nu\xi^2 - 24\omega^3$$
$$- 14\omega^2\nu + 9\omega\nu^2 - \nu^3)(3k^3\nu - 2k^2\nu\xi - 7k^2\omega\xi - k\nu\xi^2 - 2k\omega\xi^2 - 5k\omega^2\nu$$
$$- 2\omega^2\nu\xi + \omega^3\xi - 4k\omega\nu^2 - \omega\nu^2\xi)\}/4\{k^2 + (3\omega - \nu)^2\}\{(2k+\xi)^2 + (2\omega - \nu)^2\}$$
$$(k^2 + \omega^2)(\xi^2 + \nu^2)\{(k^2 + (\nu+\omega)^2)((3k+\xi)^2 + \omega^2)\}$$

$$M_4 = -6\{(16k^2\omega - 8k^2\nu + 16k\omega\xi - 6k\nu^2\xi + 3\omega\xi^2 - \nu\xi^2 - 24\omega^3$$
$$- 14\omega^2\nu + 9\omega\nu^2 - \nu^3)(3k^3\xi + 7k^2\omega\nu + 3k^2\nu^2 + k^2\xi^2 - 2k\omega\nu\xi + k\nu^2\xi$$
$$- 5k\omega^2\xi - \omega^3\nu - \omega^2\nu^2 - \omega\nu\xi^2 - \omega^2\xi^2) - (8k^3 + 4k^2\xi + 8k\omega^2 - k\nu^2$$
$$+ 2k^2\xi + k\xi^2 + 6\omega^2\xi - 2\omega\nu\xi)(3k^3\nu - 2k^2\nu\xi - 7k^2\omega\xi - k\nu\xi^2 - 2k\omega\xi^2 -$$
$$5k\omega^2\nu - 2\omega^2\nu\xi + \omega^3\xi - 4k\omega\nu^2 - \omega\nu^2\xi)\}/4\{k^2 + (3\omega - \nu)^2\}\{(2k+\xi)^2$$
$$+ (2\omega - \nu)^2\}(k^2 + \omega^2)(\xi^2 + \nu^2)\{k^2 + (\nu+\omega)^2\}\{(3k+\xi)^2 + \omega^2\}$$

$$L_5 = 18\{(k^3\xi + 3k^2\nu^2 - k^2\omega\nu + k^2\xi^2 + k\nu^2\xi + 4k\omega\nu\xi + 5k\omega^2\xi$$
$$-\omega^2\nu^2 - \omega^3\nu^2)(8k^3 + 4k^2\xi - 4k\nu^2 + 4k\nu\omega + 2k^2\xi$$
$$+k\xi^2 + 2\omega\nu\xi + 4k\omega^2 + 2\omega^2\xi) - (3k^3\xi - 2k^2\nu\xi - 3k^2\omega\xi + 4k^2\omega\nu$$
$$+4k\omega\nu^2 + 3k\omega^2\nu - \omega^3\xi)(3k^2\nu + 4k^2\omega + 6k\nu\xi$$
$$+4k\omega\xi + \nu\xi^2 + \omega\xi^2 + 3\omega\nu^2 + 2\omega^2\nu + \nu^3)\}$$
$$/2\{(3k+\xi)^2 + \omega^2\}\{k^2 + (\nu+\omega)^2\}^2\{(2k+\xi)^2 + \nu^2\}(k^2+\omega^2)(\xi^2+\nu^2)$$
$$M_5 = 18\{(k^3\xi + 3k^2\nu^2 - k^2\omega\nu + k^2\xi^2 + k\nu^2\xi + 4k\omega\nu\xi + 5k\omega^2\xi$$
$$-\omega^2\nu^2 - \omega^3\nu^2)(4k^2\nu + 4k^2\omega + 6k\nu\xi + 4k\omega\xi + \nu\xi^2$$
$$+\omega\xi^2 + 3\omega\nu^2 + 2\omega^2\nu + \nu^3) - (3k^3\xi - 2k^2\nu\xi - 3k^2\omega\xi + 4k^2\omega\nu + 4k\omega\nu^2$$
$$+3k\omega^2\nu - \omega^3\xi)(8k^3 + 4k^2\xi - 4k\nu^2 + 4k\nu\omega + 2k^2\xi$$
$$+k\xi^2 + 2\omega\nu\xi + 4k\omega^2 + 2\omega^2\xi)\}/2\{(3k+\xi)^2 + \omega^2\}\{k^2 + (\nu+\omega)^2\}^2\{(2k+\xi)^2 + \nu^2\}$$
$$(k^2+\omega^2)(\xi^2+\nu^2)$$
$$L_6 = -9\{(27k^4 + 16k^3\xi + 3k^2\xi^2 - 16k^2\omega^2 - 8k\omega^2\xi - \omega^2\xi^2)$$
$$(70k^2 + 14k^2\xi + 10k\xi + 2k\xi^2 - 27k\omega^2 - 3\omega^2\xi) - (48k^3\omega + 28k^2\omega\xi + 16\omega\xi^2)$$
$$(79k^2\omega + 20k\omega\xi + \omega\xi^2 - \omega^2)\}/8(k^2+\omega^2)^2(4k^2+\omega^2)\{(3k+\xi)^2 + \omega^2\}^2$$
$$\{(5k+\xi)^2 + \omega^2\} + 3(30k^5 + 16k^4\xi - 237k^3\omega^2 + 63k^2\omega^2\xi + 2k^3\xi^2 - 4k\omega^2\xi^2$$
$$+57\omega^4 k + 8\omega^4\xi - 4k^3\omega^2\xi - 3k^2\omega^2\xi^2)/8(k^2+\omega^2)(4k^2+\omega^2)$$
$$\{(3k+\xi) + \omega^2\}^2\{(5k+\xi)^2 + \omega^2\}$$
$$M_6 = -9\{(27k^4 + 16k^3\xi + 3k^2\xi^2 - 16k^2\omega^2 - 8k\omega^2\xi - \omega^2\xi^2)$$
$$(79k^2\omega + 20k\omega\xi + \omega\xi^2 - \omega^2) + (48k^3\omega + 28k^2\omega\xi + 16\omega\xi^2)$$
$$(70k^2 + 14k^2\xi + 10k\xi + 2k\xi^2 - 27k\omega^2 - 3\omega^2\xi)\}/8(k^2+\omega^2)^2$$
$$(4k^2+\omega^2)\{(3k+\xi)^2 + \omega^2\}^2\{(5k+\xi)^2 + \omega^2\} + 3(141k^4\omega + 30k^4\omega\xi -$$
$$60\omega^3 + 34k^3\omega\xi + 6k^3\omega\xi^2 - 41k\omega^3\xi - 117k^2\omega^3 - 3\omega^3k^2\xi + 6\omega^5$$
$$+k^2\omega\xi^2 - 2\omega^3\xi^2)/8(k^2+\omega^2)(4k^2+\omega^2)\{(3k+\xi) + \omega^2\}^2\{(5k+\xi)^2 + \omega^2\}$$
$$L_7 = -18\{(72k^4 + 16k^3\xi + 3k^2\xi^2 - 32k^2\omega^2 - 16k\omega^2\xi - 8\omega^2\xi^2)$$
$$(90k^2 + 14k^2\xi + 10k\xi + 2k\xi^2 - 27k\omega^2 - 3\omega^2\xi) - (48k^3\omega + 28k^2\omega\xi + 16\omega\xi^2)$$
$$(79k^2\omega + 20k\omega\xi + \omega\xi^2 - \omega^2)\}/8(k^2+\omega^2)^2(4k^2+\omega^2)\{(3k+\xi)^2 + \omega^2\}^2$$
$$\{(5k+\xi)^2 + \omega^2\} + 3(30k^5 + 16k^4\xi - 237k^3\omega^2 + 63k^2\omega^2\xi + 2k^3\xi^2 - 4k\omega^2\xi^2$$
$$+57\omega^4 k + 8\omega^4\xi - 4k^3\omega^2\xi - 3k^2\omega^2\xi^2)/16(k^2+\omega^2)(4k^2+\omega^2)$$
$$\{(3k+\xi) + \omega^2\}^2\{(5k+\xi)^2 + \omega^2\}$$

$$M_7 = -18\{(72k^4 + 16k^3\xi + 3k^2\xi^2 - 32k^2\omega^2 - 16k\omega^2\xi - 8\omega^2\xi^2)$$
$$(90k^2\omega + 20k\omega\xi + \omega\xi^2 - \omega^2) + (48k^3\omega + 28k^2\omega\xi + 16\omega\xi^2)$$
$$(70k^2 + 14k^2\xi + 10k\xi + 2k\xi^2 - 27k\omega^2 - 3\omega^2\xi)\}/16(k^2 + \omega^2)^2$$
$$(4k^2 + \omega^2)\{(3k + \xi)^2 + \omega^2\}^2\{(5k + \xi)^2 + \omega^2\} + 3(141k^4\omega + 30k^4\omega\xi -$$
$$60\omega^3 + 34k^3\omega\xi + 6k^3\omega\xi^2 - 41k\omega^3\xi - 117k^2\omega^3 - 3\omega^3k^2\xi + 6\omega^5$$
$$+ k^2\omega\xi^2 - 2\omega^3\xi^2)/8(k^2 + \omega^2)(4k^2 + \omega^2)\{(3k + \xi) + \omega^2\}^2\{(5k + \xi)^2 + \omega^2\}$$

$$L_8 = -3\{(3k^3\xi + 7k^2\omega\nu + 3k^2\nu^2 + k^2\xi^2 - 2k\omega\nu\xi + k\nu^2\xi - 5k\omega^2\xi - \omega^3\nu$$
$$- \omega^2\nu^2 - \omega\nu\xi^2 - \omega^2\xi^2)(8k^3 + 4k^2\xi + 8k\omega^2 - k\nu^2 + 2k^2\xi + k\xi^2 + 6\omega^2\xi$$
$$- 2\omega\nu\xi) + (3k^3\nu - 2k^2\nu\xi - 7k^2\omega\xi - k\nu\xi^2 - 2k\omega\xi^2 - 5k\omega^2\nu - 2\omega^2\nu\xi + \omega^3\xi$$
$$- 4k\omega\nu^2 - \omega\nu^2\xi)(16k^2\omega - 8k^2\nu + 16k\omega\xi - 6k\nu^2\xi + 3\omega\xi^2 - \nu\xi^2 - 24\omega^3$$
$$- 14\omega^2\nu + 9\omega\nu^2 - \nu^3)\}/4\{k^2 + (3\omega - \nu)^2\}\{(2k + \xi)^2 + (2\omega - \nu)^2\}(k^2 + \omega^2)$$
$$(\xi^2 + \nu^2)\{(3k + \xi)^2 + \omega^2\}\{k^2 + (\nu + \omega)^2\}$$

$$M_8 = -3\{(3k^3\xi + 7k^2\omega\nu + 3k^2\nu^2 + k^2\xi^2 - 2k\omega\nu\xi + k\nu^2\xi - 5k\omega^2\xi - \omega^3\nu$$
$$- \omega^2\nu^2 - \omega\nu\xi^2 - \omega^2\xi^2)(16k^2\omega - 8k^2\nu + 16k\omega\xi - 6k\nu^2\xi + 3\omega\xi^2 - \nu\xi^2 - 24\omega^3$$
$$- 14\omega^2\nu + 9\omega\nu^2 - \nu^3) - (3k^3\nu - 2k^2\nu\xi - 7k^2\omega\xi - k\nu\xi^2 - 2k\omega\xi^2 -$$
$$5k\omega^2\nu - 2\omega^2\nu\xi + \omega^3\xi - 4k\omega\nu^2 - \omega\nu^2\xi)(8k^3 + 4k^2\xi + 8k\omega^2 - k\nu^2$$
$$+ 2k^2\xi + k\xi^2 + 6\omega^2\xi - 2\omega\nu\xi)\}/4\{k^2 + (3\omega - \nu)^2\}\{(2k + \xi)^2 + (2\omega - \nu)^2\}$$
$$(k^2 + \omega^2)(\xi^2 + \nu^2)\{(3k + \xi)^2 + \omega^2\}\{k^2 + (\nu + \omega)^2\}$$

$$L_9 = 3\{(3k^3\xi + 3k^2\nu^2 - k^2\omega\nu + k^2\xi^2 + k\nu^2\xi + 4k\omega\nu\xi + 5k\omega^2\xi$$
$$- \omega^2\nu^2 - \omega^3\nu^2 + \omega\nu\xi^2 + \omega^2\xi^2)(8k^3 + 4k^2\xi - 4k\nu^2 + 4k\nu\omega + 2k^2\xi$$
$$+ k\xi^2 + 2\omega\nu\xi + 4k\omega^2 + 2\omega^2\xi) - (3k^3\xi - 2k^2\nu\xi - 3k^2\omega\xi + 4k^2\omega\nu$$
$$+ 4k\omega\nu^2 + 3k\omega^2\nu - \omega^3\xi + \omega\xi\nu^2 - k\nu\xi^2)(10k^2\nu + 4k^2\omega + 6k\nu\xi$$
$$+ 4k\omega\xi + \nu\xi^2 + \omega\xi^2 + 3\omega\nu^2 + 2\omega^2\nu + \nu^3)\}$$
$$/2\{k^2 + (\nu + \omega)^2\}^2\{(2k + \xi)^2 + \nu^2\}(k^2 + \omega^2)(\xi^2 + \nu^2)\{(3k + \xi)^2 + \omega^2\}$$

$$M_9 = 3\{(3k^3\xi + 3k^2\nu^2 - k^2\omega\nu + k^2\xi^2 + k\nu^2\xi + 4k\omega\nu\xi + 5k\omega^2\xi$$
$$- \omega^2\nu^2 - \omega^3\nu^2 + \omega\nu\xi^2 + \omega^2\xi^2)(10k^2\nu + 4k^2\omega + 6k\nu\xi + 4k\omega\xi + \nu\xi^2$$
$$+ \omega\xi^2 + 3\omega\nu^2 + 2\omega^2\nu + \nu^3) - (3k^3\xi - 2k^2\nu\xi - 3k^2\omega\xi + 4k^2\omega\nu + 4k\omega\nu^2$$
$$+ 3k\omega^2\nu - \omega^3\xi + \omega\xi\nu^2 - k\nu\xi^2)(8k^3 + 4k^2\xi - 4k\nu^2 + 4k\nu\omega + 2k^2\xi$$
$$+ k\xi^2 + 2\omega\nu\xi + 4k\omega^2 + 2\omega^2\xi)\}/2\{k^2 + (\nu + \omega)^2\}^2\{(2k + \xi)^2 + \nu^2\}$$
$$(k^2 + \omega^2)(\xi^2 + \nu^2)\{(3k + \xi)^2 + \omega^2\}$$

The variational equations (6.71), (6.72) and (6.73) are in the form of the KBM solution. The variational equations for amplitude and phase are usually appeared in a set of first order differential equations and solved by the numerical technique

Therefore, the third second approximate solution of the equation (6.47) is

$$x(t,\, \varepsilon) = a\, e^{-\xi\, t} + be^{-kt}\cos(\omega t + \varphi) + \varepsilon\, u_1 + \varepsilon^2 u_2 \cdots \tag{6.74}$$

where a, b and φ are the solutions of the equations (6.71), (6.72)and (6.73) respectively.

For the damped forced vibration, *i.e.*, to investigate the stationary regime of vibration or examine the stability of the stationary regime of oscillations, we have to substitute $c = be^{-kt}$, and $\psi = \omega t - \nu t + \phi$, which leads to

$$\begin{aligned}\dot{c} = -kc + \varepsilon L_2 c^3/4 + 2\varepsilon E\{L_3\cos\psi + M_3\sin\psi\} + \varepsilon^2 EL_6 c^5/16 + \varepsilon^2 Ec^2\{L_8\cos\psi \\ + M_8\sin\psi\}/2 + \varepsilon^2 Ec^2\{L_9\cos\psi - M_9\sin\psi\}/2\end{aligned} \tag{6.75}$$

and

$$\begin{aligned}\dot{\psi} = (\omega - \nu) + \varepsilon M_2 c^2/4 + 2\varepsilon E\{M_3\cos\psi - L_3\sin\psi\}/c + \varepsilon^2 EL_6 c^4/16 \\ + \varepsilon^2 Ec\{M_8\cos\psi - L_8\sin\psi\}/2 + \varepsilon^2 Ec\{M_9\cos\psi + L_9\sin\psi\}/2\end{aligned} \tag{6.76}$$

For steady state solution, setting $\dot{c} = \dot{\psi} = 0$, and neglecting the non-oscillatory part of the solution, *i.e*, the terms L_9 and M_9, equations (6.72) and (6.73) become

$$\begin{aligned}kc - \varepsilon c^3 L_2/4 - \varepsilon^2 Ec^5 L_6/16 = E(2\varepsilon L_3 + \varepsilon^2 c^2 L_8/2)\cos\psi \\ + E(2\varepsilon M_3 + \varepsilon^2 c^2 M_8/2)\sin\psi\end{aligned} \tag{6.77}$$

$$\begin{aligned}-(\omega - \nu)c - \varepsilon M_2 c^3/4 - \varepsilon^2 EM_6 c^5/16 = E(2\varepsilon M_3 + \varepsilon^2 M_6 c^2/2)\cos\psi \\ - E(2\varepsilon L_3 + \varepsilon^2 L_8 c^2/2)\sin\psi\end{aligned} \tag{6.78}$$

In the case of the stationary regime, eliminating ψ from equations (6.77) and (6.78) gives the equation of the resonance curve (see [57]) as

$$\begin{aligned}&\varepsilon^4 c^{10} E^2 (L_6^2 + M_6^2)/256 + 2\varepsilon^3 c^8 E(L_2 L_6 + M_2 M_6)/32 + \varepsilon^2 c^6 \{L_2^2 - 2kEL_6 \\ &+ E^2 M_2^2 + 2(\omega - \nu)M_6\}/16 + c^4\{-2\varepsilon L_2 k + 2\varepsilon E M_2(\omega-\nu) - E^2\varepsilon^4 L_8^2 \\ &+ E^2\varepsilon^4 M_8^2)/4 + c^2\{k^2 + (\omega-\nu)^2 - 2E^2\varepsilon^3 L_3 L_8 \\ &- 2E^2\varepsilon^3 M_3 M_8\} - 4E^2\varepsilon^2(L_3^2 + M_3^2) = 0\end{aligned} \tag{6.79}$$

which relates the amplitude of the response c, to the frequency ν, of the forcing term.

6.5 Results and Discussion

A technique has been derived to solve an nth order, $n = 2,3,\ldots,$ weakly non-linear differential systems based of the KBM perturbation method. In this paper the solution is determine in complex form, since the determination of real form is a laborious task. The variational equations do not form a simple nonlinear algebraic equation of amplitude; since the elimination of phase is not possible. The numerical integration show that it has a steady-state solution. The advantage is that it changes slowly with time, t; so it requires a few steps of calculations to find the steady-state. On the contrary, it requires many step of calculation to find steady-state when one solves directly solve the second and third order problems congaing harmonic terms, $\cos\omega t$ using a numerical technique.

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one can easily compare the approximate solution to the numerical solution (considered to be exact). Due to such a comparison concerning the presented method of this paper, we refer to the works of Murty *et al* [57], and Shamsul [98,109]. In this paper, we have also compared the perturbation solution of *Duffing's* equation (6.20) and (6.47) to those obtained by Runge-Kutta (Fourth-order) procedure.

First of all, $x(t,\varepsilon)$ has been computed by perturbation solution (6.20) with first approximation together with initial conditions $[x(0)=0.918484, \dot{x}(0)=.633118]$ or $a=1.11555$, $\varphi=-.603514$ for $\varepsilon=.25$, $E=1.3$. The corresponding numerical solution has also been computed by fourth order Runge-Kutta method. All the results are shown in Fig. 6.1. From Fig. 6.1 we observe that the perturbation solution is differing from the numerical solution. For this reason we have computed $x(t,\varepsilon)$ by perturbation method with second approximation together with initial conditions $[x(0)=0..855399, \quad \dot{x}(0)=.639636]$ or $a=1.05161$, $\varphi=-.6312$ for $\varepsilon=.25$, $E=1.3$. The corresponding numerical solution has also been computed in this case and all the results are plotted in Fig. 6.2. From Fig. 6.2 we see that the perturbation solution shows a good coincidence with the numerical solution. We sketched the resonance curve for the conditions $\varepsilon=.1$, $k=.1$, $E=1$ in Fig.6.3 and for the condition $\varepsilon=.05, k=.01, E=1$ in Fig.6.4.

We have again computed $x(t,\varepsilon)$ by perturbation solution (6.47) with first approximation together with initial conditions $[x(0)=-0.892002, \dot{x}=.223129, \ddot{x}(0)=.895417]$ or $a=0.0000$, $b=.919571, \varphi=-2.894207$ for $\varepsilon=.1$, $k=.1, \nu=1$, $E=1.3$. The corresponding numerical solution has also been computed by fourth order Runge-Kutta method. . All the results are shown in Fig. 6.5. From Fig. 6.5 we find that the perturbation solutions are varies from the numerical solution. For this reason we have again computed $x(t,\varepsilon)$ by perturbation method with second approximation together with initial condition $[x(0)=-.825913, \dot{x}(0)=.212056, \ddot{x}(0)=.829565,]$ or $a=0.$, $b=.8525, \varphi=-2.888851$ for $\varepsilon=.1$, $\nu=1$, $E=1.3$. The corresponding numerical solution has also been computed in this case and all the results are plotted in Fig. 6.6. From Fig. 6.6 we see that the perturbation solution shows a good coincidence with the numerical solution. We sketched the resonance curve for the conditions $\varepsilon = .1$, $k = .1$, $\xi = .2$, $k = .1$, $E = 1$ in Fig.6.7.

6.6 Conclusion

Higher approximate solution of an n-th order time dependent nonlinear deferential system has been found and resonance curve is sketched. In general, the variational equations for the amplitude and the phase are solved numerically. In this case, the perturbation method facilitates the numerical method. The variational equations of amplitude and phase are important to investigate the stability of a differential system.

Fig 6.1

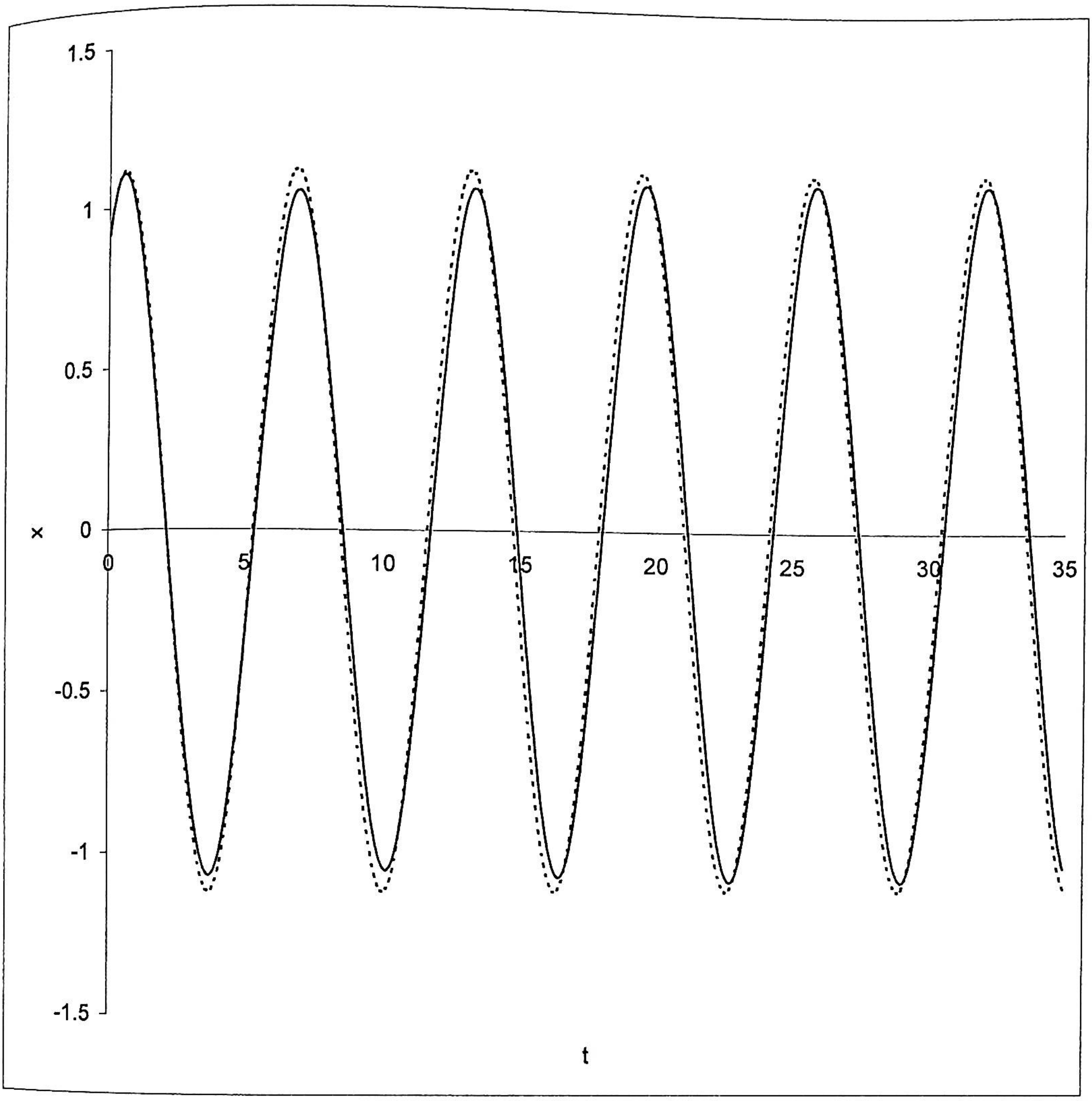

Fig 6.1: First approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when the damping coefficient is $k = 0.1$, $\nu = 1$ together with initial conditions $a = 1.11555, \varphi = -.603514$ $[x(0) = 0.918484, \quad \dot{x}(0) = .633118]$ for $\varepsilon = 0.25$, $E = 1.3$.

Fig 6.2

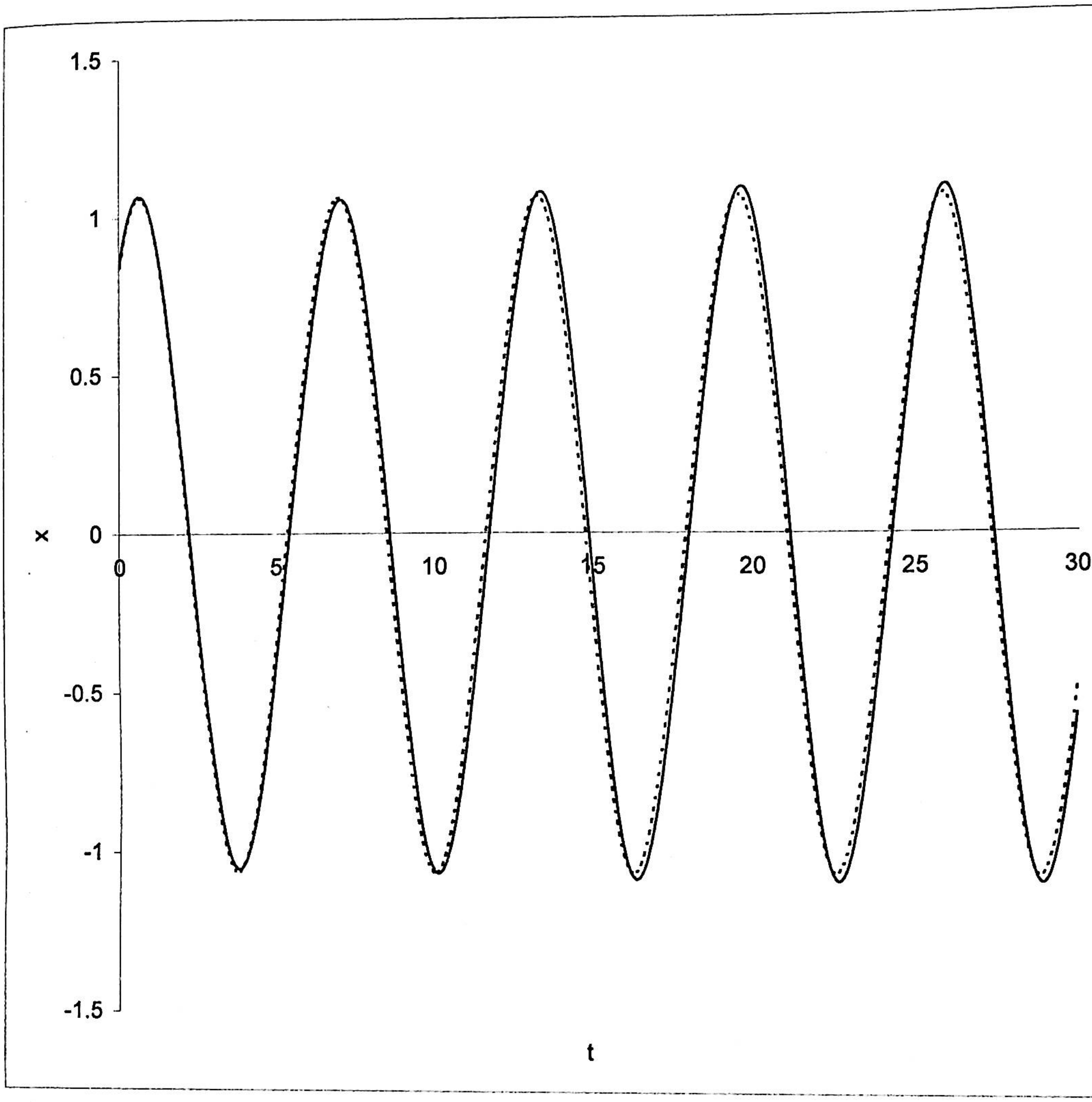

Fig 6.2: Second approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when the damping coefficient is $k = 0.1$, $\nu = 1$ together with initial conditions $a = 1.05161, \varphi = -.6312$ $[x(0) = 0.855399, \quad \dot{x}(0) = .639636]$ and $\varepsilon = 0.25, \; E = 1.3$.

Fig 6.3

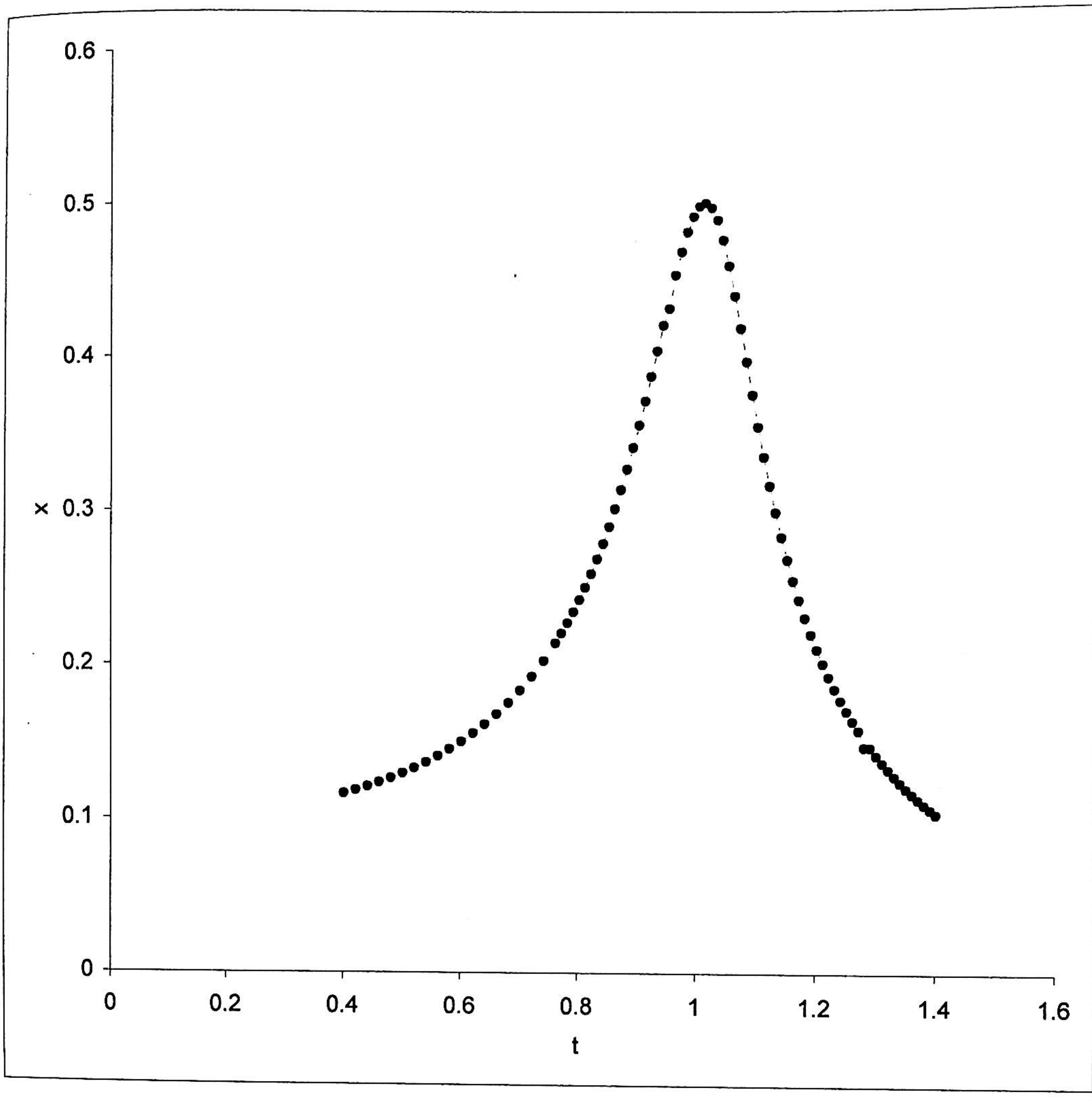

Fig 6.3: Resonance curve for $\varepsilon = .1$, $k = .1$, $E = 1$.

Fig 6.4

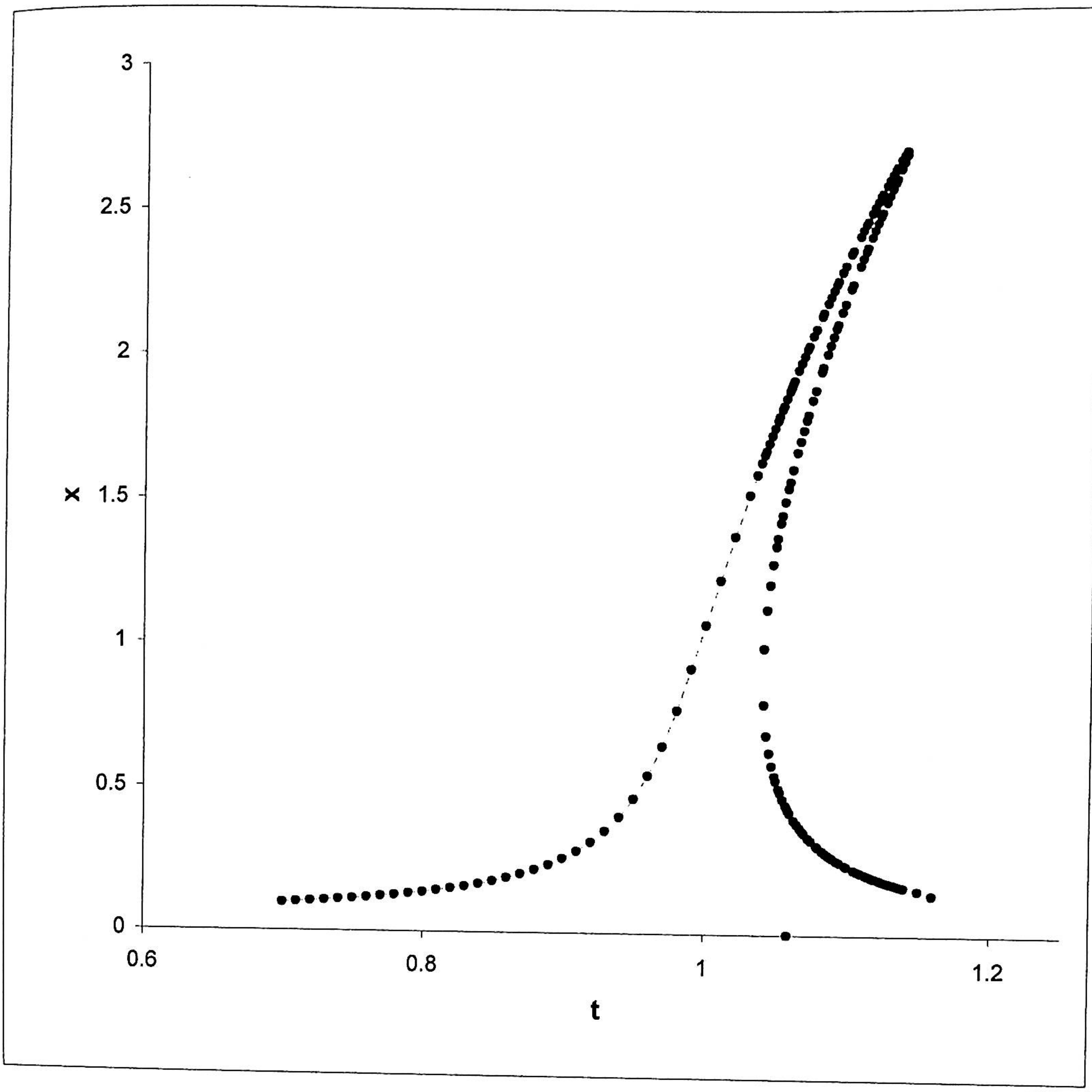

Fig 6.4: Resonance curve for $\varepsilon = .05$, $k = .01$, $E = 1$.

Fig 6.5

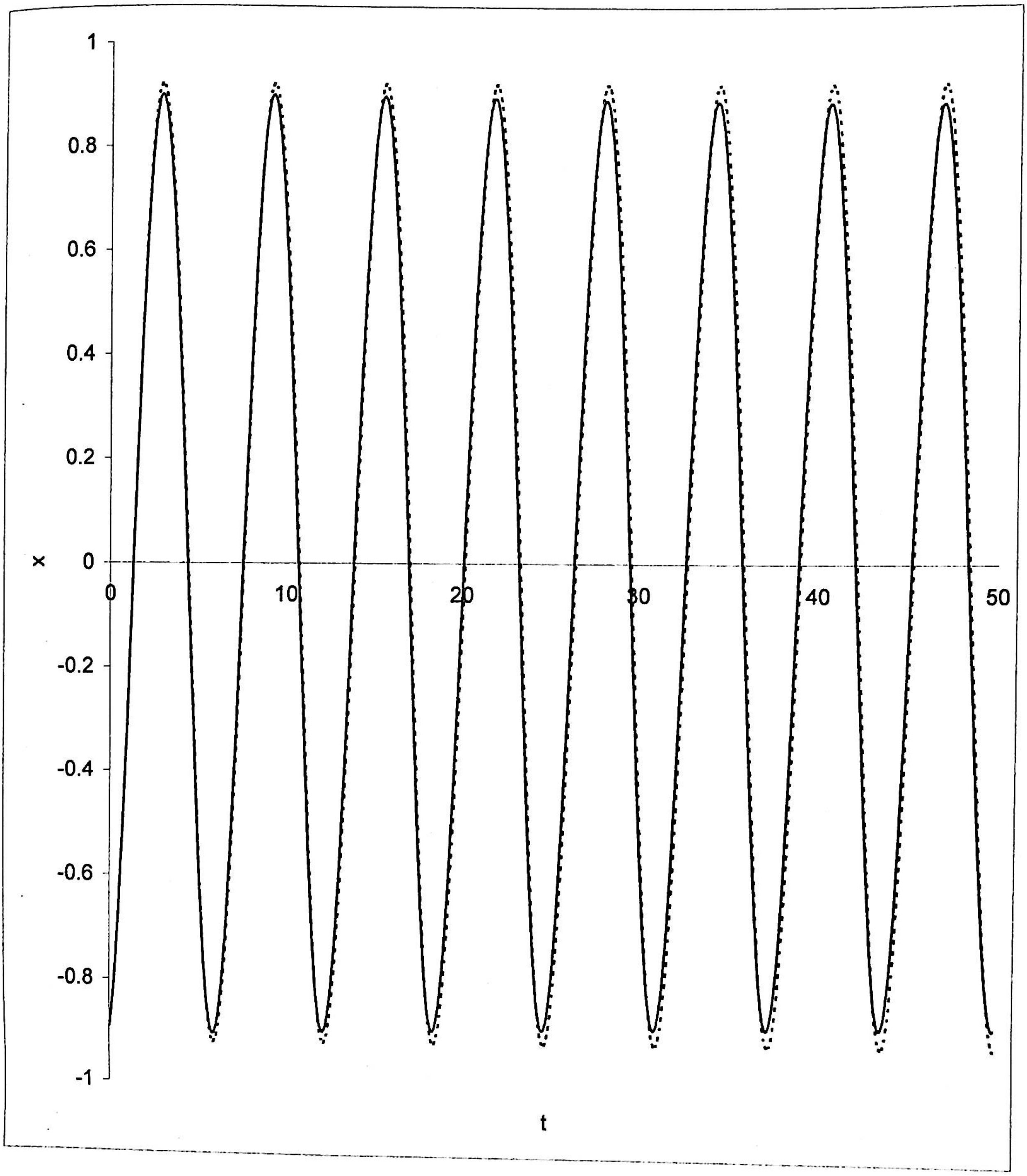

Fig 6.5: First approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when the damping coefficient is $\xi = .2$, $k = 0.1$, $\nu = 1$ together with initial conditions $a = 0.0, b = .919571, \varphi = -2.894207$ $[x(0) = -.892002,$ $\dot{x}(0) = .223129, \ddot{x}(0) = .895417]$ and $\varepsilon = 0.1$, $E = 1.3$

Fig 6.6

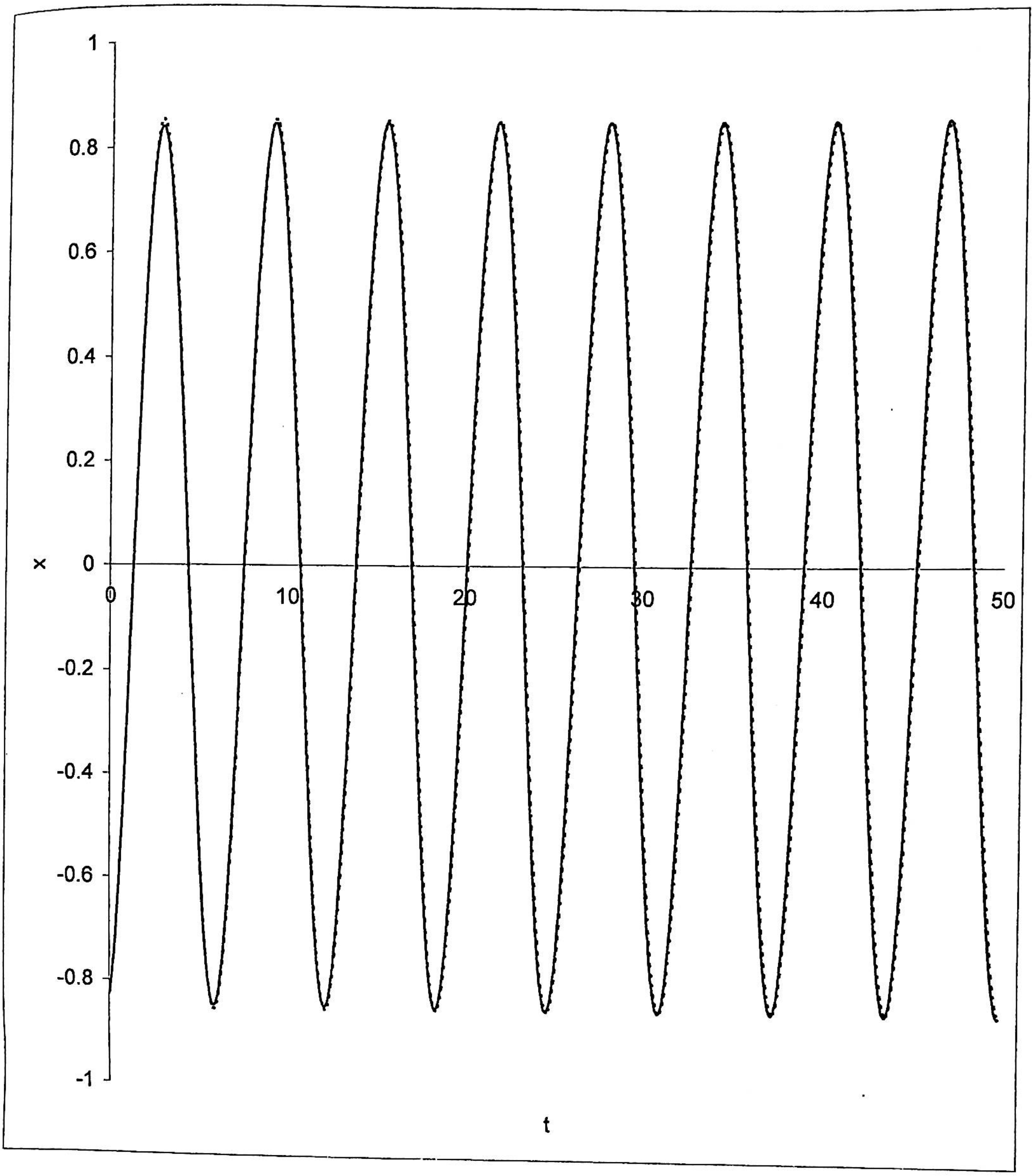

Fig 6.6: Second approximate solution (dotted line) with corresponding numerical solution (solid line) are plotted when the damping coefficient is $\xi = .2$, $k = .1$, $\nu = 1$ together with initial conditions $a = 0.0, b = .8525, \varphi = -2.888851$ or $[x(0) = -.825913$, $\dot{x}(0) = .212056$, $\ddot{x}(0) = .829565]$ and $\varepsilon = 0.1$, $E = 1.3$

Fig 6.7

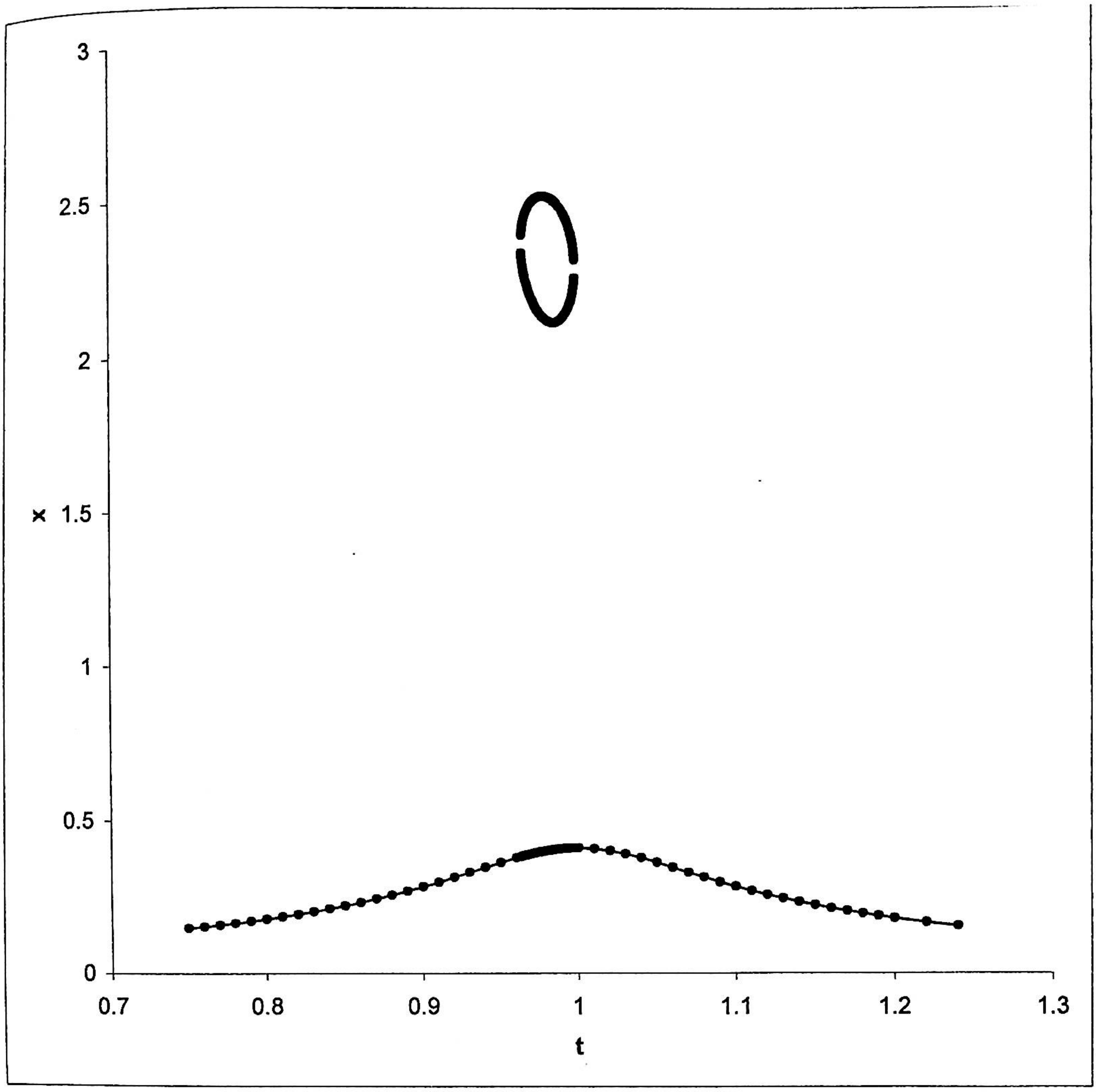

Fig 6.7: Resonance curve for $\xi = .2$, $k = .1$, $\varepsilon = .1$, $E = 1$, $\nu = 1$.

References

1. Ali Akbar, M., A. C. Paul and M. A. Sattar, An Asymptotic Method of Krylov-Bogoliubov for Fourth Order Over-damped Nonlinear Systems, Ganit, J. Bangladesh Math. Soc., Vol. **22**, pp. 83-96, 2002.

2. Ali Akbar, M., M. Shamsul Alam and M. A. Sattar, Asymptotic Method for Fourth Order Damped Nonlinear Systems, Ganit, J. Bangladesh Math. Soc. Vol. **23**, pp. 72-80, 2003.

3. Ali Akbar, M., M. Shamsul Alam and M. A. Sattar, A Simple Technique for Obtaining Certain Over-damped Solutions of an n-th Order Nonlinear Differential Equation, Soochow Journal of Mathematics, Vol. 31(2), pp.291-299, 2005.

4. Andrianov, I. and J. Awrejcewicz, New trends in asymptotic approaches: Summation and interpolation methods, Applied Mechanics Reviews, Vol. **54**, pp. 69-92, 2001.

5. Andronov, A., Les Cycles de Poincare et la Theorie des Oscillations Auto-entretenues (Limit Cycles of Poincare and the Theory of Self-excited Oscillations) Comptes Rendus, Paris, Vol. **189**, 1929.

6. Andronov, A. and S. Chaikin, Theory of Oscillations, Moscow, (Russian), 1937.

7. Arya, J. C. and G. N. Bojadziev, Damped Oscillating Systems Modeled by Hyperbolic Differential Equations with Slowly Varying Coefficients, Acta Mechanica, Vol. **35**, pp. 215-221, 1980.

8. Arya, J. C. and G. N. Bojadziev, Time Dependent Oscillating Systems with Damping, Slowly Varying Parameters, and Delay, Acta Mechanica, Vol. **41**, pp. 109-119, 1981.

9. Awrejcewicz, J., I. Andrianov and L. Manevitch, Asymptotic Approaches in Nonlinear Dynamics: New Trends and Applications, Springer-Verlag, Berlin 1998.

10. Bellouin L., Compt. Rend. **24**, 183, 1962.

11. Bellman, R., Perturbation Techniques in Mathematics, Physics and Engineering, Holt, Rinehart and Winston, New York, 1966.

12. Bogoliubov, N. N. and Yu. A. Mitropolskii, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordan and Breach, New York, 1961.

13. Bogoliubov, N. N. and Yu. A. Mitropolskii, Asymptotic Methods in the Theory of Nonlinear Oscillations (in Russian) State Press for Physics and Mathematical Literature, Moscow, 1963.

14. Bojadziev, G. N., On Asymptotic Solutions of Nonlinear Differential Equations with Time Lag, Delay and Functional Differential Equations and Their Applications (edited by K. Schmit), 299-307, New York and London: Academic Press, 1972.

15. Bojadziev, G. N., R. W. Lardner and J. C. Arya, On the Periodic Solutions of Differential Equations Obtained by the Method of Poincare and Krylov-Bogoliubov, J. Utilitas Mathematica, Vol. **3**, pp. 49-64, 1973.

16. Bojadziev, G. N. and R. W. Lardner, Monofrequent Oscillations in Mechanical Systems Governed by Hyperbolic Differential Equation with Small Nonlinearities, Int. J. Nonlinear Mech. Vol. **8**, pp. 289-302, 1973.

17. Bojadziev, G. N. and R. W. Lardner, Second Order Hyperbolic Equations with Small Nonlinearities in the Case of Internal Resonance, Int. J. Nonlinear Mech. Vol. **9**, pp. 397-407, 1974.

18. Bojadziev, G. N. and R. W. Lardner, Asymptotic Solution of a Nonlinear Second Order Hyperbolic Differential Equation with Large Time Delay, J. Inst. Math. Applics Vol. **14**, pp. 203-210, 1974.

19. Bojadziev, G. N., Damped Forced Nonlinear Vibrations of Systems with Delay, J. Sound and Vibration, Vol. **46**, pp. 113-120, 1976.

20. Bojadziev, G. N., The Krylov-Bogoliubov-Mitropolskii Method Applied to Models of Population Dynamics, Bull. Math. Biol. Vol. **40**, pp. 335-346, 1977.

21. Bojadziev, G. N., The Krylov-Bogoliubov-Mitropolskii Method Applied to Models of Population Dynamics, Bulletin of Mathematical Biology, Vol. **40**, pp. 335-345, 1978.

22. Bojadziev, G. N. and S. Chan, Asymptotic Solutions of Differential Equations with Time Delay in Population Dynamics, Bull. Math. Biol. Vol. **41**, pp. 325-342, 1979.

23. Bojadziev, G. N., Damped Oscillating Processes in Biological and Biochemical Systems, Bull. Math. Biol. Vol. **42**, pp. 701-717, 1980.

24. Bojadziev, G. N. and J. Edwards, On Some Method for Non-oscillatory and Oscillatory Processes, J. Nonlinear Vibration Probs. Vol. **20**, pp. 69-79, 1981.

25. Bojadziev, G. N., Damped Nonlinear Oscillations Modeled by a 3-dimensional Differential System, Acta Mechanica, Vol. **48**, pp. 193-201, 1983.

26. Bojadziev, G. N. and C. K. Hung, Damped Oscillations Modeled by a 3-dimensional Time Dependent Differential Systems, Acta Mechanica, Vol. **53**, pp. 101-114, 1984.

27. Cap, F. F., Averaging Method for the Solution of Nonlinear Differential Equations with Periodic Non-harmonic Solutions, Int. J. Nonlinear Mech. Vol. **9**, pp. 441-450, 1974.

28. Cunningham. W.J, Introduction to Nonlinear Analysis, McGraw-Hill Book Company, 1958.

29. Duffing, G., Erzwungene Schwingungen bei Veranderlicher Eigen Frequenz und Ihre Technische Bedeutung, Ph. D. Thesis (Sammlung Vieweg, Braunchweig) 1918.

30. Freedman, H. I., V. S. H. Rao and K. J. Lakshami, Stability, Persistence, and Extinction in a Prey-Predator System with Discrete and Continuous Time Delay, WSSIAA Vol. **1**, pp. 221-238, 1992.

31. Freedman, H. I. and S. Ruan, Hopf Bifurcation in Three-species Chain Models with Group Defense, Math. Biosci. Vol. **111**, pp. 73-87, 1992.

32. Frshcnko S. F., N. I. Shkil and L. D. Nikolenko, Asysmptotic Method in the Theory of linear differential Equation, (in Russian), Naukova Dumka, Kiev 1966 (English translation, Amer Elsevier Publishing Co., INC. New York 1967).

33. Gylden, Differentialgleichungen der Storungs Theorie (Differential Equations of the Theory of Perturbation), Petersbourg, Vol. **31**, 1883.

34. Hung Cheng and Tai Tsun Wu. An aging spring, Studies in Applied Mathematics **49**. pp. 183-185. 1970.

35. Kramers, H. A., G. Z. physik Vol.39, pp. 828, 1926.

36. Kruskal, M., Asymptotic Theory of Hamiltonian and Other Systems with all Situations Nearly Periodic, J. Math. Phys., Vol. **3**, pp. 806-828, 1962.

37. Krylov, N. N. and N. N. Bogoliubov, Introduction to Nonlinear Mechanics, Princeton University Press, New Jersey, 1947.

38. Lardner, R. W. and G. N. Bojadziev, Vibration of a Viscoelastic Rod with Small Nonlinearities, Meccanica, Vol. **12**, pp. 249-256, 1977.

39. Lardner, R. W. and G. N. Bojadziev, Asymptotic Solutions for Third Order Partial Differential Equations with Small Nonlinearities, Meccanica, pp. 249-256, 1979.

40. Lewis, A. P., An Investigation of the Stability in the Large for an Autonomous Second-Order Two-Degree-of-Freedom System, Int. J. Non-linear Mechanics, Vol. **37**, pp. 153-169, 20002.

41. Lewis, A. P., An Investigation of Stability in the Large Behavior of a Control Surface with Structural Non-linearities in Supersonic Flow, Journal of Sound and Vibration, Vol. **256**, pp. 725-754, 20002.

42. Liapounoff, M. A., Probleme General de la Stabilite du Mouvement (General Problems of Stability of Motion), Annales de la Faculte des Sciences de Toulouse, Paris, Vol. **9**, 1907.

43. Lin, J. and P. B. Khan, Averaging Methods in Prey-Predator Systems and Related Biological Models, J. Theor. Biol. Vol. **57**, pp. 73-102, 1974.

44. Lindstedt, A., Astro. Nach. 103, 211, 1882.

45. Lindstedt, A., Memoires de l, Ac. Imper, des Science de st. Petersburg 31, 1883.

46. Liouville, J., J. De Math. 2, 16, 418, 1837.

47. Mandelstam, L. and N. Papalexi, Expose des Recherches Recentes sur les Oscillations Non-lineaires (Outline of Recent Research on Non-linear Oscillations), Journal of Technical physics, USSR, 1934.

48. McLachlan, N. W., Ordinary Nonlinear Differential Equations in Engineering and Physical Science, Clarendon Press, Oxford, 1950.

49. Mendelson, K. S., Perturbation Theory for Damped Nonlinear Oscillations, J. Math. Physics, Vol. **2**, pp. 3413-3415, 1970.

50. Mickens, R. E., An Introduction to Nonlinear Oscillations, Cambridge University Press, London, 1980.

51. Minorsky, N., Nonlinear oscillations, Van Nostrand, Princeton, N. J., pp. 375, 1962.

52. Minorsky, N., Introduction to Nonlinear Mechanics, J. E. Edwards, Ann Arbor, Michigan, 1947.

53. Mitropolskii, Yu. A., Problems on Asymptotic Methods of Non-stationary Oscillations (in Russian), Izdat, Nauka, Moscow, 1964.

54. Mulholland, R. J., Nonlinear Oscillations of Third Order Differential Equation, Int. J. Nonlinear Mechanics, Vol. **6**, pp. 279-294, 1971.

55. Murty, I. S. N., and B. L. Deekshatulu, Method of Variation of Parameters for Over-Damped Nonlinear Systems, J. Control, Vol. **9**, no. 3, pp. 259-266,1969.

56. Murty, I. S. N., B. L. Deekshatulu and G. Krishna, On an Asymptotic Method of Krylov-Bogoliubov for Over-damped Nonlinear Systems, J. Frank. Inst., Vol. **288**, pp. 49-65, 1969.

57. Murty, I. S. N., A Unified Krylov-Bogoliubov Method for Solving Second Order Nonlinear Systems, Int. J. Nonlinear Mech. Vol. **6**, pp. 45-53, 1971.

58. Museenkov, P., On the Higher Order Effects in the Methods of Krylov-Bogoliubov and Poincare, J. Astron. Sci. Vol. **12**, pp. 129-134, 1965.

59. Nayfeh, A. H., Methods in Astrodynamics and Celestial Mechanics, Edited by R. L. Duncobe and V. G. Szebehely, Astronauties and Aeronautics, Vol. **17**, Academic Press, pp. 139-157, New York, 1966.

60. Nayfeh, A. H., Perturbation Methods, John Wiley and Sons, New York, 1973.

61. Nayfeh, A. H., Introduction to Perturbation Techniques, John Wiley and Sons, New York, 1981.

62. Nguyen Van Dinh, On a Variant of the asymptotic procedure, Vietnum J. of Mechanics, VAST Vol. **26**, No. 3 (139-147), 2004.

63. Osiniskii, Z., Longitudinal, Torsional and Bending Vibrations of a Uniform Bar with Nonlinear Internal Friction and Relaxation, Nonlinear Vibration Problems, Vol. **4**, pp. 159-166, 1962.

64. Osiniskii, Z., Vibration of a One Degree Freedom System with Nonlinear Internal Friction and Relaxation, Proceedings of International Symposium of Nonlinear Vibrations, Vol. **111**, pp. 314-325, Kiev, Izadt, Akad, Nauk Ukr. SSR, 1963.

65. O'Malley, R. E. Jr. and C. Schmeiser, The Asymptotic Solution of a Semiconductor Device Problem Involving Reverse Bias, SIAM J. Appl. Math., Vol. **50**, pp. 504-520, 1990.

66. O'Malley, R. E. Jr., Singular Perturbation Methods for Ordinary Differential Equations, Springer-Verlag, New York, 1991.

67. O'Malley, R. E. Jr., On Singular Perturbations, Especially Matching, SIAM Review, Vol. **36**, pp. 421-414, 1995.

68. O'Malley, R. E. Jr. and M. J. Ward, Exponential Asymptotics, Boundary Layer Resonance, and Dynamic Metastability, *Mathematics is for Solving Problems*, L. P. Cook, V. Royburd, and M. Tulin, Editors, SIAM, pp. 189-203, 1996.

69. O'Malley, R. E. Jr., Stiff Differential Equations and Singular Perturbations, Technical Report, Australian National University, 1996.

70. O'Malley, R. E. Jr., *Thinking About Ordinary Differential Equations*, Cambridge University Press, 1997.

71. Pinakee Dey, M. Zulfikar Ali, M. Shamsul Alam, An Asymptotic Method for Time Dependent Non-linear Over-damped Systems, J. Bangladesh Academy of sciences., Vol. **31**, pp. 103-108, 2007.

72. Pinakee Dey, M. Zulfikar Ali and M. Shamsul Alam, 'An Asymptotic Method for Time Dependent Non-linear Systems with Varying Coefficients, J. Mech. Cont. & Math. Sci. Vol-3, No-2, Dec 2008 (To Appear)

73. Pinakee Dey, M. Zulfikar Ali, Ali Akbar and M. Shamsul Alam, Second Approximate Solution of Second Order Damped Forced Nonlinear Systems, J. Appl. Sci. Res, 4(6) pp731-741, 2008.

74. Pinakee Dey, M. A. Sattar and M. Shamsul Alam, Perturbation Theory for Damped Forced Vibrations with Slowly Varying Coefficients , Advances in Vibration Engineering, India (Accepted).

75. Pinakee Dey, M. A. Sattar, Ali Akbar and M. Shamsul Alam, Higher Approximation of *n*-th Order Weakly Nonlinear Non-Autonomous Differential Equation With Damping,, Indian J. pure appl. Math. (Submitted).

76. Pinakee Dey, M. Zulfikar Ali and M.A. Sattar, Second Approximate Solution of Forced Vibrations, Acta Ciencia Indica (Accepted)

77. , Poincare, H., Les Methods Nouvelles de la Mecanique Celeste, Paris, 1892.

78. Popov, I. P., A Generalization of the Bogoliubov Asymptotic Method in the Theory of Nonlinear Oscillations (in Russian), Dokl. Akad. USSR Vol. **3**, pp. 308-310, 1956.

79. Proskurjakov A. P., Comparison of the Periodic Solutions of Quasi-linear Systems Constructed by the Method of Poincare and Krylov-Bogoliubov (in Russian), Applied Math. and Mech., 28, 1964.

80. Rauch L. L., Oscillations of a Third Order Nonlinear Autonomous System, *in Contribution to the Theory of Nonlinear Oscillations*, pp. 39-88, New Jersey, 1950.

81. Raymond P. Vito and G. Cabak, The Effects of Internal Resonance on Impulsively Forced Nonlinear Systems with Two Degree of Freedom, Int. J. Nonlinear Mechanics, Vol. **14**, pp. 93-99, 1979.

82. Roy. K. C., M. Shamsul Alam, Effect of higher approximation of Krylov-Bogoliubov-Mitropolskii solution and matched asymptotic solution of a differential system with slowly varying coefficients and damping near to a turning point, Vietnam Journal of Mechanics, VAST, Vol.26, pp.182-192, 2004.

83. Sattar, M. A., An asymptotic Method for Second Order Critically Damped Nonlinear Equations, J. Frank. Inst., Vol. **321**, pp. 109-113, 1986.

84. Sattar, M. A., An Asymptotic Method for Three-dimensional Over-damped Nonlinear Systems, Ganit, J. Bangladesh Math. Soc., Vol. **13**, pp. 1-8, 1993.

85. Sethna, P. R., Coupling in Certain Classes of Weakly Nonlinear Vibrating Systems, Int. Symp. Nonlinear Differential Equation and Nonlinear Mechanics, pp. 58-70, Academic Press, New York, 1963.

86. Shamsul Alam, M. and M. A. Sattar, An Asymptotic Method for Third Order Critically Damped Nonlinear Equations, J. Mathematical and Physical Sciences, Vol. **30**, pp. 291-298, 1996.

87. Shamsul Alam, M. and M. A. Sattar, A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving Third Order Nonlinear Systems, Indian J. pure appl. Math. Vol. **28**, pp. 151-167, 1997.

88. Shamsul Alam, M., Mono-frequent Damped Oscillation Modeled By 3-order Non-linear P.D.E, GANIT (J. Bangladesh Mathematical Society), Vol. **19**, pp. 39-47, 1999.

89. Shamsul Alam, M. and M. A Sattar, An Asymptotic Method for nth Dimensional Over-damped Non-linear Systems, GANIT (J. Bangladesh Mathematical Society), Vol. **19**, pp. 21-27, 1999.

90. Shamsul Alam, M., M. A Sattar and N. Akhter, Asymptotic method for Critically Damped Processes in Biological and Biochemical Systems, GANIT (J. Bangladesh Mathematical Society), Vol. **20**, pp. 37-44, 2000.

91. Shamsul Alam, M., Asymptotic Methods for Second Order Over-damped and Critically Damped Nonlinear Systems, Soochow Journal of Math. Vol. **27**, pp. 187-200, 2001.

92. Shamsul Alam, M. and M. A. Sattar, Time Dependent Third-order Oscillating Systems with Damping, J. Acta Ciencia Indica, Vol. **27**, pp. 463-466, 2001.

93. Shamsul Alam, M., Perturbation Theory for Nonlinear Systems with Large Damping, Indian J. pure appl. Math. Vol. **32**, pp. 1453-1461, 2001.

94. Shamsul Alam, M., M. F. Alam and S. S. Shanta, Approximate Solution of Non-Oscillatory Systems with Slowly Varying Coefficients, Ganit, J. Bangladesh Math. Soc. Vol. **21**, pp. 55-59, 2001.

95. Shamsul Alam, M., Asymptotic Solution of P.D.E with Large Damping Effects, Bangladesh .J. of Scientific Research, Vol. **18**, pp. 277-288, 2001.

96. Shamsul Alam, M., Krylov-Bogoliubov-Mitropolskii Method for Certain Non-Oscillatory Non-linear Systems, J. Bangladesh Academy of sciences., Vol. **25**, pp. 193-202, 2001.

97. Shamsul Alam, M., B. Hossain and S. S. Shanta, Krylov-Bogoliubov-Mitropolskii Method for Time Dependent Nonlinear Systems with Damping, Mathematical Forum, Vol. **14**, pp. 53-59, 2001-2002.

98. Shamsul Alam, M., A Unified Krylov-Bogoliubov-Mitropolskii Method for Solving *n*-th Order Nonlinear Systems, J. Frank. Inst. Vol. **339**, pp. 239-248, 2002.

99. Shamsul Alam, M., Perturbation Theory for *n*-th Order Nonlinear Systems with Large Damping, Indian J. pure appl. Math. Vol. **33**, pp. 1677-1684, 2002.

100. Shamsul Alam, M., Bogoliubov's Method for Third Order Critically Damped Nonlinear Systems, Soochow J. Math. Vol. **28**, pp. 65-80, 2002.

101. Shamsul Alam, M., Method of Solution to the *n*-th Order Over-damped Nonlinear Systems Under Some Special Conditions, Bull. Cal. Math. Soc., Vol. **94**, pp. 437-440, 2002.

102. Shamsul Alam, M., Approximate Solutions of Non-oscillatory Systems, Mathematical Forum, Vol. **14**, pp. 7-16, 2001-2002.

103. Shamsul Alam, M., On Some Special Conditions of Third Order Over-damped Nonlinear Systems, Indian J. pure appl. Math. Vol. **33**, pp. 727-742, 2002.

104. Shamsul Alam, M., Asymptotic Method for Non-oscillatory Nonlinear Systems, Far East J. Appl. Math., Vol. **7**, pp. 119-128, 2002.

105. Shamsul Alam, M., Oscillating process in third-order nonlinear differential system, J. Theoretical Physics, Vol. **50**, pp. 99-108, 2002.

106. Shamsul Alam, M., Method of solution to an n-th order Over-damped Nonlinear System under Some Special Conditions, Bulletin of Calcutta Mathematical Society, Vol. **96**, pp. 437-440, 2002.

107. Shamsul Alam, M., On Some Special Conditions of Over-damped Nonlinear Systems, Soochow J. Math. Vol. **29**, pp. 181-190, 2003.

108. Shamsul Alam, M. and M. B. Hossain, On Some Special Conditions of n-th Order Non-oscillatory Nonlinear Systems, Communication of Korean Math. Soc. Vol. **18**, pp. 755-765, 2003.

109. Shamsul Alam, M., A Unified KBM Method for Solving n-th Order Nonlinear Differential Equation with Varying Coefficients, J. Sound and Vibration, Vol. **265**, pp. 987-1002, 2003.

110. Shamsul Alam, M., Asymptotic Method for Certain Third-order Non-oscillatory Nonlinear Systems, J. Bangladesh Academy of Sciences, Vol. **27**, pp. 141-148, 2003.

111. Shamsul Alam, M., M. Bellal Hossain and S. S. Shanta, Perturbation Theory for Damped Nonlinear Systems with Varying Coefficients, Indian J. pure appl. Math., Vol. **34**, pp. 1359-1368, 2003.

112. Shamsul Alam, M., A Modified and Compact Form of Krylov-Bogoliubov-Mitropolskii Unified Method for an n-th Order Nonlinear Differential Equations, Int. J. Nonlinear Mechanics, Vol. **39**, pp. 1343-1357, 2004.

113. Shamsul Alam, M., Damped Oscillations Modeled by an *n*-th Order Time Dependent Quasi-linear Differential System, Acta Mechanica, Vol. **169**, pp. 111-122, 2004.

114. Shamsul Alam, M., M. B. Hossain and M. Ali Akbar, On a Special Condition of Over-damped Nonlinear Systems with Slowly Varying Coefficients, Journal of Interdisciplinary Mathematics, Vol. **7**, pp. 255-260, 2004.

115. Shamsul Alam, M., Unified KBM Method Under a Critical Condition, J. Franklin Inst., Vol. **341**, pp. 533-542, 2004.

116. Shamsul Alam, M. and M. A Sattar, Asymptotic Method for third-order Non-linear System with Varying Coefficients,, Southeast Asian Bull. Of Mathematics, Vol. **341**, pp. 533-542, 2004.

117. Shamsul Alam, M. M. Abul Kalam Azad and M.A. Hoque. "A general Struble's technique for solving an *n*-th order weakly non-linear differential system with damping", Journal of Non-Linear Mechanics Vol.**41**: pp.905-918, 2006..

118. Stoker, J. J., Nonlinear Vibrations in Mechanical and Electrical Systems, Interscience, New York, 1950.

119. Stuble, R. A, The geometry of the orbits of artificial satellites, Arch. Rational Mech. Anal. Vol. **7**, pp. 87-104. 1961

120. Van der Pol, B., On Relaxation Oscillations, Philosophical Magazine, 7-th series, Vol. **2**, 1926.

121. Volosov, V. M., Higher Approximation in Averaging, Sovier Math. Dokl, Vol. **2**, pp. 221-224, 1961.

122. Volosov, V. M., Averaging in Systems of Ordinary Differential Equations, Russian Math. Surveys, 7, pp. 1-126, 1962.

123. Wetzel, G. Z, Physik, Vol. **38**, pp. 518, 1926.

124. Zabrieko, P. P., Higher Approximation of the Krylov-Bogoliubov Averaging Method, Dokl, Akad. Nauk. USSR, **176**, pp. 1453-1456, 1966.