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On Some Approximate Solutions of Nonlinear Physical and Biological Problems

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University of Rajshahi

ON SOME APPROXIMATE SOLUTIONS OF NONLINEAR PHYSICAL AND BIOLOGICAL PROBLEMS

Ph.D. Thesis

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

SUBMITTED BY

MD. ALHAZ UDDIN

DEPARTMENT OF MATHEMATICS
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UNIVERSITY OF RAJSHAHI
RAJSHAHI-6205, BANGLADESH

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Session: July **2007-08**

DEPARTMENT OF MATHEMATICS
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DECLARATION

The thesis entitled **"On Some Approximate Solutions of Nonlinear Physical and Biological Problems"** is written by me and has been submitted in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh. Here I confirm that this research work is an original one and it has not been submitted elsewhere for any degree.

24-06-2010

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Dedicated to My
Beloved Parents
and
Better Half and Affectionate Son

CERTIFICATE

This is to certify that the thesis entitled "**On Some Approximate Solutions of Nonlinear Physical and Biological Problems**" is based on the study carried out by Md. Alhaz Uddin, Roll No. **07309**, Registration No. **580**, Session: July **2007-08** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh, has been completed under our joint supervision. We strongly believe that this research work is an original one and it has not been submitted elsewhere for any degree.

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Abstract

Most of the perturbation methods are developed to find the periodic solutions of nonlinear systems with small nonlinearities, transients are not considered. In 1947, first Russian scientists Krylov and Bogoliubov introduced a perturbation method to discuss the transient's response in the second order autonomous differential systems with small nonlinearities and this method is well known as "an asymptotic averaging method" in the theory of nonlinear oscillations. Later, this method has been amplified and justified by Bogoliubov and Mitropolskii in 1961 and this extended method is known as the KBM method in literature. In this dissertation, we have presented an analytical technique based on He's homotopy perturbation technique and the extended form of the KBM method to investigate the solutions of second order strongly nonlinear physical and oscillating processes in biological systems with significant damping effects. Also we have extended the KBM method to investigate the weakly third and fourth order nonlinear systems with slowly varying coefficients and damping effects.

Firstly, second order damped nonlinear autonomous differential systems are considered and He's homotopy perturbation and the KBM methods have been extended to Duffing type strongly nonlinear physical problems with small damping effects. Then the method has been applied to find the analytical approximate solution of damped oscillatory nonlinear systems with slowly varying coefficients with strong nonlinearity. Further, this method has been developed to solve second order strongly nonlinear oscillating processes in biological system with small damping effects. We have also extended the homotopy perturbation technique to find the second approximation of second order strongly nonlinear differential systems with damping effects. We have extended the KBM method to determine the second approximation of third order weakly nonlinear damped oscillatory systems under some special conditions. Lastly, a unified KBM method has been presented to obtain the analytical approximate solution of a fourth order ordinary weakly nonlinear differential equation with varying coefficients and large damping, when a pair of eigen-values of the unperturbed equation is a multiple of the other pair or pairs. The methods have been illustrated by several examples.

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Introduction

Numerous physical, mechanical, chemical, biochemical, biological, and some economic laws and relations appear mathematically in the form of differential equations which are linear or nonlinear, autonomous or non-autonomous. Practically, all differential equations involving physical phenomena are nonlinear. The methods of solutions of linear differential equations are comparatively essay and well established. On the contrary, the techniques of solutions of nonlinear differential equations are less available and, in general, linear approximations are frequently used. The method of small oscillations is a well-known example of the linearization of problems, which are essentially nonlinear. With the discovery of numerous phenomena of self- excitation of circuits containing nonlinear conductors of electricity, such as electron tubes, gaseous discharge, etc. and in many cases of nonlinear mechanical vibrations of special types, the methods of small oscillations become inadequate for their analytical treatment. There exists an important difference between the phenomena which oscillate in steady state and the phenomena governed by the linear differential equations with constant coefficients, e.g., oscillation of a pendulum with small amplitude, in that the amplitude of the ultimate stable oscillation seems to be entirely independent of initial conditions, whereas in oscillations governed by the linear differential equations with constant coefficients, it depends upon the initial conditions.

Van der Pol first paid attention to the new (self-excitation) oscillation and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the processes. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing differential equations in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations there exist some methods. Among the methods, the methods of perturbation, *i.e.,* asymptotic expansions in terms of small parameter are foremost. According to these techniques, the solutions are presented by the first two terms to avoid rapidly growing algebraic complexity. Although these perturbation expansions may be divergent, they can be more useful for qualitative and quantitative representations than the expansions that are uniformly convergent.

Now the perturbation methods are widely used in science and engineering problems to obtain the analytical approximate solutions based on known exact solutions to nearby problems. Such asymptotic techniques can also be used to provide initial guesses for numerical approximations, and they can now be generated through clever use of symbolic computations. The perturbation methods are most effectively used to analyze the problems in fluid and solid mechanics, control theory and celestial mechanics, a variety of nonlinear oscillations, nonlinear wave propagation and reaction-diffusion systems arising in numerous physical and biological contexts. Usually the physical and the biological problems occur with cubic and quadratic nonlinearities respectively.

The applications of the perturbation methods have been extended to nonlinear oscillators with strong nonlinearity. However, the algebraic manipulation of the perturbation procedures involves excessive labour. Recently, He has investigated a novel homotopy perturbation method for solving strongly nonlinear differential systems without damping effects. But most of the physical and biological problems appear in presence of damping with strong nonlinearity in nature and it keeps an important role to the systems. But He's homotopy perturbation method is able to handle the nonlinear systems without damping and the KBM method is also able to handle nonlinear systems with small nonlinearities. To overcome these limitations, we have presented an analytical technique based on He's homotopy perturbation technique and the extended form of the Krylov-Bogoliubov-Mitropolskii (KBM) method for solving strongly nonlinear differential systems in presence of significant damping effects. Our presented method, requiring no small parameters in the equations, can readily eliminate the limitations of the classical perturbation techniques. In the other hand, this technique can take full advantage of the classical perturbation techniques. Homotopy is an important part of differential topology and this technique is widely applied to determine all roots of nonlinear algebraic equations. Some interesting results have been achieved by this method. The confluence of modern mathematics has posed a challenge for developing technologies capable of handling strongly nonlinear equations which can not be successfully dealt by the classical perturbation methods. Homotopy perturbation method is uniquely qualified to address this challenge. The approximations obtained by this method are valid not only for large parameters, but also for small parameters. The homotopy perturbation method is proposed wherein the results at the first order of approximations are much more accurate than the classical perturbation solutions at second order of approximations

In this dissertation, we shall discuss the problems that can be described by the dynamical systems of the second order nonlinear autonomous differential equations with strong nonlinearities by coupling the He's homotopy perturbation technique and the extended form of the KBM method. We shall also study the third and fourth order weakly nonlinear differential systems by the classical KBM method. An important approach to study such nonlinear oscillatory problems is the small parameter (homotopy parameter) expansion according to the homotopy perturbation and the KBM methods. Two widely spread methods in the theory of nonlinear oscillations are mainly used; one is homotopy perturbation method and the other is averaging, particularly the classical KBM method. According to the homotopy perturbation and the KBM techniques, the solutions start with the solutions of linear equations, termed as generating solutions, assuming that, in the nonlinear cases, the amplitude(s) and the phase(s) variables of the solutions of linear differential equations are time dependent functions rather than constants. These methods introduce an additional condition on the first order derivative of the generating solutions for determining the solutions of nonlinear differential systems. Originally, the homotopy perturbation and the KBM methods were developed to obtain the periodic solutions of second order nonlinear differential equations. Now-a-days, these methods are used to obtain oscillatory, damped-oscillatory and non-oscillatory solutions of second, third and fourth order nonlinear differential systems by imposing some restrictions to obtain uniformly valid solutions.

Most of the authors have found the solutions of second order nonlinear systems for conservative cases by the homotopy perturbation and the KBM methods. A few number of authors have investigated the solutions of second, third, fourth order weakly nonlinear differential systems for non-conservative cases by the KBM method. In this dissertation, some second order strongly nonlinear differential equations have been studied with significant damping effects and their solutions are investigated by coupling the homotopy perturbation and the KBM methods and the third and fourth order weakly nonlinear differential systems have been studied by the well known KBM method with significant damping effects. The results obtained by the presented method may be used in mechanics, nonlinear wave equations, nonlinear oscillations, mathematical physics, plasma physics, nonlinear problems arising in various engineering applications, circuit theory, control theory, biology and bio-chemical systems, population dynamics, boundary layer theory, reaction-diffusion equations, etc.

Chapter 1

The Survey and The Proposal

1.1 The Survey

The characteristics of nonlinear differential equations are peculiar. But mathematical formulations of physical and engineering problems often results in differential equations that are nonlinear. However, in many cases it is possible to replace a nonlinear differential equation with a related linear differential equation that approximates the actual equations closely enough to give useful results. Often such linearization is not possible or feasible; when it is not, the original nonlinear equation itself must be handled.

During the last several decades a number of famous Russian scientists, Andronov [38], Andronov and Chaikin [39], Mandelstam and Papalexi [110], Krylov and Bogoliubov [94], Bogoliubov and Mitropolskii [45] worked jointly and investigated nonlinear mechanics. Among them, Krylov and Bogoliubov are certainly to be found most active scientists in nonlinear mechanics. Now-a-days, the scientists and the researchers use their concept to study the nonlinear differential systems.

Firstly, Krylov and Bogoliubov (KB) [94] considered the following nonlinear differential equation of the following form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(x.\dot{x}, t, \varepsilon), \tag{1.1}$$

where ε is a small positive parameter which characterises the nonlinearity, ω is known as the frequency of the nonlinear system and f is a given nonlinear function and it can be expanded as a power series in ε, whose coefficients are polynomials in $x, \dot{x}, \sin t, \cos t$. In fact, f contains neither ε nor t. Similar equations are well known in astronomy and have been investigated by Lindstedt [96,97], Gylden [73], Liapounoff [98] and above all by Poincare [134]. In general, it seems that, Krylov and Bogoliubov [94] applied the same method. However, the applications in which they view are quite different, being mainly in engineering, technology or physics, notably electrical circuit theory. The method has also been used in plasma physics, theory of oscillations and control theory. In the treatment of nonlinear oscillations by perturbation method, Lindstedt [96,97], Gylden [73], Liapounoff [98] and Poincare [134] discussed only the periodic solutions, but they did not consider the transients response of the systems. Firstly, Krylov and Bogoliubov (KB) [94] discussed the transients response of the systems. The KB method starts with the solution of the linear equation, assuming that in nonlinear case, the amplitude and phase variables in the solution of the linear equation are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the solution. Extensive uses have been made and some important works are done by Stoker [143], McLachlan [111], Minorsky [112], Nayfeh [123,124] and Bellman [46].

Most probably, Poisson initiated to determine the analytical approximate solutions of nonlinear differential equations around 1830 and the technique was introduced by Liouville [99]. Duffing [68] investigated many significant results concerning the periodic solutions of the following nonlinear differential equation

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = -\varepsilon x.^3 \tag{1.2}$$

Somewhat different nonlinear phenomena occur when the amplitude of the dependent variable of a dynamical system is less or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation, like these phenomena is

$$\frac{d^2x}{dt^2} - \varepsilon(1 - x^2)\frac{dx}{dt} + x = 0. \tag{1.3}$$

The Eq. (1.3) is known as Van der Pol [150] equation in literature. This equation has a very extensive field of application in connection with self-excited oscillations in electron-tube circuits. In general, f contains neither ε nor t, hence the Eq. (1.1) yields the following form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon f(x, \dot{x}). \tag{1.4}$$

The KB method is very similar to that of Van der Pol and related to it. Van der Pol applied the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of the linear equation $\ddot{x} + \omega^2 x = 0$. On the other hand KB applied the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus in the KB method, the varied constants are a and φ while in the Van der Pol's method the constants are a and b. The KB method is more interesting, convenient and widely used technique from the point of view of applications, since it deals directly with the amplitude and the phase of the quasi-harmonic oscillation.

If $\varepsilon = 0$, then the Eq. (1.4) becomes a linear equation and its solution is obtained in the following form

$$x = a\cos(\omega t + \varphi), \tag{1.5}$$

where a and φ are known as arbitrary constants to be obtained from the given initial conditions.

If $\varepsilon \neq 0$, but is sufficiently small, *i.e.*, $\varepsilon << 1$, then KB assumed that the solution is still obtained by Eq. (1.5) with the first derivative of the following form

$$\frac{dx}{dt} = -a\omega\sin(\omega t + \varphi), \tag{1.6}$$

where a and φ are functions of time t, rater than being constants. Thus the desired solution of Eq. (1.4) is obtained in the following form

$$x = a(t)\cos(\omega t + \varphi(t)), \tag{1.7}$$

and the first derivative of the solution Eq. (1.7) takes the following form

$$\frac{dx}{dt} = -a(t)\omega\sin(\omega t + \varphi(t)), \tag{1.8}$$

Now differentiating Eq. (1.7) with respect to time t, it leads to

$$\frac{dx}{dt} = \frac{da}{dt}\cos\psi - a\omega\sin\psi - a\frac{d\varphi}{dt}\sin\psi, \qquad \psi = \omega t + \varphi(t). \tag{1.9}$$

Therefore, for Eq. (1.6), one obtains

$$\frac{da}{dt}\cos\psi - a\frac{d\varphi}{dt}\sin\psi = 0. \tag{1.10}$$

Again differentiating Eq. (1.8) with respect to time t, then it yields

$$\frac{d^2x}{dt^2} = -\frac{da}{dt}\omega\sin\psi - a\omega^2\cos\psi - a\omega\frac{d\varphi}{dt}\cos\psi. \tag{1.11}$$

Substituting Eq. (1.11) into Eq. (1.4) and then using Eqs. (1.7)- (1.8), it gives

$$\frac{da}{dt}\omega\sin\psi + a\omega\frac{d\varphi}{dt}\cos\psi = -\varepsilon f(a\cos\psi, -a\omega\sin\psi). \tag{1.12}$$

By solving Eq. (1.10) and Eq. (1.11) for $\frac{da}{dt}$ and $\frac{d\varphi}{dt}$ one obtains

$$\begin{aligned} \frac{da}{dt} &= -\varepsilon f(a\cos\psi, -a\omega\sin\psi)\sin\psi / \omega, \\ \frac{d\varphi}{dt} &= -\varepsilon f(a\cos\psi, -a\omega\sin\psi)\cos\psi / a\omega. \end{aligned} \tag{1.13}$$

Thus instead of the second order single differential Eq. (1.4) with the unknown x, we obtain two first order differential equations with the unknown amplitude a and phase φ. Since $\frac{da}{dt}$ and $\frac{d\varphi}{dt}$ are proportional to the small parameter ε, the amplitude a and the phase φ are slowly varying functions with respect to time t with the period $T = 2\pi/\omega$ and for the first approximation they are assumed as constants.

Expanding $f(a\cos\psi, -a\omega\sin\psi)\sin\psi$ and $f(a\cos\psi, -a\omega\sin\psi)\cos\psi$ in a Fourier series, the first approximate solution of Eq. (1.4) by averaging Eq. (1.13) over one period is given by

$$
\begin{aligned}
&< \frac{da}{dt} > = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\, d\psi, \\
&< \frac{d\varphi}{dt} > = -\frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\, d\psi,
\end{aligned}
\tag{1.14}
$$

where the amplitude a and the phase φ are independent of time t under the integrals.

KB called their method asymptotic series in the sense that $\varepsilon \to 0$. In fact, an asymptotic series itself is not convergent, but for fixed number of terms, the approximate solution tends to the exact solution as $\varepsilon \to 0$. It is noticed that the term asymptotic is frequently used in the theory of oscillations, also in the sense that $\varepsilon \to \infty$. But in the case of the mathematical model is quite different.

Later, this method has been amplified and justified mathematically by Bogoliubov and Mitropolskii [45], and extended to non-stationary vibrations by Mitropolskii [113]. They assumed the solution of Eq. (1.4) in the following form

$$
x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \cdots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{1.15}
$$

where u_i, $i = 1,2,3,\cdots,n$ are periodic functions of ψ with a period 2π and the amplitude a and the phase ψ are functions of time t, and they satisfy the following first order differential equations

$$
\begin{aligned}
\frac{da}{dt} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \varepsilon^3 A_3(a) + \cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \\
\frac{d\psi}{dt} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \varepsilon^3 B_3(a) + \cdots + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}).
\end{aligned}
\tag{1.16}
$$

The functions u_i, A_i and B_i, $i = 1,2,3,\cdots n$ are to be chosen in such a way that the Eq. (1.15), after replacing a and ψ by the functions defined in Eq. (1.16) is a solution of the Eq. (1.4). Since there is no restriction in choosing the functions A_i and

B_i, that generate the arbitrariness in definitions of the functions u_i [45]. To remove this arbitrariness, the following additional conditions are imposed

$$\int_0^{2\pi} u_i(a,\psi)\cos\psi\, d\psi = 0,$$
$$\int_0^{2\pi} u_i(a,\psi)\sin\psi\, d\psi = 0. \tag{1.17}$$

These conditions guarantee the absence of the secular terms in arising all successive approximations. Differentiating Eq. (1.15) twice with respect to time t, utilizing the relations Eq. (1.16), substituting $\frac{d^2x}{dt^2}, \frac{dx}{dt}$ together with x into the original Eq. (1.4) and equating the coefficients of like powers of $\varepsilon^i, i = 1,2,3,\cdots,n$ results are obtained in a recursive system as the following form

$$\omega^2\left(\frac{\partial^2 u_i}{\partial\psi^2} + u_i\right) = f^{(i-1)}(a,\psi) + 2\omega(aB_i\cos\psi + A_i\sin\psi), \tag{1.18}$$

where

$$f^{(0)}(a,\psi) = f(a\cos\psi, -a\omega\sin\psi),$$
$$f^{(1)}(a,\psi) = u_1 f_x(a\cos\psi, -a\omega\sin\psi) + (A_1\cos\psi - a B_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi})$$
$$\times f_{\dot{x}}(a\cos\psi, -a\omega\sin\psi) + (a B_1^2 - A_1\frac{dA_1}{d\psi})\cos\psi \tag{1.19}$$
$$+ (2A_1 B_1 - a A_1\frac{dB_1}{d\psi})\sin\psi - 2\omega(A_1\frac{\partial^2 u_1}{\partial a\partial\psi} + \frac{\partial^2 u_1}{\partial\psi^2}).$$

It is clear that $f^{(i-1)}$ is a periodic function of the variable ψ with period 2π, which also depends on the amplitude a. Therefore $f^{(i-1)}$ as well as u_i can be expanded in a Fourier series in the following forms

$$f^{(i-1)}(a,\psi) = g_0^{(i-1)}(a) + \sum_{n=1}^{\infty}(g_n^{(i-1)}(a)\cos n\psi + h_n^{(i-1)}(a)\sin n\psi),$$
$$u_i(a,\psi) = v_0^{(i-1)}(a) + \sum_{n=1}^{\infty}(v_n^{(i-1)}(a)\cos n\psi + w_n^{(i-1)}(a)\sin n\psi), \tag{1.20}$$

where

$$g_0^{(i-1)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(i-1)}(a\cos\psi, -a\omega\sin\psi)\,d\psi,$$

$$g_n^{(i-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(i-1)}(a\cos\psi, -a\omega\sin\psi)\cos n\psi\,d\psi, \tag{1.21}$$

$$h_n^{(i-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(i-1)}(a\cos\psi, -a\omega\sin\psi)\sin n\psi\,d\psi, \qquad n \geq 1.$$

Here $v_1^{(i-1)} = w_1^{(i-1)} = 0$ for all values of i, since the integrations in Eq. (1.17) vanish. Substituting these values into Eq. (1.18), it becomes

$$\begin{aligned}
&\omega^2 v_0^{(i-1)}(a) + \sum_{n=1}^{\infty} \omega^2(1-n^2)[v_n^{(i-1)}(a)\cos n\psi + w_n^{(i-1)}(a)\sin n\psi] \\
&= g_0^{(i-1)}(a) + (g_1^{(i-1)}(a) + 2\omega a B_i)\cos n\psi + (h_1^{(i-1)}(a) + 2\omega A_i)\sin n\psi) \\
&+ \sum_{n=2}^{\infty}(g_n^{(i-1)}(a)\cos n\psi + h_n^{(i-1)}(a)\sin n\psi).
\end{aligned} \tag{1.22}$$

Now equating the coefficients of the same order of harmonics, we obtain

$$\begin{aligned}
&(g_1^{(i-1)}(a) + 2\omega a B_i) = 0, \qquad (h_1^{(i-1)}(a) + 2\omega A_i) = 0, \\
&v_0^{(i-1)}(a) = \frac{g_0^{(i-1)}(a)}{\omega^2}, \qquad v_n^{(i-1)}(a) = \frac{g_n^{(i-1)}(a)}{\omega^2(1-n^2)}, \\
&w_n^{(i-1)}(a) = \frac{h_n^{(i-1)}(a)}{\omega^2(1-n^2)}, \qquad n > 1.
\end{aligned} \tag{1.23}$$

These are the sufficient conditions to find the desired order of analytical approximations. For the first order analytical approximation, we can find

$$\begin{aligned}
A_1 &= -\frac{h_1^{(1)}(a)}{2\omega} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\,d\psi, \\
B_1 &= -\frac{g_1^{(1)}(a)}{2\omega a} = -\frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\,d\psi.
\end{aligned} \tag{1.24}$$

Therefore, the variational Eq. (1.16) yields

$$\begin{aligned}
\frac{da}{dt} &= -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi\,d\psi, \\
\frac{d\psi}{dt} &= \omega - \frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi\,d\psi.
\end{aligned} \tag{1.25}$$

The Eq. (1.25) is similar to the Eq. (1.14). Thus the first order analytical approximate solution obtained by Bogoliubov and Mitropolskii [45] is identical to the original solution obtained by KB [94]. In literature, this method is well known as Krylov-Bogoliubov-Mitropolskii (KBM) [45,94,113] method. In the second case, higher order analytical approximate solution can be found easily. The correction term u_1 is obtained from Eq. (1.23) in the following form

$$u_1 = \frac{g_0^{(1)}(a)}{\omega^2} + \sum_{n=2}^{\infty} \frac{(g_n^{(1)}(a)\cos n\psi + h_n^{(1)}(a)\sin n\psi)}{\omega^2(1-n^2)}, \; n > 1. \qquad (1.26)$$

The solution Eq. (1.15) combining with u_1 is known as the first order analytical (improved) solution in which the amplitude a and the phase φ are the solutions of Eq. (1.25). If the values of the functions A_1, B_1 and u_1 are substituted from the relation Eqs. (1.24) – (1.25) into the second relation Eq. (1.19), the function $f^{(1)}$ and in a similar way, the unknown functions A_2, B_2 and u_2 can be found. Thus the determination of higher order analytical approximation is completed.

Volosov [152,153], Museenkov [114] and Zabreiko [154] also obtained higher order effects of the nonlinear differential systems. The KBM method has been extended by Kruskal [95] to solve the fully nonlinear ordinary differential equation of the following form

$$\frac{d^2x}{dt^2} = F(x, \frac{dx}{dt}, \varepsilon). \qquad (1.27)$$

The solution of this equation is based on the recurrent relations and is given as a power series of the small parameter.

Cap [64] has studied the nonlinear differential system of the following form

$$\frac{d^2x}{dt^2} + \omega^2 x = \varepsilon F(x, \frac{dx}{dt}). \qquad (1.28)$$

The solution of Eq. (1.28) has been obtained by using the elliptical functions in the sense of KBM [45,94,113] method.

Struble [140] has developed a technique for solving weakly nonlinear oscillatory systems governed by the following equation

$$\frac{d^2x}{dt^2} + \omega_0^{\,2} x = \varepsilon F(x, \frac{dx}{dt}, t). \qquad (1.29)$$

He has assumed the asymptotic solution of Eq. (1.29) in the form

$$x = a\cos(\omega_0 t - \varphi) + \sum_{i=1}^{n} \varepsilon^i x_i(t) + O(\varepsilon^{i+1}), \qquad (1.30)$$

where the amplitude a and the phase φ are slowly varying functions of time t.

Later, the KBM method has been extended by Popov [135] to damped nonlinear differential systems of the following from

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega^2 x=\varepsilon F(x,\frac{dx}{dt}), \tag{1.31}$$

where $-2k\frac{dx}{dt}$ is the linear damping force and $0<k<\omega$. It is noteworthy that, because of the importance of the method [135] in the physical systems, involving damping force, Meldenlson [118] and Bojadziev [57] rediscovered Popov's results. In the case of damped nonlinear systems, the first equation of Eq. [1.16] has been replaced by

$$\frac{da}{dt}=-ka+\varepsilon A_1(a)+\varepsilon^2 A_2(a)+\varepsilon^3 A_3(a)+\cdots+\varepsilon^n A_n(a)+O(\varepsilon^{n+1}). \tag{1.16a}$$

Murty *et al.* [115] have found a hyperbolic type asymptotic solution of an over-damped system represented by the nonlinear differential Eq. (1.31) in the sense of KBM method; *i.e.*, in the case $k>\omega$. They have used $\cosh\varphi$ or $\sinh\varphi$ instead of $\cos\varphi$, which is used in [45,94,113,118,135]. In the case of oscillatory or damped oscillatory processes $\cos\varphi$ may be used arbitrarily for all kinds of initial conditions. But in the case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the set of initial conditions [45,94,113,115,117]. Murty and Deekshatulu [116] have found another asymptotic solution of the over-damped nonlinear system represented by the Eq. (1.31), by the method of variation of parameters. Alam [18] has extended the KBM method to find the solutions of over-damped nonlinear systems, when one root becomes much smaller than the other root. Murty [117] has presented a unified KBM method for solving nonlinear systems represented by the Eq. (1.31). Bojadziev and Edwards [56] have investigated the solutions of oscillatory and non-oscillatory systems represented by the Eq. (1.31), when k and ω are slowly varying functions of time t. Arya and Bojadziev [41,42] have examined damped oscillatory and time dependent oscillatory systems with slowly varying parameters and delay. Alam *et al.* [4] have extended the KBM method to certain non-oscillatory systems with slowly varying coefficients. Later, Alam [19] has unified the KBM method for solving an *nth* order nonlinear differential systems with slowly varying coefficients. Sattar [141] has developed an asymptotic method to solve a critically damped nonlinear system represented by Eq. (1.31). He has found the asymptotic solution of the Eq. (1.31) in the following form

$$x=a(1+\psi)+\varepsilon u_1(a,\psi)+\varepsilon^2 u_2(a,\psi)+\cdots+\varepsilon^n u_n(a,\psi)+O(\varepsilon^{n+1}), \tag{1.32}$$

where a is defined in the Eq. (1.16a) and ψ is defined as

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + \cdots + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}). \tag{1.16b}$$

Alam [5] has developed an asymptotic method for the second order over-damped and a critically damped nonlinear differential system. Alam [9,20] has also extended the KBM method for certain non-oscillatory systems when the eigen-values of the unperturbed equation are real and non-positive. Alam [6] has presented a new perturbation method based on the work of Krylov-Bogoliubov-Mitropolskii [45,94,113] to find the analytical approximate solutions of the nonlinear differential systems with large damping. Later he [11] has extended it to *nth* order nonlinear differential systems with large damping effects. Alam *et al.* [12] have investigated the perturbation solution of a second order time dependent nonlinear system based on the modified Krylov-Bogoliubov-Mitropolskii method.

Making use of the KBM method, Bojadziev [47] has investigated nonlinear damped oscillatory systems with small time lag. Bojadziev [52] has also found the solutions of the damped forced nonlinear vibrations with small time delay. Bojadziev [53], Bojadziev and Chan [54] have applied the KBM method to the problems of population dynamics. Bojadziev [55] has used the KBM method to investigate the nonlinear biological and biochemical systems. Lin and Khan [101] have also used the KBM method to study some biological problems. Bojadziev *et al.* [48], Proskurjakov [136] have investigated the periodic solutions of nonlinear systems by the KBM and the Poincare methods and compared the two solutions. Bojadziev and Lardner [49,50] have investigated monofrequent oscillations in mechanical systems including the case of internal resonance, governed by hyperbolic differential equation with small nonlinearity. Bojadziev and Lardner [51] have also investigated hyperbolic differential equations with large time delay. Freedman *et al.* [70] have used the KBM method to study the stability, persistence and extinction in a prey-predator system with discrete and continuous time delay. Freedman and Ruan [71] have also used the KBM method in three-species food chain models with group defense. Murty [117] has presented a unified KBM method for solving the differential Eq. (1.31) by using their previous solution [115] as a general solution for the un-damped, damped and over-damped cases, which is the basis of the unified theory and assumed a solution of Eq. (1.31) according to the asymptotic method in the following form

$$x = \frac{a}{2}e^{\psi} - \frac{a}{2}e^{-\psi} + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \cdots, \tag{1.33}$$

where the amplitude a and the phase ψ satisfy the following first order differential equations

$$\begin{aligned} \frac{da}{dt} &= -ka + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \varepsilon^3 A_3(a) + \cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}), \\ \frac{d\psi}{dt} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \varepsilon^3 B_3(a) + \cdots + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}), \end{aligned} \tag{1.34}$$

with $\lambda_1 - \lambda_2 = 2\omega_1$ and ω_1 is an unknown function of a and ψ, where λ_1 and λ_2 are the eigen-values of the corresponding linear equation of Eq. (1.31). In his paper, Murty [117] restricted by himself to only the first approximation. When the eigen-values of the corresponding linear system are real, ψ being a real quantity and the first two terms on the right hand sides of Eq. (1.33) can be combined as

$$x = a\sinh\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \cdots, \tag{1.35}$$

which corresponds to over-damped solution of Eq. (1.33). When the eigen-values of the corresponding linear system are complex conjugates (*i.e.*, for un-damped and under damped cases), instead of real, inserting $a = -ia$, $\psi = i\psi$, $\cosh i\psi = \cos\psi$ and $\sinh i\psi = -i\sin\psi$, then the solution of Eq. (1.31) yields

$$x = a\sin\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \cdots, \tag{1.36}$$

which corresponds to the periodic and under damped solution of Eq. (1.31). Murty's [117] technique is a generalization of the KBM method. Many authors have extended this technique in various oscillatory and non- oscillatory nonlinear physical and biological systems. Bojadziev and Edwards [56] have investigated nonlinear damped oscillatory and non- oscillatory systems with slowly varying coefficients by following the Murty's [117] unified method.

Most probably, first Osiniskii [132] has extended the KBM method to handle a third order nonlinear differential equation of the following form

$$\frac{d^3x}{dt^3} + k_1\frac{d^2x}{dt^2} + k_2\frac{dx}{dt} + k_3 x = \varepsilon f(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}), \tag{1.37}$$

where ε is a small positive parameter, $k_j, j = 1,2,3$ are arbitrary constants and f is a given nonlinear function and he has assumed the asymptotic solution of the following form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \varepsilon^2 u_2(a,b,\psi) + \cdots + \varepsilon^n u_n(a,b,\psi) + O(\varepsilon^{n+1}), \tag{1.38}$$

where u_i, $i = 1,2,3,\cdots n$ are periodic functions of ψ with period 2π and the amplitudes a, b and the phase ψ are functions of time t and they satisfy the following first order differential equations

$$\frac{da}{dt} = -\lambda a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \varepsilon^3 A_3(a) + \cdots + \varepsilon^n A_n(a) + O(\varepsilon^{n+1}),$$
$$\frac{db}{dt} = -\mu a + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \varepsilon^3 B_3(a) + \cdots + \varepsilon^n B_n(a) + O(\varepsilon^{n+1}), \quad (1.39)$$
$$\frac{d\psi}{dt} = \omega + \varepsilon C_1(a) + \varepsilon^2 C_2(a) + \varepsilon^3 C_3(a) + \cdots + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}),$$

where $-\lambda$ and $-\mu \pm i\omega$ are the eigen-values of Eq. (1.37) for unperturbed case, *i.e.*, $\varepsilon = 0$.

Alam and Sattar [2] have extended Murty's [117] unified technique for obtaining the transient response of a third order nonlinear system. Alam [16] has presented a unified KBM method to find a general solution of an *nth* order differential equation with constant coefficients, which is not the formal form of the original KBM method. In his paper, the solution contains some unusual variables. Yet this solution is very important. He [16] has assumed a weakly nonlinear system of the following form

$$\frac{d^{(n)}x}{dt^{(n)}} + k_1 \frac{d^{(n-1)}x}{dt^{(n-1)}} + \cdots + k_n x = \varepsilon f(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \cdots), \quad (1.40)$$

where over-dots denote differentiation with respect to time t and k_j, $j = 1,2,\cdots,n$ are arbitrary constants. He [16] has assumed the solution of Eq. (1.40) in the following form

$$x(t,\varepsilon) = \sum_{j=1}^{n} a_j e^{\lambda_j t} + \varepsilon u_1(a_1, a_2, \cdots, a_n, t) + \varepsilon^2 u_2(a_1, a_2, \cdots, a_n, t) + \cdots, \quad (1.41)$$

where λ_j, $j = 1,2\cdots,n$ are the eigenvalues of the corresponding linear equation of Eq. (1.40) and each a_j satisfies a first order differential equation

$$\dot{a}_j = \varepsilon A_j(a_1, a_2, \cdots, a_n) + \varepsilon^2 A_j(a_1, a_2, \cdots, a_n) + \cdots \quad (1.42)$$

In most of the perturbation techniques an approximate solution is determined in terms of the amplitude and the phase variables. But the solution of Eq. (1.40) starts with some unusual variables $a_1, a_2, \cdots, a_n$, such a choice of variables is important to tackle various nonlinear problems with an easier approach. This technique greatly speeds up the KBM method to determine the asymptotic solution.

Osiniskii [132] has also extended the KBM method to a third order nonlinear partial differential equation with internal friction and relaxation. Mulholland [119] has studied nonlinear oscillations governed by a third order ordinary differential equation. Lardner and Bojadziev [103] have investigated nonlinear damped oscillations governed by a third order partial differential equation. They have introduced the concept of the "couple amplitude" where the unknown functions A_i, B_i and C_i depend on both the amplitudes a and b. Rauch [137] has studied the oscillations of a third order nonlinear autonomous system. Bojadziev [57], Bojadziev and Hung [58] have used the KBM method to investigate a 3-dimensional time dependent differential systems. Sattar [142] has extended the KBM asymptotic method for three-dimensional over-damped nonlinear differential systems. Alam and Sattar [1] have developed a method to solve third order critically damped nonlinear differential systems. Alam [13] have redeveloped the method presented in [1] to find the approximate solutions of critically damped nonlinear systems in presence of different damping forces. Later he has unified the KBM method for solving critically damped nonlinear differential systems [22]. Alam and Sattar [7] have studied the time dependent third order oscillating systems with damping based on the extension of the KBM asymptotic method. Alam [10,14], Alam *et al.* [23] have developed a simple method to obtain the time response of second order over-damped nonlinear differential systems with slowly varying coefficients under some special conditions. Later Shamsul [14], Alam and Hossain [21] have extended the method [10,18] to obtain the time response of *nth* order $(n \geq 2)$, over-damped systems. Alam [15] has also developed an asymptotic method for obtaining non-oscillatory solution of the third order nonlinear ordinary differential systems. Alam and Sattar [2] have presented a unified KBM method for solving third order oscillating systems. Alam [24] has also presented a modified and compact form of a unified KBM method for solving *nth* order nonlinear ordinary differential systems. The formula presented in [24] is compact, systematic, practical and easier than that of [16]. Alam [25] has developed a general formula based on the extended KBM method for obtaining *nth* order time dependent quasi-linear differential equation with damping. Bojadziev [57], Bojadziev and Hung [58] have used at least two trial solutions to investigate the time dependent differential systems; one is for the resonant case and the other is for the non- resonant case. But Alam [25] has used only one set of variational equations, arbitrary for resonant and non- resonant cases. Alam *et al.* [28] have also presented a general Struble's technique for solving an *nth* order weakly nonlinear ordinary differential system with damping effects. They have considered the following ordinary differential equation

$$\frac{d^{(n)}x}{dt^{(n)}}+k_1\frac{d^{(n-1)}x}{dt^{(n-1)}}+\cdots+k_n x=\varepsilon f(x,\frac{dx}{dt},\frac{d^2x}{dt^2},\cdots), \tag{1.43}$$

and assumed the solution of Eq. (1.43) in the following form

$$x(t,\varepsilon)=\sum_{j=1}^{n}a_j e^{\lambda_j t}+\varepsilon u_1(a_1,a_2,\cdots,a_n,t)+\varepsilon^2 u_2(a_1,a_2,\cdots,a_n,t)+\cdots. \tag{1.44}$$

Then they have written the Eq. (1.43) as the following form

$$\sum_{j=1}^{n}(\prod_{k=1,k\neq j}^{n}(D-\lambda_k)(\dot{a}_j e^{\lambda_j t}))+\prod_{j=1}^{n}(D-\lambda_j)(\varepsilon u_1+\cdots)=\varepsilon f, \tag{1.45}$$

since $(D-\lambda_k)(a_j e^{\lambda_j t})=\dot{a}_j e^{\lambda_j t}$ and $D=\frac{d}{dt}$. Finally, they have used the following transformations to obtain the formal form of the KBM method

$$a_{2l-1}=\alpha_l e^{i\theta_l}/2,\ a_{2l}=\alpha_l e^{-i\theta_l}/2,\ l=1,2,3,\cdots, \tag{1.46}$$

where α_l and θ_l are the amplitude and the phase variables respectively.

Raymond and Cabak [138] have examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Lewis [104,105] has investigated stability for autonomous second order two-degree-of-freedom systems and for a control surface with structural nonlinearities in supersonic flow. Andrianov *et al.* [43], Awrejcewicz *et al.* [44] have presented some new trends of asymptotic techniques in application to nonlinear dynamical systems in terms of summation and interpolation methods. O'Malley *et al.* [126] has found an asymptotic solution of a semiconductor device problem involving reverse bias. O'Malley *et al.* [127-128] has presented singular perturbation method for ordinary differential equations with matching and used this singular perturbation method to stiff differential equations. O'Malley *et al.* [129] has also presented exponential asymptotic for boundary layer resonance and dynamical metastability. Akbar *et al.* [34,35] have found an asymptotic solution of the fourth order over-damped and under-damped nonlinear systems based on the work of [18,21]. Akbar *el al.* [36] have developed a simple technique for obtaining certain over-damped solution of an *nth* order nonlinear ordinary differential equation. Akbar *el al.* [37] have also presented the KBM unified method for solving *nth* order nonlinear differential equation under some special conditions including the case of internal resonance.

Recently, Shamsul *et al.* [29] have developed an extension of the general Struble's technique [28] for solving an *nth* order nonlinear differential equation when the corresponding unperturbed equation has some repeated eigenvlues. They have studied the following *nth* order weakly nonlinear ordinary differential equation

$$\frac{d^{(n)}x}{dt^{(n)}} + k_1\frac{d^{(n-1)}x}{dt^{(n-1)}} + \cdots + k_n x = \varepsilon f(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \cdots \nu t), \tag{1.47}$$

where $x^{(j)}, j \geq 4$ represents a *jth* derivative of x, ε is a small positive parameter, k_l, $j = 1,2,\cdots,n$ are constants and f is a given nonlinear function. They have considered that the unperturbed equation of Eq. (1.47) has $r = 1,2,\cdots$ pair repeated eigenvalues, namely $\lambda_{2l-1} = \lambda_{2l}$, $l = 1,2,\cdots,r$ and the rest are distinct. They have assumed a function of the form of Eq. (1.44) to solve Eq. (1.47). In [22] it is substituted for variables $a_j, j = 1,2,\cdots 2r$, *i.e.*, for $a_{2l-1} = a_{2l}$, $l = 1,2,\cdots r$ as

$$\begin{aligned} a_{2l-1} &= \frac{1}{2}\bar{a}_l(t) + \bar{b}_l(t)/(\lambda_{2l-1} - \lambda_{2l}), \\ a_{2l} &= \frac{1}{2}\bar{a}_l(t) - \bar{b}_l(t)/(\lambda_{2l-1} - \lambda_{2l}). \end{aligned} \tag{1.48}$$

Therefore, the solution Eq. (1.44) becomes

$$\begin{aligned} x(t,\varepsilon) &= \sum_{l=1}^{r}[\frac{1}{2}\bar{a}_l(t)(e^{\lambda_{2l-1}t} + e^{\lambda_{2l}t}) + \bar{b}_l(t)(e^{\lambda_{2l-1}t} - e^{\lambda_{2l}t})] \\ &+ \sum_{j=2r+1}^{n} a_j(t)e^{\lambda_j t} + \varepsilon u_1(\bar{a}_1,\cdots \bar{a}_r \cdots, \bar{b}_1, \cdots, \bar{b}_r, a_{2r+1}, \cdots, a_n, t) + \cdots, \end{aligned} \tag{1.49}$$

Finally they have written the solution of Eq. (1.49) in the following form

$$x(t,\varepsilon) = \sum_{l=1}^{r}\alpha_l(t)e^{\lambda_{2l-1}t} + \sum_{j=2r+1}^{n} a_j(t)e^{\lambda_j t} + \varepsilon u_1(\alpha_1, \alpha_1, \cdots \alpha_r, a_{2r+1}, \cdots, a_n, t) + \cdots, \tag{1.50}$$

where $\alpha_l(t) = \bar{a}_l(t) + t\bar{b}_l(t)$. In [29], they have used the variables $\bar{a}_l, \bar{b}_l$ or α_l, $l = 1,2,\cdots r$ to analyze the case of repeated complex eigenvalues as well as of repeated real eigenvalues.

There exists a large body of literature dealing with the problems of approximate solutions to nonlinear equations using various methodologies [1-75,93-154] but many of them are applicable only to weakly nonlinear ones. To overcome this limitation, many novel techniques have been proposed in resent years. Cheung *et* *al.* [65] have proposed a modified Lindstedt-Poincare method. Lim and Wu [107] have presented a modified Mickens procedure for a certain strongly nonlinear oscillators. Hu [77] has presented a classical perturbation technique which is valid for large parameters. Hu [78] has also developed the solution of a strongly quadratic nonlinear oscillator by the method of harmonic balance. Hu and Tang [79] have presented a classical iteration procedure valid for certain strongly nonlinear oscillator. He [80] has investigated an approximate solution of nonlinear differential equations with convolution product nonlinearities. He [81] has investigated a novel homotopy perturbation technique to find a periodic solution of a general nonlinear oscillator for conservative systems. He [81] has considered the following nonlinear differential equation in the following form

$$A(u) - f(r) = 0, \; r \in \Omega, \tag{1.51}$$

with the boundary conditions

$$B(u, \frac{\partial u}{\partial t}) = 0, \quad r \in \Gamma, \tag{1.52}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of the domain Ω.Then He [81] has written Eq. (1.51) in the following form.

$$L(u) + N(u) - f(r) = 0, \tag{1.53}$$

where L is linear part, while N is nonlinear part. He [81] has constructed a homotopy $v(r,p) : \Omega \times [0,1] \to \Re$ which satisfies

$$\mathrm{H}(v,p) = (1-p)[L(v) - L(u_0)] + p[A(u) - f(r)] = 0, \; p \in [0,1], \; r \in \Omega \tag{1.54a}$$

or

$$\mathrm{H}(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{1.54b}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (1.51), which satisfies the boundary conditions. Obviously, from Eq. (1.54), it becomes

$$\mathrm{H}(v,0) = L(v) - L(u_0) = 0, \tag{1.55}$$

$$\mathrm{H}(v,1) = A(v) - f(r) = 0. \tag{1.56}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. He [81] has assumed the solution of Eq. (1.54) as a power series of p in the following form

$$v = v_0 + pv_1 + p^2 v_2 + \cdots. \tag{1.57}$$

The approximate solution of Eq. (1.51) is given by setting $p = 1$ in the form

$$u = v_0 + v_1 + v_2 + \cdots. \tag{1.58}$$

He [82] have presented some new approaches to Duffing equation with strongly and higher order nonlinearity (I) linearized perturbation method. In this paper, He [82] has considered a typical nonlinear equation, Duffing equation with higher order nonlinearity of following form

$$\frac{d^2u}{dt^2} + u = -\varepsilon u^5, \qquad u(0) = A, \dot{u}(0) = 0. \tag{1.59}$$

He has constructed a homotopy $\Omega \times [0,1] \to \Re$ which satisfies

$$L(v) - L(u_0) + pL(u_0) + p\varepsilon v^5 = 0, \tag{1.60}$$

where

$$Lu = \frac{d^2u}{dt^2} + u. \tag{1.61}$$

He [82] has assumed the initial solution of Eq (1.59) in the following form

$$u_0(t) = A\cos\alpha t, \tag{1.62}$$

where $\alpha(\varepsilon)$ is a nonzero unknown constant with $\alpha(0) = 1$. Supposing the approximate solution of Eq. (1.60) has the form of Eq. (1.57), by the same manipulation, He [82] has found the following equations

$$L(v_0) - L(u_0) = 0, \tag{1.63}$$

$$L(v_1) + L(u_1) + \varepsilon v_0^5 = 0, \qquad \dot{v}_1(0) = v_1(0) = 0. \tag{1.64}$$

Finally, He [82] has obtained the solution of Eq. (1.59) in the following form

$$u(t) = A\cos(1 + 5\varepsilon A^4/16)t. \tag{1.65}$$

Belendez *et al.* [59] have developed the approximate solution of a nonlinear oscillator typified as a mass attached to a stretched elastic wire by homotopy perturbation method. They have studied the governing non-dimensional equation of motion for a mass attached to a stretched elastic wire in the following form

$$\frac{d^2x}{dt^2} + x - \frac{\lambda x}{\sqrt{1+x^2}} = 0, \; 0 \le \lambda \le 1. \tag{1.66}$$

According to the homotopy perturbation method, they have re-written Eq. (1.66) as

$$\frac{d^2x}{dt^2}+\omega^2 x=\omega^2 x-x+\frac{\lambda x}{\sqrt{1+x^2}}, \tag{1.67}$$

where ω is the unknown frequency of the nonlinear oscillator. For Eq. (1.67), they have established the following homotopy

$$\frac{d^2x}{dt^2}+\omega^2 x=p(\omega^2 x-x+\frac{\lambda x}{\sqrt{1+x^2}}), \tag{1.68}$$

where p is the homotopy parameter. When $p=0$, Eq. (1.68) becomes the linearized equation and for the case $p=1$, Eq. (1.68) returns the original problem. Now the homotopy parameter is used to expand the solution $x(t)$ in powers of the parameter p in the following form

$$x(t)=x_0(t)+p\,x_1(t)+p^2x_2(t)+\cdots. \tag{1.69}$$

Substituting Eq. (1.69) into Eq. (1.68) and equating the terms with like powers of p, they found a series of linear equations, of which they have written only the first two as

$$\frac{d^2x_0}{dt^2}+\omega^2 x_0=0,\ x_0(0)=A,\ \frac{dx_0(0)}{dt}=0, \tag{1.70}$$

and

$$\frac{d^2x_1}{dt^2}+\omega^2 x_1=(\omega^2-1)\,x_0+\frac{\lambda\,x_0}{\sqrt{1+{x_0}^2}}),\ x_1(0)=0,\ \frac{dx_1(0)}{dt}=0. \tag{1.71}$$

They have obtained the solution of Eq. (1.70) in the following form

$$x_0=A\cos\omega t, \tag{1.72}$$

where A is the constant amplitude. Substitution of Eq. (1.72) into Eq. (1.71), yields the following differential equation for x_1 as

$$\frac{d^2x_1}{dt^2}+\omega^2 x_1=(\omega^2-1)\,A\cos\omega t+\frac{\lambda\,A\cos\omega t}{\sqrt{1+A^2\cos^2\omega t}}). \tag{1.73}$$

It is possible to do the following Fourier series expansion

$$\frac{A\cos\omega t}{\sqrt{1+A^2\cos^2\omega t}}=a_1\cos\omega t+a_3\cos 3\omega t+\cdots, \tag{1.74}$$

where the first term of this expansion can be obtained by means of the following equation

$$a_1 = \int_0^{\pi/2} \frac{A\cos\theta}{\sqrt{1+A^2\cos^2\theta}} \cos\theta d\theta,\ \theta = \omega t. \tag{1.75}$$

Substituting Eq. (1.67) into Eq. (1.66) gives

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = (\omega^2 - 1 + \frac{\lambda a_1}{A}) A\cos\omega t + \lambda \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t]. \tag{1.76}$$

The requirement of no secular terms in particular solution of Eq. (1.76) implies that the coefficient of the $\cos\omega t$ term is zero, *i.e.*,

$$\omega^2 - 1 + \frac{\lambda a_1}{A} = 0 \tag{1.77}$$

Substituting Eq. (1.77) into Eq. (1.76) and reordering, they have obtained the frequency ω as the following form

$$\omega = \sqrt{1 - \frac{4\lambda_1}{\pi A^2}[E(-A^2) - K(-A^2)]}, \tag{1.78}$$

where $K(m)$ and $E(m)$ are the complete integrals of the first and second kind, respectively and defined as follows

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\cos^2\theta}},$$
$$E(m) = \int_0^{\pi/2} \sqrt{1-m\cos^2\theta} d\theta. \tag{1.79}$$

Then Eq. (1.76) has been written as

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = \lambda \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \tag{1.80}$$

The solution of Eq. (1.80) is given by

$$x_1 = \sum_{n=0}^{\infty} c_{2n+1} \cos[(2n+1)\omega t], \tag{1.81}$$

$$c_{2n+1} = -\frac{\lambda a_{2n+1}}{4n(n+1)\omega^2},\ n \geq 1. \tag{1.82}$$

Thus the first order approximate solution of Eq. (1.66) has been obtained as

$$x(t) = x_0(t) + x_1(t) + \cdots,\ p = 1 \tag{1.83}$$

Belendez *et al.* [60] have developed the application of modified He's homotopy perturbation method to obtain higher-order approximations to a nonlinear oscillator with discontinuities. In another article, Belendez, *et al.* [61] have determined the solution for anti- symmetric quadratic oscillator by a modified He's homotopy perturbation method. Belendez *et al.* [62] have used a modified He's homotopy perturbation method to calculate the periodic solutions of a conservative nonlinear oscillator for which the elastic force term is proportional to $x^{1/3}$.

From our study it is seen that, most of the authors have developed the analytical method only for conservative systems with strongly nonlinearities by the homotopy perturbation method and the KBM method for weakly nonlinear differential systems with constant coefficients with damping effects.

1.2 The Proposal

We propose an analytical technique based on the He's homotopy perturbation technique and the extended form of the KBM method modeled by second order strongly nonlinear autonomous differential equations in presence of damping in the following form

$$\ddot{x} + 2k\,\dot{x} + \nu^2 x = \varepsilon f(x, \dot{x}), \tag{1.84}$$

and the perturbation method has been developed to obtain the analytical approximate solutions of third and fourth order weakly nonlinear autonomous differential systems with varying coefficients and significant damping effects in the following forms respectively

$$\dddot{x} + c_1(\tau)\ddot{x} + c_2(\tau)\dot{x} + c_3(\tau)x = \varepsilon f(x, \dot{x}, \ddot{x}, \tau) \tag{1.85}$$

and

$$x^{(4)} + c_1(\tau)\dddot{x} + c_2(\tau)\ddot{x} + c_3(\tau)\dot{x} + c_4(\tau)x = \varepsilon f(x, \dot{x}, \ddot{x}, \dddot{x}, \tau) \tag{1.86}$$

In **Chapter 2,** [145] He's homotopy perturbation method has been extended to second order strongly nonlinear differential systems with cubic nonlinearity and significant damping effects. Also this analytical technique is valid for weakly nonlinear systems. **Chapter 3,** [146] deals with a homotopy perturbation and the KBM methods for second order strongly nonlinear differential systems with slowly varying coefficients and significant damping effects. In **Chapter 4,** an extended approximate technique has been found for solving strongly nonlinear oscillating processes in biological systems with small damping effects. In **Chapter 5,** [147] homotopy perturbation method has been developed to obtain the second approximate solution of a second order strongly nonlinear differential system with significant damping effects. In **Chapter 6,** [148] an asymptotic method has been developed to obtain the second order approximate solution of a third order weakly nonlinear differential equation with slowly varying coefficients and small damping effects. A unified KBM method has been presented to obtain the analytical approximate solution of a fourth order weakly nonlinear differential equation with large damping effects and slowly varying coefficients [149] in **Chapter 7.**

Chapter 2

An Approximate Technique for Solving Strongly Nonlinear Differential Systems with Damping Effects

2.1 Introduction

The most common methods for constructing the analytical approximate solutions to the nonlinear oscillator equations are the perturbation methods. Some well known perturbation methods are the Krylov-Bogoliubov-Mitropolskii (KBM) [45,94,113] method, the Lindstedt-Poincare (LP) method [122,125] and the method of multiple time scales [125]. Almost all perturbation methods are based on an assumption that small parameter must exist in the equations. In general, the perturbation approximations are valid only for weakly nonlinear problems. Lim *et al.* [107] have presented a new analytical approach to the Duffing-harmonic oscillator. Recently, He [91] has presented a new interpretation of homotopy perturbation method for strongly nonlinear differential systems. Belendez *et al.* [63] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator. Alam *et al.* [28] and Ludeke *et al.* [108] have obtained the solutions of strongly cubic nonlinear oscillators with large damping effects. Chatterjee [66] has also presented the solution of a strongly cubic nonlinear oscillator with damping effects by the harmonic balance based on averaging. But numerous physical and oscillating systems encounter in presence of small damping in nature. In this chapter, we have presented an analytical technique by coupling the He's [63,80-92] homotopy perturbation technique and the extended KBM [45,94,113] method to solve the second order strongly nonlinear ordinary differential system with small damping effects. The advantage of the presented method is that the first order approximate solutions show a good agreement with the corresponding numerical solutions.

2.2 The Method

Let us consider a conservative nonlinear oscillator in the following form

$$\ddot{x} + \nu^2 x = -\varepsilon f(x), \qquad (2.1)$$

where over dots denote differentiation with respect to time t, ν is a constant, ε is a positive parameter, not necessarily small and $f(x)$ is a given nonlinear function which satisfies the following condition

$$f(-x) = -f(x). \tag{2.2}$$

According to the homotopy perturbation technique [63,80-92], we can write the above equation in the following form

$$\ddot{x} + (\nu^2 + \lambda)x = \lambda x - \varepsilon x^3, \tag{2.3}$$

where λ is an unknown constant which can be found by eliminating the secular terms and $f(x) = x^3$. Now Eq. (2.3) can be re-written as

$$\ddot{x} + \omega^2 x = \lambda x - \varepsilon x^3, \tag{2.4}$$

where

$$\omega^2 = \nu^2 + \lambda. \tag{2.5}$$

Herein ω is a constant and known as the frequency of the nonlinear oscillator. According to the He's [63,80-92] homotopy perturbation method, we have constructed the following homotopy

$$\ddot{x} + \omega^2 x = p(\lambda x - \varepsilon x^3), \tag{2.6}$$

where p is the homotopy parameter. When $p = 0$, Eq. (2.6) becomes the following linearized equation

$$\ddot{x} + \omega^2 x = 0, \tag{2.7}$$

and for the case $p = 1$, Eq. (2.6) becomes the original problem. According to the homotopy perturbation method, parameter p is used to expand the solution $x(t)$ in powers of the parameter p in the following form

$$x(t) = x_0(t) + p x_1(t) + p^2 x_2(t) + \cdots \tag{2.8}$$

Substituting Eq. (2.8) into Eq. (2.6) and then equating the coefficients of the like powers of p, we obtain the following linear differential equations

$$\ddot{x}_0 + \omega^2 x_0 = 0, \; x_0(0) = a_0, \; \dot{x}_0(0) = 0 \tag{2.9}$$

$$\ddot{x}_1 + \omega^2 x_1 = \lambda x_0 - \varepsilon x_0^3, \; x_1(0) = 0 \; \dot{x}_1(0) = 0, \tag{2.10}$$

where a_0 is a positive constant. The solution of Eq. (2.9) is obtained as

$$x_0(t) = a_0 \cos \omega t, \tag{2.11}$$

Substituting Eq. (2.11) into Eq. (2.10), we obtain

$$\ddot{x}_1 + \omega^2 x_1 = (\lambda a_0 - \frac{3}{4}\varepsilon a_0^3)\cos\omega t - \frac{1}{4}\varepsilon a_0^3 \cos 3\omega t \tag{2.12}$$

To remove the secular terms from Eq. (2.12), we can set

$$\lambda a_0 - \frac{3}{4}\varepsilon a_0^3 = 0 \tag{2.13}$$

which leads to

$$\lambda = \frac{3\varepsilon a_0^2}{4}. \tag{2.14}$$

By inserting the value of λ from Eq. (2.14) into Eq. (2.5), we obtain the following solution for ω as

$$\omega(a_0) = \sqrt{\upsilon^2 + \frac{3\varepsilon a_0^2}{4}}. \tag{2.15}$$

From Eq. (2.15), it is seen that the frequency depends on the initial amplitude a_0 and independent of time t for conservative nonlinear systems. Now Eq. (2.12) can be rewritten in the following form

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{1}{4}\varepsilon a_0^3 \cos 3\omega t, \tag{2.16}$$

with the initial conditions

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0. \tag{2.17}$$

The solution of Eq. (2.16) is then

$$x_1 = -\frac{1}{32\omega^2}\varepsilon a_0^3(\cos\omega t - \cos 3\omega t). \tag{2.18}$$

Thus we obtain the first order analytical approximate solution of Eq. (2.1) by setting $p = 1$ in the following form

$$x = x_0 + x_1 = \left(\frac{32\omega^2 - \varepsilon a_0^2}{32\omega^2}\right) a_0 \cos\omega t + \frac{\varepsilon a_0^3}{32\omega^2}\cos 3\omega t, \tag{2.19}$$

where the frequency $\omega(a_0)$ is given by Eq. (2.15).

But most of the physical and oscillating systems encounter in presence of small damping in nature. So we are interested to consider a nonlinear oscillator with small damping effects in the following form [28,66,108]

$$\ddot{x} + 2k\dot{x} + \upsilon^2 x = -\varepsilon f(x), \quad k << 1, \tag{2.20}$$

where $2k$ is the linear damping coefficients. We can easily return to Eq. (1) from Eq. (2.20) by setting $k = 0$. Now we are going to consider the following transformation

$$x = y(t)e^{-kt}. \tag{2.21}$$

Differentiating Eq. (2.21) twice with respect to time t and substituting $\ddot{x}$, $\dot{x}$ together with x into the original Eq. (2.20) and then simplifying them, we obtain

$$\ddot{y} + (\upsilon^2 - k^2)y = -\varepsilon e^{kt} f(ye^{-kt}). \tag{2.22}$$

According to the homotopy perturbation method [63,80-92], Eq. (2.22) can be written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon e^{kt} f(ye^{-kt}), \tag{2.23}$$

where

$$\omega^2 = \upsilon^2 - k^2 + \lambda. \tag{2.24}$$

Herein ω is a constant for undamped nonlinear oscillator and λ is an unknown constant which can be determined by eliminating the secular terms (as it is eliminated for the undamped problem). However, for a damped nonlinear differential system ω is a time dependent function and it varies slowly with time t. To handle this situation, we can use the extended KBM [45,94] method by Mitropolskii [113]. According to this technique, we choose the first approximate solution of Eq. (2.23) in the following form

$$y = a\cos\varphi, \tag{2.25}$$

where the amplitude a and the phase φ vary slowly with time t. When damping is present the amplitude a is a function of time t, and approaches zero as $t \to \infty$. The amplitude a and the phase φ satisfy the following first order ordinary differential equations

$$\begin{aligned} \dot{a} &= k A_1(a,\tau) + k^2 A_2(a,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + k B_1(a,\tau) + k^2 B_2(a,\tau) + \cdots, \end{aligned} \tag{2.26}$$

where k is a small positive parameter and $\tau = kt$ is the slowly varying time. It is clear that this solution is similar to the undamped solution if $k \to 0$ and $a \to a_0$, $\varphi \to \omega t$. Differentiating Eq. (2.25) twice with respect to time t, utilizing the relations Eq. (2.26) and by substituting $\ddot{y}$ and y into Eq. (2.23) and then by equating the coefficients of $\sin\varphi$ and $\cos\varphi$, we obtain

$$A_1 = -\omega' a/(2\omega), \; B_1 = 0, \tag{2.27}$$

where a prime denotes differentiation with respect to τ. Now putting Eq. (2.25) into Eq. (2.21) and Eq. (2.27) into Eq. (2.26), we obtain the following equations

$$x = ae^{-kt}\cos\varphi, \tag{2.28}$$

$$\dot{a} = -k\omega' a/(2\omega), \quad \dot{\varphi} = \omega(\tau). \tag{2.29}$$

Eq. (2.28) represents the first order analytical approximate solution of Eq (2.20) by the presented method with small damping effects. Usually the integration of Eq. (2.29) is performed by well-known techniques of calculus [125], but sometimes they are solved by a numerical procedure [28,57,66,108]. Thus the determination of the first order analytical approximate solution of Eq. (2.20) is completed.

2.3 Example

As an example of the above procedure, let us consider the Duffing equation with small damping effects and strongly cubic nonlinearity [28,66,108] in the following form

$$\ddot{x} + 2k\dot{x} + \upsilon^2 x = -\varepsilon x^3, \tag{2.30}$$

where $f(x) = x^3$. Now using the transformation Eq. (2.21) into Eq. (2.30) and then simplifying them, we obtain

$$\ddot{y} + (\upsilon^2 - k^2)y = -\varepsilon e^{-2kt} y^3. \tag{2.31}$$

According to the homotopy perturbation [63,80-92] method, Eq. (2.31) can be re-written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon e^{-2kt} y^3, \tag{2.32}$$

where ω is given by Eq. (2.24). Now according to the extended form of the KBM [45,94,113] method, the solution of Eq. (2.32) is given by Eq. (2.25) and the amplitude a and the phase φ are obtained by Eq. (2.29).

The requirement of no secular terms in particular solution of Eq. (2.32) implies that the coefficient of the $\cos\varphi$ term is zero. Setting this term to zero, we obtain

$$\lambda a - \frac{3\varepsilon a^3 e^{-2kt}}{4} = 0, \tag{2.33}$$

which leads to

$$\lambda = \frac{3\varepsilon a^2 e^{-2kt}}{4}. \tag{2.34}$$

Putting the value of λ from Eq. (2.34) into Eq. (2.24), yields

$$\omega^2 = \upsilon^2 - k^2 + \frac{3\varepsilon a^2 e^{-2kt}}{4}. \tag{2.35}$$

This is a time dependent frequency equation of the given nonlinear differential system. As $t \to 0$, Eq. (2.35) gives

$$\omega_0 = \omega(0) = \sqrt{\upsilon^2 - k^2 + \frac{3\varepsilon a_0^2}{4}}. \tag{2.36}$$

By integrating the first equation of Eq. (2.29), we get

$$a = a_0\sqrt{\frac{\omega_0}{\omega}}, \tag{2.37}$$

where a_0 is a constant of integration which represents the initial amplitude of the nonlinear system. Now putting Eq. (2.37) into Eq. (2.35), we obtain the following equation

$$\omega^3 + q\omega + r = 0, \tag{2.38}$$

where

$$q = -(\upsilon^2 - k^2),\ r = -\frac{3\varepsilon\omega_0 a_0^2 e^{-2kt}}{4}. \tag{2.39}$$

Eq. (2.38) is a cubic equation in ω. It has an analytical solution for every real value of ν. When $\nu > 0$ (especially $\nu > k$), then the solution of Eq. (2.38) becomes (see also [76] for details)

$$\omega = \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{1/3} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{1/3} \tag{2.40}$$

Now substituting $r = -2R, q = -3Q$ into Eq. (2.40), we get

$$\omega = \left(R + \sqrt{R^2 - Q^3}\right)^{1/3} + \left(R - \sqrt{R^2 - Q^3}\right)^{1/3},\quad R > Q, \tag{2.41}$$

or,

$$\omega = \left(R + i\sqrt{Q^3 - R^2}\right)^{1/3} + \left(R - i\sqrt{Q^3 - R^2}\right)^{1/3},\quad R < Q. \tag{2.42}$$

Herein the relations among Q, R, ν, k, ω_0 and a_0 are given by

$$Q = \frac{(\upsilon^2 - k^2)}{3},\ R = \frac{3\varepsilon\omega_0 a_0^2 e^{-2kt}}{8}. \tag{2.43}$$

According to [76], the real form of Eq. (2.42) is obtained as

$$\omega = 2\sqrt{Q}\cos\left(\frac{\tan^{-1}(V/R)}{3}\right), \tag{2.44}$$

where

$$V = \sqrt{Q^3 - R^2}. \tag{2.45}$$

The solution of the second equation of Eq. (2.29) becomes

$$\varphi = \varphi_0 + \int_0^t \omega(t)dt, \tag{2.46}$$

where φ_0 is the initial phase and ω is given by Eqs. (2.41) or (2.44). Thus the first order analytical approximate solution of Eq. (2.30) is obtained by Eq. (2.29) and the amplitude a and the phase φ are calculated by Eqs. (2.37) and (2.46) respectively.

The presented method also gives the desired results for second order strongly cubic nonlinear oscillator in presence of damping without linear term in equation, *i.e.*, $\ddot{x} + 2k\dot{x} + \varepsilon x^3 = 0$ (see also [66]). To handle this case, we had to put $\nu = 0$ in the solution of Eq. (2.30). Now substituting $r = -2R, q = 3Q$ into Eq. (2.40), and then it yields

$$\omega = \left(R + \sqrt{R^2 + Q^3}\right)^{1/3} - \left(-R + \sqrt{R^2 + Q^3}\right)^{1/3}. \tag{2.47}$$

Herein the relations among Q, R, k, ω_0 and a_0 are

$$Q = \frac{k^2}{3},\ R = \frac{3\varepsilon\omega_0 a_0^2 e^{-2kt}}{8}. \tag{2.48}$$

To obtain more corrected results for the cubic nonlinear oscillator $\ddot{x} + 2k\dot{x} + \varepsilon x^3 = 0$, we obtain the following solutions

$$x = a e^{-kt}(21\cos\varphi + \cos 3\varphi)/22, \tag{2.49}$$

$$\dot{\varphi} = \frac{21\omega(\tau)}{22}, \tag{2.50}$$

$$\omega_0 = \sqrt{\frac{11\varepsilon a_0{}^2}{14} - k^2}, \tag{2.51}$$

and a, ω are given by Eqs. (2.37) and (2.47) respectively.

where

$$R = \frac{11\varepsilon\omega_0 a_0{}^2 e^{-2kt}}{28}. \tag{2.52}$$

2.4 Results and Discussions

In this chapter, an analytic technique has been presented to obtain the first order analytical approximate solutions for second order strongly cubic nonlinear oscillators with small damping effects and the method has been successfully implemented to illustrate the effectiveness and convenience of the presented method. The first order analytical approximate solutions of Eq. (2.30) are computed by Eq. (2.28) and the corresponding numerical solutions are obtained by fourth order ***Runge-Kutta*** method.

This method can also be used to solve the second order strongly cubic nonlinear system in absence of linear term (*i.e.*, $\nu = 0$) in the equation. To justify the effectiveness of this method, we have compared the result obtained by the presented method to the result obtained by Chatterjee [66] and we have obtained better result than his [66] result. The solution obtained by the presented method coincides with the corresponding numerical but the Chatterjee's [66] solution deviate from the numerical solution for the same initial conditions (see also **Fig.2.5**).

In summary, the He's homotopy perturbation method [63,80-92] is able to handle the nonlinear systems without damping and the KBM method is also able to handle nonlinear systems with small nonlinearities [1-37,45,94,113]. The second order perturbation solution obtained by Alam *et al.* [28] shows a good coincidence with the numerical solution for strongly nonlinear oscillator (when $\upsilon \neq 0$) with strong damping effects but it does not give the desired result when the damping effect is small. Ludeke *et al.* [108] have also presented the generalized Duffing equation with large damping effects (when $\upsilon \neq 0$), but it does not give the desired result when the damping effect is small. But most of the physical and oscillating systems encounter in presence of small damping in nature. Furthermore, the presented method is simple and the advantage of this method is that the first order approximations show good agreement (see also **Figs. 1-5**) with the corresponding numerical solutions. The initial approximations can be freely chosen, which is identified via various methods [28,42,45,57,63,66,80-94,107,108,113]. The approximations obtained by the presented method are valid not only for strongly nonlinear differential systems, but also for weakly one with small damping effects. **Figs. 1-5** are provided to compare the solutions obtained by the presented method to the corresponding numerical solutions with small damping effects and strong nonlinearity.

To obtain the numerical solution of $\ddot{x} + 2k\dot{x} + \upsilon^2 x = -\varepsilon x^3$, the initial conditions $x(0), \dot{x}(0)$ are computed from the following equations

$$x(0) = a_0 \cos\varphi_0,$$

$$\dot{x}(0) = \left(\frac{3\varepsilon k a_0^2 \omega_0}{2(\omega_0^2 + k^2 - \nu^2)} - k a_0\right)\cos\varphi_0 - a_0\,\omega_0 \sin\varphi_0.$$

In general, the initial conditions $[x(0), \dot{x}(0)]$ are specified. Then one has to solve nonlinear algebraic equation in order to determine the initial amplitude a_0 and the initial phase φ_0 that appear in the solution, from the initial conditions equation.

2.5 Conclusion

In this chapter, we have presented an analytical technique by coupling the He's homotopy perturbation technique and the extended form of the KBM method. The presented method has eliminated some limitations of the homotopy perturbation technique and the KBM method.

Fig. 2.1

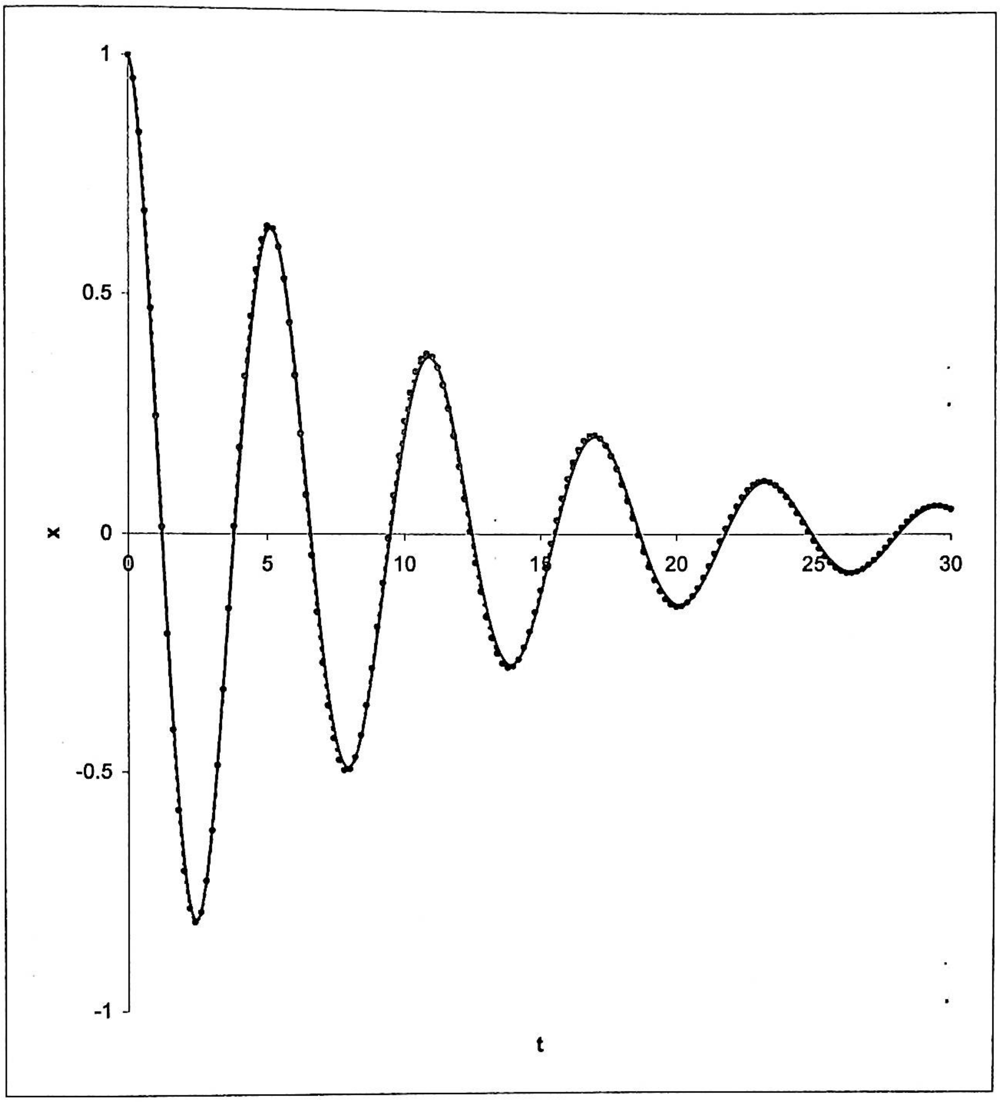

Fig. 2.1 First approximate solution of Eq. (2.30) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $[x(0)=1.0,\ \dot{x}(0)=-0.08227]$ or $a_0=1.0,\ \varphi_0=0$ when $\upsilon=1.0,\ k=0.1,\ \varepsilon=1.0$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Fig. 2.2

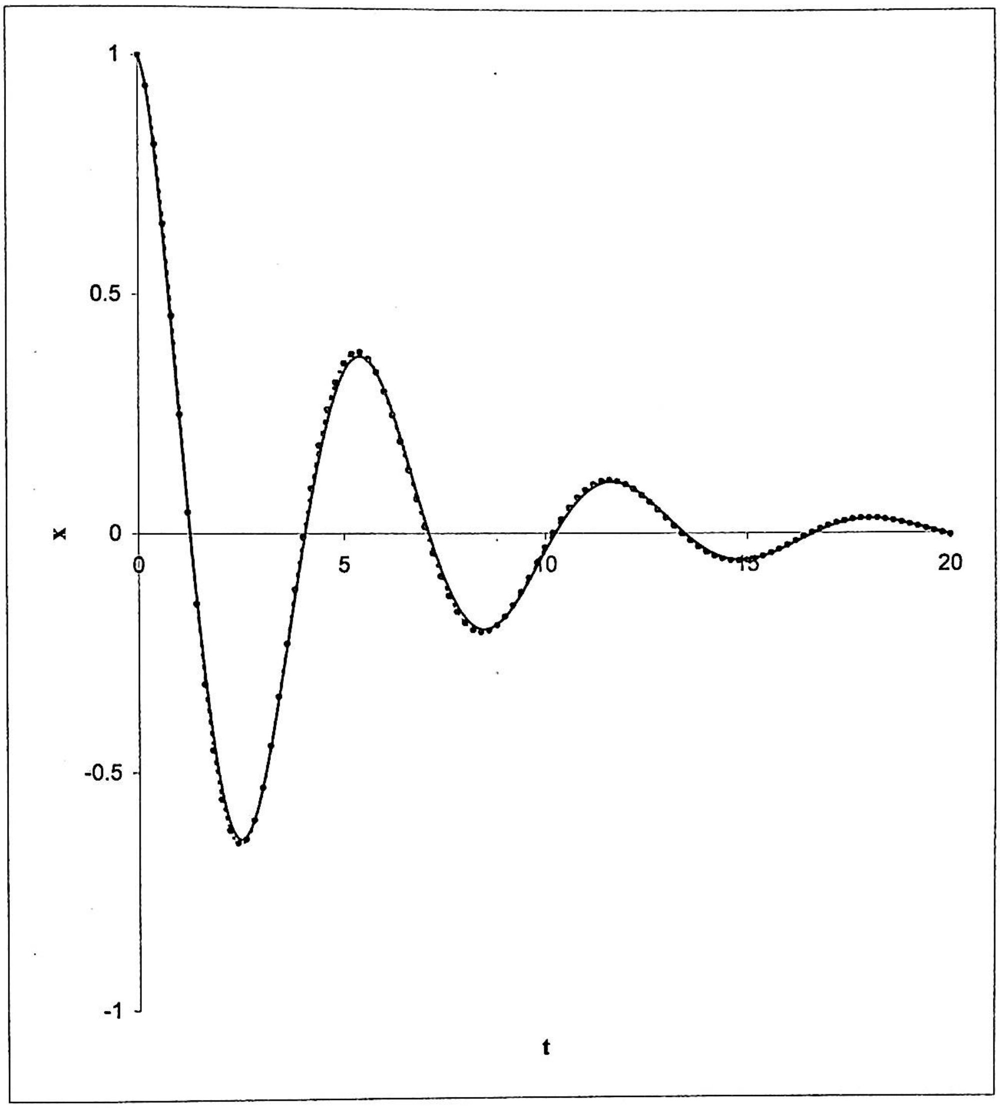

Fig. 2.2 First approximate solution of Eq. (2.30) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $[x(0)=1.0,\ \dot{x}(0)=-0.18320]$ or $a_0=1.0,\ \varphi_0=0$ when $\upsilon=1.0,\ k=0.2,\ \varepsilon=1.0$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Fig. 2.3

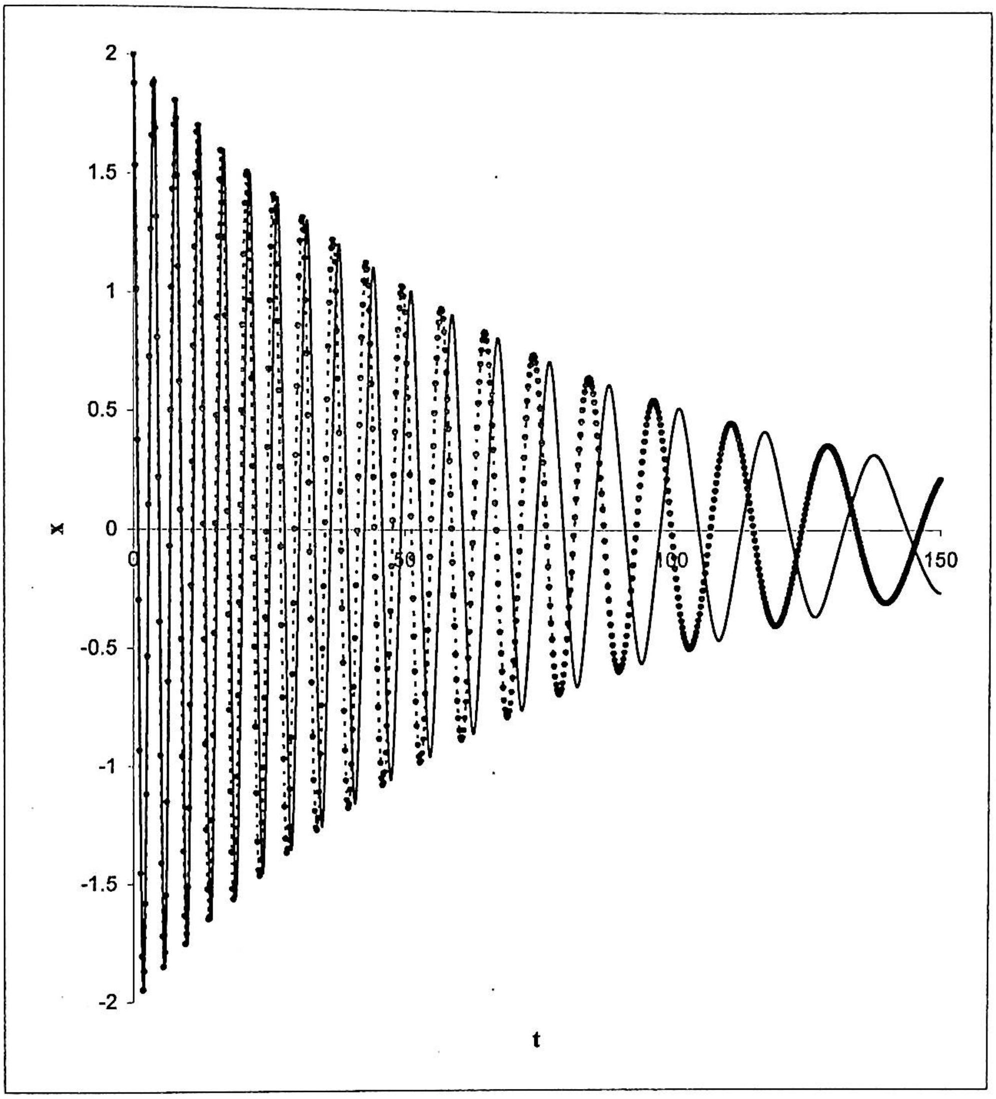

Fig. 2.3 First approximate solution of Eq. (2.30) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $[x(0)=2.0,\ \dot{x}(0)=-0.02667]$ or $a_0=2.0,\ \varphi_0=0$ when $\upsilon=0.0,\ k=0.02,\ \varepsilon=1.0$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Fig. 2.4

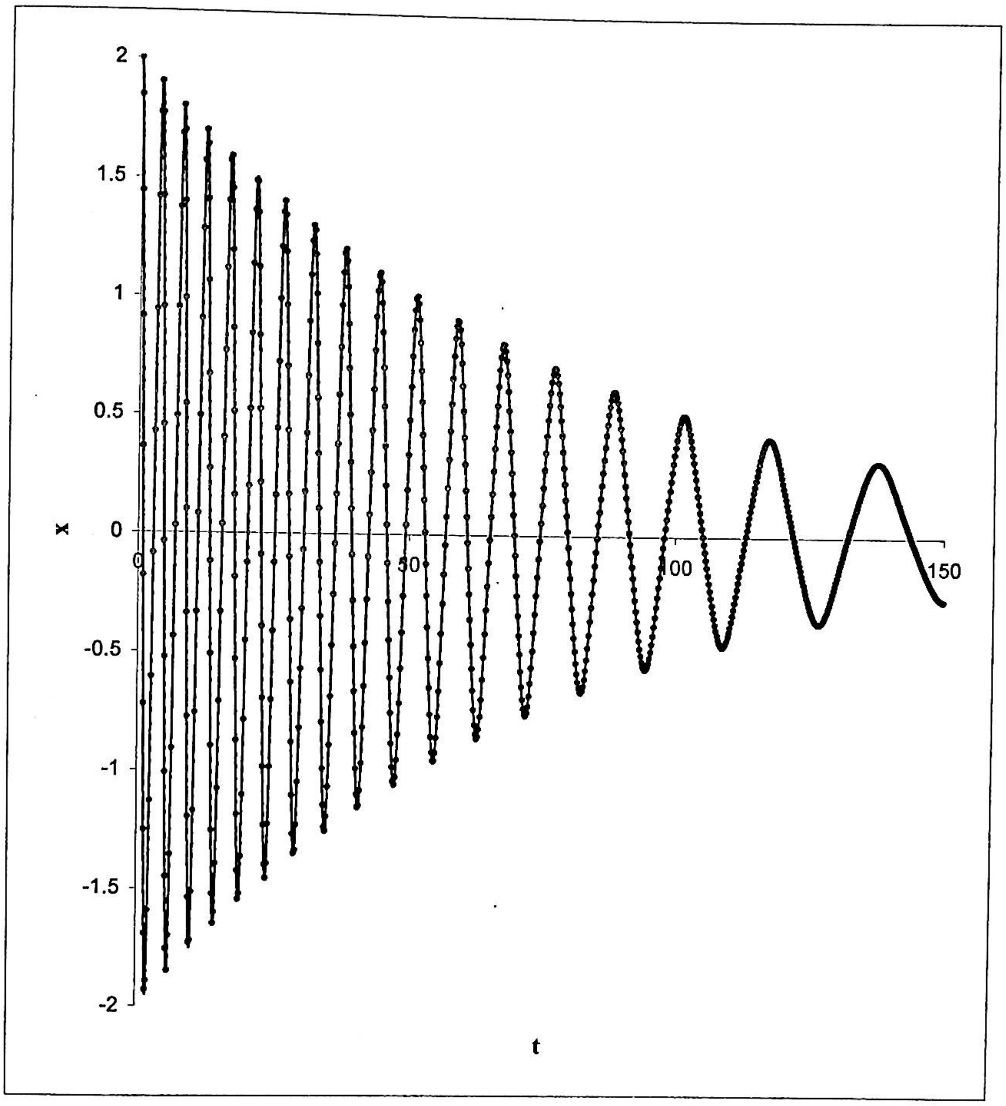

Fig. 2.4 First approximate solution of Eq. (2.30) is denoted by $-\bullet-$ (dashed lines) by the presented method with the initial conditions $[x(0)=2.0,\ \dot{x}(0)=-0.02667]$ or $a_0=2.0,\ \varphi_0=0$ when $\upsilon=0.0,\ k=0.02,\ \varepsilon=1.0$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Fig. 2.5

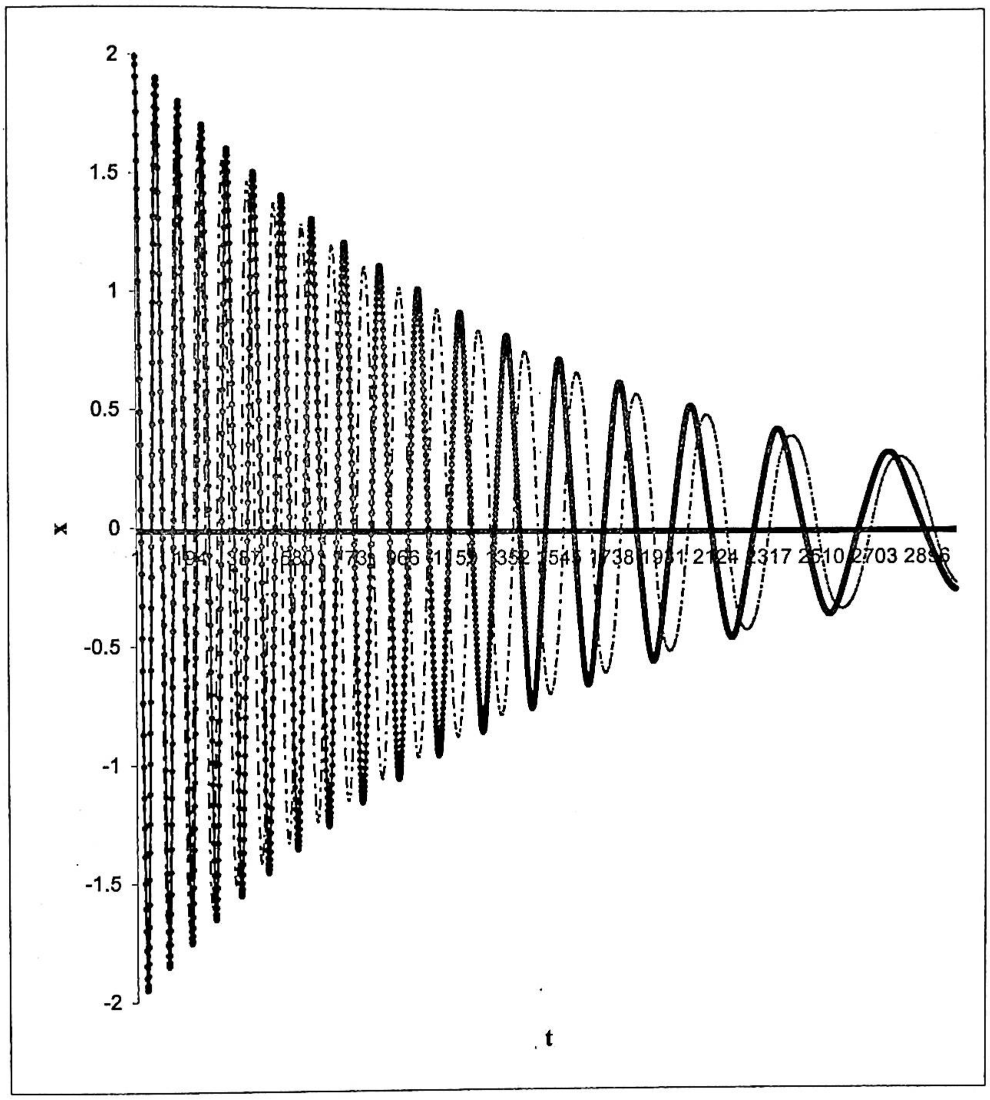

Fig. 2.5 First approximate solution of Eq. (2.30) is denoted by $-\bullet-$ (dashed lines) by the present method and the Chatterjee's [66] solution is denoted by ---- (dashed lines) with the same initial conditions $[x(0)=2.0,\ \dot{x}(0)=0]$ when $\upsilon=0.0$, $k=0.02$, $\varepsilon=1.0$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Chapter 3

An Approximate Technique to Duffing Equation with Small Damping and Slowly Varying Coefficients

3.1 Introduction

Most of the physical phenomena and engineering problems occur in nature in the forms of nonlinear differential systems with damping effects. The common methods for constructing the analytical approximate solutions to the nonlinear oscillator equations are the perturbation methods. Some well known perturbation methods are the Krylov-Bogoliubov- Mitropolskii (KBM) [45,94,113] method, the Lindstedt-Poincare (LP) method [122,125] and the method of multiple time scales [125]. Almost all perturbation methods are based on an assumption that small parameter must exist in the equations. Lim *et al.* [107] have presented a new analytical approach to the Duffing- harmonic oscillator. In recent years, He [81] has investigated the homotopy perturbation technique. In another paper, He [83] has developed a coupling method of a homotopy perturbation technique and a perturbation technique for strongly nonlinear problems. Recently, He [91] has also presented a new interpretation of homotopy perturbation method for strongly nonlinear differential systems. Belendez *et al.* [63] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator. Recently Roy *et al.* [139] have presented the effect of higher approximation of Krylov-Bogoliubov-Mitropolskii solution and matched asymptotic differential system with slowly varying coefficients and damping near to a turning point for weakly nonlinear system. The authors [63,81,83,91] have studied the nonlinear systems without considering any damping effects. But most of the physical and oscillating systems encounter in presence of damping in nature. In this chapter, we have presented an analytical technique to solve second order strongly nonlinear ordinary differential systems with small damping and slowly varying coefficients. Figures are provided to compare between the solutions obtained by the presented method and the corresponding numerical (considered to be exact) solutions.

3.2 The Method

Consider a nonlinear oscillator [139] modeled by the following equation

$$\ddot{x}+e^{-\tau}x+\varepsilon f(x)=0,\ x(0)=a_0,\qquad \dot{x}(0)=0, \tag{3.1}$$

where over dots denote differentiation with respect to time t, τ is a slowly varying time, a_0 is a given positive constant and $f(x)$ is a given nonlinear function which satisfies the following condition

$$f(-x)=-f(x). \tag{3.2}$$

According to the homotopy perturbation [63,80-92] technique, Eq. (3.1) can be re-written as

$$\ddot{x}+(e^{-\tau}+\lambda)x=\lambda x-\varepsilon f(x) \tag{3.3}$$

Eq. (3.3) yields,

$$\ddot{x}+\omega^2 x=\lambda x-\varepsilon f(x), \tag{3.4}$$

where

$$\omega^2=e^{-\tau}+\lambda. \tag{3.5}$$

Herein ω is a constant for undamped nonlinear oscillator and known as the frequency in literature and λ is an unknown function which can be determined by eliminating the secular terms. But for a damped nonlinear differential system ω is a time dependent function and it varies slowly with time t. To handle this situation, we are going to use the extended KBM [45,94] method by Mitropolskii [113]. According to the He's [63,80-92] homotopy perturbation method, we have constructed the following homotopy

$$\ddot{x}+\omega^2 x=p(\lambda x-\varepsilon x^3), \tag{3.6}$$

where p is the homotopy parameter and $f(x)=x^3$. When $p=0$, Eq. (3.6) becomes the linearized equation

$$\ddot{x}+\omega^2 x=0, \tag{3.7}$$

and for the case $p=1$, Eq. (3.6) becomes the original problem. The homotopy parameter p is used to expand the solution $x(t)$ in powers of p in the following form

$$x(t)=x_0(t)+px_1(t)+p^2x_2(t)+\cdots \tag{3.8}$$

Substituting Eq. (3.8) into Eq. (3.6) and then equating the coefficients of the like powers of p, we obtain the following linear differential equations

$$\ddot{x}_0+\omega^2 x_0=0,\ x_0(0)=a_0,\ \dot{x}_0(0)=0 \tag{3.9}$$

$$\ddot{x}_1+\omega^2 x_1=\lambda x_0-\varepsilon x_0^3,\ x_1(0)=0\ \dot{x}_1(0)=0, \tag{3.10}$$

where a_0 is a positive constant. The solution of Eq. (3.9) is then obtained as

$$x_0(t) = a_0 \cos \omega t. \tag{3.11}$$

Substituting Eq. (3.11) into Eq. (3.10), we obtain

$$\ddot{x}_1 + \omega^2 x_1 = (\lambda a_0 - \frac{3}{4}\varepsilon a_0^3)\cos \omega t - \frac{1}{4}\varepsilon a_0^3 \cos 3\omega t. \tag{3.12}$$

The requirement of no secular terms in particular solution of Eq. (3.12) implies that the coefficient of the $\cos \omega t$ term is zero. Setting this term to zero, we obtain

$$\lambda a_0 - \frac{3}{4}\varepsilon a_0^3 = 0 \tag{3.13}$$

For the nontrivial solution *i.e.*, $a_0 \neq 0$, Eq. (3.13) leads to

$$\lambda = \frac{3\varepsilon a_0^2}{4}. \tag{3.14}$$

By inserting the value of λ from Eq. (3.14) into Eq. (3.5), we obtain the following solution for ω as

$$\omega(a_0) = \sqrt{e^{-\tau} + \frac{3\varepsilon a_0^2}{4}}. \tag{3.15}$$

From Eq. (3.15), it is seen that the frequency depends on the initial amplitude a_0 and slowly varying time τ. Now using Eq. (3.13), Eq. (3.12) can be rewritten in the following form

$$\ddot{x}_1 + \omega^2 x_1 = -\frac{1}{4}\varepsilon a_0^3 \cos 3\omega t, \tag{3.16}$$

with the initial conditions

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0. \tag{3.17}$$

The solution of Eq. (3.16) is then obtained as

$$x_1 = -\frac{1}{32\omega^2}\varepsilon a_0^3(\cos \omega t - \cos 3\omega t). \tag{3.18}$$

Thus we obtain the first order analytical approximate solution of Eq. (3.1) is obtained by setting $p = 1$ in the following form

$$x = x_0 + x_1 = \left(\frac{32\omega^2 - \varepsilon a_0^2}{32\omega^2}\right) a_0 \cos \omega t + \frac{\varepsilon a_0^3}{32\omega^2}\cos 3\omega t, \tag{3.19}$$

where the frequency ω is given by Eq. (3.15).

But most of the physical and oscillating systems occur in presence of damping in nature and it keeps an important role to the systems. From our study, it is seen that the most of the authors [63,80-92,107] have presented the analytical technique for solving nonlinear oscillators without considering damping effects. So in chapter 3, we are interested to consider a strongly nonlinear oscillator [139] with small damping and slowly varying coefficients in the following form

$$\ddot{x} + 2k(\tau)\dot{x} + e^{-\tau}x = -\varepsilon f(x), \quad k << 1, \tag{3.20}$$

where $2k$ is the linear damping coefficient which varies slowly with time $t, \tau = kt$ is the slowly varying time.

Eq. (3.20) leads to Eq. (3.1) when $k = 0$. Let us assume the following transformation

$$x = y(t)e^{-kt}. \tag{3.21}$$

Differentiating Eq. (3.21) twice with respect to time t and substituting $\ddot{x}$, $\dot{x}$ together with x into Eq. (3.20) and then simplifying them, we obtain

$$\ddot{y} + (e^{-\tau} - k^2)y = -\varepsilon e^{kt} f(y e^{-kt}). \tag{3.22}$$

According to the homotopy perturbation method [63,80-92], Eq. (3.22) can be written as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon e^{kt} f(y e^{-kt}), \tag{3.23}$$

where

$$\omega^2 = e^{-\tau} - k^2 + \lambda. \tag{3.24}$$

Herein λ is an unknown constant which can be determined by eliminating the secular terms (as it is eliminated for the undamped problem). However, for a damped nonlinear differential system ω is a time dependent function and it varies slowly with time t. To handle this situation, we can use the extended KBM [45,94] method by Mitropolskii [113]. According to this technique, we are going to choose the analytical approximate solution of Eq. (3.23) in the following form

$$y = a\cos\varphi, \tag{3.25}$$

where a and φ vary slowly with time t, a and φ satisfy the following first order differential equations

$$\begin{aligned} \dot{a} &= k A_1(a,\tau) + k^2 A_2(a,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + k B_1(a,\tau) + k^2 B_2(a,\tau) + \cdots, \end{aligned} \tag{3.26}$$

where k is a small positive parameter and A_j, B_j are unknown functions. It is clear that, this solution is similar to the undamped solution if $k \to 0$ and $a \to a_0$, $\varphi \to \omega t$. Now differentiating Eq. (3.25) twice with respect to time t, utilizing the relations Eq. (3.26) and substituting $\ddot{y}$ and y into Eq. (3.23) and then equating the coefficients of $\sin\varphi$ and $\cos\varphi$, we obtain

$$A_1 = -\omega' a/(2\omega),\ B_1 = 0, \tag{3.27}$$

where a prime denotes differentiation with respect to τ. Now putting Eq. (3.25) into Eq. (3.21) and Eq. (3.27) into Eq. (3.26) we obtain the following equations

$$x = a e^{-kt} \cos\varphi, \tag{3.28}$$

and

$$\begin{aligned} \dot{a} &= -k\,\omega' a/(2\omega), \\ \dot{\varphi} &= \omega(\tau). \end{aligned} \tag{3.29}$$

Eq. (3.28) represents the first order analytical approximate solution of Eq (3.20) by the presented method. Usually, the integration of Eq. (3.29) is performed by well-known techniques of calculus [122,125], but sometimes they are solved by a numerical procedure [28,42,57,78,139,108]. Thus the determination of the first order analytical approximate solution of Eq. (3.20) is completed.

3.3 Example

As an example of the above procedure, we are going to consider the Duffing equation in the following form [139]

$$\ddot{x} + 2k(\tau)\dot{x} + e^{-\tau}x = -\varepsilon x^3, \tag{3.30}$$

where $f(x) = x^3$. Now using the transformation Eq. (3.21) into Eq. (3.30) and then simplifying them, we obtain

$$\ddot{y} + (e^{-\tau} - k^2)y = -\varepsilon e^{-2kt} y^3. \tag{3.31}$$

According to the homotopy perturbation [63,80-92] method, Eq. (3.31) can be rewritten as

$$\ddot{y} + \omega^2 y = \lambda y - \varepsilon e^{-2kt} y^3, \tag{3.32}$$

where

$$\omega^2 = e^{-\tau} - k^2 + \lambda. \tag{3.33}$$

According to the extended form of the KBM [45,94,113] method, the solution of Eq. (3.32) is given Eq. (3.25). The requirement of no secular terms in particular solution of Eq. (3.32) implies that the coefficient of the $\cos\omega t$ term is zero. Setting this term to zero, we obtain

$$\lambda a - \frac{3\varepsilon a^3 e^{-2kt}}{4} = 0, \tag{3.34}$$

For the nontrivial solution *i.e.*, $a \neq 0$, Eq. (3.34) leads to

$$\lambda = \frac{3\varepsilon a^2 e^{-2kt}}{4}. \tag{3.35}$$

Inserting the value of λ from Eq. (3.35) into Eq. (3.33), it yields

$$\omega^2 = e^{-\tau} - k^2 + \frac{3\varepsilon a^2 e^{-2kt}}{4}. \tag{3.36}$$

This is a time dependent frequency equation of the given nonlinear system. As $t \to 0$, Eq. (3.36) yields

$$\omega_0 = \omega(0) = \sqrt{1 - k^2 + \frac{3\varepsilon a_0^2}{4}}. \tag{3.37}$$

Integrating the first equation of Eq. (3.29), we get

$$a = a_0 \sqrt{\frac{\omega_0}{\omega}}, \tag{3.38}$$

where a_0 is a constant of integration which represents the initial amplitude of the nonlinear system. Now putting Eq. (3.38) into Eq. (3.36), we obtain the following equation

$$\omega^3 + q\omega + r = 0, \tag{3.39}$$

where

$$q = -(e^{-\tau} - k^2), \quad r = -\frac{3\varepsilon a_0^2 \omega_0 e^{-2kt}}{4}. \tag{3.40}$$

Eq. (3.39) is a cubic equation in ω. It has an analytical solution for every real value of $e^{-\tau}$. When $e^{-\tau} > k$, then the solution of Eq. (3.39) becomes (see also [6] for details)

$$\omega = \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{1/3} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{1/3} \tag{3.41}$$

Now substituting $r = -2R, q = -3Q$ into Eq. (3.41), then it becomes

$$\omega = \left(R + \sqrt{R^2 - Q^3}\right)^{1/3} + \left(R - \sqrt{R^2 - Q^3}\right)^{1/3}, \quad R > Q, \tag{3.42}$$

or,

$$\omega = \left(R + i\sqrt{Q^3 - R^2}\right)^{1/3} + \left(R - i\sqrt{Q^3 - R^2}\right)^{1/3}, \quad R < Q. \tag{3.43}$$

Herein the relations among Q, R, v, k, ω_0 and a_0 are obtained as

$$Q = \frac{(e^{-\tau} - k^2)}{3}, \quad R = \frac{3\varepsilon\omega_0 a_0^2 e^{-2kt}}{8}. \tag{3.44}$$

According to [76], the real form of Eq. (3.43) is given by

$$\omega = 2\sqrt{Q}\cos\left(\frac{\pi + \tan^{-1} V/R}{3}\right) = 2\sqrt{Q}\cos\left(\frac{\tan^{-1} V/R}{3}\right), \tag{3.45}$$

where

$$V = \sqrt{Q^3 - R^2}. \tag{3.46}$$

The solution of the second equation of Eq. (3.29) becomes

$$\varphi = \varphi_0 + \int_0^t \omega(t)dt, \tag{3.47}$$

where φ_0 is the initial phase and ω is given by Eqs. (3.42) or (3.45). Therefore, the first order analytical approximate solution of Eq (3.30) is obtained by Eq. (3.28) and the amplitude a and the phase φ are calculated from Eq. (3.38) and Eq. (3.47) respectively. Thus the determination of the first order analytical approximate solution of Eq. (3.30) is completed by the presented analytical technique.

3.4 Results and Discussions

In this chapter, an extended form of He's homotopy perturbation technique has been presented to obtain the analytical approximate solutions of second order strongly cubic nonlinear oscillators with small damping and slowly varying coefficients and the method has been successfully implemented to illustrate the effectiveness and convenience of the presented method. From our results, it is seen that the first order analytical approximate solutions (without any correction term) show a good agreement with the corresponding numerical solutions for the several damping effects. The analytical approximate solutions of Eq. (3.30) is computed by Eq. (3.28) with small damping effects and slowly varying coefficients and the corresponding numerical solutions are obtained by using fourth order ***Runge-Kutta*** method. This method can also be used to solve the second order strongly nonlinear system without damping (as $k \to 0$). The presented method is very simple in its principle, and is very easy to be applied to the nonlinear systems. The variational equations of the amplitude and phase variables appeared in a set of first order nonlinear ordinary differential equations. The integrations of these variational equations are obtained by well-known techniques of calculus [122,125]. In lack of analytical solutions, they are solved by numerical procedure [1-37,42,57,78,108,113, 125]. The amplitude and phase variables change slowly with time t. So, it requires the numerical calculation of a few number of points. On the contrary, a direct attempt to solve a strongly nonlinear differential equation dealing with some harmonic terms requires the numerical calculation of a great number of points. The behavior of amplitude and phase variables characterizes the oscillating processes. Moreover, the variational equations of amplitude and phase variables are used to investigate the stability of nonlinear differential equations. He's homotopy perturbation technique is able to handle nonlinear systems without damping and the KBM method is valid only for weakly nonlinear systems. The presented method has overcome some limitations of He's homotopy perturbation and the perturbation techniques. The advantage of this method is that the first order analytical approximate solutions show a good agreement with the corresponding numerical solutions. The method has been successfully implemented to solve for both strongly and weakly cubic nonlinear oscillators with small damping effects and slowly varying coefficients. Comparison is made between the solutions obtained by the presented analytical technique and those obtained by the numerical solutions graphically in figures.

To determine the numerical solutions of $\ddot{x}+2k(\tau)\dot{x}+e^{-\tau}x=-\varepsilon x^3$, the initial conditions $[x(0),\dot{x}(0)]$ are obtained as

$$x(0)=a_0\cos\varphi_0,$$
$$\dot{x}(0)=\left(\frac{k a_0(4+3\varepsilon a_0^2)}{8(3\omega_0^2+k^2-1)}-k a_0\right)\cos\varphi_0-a_0\,\omega_0\sin\varphi_0.$$

In general, the initial conditions $[x(0),\dot{x}(0)]$ are specified. Then one has to solve nonlinear algebraic equations in order to determine the initial amplitude a_0 and the initial phase φ_0 that appear in the solutions, from the initial conditions.

3.5 Conclusion

It may be concluded that the presented method is very efficient and powerful in finding the analytical approximate solutions for strongly nonlinear systems with small damping and slowly varying coefficients rather than the classical one.

Fig.3.1 (a)

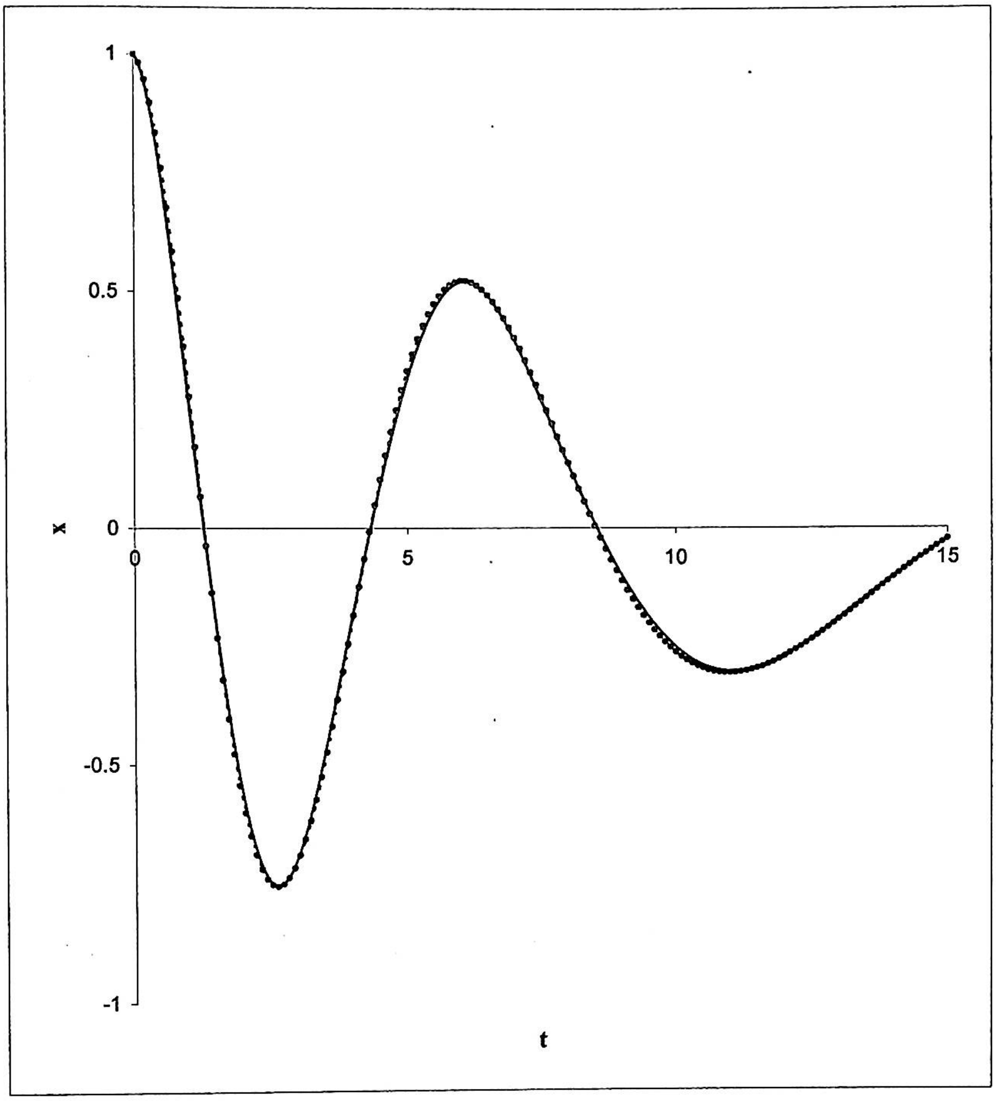

Fig.3.1 (a) First approximate solution of Eq. (3.30) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $[x(0)=1.0,\ \dot{x}(0)=-0.118879]$ or $a_0=1.0,\ \varphi_0=0$ with $k=0.15,\ \varepsilon=1.0$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Fig.3.1 (b)

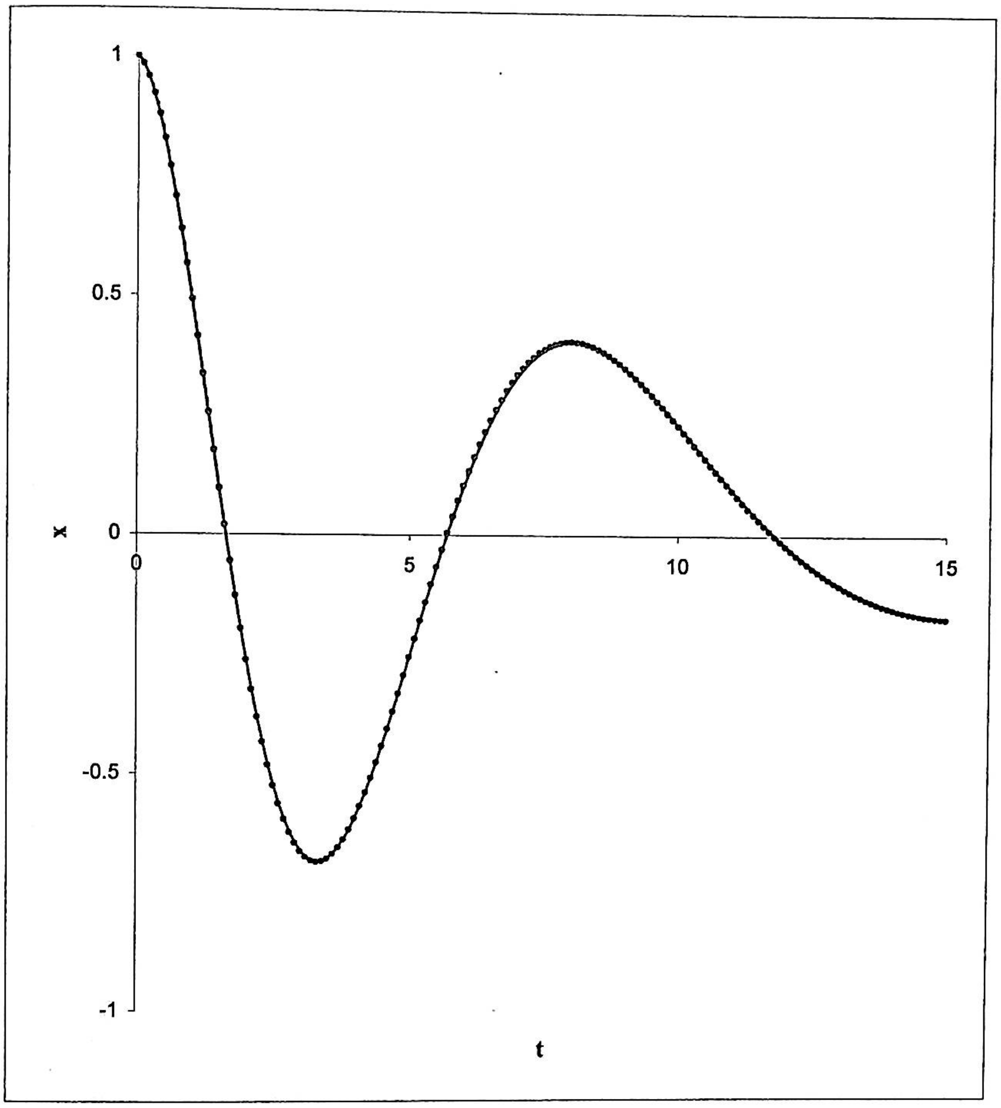

Fig.3.1 (b) First approximate solution of Eq. (3.30) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $[x(0)=1.0,\ \dot{x}(0)=-0.11302]$ or $a_0=1.0,\ \varphi_0=0$ with $k=0.15,\ \varepsilon=0.1$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Fig.3.2 (a)

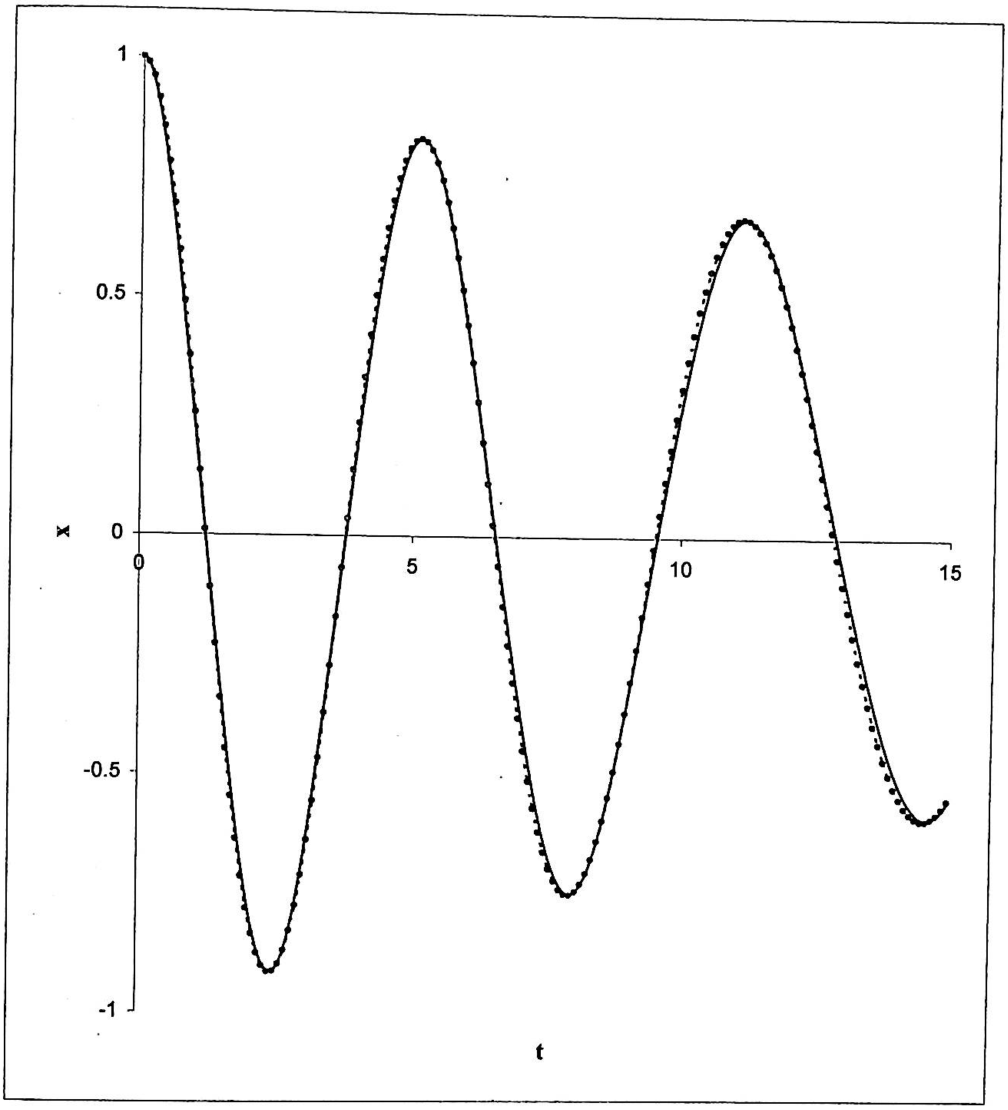

Fig.3.2 (a) First approximate solution of Eq. (3.30) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $[x(0)=1.0,\ \dot{x}(0)=-0.03969]$ or $a_0=1.0,\ \varphi_0=0$ with $k=0.05,\ \varepsilon=1.0$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Fig.3.2 (b)

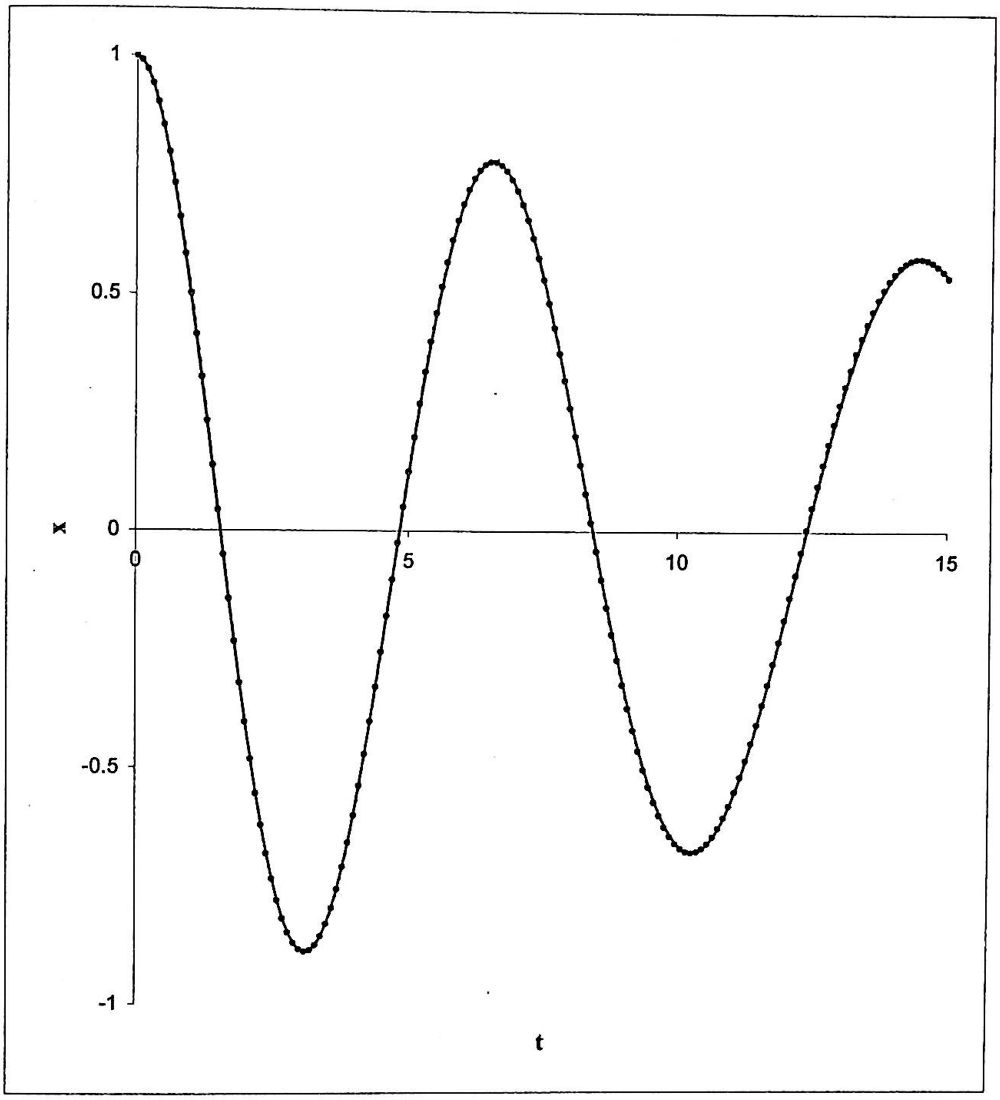

Fig.3.2 (b) First approximate solution of Eq. (3.30) is denoted by $-\bullet-$ (dashed lines) by the presented analytical technique with the initial conditions $[x(0)=1.0,\ \dot{x}(0)=-0.03789]$ or $a_0=1.0,\ \varphi_0=0$ with $k=0.05,\ \varepsilon=0.1$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Chapter 4

An Approximate Technique for Solving Strongly Nonlinear Biological Systems with Small Damping Effects

4.1 Introduction

Nonlinear oscillating processes in nature are of great importance. In the last several decades there has been increased interest in oscillating processes in physics, mechanics, circuit and control theory, biology and biochemistry. In particular, periodic events with constant repetitions have provided a foundation for the establishment of the very important concept of periodicity. Great achievements in science have to be attributed to the theory of periodic oscillations. In this connection, among many branches of science, astronomy has played a significant role. In reality a purely periodic process is an idealization which in many cases approximates an event closely enough, either from practical or philosophical point of view. However, there are also events whose study based on the assumption of no damping or small damping effects with weak nonlinearity may severely limit their closeness to reality. For the correct treatment of these events in presence of small damping and strong nonlinearity has to be incorporated.

The most common methods for constructing the analytical approximate solutions to the nonlinear oscillator equations are the perturbation methods. Some well known perturbation methods are the Krylov-Bogoliubov-Mitropolskii (KBM) [1-37,45,94,113] method, the Lindstedt-Poincare (LP) method [122-125] and the method of multiple time scales [125]. Almost all perturbation methods are based on an assumption that small parameter must exist in the equations. In general, the perturbation approximations are valid only for weakly nonlinear problems. Recently, He [91] has presented a new interpretation of homotopy perturbation method for strongly nonlinear differential systems. Belendez *et al.* [63] have presented the application of He's homotopy perturbation method to Duffing harmonic oscillator. Alam *et al.* [28] and Ludeke *et al.* [108] have obtained the solutions of strongly cubic nonlinear oscillators with large damping effects. Chatterjee [66] has also presented the solution of a strongly cubic nonlinear oscillator with damping effects by the harmonic balance based on averaging. Bojadziev [55] have presented the weakly nonlinear damped oscillating processes in biological and biochemical systems. Akhter *et al.* [30] have developed an asymptotic method for over-damped processes in biological and biochemical systems Azad *et al.* [32] have developed the KBM asymptotic method for over-damped processes in biological and biochemical systems. Recently Uddin *et al.* [145] have presented an approximate technique for solving strongly cubic nonlinear differential systems with small damping effects.

Four well-known biological and biochemical models are mentioned bellow:

(i) **A modified Lotka-Volterra model.** In absence of predation and logistic growth for prey one obtains the well known equations

$$\dot{N}_1 = N_1(\alpha_1 - \beta_1 N_2 - \gamma N_1),\ \ \dot{N}_2 = N_2(-\alpha_2 + \beta_2 N_1), \tag{4.1}$$

where N_1 and N_2 are two populations, $\alpha_1, \beta_1, \alpha_2, \beta_2$ and γ are positive constants and $\alpha_1 \beta_2 > \gamma \alpha_2$.

(ii) **A generalized Lotka-Volterra model.** Consider the following equations

$$\dot{N}_1 = N_1[-(N_1 - \alpha)(N_1 - \beta) - \gamma N_2],\ \dot{N}_2 = N_2(-c + N_1), \tag{4.2}$$

where N_1 and N_2 are two populations, α, β, γ and c are positive parameters which are supposed to satisfy the relations $\alpha < c < \beta$ and $(\alpha + \beta)/2 < c$. The model Eq. (4.2) describes a situation where the prey N_1 in absence of predators N_2 has an asymptotic carrying capacity β and a minimum density α, below which successful mating cannot overcome the death processes. The stability properties of the nontrivial equilibrium position of Eqs. (4.1) and (4.2) have been studied by Gatto and Rinaldi [74].

(iii) The FitzHugh equations. To investigate the physiological sates of nerve membranes, FitzHugh [72] introduced a theoretical model described by the following equations

$$\dot{x}_1 = \alpha + x_1 + x_2 - x_1^3/3, \;\; \dot{x}_2 = \rho(\gamma - x_1 - \beta x_2), \tag{4.3}$$

where it is assumed that $\alpha, \gamma \in (-\infty, \infty)$ and $\beta, \rho \in (0,1)$. If $\alpha = \beta = \gamma = 0$, then Eq. (4.3) becomes to a Van der Pol equation. Recently this model has been studied by Troy [144] and Hsu and Kazarinoff [93]. FitzHugh investigated the model qualitatively in the phase plane while Hsu and Kazarinoff [93] dealt with periodic solutions using Poincare-Hopf bifurcation theory.

(iv) Oscillating chemical reactions. Lefever and Nicolis [102] have considered a set of chemical reactions modeled by the following chemical kinetic equations

$$\dot{X} = A + X^2Y - BX - X, \;\; \dot{Y} = BX - X^2Y, \tag{4.4}$$

where X and Y are two concentrations, A and B are initial product concentrations. Lefever and Nicolis [102] have studied the phase portrait in the phase plane (X, Y) both analytically and numerically, and shown the existence of a limit cycle.

It has been shown [30,32,55,72,74,75,93,101,102,145,144,150] that all modeling Eqs (4.1)-(4.4) can be represented in the neighborhood of the equilibrium position by a second order nonlinear differential system of the following form

$$\ddot{x} + 2b\dot{x} + c^2x = \varepsilon f^{(1)}(x,\dot{x}) + \varepsilon^2 f^{(2)}(x,\dot{x}) + \cdots, \tag{4.5}$$

where c is a constant, $2b$ is the significant linear damping coefficients, ε is a small positive parameter, $f^{(1)}(x,\dot{x})$ and $f^{(2)}(x,\dot{x})$ are given functions with quadratic nonlinearity. In particular for $\varepsilon = 0$ in Eq. (4.5), one obtains the unperturbed equations $\ddot{x}+2b\dot{x}+c^2x=0$ with two complex eigenvalues $\lambda_{1,2}=-b\pm i\omega$, $\omega=\sqrt{c^2-b^2}$. Then the solution of the linearized equation of Eq. (4.5) is $x_0=a_0\exp(-bt)\cos(\omega t+\varphi_0)$, where a_0 and φ_0 are constants of integration which are also known as the initial amplitude and phase variables respectively. This solution describes the oscillating processes, decreasing for $b>0$ and increasing for $b<0$.

In our observation, based on various known models, it is noticed that the most of the authors [30,32,55,72,74,75,93,101,102,144,150] have studied the weakly nonlinear biological oscillating and non-oscillating systems. But numerous of biological oscillating systems encounter in presence of small damping with strong nonlinearity in nature. The aim of this chapter is to fill this gap. So we have presented an analytical technique by coupling the He's [80-92] homotopy perturbation technique and the extended KBM [1-38,45,94,113] method to solve second order strongly nonlinear oscillating processes in biological system with small damping effects.

4.2 The Method

We are interested to consider strongly nonlinear biological oscillating systems with small damping effects in the following form [78]

$$\ddot{x}+2k\dot{x}+\upsilon^2x=-\varepsilon f(x,\dot{x}), \quad k<<1, \tag{4.6}$$

where over dots denote differentiation with respect to time t, ν is a constant, $2k$ is the linear damping coefficients, ε is a positive parameter which is not necessarily small and $f(x,\dot{x})$ is a given nonlinear function which satisfies the following condition

$$f(-x,-\dot{x})=f(x,\dot{x}). \tag{4.7}$$

Now we are going to use the following transformation

$$x=y(t)e^{-kt}. \tag{4.8}$$

Differentiating Eq. (4.8) twice with respect to time t and substituting the values of $\ddot{x}$, $\dot{x}$ and x into Eq. (4.6) and then simplifying them, we obtain

$$\ddot{y}+(\upsilon^2-k^2)y=-\varepsilon e^{kt}f(y\,e^{-kt},(\dot{y}-k\,y)e^{-kt}). \tag{4.9}$$

According to the homotopy perturbation method [59-63,80-92], Eq. (4.9) can be written as

$$\ddot{y}+\omega^2 y=\lambda\,y-\varepsilon e^{kt}f(y e^{-kt},(\dot{y}-k y)e^{-kt}), \tag{4.10}$$

where

$$\omega^2=\upsilon^2-k^2+\lambda. \tag{4.11}$$

Herein ω is a constant for undamped nonlinear oscillating processes in biological systems and λ is an unknown constant which can be determined by eliminating the secular terms. However, for a damped nonlinear differential systems ω is a time dependent function and it varies slowly with time t. To handle this situation, we can use the extended KBM [45,94] method by Mitropolskii [113]. According to this technique, the first approximate solution of Eq. (4.10) can be assumed in the following form

$$y=a\cos\varphi, \tag{4.12}$$

where the amplitude a and the phase φ vary slowly with time t. In presence of damping, the amplitude a is a function of time t and approaches zero as $t\to\infty$. The amplitude a and the phase φ satisfy the following first order ordinary differential equations

$$\begin{aligned}\dot{a}&=k\,B_1(a,\tau)+k^2B_2(a,\tau)+\cdots,\\ \dot{\varphi}&=\omega(\tau)+k\,C_1(a,\tau)+k^2\,C_2(a,\tau)+\cdots,\end{aligned} \tag{4.13}$$

where k is a small positive parameter and $\tau=kt$ is the slowly varying time. It is noticed that this solution is similar to the undamped solution if $k\to 0$ and $a\to a_0$, $\varphi\to\omega t$. Differentiating Eq. (4.12) twice with respect to time t, utilizing the relations Eq. (4.13) and substituting $\ddot{y}$ and y into Eq. (4.10) and then equating the coefficients of $\sin\varphi$ and $\cos\varphi$, we obtain

$$B_1=-\omega' a/(2\omega),\ C_1=0, \tag{4.14}$$

where a prime denotes differentiation with respect to τ. Now putting Eq. (4.12) into Eq. (4.8) and Eq. (4.14) into Eq. (4.13), we obtain the following solutions of Eq. (4.6)

$$x=a\,e^{-kt}\cos\varphi, \tag{4.15}$$

$$\begin{aligned}\dot{a}&=-k\omega' a/(2\omega),\\ \dot{\varphi}&=\omega(\tau).\end{aligned} \tag{4.16}$$

Eq. (4.15) represents the first order analytical approximate solution of Eq (4.6) by the presented method with small significant damping effects. Usually, the integration of Eq. (4.16) is obtained by well-known techniques of calculus [122,125], but sometimes they are solved by a numerical procedure [1-37]. Thus the determination of the first order analytical approximate solution of Eq. (4.6) is completed by the presented method.

4.3 Example

Let us assume the strongly nonlinear oscillating processes in biological systems [78] with linear damping effects in the following form

$$\ddot{x} + 2\,k\,\dot{x} + x = -\varepsilon x^2, \tag{4.17}$$

where $f(x,\dot{x}) = x^2$ and $\nu = 1$. By using the transformation Eq. (4.8) into Eq. (4.17), we obtain

$$\ddot{y} + (1-k^2)y = -\varepsilon y^2 e^{-kt}, \tag{4.18}$$

According to the homotopy perturbation [59-63,80-92] method, Eq. (4.18) can be re-written as

$$\ddot{y} + \omega_a^2 y = \lambda y - \varepsilon y^2 e^{-kt}, \tag{4.19}$$

where

$$\omega_a^2 = 1 - k^2 + \lambda. \tag{4.20}$$

For $x > 0$, Eq. (4.19) can be written as

$$\ddot{y} + \omega_a^2 y = \lambda y - \varepsilon\,|\,y\,|\,y e^{-kt}, \tag{4.21}$$

Now according to the extended KBM [45.94,113] method, the solution of Eq. (4.21) is given by Eq. (4.12) and the amplitude a and the phase φ satisfy the following first order differential equations

$$\begin{aligned} \dot{a} &= -k\omega_a'\,a/(2\,\omega_a), \\ \dot{\varphi} &= \omega_a(\tau). \end{aligned} \tag{4.22}$$

Assume the following Fourier series expansion [78]

$$|\,a\cos\varphi\,|\,a\cos\varphi = c_1\cos\varphi + c_3\cos 3\varphi + \cdots. \tag{4.23}$$

Here,

$$c_1 = \frac{2}{\pi}\int_0^{\pi} |\,a\cos\varphi\,|\,a\cos^2\varphi\,d\varphi = \frac{4a^2}{\pi}\int_0^{\pi/2}\cos^3\varphi\,d\varphi = \frac{8a^2}{3\pi}. \tag{4.24}$$

For avoiding the secular terms from the right hand side of Eq. (4.21), we obtain

$$\lambda = \frac{8\varepsilon a e^{-2kt}}{3\pi}. \tag{4.25}$$

In this case, Eq. (4.20) yields

$$\omega_a^2 = 1 - k^2 + \frac{8\varepsilon a e^{-2kt}}{3\pi}. \tag{4.26}$$

From Eq. (4.26), it is clear that the frequency of nonlinear oscillating processes in biological systems depends on time t. As $t \to 0$, Eq. (4.26) leads to

$$\omega_{a,0} = \omega_a(0) = \sqrt{1 - k^2 + \frac{8\varepsilon a_0}{3\pi}}. \tag{4.27}$$

By integrating the first equation of Eq. (4.22), we obtain

$$a = a_0\sqrt{\omega_{a,0}/\omega_a}. \tag{4.28}$$

where a_0 is a constant of integration which represents the initial amplitude of the nonlinear systems for $x > 0$. Now substituting Eq. (4.28) into Eq. (4.26), we obtain

$$\omega_a^2 = 1 - k^2 + \frac{8\varepsilon a_0 e^{-2kt}}{3\pi}\sqrt{\omega_{a,0}/\omega_a}. \tag{4.29}$$

From Eq. (4.29), it is clear that it has no analytical solution. But for a definite value of t it can be easily solved by an iteration procedure. Now integrating the second equation of Eq. (4.22), it is found that

$$\varphi = \varphi_0 + \int_0^t \omega_a(t)dt, \tag{4.30}$$

where φ_0 is the initial phase and ω_a is given by Eq. (4.29). Herein, the integration of Eq. (4.30) is also carried on by the numerical procedure. Thus the first order analytical approximate solution of Eq. (4.17) for $x > 0$ is obtained by Eq. (4.15) and the amplitude a and the phase φ are calculated by Eqs. (4.28) and (4.30) respectively.

For $x < 0$, we replace x by $-x$, and then Eq. (4.17) becomes

$$\ddot{x} + 2k\dot{x} + x = -\varepsilon x^2. \tag{4.31}$$

Utilizing the transformation Eq. (4.8) into Eq. (4.31) and then simplifying them, we obtain

$$\ddot{y} + (1 - k^2)y = \varepsilon y^2 e^{-kt}. \tag{4.32}$$

According to the homotopy perturbation [59-63,80-92] method Eq. (4.32) yields

$$\ddot{y} + \omega_b^2 y = \lambda y + \varepsilon y^2 e^{-kt}, \tag{4.33}$$

where

$$\omega_b^2 = 1 - k^2 + \lambda. \tag{4.34}$$

According to the above method, we can easily determine the following solutions of Eq. (4.17) for $x < 0$ by replacing a by b, ω_a by ω_b and ε by $-\varepsilon$ in the following forms

$$x = b e^{-kt} \cos\varphi, \tag{4.35}$$

$$b = b_0 \sqrt{\frac{\omega_{b,0}}{\omega_b}}, \tag{4.36}$$

$$\omega_b^2 = 1 - k^2 - \frac{8\varepsilon b_0 e^{-2kt}}{3\pi} \sqrt{\omega_{b,0}/\omega_b}, \tag{4.37}$$

$$\omega_{b,0} = \omega_b(0) = \sqrt{1 - k^2 - \frac{8\varepsilon b_0}{3\pi}}. \tag{4.38}$$

$$\varphi = \varphi_0 + \int_0^t \omega_b(t)dt. \tag{4.39}$$

To study the strongly nonlinear oscillating processes in biological systems, we have to change the parity. This presence has a consequence a shift of the amplitude. Now we are interested to determine the relation between the amplitudes a and b which are varying slowly with time t. Now Eq. (4.18) can be written as

$$\dot{y} d\dot{y} + \{(1 - k^2) y + \varepsilon y^2 e^{-kt}\} dy = 0. \tag{4.40}$$

By integrating Eq. (4.40), it yields

$$\frac{\dot{y}^2}{2} + \frac{(1 - k^2) y^2}{2} + \frac{\varepsilon e^{-kt} y^3}{3} = h, \tag{4.41}$$

where h is a constant of integration. Let us assume that the systems oscillate [78] between asymmetric limits $[a, -b]$, $b > 0$. Then from Eq. (4.41) we obtain

$$\frac{(1 - k^2)(a^2 - b^2)}{2} + \frac{\varepsilon e^{-kt}(a^3 + b^3)}{3} = 0. \tag{4.42}$$

By solving Eq. (4.42) for b, we obtain

$$b = -a, \tag{4.43}$$

$$b_{1,2} = \frac{e^{kt}}{4\varepsilon}\left[3(1-k^2) + 2\varepsilon a e^{-kt} \pm 3\sqrt{(1-k^2)^2 - \frac{4}{3}\varepsilon a e^{-kt}(1-k^2+\varepsilon a e^{-kt})}\right]. \quad (4.44)$$

Since it is assumed that $b > 0$ and $b \to 0$ for $a \to 0,\ k \to 0$ [78], then we get the relation between the amplitudes a and b as the form

$$b = \frac{e^{kt}}{4\varepsilon}\left[3(1-k^2) + 2\varepsilon a e^{-kt} - 3\sqrt{(1-k^2)^2 - \frac{4}{3}\varepsilon a e^{-kt}(1-k^2+\varepsilon a e^{-kt})}\right], \quad (4.45)$$

where

$$\frac{4}{3}\varepsilon a e^{-kt}(1-k^2+\varepsilon a e^{-kt}) < (1-k^2)^2. \quad (4.46)$$

When $k \to 0$, Eq. (45) agrees to Hu's [78] results.

4.4 Results and Discussions

In this chapter, He's [80-92] homotopy perturbation method has been extended to obtain the approximate solutions for second order strongly nonlinear oscillating processes in biological systems with small damping effects and the method has been successfully implemented to illustrate the effectiveness and convenience of the presented method. The complete analytical approximate solutions of Eq. (4.17) are computed by Eqs. (4.15) and (4.35) according to the cases respectively and the corresponding numerical solutions are obtained by fourth order ***Runge-Kutta*** method.

We have also compared the results obtained by the presented method to the results obtained by the perturbation method for strongly nonlinear oscillating processes in biological systems with damping effects and it is presented graphically. **Figs. 1-2** are provided to compare the results obtained by the presented method to the corresponding numerical solutions. The solution of Eq. (4.17) agrees with Hu's [78] solution when $k \to 0$. The analytical approximate solutions deviate from the numerical solutions as t increases, while the second order classical perturbation solutions approach toward the numerical solutions for that time. On the contrary, the perturbation solutions quickly deviate from the numerical solutions when t increases from 1 to 2 or 3. Thus the matched solutions of Eq. (4.17) (by the presented method) and the second order perturbation solutions of that equation can be successfully used in this situation (see **Fig. 2**) [139]. The presented method gives much better approximations for strongly nonlinear oscillating processes in biological systems with small damping effects than the perturbation solutions.

In summary, He's homotopy perturbation method [59-63,80-92] is able to handle the nonlinear systems without damping and the KBM method [1-38,45,94,113] is also able to handle nonlinear systems with small nonlinearity. Hu [78] has also obtained the solution for strongly nonlinear oscillating processes in biological systems without damping effects by the harmonic balance method. But most of the physical and biological oscillating systems encounter in presence of damping in nature. It is observed that there is no suitable method [30,32,55,72,74,75,93,101,102,145,144,150] for solving strongly nonlinear oscillating processes in biological systems with small damping effects. The proposed coupling method has eliminated these limitations and plays an important role in the solutions. Furthermore, the presented method is as simple as the straightforward expansion and the first order approximations show good accuracy (see also **Figs. 1-2**) with the corresponding numerical solutions. The initial approximations can be freely chosen, which is identified via various methods [1-37,59-63,78,80-92]. The approximations obtained by the presented method are valid not only for strong nonlinearity, but also for weak one with small damping effects. Therefore, the presented method is suitable for strongly nonlinear oscillating processes in biological systems with small damping effects where the He's homotopy perturbation and the KBM methods fail to give the desired results.

4.5 Conclusion

The determination of amplitude and phase variables is a crucial question in strongly nonlinear oscillating processes in biological systems. In this chapter, we have presented an analytical technique based on He's homotopy perturbation technique and the extended form of the KBM method to tackle the strongly nonlinear oscillating processes in biological systems with small damping effects. It is also noted that some limitations of He's homotopy perturbation technique and the KBM method have been overcome by the presented method.

Appendix 4.A

Perturbation solution by extended Struble's technique [8]

In this appendix, we have extended the perturbation method to obtain the second order approximate solutions of second order strongly nonlinear oscillating processes in biological systems with damping of a point of view of a general Struble's technique [28].

Let us assume that the unperturbed ($\varepsilon = 0$) equation of Eq. (4.17) has two eigenvalues, say $\lambda_{1,2} = -k \pm i\omega$, $\omega^2 = 1 - k^2$, $k < 1$ and then the solution of Eq. (4.17) becomes

$$x(t,0) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}, \tag{4.A.1}$$

where a_1 and a_2 are arbitrary constants. If $\varepsilon \neq 0$, then the analytical approximate solution of Eq. (4.17) has been chosen in the following form

$$x(t,\varepsilon) = a_1(t) e^{\lambda_1 t} + a_2(t) e^{\lambda_2 t} + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots. \tag{4.A.2}$$

In this situation, Eq. (4.17) can be written as [28]

$$\begin{aligned} &(D-\lambda_1)(\dot{a}_1 e^{\lambda_1 t}) + (D-\lambda_2)(\dot{a}_2 e^{\lambda_2 t}) + (D-\lambda_1)(D-\lambda_2)(\varepsilon u_1 + \varepsilon^2 u_2 + \cdots) \\ &= -\varepsilon(a_1^2 e^{2\lambda_1 t} + a_2^2 e^{2\lambda_2 t} + 2a_1 a_2 e^{(\lambda_1+\lambda_2)t} + 2\varepsilon u_1(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}) + \cdots), \end{aligned} \tag{4.A.3}$$

where $D = \dfrac{d}{dt}$.

According to the separation rules [details can be found in [28]), we obtain

$$\begin{aligned} &(D-\lambda_1)(\dot{a}_1 e^{\lambda_1 t}) = 0, \\ &(D-\lambda_2)(\dot{a}_2 e^{\lambda_2 t}) = 0, \end{aligned} \tag{4.A.4}$$

$$(D-\lambda_1)(D-\lambda_2)u_1 = -(a_1^2 e^{2\lambda_1 t} + a_2^2 e^{2\lambda_2 t} + 2a_1 a_2 e^{(\lambda_1+\lambda_2)t}). \tag{4.A.5}$$

In order to determine the first order approximate solution, it can be assumed that a_1 and a_2 are constants; so that (4.A.4) has no non trivial solution *i.e.* $\dot{a}_1 = 0$, $\dot{a}_2 = 0$ and the particular solution of (4.A.5) is then

$$u_1 = C_1 a_1^2 e^{2\lambda_1 t} + C_1^* a_2^2 e^{2\lambda_2 t} + C_2 a_1 a_2 e^{(\lambda_1+\lambda_2)t}, \tag{4.A.6}$$

where

$$C_1 = -\frac{1}{\lambda_1(2\lambda_1 - \lambda_2)}, \quad C_1^* = -\frac{1}{\lambda_2(2\lambda_2 - \lambda_1)}, \quad C_2 = -\frac{2}{\lambda_1\lambda_2}. \tag{4.A.7}$$

Now putting the value of u_1 from Eq. (4.A.6) into Eq. (4.A.3) and then simplifying them, we obtain

$$(D-\lambda_1)(\dot{a}_1 e^{\lambda_1 t}) + (D-\lambda_2)(\dot{a}_2 e^{\lambda_2 t}) + \varepsilon\lambda_1(2\lambda_1-\lambda_2)C_1 a_1^2 e^{2\lambda_1 t}$$
$$+\varepsilon\lambda_2(2\lambda_2-\lambda_1)C_1^* a_2^2 e^{2\lambda_2 t} + \varepsilon\lambda_1\lambda_2 C_2 a_1 a_2 e^{(\lambda_1+\lambda_2)t} + (D-\lambda_1)(D-\lambda_2)(\varepsilon^2 u_2 + \cdots)$$
$$= -\varepsilon(a_1^2 e^{2\lambda_1 t} + a_2^2 e^{2\lambda_2 t} + 2a_1 a_2 e^{(\lambda_1+\lambda_2)t}) - 2\varepsilon^2(C_1 a_1^3 e^{3\lambda_1 t} + C_1^* a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} \tag{4.A.8}$$
$$+ C_2 a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + C_1 a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + C_1^* a_2^3 e^{3\lambda_2 t} + C_2 a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + \cdots).$$

All the terms with ε of the left side and the terms $a_1^2 e^{2\lambda_1 t}, a_2^2 e^{2\lambda_2 t}, 2a_1 a_2 e^{(\lambda_1+\lambda_2)t}$ of the right sides of Eq. (4.A.8) are cancelled, since C_1, C_1^*, C_2 satisfy Eq. (4.A.7). Thus we obtain

$$(D-\lambda_1)(\dot{a}_1 e^{\lambda_1 t}) = -2\varepsilon^2(C_1 + C_2)a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t},$$
$$(D-\lambda_2)(\dot{a}_2 e^{\lambda_2 t}) = -2\varepsilon^2(C_1^* + C_2)a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t}, \tag{4.A.9}$$

and

$$(D-\lambda_1)(D-\lambda_2)u_2 = -2(C_1 a_1^3 e^{3\lambda_1 t} + C_1^* a_1^2 e^{3\lambda_2 t}). \tag{4.A.10}$$

Now the particular solutions of Eqs. (4.A.9)- (4.A.10) are given by

$$\dot{a}_1 = \varepsilon^2 l a_1^2 a_2 e^{(\lambda_1+\lambda_2)t},$$
$$\dot{a}_2 = \varepsilon^2 l^* a_1 a_2^2 e^{(\lambda_1+\lambda_2)t}, \tag{4.A.11}$$

where

$$l = -(C_1 + C_2)/\lambda_1, \qquad l^* = -(C_1^* + C_2)/\lambda_2, \tag{4.A.12}$$

and

$$u_2 = E_1 a_1^3 e^{3\lambda_1 t} + E_1^* a_1^3 e^{3\lambda_2 t}, \tag{4.A.13}$$

where

$$E_1 = -\frac{C_1}{\lambda_1(3\lambda_1-\lambda_2)}, \; E_1^* = -\frac{C_1^*}{\lambda_2(3\lambda_2-\lambda_1)}. \tag{4.A.14}$$

By using the following transformation equations

$$\dot{a}_1 = b e^{i\varphi}/2, \quad \dot{a}_2 = b e^{-i\varphi}/2 \tag{4.A.15}$$

into Eq. (4.A.11) and then equating the real and imaginary parts, we obtain the following variational equations of amplitude and phase variables

$$\dot{b} = \varepsilon^2 m_1 b^3 e^{-2kt},$$
$$\dot{\varphi} = \varepsilon^2 n_1 b^2 e^{-2kt}, \tag{4.A.16}$$

where

$$m_1 = -\frac{k(3k^2+11\omega^2)}{4(k^2+\omega^2)(k^2+9\omega^2)}, \; n_1 = -\frac{\omega(7k^2+15\omega^2)}{4(k^2+\omega^2)(k^2+9\omega^2)}. \tag{4.A.17}$$

Now integrating Eq. (4.A.16), it yields

$$b = b_0\sqrt{\left(\frac{kb_0^2}{k + \varepsilon^2 m_1 b_0^2 (e^{-2kt} - 1)}\right)},$$
$$\varphi = \omega t - \frac{n_1}{2m_1}\ln\left(\frac{k + \varepsilon^2 m_1 b_0^2 (e^{-2kt} - 1)}{k}\right). \quad (4.A.18)$$

Also the correction terms are reduced in the following forms

$$u_1 = b^2 e^{-2kt}(P_1 \cos 2\psi + Q_1 \sin 2\psi + P_2), \ \psi = \omega t + \varphi,$$
$$u_2 = b^3 e^{-3kt}(P_3 \cos 3\psi + Q_3 \sin 3\psi), \quad (4.A.19)$$

where

$$P_1 = \frac{(3\omega^2 - k^2)}{2(k^2 + \omega^2)(k^2 + 9\omega^2)}, \quad Q_1 = \frac{4k\omega}{2(k^2 + \omega^2)(k^2 + 9\omega^2)}, \quad P_2 = -\frac{1}{2(k^2 + \omega^2)},$$
$$P_3 = \frac{(\omega^2 - k^2)(k^2 - 6\omega^2)}{8(k^2 + \omega^2)(k^2 + 4\omega^2)(k^2 + 9\omega^2)}, \ Q_1 = -\frac{k\omega(k^2 - 17\omega^2)}{8(k^2 + \omega^2)(k^2 + 4\omega^2)(k^2 + 9\omega^2)}. \quad (4.A.20)$$

Thus the second order approximate solutions of Eq. (4.17) by the general Struble's technique [28] is given by

$$x = be^{-kt}\cos(\omega t + \varphi) + \varepsilon u_1 + \varepsilon^2 u_2, \quad (4.A.21)$$

where b, φ are calculated from Eq. (4.A.18) and u_1, u_2 are given by Eq. (4.A.19) respectively.

Further, we assume that a_0 and b_0 are the initial amplitudes for the solutions by the presented method and the classical perturbation method respectively. To establish the relation between these amplitudes, we consider that

$$x_{per}(0) = x_{pre}(0), \quad (4.A.22)$$

where x_{pre} and x_{per} represent the solutions by the presented method and the perturbation method respectively. Then Eq. (4.A.22) leads to

$$b_0 + (P_1 + P_2)b_0^2 + P_3 b_0^3 = a_0. \quad (4.A.23)$$

To solve Eq. (A.23), let us consider

$$b_0 = a_0 + h_1 a_0^2 + h_1 a_0^3, \quad (A.24)$$

where h_1 and h_2 are unknown coefficients. Now putting Eq. (4.A.24) into Eq. (4.A.23) and then simplifying them, we obtain

$$h_1 = -(P_1 + P_2), \ h_2 = 2(P_1 + P_2)^2 - P_3. \quad (4.A.25)$$

Thus we obtain the following relation between the amplitudes a_0 and b_0 as

$$b_0 = a_0 - (P_1 + P_2)a_0^2 + (2(P_1 + P_2)^2 - P_3)a_0^3. \quad (4.A.26)$$

Fig.4.1

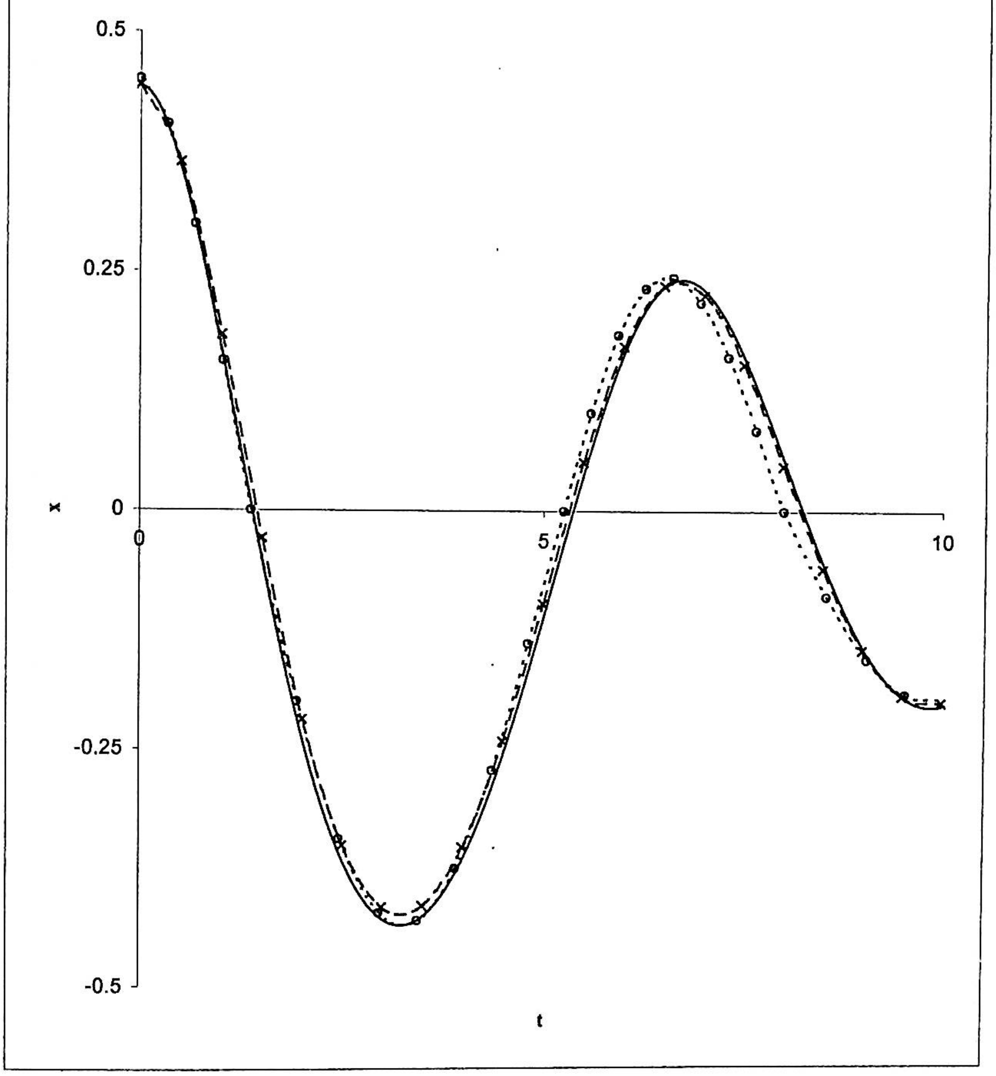

Fig. 4.1 First approximate solution is denoted by $\cdots\bullet\cdots$ of Eq. (4.17) by the presented method with the initial conditions $[x(0)=0.45,\ \dot{x}(0)=-0.03874]$ or $a_0=0.45,\ \ \varphi_0=0$ and the second approximate solution is denoted by $-\times-$ of Eq. (2.54) by the perturbation method with the initial conditions $[x(0)=0.44322,\ \dot{x}(0)=-0.02459]$ or $a_0=0.53615,\ \varphi_0=0$ with $k=0.1,\ \varepsilon=1.0$ and $f=x^2$. Corresponding numerical solution is denoted by – (solid line).

Fig. 4.2

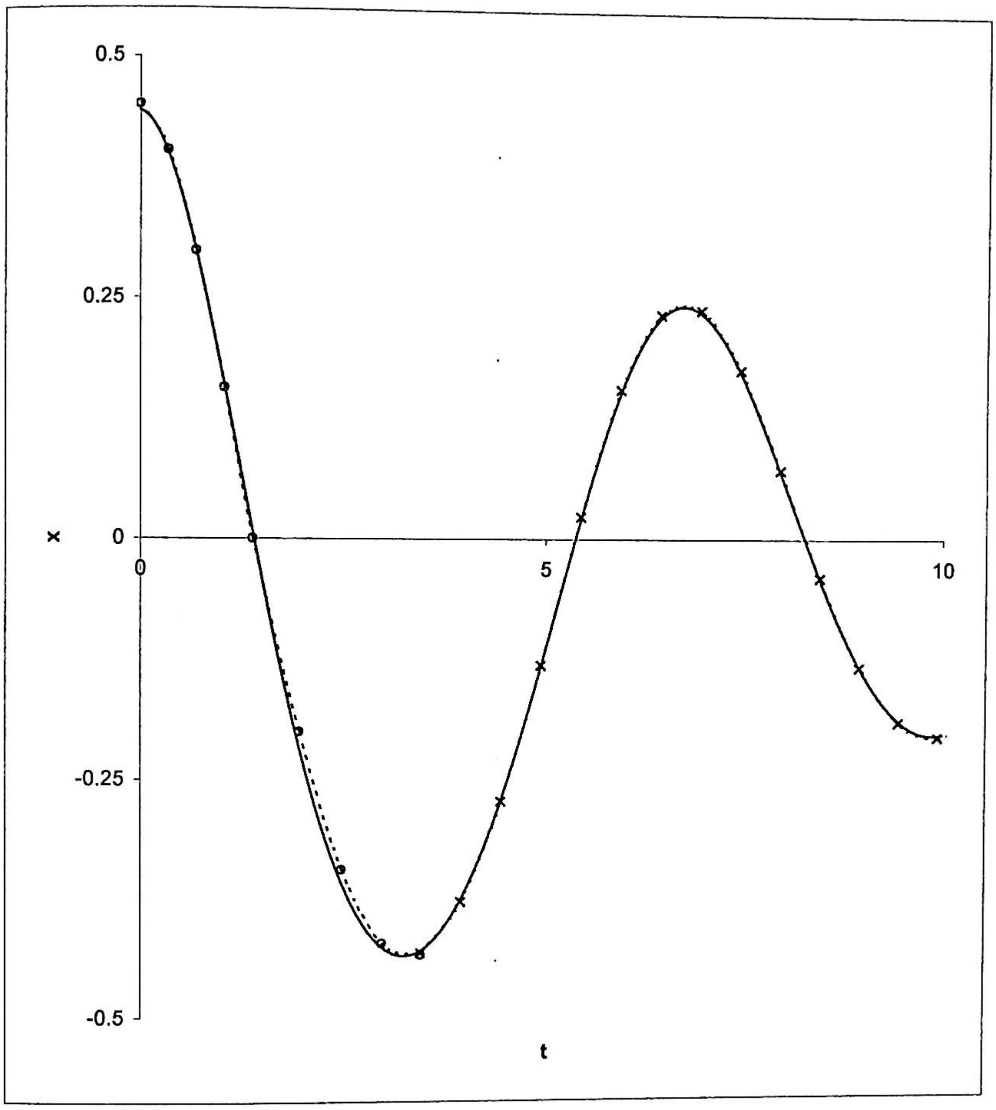

Fig. 4.2 First approximate solution is denoted by $\cdots\bullet\cdots$ of Eq. (4.17) by the present method with the initial conditions $[x(0)=0.45,\ \dot{x}(0)=-0.03874]$ or $a_0=0.45$, $\varphi_0=0$ from $t=0$ to $t=3.44339$ and the second approximate solution is denoted by $\cdots\times\cdots$ of Eq. (4.17) by the perturbation method with the initial conditions $[x(0)=0.44322,\ \dot{x}(0)=-0.02459]$ or $a_0=0.55337$, $\varphi_0=-0.601258$ from $t=3.44339$ to $t=10.0$ with $k=0.1$, $\varepsilon=1.0$ and $f=x^2$. Corresponding numerical solution is denoted by – (solid line).

Chapter 5

Second Approximate Solution of Duffing Equation with Strong Nonlinearity by Homotopy Perturbation Method

5.1 Introduction

The study of nonlinear differential system is of great interest in engineering and physical sciences and many other branches of applied mathematics. The solutions of nonlinear problems are very complicated and, in general, it is more difficult to get an analytical approximation than a numerical one to a given nonlinear problem. There exists a wide body of literature dealing with the problem of approximate solutions to nonlinear differential equations with various different methodologies. Many different approaches have been proposed, such as Struble's techniques [28,140], Kryloff-Bogoliuboff-Mitropolskii (KBM) [45,94,113] method, multiple time-scales [124] procedure, the modified Lindstedt-Poincare method [85], He's homotopy perturbation method [63,80-92], etc. Most of these methods have been originally formulated to get the periodic solution of second order nonlinear differential systems for weak or strong nonlinearity without considering any damping effects in the following form

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \;\; \varepsilon << 1, \varepsilon > 0. \tag{5.1}$$

Several authors have extended these methods to investigate similar nonlinear problems with a strong linear damping effect $-2k\dot{x}$, $k = O(1)$ and $k > 0$ modeled by the following equation

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon f(x, \dot{x}), \;\; \varepsilon << 1, \varepsilon > 0. \tag{5.2}$$

Popov [135] was well known among them. He extended the KBM method and investigated the under-damped case of Eq. (5.2).Then Mendelson [118] reproduced Popov's results. Bojadziev [57] investigated a third order nonlinear problem with internal friction and relaxation based on the KBM technique. Following Popov [135], Murty *et al.* [115] investigated the over damped case of Eq. (5.2).They used Popov's formula by replacing the trigonometric functions with the corresponding hyperbolic functions. In their investigation, they also examined a fourth order over-damped system. Murty [117] presented a unified method for solving Eq. (5.2). Such a unified solution is a general one and covers the three cases viz. under-damped, undamped and over-damped situations. It is seen that the unified solution represents the original KBM solution [45,94] as the limit $k \to 0$. Alam [16] has generalized Murty's [117] technique for solving an *nth*, $n = 2,3....$ order nonlinear differential equation. Recently, Alam *et al.* [28] have presented a generalized Struble's technique for solving an *nth* order weakly nonlinear differential system with damping effects. Thus we observe that a considerable amount of research activities have been carried out by several authors [1-37,45,57,94,113115,117,140] for the solution of the damped or undamped nonlinear systems with small nonlinearity. Therefore, the small parameter plays a very important role in the perturbation methods. It determines not only the accuracy of the perturbation approximations, but also the validity of the perturbation technique itself. In Ref. [85,87], He has presented Modified Lindstedt-Poincare method for some strongly non-linear oscillations and the homotopy perturbation method for some strongly nonlinear oscillations without damping effects. But in science and engineering, there exist many nonlinear problems in presence of damping effects which do not contain any small parameter, especially those with strong nonlinearity. Thus it is necessary to develop and improve some nonlinear analytical techniques which are independent of small parameters. The main goal of this chapter is to find the second order approximate solutions for general nonlinear systems with strong nonlinearity in presence of damping effects. The method has been illustrated by applying it to a typical nonlinear problem of practical importance in this chapter. To get our desired result, we have re-written Eq. (5.2) in the following form:

$$\ddot{x} + 2k\dot{x} + (\omega^2 + \varepsilon_1)x = \varepsilon(\frac{\varepsilon_1}{\varepsilon}x - f(x,\dot{x})), \quad \varepsilon > 0, \tag{5.3}$$

where ε is a positive parameter which measures the strength of nonlinearity of the system, ε_1 is an artificial constant, $0 \le \varepsilon_1 \le 1$, the significant damping term is expressed by the linear term $2k\dot{x}$. The damping coefficient $2k$ which is of the order of unity and also the nonlinear frequency ω of the system are constants. The assumption $\omega^2 > k^2$ or $\omega^2 < k^2$ guarantees the oscillating or non-oscillating character of the systems. In most of the nonlinear dynamical systems, the quantity ε is small compared with ω^2 and its solutions may be shown to converge with the numerical results.

5.2 The Method

In this chapter, we are going to consider a general second order nonlinear ordinary differential equation in following form

$$\ddot{x} + c_1\dot{x} + (c_2 + \varepsilon_1)x = \varepsilon(\frac{\varepsilon_1}{\varepsilon}x + f(x,\dot{x})), \tag{5.4}$$

where over dots denote derivatives with respect to time t, ε is a positive parameter which plays an important role to the nonlinear systems, ε_1 is an artificial constant, $0 \le \varepsilon_1 \le 1$, the coefficients c_j, $j = 1,2$ are constants and f is a given nonlinear function.

When $\varepsilon \to 0$, then the corresponding linear equation of Eq. (5.4) has two eigen values, say λ_j, $j = 1,2$. Hence the general solution of the unperturbed equation of Eq. (5.4) leads to

$$x(t,0) = \sum_{j=1}^{2} a_j e^{\lambda_j t}, \tag{5.5}$$

where a_j, $j = 1,2$ are arbitrary constants. For $\varepsilon \neq 0$, we are seeking an approximate solution of Eq. (4) in the following form

$$x(t,\varepsilon) = \sum_{j=1}^{2} a_j e^{\lambda_j t} + \varepsilon u_1(t,\varepsilon) + \varepsilon^2 u_2(t,\varepsilon) + \cdots \tag{5.6}$$

According to both Struble's [28,140] technique and KBM [45,94,113] method, the solution Eq. (5.6) is differentiated twice with respect to time t, to obtain the derivatives of x, *i.e.*, $\dot{x}$ and $\ddot{x}$. Then inserting the values of $\dot{x}$, $\ddot{x}$ together with x into Eq. (5.4) and after simplifying one obtains a needful formula. Clearly, this is a very difficult and tedious task. On the basis of mathematical induction, Alam [16] has presented such a general formula in terms of the variables $a_j(t)$, $j = 1,2,\cdots n$, for determining the KBM type solution. Further, Alam [24] has investigated a simple technique to derive the noted general formula. In this chapter, we are going to present the generalized formula for the second order nonlinear differential systems with strong nonlinearity in presence of significant damping effects. This formula is used arbitrarily for the different damping effects. Now the Eq. (5.4) can be re-written as

$$\prod_{j=1}^{2}(D-\lambda_j)x = \varepsilon(\frac{\varepsilon_1}{\varepsilon}x + f), \quad D \equiv \frac{d}{dt}. \tag{5.7}$$

By substituting Eq. (5.6) into Eq. (5.7), we obtain

$$\prod_{j=1}^{2}(D-\lambda_j)(\sum_{j=1}^{2} a_j e^{\lambda_j t} + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots) = \varepsilon(\frac{\varepsilon_1}{\varepsilon}x + f),$$

or

$$\sum_{j=1}^{2}(\prod_{k=1,k\neq j}^{2}(D-\lambda_k))((D-\lambda_j)a_j e^{\lambda_j t}) + \prod_{j=1}^{2}(D-\lambda_j)(\varepsilon u_1 + \varepsilon^2 u_2 + \cdots) = \varepsilon(\frac{\varepsilon_1}{\varepsilon}x + f)$$

or

$$\sum_{j=1}^{2}(\prod_{k=1,k\neq j}^{2}(D-\lambda_k)(\dot{a}_j e^{\lambda_j t}) + \prod_{j=1}^{2}(D-\lambda_j)(\varepsilon u_1 + \varepsilon^2 u_2 + \cdots) = \varepsilon(\frac{\varepsilon_1}{\varepsilon}x + f), \tag{5.8}$$

since $(D-\lambda_j)(a_j e^{\lambda_j t}) = \dot{a}_j e^{\lambda_j t}$.

5.3 Example

As an example of the above procedure, let us consider the following autonomous nonlinear differential equation [28]

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon x^3 . \tag{5.9}$$

The Eq. (5.9) occurs in the theory of nonlinear vibrating systems and in a certain type of nonlinear electrical circuit theory. We have re-written Eq. (5.9) in the following form

$$\ddot{x} + 2k\dot{x} + (\omega^2 + \varepsilon_1)x = \varepsilon(\frac{\varepsilon_1}{\varepsilon}x - x^3). \tag{5.10}$$

In particular, when $\varepsilon \to 0$ from Eq. (5.10) one obtains the unperturbed equation $\ddot{x} + 2k\dot{x} + (\omega^2 + \varepsilon_1)x = 0$ with two eigen-values, say $\lambda_{1,2} = -k \pm i\omega_0$, where $\omega_0 = \sqrt{(\omega^2 + \varepsilon_1 - k^2)}$ or $\lambda_{1,2} = -k \pm \omega_0$, where $\omega_0 = \sqrt{k^2 - (\omega^2 + \varepsilon_1)}$. Here ω_0 is known as the reduced frequency of the systems and the physical character of the motion depend on the nature of it. Thus depending on the values of k and $(\omega^2+\varepsilon_1)$, the solution becomes under-damped, over-damped or critically damped. However, we are able to find a general solution in terms of the variables a_1 and a_2 as well as of the eigen-values λ_1 and λ_2. Then by putting the values of λ_1 and λ_2, the desired solution can be found for all real or complex values of λ_1 and λ_2. Then the solution of the linearized equation of Eq. (5.10) is obtained by

$$x(t,0)=a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}, \tag{5.11}$$

where a_1 and a_2 are arbitrary constants. When $\varepsilon\neq 0$, we seek a general solution of Eq. (5.10) in the following form

$$x(t,\varepsilon)=a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+\varepsilon u_1(a_1,a_2,t)+\varepsilon^2u_2(a_1,a_2,t)+\varepsilon^3\cdots. \tag{5.12}$$

Here

$$\begin{aligned} f(x)=\varepsilon_1x-\varepsilon x^3&=\varepsilon_1(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+\varepsilon u_1)-\varepsilon(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+\varepsilon u_1)^3\\ &=\varepsilon_1(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+\varepsilon u_1)-\varepsilon(a_1^3e^{3\lambda_1 t}+3a_1^2a_2e^{(2\lambda_1+\lambda_2)t}\\ &\quad+3a_1a_2^2e^{(\lambda_1+2\lambda_2)t}+a_2^3e^{3\lambda_2 t}+3\varepsilon u_1(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t})^2+\cdots). \end{aligned} \tag{5.13}$$

Eq. (5.10) can be re-written as

$$\begin{aligned} &(D-\lambda_2)(\dot a_1e^{\lambda_1 t})+(D-\lambda_1)(\dot a_2e^{\lambda_2 t})+(D-\lambda_1)(D-\lambda_2)(\varepsilon u_1+\varepsilon^2u_2+\cdots)\\ &=\varepsilon_1(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+\varepsilon u_1)-\varepsilon(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+\varepsilon u_1)^3\\ &=\varepsilon_1(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+\varepsilon u_1)-\varepsilon(a_1^3e^{3\lambda_1 t}+3a_1^2a_2e^{(2\lambda_1+\lambda_2)t}\\ &\quad+3a_1a_2^2e^{(\lambda_1+2\lambda_2)t}+a_2^3e^{3\lambda_2 t}+3\varepsilon u_1(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t})^2+\cdots). \end{aligned} \tag{5.14}$$

Now we are going to consider the terms up to $O(\varepsilon)$. According to the separation rules (details can be found in [28]), we can equate the various terms of Eq. (5.14) and we get the following equations

$$(D-\lambda_2)(\dot a_1e^{\lambda_1 t})=\varepsilon_1a_1e^{\lambda_1 t}-3\varepsilon a_1^2a_2e^{(2\lambda_1+\lambda_2)t}, \tag{5.15}$$

$$(D-\lambda_1)(\dot a_2e^{\lambda_2 t})=\varepsilon_1a_2e^{\lambda_2 t}-3\varepsilon a_1a_2^2e^{(\lambda_1+2\lambda_2)t}. \tag{5.16}$$

This leaves the following perturbational equation

$$(D-\lambda_1)(D-\lambda_2)u_1=-(a_1^3e^{3\lambda_1 t}+a_2^3e^{3\lambda_2 t}). \tag{5.17}$$

To determine the first order approximate solution, it can be considered that a_1 and a_2 are constants. Hence the particular solutions of Eqs. (5.15)- (5.17) yield,

$$\dot{a}_1 = \varepsilon_1 l_0\, a_1 + \varepsilon l_1 a_1^2 a_2 e^{(\lambda_1+\lambda_2)t},$$
$$\dot{a}_2 = \varepsilon_1 l_0^*\, a_2 + \varepsilon l_1^*\, a_1\, a_2^2\, e^{(\lambda_1+\lambda_2)t}, \tag{5.18}$$

where

$$l_0 = \frac{1}{(\lambda_1-\lambda_2)}, \quad l_1 = -\frac{3}{2\lambda_1}, \quad l_0^* = \frac{1}{(\lambda_2-\lambda_1)}, \quad l_1^* = -\frac{3}{2\lambda_2}$$

and

$$u_1 = C_1 a_1^3 e^{3\lambda_1 t} + C_1^* a_2^3 e^{3\lambda_2 t}, \tag{5.19}$$

where

$$C_1 = -\frac{1}{2\lambda_1(3\lambda_1-\lambda_2)}, \quad C_1^* = -\frac{1}{2\lambda_2(3\lambda_2-\lambda_1)}.$$

Now substituting the values of u_1 from Eq. (5.19) into Eq. (5.14) and then simplifying, we obtain

$$\begin{aligned}
&(D-\lambda_2)(\dot{a}_1 e^{\lambda_1 t}) + (D-\lambda_1)(\dot{a}_2 e^{\lambda_2 t}) + \varepsilon(D-\lambda_1)(D-\lambda_2)(C_1 a_1^3 e^{3\lambda_1 t} + C_1^* a_2^3 e^{3\lambda_2 t}) \\
&+ (D-\lambda_1)(D-\lambda_2)(\varepsilon^2 u_2 + \cdots) \\
&\quad = \varepsilon_1(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \varepsilon u_1) - \varepsilon(a_1^3 e^{3\lambda_1 t} + 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} \\
&\quad + a_2^3 e^{3\lambda_2 t}) - 3\varepsilon^2(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t})^2 (C_1 a_1^3 e^{3\lambda_1 t} + C_1^* a_2^3 e^{3\lambda_2 t}).
\end{aligned} \tag{5.20}$$

Eq. (5.20) can be written as

$$\begin{aligned}
&(D-\lambda_2)(\dot{a}_1 e^{\lambda_1 t}) + (D-\lambda_1)(\dot{a}_2 e^{\lambda_2 t}) + 2\varepsilon\lambda_1(3\lambda_1-\lambda_2)\, C_1 a_1^3 e^{3\lambda_1 t} \\
&+2\varepsilon\lambda_2(3\lambda_2-\lambda_1) C_1^* a_2^3 e^{3\lambda_2 t} - 27\varepsilon^2 C_1 a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t} \\
&- 27\varepsilon^2 C_1^* a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t} + 3\varepsilon\varepsilon_1(5\lambda_1-\lambda_2) l_0 C_1 a_1^3 e^{3\lambda_1 t} \\
&+ 3\varepsilon\varepsilon_1(5\lambda_2-\lambda_1) l_0^* C_1^* a_2^3 e^{3\lambda_2 t} + (D-\lambda_1)(D-\lambda_2)(\varepsilon^2 u_2 + \ldots) \\
&\quad = \varepsilon_1(a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}) + \varepsilon\varepsilon_1(C_1 a_1^3 e^{3\lambda_1 t} + C_1^* a_2^3 e^{3\lambda_2 t}) \\
&\quad - \varepsilon(a_1^3 e^{3\lambda_1 t} + 3a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} + a_2^3 e^{3\lambda_2 t}) \\
&\quad - 3\varepsilon^2(C_1 a_1^5 e^{5\lambda_1 t} + 2C_1 a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t} + C_1 a_1^3 a_2^2 e^{(3\lambda_1+2\lambda_2)t} \\
&\quad + C_1^* a_1^2 a_2^3 e^{(2\lambda_1+3\lambda_2)t} + 2C_1^* a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t} + C_1^* a_2^5 e^{5\lambda_2 t}).
\end{aligned} \tag{5.21}$$

All the terms with ε of the left side and the terms $a_1^3 e^{3\lambda_1 t}$, $a_2^3 e^{3\lambda_2 t}$ of the right side of Eq. (5.21) are cancelled since C_1 and C_1^* satisfy Eq. (5.19). According to the separation rules (in Ref [28]), u_2 excludes the terms $a_1^3 a_2^2 e^{(3\lambda_1+2\lambda_2)t}$ and $a_1^2 a_2^3 e^{(2\lambda_1+3\lambda_2)t}$ and they will be added to the equations of $\dot{a}_1$ and $\dot{a}_2$ respectively. Therefore, we obtain the equations for $\dot{a}_1$ and $\dot{a}_2$ up to $O(\varepsilon^2)$ and u_2 in the following forms

$$(D-\lambda_2)(\dot{a}_1 e^{\lambda_1 t}) = \varepsilon_1 a_1 e^{\lambda_1 t} - 3\varepsilon a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} - 3\varepsilon^2 C_1 a_1^3 a_2^2 e^{(3\lambda_1+2\lambda_2)t}, \tag{5.22}$$

$$(D-\lambda_1)(\dot{a}_2 e^{\lambda_2 t}) = \varepsilon_1 a_2 e^{\lambda_2 t} - 3\varepsilon a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t} - 3\varepsilon^2 C_1^* a_1^2 a_2^3 e^{(2\lambda_1+3\lambda_2)t}, \tag{5.23}$$

and

$$\begin{aligned}
&(D-\lambda_1)(D-\lambda_2)u_2 \\
&= 27C_1 a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t} + 27C_1^* a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t} - \frac{3\varepsilon_1}{\varepsilon}(5\lambda_1-\lambda_2)l_0 C_1 a_1^3 e^{3\lambda_1 t} \\
&- \frac{3\varepsilon_1}{\varepsilon}(5\lambda_2-\lambda_1)l_0^* C_1^* a_2^3 e^{3\lambda_2 t} + \frac{\varepsilon_1}{\varepsilon}(C_1 a_1^3 e^{3\lambda_1 t} + C_1^* a_2^3 e^{3\lambda_2 t}) \\
&- 3(C_1 a_1^5 e^{5\lambda_1 t} + 2C_1 a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t} + C_1 a_1^3 a_2^2 e^{(3\lambda_1+2\lambda_2)t} + C_1^* a_1^2 a_2^3 e^{(2\lambda_1+3\lambda_2)t} \\
&+ 2C_1^* a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t} + C_1^* a_2^5 e^{5\lambda_2 t}).
\end{aligned} \tag{5.24}$$

Now we are going to assume that the particular solutions of Eqs. (5.22)- (5.23) in the following forms

$$\begin{aligned}
\dot{a}_1 &= \varepsilon_1 l_0 a_1 + \varepsilon l_1 a_1^2 a_2 e^{(\lambda_1+\lambda_2)t} + \varepsilon^2 l_2 a_1^3 a_2^2 e^{2(\lambda_1+\lambda_2)t} + \varepsilon\varepsilon_1 l_3 a_1^2 a_2 e^{(\lambda_1+\lambda_2)t} + \varepsilon_1^2 l_4 a_1, \\
\dot{a}_2 &= \varepsilon_1 l_0^* a_2 + \varepsilon l_1^* a_1 a_2^2 e^{(\lambda_1+\lambda_2)t} + \varepsilon^2 l_2^* a_1^2 a_2^3 e^{2(\lambda_1+\lambda_2)t} + \varepsilon\varepsilon_1 l_3^* a_1 a_2^2 e^{(\lambda_1+\lambda_2)t} + \varepsilon_1^2 l_4^* a_2,
\end{aligned} \tag{5.25}$$

where l_0, l_1, l_0^* and l_1^* are given in Eq. (5.18) while l_2, l_3, l_4, l_2^* l_3^* and l_4^* are to be determined. Now substituting the values of $\dot{a}_1$ and $\dot{a}_2$ from Eq. (5.25) into Eqs. (5.22)- (5.23), and then simplifying them, we get

$$\begin{aligned}
&\varepsilon_1(\lambda_1-\lambda_2)l_0 a_1 e^{\lambda_1 t} + 2\varepsilon\lambda_1 l_1 a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} + \varepsilon^2\{l_1(2l_1+l_1^*) \\
&+ (3\lambda_1+\lambda_2)l_2\} a_1^3 a_2^2 e^{(3\lambda_1+2\lambda_2)t} + \varepsilon\varepsilon_1\{l_1(3l_0+l_0^*) + 2\lambda_1 l_3\} a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} \\
&+ \varepsilon_1^2\{l_0^2 + (\lambda_1-\lambda_2)l_4\} a_1 e^{\lambda_1 t} \\
&= \varepsilon_1 a_1 e^{\lambda_1 t} - 3\varepsilon a_1^2 a_2 e^{(2\lambda_1+\lambda_2)t} - 3\varepsilon^2 C_1 a_1^3 a_2^2 e^{(3\lambda_1+2\lambda_2)t},
\end{aligned} \tag{5.26}$$

and

$$\varepsilon_1(\lambda_1-\lambda_2)l_0^* a_2 e^{\lambda_2 t}+2\varepsilon\lambda_2 l_1^* a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t}+\varepsilon^2\{l_1^*(2l_1^*+l_1)$$
$$+(\lambda_1+3\lambda_2)l_2^*\}a_1^2 a_2^3 e^{(2\lambda_1+3\lambda_2)t}+\varepsilon\varepsilon_1\{l_1^*(3l_0^*+l_0)+2\lambda_2 l_3^*\}a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t}$$
$$+\varepsilon_1^2\{l_0^{*2}+(\lambda_2-\lambda_1)l_4^*\}a_2 e^{\lambda_2 t} \tag{5.27}$$
$$=\varepsilon_1 a_2 e^{\lambda_2 t}-3\varepsilon a_1 a_2^2 e^{(\lambda_1+2\lambda_2)t}-3\varepsilon^2 C_1^* a_1^2 a_2^3 e^{(2\lambda_1+3\lambda_2)t}.$$

All the terms with ε of Eqs. (5.26) – (5.27) are cancelled since l_0, l_1, l_0^* and l_1^* satisfy Eq. (5.18).By comparing the coefficients of ε^2, $\varepsilon\varepsilon_1$ and ε_1^2 from both sides of Eqs. (5.26)- (5.27), we get the following algebraic equations

$$l_1(2l_1+l_1^*)+(3\lambda_1+\lambda_2)l_2=-3C_1,$$
$$l_1(3l_0+l_0^*)+2\lambda_1 l_3=0, \tag{5.28}$$
$$l_1^2+(\lambda_1-\lambda_2)l_4=0,$$

and

$$l_1^*(2l_1^*+l_1)+(\lambda_1+3\lambda_2)l_2^*=-3C_1^*,$$
$$l_1^*(3l_0^*+l_0)+2\lambda_2 l_3^*=0, \tag{5.29}$$
$$l_1^{*2}+(\lambda_2-\lambda_1)l_4^*=0.$$

By solving Eqs. (5.28)- (5.29) and then substituting the values of l_1, l_1^*, C_1 and C_1^* from Eqs. (5.18)- (5.19) and then by simplifying, we have

$$l_2=\frac{3(-9\lambda_2^2-13\lambda_1\lambda_2+6\lambda_1^2)}{4\lambda_1^2\lambda_2(9\lambda_1^2-\lambda_2^2)}, l_3=\frac{3}{2\lambda_1^2(\lambda_1-\lambda_2)}, l_4=-\frac{1}{(\lambda_1-\lambda_2)^3},$$
$$l_2^*=\frac{3(-9\lambda_1^2-13\lambda_1\lambda_2+6\lambda_2^2)}{4\lambda_1\lambda_2^2(9\lambda_2^2-\lambda_1^2)}, l_3^*=\frac{3}{2\lambda_2^2(\lambda_2-\lambda_1)}, l_4^*=-\frac{1}{(\lambda_2-\lambda_1)^3}. \tag{5.30}$$

On simplification Eq. (5.24) gives

$$(D-\lambda_1)(D-\lambda_2)u_2$$
$$=\frac{\varepsilon_1}{\varepsilon}\{1-3(5\lambda_1-\lambda_2)l_0\}C_1 a_1^3 e^{3\lambda_1 t}+\frac{\varepsilon_1}{\varepsilon}\{1-3(5\lambda_2-\lambda_1)l_0^*\}C_1^* a_2^3 e^{3\lambda_2 t} \tag{5.31}$$
$$+21C_1 a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t}+21C_1^* a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t}-3C_1 a_1^5 e^{5\lambda_1 t}-3C_1^* a_2^5 e^{5\lambda_2 t}.$$

By solving Eq. (5.31) for u_2, we obtain the particular solution as

$$u_2 = \frac{\varepsilon_1}{\varepsilon}\{\frac{1-3(5\lambda_1-\lambda_2)l_0}{2\lambda_1(3\lambda_1-\lambda_2)}\}C_1 a_1^3 e^{3\lambda_1 t} + \frac{\varepsilon_1}{\varepsilon}\{\frac{1-3(5\lambda_2-\lambda_1)l_0^*}{2\lambda_2(3\lambda_2-\lambda_1)}\}C_1^* a_2^3 e^{3\lambda_2 t}$$
$$+\frac{21C_1 a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t}}{4\lambda_1(3\lambda_1+\lambda_2)} + \frac{21C_1^* a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t}}{4\lambda_2(\lambda_1+3\lambda_2)} - \frac{3C_1 a_1^5 e^{5\lambda_1 t}}{4\lambda_1(5\lambda_1-\lambda_2)} \tag{5.32}$$
$$-\frac{3C_1^* a_2^5 e^{5\lambda_2 t}}{4\lambda_2(5\lambda_2-\lambda_1)}.$$

Finally, inserting the values of l_0, l_0^*, C_1 and C_1^* from Eqs. (5.18)- (5.19) into Eq. (5.32) and simplifying, we obtain

$$u_2 = \frac{\varepsilon_1}{\varepsilon}(E_0 a_1^3 e^{3\lambda_1 t} + E_0^* a_2^3 e^{3\lambda_2 t}) + E_2 a_1^4 a_2 e^{(4\lambda_1+\lambda_2)t}$$
$$+ E_2^* a_1 a_2^4 e^{(\lambda_1+4\lambda_2)t} + C_2 a_1^5 e^{5\lambda_1 t} + C_2^* a_2^5 e^{5\lambda_2 t}, \tag{5.33}$$

where

$$E_0 = \frac{7\lambda_1-\lambda_2}{[2\lambda_1^2(\lambda_1-\lambda_2)(3\lambda_1-\lambda_2)^2]}, \quad E_0^* = \frac{7\lambda_2-\lambda_1}{[2\lambda_2^2(\lambda_2-\lambda_1)(3\lambda_2-\lambda_1)^2}, $$
$$E_2 = -\frac{21}{[8\lambda_1^2(9\lambda_1^2-\lambda_2^2)]}, \quad E_2^* = -\frac{21}{[8\lambda_2^2(9\lambda_2^2-\lambda_1^2)]}, \tag{5.34}$$
$$C_2 = \frac{3}{[8\lambda_1^2(3\lambda_1-\lambda_2)(5\lambda_1-\lambda_2)]}, \quad C_2^* = \frac{3}{[8\lambda_2^2(3\lambda_2-\lambda_1)(5\lambda_2-\lambda_1)]}.$$

Thus the second order approximate solution of Eq. (5.10) is obtained as

$$x(t,\varepsilon) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + \varepsilon u_1(a_1,a_2,t) + \varepsilon^2 u_2(a_1,a_2,t), \tag{5.35}$$

where a_1 and a_2 are the solutions of Eq. (5.25) and u_1 , u_2 are respectively given by Eqs. (5.19) and (5.33).This solution can be carried out to the usual form by using the suitable transformations. For the under-damped system, the variables should be transformed by

$$a_1 = \frac{1}{2}ae^{i\theta}, \quad a_2 = \frac{1}{2}ae^{-i\theta}. \tag{5.36}$$

Now inserting Eq. (5.36) into Eq. (5.25) and simplifying them, we obtain the following variational equations for the amplitude and phase variables respectively

$$\dot{a} = \varepsilon m_1 a^3 e^{-2kt} + \varepsilon^2 m_2 a^5 e^{-4kt} + \varepsilon\varepsilon_1 m_3 a^3 e^{-2kt},$$
$$\dot{\theta} = \varepsilon_1 n_0 + \varepsilon n_1 a^2 e^{-2kt} + \varepsilon^2 n_2 a^4 e^{-4kt} + \varepsilon\varepsilon_1 n_3 a^2 e^{-2kt} + \varepsilon_1^2 n_4, \tag{5.37}$$

Where

$$m_1 = \frac{3k}{8(k^2+\omega_0^2)}, \quad m_2 = \frac{3k\{15\omega_0^2(7k^2-2\omega_0^2)-(7\omega_0^2-k^2)(8k^2+5\omega_0^2)\}}{128(k^2+\omega_0^2)^2(k^2+4\omega_0^2)(4k^2+\omega_0^2)},$$

$$m_3 = \frac{3k}{8(k^2+\omega_0^2)^2}, \quad n_0 = -\frac{1}{2\omega_0}, \quad n_1 = \frac{3\omega_0}{8(k^2+\omega_0^2)}, \quad n_0 = -\frac{1}{2\omega_0},$$

$$n_1 = \frac{3\omega_0}{8(k^2+\omega_0^2)}, \quad n_2 = \frac{3\omega_0\{15k^2(7\omega_0^2-2k^2)+(7k^2-2\omega_0^2)(8k^2+5\omega_0^2)\}}{128(k^2+\omega_0^2)^2(k^2+4\omega_0^2)(4k^2+\omega_0^2)},$$

$$n_3 = -\frac{3(k^2-\omega_0^2)}{16\omega_0(k^2+\omega_0^2)^2}, \quad n_4 = -\frac{1}{2\omega_0^2}. \tag{5.38}$$

By using Eq. (5.36) and assuming that $\varphi = \omega_0 t + \theta$, the correction terms u_1 and u_2 can be written in the following forms

$$u_1 = a^3 e^{-3kt}(P_3\cos 3\varphi + Q_3 \sin 3\varphi), \tag{5.39}$$

and

$$\begin{aligned} u_2 = {} & \frac{\varepsilon_1}{\varepsilon} a^3 e^{-3kt}(P_4\cos 3\varphi + Q_4\sin 3\varphi) + a^5 e^{-5kt}(P_5\cos 3\varphi + Q_5\sin 3\varphi) \\ & + a^5 e^{-5kt}(P_6\cos 5\varphi + Q_6 \sin 5\varphi), \end{aligned} \tag{5.40}$$

where

$$P_3 = -\frac{(k^2-2\omega_0^2)}{16(k^2+\omega_0^2)(k^2+4\omega_0^2)}, \; Q_3 = \frac{3k\omega_0}{16(k^2+\omega_0^2)(k^2+4\omega_0^2)}, \tag{5.41}$$

and

$$\begin{aligned} P_4 &= -\frac{(7k^4+8k^2\omega_0^2-8\omega_0^4)}{16(k^2+\omega_0^2)^2(k^2+4\omega_0^2)^2}, \\ Q_4 &= -\frac{3k(k^4-5k^2\omega_0^2-12\omega_0^4)}{32\omega_0(k^2+\omega_0^2)^2(k^2+4\omega_0^2)^2}, \\ P_5 &= -\frac{21(k^4-7k^2\omega_0^2+\omega_0^4)}{256(k^2+\omega_0^2)^2(k^2+4\omega_0^2)(4k^2+\omega_0^2)}, \\ Q_5 &= \frac{189k\omega_0(k^2-\omega_0^2)}{512(k^2+\omega_0^2)^2(k^2+4\omega_0^2)(4k^2+\omega_0^2)}, \\ P_6 &= \frac{3(k^4-11k^2{\omega_0}^2+3{\omega_0}^4)}{256(k^2+{\omega_0}^2)^2(k^2+4{\omega_0}^2)(4k^2+9{\omega_0}^2)}, \\ Q_6 &= -\frac{3k\omega_0(11k^2-19\omega_0^2)}{512(k^2+\omega_0^2)^2(k^2+4\omega_0^2)(4k^2+9\omega_0^2)}. \end{aligned} \tag{5.42}$$

Hence the second order approximate solution of Eq. (5.10) for damped oscillatory processes is given by

$$x = ae^{-kt}\cos\varphi + \varepsilon u_1 + \varepsilon^2 u_2, \quad \varphi = \omega_0 t + \theta. \tag{5.43}$$

where a and θ are the solutions of Eq. (5.37) and u_1 and u_2 are respectively obtained from Eqs. (5.39) and (5.40).

5.4 Results and Discussions

The obtained approximate solutions are compared with the numerical solutions graphically. Also to show the effect of second order approximate solutions, graphs are drawn for both first and second order approximations.

In **Fig. 5.1 (a)**, comparison is made between the first order approximate solution and the numerical solution obtained by using ***Runge-Kutta*** fourth order formula for strong nonlinearity with large damping effects. Here we notice that with the increase of time t the analytical result deviates from the numerical one. **Fig. 5.1 (b)** represents the same for the second order approximate solution within the same time domain and it is observed that the deviation from the numerical result is very small in the case of second order approximate solution. In **Fig. 5.2 (a)**, comparison is made between the first order approximate solution and the numerical solution obtained by using ***Runge-Kutta*** fourth order formula for strong nonlinearity with small damping effects. Here we notice that with the increase of time t the analytical result deviates from the numerical one. **Fig. 5.2 (b)** represents the same for the second order approximate solution within the same time domain and it is observed that the deviation from the numerical result is very small in case of the second order approximate solution. In **Fig. 5.3 (a),** comparison is made between the first order approximate solution and the numerical solution obtained by using ***Runge-Kutta*** fourth order formula for strong nonlinearity without damping effects. From this figure we notice that with the increase of time t, the analytical result deviates from the numerical one. **Fig. 5.3 (b)** represents the same for the second order approximate solution within the same time domain and it is observed that the analytical solution has good agreement with the numerical result in the case of second order approximate solution. It may be mentioned that if we consider $\varepsilon_1 = 0$, then our result becomes the same as that of Alam *et al.* [28]. To check this, we have plotted the **Figs. 5.4 (a, b)**. In

Fig. 5.4 (a), comparison is made between the first order approximate solution and the numerical solution obtained by using ***Runge-Kutta*** fourth order formula for strong nonlinearity with large damping effects. Here we notice that with the increase of time t the analytical result deviates from the numerical one. **Fig. 5.4 (b)** represents the same for the second order approximate solution within the same time domain and it is observed that the deviation from the numerical result is very small in the case of second order approximate solution by setting $\varepsilon_1 = 0$. From the **Figs. 5.1 (a, b)** and **Figs. 5.4 (a, b)**, it is notified that, our new homotopy perturbation technique gives better results than that of Alam *et al.* [28].

5.5 Conclusion

In this chapter, a new kind of analytical technique for a general second order nonlinear differential systems with constant coefficients is presented. From the figures, it is clear to us that the first order approximate solutions continuously deviate from the numerical solutions with the increase of time t. Thus we are forced to determine the second or higher order approximate solutions. The approximate solutions and the numerical solutions of Eq. (5.10) are obtained for the different damping effects and for several artificial constants with $\varepsilon = 1.0$. Comparison is made between the solutions obtained by the homotopy perturbation method (dashed lines) and those obtained by the numerical procedure (solid line) in figures. This method shows effectively and accurately that large classes of second order approximate solutions converge rapidly to the numerical solutions in presence of significant damping effects with strong nonlinearity. Also this new homotopy perturbation technique is valid for strongly damped, weakly damped and undamped cases with strong nonlinearity. Moreover, it is also valid for weakly nonlinear differential systems. The variational equations are very important in a homotopy perturbation solution whatever the relations of them with ε. We conclude that, this new homotopy perturbation method is effective and accurate for nonlinear problems where the approximate solutions converge rapidly to the exact solutions. In a similar way, the method can be used to determine the higher order approximate solutions to the nonlinear systems.

Fig. 5.1 (a)

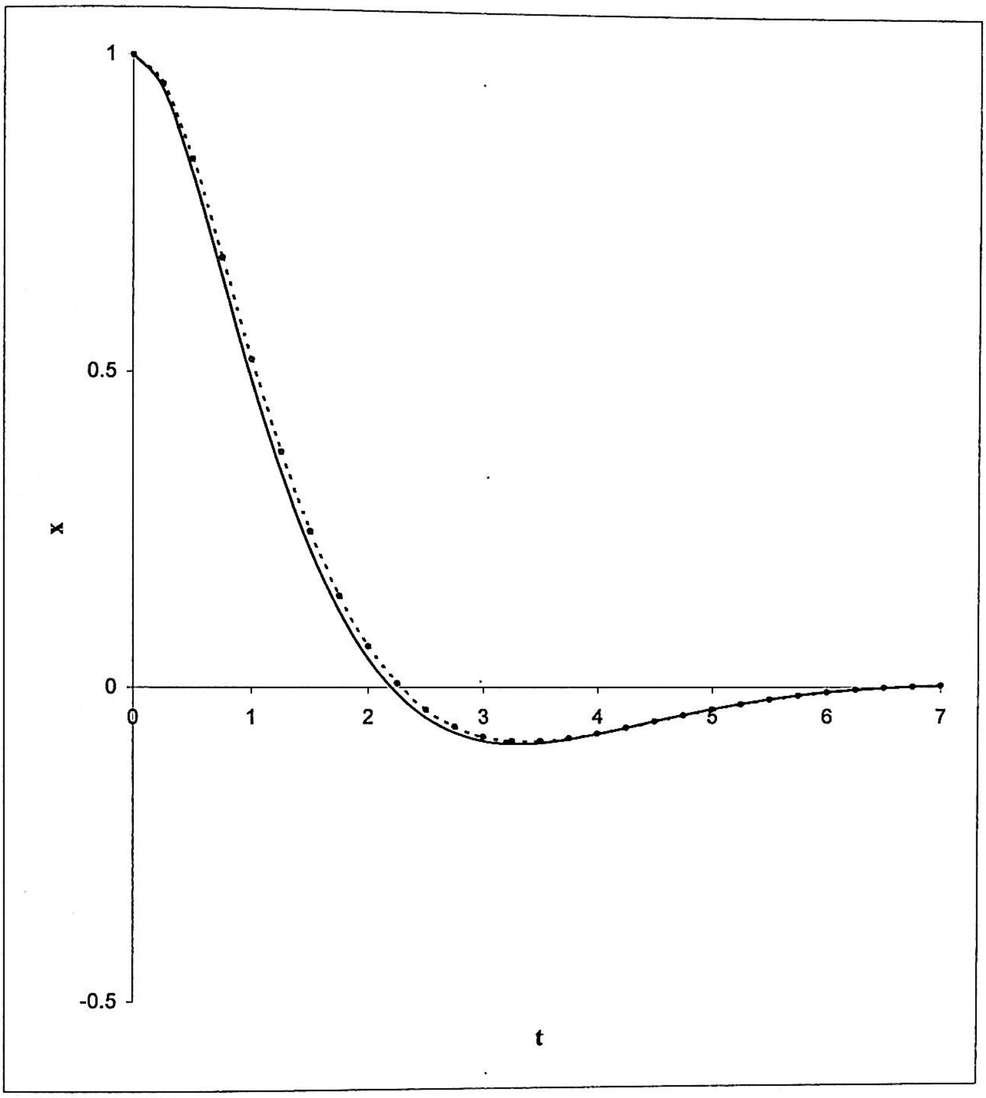

Fig. 5.1 (a) First order approximate solution $-\bullet-$ (dashed lines) of Eq. (5.10) is compared with the corresponding numerical solution (solid line) obtained by *Runge-Kutta* fourth-order formula when $a_0 = 1.07073$, $\varphi_0 = -0.31590$, $k = \sqrt{.5}$, $\omega = 1.0$, $\varepsilon_1 = 0.2$ and $\varepsilon = 1.0$.

Fig. 5.1 (b)

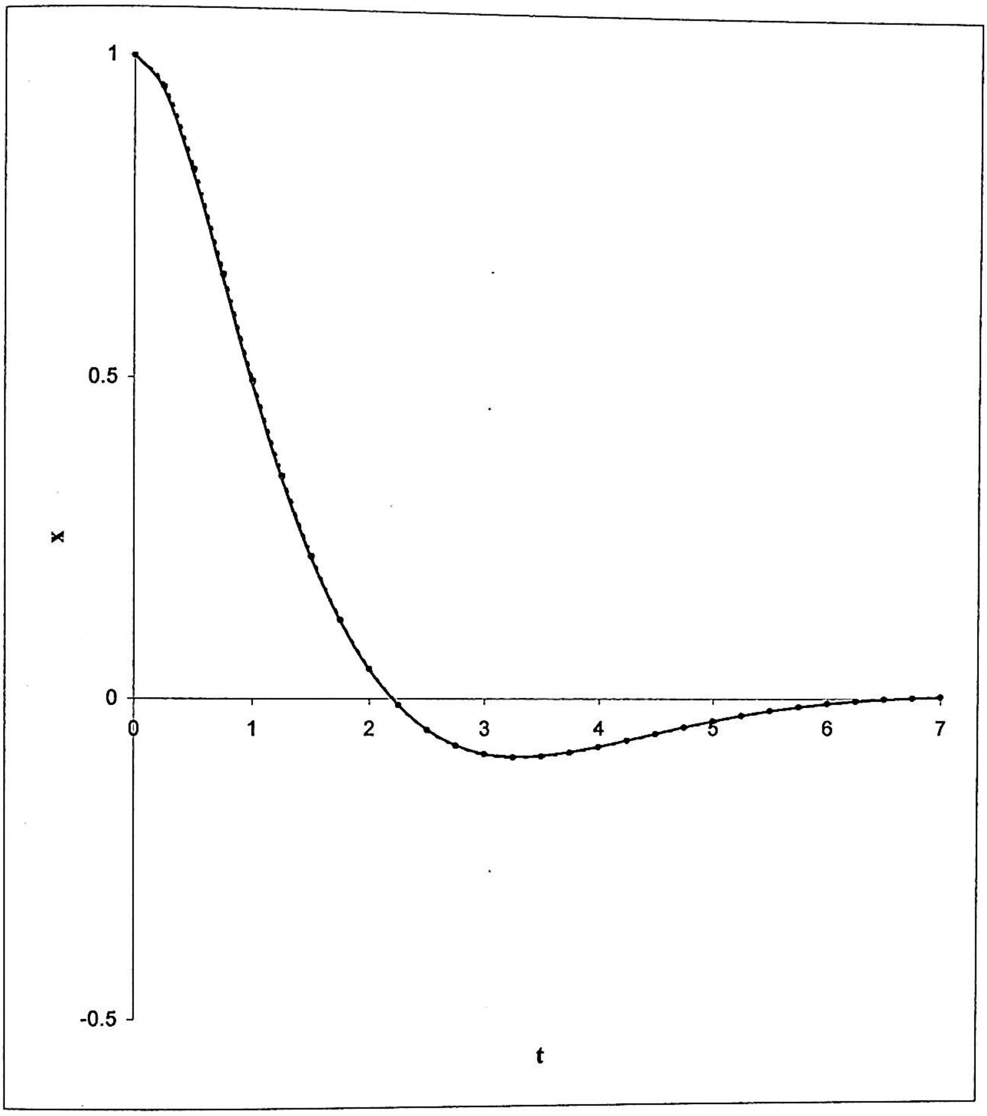

Fig. 5.1 (b) Second order approximate solution –•– (dashed lines) of Eq. (5.10) is compared with the corresponding numerical solution - (solid line) obtained by *Runge-Kutta* fourth-order formula when $a_0 = 1.04923$, $\varphi_0 = -0.27983$, $k = \sqrt{.5}$, $\omega = 1.0$, $\varepsilon_1 = 0.2$ and $\varepsilon = 1.0$.

Fig. 5.2 (a)

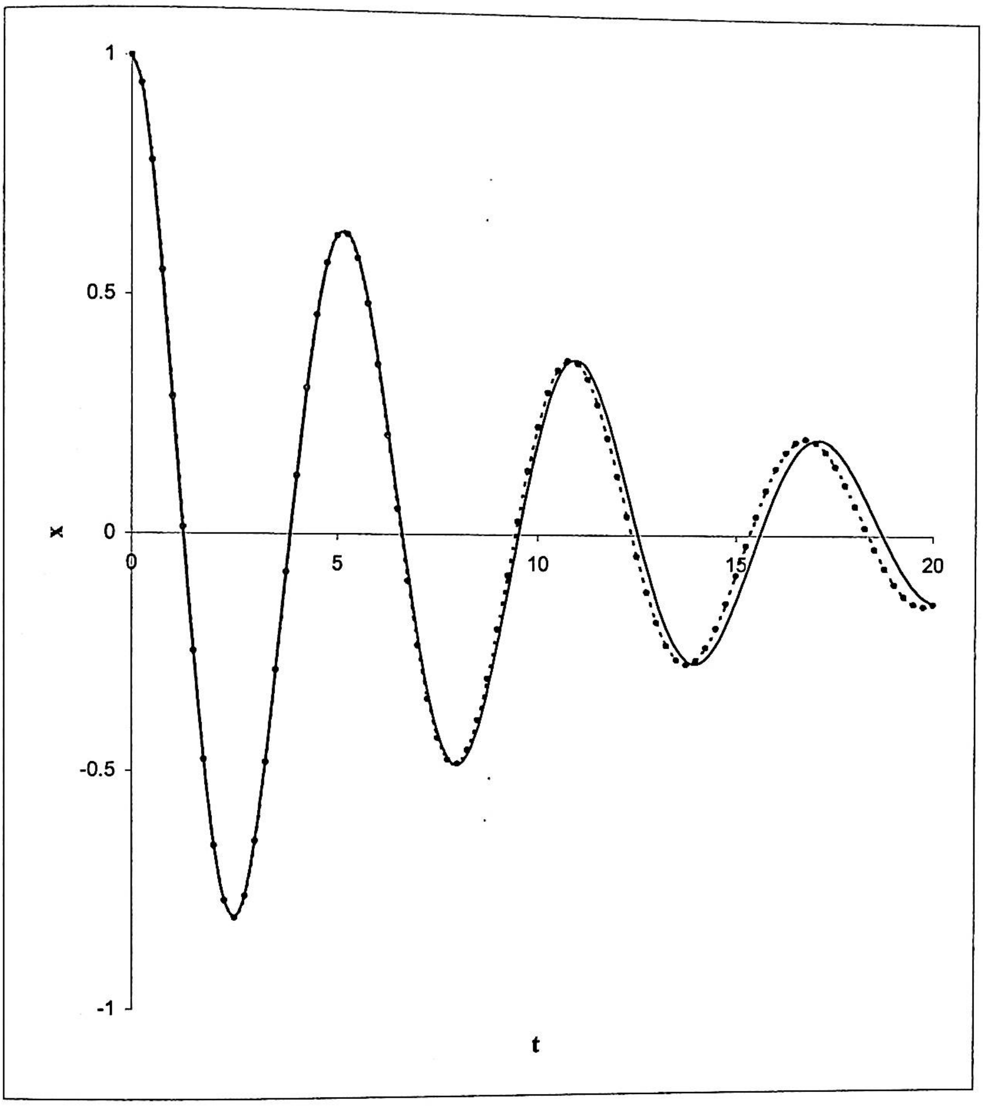

Fig. 5.2 (a) First order approximate solution $-\bullet-$ (dashed lines) of Eq. (5.10) is compared with the corresponding numerical solution - (solid lines) obtained by *Runge-Kutta* fourth-order formula when $a_0 = .98477$, $\varphi_0 = -0.05055$, $k = .1$, $\omega = 1.0$, $\varepsilon_1 = 0.75$ and $\varepsilon = 1.0$.

Fig.5.2(b)

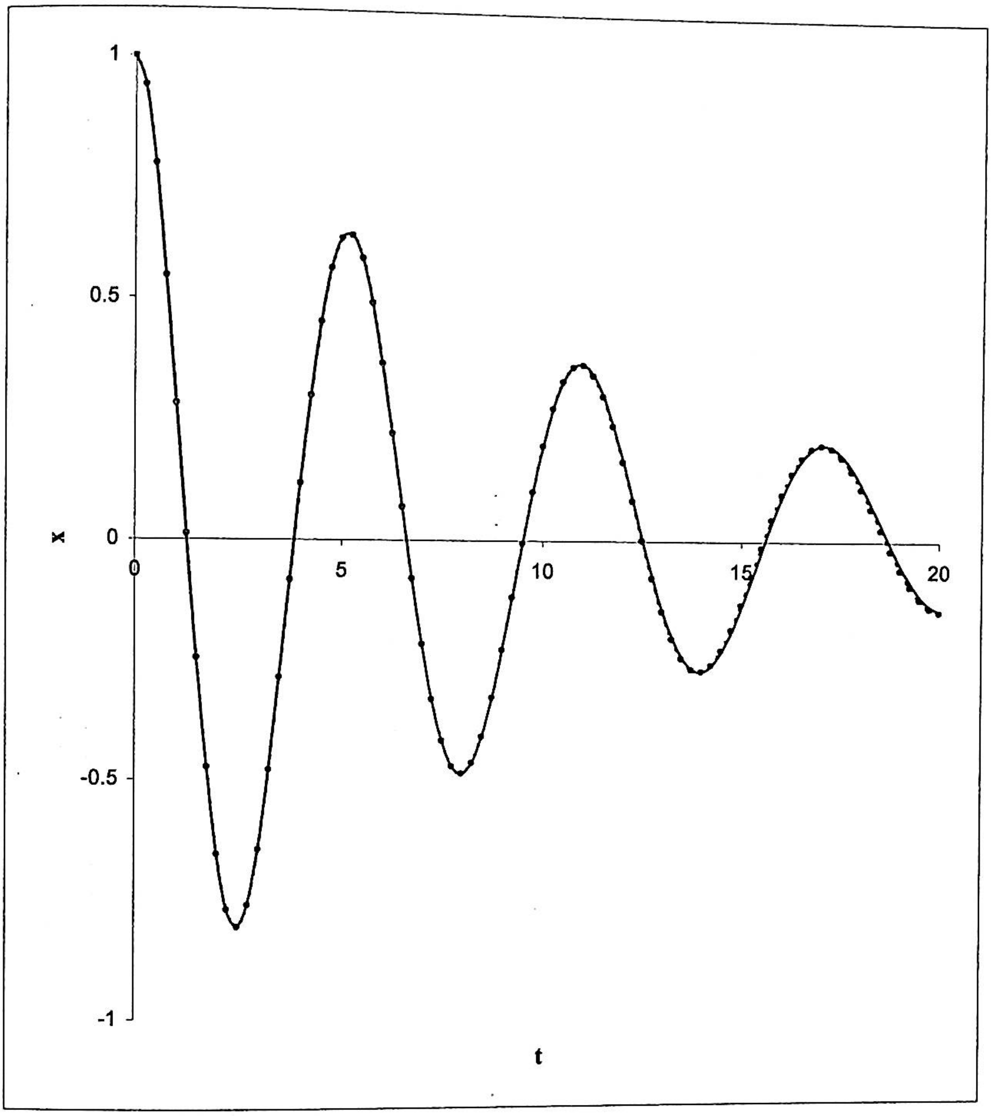

Fig. 5.2 (b) Second order approximate solution $-\bullet-$ (dashed lines) of Eq. (5.10) is compared with the corresponding numerical solution - (solid line) obtained by *Runge-Kutta* fourth-order formula for $a_0 = 0.98313$, $\varphi_0 = -0.05200$, $k = 0.1$, $\omega = 1.0$, $\varepsilon_1 = 0.75$ and $\varepsilon = 1.0$.

Fig. 5.3 (a)

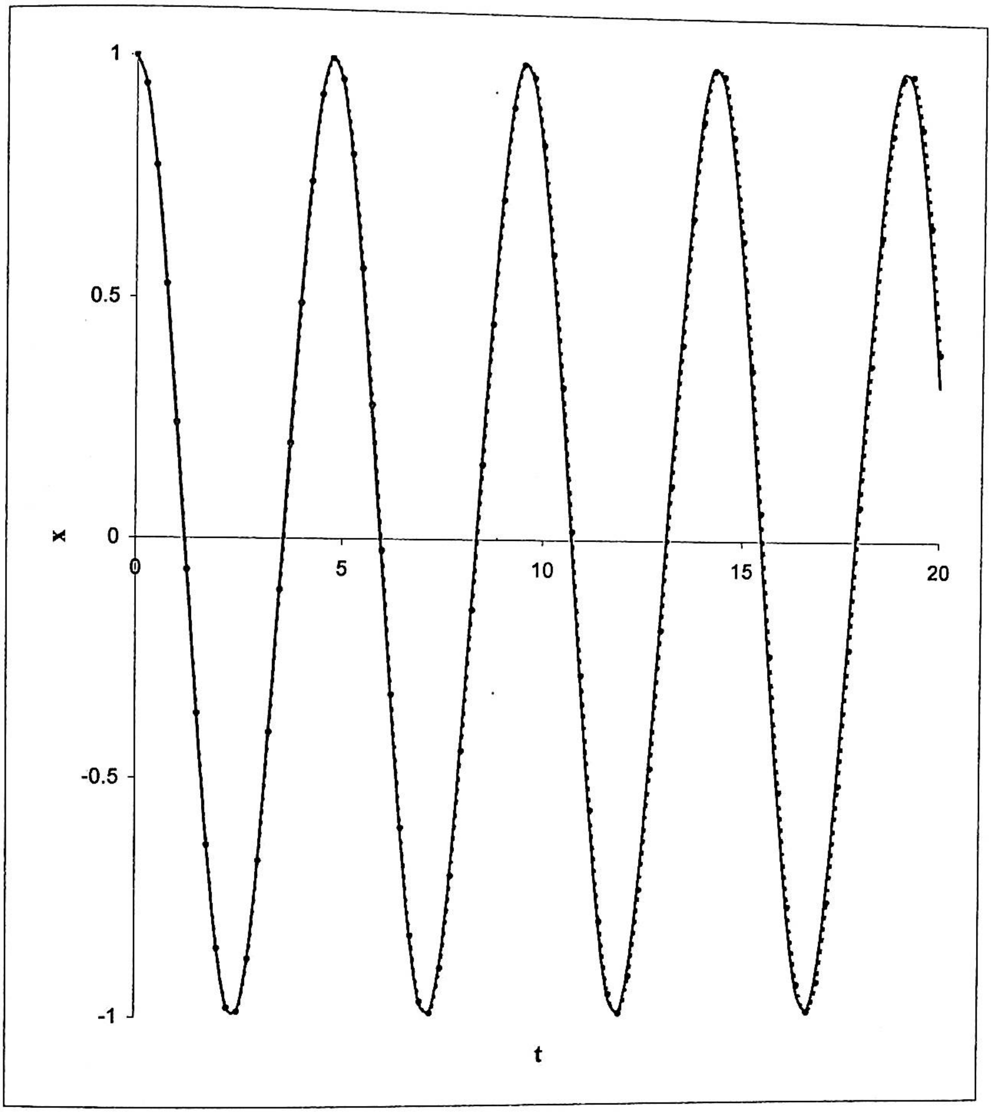

Fig. 5.3 (a) First order approximate solution $-\bullet-$ (dashed lines) of Eq. (5.10) is compared with the corresponding numerical solution - (solid line) obtained by *Runge-Kutta* fourth-order formula when $a_0 = 0.98304$, $\varphi_0 = 0.0$, $k = 0.0$, $\omega = 1.0$, $\varepsilon_1 = 0.75$ and $\varepsilon = 1.0$.

Fig. 5.3 (b)

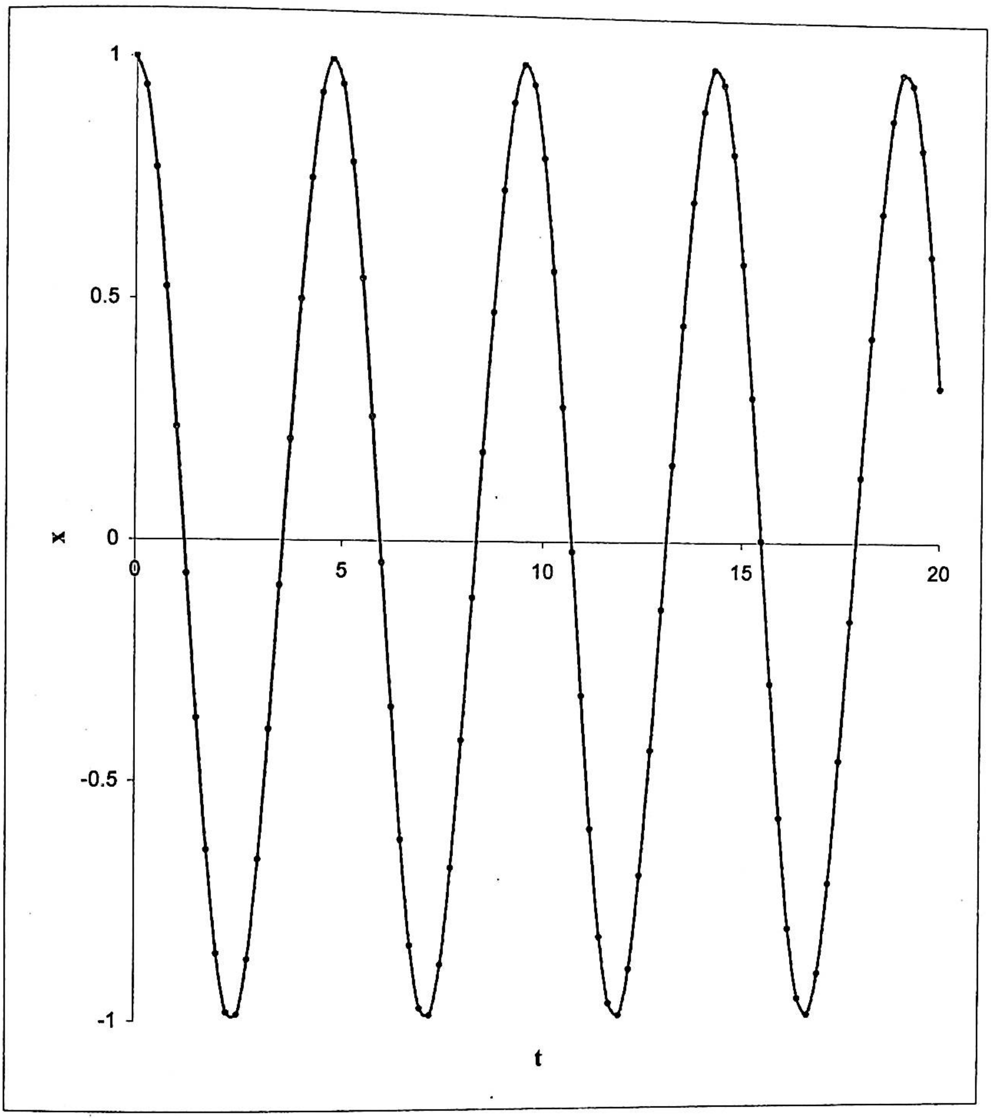

Fig. 5.3 (b) Second order approximate solution $-\bullet-$ (dashed lines) of Eq. (5.10) is compared with the corresponding numerical solution - (solid line) obtained by *Runge-Kutta* fourth-order formula when $a_0 = 0.98164$, $\varphi_0 = 0.0$, $k = 0.0$, $\omega = 1.0$, $\varepsilon_1 = 0.75$ and $\varepsilon = 1.0$.

Fig. 5.4 (a)

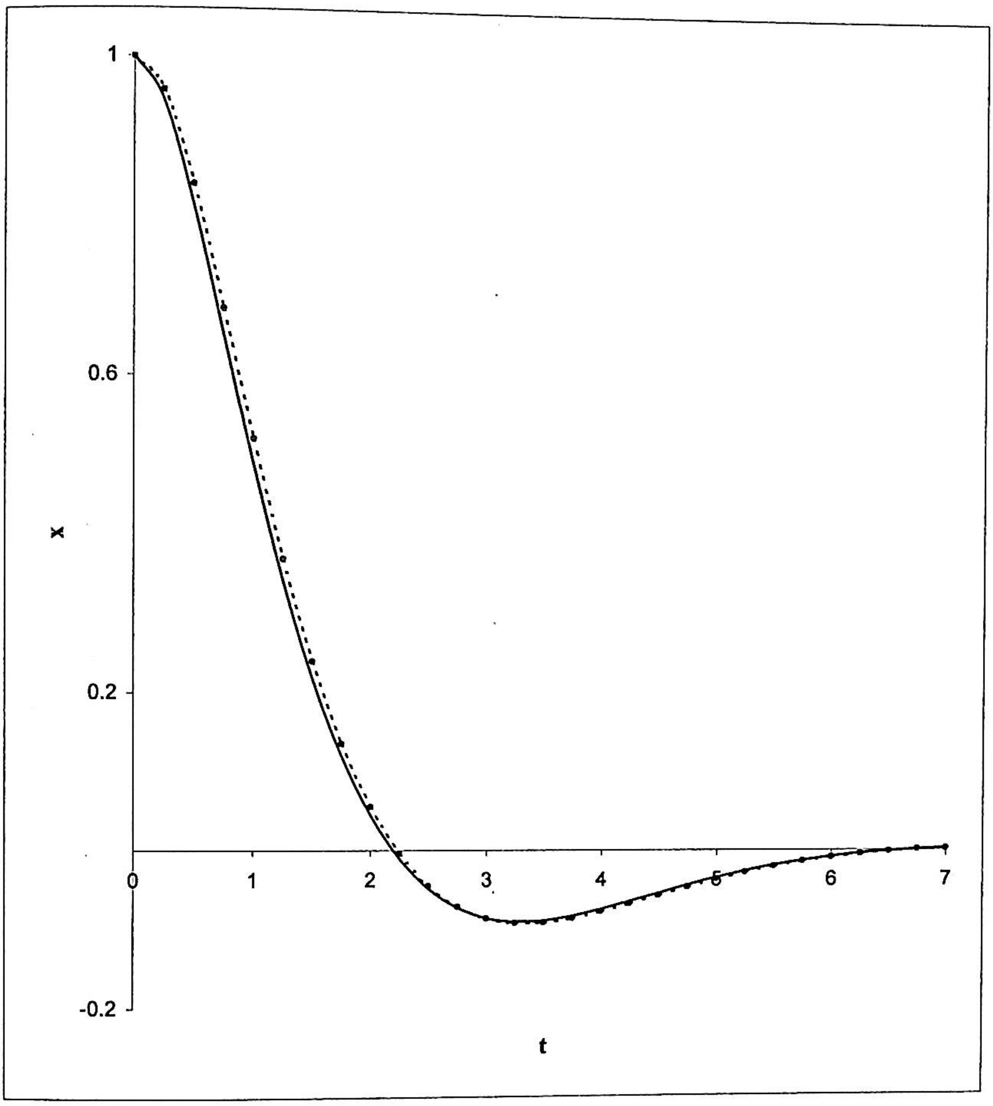

Fig. 5.4 (a) First order approximate solution –•– (dashed lines) of Eq. (5.10) is compared with the corresponding numerical solution - (solid line) obtained by *Runge-Kutta* fourth-order formula when $a_0 = 1.06008$, $\varphi_0 = -0.26899$, $k = \sqrt{.5}$, $\omega = 1.0$, $\varepsilon_1 = 0.0$ and $\varepsilon = 1.0$.

Fig. 5.4 (b)

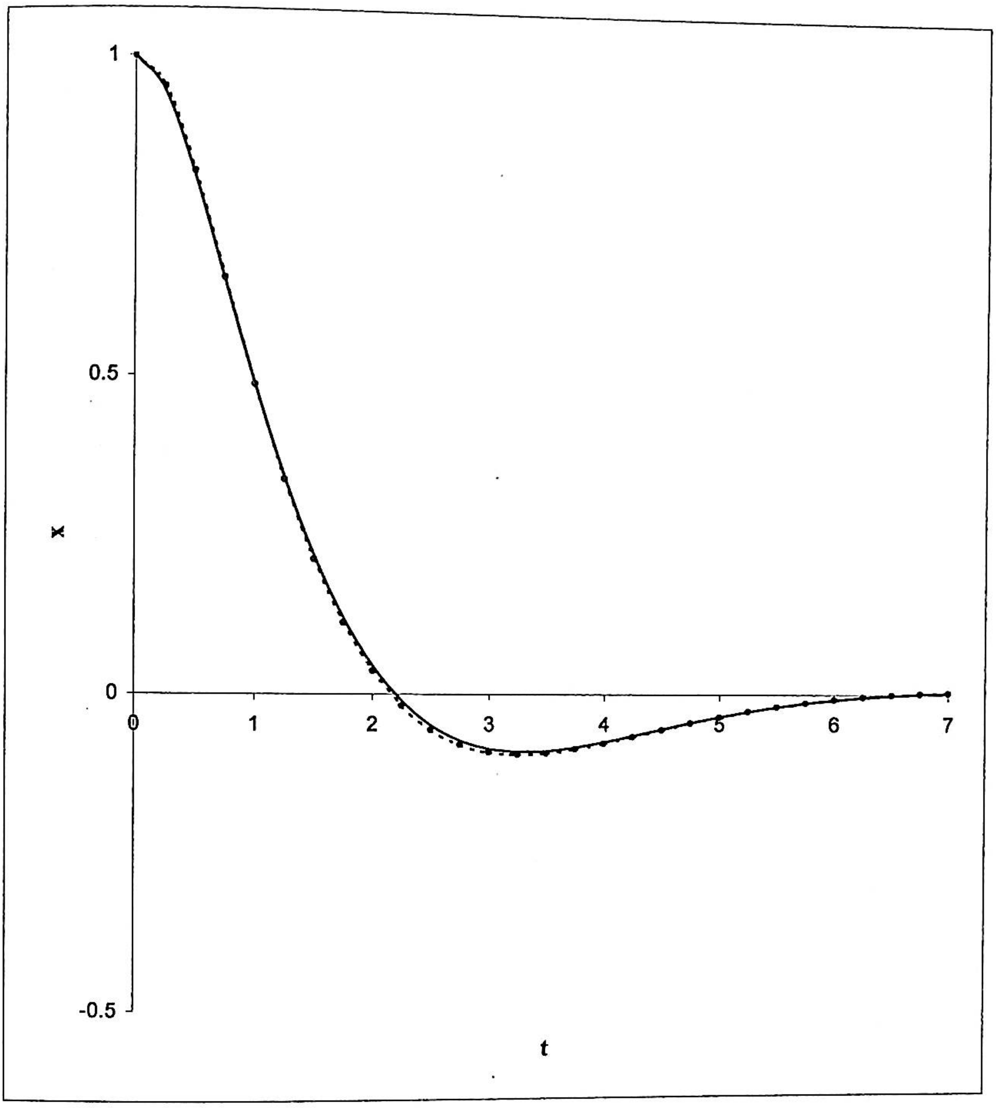

Fig. 5.4 (b) Second order approximate solution $-\bullet-$ (dashed lines) of Eq. (5.10) is compared with the corresponding numerical solution - (solid line) obtained by *Runge-Kutta* fourth-order formula when $a_0 = 1.04049$, $\varphi_0 = -0.25712$, $k = \sqrt{.5}$, $\omega = 1.0$, $\varepsilon_1 = 0.0$ and $\varepsilon = 1.0$.

Chapter 6

Second Approximation of Third-Order Nonlinear Differential Systems with Slowly Varying Coefficients

6.1 Introduction

From the beginning of modern developments in the theory of oscillations, it is seen that the asymptotic method has not been used in the investigations of second approximate solution of non-stationary phenomena, particularly, in all the cases in which the coefficients of differential equations are varying slowly with time t. The oscillations of this nature are not necessarily periodic. The asymptotic method in nonlinear problems has been developed by scientists and engineers day by day, because this method transforms the difficult problem under study into a simple problem which is easy to solve. The method has been extended to damped oscillatory and purely non oscillatory systems with slowly varying coefficients by Bojadziev and Edward [56]. Arya and Bojadziev [42] have studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Feshchenko, Shkil and Nikolenko [69] have presented a brief way to determine Krylov-Bogoliubov-Mitropolskii (KBM) [45,94,113] solution (first order) of an nth, $n = 2, 3, \ldots$ order differential systems. Arya and Bojadziev [41] have also studied a system of second order nonlinear hyperbolic differential equations with slowly varying coefficients. Alam [19] has investigated a unified Krylov-Bogoliubov-Mitropolskii method for solving nth order nonlinear systems with slowly varying coefficients. In another paper, Alam and Sattar [27] have presented an asymptotic method for third order nonlinear system with varying coefficients. Recently, Roy and Alam [139] have studied the effect of higher approximation of Krylov-Bogoliubov-Mitropolskii's [45,94,113] solution and matched asymptotic differential systems with slowly varying coefficients and damping. Sometimes the first approximate solution obtained in [1-37,45,56,57,69,94,113] gives desired result when the linear damping effect is absent or very small. Otherwise the solution gives incorrect result after a long time $t >> 1$ where the reduced frequency becomes small. The more difficult and no less important case, the second approximate solution of a third order nonlinear ordinary differential equation with slowly varying coefficients has remained almost untouched. The main goal of this chapter is to fill this gap. In this study, a new kind of analytical technique has been presented for a third order nonlinear systems with slowly varying coefficients. The method has been illustrated by applying it to a typical nonlinear problem of practical importance.

6.2 The Method

We have considered the following weakly nonlinear ordinary differential equation with slowly varying coefficients governed by [19]

$$\dddot{x} + k_1(\tau)\ddot{x} + k_2(\tau)\dot{x} + k_3(\tau)x = \varepsilon f(\ddot{x},\dot{x},x,\tau), \tag{6.1}$$

where the over-dots denote differentiations with respect to time t, ε is a small positive parameter which plays a very important role in the perturbation method, $k_1(\tau)$, $k_2(\tau)$ and $k_3(\tau)$ are slowly varying coefficients, $\tau = \varepsilon t$ is the slowly varying time and, in general, f is assumed to be a nonlinear function of $x, \dot{x}, \ddot{x}$ and τ, which may be expanded in Fourier series. The coefficients of Eq. (6.1) are slowly varying in the sense that their time derivatives are proportional to ε. In the case of variable coefficients, an extended form of KBM [1-37,45,56,57,69,94,113] solution is needed even if the damping is very small, especially when $\omega(\tau)$ is in a decreasing order.

Let us assume that the auxiliary equation of the unperturbed equation of Eq. (6.1) has three roots $\lambda_1 = -\lambda$, $\lambda_{2,3} = -\mu \pm i\omega$, where λ, μ and ω are constants. But if $\varepsilon \neq 0$, λ and μ are constants and $\omega(\tau)$ is assumed to be the reduced frequency of the nonlinear problem which varies slowly with time t. Hence the solution of the unperturbed equation of Eq. (6.1) can be written as

$$x(t,0) = a_0 e^{-\lambda t} + b_0 e^{-\mu t}\cos(\omega t + \varphi_0), \tag{6.2}$$

where a_0, b_0 and φ_0 are arbitrary constants.

Again Eq. (6.1) can be re-written as

$$\dddot{x} + 2\mu\ddot{x} + (\mu^2 + \omega^2)\dot{x} + \lambda(\ddot{x} + 2\mu\dot{x} + (\mu^2 + \omega^2)x) = \varepsilon f(\ddot{x},\dot{x},x,\tau), \tag{6.3}$$

where $k_1(\tau) = \lambda + 2\mu$, $k_2(\tau) = 2\lambda\mu + \mu^2 + \omega^2$ and $k_3(\tau) = \lambda(\mu^2 + \omega^2)$. For $\varepsilon \neq 0$, we are going to choose the asymptotic solution of Eq. (6.1) in the following form [45,94,113]

$$x(t,\varepsilon) = b\cos\varphi, \tag{6.4}$$

where the amplitude $b(t)$ and the phase $\varphi(t)$ are slowly varying functions of time t and these are given by the following first order differential equations

$$\begin{aligned} \dot{b} &= -\mu b + \varepsilon B_1(b,\tau) + \varepsilon^2 B_2(b,\tau) + \cdots, \\ \dot{\varphi} &= \omega(\tau) + \varepsilon C_1(b,\tau) + \varepsilon^2 C_2(b,\tau) + \cdots. \end{aligned} \tag{6.5}$$

In order to find the unknown functions $B_1, B_2, \cdots, C_1, C_2, \cdots$, it was early restricted in KBM method [45, 94,113] that the functions $B_1, B_2, \cdots, C_1, C_2, \cdots$ are independent of the phase variable φ. Now differentiating Eq. (6.4) three times with respect to time t and by using the relations Eq. (6.5) and by equating the coefficients of ε and ε^2 respectively, we obtain the following equations

$$\begin{aligned} &(\mu^2 b^2 \frac{\partial^2 B_1}{\partial b^2} - 2\omega^2 B_1 + 3\mu\omega b^2 \frac{\partial C_1}{\partial b} + 2\mu\omega b C_1 - 3\omega\omega' b)\cos\varphi \\ &+ (3\mu\omega b \frac{\partial B_1}{\partial b} - \mu\omega B_1 - \mu^2 b^3 \frac{\partial^2 C_1}{\partial b^2} - 2\mu^2 b^2 \frac{\partial C_1}{\partial b} + 2\omega^2 b C_1 + \mu\omega' b)\sin\varphi \\ &+ \lambda((-\mu b \frac{\partial B_1}{\partial b} + \mu B_1 - 2\omega b C_1)\cos\varphi + (\mu b^2 \frac{\partial C_1}{\partial b} - 2\omega B_1 - \omega' b)\sin\varphi) \\ &= f^{(0)}(b,\varphi,\tau) \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} &(-3\omega' b C_1 - 6\omega B_1 C_1 - 3\omega b B_1 \frac{\partial C_1}{\partial b} - 3\omega b C_1' - 2\mu b B_1 \frac{\partial^2 B_1}{\partial b^2} - 2\mu b \frac{\partial^2 B_1}{\partial \tau \partial b} \\ &+ 3\mu b^2 C_1 \frac{\partial C_1}{\partial b} + \mu B_1' - 2\omega^2 B_2 + 3\mu\omega b^2 \frac{\partial C_2}{\partial b} + 2\mu\omega b C_2 - \mu b (\frac{\partial B_1}{\partial b})^2 \\ &+ \mu b C_1^2 + \mu^2 b^2 \frac{\partial^2 B_2}{\partial b^2})\cos\varphi + (-\omega'' b - 3\omega' B_1 - 3\omega B_1 \frac{\partial B_1}{\partial b} - 3\omega B_1' + 3\omega b C_1^2 \\ &+ 3\mu b C_1 \frac{\partial B_1}{\partial b} + 2\mu b^2 B_1 \frac{\partial^2 C_1}{\partial b^2} + 2\mu b^2 \frac{\partial^2 C_1}{\partial \tau \partial b} + 5\mu b B_1 \frac{\partial C_1}{\partial b} - \mu B_1 C_1 + 3\mu\omega b \frac{\partial B_2}{\partial b} \\ &+ 2\omega^2 b C_2 + \mu b^2 \frac{\partial B_1}{\partial b}\frac{\partial C_1}{\partial b} + \mu b C_1' - \mu^2 b^3 \frac{\partial^2 C_2}{\partial b^2} - 2\mu^2 b^2 \frac{\partial C_2}{\partial b} - \mu\omega B_2)\sin\varphi \\ &+ \lambda((\mu B_2 - 2\omega b C_2 + B_1 \frac{\partial B_1}{\partial b} + B_1' - b C_1^2 - \mu b \frac{\partial B_2}{\partial b})\cos\varphi + (-2\omega B_2 - 2B_1 C_1 \\ &- b B_1 \frac{\partial C_1}{\partial b} - b C_1' + \mu b^2 \frac{\partial C_2}{\partial b})\sin\varphi) = 0, \end{aligned} \tag{6.7}$$

where $f^{(0)} = f(\ddot{x}_0, \dot{x}_0, x_0, \tau)$ and $x_0 = b\cos\varphi$ and the primes denote the differentiations with respect to slow varying time τ, *i.e.*, $\omega' = \frac{\partial \omega}{\partial \tau}$, $B_1' = \frac{\partial B_1}{\partial \tau}$, and $C_1' = \frac{\partial C_1}{\partial \tau}$. When the coefficients of Eq. (6.1) are assumed to be constants, then $\omega' = \omega'' = B_1' = C_1' = 0$. Sometimes the first approximate solution of a nonlinear differential equation with constant coefficients nicely agrees with the numerical solution, while the corresponding first approximate solution with variable coefficients gives desired results for a particular time interval, so that the problem is in its linear part only. A similar problem has arisen in [139]. Hence our investigation may be limited to the linear part of Eq. (6.1), *i.e.*, we are interested to find the terms of a second approximate solution which has appeared for varying the coefficients of Eq. (6.1).

Let us assume that the function $f^{(0)}$ can be expanded in Fourier series as

$$f^{(0)}(b,\varphi,\tau) = F_0(b,\varphi,\tau) + \sum_{n=1}^{\infty}(F_n(b,\varphi,\tau)\cos n\varphi + G_n(b,\varphi,\tau)\sin n\varphi). \tag{6.8}$$

Inserting Eq. (6.8) into Eq. (6.6) and by equating the coefficients of $\cos\varphi$ and $\sin\varphi$ and transposing to the right, we obtain the following partial differential equations

$$\begin{aligned}&\mu^2 b^2 \frac{\partial^2 B_1}{\partial b^2} - \lambda\mu b \frac{\partial B_1}{\partial b} + (\lambda\mu - 2\omega^2)B_1 + 3\mu\omega b^2 \frac{\partial C_1}{\partial b} - 2(\lambda - \mu)\omega b\, C_1 \\ &= 3\omega\omega' b + F_1,\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}&3\mu\omega b \frac{\partial B_1}{\partial b} - (2\lambda + \mu)\omega B_1 - \mu^2 b^3 \frac{\partial^2 C_1}{\partial b^2} + \mu b^2(\lambda - 2\mu)\frac{\partial C_1}{\partial b} + 2\omega^2 b C_1 \\ &= (\lambda - \mu)\omega' b + G_1.\end{aligned} \tag{6.10}$$

Also by equating the coefficients of $\cos\varphi$ and $\sin\varphi$ from Eq. (6.7) and then transposing to the right, we obtain the following partial differential equations

$$\mu^2 b^2 \frac{\partial^2 B_2}{\partial b^2} - \lambda \mu b \frac{\partial B_2}{\partial b} + (\lambda \mu - 2\omega^2) B_2 + 3\mu \omega b^2 \frac{\partial C_2}{\partial b} - 2(\lambda - \mu)\omega b C_2$$
$$= -(\lambda + \mu) B_1' + (\lambda - \mu) b C_1^2 + 3\omega' b C_1 + 3\omega b C_1' + 2\mu b B_1 \frac{\partial^2 B_1}{\partial b^2} \tag{6.11}$$
$$-(\lambda B_1 - \mu b \frac{\partial B_1}{\partial b}) \frac{\partial B_1}{\partial b} + 2\mu b \frac{\partial^2 B_1}{\partial \tau \partial b} + 6\omega B_1 C_1 - 3(\mu b C_1 - \omega B_1) b \frac{\partial C_1}{\partial b},$$

and

$$3\mu\omega b \frac{\partial B_2}{\partial b} - (2\lambda + \mu)\omega B_2 - \mu^2 b^3 \frac{\partial^2 C_2}{\partial b^2} + (\lambda - 2\mu)\mu b^2 \frac{\partial C_2}{\partial b} + 2\omega^2 b C_2$$
$$= \omega'' b + 3\omega' B_1 + 3\omega B_1' - 3\omega b C_1^2 + (\lambda - \mu) b C_1' + (3\omega B_1 - 3\mu b C_1$$
$$-\mu b^2 \frac{\partial C_1}{\partial b}) \frac{\partial B_1}{\partial b} + (2\lambda + \mu) B_1 C_1 - 2\mu b^2 B_1 \frac{\partial^2 C_1}{\partial b^2} - 2\mu b^2 \frac{\partial^2 C_1}{\partial \tau \partial b} \tag{6.12}$$
$$+ (\lambda - 5\mu) b B_1 \frac{\partial C_1}{\partial b}.$$

In general, the Fourier coefficients F_n, G_n, $n = 0, 1, 2\cdots$ in Eq. (6.8) can be expanded in powers of b. Hence Eqs. (6.9)- (6.10) have particular solutions of the forms

$$B_1 = m_1 b + m_3 b^3 + \cdots, \qquad C_1 = n_1 + n_3 b^2 + \cdots. \tag{6.13}$$

Now inserting these expressions of B_1 and C_1 into Eqs. (6.9)- (6.10) and by equating the coefficients of like powers of b, we get a set of algebraic equations which are able to give us the unknown coefficients m_1, m_3, n_1 and n_3 in terms of λ, μ and ω. To determine the second correction terms of the amplitude b and the phase φ, we assume that λ and μ are very small constants and ω is varying slowly with time t. So we can ignore the product terms of λ and μ from B_1 and C_1 to overcome the difficulty arising in the subsequent calculations. Now if we substitute the reduced values of B_1 and C_1 into the right hand sides of Eqs. (6.11)- (6.12), then the right hand sides of these equations appear in polynomials in b and the choice of particular solution is dependent on the right hand sides. So, Eqs. (6.11)- (6.12) have particular solutions of the forms

$$B_2 = p_1 b + p_3 b^3 + p_5 b^5 + \cdots, \; C_2 = q_1 + q_3 b^2 + q_5 b^4 + \cdots. \tag{6.14}$$

Substituting these expressions of B_2 and C_2 into the resulting equations of (6.11)- (6.12) and by equating the coefficients of like powers of b, we obtain another set of algebraic equations which are able to give us the unknown coefficients p_1, p_3, p_5, q_1, q_3 and q_5. Then inserting the values of B_1, C_1, B_2 and C_2 into Eq.(6.5) and integrating it, we are able to know the amplitude b and the phase φ in terms of time t. Hence the determination of second order improved solution of Eq. (6.1) is completed. The method can also be applied to find the higher order approximations in a similar way.

6.3 Example

As an example of the above procedure, we may consider the following weakly nonlinear differential equation [27]

$$\dddot{x} + k_1(\tau)\ddot{x} + k_2(\tau)\dot{x} + k_3(\tau)x = \varepsilon x^3, \tag{6.15}$$

where $k_1(\tau) = \lambda + 2\mu$, $k_2(\tau) = 2\lambda\mu + \mu^2 + \omega^2$ and $k_3(\tau) = \lambda(\mu^2 + \omega^2)$. Hence $f^{(0)} = b^3(\frac{3}{4}\cos\varphi + \frac{1}{4}\cos 3\varphi)$ and the Fourier coefficients are $F_1 = \frac{3}{4}b^3$, $F_3 = \frac{1}{4}b^3$ and the rests are zero. Now inserting the values of F_1 and G_1 into Eqs. (6.9)- (6.10) and then solving them, we get

$$\begin{aligned} m_1 &= \frac{-\omega'((\lambda-\mu)^2 + 3\omega^2)}{2\omega((\lambda-\mu)^2 + \omega^2)}, \quad n_1 = \frac{-\omega'(\lambda-\mu)}{(\lambda-\mu)^2 + \omega^2} \\ m_3 &= \frac{3(\mu(-\lambda+3\mu) - \omega^2)}{8(\mu^2+\omega^2)((\lambda-3\mu)^2+\omega^2)}, \quad n_3 = \frac{3\omega(-\lambda+4\mu)}{8(\mu^2+\omega^2)((\lambda-3\mu)^2+\omega^2)}. \end{aligned} \tag{6.16}$$

According to our assumption, Eq. (6.16) can be written as

$$m_1 = \frac{3h}{2}, \quad n_1 = \frac{h(\lambda-\mu)}{\omega}, \quad m_3 = -\frac{3}{8\omega^2}, \quad n_3 = \frac{3(-\lambda+4\mu)}{8\omega^3} \tag{6.17}$$

Hence by using Eqs. (6.13) and (6.17) into Eqs. (6.11)- (6.12), we obtain

$$\begin{aligned} &\mu^2 b^2 \frac{\partial^2 B_2}{\partial b^2} - \lambda\mu b \frac{\partial B_2}{\partial b} + (\lambda\mu - 2\omega^2)B_2 + 3\mu\omega b^2 \frac{\partial C_2}{\partial b} - 2(\lambda-\mu)\omega b C_2 \\ &= h^2(\lambda-\mu)(\frac{27}{4} + \frac{(\lambda-\mu)^2}{\omega^2})b - 3h(\frac{(22\lambda - 65\mu)}{8\omega^2} + \frac{2(\lambda-\mu)(-\lambda+4\mu)^2}{8\omega^4})b^3 \\ &+ 9(\frac{9(\lambda-3\mu)}{64\omega^4} + \frac{(\lambda-7\mu)(-\lambda+4\mu)^2}{64\omega^6})b^5, \end{aligned} \tag{6.18}$$

and

$$3\mu\omega b\frac{\partial B_2}{\partial b}-(2\lambda+\mu)\omega B_2-\mu^2 b^3\frac{\partial^2 C_2}{\partial b^2}+(\lambda-2\mu)\mu b^2\frac{\partial C_2}{\partial b}+2\omega^2 bC_2$$
$$=h^2(\frac{13\omega}{4}+\frac{(\lambda-\mu)^2}{\omega})b-(\frac{63h}{8\omega}+\frac{15(-\lambda+4\mu)}{8\omega^3})b^3 \tag{6.19}$$
$$+9(\frac{9}{64\omega^3}-\frac{(\lambda-16\mu)(-\lambda+4\mu)}{64\omega^5})b^5.$$

Finally, by solving Eqs. (6.18)- (6.19), we obtain the unknown coefficients of B_2 and C_2 in the following forms

$$p_1=h^2(\lambda-\mu)(-\frac{5}{\omega^2}+\frac{19(\lambda-\mu)^2}{8\omega^4}-\frac{(\lambda-\mu)^5}{2\omega^6}),$$
$$q_1=h^2(\frac{13}{8\omega}-\frac{23(\lambda-\mu)^2}{8\omega^3}+\frac{(\lambda-\mu)^4}{2\omega^5}),$$
$$p_3=h(\frac{(129\lambda-447\mu)+3(\lambda-\mu)(-\lambda+4\mu)(22\lambda-65\mu)}{16\omega^4})$$
$$+3h(-\lambda+4\mu)^2(\frac{-5+2(\lambda-\mu)^2(-\lambda+4\mu)+2(\lambda-\mu)}{16\omega^6}),$$
$$q_3=h(\frac{-63+3(\lambda-\mu)(22\lambda-65\mu)}{16\omega^3})$$
$$+3h(-\lambda+4\mu)(\frac{-5+2(\lambda-\mu)^2(-\lambda+4\mu)}{16\omega^5}),$$
$$p_5=-\frac{81(\lambda-5\mu)}{64\omega^6}+\frac{9(\lambda-5\mu)(7\lambda^2-62\lambda\mu+109\mu^2)}{128\omega^8}$$
$$+\frac{9(\lambda-7\mu)^2(-\lambda+4\mu)^2}{128\omega^{10}},$$
$$q_5=\frac{81}{128\omega^5}-\frac{9(8\lambda^2-70\lambda\mu+125\mu^2)}{128\omega^7}-\frac{9(\lambda-7\mu)^2(-\lambda+4\mu)^2}{128\omega^9}. \tag{6.20}$$

Since the response of the product terms of λ and μ are very small according to our assumption, so Eq. (6.20) can be written as

$$p_1=-\frac{5h^2(\lambda-\mu)}{\omega^2},\qquad q_1=\frac{13h^2}{8\omega},$$
$$p_3=\frac{h(129\lambda-447\mu)}{16\omega^4},\qquad q_3=-\frac{63h}{16\omega^3}-\frac{15h(-\lambda+4\mu)}{16\omega^5}, \tag{6.21}$$
$$p_5=-\frac{81(\lambda-5\mu)}{64\omega^6},\qquad q_5=\frac{81}{128\omega^5}.$$

Now inserting the values of B_1, C_1, B_2 and C_2 into Eq. (6.5), we obtain the variational equations for the amplitude b and the phase φ in the following forms

$$\dot{b} = -\mu b + \varepsilon(m_1 b + m_3 b^3) + \varepsilon^2 (p_1 b + p_3 b^3 + p_5 b^5) + \cdots,$$
$$\dot{\varphi} = \omega(\tau) + \varepsilon(n_1 + n_3 b^2) + \varepsilon^2 (q_1 + q_3 b^2 + q_3 b^4) + \cdots, \tag{6.22}$$

where m_1, m_3, n_1, n_3, p_1, p_3, p_5, q_1, q_3 and q_5 are given by Eqs. (6.17) and (6.21). The variational Eq. (6.22) has not an exact solution. In general, it is solved by the numerical procedure [1-37,45,56,57,69,94,113]. In this situation the perturbation method facilitates the numerical technique. Numerically, it is advantageous to solve the variational equations instead of original equations because a large step size can be used in the integration. In the case of linear equation where $f = 0$ or $m_3 = n_3 = p_3 = p_5 = q_3 = q_5 = 0$, Eq. (6.22) may have an analytical solution. We are going to consider the special case of Eq. (6.15), where λ and μ are very small constants and ω is assumed to vary slowly with time t as $\omega(\tau) = \omega_0 e^{-h\tau}$, where ω_0 and h are constants. Hence Eq. (6.22) has the following solutions

$$b = b_0 \exp[(-\mu t + 1.5h\tau) + \frac{5\varepsilon h(\lambda - \mu)(1 - \exp(2h\tau))}{2\omega_0^{\;2}}],$$
$$\varphi = \varphi_0 + \frac{1}{\varepsilon h}\omega_0(1 - \exp(-h\tau)) + \frac{(8(\lambda - \mu) + 13\varepsilon h)(\exp(h\tau) - 1)}{8\omega_0}. \tag{6.23}$$

It is obvious to us that Eq. (6.23) can be reduced to generate solution (6.2) as $h \to 0+$, *i.e.*, in the case of constant coefficients. But if $\mu \to 0+$ and $h > 0$, the amplitude b increases with time t. Hence the solution of a linear differential equation with the variable coefficients may be unstable though its unperturbed solution is stable. Thus the second order improved solution of Eq. (6.15) for both linear and nonlinear cases is

$$x(t, \varepsilon) = b \cos \varphi, \tag{6.24}$$

where b and φ are given by Eq. (6.23) and $\mu > 0$ for the linear case, and b and φ are computed from Eq. (6.22) by numerical procedure [1-37,45,56,57,69,94,113] for the nonlinear case. The amplitude b and the phase φ change slowly with time t. Hence it requires the numerical calculation of a few numbers of points. On the contrary, a direct attempt to solve the Eq. (6.15) dealing with harmonic term in solution (6.24), it requires the numerical calculation of a great number of points. Often one is not interested in only the oscillating processes itself, *i.e.*, finding the x

in terms of t, but mainly in the behavior of the amplitude b and the phase φ, which as t increases characterize the oscillating processes [1-45].

6.4 Results and Discussions

For certain special cases, very simple analytical method has been developed to obtain the time response of a third order weakly nonlinear ordinary differential equation with slowly varying coefficients. In most of the nonlinear cases, the method depends on the numerical techniques. It is very well known that the order of errors of first approximate solution of a nonlinear differential equation with constant coefficients is ε^2. But in the case of varying coefficients, the order of errors also depends on time t. However, when the coefficients vary slowly with time t, the perturbation solution shows a good coincidence with the numerical solution (assumed to be exact). To illustrate this, we have already computed $x(t,\varepsilon)$ from Eq. (6.24) by taking different set of values of λ , μ, h and $\omega(\tau)=\omega_0 e^{-h\tau}$, $\omega_0=1$, $\varepsilon=0.1$ with the initial conditions $b_0=1.0$ and $\varphi_0=0$. A second solution of Eq. (6.15) for $f=x^3$ and the same values of λ, μ, ω and with the same initial conditions is computed by ***Runge-Kutta*** fourth order formula. All the results are shown in **Fig. 6.1 -Fig. 6.3** for $h=0.5$. From **Fig. 6.1** and Fig. **6.2,** it is seen that the perturbation results almost coincide with the numerical results. Moreover, in **Fig. 6.3**, the perturbation and the numerical solutions are compared for the corresponding linear case.

It is seen from Fig. **6.1** and Fig. **6.2** that the second approximate solutions obtained by the presented method are nearly identical with those obtained by the numerical procedure. From **Fig. 6.3**, it is again clear to us that when ω change rapidly, the perturbation results deviate greatly from numerical results. Hence our derived perturbation solutions show a good agreement with the numerical results when the coefficients of Eq. (6.15) are varying slowly with time t.

In order to find the numerical solution, we have used the following initial conditions to determine $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$

$$
\begin{aligned}
x(0) &= b_0 \cos\varphi_0, \\
\dot{x}(0) &= -\mu b_0 \cos\varphi_0 - \omega_0 b_0 \sin\varphi_0 + \varepsilon((m_1 + m_3 b_0^{\,2}) b_0 \cos\varphi_0 - (n_1 + n_3 b_0^{\,2}) b_0 \sin\varphi_0) \\
&\quad + \varepsilon^2((p_1 + p_3 b^2 + p_5 b^4) b_0 \cos\varphi_0 - (q_1 + q_3 b^2 + q_5 b^4) b_0 \sin\varphi_0), \\
\ddot{x}(0) &= (\mu^2 - \omega_0^{\,2}) b_0 \cos\varphi_0 + 2\mu\omega_0 b_0 \sin\varphi_0 + \varepsilon(-2(\mu(m_1 + 2m_3 b_0^{\,2}) \\
&\quad + \omega_0 (n_1 + n_3 b_0^{\,2})) b_0 \cos\varphi_0 + (h\omega_0 - 2\omega_0 (m_1 + m_3 b_0^{\,2}) \\
&\quad + 2\mu(n_1 + 2n_3 b_0^{\,2})) b_0 \sin\varphi_0) + \varepsilon^2((2hm_3 b_0^{\,2} + (m_1 + m_3 b_0^{\,2})(m_1 + 3m_3 b_0^{\,2}) \\
&\quad - (n_1 + n_3 b_0^{\,2})^2 - 2\mu(p_1 + 2p_3 b_0^{\,2} + 3p_5 b_0^{\,4}) - 2\omega_0 (q_1 + q_3 b^2 + q_5 b^4)) \\
&\quad \times b_0 \cos\varphi_0 - (h(n_1 + 3n_3 b_0^{\,2}) + 2(m_1 + m_3 b_0^{\,2})(n_1 + 2n_3 b_0^{\,2}) \\
&\quad + 2\omega_0 (p_1 + p_3 b^2 + p_5 b^4) - 2\mu(q_1 + 2q_3 b^2 + 3q_5 b^4)) b_0 \sin\varphi_0),
\end{aligned}
\tag{6.25}
$$

where b_0 and φ_0 are the initial values of b and φ. Usually $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$ are given, so that one can determine the initial amplitude b_0 and the initial phase φ_0 by solving the equations of initial conditions Eq. (6.25). On the other hand, if b_0 and φ_0 are given, one can also find $x(0)$, $\dot{x}(0)$ and $\ddot{x}(0)$ from the above equations.

6.5 Conclusion

Usually, second or higher order approximate solution is used for obtaining better results. In this chapter, a new kind of analytical technique has been presented for a third order nonlinear differential equation with slowly varying coefficients. A general formula is presented by the unified KBM method [1-37,45,56,57,69,94,113] to obtain the first approximate solution. It is very difficult to find the formula for second or higher order approximation and it is a laborious and tedious task. But sometimes the first approximate solution does not give us good results. Then we need to determine the second or higher order approximations. Here we have developed a new technique which is easy to carry on the subsequent calculations for the second or higher order approximate solutions and it also gives us better results than the first approximate solutions. Also this solution converges nicely to those obtained by the numerical procedure when the coefficients become constant (*i.e.*, $h \to 0$).

However, the second or higher order solution diverges faster than the lower order solution when the reduced frequency becomes small. It has been shown effectively and accurately that large classes of second approximations converge rapidly to the numerical solutions. Comparison is made between the solutions obtained by the perturbation method (dashed lines) and those obtained by the numerical procedure (solid line) in figures.

Fig. 6.1(a)

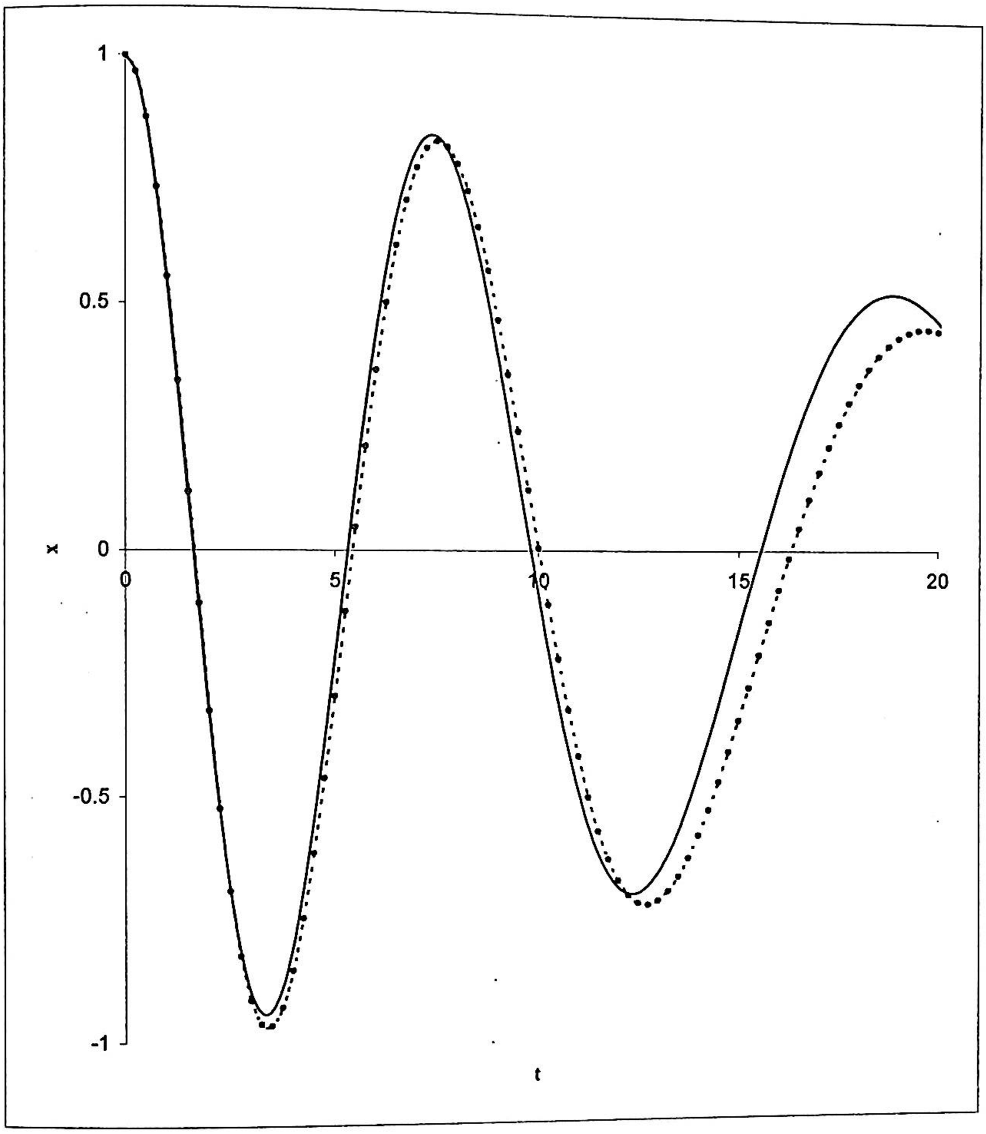

Fig. 6.1 (a) First approximate solution $-\bullet-$ (dashed lines) of Eq. (6.15) is compared with the corresponding numerical solution - (solid line) obtained by *Runge-Kutta* fourth-order formula for $\lambda = 0.15$, $\mu = 0.05$, $\omega_0 = 1.0$, $h = 0.5, \varepsilon = 0.1$ and $f = x^3$ when $[x(0) = 1.00000,\ \dot{x}(0) = -0.01250,\ \ddot{x}(0) = -1.01125]$ or $b_0 = 1.0$ and $\varphi_0 = 0$.

Fig. 6.1(b)

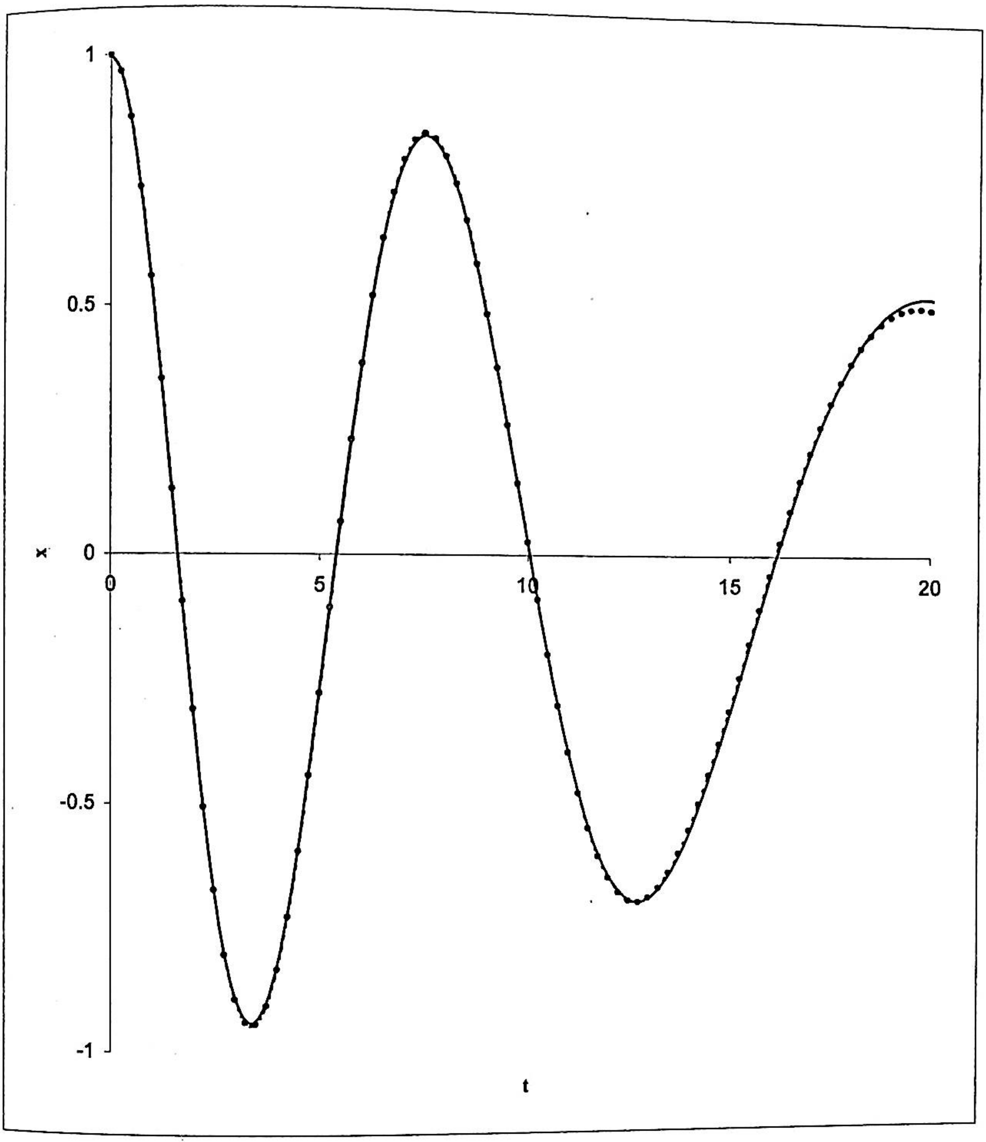

Fig. 6.1 (b) Second approximate solution $-\bullet-$ (dashed lines) of Eq. (6.15) is compared with the corresponding numerical solution - (solid lines) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.15$, $\mu = 0.05$, $\omega_0 = 1.0$, $h = 0.5$, $\varepsilon = 0.1$ and $f = x^3$ when $[x(0) = 1.00000,\ \dot{x}(0) = -0.01342,\ \ddot{x}(0) = -0.99746]$ or $b_0 = 1.0$ and $\varphi_0 = 0$.

Fig. 6.2(a)

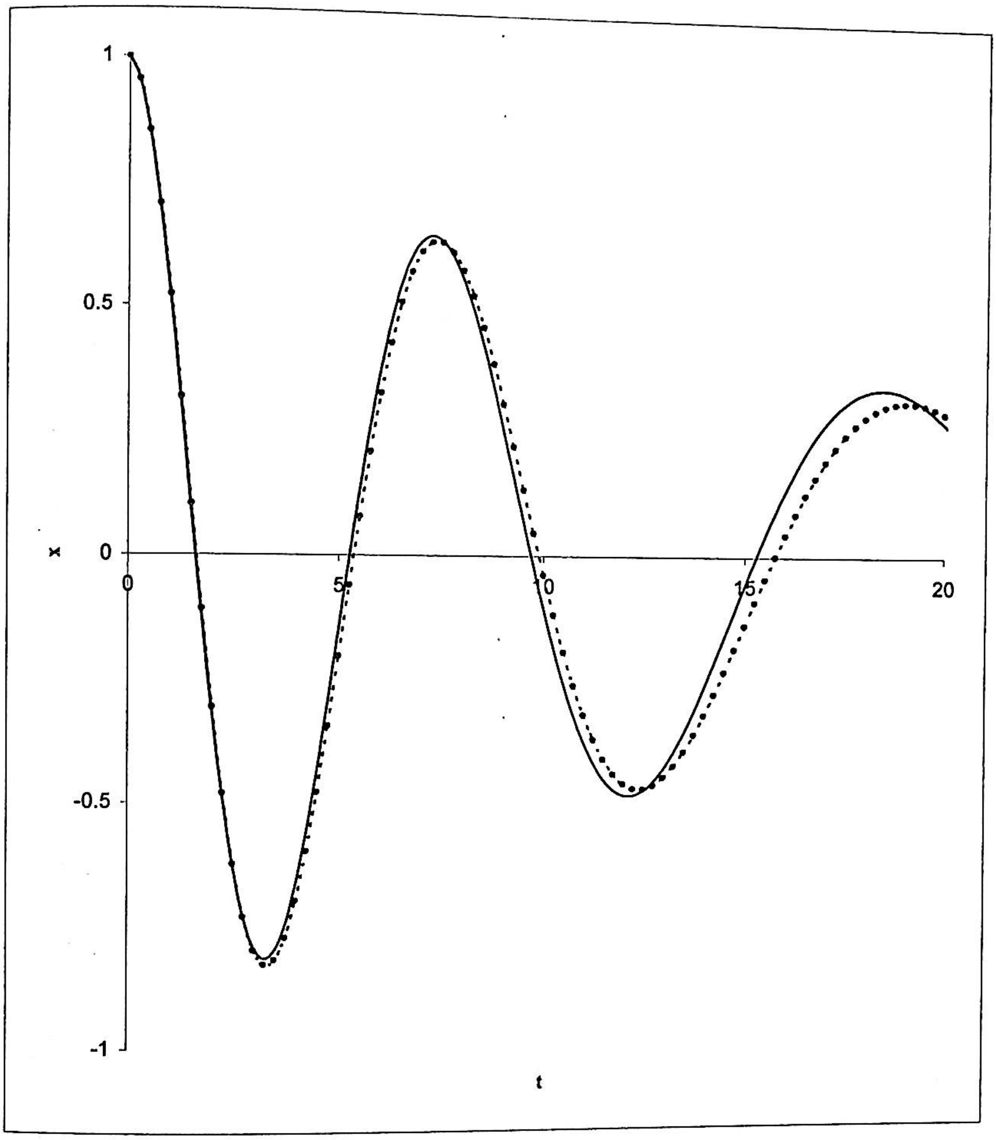

Fig. 6.2 (a) First approximate solution –•– (dashed lines) of Eq. (6.15) is compared with the corresponding numerical solution - (solid lines) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.3$, $\mu = 0.1$, $\omega_0 = 1.0$, $h = 0.5$, $\varepsilon = 0.1$ and $f = x^3$ when $[x(0) = 1.00000,\ \dot{x}(0) = -0.06250,\ \ddot{x}(0) = -1.01750]$ or $b_0 = 1.0$ and $\varphi_0 = 0$.

Fig. 6.2 (b)

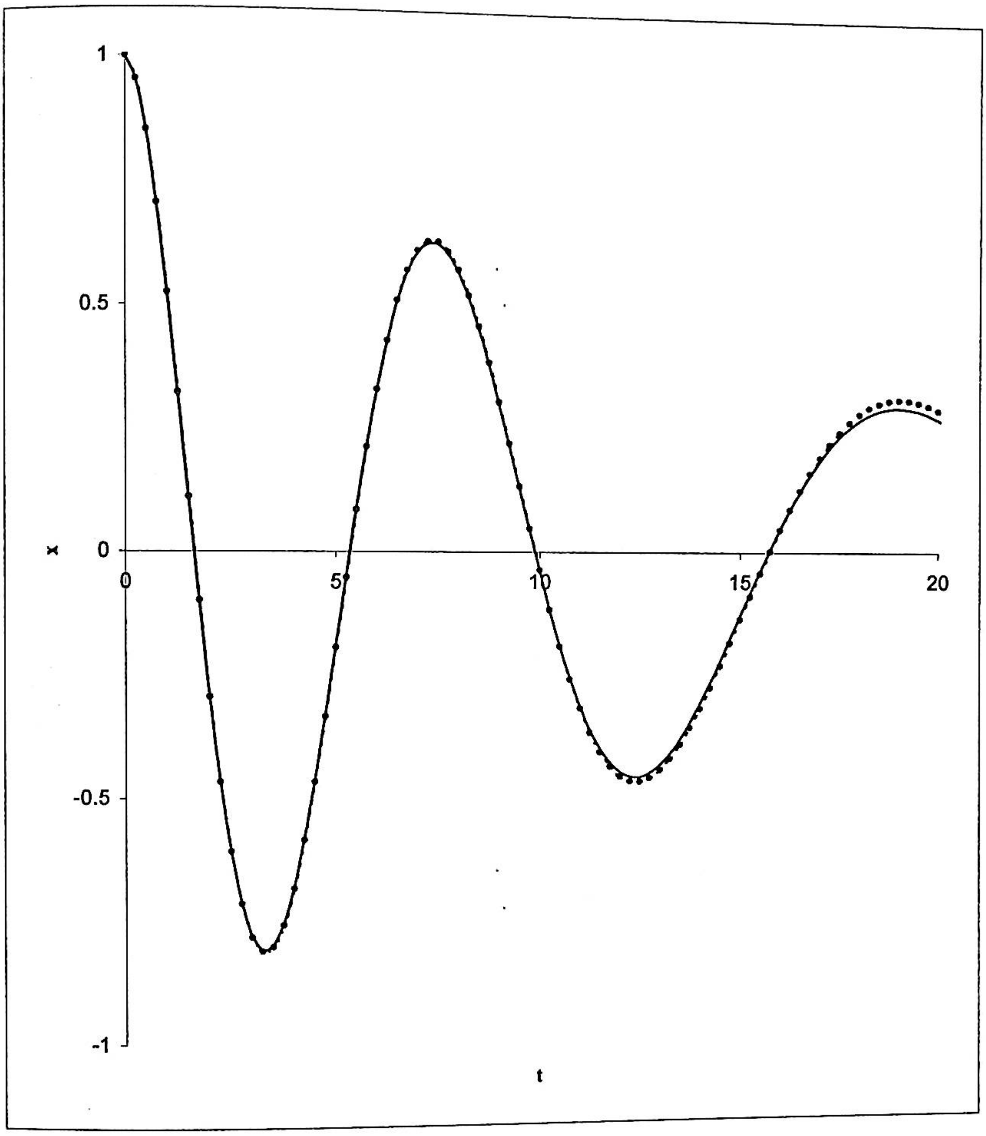

Fig. 6.2 (b) Second approximate solution –•– (dashed lines) of Eq. (6.15) is compared with the corresponding numerical solution - (solid lines) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.3$, $\mu = 0.1$, $\omega_0 = 1.0$, $h = 0.5, \varepsilon = 0.1$ and $f = x^3$ when $[x(0) = 1.00000,\ \dot{x}(0) = -0.06434,\ \ddot{x}(0) = -1.00358]$ or $b_0 = 1.0$ and $\varphi_0 = 0$.

Fig. 6.3 (a)

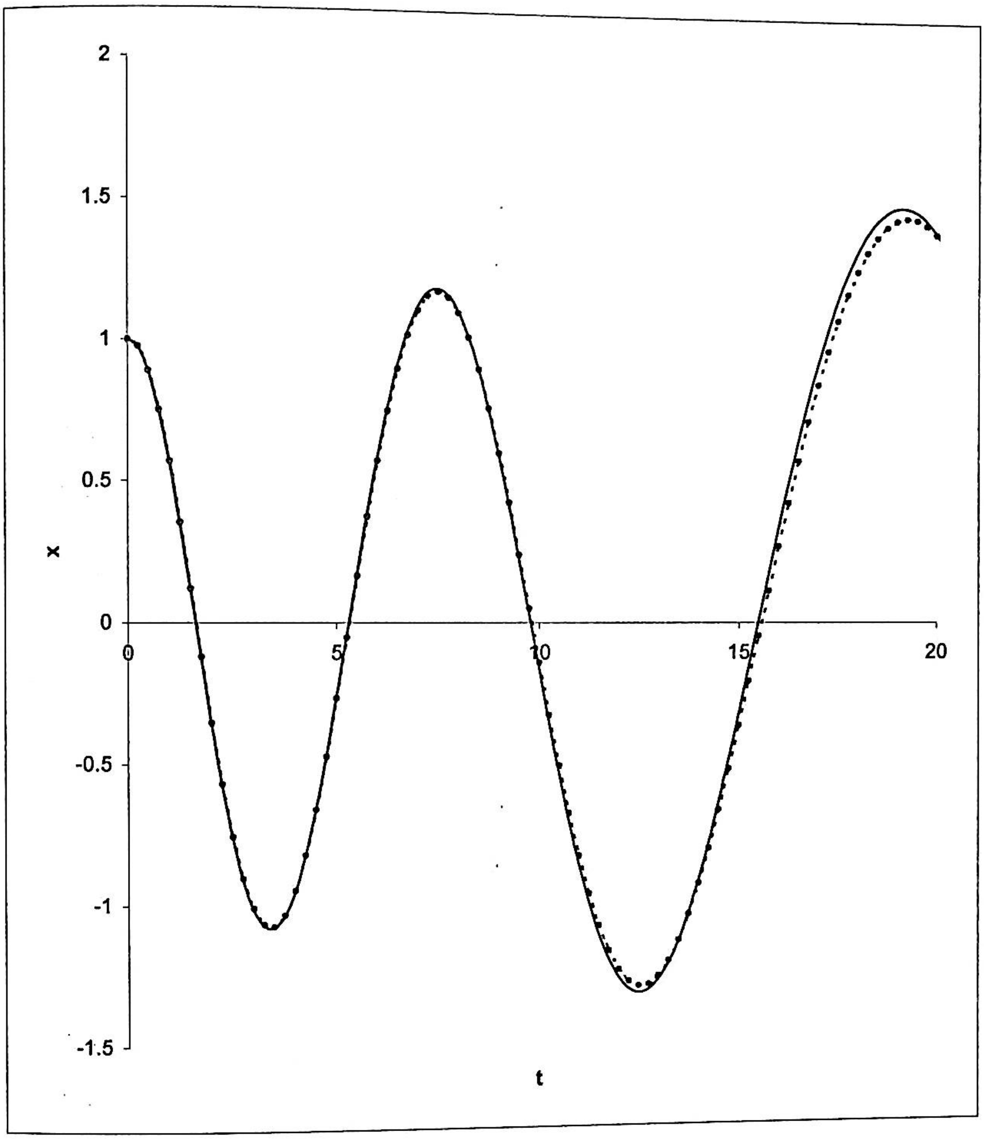

Fig. 6.3 (a) First approximate solution $-\bullet-$ (dashed lines) of Eq. (6.15) is compared with the corresponding numerical solution - (solid lines) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.15$, $\mu = 0.05$, $\omega_0 = 1.0$, $h = 0.5$, $\varepsilon = 0.1$ and $f = x^3$ when $[x(0) = 1.00000, \dot{x}(0) = 0.02500, \ddot{x}(0) = -1.01500]$ or $b_0 = 1.0$ and $\varphi_0 = 0$.

Fig. 6.3(b)

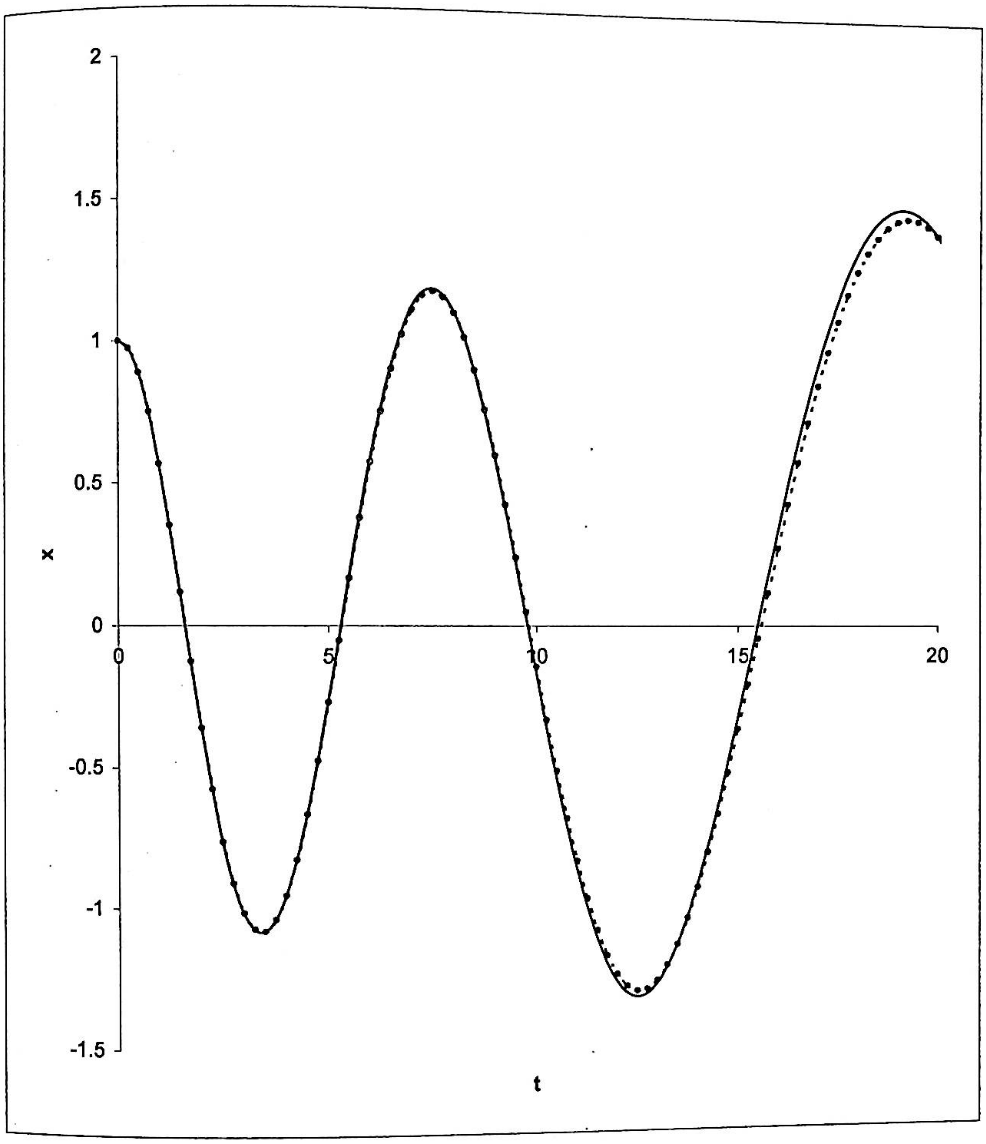

Fig. 6.3 (b) Second approximate solution $-\bullet-$ (dashed lines) of Eq. (6.15) is compared with the corresponding numerical solution - (solid lines) obtained by Runge-Kutta fourth-order formula for $\lambda = 0.15$, $\mu = 0.05$, $\omega_0 = 1.0$, $h = 0.5$, $\varepsilon = 0.1$ and $f = x^3$ when $[x(0) = 1.00000, \dot{x}(0) = 0.02375, \ddot{x}(0) = -1.01740]$ or $b_0 = 1.0$ and $\varphi_0 = 0$.

Chapter 7

An Approximate Solution of a Fourth Order Weakly Nonlinear Differential System with Strong Damping and Slowly Varying Coefficients by Unified KBM Method

7.1 Introduction

The method of KBM is convenient and one of the widely used techniques to obtain the analytical approximate solutions of nonlinear differential systems. It is perhaps noteworthy that because of importance of physical process involving damping, Popov [135] extended this method to damped oscillatory systems. Murty *et al.* [117] used Popov's method to obtained over-damped solutions of nonlinear differential equations, which were the basis of unified theory of Murty [115]. Later this method has been extended to damped oscillatory and purely non oscillatory systems with slowly varying coefficients by Bojadziev and Edwards [56]. Arya and Bojadziev [42] have studied a time-dependent nonlinear oscillatory system with damping, slowly varying coefficients and delay. Feshchenko *et al.* [69] have presented a brief way to determine KBM [45,94,113] solution (first order) of second and third order nonlinear differential systems. Arya and Bojadziev [41] have also studied a system of second order nonlinear hyperbolic partial differential equation with slowly varying coefficients. Alam [2] has investigated a unified Krylov-Bogoliubov-Mitropolskii method for solving nonlinear system of order $n \geq 2$. Further, Alam [16] has investigated a unified Krylov-Bogoliubov-Mitropolskii method for solving of second and third order nonlinear systems with constant coefficients. In another paper, Alam [19] has also investigated a unified Krylov-Bogoliubov-Mitropolskii method for solving nonlinear systems of order $n \geq 3$ with slowly varying coefficients. Recently Alam and Sattar [27] have also presented an asymptotic method for the third order nonlinear systems with slowly varying coefficients. Recently Akbar *et al.* [37] have studied a fourth order nonlinear differential equation with constant coefficients. Most of the authors have studied the second and third order nonlinear differential systems for both constant and varying coefficients to obtain the first order analytical approximate solutions. The complicated and no less important case of a fourth order nonlinear differential equation with strong damping and slowly varying coefficients has remained almost untouched. The aim of this chapter is to fill this gap.

7.2 The Method

Let us consider a fourth order weakly nonlinear ordinary differential equation with slowly varying coefficients in the following form

$$x^{(4)} + c_1(\tau)\dddot{x} + c_2(\tau)\ddot{x} + c_3(\tau)\dot{x} + c_4(\tau)x = \varepsilon f(x,\dot{x},\ddot{x},\dddot{x},\tau) \tag{7.1}$$

where the over dots represent the time derivatives, ε is a small positive parameter which measures the strength of the nonlinearity, $\tau = \varepsilon t$ is the slowly varying time, $c_j(\tau) \geq 0,\quad j = 1, 2, 3, 4$ are slowly varying coefficients and f is a given nonlinear function. The coefficients are slowly varying in the sense that their time derivatives are proportional to ε [19].

By setting $\varepsilon = 0$, $\tau = \tau_0 =$constant in Eq. (7.1), then we obtain the solution of the unperturbed equation. We assume that the unperturbed equation of Eq. (7.1) has four eigenvalues $\lambda_j(\tau_0)$, $j = 1, 2, 3, 4$, where $\lambda_j(\tau_0)$ are constants, but if $\varepsilon \neq 0$ then $\lambda_j(\tau)$ are varying slowly with time t. The solution of the linearized equation of Eq. (7.1) has the following form

$$x(t,0) = \sum_{j=1}^{4} a_{j,0} e^{\lambda_j(\tau_0)t}, \tag{7.2}$$

where $a_{j,0}$, $j = 1, 2, 3, 4$ are arbitrary constants.

Now we are going to choose a solution of Eq. (7.1) that reduces to Eq. (7.2) as a limit $\varepsilon \to 0$ in the following form according to the KBM [45,94,113] method

$$x(t,\varepsilon) = \sum_{j=1}^{4} a_j(t) + \varepsilon u_1(a_1,a_2,a_3,a_4,\tau) + \varepsilon^2 u_2(a_1,a_2,a_3,a_4,\tau) + \varepsilon^3 \cdots, \tag{7.3}$$

where u_1 is a function of a_j, $j = 1, 2, 3, 4$ and each a_j satisfies the following first order differential equation

$$\dot{a}_j = \lambda_j a_j + \varepsilon A_j(a_1,a_2,a_3,a_4,\tau) + \varepsilon^2 B_j(a_1,a_2,a_3,a_4,\tau) + \varepsilon^3 \cdots. \tag{7.4}$$

Confining only to the first few terms, 1,2,3... in the series expansions of Eq.(7.3) and Eq.(7.4), we evaluate the functions u_1, u_2 and $A_j, B_j, \cdots, j = 1, 2, 3, 4$

such that each $a_j(t)$ appearing in Eq.(7.3) and Eq.(7.4) satisfies the given differential equation (7.1) with an accuracy of ε^{m+1} [19]. In order to determine these functions it is assumed that the functions u_1, $u_2,\cdots$ do not contain the fundamental terms [2,19,56,117] which are included in the series expansions (7.3) at order ε^0. Now differentiating Eq. (7.3) four-times with respect to time t and using the relations Eq. (7.4) and by substituting the values of $x^{(4)}$, $\dddot{x}$, $\ddot{x}$, $\dot{x}$ and x into the original Eq. (7.1) with the slowly varying coefficients $c_1(\tau) = -(\lambda_1(\tau)+\lambda_2(\tau)+\lambda_3(\tau)+\lambda_4(\tau))$,
$c_2(\tau) = \lambda_1(\tau)\lambda_2(\tau)+\lambda_1(\tau)\lambda_3(\tau)+\lambda_1(\tau)\lambda_4(\tau)+\lambda_2(\tau)\lambda_3(\tau)+\lambda_2(\tau)\lambda_4(\tau)+\lambda_3(\tau)\lambda_4(\tau)$,
$c_3(\tau) = -(\lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau)+\lambda_1(\tau)\lambda_2(\tau)\lambda_4(\tau)+\lambda_1(\tau)\lambda_3(\tau)\lambda_4(\tau)+\lambda_2(\tau)\lambda_3(\tau)\lambda_4(\tau))$
and $c_4(\tau) = \lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau)\lambda_4(\tau)$ and expanding the right hand side of Eq. (7.1) by Taylor series and equating the coefficients of ε on both sides, we obtain the following equation

$$\prod_{j=1}^{4}(\Omega-\lambda_j)u_1+\sum_{j=1}^{4}\Big(\prod_{k=1,k\neq j}^{4}(\Omega-\lambda_k)A_j\Big)+\sum_{j=1}^{4}\frac{1}{2}\Big(\sum_{k=0}^{2}(4-k)(3-k)c_k\lambda_j^{(2-k)}\Big)\lambda_j' a_j \qquad (7.5)$$
$$= f^{(0)}(a_1,a_2,a_3,a_4,\tau),$$

where

$$\Omega=\sum_{j=1}^{4}\lambda_j a_j\frac{\partial}{\partial a_j},\ \lambda_j'=\frac{d\lambda_j}{d\tau}, j=1.2,\cdots 4,\ f^{(0)}(a_1,a_2,a_3,a_4,\tau)= f(x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0,\tau) \text{ and}$$

$$x_0=\sum_{j=1}^{4}a_j.$$

We have already assumed that u_1 does not contain the fundamental terms and for this reason the solution will be free from secular terms, namely $t\cos t$, $t\sin t$ and te^{-t}. Under these restrictions, we are able to solve Eq. (7.5) by separating this into five individual equations for the unknown functions u_1 and A_j. In general, the functions $f^{(0)}$ and u_1 are expanded in Taylor's series in the following forms [19]

$$f^{(0)} = \sum_{m_1=0,m_2=0,m_3=0m_4=0}^{\infty,\infty,\infty,\infty} F_{m_1,m_2,m_3,m_4}(\tau)a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}, \qquad (7.6)$$

$$u_1 = \sum_{m_1=0,m_2=0,m_3=0m_4=0}^{\infty,\infty,\infty,\infty} U_{m_1,m_2,m_3,m_4}(\tau)a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}. \qquad (7.7)$$

The eigen-values of the unperturbed equation can be written as $-\mu_l(\tau_0) \pm i\omega_l(\tau_0)$, where $l = 1, 2$. For the above restrictions, it guaranties that u_1 must exclude all terms with $a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}$ of $f^{(0)}$,.where $m_{2l-1} - m_{2l} = \pm 1$. Since according to the linear approximation (*i.e.* $\varepsilon \to 0$), $a_{2l-1}{}^{m_{2l-1}} a_{2l}{}^{m_{2l}}$ becomes $e^{\omega_l t}$ when $m_{2l-1} - m_{2l} = 1$ or $e^{-\omega_l t}$ when $m_{2l-1} - m_{2l} = -1$. It is noticed that $e^{\pm\omega_l t}$ are known as the fundamental terms [2,19,56,117]. Usually these are included in equations A_j. Also, it is restricted [45,94,113] that the functions A_j are independent of the fundamental terms.

Then the equations for u_1 and A_j, $j = 1,2,3,4$ are written as

$$\prod_{j=1}^{4}(\Omega - \lambda_j)u_1 = \sum_{m_{2l-1}=0, m_{2l}=0}^{\infty,\infty} F_{,m_{2l-1},m_{2l}}(\tau) a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}, \quad m_{2l-1} - m_{2l} \neq 0, \pm 1 \tag{7.8}$$

and

$$\begin{aligned}(\prod_{k=1,k\neq 2l-1}^{4}(\Omega - \lambda_k))A_{2l-1} + \frac{1}{2}(\sum_{k=0}^{2}(4-k)(3-k)c_k \lambda_{2l-1}{}^{(2l-k-2)})\lambda_{2l-1}' a_{2l-1} \\ = \sum_{m_{2l-1}=0, m_{2l}=0}^{\infty,\infty} F_{m_{2l-1}m_{2l}} a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}, \quad m_{2l-1} - m_{2l} = 1,\end{aligned} \tag{7.9}$$

$$\begin{aligned}(\prod_{k=1,k\neq 2l}^{4}(\Omega - \lambda_k))A_{2l} + \frac{1}{2}(\sum_{k=0}^{2}(4-k)(3-k)c_k \lambda_{2l}{}^{(2l-k-2)})\lambda_{2l}' a_{2l} \\ = \sum_{m_{2l-1}=0, m_{2l}=0}^{\infty,\infty} F_{m_{2l-1}m_{2l}} a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}, \quad m_{2l-1} - m_{2l} = -1.\end{aligned} \tag{7.10}$$

To determine the particular solutions of Eqs. (7.8) - (7.10), we have to replace the operator Ω by $\sum_{j=1}^{4} m_j \lambda_j$, since we know that $\Omega(a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}}) = \sum_{j=1}^{n} m_j \lambda_j (a_{2l-1}^{m_{2l-1}} a_{2l}^{m_{2l}})$. Hence the determination of the first order analytical approximate solution of Eq. (7.1) is obtained. We notice that the solution Eq. (7.3) is not a standard form of the KBM method and is presented in terms of some unusual variables. Therefore, the solution obtained by formula of Eq. (7.1) is transformed to the formal form by replacing the unusual variables by amplitudes and phases variables in the forms

$$a_{2l-1} = \frac{1}{2} b_l e^{i\varphi_l}, \qquad a_{2l} = \pm\frac{1}{2} b_l e^{-i\varphi_l}, \; l = 1, 2 \tag{7.11}$$

Thus the first order approximate solution of Eq. (7.1) can be found as a standard form of the KBM method. The method can be carried out to higher order approximations in a similar way. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first order [1-37,45,94,113].

7.3 Example

To obtain the practical working of the above method, we consider the following fourth order weakly nonlinear differential equation with slowly varying coefficients in the following form

$$x^{(4)} + c_1(\tau)\dddot{x} + c_2(\tau)\ddot{x} + c_3(\tau)\dot{x} + c_4(\tau)x = \varepsilon x^3, \tag{7.12}$$

where $f(x, \dot{x}, \ddot{x}, \dddot{x}, \tau) = x^3$ and $x_0 = a_1 + a_2 + a_3 + a_4$.

Now

$$\begin{aligned} f^{(0)} = {} & a_1^3 + a_2^3 + a_3^3 + a_4^3 + 3(a_1^2 a_2 + a_1 a_2^2 + a_1^2 a_3 + 2a_1 a_2 a_3 + a_2^2 a_3 + a_1^2 a_4 + 2a_1 a_2 a_4 \\ & + a_2^2 a_4 + a_1 a_3^2 + 2a_1 a_3 a_4 + a_1 a_4^2 + a_2 a_3^2 + 2a_2 a_3 a_4 + a_2 a_4^2 + a_3^2 a_4 + a_3 a_4^2). \end{aligned} \tag{7.13}$$

Substituting the values of $f^{(0)}$ in Eq. (7.5) and according to the above restrictions, we obtain five equations for A_1, A_2, A_3, A_4 and u_1 whose solutions are respectively given by

$$\begin{aligned} A_1 = {} & -\frac{(3\lambda_1^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} \\ & + \frac{3a_1^2 a_2}{2\lambda_1(2\lambda_1 + \lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - \lambda_4)} + \frac{3a_1 a_3^2}{(\lambda_1 + \lambda_3)(\lambda_1 + 2\lambda_3 - \lambda_2)(\lambda_1 + 2\lambda_3 - \lambda_4)} \\ & + \frac{3a_1 a_4^2}{(\lambda_1 + \lambda_4)(\lambda_1 + 2\lambda_4 - \lambda_2)(\lambda_1 + 2\lambda_4 - \lambda_3)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2)}, \end{aligned}$$

$$\begin{aligned} A_2 = {} & -\frac{(3\lambda_2^2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 - 2\lambda_2\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4)\lambda_2' a_2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} \\ & + \frac{3a_1 a_2^2}{2\lambda_2(\lambda_1 + 2\lambda_2 - \lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_4)} + \frac{3a_2 a_3^2}{(\lambda_2 + \lambda_3)(\lambda_2 + 2\lambda_3 - \lambda_1)(\lambda_2 + 2\lambda_3 - \lambda_4)} \\ & + \frac{3a_2 a_4^2}{(\lambda_2 + \lambda_4)(\lambda_2 + 2\lambda_4 - \lambda_1)(\lambda_2 + 2\lambda_4 - \lambda_3)} + \frac{6a_2 a_3 a_4}{(\lambda_2 + \lambda_3)(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4 - \lambda_1)}, \end{aligned}$$

$$A_3 = -\frac{(3\lambda_3^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\lambda_3' a_3}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)}$$
$$+ \frac{3a_1^2 a_3}{(\lambda_1 + \lambda_3)(2\lambda_1 + \lambda_3 - \lambda_2)(2\lambda_1 + \lambda_3 - \lambda_4)} + \frac{3a_2^2 a_3}{(\lambda_2 + \lambda_3)(2\lambda_2 + \lambda_3 - \lambda_1)(2\lambda_2 + \lambda_3 - \lambda_4)}$$
$$+ \frac{3a_3^2 a_4}{2\lambda_3(2\lambda_3 + \lambda_4 - \lambda_1)(2\lambda_3 + \lambda_4 - \lambda_2)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)},$$

$$A_4 = -\frac{(3\lambda_4^2 - 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda_4' a_4}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)}$$
$$+ \frac{3a_1^2 a_4}{(\lambda_1 + \lambda_4)(2\lambda_1 + \lambda_4 - \lambda_2)(2\lambda_1 + \lambda_4 - \lambda_3)} + \frac{3a_2^2 a_4}{(\lambda_2 + \lambda_4)(2\lambda_2 + \lambda_4 - \lambda_1)(2\lambda_2 + \lambda_4 - \lambda_3)}$$
$$+ \frac{3a_3 a_4^2}{2\lambda_4(\lambda_3 + 2\lambda_4 - \lambda_1)(\lambda_3 + 2\lambda_4 - \lambda_2)} + \frac{6a_1 a_2 a_4}{(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 + \lambda_4 - \lambda_3)},$$

(7.14)

and

$$u_1 = C_1 a_1^3 + D_1 a_2^3 + C_2 a_3^3 + D_2 a_4^3, \tag{7.15}$$

where

$$C_1 = \frac{1}{2\lambda_1(3\lambda_1 - \lambda_2)(3\lambda_1 - \lambda_3)(3\lambda_1 - \lambda_4)}, \quad D_1 = \frac{1}{2\lambda_2(3\lambda_2 - \lambda_1)(3\lambda_2 - \lambda_3)(3\lambda_2 - \lambda_3)},$$
$$C_2 = \frac{1}{2\lambda_3(3\lambda_3 - \lambda_1)(3\lambda_3 - \lambda_2)(3\lambda_3 - \lambda_4)}, \quad D_2 = \frac{1}{2\lambda_4(3\lambda_4 - \lambda_1)(3\lambda_4 - \lambda_2)(3\lambda_4 - \lambda_3)}.$$

(7.16)

Now by substituting the values of A_1, A_2, A_3 and A_4 from Eq. (7.14) into Eq. (7.4), we get

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\Big(-\frac{(3\lambda_1^2 - 2\lambda_1\lambda_2 - 2\lambda_1\lambda_3 - 2\lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4)\lambda_1' a_1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}$$
$$+ \frac{3a_1^2 a_2}{2\lambda_1(2\lambda_1 + \lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - \lambda_4)} + \frac{3a_1 a_3^2}{(\lambda_1 + \lambda_3)(\lambda_1 + 2\lambda_3 - \lambda_2)(\lambda_1 + 2\lambda_3 - \lambda_4)}$$
$$+ \frac{3a_1 a_4^2}{(\lambda_1 + \lambda_4)(\lambda_1 + 2\lambda_4 - \lambda_2)(\lambda_1 + 2\lambda_4 - \lambda_3)} + \frac{6a_1 a_3 a_4}{(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2)}\Big),$$

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon(-\frac{(3\lambda_2^2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 - 2\lambda_2\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4)\lambda_2' a_2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}$$
$$+\frac{3a_1a_2^2}{2\lambda_2(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_4)}+\frac{3a_2a_3^2}{(\lambda_2+\lambda_3)(\lambda_2+2\lambda_3-\lambda_1)(\lambda_2+2\lambda_3-\lambda_4)}$$
$$+\frac{3a_2a_4^2}{(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4-\lambda_1)(\lambda_2+2\lambda_4-\lambda_3)}+\frac{6a_2a_3a_4}{(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)(\lambda_2+\lambda_3+\lambda_4-\lambda_1)}),$$

$$\dot{a}_3 = \lambda_3 a_3 + \varepsilon(-\frac{(3\lambda_3^2 - 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4)\lambda_3' a_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)}$$
$$+\frac{3a_1^2a_3}{(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)(2\lambda_1+\lambda_3-\lambda_4)}+\frac{3a_2^2a_3}{(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)(2\lambda_2+\lambda_3-\lambda_4)}$$
$$+\frac{3a_3^2a_4}{2\lambda_3(2\lambda_3+\lambda_4-\lambda_1)(2\lambda_3+\lambda_4-\lambda_2)}+\frac{6a_1a_3a_4}{(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)}),$$

$$\dot{a}_4 = \lambda_4 a_4 + \varepsilon(-\frac{(3\lambda_4^2 - 2\lambda_1\lambda_4 - 2\lambda_2\lambda_4 - 2\lambda_3\lambda_4 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda_4' a_4}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}$$
$$+\frac{3a_1^2a_4}{(\lambda_1+\lambda_4)(2\lambda_1+\lambda_4-\lambda_2)(2\lambda_1+\lambda_4-\lambda_3)}+\frac{3a_2^2a_4}{(\lambda_2+\lambda_4)(2\lambda_2+\lambda_4-\lambda_1)(2\lambda_2+\lambda_4-\lambda_3)}$$
$$+\frac{3a_3a_4^2}{2\lambda_4(\lambda_3+2\lambda_4-\lambda_1)(\lambda_3+2\lambda_4-\lambda_2)}+\frac{6a_1a_2a_4}{(\lambda_1+\lambda_4)(\lambda_2+\lambda_4)(\lambda_1+\lambda_2+\lambda_4-\lambda_3)}).$$

(7.17)

For a damped solution of Eq. (7.12), we may substitute $\lambda_{1,2} = -\mu_1(\tau) \pm i\omega_1(\tau)$, $\lambda_{3,4} = -\mu_2(\tau) \pm i\omega_2(\tau)$ and using the transformation equations $a_1 = \frac{1}{2}ae^{i\varphi_1}$, $a_2 = \frac{1}{2}ae^{-i\varphi_1}$, $a_3 = \frac{1}{2}be^{i\varphi_2}$, $a_4 = \frac{1}{2}be^{-i\varphi_2}$ into Eq. (7.17) and by separating the real and imaginary parts, we obtain the following variational equations for the amplitudes and phases variables:

$$\begin{aligned}
\dot{a} &= -\mu_1 a + \varepsilon(l_1 a + l_2 a^3 + l_3 ab^2 + ab^2(E_1\cos 2\varphi_2 + F_1 \sin 2\varphi_2)),\\
\dot{\varphi}_1 &= \omega_1(\tau) + \varepsilon(m_1 + m_2 a^2 + m_3 b^2 + b^2(E_2 \cos 2\varphi_2 + F_2 \sin 2\varphi_2)),\\
\dot{b} &= -\mu_2 b + \varepsilon(p_1 b + p_2 a^2 b + p_3 b^3 + a^2 b(E_3 \cos 2\varphi_1 + F_3 \sin 2\varphi_1)),\\
\dot{\varphi}_2 &= \omega_2(\tau) + \varepsilon(q_1 + q_2 a^2 + q_3 b^2 + a^2(E_4 \cos 2\varphi_1 + F_4 \sin 2\varphi_1)),
\end{aligned} \quad (7.18)$$

and the first correction term u_1 is obtained as

$$u_1 = a^3(c_1 \cos 3\varphi_1 + d_1 \sin 3\varphi_1) + b^3(c_2 \cos 3\varphi_2 + d_2 \sin 3\varphi_2), \quad (7.19)$$

where

$$l_1=-\frac{\begin{array}{l}(\omega_1'(((\mu_1-\mu_2)^2-5\omega_1^2+\omega_2^2)((\mu_1-\mu_2)^2-\omega_1^2+\omega_2^2)\\+12\omega_1^2(\mu_1-\mu_2)^2)+4\mu_1'\omega_1(\mu_1-\mu_2)((\mu_1-\mu_2)^2+\omega_1^2+\omega_2^2))\end{array}}{2\omega_1((\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$m_1=\frac{\begin{array}{l}4\omega_1\omega_1'(\mu_1-\mu_2)((\mu_1-\mu_2)^2+\omega_1^2+\omega_2^2)-\mu_1'(((\mu_1-\mu_2)^2\\-5\omega_1^2+\omega_2^2)((\mu_1-\mu_2)^2-\omega_1^2+\omega_2^2)+12\omega_1^2(\mu_1-\mu_2)^2)\end{array}}{2\omega_1((\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$l_2=-\frac{3(\mu_1((3\mu_1-\mu_2)^2-\omega_1^2+\omega_2^2)-2\omega_1^2(3\mu_1-\mu_2))}{8(\mu_1^2+\omega_1^2)((3\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$m_2=-\frac{3\omega_1((3\mu_1-\mu_2)(5\mu_1-\mu_2)-\omega_1^2+\omega_2^2)}{8(\mu_1^2+\omega_1^2)((3\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$l_3=-\frac{3(\mu_2((\mu_1+\mu_2)^2-\omega_1^2+\omega_2^2)-2\omega_2^2(\mu_1+\mu_2))}{4(\mu_2^2+\omega_1^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$m_3=-\frac{3\omega_1((\mu_1+\mu_2)(\mu_1+3\mu_2)-\omega_1^2+\omega_2^2)}{4(\mu_2^2+\omega_1^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$l_4=-\frac{3(\mu_2((\mu_1+\mu_2)^2-(\omega_1+\omega_2)(\omega_1+3\omega_2))-2(\mu_1+\mu_2)(\omega_1+\omega_2)(\omega_1+2\omega_2))}{8(\mu_2^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+3\omega_2)^2)},$$

$$l_4^*=\frac{3(2\mu_2(\mu_1+\mu_2)(\omega_1+2\omega_2)+(\omega_1+\omega_2)((\mu_1+\mu_2)^2-(\omega_1+\omega_2)(\omega_1+3\omega_2)))}{8(\mu_2^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+3\omega_2)^2)},$$

$$m_4=-l_4^*, m_4^*=l_4,$$

$$l_5=-\frac{3(\mu_2((\mu_1+\mu_2)^2-(\omega_1-\omega_2)(\omega_1-3\omega_2))-2(\mu_1+\mu_2)(\omega_1-\omega_2)(\omega_1-2\omega_2))}{8(\mu_2^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-3\omega_2)^2)},$$

$$l_5^*=-\frac{3(2\mu_2(\mu_1+\mu_2)(\omega_1-2\omega_2)+(\omega_1-\omega_2)((\mu_1+\mu_2)^2-(\omega_1-\omega_2)(\omega_1-3\omega_2)))}{8(\mu_2^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-3\omega_2)^2)},$$

$$m_5=l_5^*, m_4^*=-l_5,$$

$$p_1=-\frac{\begin{array}{l}(\omega_2'(((\mu_1-\mu_2)^2+\omega_1^2-\omega_2^2)((\mu_1-\mu_2)^2+\omega_1^2-5\omega_2^2)+12\omega_2^2(\mu_1-\mu_2)^2)\\-4\mu_2'\omega_2(\mu_1-\mu_2)((\mu_1-\mu_2)^2+\omega_1^2+\omega_2^2))\end{array}}{2\omega_2((\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$q_1=-\frac{\begin{array}{l}(4\omega_2\omega_2'(\mu_1-\mu_2)((\mu_1-\mu_2)^2+\omega_1^2+\omega_2^2)+\mu_2'(((\mu_1-\mu_2)^2\\+\omega_1^2-\omega_2^2)((\mu_1-\mu_2)^2+\omega_1^2-5\omega_2^2)+12\omega_2^2(\mu_1-\mu_2)^2)\end{array}}{2\omega_2((\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$p_2 = -\frac{3(\mu_1((\mu_1+\mu_2)^2+\omega_1^2-\omega_2^2)-2\omega_2^2(\mu_1+\mu_2))}{4(\mu_1^2+\omega_2^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)},$$

$$q_2 = -\frac{3\omega_2((\mu_1+\mu_2)(3\mu_1+\mu_2)+\omega_1^2-\omega_2^2)}{4(\mu_1^2+\omega_2^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)},$$

$$p_3 = -\frac{3(\mu_2((\mu_1-3\mu_2)^2+\omega_1^2-\omega_2^2)+2\omega_2^2(\mu_1-3\mu_2))}{8(\mu_2^2+\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$q_3 = -\frac{3\omega_2((\mu_1-3\mu_2)(\mu_1-5\mu_2)+\omega_1^2-\omega_2^2)}{8(\mu_2^2+\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+\omega_2)^2)},$$

$$p_4 = -\frac{3(\mu_1((\mu_1+\mu_2)^2-(\omega_1+\omega_2)(3\omega_1+\omega_2))-2(\mu_1+\mu_2)(\omega_1+\omega_2)(2\omega_1+\omega_2))}{8(\mu_1^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$p_4^* = \frac{3(2\mu_1(\mu_1+\mu_2)(2\omega_1+\omega_2)+(\omega_1+\omega_2)((\mu_1+\mu_2)^2-(\omega_1+\omega_2)(3\omega_1+\omega_2)))}{8(\mu_1^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2)((\mu_1+\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$q_4 = -p_4^*, q_4^* = p_4,$$

$$p_5 = -\frac{3(\mu_1((\mu_1+\mu_2)^2-(\omega_1-\omega_2)(3\omega_1-\omega_2))-2(\mu_1+\mu_2)(\omega_1-\omega_2)(2\omega_1-\omega_2))}{8(\mu_1^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(3\omega_1-\omega_2)^2)},$$

$$p_5^* = \frac{3(2\mu_1(\mu_1+\mu_2)(2\omega_1-\omega_2)+(\omega_1-\omega_2)((\mu_1+\mu_2)^2-(\omega_1-\omega_2)(3\omega_1-\omega_2)))}{8(\mu_1^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2)((\mu_1+\mu_2)^2+(3\omega_1-\omega_2)^2)},$$

$$q_5 = p_5^*, q_5^* = -p_5,$$

$$E_1 = l_4 + l_5,\ F_1 = l_4^* + l_5^*,\ E_2 = m_4 + m_5,\ F_2 = m_4^* + m_5^*,$$
$$E_3 = p_4 + p_5,\ F_3 = p_4^* + p_5^*,\ E_4 = q_4 + q_5,\ F_4 = q_4^* + q_5^*, \tag{7.20}$$

and

$$c_1 = \frac{(\mu_1^2-2\omega_1^2)((3\mu_1-\mu_2)^2-9\omega_1^2+\omega_2^2)-18\mu_1\omega_1^2(3\mu_1-\mu_2)}{16(\mu_1^2+\omega_1^2)(\mu_1^2+4\omega_1^2)((3\mu_1-\mu_2)^2+(3\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$d_1 = -\frac{3\omega_1(\mu_1((3\mu_1-\mu_2)^2-9\omega_1^2+\omega_2^2)+2(3\mu_1-\mu_2)(\mu_1^2-2\omega_1^2))}{16(\mu_1^2+\omega_1^2)(\mu_1^2+4\omega_1^2)((3\mu_1-\mu_2)^2+(3\omega_1-\omega_2)^2)((3\mu_1-\mu_2)^2+(3\omega_1+\omega_2)^2)},$$

$$c_2 = \frac{(\mu_2^2-2\omega_2^2)((\mu_1-3\mu_2)^2+\omega_1^2-9\omega_2^2)+18\mu_2\omega_2^2(\mu_1-3\mu_2)}{16(\mu_2^2+\omega_2^2)(\mu_2^2+4\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-3\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+3\omega_2)^2)},$$

$$d_2 = -\frac{3\omega_2(\mu_2((\mu_1-3\mu_2)^2+\omega_1^2-9\omega_2^2)-2(\mu_1-3\mu_2)(\mu_2^2-2\omega_2^2))}{16(\mu_2^2+\omega_2^2)(\mu_2^2+4\omega_2^2)((\mu_1-3\mu_2)^2+(\omega_1-3\omega_2)^2)((\mu_1-3\mu_2)^2+(\omega_1+3\omega_2)^2)}. \tag{7.21}$$

Thus the first order analytical approximate solution (improved) of Eq. (7.12) is obtained by

$$x(t,\varepsilon) = a\cos\varphi_1 + b\cos\varphi_2 + \varepsilon u_1, \tag{7.22}$$

where the amplitudes a, b and the phases φ_1, φ_2 are the solutions of Eq. (7.18) and u_1 is given by Eq. (7.19).

7.4 Results and Discussions

A standard form of the KBM method is presented to obtain the analytical approximate solution of a fourth order nonlinear differential equation with strong damping and slowly varying coefficients with small nonlinearity. The KBM method was originally developed for obtaining the periodic solutions of second-order nonlinear systems by Krylov and Bogoliubov [45] and later it was amplified and justified by Bogoliubov and Mitropolskii [94]. The method is not only limited to second-order nonlinear problems, but also useful in third-order [16,19] and fourth order [37] nonlinear systems. A general solution has been found for the damped nonlinear differential equation with slowly varying coefficients based on the unified KBM [2,16,19,45,94,113,117] method.

We have solved four simultaneous differential equations for amplitude(s) and phase(s) variables and a partial differential equation for u_1 involving four independent variables of amplitudes and phases. Also we are able to solve all the equations of A_j, $j = 1, 2, 3, 4$ and u_1 by a unified formula. In a particular case, we are forced to assume that $\mu_l(\tau)$, $l = 1,2$ are constants, $\omega_1(\tau) = 2\,\omega_2(\tau)$ and $\omega_2(\tau) = \omega_0 e^{-h\tau}$ are varying slowly with time t, where ω_0 and h are constants. Figures are drawn to compare between the first order analytical approximate solutions obtained by the perturbation method and those obtained by the fourth-order ***Runge-Kutta*** method for several damping effects. Moreover this method is able to give the required results when the coefficients of the given nonlinear differential equation become constants $(h = 0)$. From the **Figs. (1)- (2)**, it is seen that the new analytical approximate solutions show a good agreement with the corresponding numerical solutions (considered to be exact).

7.5 Conclusion

A unified KBM [2,16,19,45,94,113,117] method is presented to obtain the analytical approximate solutions of fourth order nonlinear differential systems with strong damping and slowly varying coefficients with small nonlinearity. The later form of the solution of Eq. (7.12) is presented in terms of amplitudes and phases variables. This form is very important in physical problems, since amplitudes and phases variables characterize the oscillating processes. Moreover the variational equations of amplitudes and phases variables are important to investigate the stability of differential systems. In general, the variational equations for the amplitudes and phases variables, namely Eq. (7.18) is solved numerically. In this case, the perturbation method facilitates the numerical method. The variables a, φ_1, b and φ_2 change slowly with time t. So it requires the numerical calculation of a few numbers of points. On the contrary, a direct attempt to solve Eq. (7.12) dealing with some harmonic terms in the solution Eq. (7.22), requires the numerical calculation of a great number of points.

Fig. 7.1

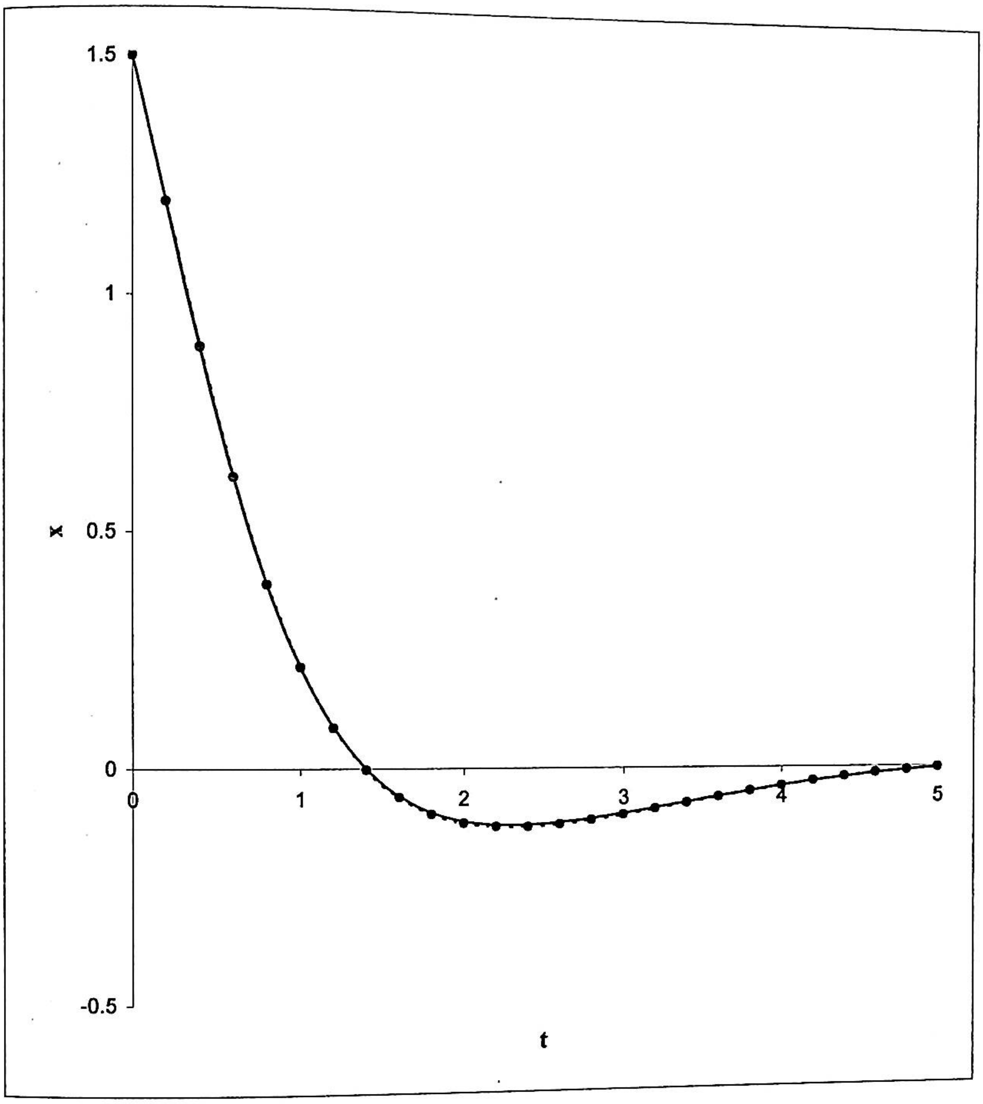

Fig. 7.1 First approximate solution is denoted by $-\bullet-$ (dashed lines) of Eq. (7.13) with the initial conditions $[x(0)=1.49992,\ \dot{x}(0)=-1.43242,\ \ddot{x}(0)=-1.43652,\ \dddot{x}(0)=8.76772]$ or $a_0=.5,\ \varphi_1=0,\ b=1.0,\ \varphi_2=0$, with $\mu_1=1.5,\ \mu_2=.75,\ \omega_0=1.0,\ h=0.5,\ \varepsilon=0.1, \omega_1=2\omega_2,\ \omega_2=\omega_0 e^{-h\tau},\ \tau=\varepsilon t$ and $f=x^3$. Corresponding numerical solution is denoted by - (solid line).

Fig. 7.2

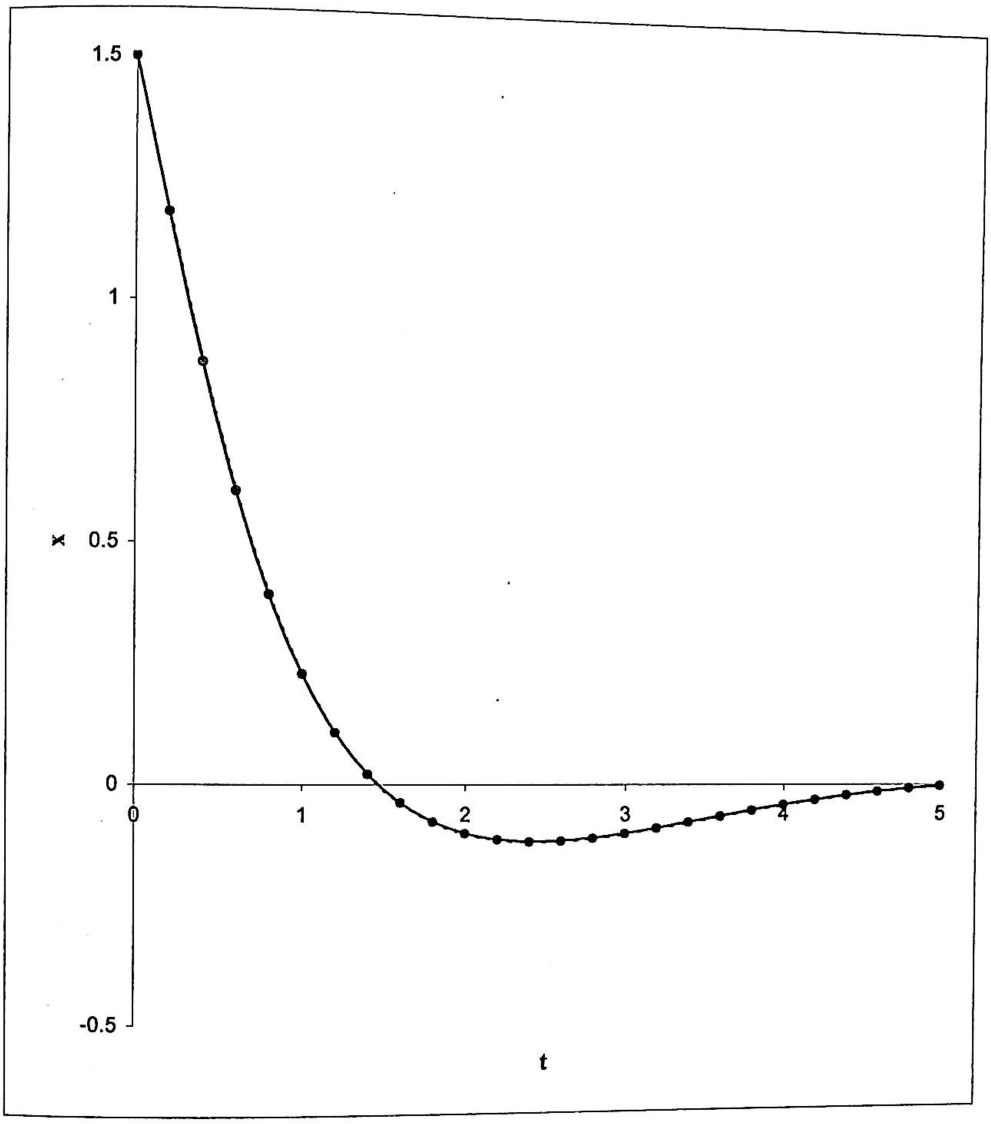

Fig. 7.2 First approximate solution is denoted by $-\bullet-$ (dotted lines) of Eq. (7.13) with the initial conditions $[x(0) = 1.50000,\ \dot{x}(0) = -1.55988,\ \ddot{x}(0) = -1.01827,$ $\dddot{x}(0) = 9.27141]$ or $a_0 = .5$, $\varphi_1 = 0$, $b = 1.0$, $\varphi_2 = 0$, with $\mu_1 = 1.75$, $\mu_2 = .75$, $\omega_0 = 1.0$, $h = .5$, $\varepsilon = .1$, $\omega_1 = 2\omega_2$, $\omega_2 = \omega_0 e^{-h\tau}$, $\tau = \varepsilon t$ and $f = x^3$. Corresponding numerical solution is denoted by - (solid line).

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