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Characterizations of Some Radical Rings and Gamma Rings

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Characterizations of Some Radical Rings and Gamma Rings

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Dedicated
To
My Beloved Parents

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Certificate of Originality

This is certified that the thesis entitled "Characterizations of Some Radical Rings and Gamma Rings" submitted by Md. Mahbubur Rashid contains the fulfillment of all the requirements for the degree of Doctor of Philosophy in Mathematics under the University of Rajshahi, has been completed under my supervision. I do believe that this research work is an original one and it has not been submitted elsewhere for any degree.

Akhil Ch. Paul
(Professor Akhil Chandra Paul)
Supervisor

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(Md. Mahbubur Rashid)

Abstract

The present thesis entitled "Characterizations of Some Radical Rings and Gamma Rings" is the outcome of the research by me under the supervision of Dr. A. C. Paul, Professor, Department of Mathematics, University of Rajshahi. The main emphasis of this thesis is to find radical rings. In introduction, we introduce the concepts of the complete thesis.

The thesis is of eight chapters. In the first chapter, we discuss the preliminaries of the complete thesis. This chapter is the basic concept of our work. Here we discuss the definitions and some results of Γ-rings. We develop some characterizations of p-rings in the second chapter and in the third chapter, we generalize the p-rings. We define p-Γ-ring and prove the analogous properties of p-rings for p-Γ-rings in the fourth chapter and the fifth chapter J-Γ-rings are the generalizations of p-Γ-rings. In the sixth chapter, we define the regular Γ-rings that are more general than that of S. Kyuno, N. Nobusawa and B. Smith [14] and we also develop sufficient conditions for Γ-rings to be regular. In the seventh chapter, we define an abelian regular Γ-rings and prove that an abelian regular Γ-ring is equivalent to "strongly regular" Γ-rings. We also develop some other properties. We define unit-regular Γ-rings in chapter eight and develop a number of equivalent characterizations and also develop a lattice theoretic characterization.

Contents

Introduction

The idea of a Γ-ring as the generalization of a ring was introduced by N. Nobusawa [16] and obtained analogues of the Wedderburn Theorem for Γ-rings with minimum condition on left ideals. W.E. Barnes [4] improved the idea of N. Nobusawa and gave the definition of Γ-rings which are more general than that of N. Nobusawa [16]. The notion of Γ-homomorphism, prime and primary ideals, m-systems, the radical of an ideal were introduced by him.

The notion of Jacobson radical, nil radical and strongly nilpotent radical for Γ-rings were introduced by Coppape and Luh [7] and they developed some radical properties.

Regular rings were invented and named by von Neumann and he obtained a necessary and sufficient condition for regular rings. Brown and McCoy developed the concept of a regular ideal. Warfield, Bergman, Kaplansky, Auslander and some other renounce mathematicians made deeper studies on regular rings.

The general radical theory for rings had been introduced by A. Kurosh [11] and S. A. Amitsur[1]. They studied the characterizations of a general radical. Divinsky [8] studied the general radical theory, the upper radical and the lower radical. Various kinds of radicals were studied here and he had also shown that these radicals are equal by minimum condition. He also characterized special class of rings and special radicals.

Shoji KYUNO, Nobuo NOBUSAWA and Mi-Soo B. Smith [14] introduced regular Γ_N -rings and they developed various properties of regular Γ_N -rings. They obtained a couple of necessary and sufficient conditions that Γ_N -rings are regular and then characterized a commutative regular Γ_N -rings as a subdirect sum of gamma fields.

Strongly regular rings were invented and named by Arens and Kaplansky, while the term "abelian' came into use later via operator algebras and Bear rings. Arens and Kaplansky proved that every strongly regular ring is regular, and that, in a strongly regular ring, every one-sided ideal is two-sided. Forsythe and McCoy showed that a regular ring is strongly regular if and only if it has no nonzero nilpotent elements, and that in a ring with no nonzero nilpotent elements, all idempotents are central. Most of the basic characterizations of abelian regular rings have been re-proved in a number of papers, along with innumerable variations and alternative characterizations.

Unit-regular rings were invented by G. Ehrlich [9] and he saw that this unit-regularity was equivalent to various properties for direct sums of finitely generated projective modules. K. R. Goodearl [10] developed a number of equivalent characterizations of the unit-regularity of a regular ring and also developed a lattice–theoretic characterization.

In our work, we have obtained the general radical theory for various type of Γ-rings. We have studied some of the characterizations of general radicals for J-Γ-rings, regular Γ-rings, unit-regular Γ-rings and abelian regular Γ-rings. In this connection we have also discussed particular radicals such as Jacobson radicals.

The main body of this thesis is divided into eight chapters.

The first chapter is the fundamental concepts relevant to our works. Here we have given some basics concepts of Γ-rings, Γ-rings M-modules, radical classes and other concepts that are needed to our research works.

In the second chapter, we have studied p-rings and developed some basic properties. We have proved that the class of all p-rings is a radical class. We have also developed a number of equivalent characterizations of p-rings.

We have studied J-rings and developed some properties in chapter three. We have showed that the class of all J-rings is a radical class. We have also proved that the Jacobson radical of J-ring is zero.

The concepts of p-Γ-rings have given in the third chapter. Here we have proved the analogous properties of p-rings for p-Γ-rings. We have proved that the class of all p-Γ-rings is a radical class. We have also developed some other properties of this ring.

The purpose of chapter five is to introduce the notion of J-Γ-rings and obtain the analogous properties of J-rings for J-Γ-rings. We also develop some other properties for J-Γ-rings.

In chapter six, we have defined a regular Γ-ring that is more general than that of S. Kyuno [14]. Here we have developed a few of their most basic properties. The main emphasis is on developing sufficient conditions for Γ-rings to be regular. We also have proved that the class of all regular Γ-rings is a radical class.

Chapter seven is one of several in which we have developed the basic properties of a class of regular Γ-rings of some "classical" type. Those considered in the present chapter are somewhat commutative, in that all idempotents are central, and also closely connected to division Γ-rings. Abelian regular Γ-rings are also known as strongly regular Γ-rings, which is, however, a more indirect concept. In that a nontrivial theorem is required to show that strongly regular Γ-rings are actually regular. For this reason, we view abelianness as the more general property. We have first developed a number of equivalent characterizations of abelian regular Γ-rings before proving that "abelian regular" is equivalent to "strongly regular".

The last chapter eight is concerned with unit-regular Γ-rings. We have developed a number of equivalent characterizations of the unit regularity of regular Γ-rings, mostly in the form of cancellation properties, either internal (within the lattice $L(M_R)$) or external (for finitely generated projective M-modules). These cancellation properties are then used to derive further properties of finitely generated projective M-modules over unit-regular Γ-rings. We have also developed a lattice theoretic characterization of the unit-regularity of M, namely transitivity of the relation of perspectivity in the lattice $L(2M_R)$

Chapter-One

Preliminaries

Let M and Γ be additive abelian groups. If there is a mapping $M\times\Gamma\times M\to M$ satisfying, for all a, b, $c\in M$; α, β, $\gamma\in\Gamma$

(i) $(a+b)\alpha c = a\alpha c + b\alpha c$

(ii) $a(\alpha+\beta)b = a\alpha b + a\beta b$

(iii) $a\alpha(b+c) = a\alpha b + a\alpha c$ and

(iv) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then M is called a Γ-ring. This Γ-ring is due to Barnes [4].

If the defining conditions for a Γ-ring are strengthened to

(i$'$) $a\alpha b$ is an element of M, $\alpha a\beta$ is an element of Γ,

(ii$'$) and (iii$'$) are same as (ii) and (iii),

(iv$'$) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$

(v$'$) $a\alpha b = 0$ for all a, $b\in M$ implies $\alpha = 0$,

then we have a Γ-ring in the sense of Nobusawa [16]. This Γ-ring is denoted by Γ_N-ring.

Throughout this thesis we consider the Γ-rings due to Barnes.

If A and B are subsets of a Γ-ring M and θ, $\Phi\subseteq\Gamma$, then we denote by $A\theta B$, the subset of M consisting of all finite sums of the form $\sum a_i\alpha_i b_i$, where a_i, $b_i\in M$ and $\alpha_i\in\theta$.

Ideal of Γ-rings: A right (left) ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M = \{a\alpha b | a\in A \;\; \alpha\in\Gamma,\, b\in M\}\subseteq I (M\Gamma I\subseteq I)$. If I is both a right ideal and a left ideal then we say that I is an ideal, or redundantly, a two-sided ideal of M.

If I and J are both left (respectively right or two-sided) ideals of M, then $I + J = \{a + b \mid a \in I,\ b \in J\}$ is clearly a left (respectively right or two-sided) ideal, called the sum of I and J. We can say every finite sum of left (respectively right or two-sided) ideal of a Γ-ring is also a left (respectively right or two-sided) ideal.

It is clear that the intersection of any number of left (respectively right or two-sided) ideal of M is also a left (respectively right or two-sided) ideal of M.

If A is a left ideal of M, B is a right ideal of M and S is any non empty subset of M, then the set $A\Gamma S = \{ \sum_{i=1}^{n} a_i\gamma_i s_i \mid a_i \in A,\ \gamma_i \in \Gamma,\ s_i \in S,$ n is a positive integer$\}$ is a left ideal of M and $S\Gamma B$ is a right ideal of M. $A\Gamma B$ is a two-sided ideal of M.

If $a \in M$, then the **principal ideal** generated by a denoted by $\langle a \rangle$ is the intersection of all ideals containing a and is the set of all finite sum of elements of the form $na + x\alpha a + a\beta y + u\gamma a\mu v$, where n is an integer, x, y, u, v are elements of M and $\alpha, \beta, \gamma, \mu$ are elements of Γ. This is the smallest ideal generated by a. Let $a \in M$. The smallest right ideal containing a is called the principal right ideal generated by a is denoted by $|a\rangle$. We similarly define $\langle a|$ and $\langle a \rangle$, the principal left and two-sided ideal generated by a respectively. We have $|a\rangle = Za + a\Gamma M$, $\langle a| = Za + M\Gamma a$, and $\langle a \rangle = Za + a\Gamma M + M\Gamma a + M\Gamma a\Gamma M$, where $Za = \{na\text{: } n \text{ is an integer}\}$. The smallest left (right) ideal generated by a is called the **principal left (right) ideal** and is denoted by $\langle a|$ $(|\,a\rangle)$.

Semiprime ideal: An ideal P of a Γ-ring M is said to be semiprime if for any ideal Q of M, $Q\Gamma Q \subseteq P$ implies $Q \subseteq P$. A Γ-ring M is semiprime if the zero ideal is semiprime.

If A is semiprime ideal and B an ideal of M with $B \subseteq A$, then $(B\Gamma)^n B = (B\Gamma B\Gamma B\Gamma \ldots\ldots\ldots \Gamma B\Gamma)B \subseteq A$ for any positive integer n.

Prime ideal: An ideal P of a Γ-ring M is said to be prime if for any ideals A and B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. A Γ-ring M is said to be **prime** if the zero ideal is prime.

***The Descending Chain Condition* (*DCC*)**: A Γ-ring M is said to have the descending chain condition on left ideals or DCC on left ideals if every descending sequence of left ideals $M \supseteq L_1 \supseteq L_2 \supseteq \ldots\ldots\ldots\ldots \supseteq L_n \supseteq \ldots\ldots$ terminates after a finite number steps, i.e. there exists an integer n such that $L_n = L_{n+1} = L_{n+2} = \ldots\ldots$.

***The Ascending Chain Condition* (*ACC*)**: A Γ-ring M is said to have the **ascending chain condition on left ideals** or **ACC on left ideals** if every ascending sequence of left ideals $L_1 \subseteq L_2 \subseteq \ldots\ldots\ldots \subseteq L_n \subseteq \ldots\ldots\ldots$ terminates after a finite number of steps, i. e. there exists an integer n such that $L_n = L_{n+1} = L_{n+2} = \ldots\ldots\ldots\ldots\ldots$.

Γ-homomorphism: If M and M' are Γ-rings and f: $M \rightarrow M'$ is a group homomorphism with extra property that $f(a\gamma b) = f(a)\gamma f(b)$, for all $a, b \in M$ and $\gamma \in \Gamma$, then f is a Γ-homomorphism.

Quotient Γ-ring: Let I be an ideal of a Γ-ring M. If for each $a + I$, $b + I$ are in the factor group M/I and each $\gamma \in \Gamma$, we define $(a + I)\gamma(b + I) = a\gamma b + I$, then M/I is a Γ-ring which we call the **quotient Γ-ring**

of M with respect to I. The mapping Φ: $a \rightarrow (a + I)$ is a Γ-homomorphism of M onto M/I, called the **natural Γ-homomorphism**.

Idempotent element: An element x of a Γ-ring M is called **idempotent** if $x\gamma x = x$ for some $\gamma \in \Gamma$.

Nilpotent element: An element x of a Γ-ring M is called **nilpotent** if for every $\gamma \in \Gamma$ there exists a positive integer $n = n(\gamma)$ such that $(x\gamma)^n x = 0$ and an ideal A of M is called **nil** if every element of A is nilpotent.

Strongly nilpotent element: An element x of a Γ-ring M is called strongly nilpotent if there exists a positive integer n such that $(x\Gamma)^n x = 0$ and an ideal A of M is called **strongly nil** if every element of A is strongly nilpotent.

Strongly nilpotent ideal: An ideal I of a Γ-ring M is called **strongly nilpotent** if there exists a positive integer n such that $(A\Gamma)^n A = 0$. Clearly a strongly nilpotent set is also strongly nil.

Locally nilpotent element: An element x of a Γ-ring M is called **locally nilpotent** if for every finite subset $F \subseteq M$ and finite subset $\Phi \subseteq \Gamma$, there exists a positive integer n such that $(F\Phi)^n F = 0$.

Weakly nilpotent element: An element a of Γ-ring M is said to be **weakly nilpotent** if there exists a non-zero element $\gamma \in \Gamma$ and an integer $n > 1$ such that $(a\gamma)^{n-1} a = 0$. A Γ-ring M is weakly nilpotent if every element of M is weakly nilpotent.

Orthogonal idempotent elements: Let M be a Γ-ring. A set of elements $\{e_i\}$ of M is called orthogonal idempotent if $e_i\gamma e_j = 0$ for $i \neq j$ and $e_i\gamma e_i = e_i = 0$ for some $\gamma\in\Gamma$.

Subdirect Sum: By the direct product (or complete direct sum) of Γ-rings M_r, r is in some index set I, we mean the set $\prod_{r\in I}M_r = \{f: I \rightarrow \cup_{r\in I}M_r \mid f(r)\in M_r \text{ all } r\in I\}$. We give a Γ-ring structure to $\prod_{r\in I}M_r$ by defining $(f+g)(r) = f(r) + g(r)$ and $(f\gamma g)(r) = f(r)\gamma g(r)$.

Let π_r be the projection of $\prod_{r\in I}M_r$ onto M_r. A Γ-ring M is said to be a **subdirect sum** of Γ-rings $\{M_r\}_{r\in I}$ if there is a monomorphism $\Phi: I \rightarrow \prod_{r\in I}M_r$ such that $M\Phi\pi_r = M_r$ for some $r\in I$.

Internal direct sum: Let M be a Γ-ring and N_1 and N_2 be two left ideals of M such that

(i) $M = N_1 + N_2 = \{n_1+n_2 \mid n_1\in N_1, n_2\in N_2\}$

(ii) $N_1 \cap N_2 = 0$

then we say M is the internal direct sum or simply direct sum of N_1 and N_2 and we write $M = N_1 \oplus N_2$.

Subdirect Product: A Γ-ring M is said to be a subdirect product of the family $\{M_r\}_{r\in I}$ of Γ-rings if there is a **natural projection** p_i such that $p_i(M) = M_i$ for every $r\in I$.

Sub-directly irreducible: A Γ-ring M is said to be subdirectly irreducible if it has no nontrivial representation as a sub direct sum of any Γ-ring M.

Clearly, a ring with only one element is subdirectly irreducible.

A Γ-ring M has a nontrivial representation as a subdirect sum of Γ-rings if and only if there exists in M a set of nonzero ideals with zero intersection. Thus every representation is trivial if and only if every set of nonzero ideals has nonzero intersection. Hence, a nonzero Γ-ring M is **sub-directly irreducible** if and only if the intersection of all of its nonzero ideals of M is different from zero.

***Right operator ring*:** Let M be a Γ-*ring* and F be a free abelian group generated by $\Gamma\times M$. Then $A = \{\sum_i n_i(\gamma_i, x_i)\in F \mid a\in M \Rightarrow \sum_i n_i a\gamma_i x_i = 0\}$ is a subgroup of F. Let $K = F/A$, the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. Then $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$ and $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ for all $\alpha, \beta\in\Gamma$.

We define a multiplication in K by $(\sum_i[\alpha_i, x_i])(\sum_j[\beta_j, y_j]) = \sum_{i,j}[\alpha_i, x_i\beta_j y_j]$. Then K forms a ring. Now we define a composition on $M\times K$ into M by $x(\sum_i[\alpha_i, x_i]) = \sum_i x\alpha_i x_i$ for $x\in M$, $\sum_i[\alpha_i, x_i]\in K$. Then M is a right K-module and we call K the **right operator ring** of the Γ-ring M. Similarly we can define the left operator ring of the Γ-ring M.

***Quasi-regular Γ-ring*:** An element x of a Γ-ring M is said to be **right quasi-regular** (abbreviated rqr) if for any $\gamma\in\Gamma$, there exists $\delta_i\in\Gamma$, $x_i\in M$, $i = 1, 2, 3, \ldots, n$ such that $x\gamma a + \sum_i x\delta_i x_i - \sum_i x\gamma a\delta_i x_i = 0$.

A Γ-ring M is called **right-quasi regular** if every element of M is right-quasi regular. The class of all right-quasi-regular Γ-rings is a radical class. This radical is called Jacobson radical and is denoted by $\mathfrak{J}$.

Completely prime ideal in a Γ-ring: A completely prime ideal in a *Γ-ring* M is a proper two-sided ideal P such that M/P is an integral domain (not necessarily commutative).

Division Γ-ring: A Γ-ring M is called **division Γ-ring** if for every $x \in M$ there exists $y \in M$ such that $x\gamma y = y\gamma x = 1$.

Γ-Field: A commutative division Γ-ring is called a Γ-field.

Semi-hereditary: A Γ-ring M is said to be **semi-hereditary** if every finitely generated right ideal of M is projective M-module.

Non-singular Γ-ring: An ideal I of a Γ-ring M is called essential if for every nonzero ideal A in M, $I \cap A \neq 0$. Let $\varphi(M)$ be the class of all essential ideals in M and $Z_r(M) = \{x \in M \mid x\Gamma I = 0$ for some $I \in \varphi(M)\}$. M is called a **non-singular** Γ-ring if $Z_r(M) = 0$. For the case of a classical ring R, we define $Z_r(R) = \{x \in R \mid xI = 0$ for some $I \in \varphi(R)$. Then R is called a non-singular if $Z_r(R) = 0$.

m-system: A subset S of a Γ-ring M is an m-system if $S = \Phi$ or if a, b∈S implies $\langle a\rangle\Gamma\langle a\rangle \cap S \neq \Phi$. A subset N of M is said to be n-system in M if $N = \Phi$ or if $a \in N$ implies $\langle a\rangle \cap N \neq \Phi$.

Radical Class: A class of rings (Γ-rings) $\mathfrak{R}$ is called a radical class if

a) $\mathfrak{R}$ is homomorphically closed, i.e. if $R \in \mathfrak{R}$ and I is an ideal of R, then $R/I \in \mathfrak{R}$.

ii) $\mathfrak{R}$ is closed under extension, i.e. if R/I and $I \in \mathfrak{R}$, then $R \in \mathfrak{R}$. Here I is an ideal of R.

iii) If $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ .... is an ascending chain of $\mathfrak{R}$-ideals, then $\cup_\alpha I_\alpha \in \mathfrak{R}$.

Let A be any ring (Γ-ring) and let $\mathfrak{R}$ be a radical class. Then there is a unique largest $\mathfrak{R}$ -ideal $R(A)$ such that $A/R(A)$ has no non-zero $\mathfrak{R}$-ideals. This $R(A)$ is called a radical of A. If $R(A) = A$, then A is called an $\mathfrak{R}$-radical ring (Γ-ring). If $R(A) = 0$, then A is called $\mathfrak{R}$-semisimple.

***Jacobson radical*:** Let M be a commutative Γ-ring with 1. Then Jacobson radical of the Γ-ring M is the intersection of all maximal ideals of M.

***M-module*:** Let M be a Γ-ring and let $(E, +)$ be an abelian group. Then E is called a M-module if there exists a Γ-mapping (Γ-composition) from $M\times\Gamma\times E$ to E sending (m, α, p) to $m\alpha p$ such that

i) $(m_1 + m_2)\alpha p = m_1\alpha p + m_2\alpha p$

ii) $m\alpha(p_1 + p_2) = m\alpha p_1 + m\alpha p_2$

iii) $(m_1\alpha m_2)\beta p = m_1\alpha(m_2\beta p)$,

for all $p, p_1, p_2 \in E$, $m, m_1, m_2 \in M$, $\alpha, \beta \in \Gamma$.

If in addition, M has an identity 1 and $1\gamma p = p$ for all $p \in E$ and some $\gamma \in \Gamma$, then E is called a unital M-module.

***Sub module*:** Let M be a Γ-ring. Let E be a M-module. Let $(Q, +)$ be a subgroup of $(E, +)$. We call Q, a sub module of E if $m\gamma q \in Q$ for all $m \in M$, $q \in Q$ and $\gamma \in \Gamma$.

Quotient module: Let Q be submodule of a *M-module* E. If for each $a + Q$, $b + Q$ are in the factor group $E/Q = \{p + Q \mid p \in E\}$ and each $\gamma \in \Gamma$, we define $m\gamma(p + Q) = m\gamma p + Q$ for all $m \in M$, $p \in P$ and $\gamma \in \Gamma$, and $(p_1 + Q) + (p_2 + Q) = (p_1 + p_2) + Q$ for all $p_1, p_2 \in P$, then E/Q is a *M-module* which we call the quotient *module* of M with respect to Q.

Irreducible module: An M-module is said to be irreducible if it has exactly two submodules. These must be itself and 0; the module 0 is not irreducible according to this definition.

Annihilator: Annihilator of a subset S of an M-module E is defined as

$$Ann(S) = \{a \in E \mid a\gamma x = 0 \text{ for all } x \in S, \gamma \in \Gamma\}.$$

It is a left ideal of E. If S is submodule of E, then $Ann(S)$ is a two-sided ideal of E.

If $S = E$, then $Ann(S) = Ann(E) = 0$

If $Ann(E) = 0$, then E is called a **faithful** module.

Indecomposable Submodules: Submodules that are not the direct sum of two nonzero submodules are known as **indecomposable Submodules.**

Chapter - Two

Some Characterizations of p-Rings

In this chapter, we study various properties of p-rings. We proved a basic theorem like: If R is a ring and I be an ideal of R, then R is a p-ring if and only if I and R/I are p-rings; if $\mathfrak{R}$ is a class of all p-rings, then this theorem shows that $\mathfrak{R}$ is a radical class. We also develop some other properties of p-rings.

Definition: *A ring R is said to be a p-ring if for each $x \in R$ there exists a prime integer $p > 1$ with $px = 0$ such that $x^p = x$.*

This p-ring R has no nonzero nilpotent elements; for any $a \in R$, a^{p-1} is an idempotent element; every ideal is two-sided and R is commutative.

***Lemma* 2.01.** *Let R be a commutative ring. If I is an ideal of R such that I is a p-ring, then $e(y - y^p) = 0$ for all $y \in R$ and e is an idempotent of I.*

Proof. Let $x \in I$ and $y \in R$. Then $xy \in I$. Since I is a p-ring, $x^p = x$ and $(xy)^p = xy$ for some prime $p > 1$. Now, $x^p y^p = xy$ implies $xy^p = xy$ implies $x(y^p - y) = 0$, so $x^{p-1}(y - y^p) = 0$ and hence, $e(y - y^p) = 0$, where $e = x^{p-1}$ is an idempotent of I. ■

***Lemma* 2.02.** *Let R be a commutative ring and I an ideal of R. Then R is a p-ring if and only if R/I and I are both p-rings.*

Proof. Suppose that R is a p-ring. Then, obviously, I is a p-ring. Now, let $x \in R/I$, then $x = r + I$ for some $r \in R$ with $r^p = r$. Now, $x^p = (r + I)^p = r^p + I = r + I = x$. Thus, R/I is a p-ring.

Conversely, let R/I and I be p-rings. Let $x \in R$, then $x + I \in R/I$ and so $(x + I)^p = x + I = x^p + I$. Thus, $x^p - x \in I$ and since I is a p-ring, $(x^p - x)^p = x^p - x$ for some prime $p > 1$. Let $e' = (x^p - x)^{p-1}$. Then e' is an idempotent of I. By *Lemma* 2.01, $e'(x^p - x) = 0$ for every $x \in R$. Now, $0 = e'(x^p - x) = (x^p - x)^{p-1}(x^p - x) = (x^p - x)^p = x^p - x$. Hence, $x^p = x$. Therefore R is a p-ring. ■

***Lemma* 2.03.** *Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ .... be ascending chain of ideals which are all p-rings. Then $\cup_\alpha I_\alpha$ is also a p-ring.*

Proof. Let $x \in \cup_\alpha I_\alpha$, then $x \in I_\alpha$ for some α. Since I_α is a p-ring, then $x^p = x$ for some prime $p > 1$. Hence, $\cup_\alpha I_\alpha$ is a p-ring. ■

Thus, by *Theorem* 2.02 and *Theorem* 2.03, we have the following theorem:

***Theorem* 2.04.** *The class $\mathfrak{R}$ of all p-rings is a radical class.* ■

Characterizations of p-rings

***Theorem* 2.05.** *Let R be a ring with* 1. *Let $a, x \in R$ such that $a = x^{p-2}$. Then the following statements are equivalent:*

a) R is a p-ring.

b) Every principal ideal Ra is generated by an idempotent element e where $e = xa = x^{p-1}$, $x \in R$.

c) For every principal ideal Ra of R, there exists an element $b \in R$ such that $R = Ra \oplus Rb$.

d) Every principal ideal Ra is a direct summand of R.

Proof. ***a*** $\Rightarrow$ ***b***) Let $x \in R$. Then $x^p = x$ for some prime $p > 1$. Let $a \in R$. and Let $a = x^{p-2}$. Now, the principal ideal Ra is generated by the element xa which is idempotent; for $(xa)(xa) = xx^{p-2}xx^{p-2} = x^p x^{p-2} = xa$.

b $\Rightarrow$ ***c***) Let $Ra = Re$, where $e^2 = e$ and $a = x^{p-1}$, $x \in R$. Since $1 = e + (1 - e)$, and if there exists $b \in R$ such that $ae = b(1 - e)$, then $ae = ae^2 = b(1 - e)e = 0$. So $R = Re \oplus R(1-e)$.

c $\Rightarrow$ ***d***) Trivial.

D $\Rightarrow$ ***a***) Let $a \in R$. Then there exists an ideal I of R such that $R = Ra \oplus I$. Hence, $1 = xa + b$, where $b \in I$, so $x = xax + bx$. Since $a = x^{p-2}$, $bx = x - xax \in Ra \cap I = 0$, and therefore $x = xax = x^p$. Hence, R is a p-ring. ■

***Theorem* 2.06.** *Let R be a p-ring with 1. Then*

1) Every finitely generated ideal is principal.

2) The intersection of any two principal ideals of R is principal.

Proof. 1) It is enough to prove that if a, $b \in R$, then $Ra + Rb$ is principal. Since R is a p-ring, by *Theorem* 2.05, there exist elements $x, y \in R$ with $a = x^{p-2}$ and $b = y^{p-2}$ for some prime $p > 1$, such that the elements $e_1 = xa$ and $e_2 = ya$ are the idempotent elements of Ra and Rb respectively and also $Ra = Re_1$ and $Rb = Re_2$.

Now, $Ra + Rb = Re_1 + Re_2 = Re_1 + R(e_2 - e_2e_1)$ because $a_1e_1 + a_2e_2 = (a_1 + a_2e_2)e_1 + a_2(e_2 - e_2e_1)$. If $s = (e_2 - e_2e_1)^{p-2} \in R$, then $(e_2 - e_2e_1)s(e_2 - e_2e_1) = (e_2 - e_2e_1)^p = (e_2 - e_2e_1)$. Then $e_2^/ = s(e_2 - e_2e_1)$ is an idempotent of Rb. Then $Re_1 + Re_2 = Re_1 + Re_2^/$ with $e_2^/e_1 = s(e_2 - e_2e_1)e_1 = 0$.

Finally, we have, $a_1e_1 + a_2e_2^/ = (a_1e_1 + a_1e_2^/)(e_1 + e_2^/ - e_2^/e_1)$, $a_1, b_1 \in R$. Thus, $Re_1 + Re_2^/ = R(e_1 + e_2^/ - e_2^/e_1)$. Therefore $Ra + Rb = R(e_1 + e_2^/ - e_2^/e_1)$ is a principal ideal. Thus, $Ra + Rb$ is a principal ideal.

2) Let Ra and Rb be two principal ideals. Since R is a p-ring by *Theorem* 2.05, there exists elements $x, y \in R$ with $a = x^{p-2}$ and $b = y^{p-2}$ for some prime $p > 1$, such that the elements $e_1 = xa$ and $e_2 = ya$ are the idempotent elements of Ra and Rb respectively and also $Ra = Re_1$ and $Rb = Re_2$. Hence, $R = Re_1 \oplus R(1 - e_1) = Re_2 \oplus R(1-e_2)$, and

$$Re_1 = Ann_R[(1 - e_1)R] = \{x \in R \mid x(1 - e_1)R = 0\},$$

$$Re_2 = Ann_R[(1 - e_2)R] = \{x \in R \mid x(1 - e_2)R = 0\}.$$

Indeed obviously $Re_1 \subseteq Ann_R[(1 - e_1)R]$.

Conversely, if $x \in R$ and $x(1 - e_1) = 0$, writing $x = a_1e_1 + b_1(1 - e_1)$, $a_1, b_1 \in R$, we have

$a_1e_1(1 - e_1) + b_1(1 - e_1)(1 - e_1) = 0$, and so

$b_1(1 - e_1) = 0$, hence, $x = a_1e_1 \in Re_1$.

Thus, $Re_1 \cap Re_2 = Ann_R[(1 - e_1)R + (1 - e_2)R]$. Now, there exists $e_3 \in R$ such that $(1 - e_1)R + (1 - e_2)R = (1 - e_3)R$, and from $Re_3 = Ann_R[(1 - e_3)R]$ we deduce that $Re_1 \cap Re_2 = Re_3$. Thus, $Re_1 \cap Re_2 = Ra \cap Rb$ is a principal ideal. ■

***Theorem* 2.07.** *Let R be a p-ring with unity 1. Then*

a) The Jacobson radical $\mathfrak{J}(R)$ of R is zero.

b) R is a semisimple ring if and only if it is a Noetherian p-ring.

c) The centre of R is also a p-ring.

d) The p-ring R without zero divisor is a field.

e) Every ideal of R is nonsingular.

f) For any idempotent element e of R, $(1 - e)Re = 0$.

g) If $(R_i)_i \in I$ is a family of p-rings, then $\prod R_i$ is a p-ring.

h) R is semihereditary.

Proof. ***a***) Let $a \in \mathfrak{J}(R)$. Then $Ra \subseteq \mathfrak{J}(R)$. Since $Ra = Re$, where $e = xa$ is an idempotent with $a = x^{p-2}$ for some prime $p > 1$, so $e \in \mathfrak{J}(R)$, it follows that $(1 - e)$ is invertible. So there exists $y \in R$ such that $1 = y(1 - e) = y - ye$. Hence, $e = ye - ye^2 = ye - ye = 0$ and therefore $a = 0$. Thus, $\mathfrak{J}(R) = 0$.

b) First suppose that R is finitely generated. Then every ideal of R is finitely generated and hence a direct summand. So R is a semisimple ring.

Conversely, let R be a semisimple ring. Then every principal ideal of R is a direct summand of R and hence, R is a p-ring (by

Theorem 2.05). Since Jacobson radical $\mathfrak{J}(R)$ is the largest ideal of R, and since, in a p-ring, $\mathfrak{J}(R) = 0$, so any ascending chain of ideals of R must be finite. Hence, R is Noetherian.

c) Since p-ring is abelian, so centre of R is R itself, i.e. $C(R) = R$.

d) Let $a \in R$ with $a \neq 0$. Then $a^p = a$ for some prime $p > 1$. Then $a^p - a = 0 \Rightarrow a(a^{p-1} - 1) = 0$. Since $a \neq 0$, so $a^{p-1} - 1 = 0$ and so a^{p-2} is the inverse of a. Since p-ring R is abelian, so R is a field.

e) Suppose that $xI = 0$ for some $x \in R$ and $I \subseteq R$ is an ideal of R. Let Rx be a principal ideal of R. Then there is an idempotent e such that $Rx = Re$.

Now, since $ReI = RxI = 0$, we see that $I \subseteq R(1 - e)$. Then $I \cap Re = 0$, whence $Re = 0$ and consequently $x = 0$. Thus, R is nonsingular.

f) Since Re is a two-sided ideal, so $(1 - e)Re = Re - Re^2 = Re - Re = 0$.

g) Proof is obvious.

h) Since a finitely generated ideal of R is a direct summand of R and so is projective. Hence, R is semihereditary. ■

Chapter-Three

Some Characterizations of J-Rings

In this chapter, we study various properties of J-rings and obtain some characterizations of J-rings. We first study the commutitivity of J-rings. Then we obtain a basic theorem like: If R is a ring and I be an ideal of R, then R is a J-ring if and only if I and R/I are both J-rings. If $\mathfrak{R}$ is the class of all J-rings then with the help of this theorem we prove that $\mathfrak{R}$ is a radical class. We also obtain a couple of necessary and sufficient conditions that R is a J-ring. We also establish some other properties.

Definition. *A ring R is said to be a J-ring if for each $x \in R$ there exists an integer $n = n(x) > 1$ such that $x^n = x$.*

***Lemma* 3.01.** *Let R be a J-ring. Then every right ideal I of R is a two-sided ideal of R.*

Proof. We first observe that R has no nonzero nilpotent elements. For if $x \neq 0$, then $x^n = x$ implies that $x^m \neq 0$ for all $m > 1$. Next, let $a \in I$ and suppose $a^n = a$ for some $n > 1$. Then $(a^{n-1})^2 = a^{2n-2} = a^n a^{n-2} = aa^{n-2} = a^{n-1}$, so a^{n-1} is an idempotent.

Now, let e be an idempotent. Then for any $x \in R$, $(xe - exe)^2 = 0 = (ex - exe)^2$. Thus, $xe - exe = ex - exe = 0$ and so $xe = ex$, i.e. e commutes with every elements of R.

Thus, for any $r \in R$, $ra = ra^n = ra^{n-1}a = a^{n-1}ra = ar^{/}$, where $r^{/} = a^{n-2}ra \in R$. Since $ar^{/} \in I$, so does ra and so I is a two-sided ideal. ■

***Theorem* 3.02.** *Let R be a J-ring and I an ideal of R. Then R/I and I are both J-rings.*

Proof. Obviously I is a J-ring. Now, let $x \in R/I$, then $x = r + I$ for some $r \in R$ with $r^n = r$. Now, $x^n = (r + I)^n = r^n + I = r + I = x$. Thus, R/I is a J-ring.

***Lemma* 3.03.** *Let D be a division ring such that for every $x \in D$, there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$. Then D is a field.*

Proof. *The proof is given in* [18, *Lemma* 7.8.13]

***Lemma* 3.04.** *Let R be a J-ring with identity 1. Then for $x, y \in R$, $xy - yx$ is in the intersection of the maximal ideals of R.*

Proof. Every ring has a maximal ideal. Let I be such a maximal ideal. Then the quotient ring R/I has an identity, and since I is a maximal right ideal of R, R/I has no maximal ideals other than *0* and R/I. Thus, R/I is a division ring. Since R is a J-ring, R/I is a J-ring (by *Lemma* 3.02). Then R/I is commutative (by *Lemma* 3.03). From this it follows that $xy - yx \in I$, for all $x, y \in R$.

***Lemma* 3.05.** *Let R be J-ring with identity 1. Then R is commutative.*

Proof. Suppose $x \neq 0$ is in every maximal ideal of R. Then $x^n = x$, and x^{n-1} is an idempotent, say $x^{n-1} = e \neq 0$ and e must also be in every maximal ideal of R. Now, $1 - e$ can not be in any proper right ideal of R, for if it were, it would be in a maximal ideal K of R. Since $e \in K$, $1 = e + (1 - e)$ would be in K and hence, $K = R$, a contradiction. Since $(1 - e)R \neq 0$ and since $(1 - e)R$ is a (right) ideal, it follows that $(1 - e)R = R$, whence $(1 - e)r = e$ for some $r \in R$. Thus, $0 = e(1 - e)r = e$, a contradiction. Thus, x can not be in every maximal ideal in R and the intersection of all the maximal ideals of R is 0. Thus, by *Lemma* 3.04, $xy - yx \in 0$, $x, y \in R$, that is, $xy = yx$, for all $x, y \in R$. ■

***Theorem* 3.06.** *If R is a J-ring, then R is commutative.*

Proof. Let e be an idempotent in R. Then, $ex = xe$ for all $x \in R$. Thus, $eR = Re = T$ is also a J-ring, but T has an identity, namely e. Hence, by *Lemma* 3.05, T is commutative. Now, for all $x, y \in R$, $xye = xye^2$ $= (xe)(ye) = (ye)(xe) = yxe$, that is $(xy - yx)e = 0$. Since $(xy - yx)^n$ $= (xy - yx)$ for some $n > 1$, so $(xy - yx)^{n-1}$ is an idempotent, say e_1. Thus, $0 = (xy - yx)e_1 = (xy - yx)^n = xy - yx$, that is $xy = yx$. ■

***Lemma* 3.07.** *Let R be a commutative ring. Suppose that I is an ideal of R such that I is a J-ring. Then* $e(y - y^n) = 0$ *for all* $y \in R$ *and e is an idempotent of I.*

Proof. Let $x \in I$ and $y \in R$. Then $xy \in I$. Since I is a J-ring, $x^n = x$. Also $(xv)^n = xy$ for some $n > 1$. Now, $x^n y^n = xy$ implies $xy = x y^n$ implies $x(y^n - y) = 0$, so $x^{n-1}(y - y^n) = 0$ and hence, $e(y - y^n) = 0$, whence $e = x^{n-1}$ is an idempotent of I. ■

***Lemma* 3.08.** *Let R be a commutative ring and I an ideal of R. If R/I and I are both J-rings, then R is also a J-ring.*

Proof. let R/I and I be J-rings. Let $x \in R$, then $x + I \in R/I$ and so $(x + I)^n = x + I = x^n + I$. Thus, $x^n - x \in I$ and since I is a J-ring, then $(x^n - x)^m = x^n - x$ for some $m > 1$. Let $e' = (x^n - x)^{m-1}$. Then e' is an idempotent of I. By *Lemma* 3.07, $e'(x^n - x) = 0$ for every $x \in R$. Now, $0 = e'(x^n - x) = (x^n - x)^{m-1}(x^n - x) = (x^n - x)^m = x^n - x$. Hence, $x^n = x$. Therefore R is a J-ring. ■

***Lemma* 3.09.** *If $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ .... is an ascending chain of ideals which are all J-rings, then $\cup_\alpha I_\alpha$ is a J-ring.*

Proof. Let $x \in \cup_\alpha I_\alpha$, then $x \in I_\alpha$ for some α. Since I_α is a J-ring, then $x^n = x$ for some $n > 1$. Hence, $\cup_\alpha I_\alpha$ is a J-ring. ■

Thus, by *Lemma* 3.02, *Lemma* 3.08, and *Lemma* 3.09, we have the following theorem:

***Theorem* 3.10.** *The class of all J-rings is a radical class.*

Some characterizations of *J*-rings

***Theorem* 3.11.** *Let R be a ring with 1. Let $a, x \in R$ such that $a = x^{n-2}$. Then the following statements are equivalent:*

a) R is a J-ring.

b) Every principal ideal Ra is generated by an idempotent.

c) For every principal ideal Ra of R, there exists an element $b \in R$ such that $R = Ra \oplus Rb$.

d) Every principal ideal Ra is a direct summand of R.

Proof. ***a*** $\Rightarrow$ ***b***) Let $x \in R$. Then $x^n = x$. Let $a \in R$ such that $a = x^{n-2}$. Now, the principal ideal Ra is generated by the element xa which is idempotent; for $(xa)(xa) = x x^{n-2} x x^{n-2} = x^n x^{n-2} = xa$.

b $\Rightarrow$ ***c***) Let $Ra = Re$, where $e^2 = e$ and $a = x^{n-2}$, $x \in R$. Since $1 = e + (1 - e)$, and if there exists $b \in R$ such that $ae = b(1 - e)$, then $ae = ae^2 = b(1 - e)e = 0$. So $R = Re \oplus R(1 - e)$.

c $\Rightarrow$ ***d***) Trivial.

d $\Rightarrow$ ***a***) Let $a \in R$. Then there exists an ideal I of R such that $R = Ra \oplus I$. Hence, $1 = xa + b$, where $b \in I$, so $x = xax + bx$. Since $a = x^{n-2}$, $bx = x - xax \in Ra \cap I = 0$, and therefore $x = xax = x^n$. Hence, R is a J-ring. ■

***Theorem* 3.12.** *Let R be a J-ring with 1. Then*

1) Every finitely generated ideal is principal.

2) The intersection of any two principal ideals of R is principal.

Proof. 1) It is enough to prove that if a, $b \in R$, then $Ra + Rb$ is principal. Since R is a J-ring by *Theorem* 3.11(b), there exists elements $x, y \in R$ with $a = x^{n-2}$ and $b = y^{n-2}$ such that the elements $e_1 = xa$ and $e_2 = yb$ are the idempotent elements of Ra and Rb respectively and also $Ra = Re_1$ and $Rb = Re_2$. Now, $Ra + Rb = Re_1 + Re_2 = Re_1 + R(e_2 - e_2e_1)$ because $a_1e_1 + a_2e_2 = (a_1 + a_2e_2)e_1 + a_2(e_2 - e_2e_1)$. If $s = (e_2 - e_2e_1)^{n-2} \in R$, then $(e_2 - e_2e_1)s(e_2 - e_2e_1) = (e_2 - e_2e_1)^n = (e_2 - e_2e_1)$. Then $e_2^/ = s(e_2 - e_2e_1)$ is an idempotent of Rb. Then $Re_1 + Re_2 = Re_1 + Re_2^/$ with $e_2^/e_1 = s(e_2 - e_2e_1)e_1 = 0$.

Finally, we have, $a_1e_1 + a_2e_2^/ = (a_1e_1 + a_2e_2^/)(e_1 + e_2^/ - e_2^/e_1)$. Thus, $Re_1 + Re_2^/ = R(e_1 + e_2^/ - e_2^/ e_1)$. Therefore $Ra + Rb = R(e_1 + e_2^/ - e_2^/ e_1)$. Thus, $Ra + Rb$ is a principal ideal.

2) Let Ra and Rb be two principal ideals. Since R is a J-ring by *Theorem* 3.11(b), there exists elements $x, y \in R$ with $a = x^{n-2}$ and $b = y^{n-2}$ such that the elements $e_1 = xa$ and $e_2 = yb$ are the idempotents of Ra and Rb respectively and also $Ra = Re_1$ and $Rb = Re_2$. Hence, $R = Re_1 \oplus R(1 - e_1) = Re_2 \oplus R(1 - e_2)$, and

$Re_1 = Ann_R[(1 - e_1)R] = \{x \in R \mid x(1 - e_1)R = 0\}$,

$Re_2 = Ann_R[(1 - e_2)R] = \{x \in R \mid x(1 - e_2)R = 0\}$.

Indeed obviously $Re_1 \subseteq Ann_R[(1 - e_1)R]$.

Conversely, if $x \in R$ and $x(1 - e_1) = 0$, writing $x = a_1e_1 + b_1(1 - e_1)$, we have

$a_1e_1(1 - e_1) + b_1(1 - e_1)(1 - e_1) = 0$, and so

$b_1(1 - e_1) = 0$, hence, $x = a_1e_1 \in Re_1$.

Thus, $Re_1 \cap Re_2 = Ann_R[(1 - e_1)R + (1 - e_2)R]$. Now, there exists $e_3 \in R$ such that $(1 - e_1)R + (1 - e_2)R = (1 - e_3)R$, and from $Re_3 = Ann_R[(1 - e_3)R]$ we deduce that $Re_1 \cap Re_2 = Re_3$. Thus, $Re_1 \cap Re_2 = Ra \cap Rb$ is a principal ideal. ■

***Theorem* 3.13.** *Let R be a J-ring with unity* 1. *Then*

a) *The Jacobson radical $\mathfrak{J}(R)$ of R is zero.*

b) *R is a semisimple ring if and only if it is a Noetherian J-ring.*

c) *The centre of R is also a J-ring.*

d) *The J-ring R without zero divisors is a field.*

e) *Every ideal of R is nonsingular.*

f) *For any idempotent element e of R, $(1 - e)Re = 0$.*

g) *If $(R_i)_i \in I$ is a family of J-rings then $\prod R_i$ is a J-ring.*

h) *R is semihereditary.*

Proof. a) Let $a \in \mathfrak{J}(R)$. Then $Ra \subseteq \mathfrak{J}(R)$. Since $Ra = Re$ where $e = xa$ is an idempotent with $a = x^{n-2}$, so $e \in \mathfrak{J}(R)$. It follows that $(1 - e)$ is inevitable. So there exists $y \in R$ such that $1 = y(1 - e) = y - ye$. Hence, $e = ye - ye^2 = ye - ye = 0$ and therefore $a = 0$. Thus, $\mathfrak{J}(R) = 0$.

b) First suppose that R is finitely generated. Then every ideal of R

is finitely generated and hence a direct summand. So R is a semi-simple ring.

Conversely, let R be a semisimple ring. Then every principal ideal of R is a direct summand of R and hence R is a J-ring by *Theorem* 3.11(*d*). Since Jacobson radical $\mathfrak{J}(R)$ is the largest ideal of R and since in a J-ring, $\mathfrak{J}(R) = 0$, so any ascending chain of ideals of R must be finite. Hence, R is Noetherian.

c) Since J-ring is abelian, so centre of R is R itself, i.e. $C(R) = R$.

d) Let $a \in R$ with $a \neq 0$. Then $a^n = a$ for some n > 1. Then $a^n - a = 0$ implies $a(a^{n-1} - 1) = 0$. Since $a \neq 0$, so $a^{n-1} - 1 = 0$ and so a^{n-2} is the inverse of a. Since R is abelian, so R is a field.

e) Suppose that $xI = 0$ for some $x \in R$ and $I \subseteq R$ is an ideal of R. Let Rx be a principal ideal of R. Then there is an idempotent $e \in R$ such that $Rx = Re$. Now, since $ReI = RxI = 0$, we see that $I \subseteq R(1 - e)$. Then $I \cap eR = 0$, whence $Re = 0$ and consequently $x = 0$. Thus, R is nonsingular.

f) Since Re is a two-sided ideal, so $(1 - e)\, Re = Re - Re^2 = Re - Re = 0$.

g) Proof is obvious.

h) Since a finitely generated ideal of R is a direct summand of R and so is projective. Hence, R is semihereditary. ■

Chapter-Four

p-Γ-Rings

The purpose of this chapter is to introduce p-Γ-rings and a few of their most basic properties. In this chapter we have proved that p-Γ-rings are commutative and also the class of all p-Γ-rings is a radical class. We also develop similar properties of p-rings for the case of p-Γ-rings.

Example. Let $M = (Z_5, +, .)$ and $\Gamma = (Z_5, +)$. Then M is a p-Γ-ring.

***Lemma* 4.01.** Let M be a p-Γ-ring. Then every right ideal I of M is a two-sided ideal of M.

Proof. We first observe that M has no nonzero nilpotent elements. For if $x \neq 0$, then $(x\gamma)^p x = x$ implies that $(x\gamma)^p x \neq 0$ for some prime p and some $\gamma \in \Gamma$. Next, let $a \in I$ and suppose $(a\gamma)^p a = a$ for some prime p. Then $\{(a\gamma)^{p-1}a\}\gamma\{(a\gamma)^{p-1}a\} = \{(a\gamma)^{p-1}a\gamma\}\{(a\gamma)^{p-1}a\} = (a\gamma)^p(a\gamma)^{p-1}a = (a\gamma)^p a\gamma(a\gamma)^{p-2}a = a\gamma(a\gamma)^{p-2}a = (a\gamma)^{p-1}a$, so $(a\gamma)^{p-1}a$ is an idempotent element.

Next, we show that an idempotent element commutes with every elements of M. To show this let e be an idempotent element of M. Then for any $x \in M$, $(x\gamma e - e\gamma x\gamma e)\gamma(x\gamma e - e\gamma x\gamma e) = 0 = (e\gamma x - e\gamma x\gamma e)\gamma(e\gamma x - e\gamma x\gamma e)$. Thus, $x\gamma e - e\gamma x\gamma e = e\gamma x - e\gamma x\gamma e = 0$ and so $x\gamma e = e\gamma x$, i.e., e commutes with every elements of M.

Now, for any $r\in M$ and $a\in I$ with $(a\gamma)^p a = a$, $r\gamma a = r\gamma(a\gamma)^p a = r\gamma(a\gamma)^{p-1}(a\gamma)a = (a\gamma)^{p-1}a\gamma r\gamma a = (a\gamma)(a\gamma)^{p-2}a\gamma r\gamma a = a\gamma(a\gamma)^{p-1}r\gamma a = a\gamma r'$, where $r' = (a\gamma)^{p-1}r\gamma a\in M$. Since $a\gamma r'\in I$, so does $r\gamma a$ and so I is a two-sided ideal. ■

***Lemma* 4.02.** *Let M be a p-Γ-ring and I an ideal of M. Then M/I is p-Γ-ring.*

Proof. Let $x\in M/I$, then $x = m + I$ for all $m\in M$ with $(m\gamma)^p m = m$, $p > 1$ and $\gamma\in\Gamma$. Now, $(x\gamma)^p x = (m + I)\gamma\}^p(m + I) = \{m\gamma + I\}^p(m + I) = \{(m\gamma)^p + I\}(m + I) = (m\gamma)^p m + I = m + I = x$. Thus, M/I is a *p-Γ-ring*. ■

***Lemma* 4.03.** *Let D be a division p-Γ-ring of characteristic $p \neq 0$ and let C be the center of D. Suppose that $a\in D$, $a\notin C$ is such that $(a\gamma)^{p^h} a = a$ for some $h > 0$. Then there exists an element $x\in D$ such that $x\gamma a\gamma x^{-1} \neq a$.*

Proof. We define the mapping f: $D \to D$ by $f(x) = x\gamma a - a\gamma x$ for every $x\in D$. Now, $f^2(x) = ff(x) = f(x\gamma a - a\gamma x) = (x\gamma a - a\gamma x)\gamma a - a\gamma(x\gamma a - a\gamma x) = x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x$.

Again, $f^3(x) = f(x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x) = (x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x)\gamma a = (x\gamma a\gamma a\gamma a - 3a\gamma x\gamma a\gamma a - 3a\gamma a\gamma x\gamma a + a\gamma a\gamma a\gamma x)$. Thus, a simple computation yields that

$$f^p(x) = x\gamma(a\gamma)^{p-1}a - (a\gamma)^{p-1}a\gamma x, \text{ where char}D = p, \text{ a prime.}$$

Continuing we obtain that

$$f^{p^k}(x) = x\gamma(a\gamma)^{p^k}a - (a\gamma)^{p^k}a\gamma x,$$

for all $k \geq 0$. Let P denote the prime field of C; since a is algebraic over P, $P(a)$ must be a finite field having p^m elements, say. Hence $(a\gamma)^{p^m}a = a$ and so

$$f^{p^m}(x) = x\gamma(a\gamma)^{p^m} - (a\gamma)^{p^m}a\gamma x = x\gamma a - a\gamma x = f(x).$$

Thus, we see that the function $f^{p^m} = f$.

If $r \in P(a)$, then $f(r\gamma x) = (r\gamma x)\gamma a - a\gamma(r\gamma x) = r\gamma(x\gamma a - a\gamma x) = r\gamma f(x)$, since r commutes with a. If I denotes the identity map on D and rI denotes the map defined by $(rI)(x) = r\gamma x$, we have that $fo(rI) = (rI)of$, for all $r \in P(a)$. Since all elements of $P(a)$ satisfy the polynomial $t^{p^m} - t$, we find that $t^{p^m} - t = \prod_{r\in P(a)}(t - r)$. Since rI commutes with f, we have that $0 = f^{p^m} - f = \prod_{r \in P(a)}(f - r)$, where $(f - rI)(x) = f(x) - r\gamma x$. Now, Let $r_1 = 0$ (one of r's must be zero), and suppose for each $r_i \neq 0$, $(f - r_iI) \neq 0$, all $x \in D$, $x \neq 0$. Then $[(f - r_2I)o(f - r_3I)o\ldots\ldots o(f - r_{p^m}I)](x) \neq 0$, for all $x \in D$, $x \neq 0$. But since

$$0 = f^{p^m} - f = fo(f - r_2I)o(f - r_3I)o\ldots\ldots\ldots o(f - r_{p^m}I),$$

it follows that $f(x) = 0$ for all $x \in D$. Thus, $0 = f(x) = x\gamma a - a\gamma x$, whence $x\gamma a = a\gamma x$ for all $x \in D$. Thus, $a \in C$, contradicting the hypothesis. Thus, there is a $r_i \neq 0$, $r_i \in P(a)$ and $x \neq 0$ in D such that $(f - r_iI)(x) = 0$,

i.e. $(f(x) - r_iI)(x) = 0$

i.e. $x\gamma a - a\gamma x - r_i\gamma x = 0$

i.e. $x\gamma a - a\gamma x = r_i\gamma x$

i.e. $x\gamma a\gamma x^{-1} - a\gamma x\gamma x^{-1} = r_i\gamma x\gamma x^{-1}$

i.e. $x\gamma a\gamma x^{-1} = r_i\gamma x\gamma x^{-1} + a\gamma x\gamma x^{-1} \neq a$, since $r_i \neq 0$.

This completes the proof. ■

Lemma 4.04. *If D is a division Γ-ring of characteristic $p \neq 0$ and $G \subseteq D$ is a finite multiplicative subgroup of D, then G is commutative.*

Proof. Let P be the prime field of D and let $A = \{r_i\gamma g_i / r_i \in D$ and $g_i \in G\}$. Clearly A is a finite subgroup of D under addition; moreover, since G is a group under multiplication, A is finite sub-Γ-ring of D. Therefore A is a finite division Γ-ring, hence is commutative. Since $G \subseteq A$, G is also commutative. ■

Lemma 4.05. *Let D be a division Γ-ring such that for every $x \in D$ there exists a prime p such that $(x\gamma)^p x = x$. Then D is commutative.*

Proof. Suppose $a, b \in D$ are such that $c = a\gamma b - b\gamma a \neq 0$. By hypothesis $(c\gamma)^m c = c$ for some prime $m > 1$. If $r(\neq 0) \in C$, the center of D, then $r\gamma c = r\gamma(a\gamma b - b\gamma a) = (r\gamma a)\gamma b - b\gamma(r\gamma a)$, hence by hypothesis, $\{(r\gamma c)\gamma\}^p (r\gamma c) = r\gamma c$. Let $q = (m-1)(p-1) + 1$, $m > 1$, $p > 1$. Then $q > 1$ and q is prime. It follows that $(c\gamma)^q c = c$ and $\{(r\gamma c)\gamma\}^q (r\gamma c) = r\gamma c$, hence

$$\{(r\gamma c)\gamma(r\gamma c)\gamma(r\gamma c)\gamma \ldots\ldots \text{up to } q \text{ times}\}(r\gamma c) = r\gamma c$$

i.e. $(r\gamma)^q (c\gamma)^q (r\gamma c) = r\gamma c$,

i.e. $(r\gamma)^q (c\gamma)^q (c\gamma r) = r\gamma c$,

i.e. $(r\gamma)^q c\gamma r = r\gamma c$,

i.e. $(r\gamma)^q r\gamma c = r\gamma c$, i.e. $\{(r\gamma)^q r - r\}\gamma c = 0$.

Since D is a division Γ-ring and $c \neq 0$, so $(r\gamma)^q r = r$ for every $r \in C$, $q > 1$ depending on r and γ. We know that C is of characteristic $p \neq 0$. Let P be the prime field of C. We claim that if D is not commutative, we could have chosen our a, b such that not only is $c = a\gamma b - b\gamma a \neq 0$ but, in fact, c is not even in C. If not, all commutators are in C; hence $c \in C$ and C contains $a\gamma(a\gamma b) - (a\gamma b)\gamma a = a\gamma(a\gamma b) - a\gamma(b\gamma a) = a\gamma(a\gamma b - b\gamma a) = a\gamma c$. This would place $a \in C$ contrary to $c = a\gamma b - b\gamma a \neq 0$. Thus, we assume that $c = a\gamma b - b\gamma a \notin C$. Since $(c\gamma)^m c = c$, c is algebraic over P hence $(c\gamma)^{p^k} c = c$ for some $k > 0$. Thus, all the hypothesis of the *Lemma* 4.03 are satisfied for C. Hence we can find $x \in D$ such that $x\gamma c\gamma x^{-1} = c_1 \neq c$, that is $x\gamma c = c_1\gamma x \neq c\gamma x$. In particular, $d = x\gamma c - c\gamma x \neq 0$; but $d\gamma c = x\gamma c\gamma c - c\gamma x\gamma c = c_1\gamma x\gamma c - c\gamma c_1\gamma x = c_1\gamma x\gamma c - c_1\gamma c\gamma x$ (since $c_1 \in C$) $= c_1\gamma(x\gamma c - c\gamma x) = c_1\gamma d$. As a commutator, $(d\gamma)^t d = d$ for some prime $t > 1$ and $d\gamma c\gamma d^{-1} = c_1$. Thus, the multiplicative subgroup of D generated by c and d is finite. Hence by *Lemma* 4.04, the multiplicative subgroup is abelian. This contradicts $c\gamma d \neq d\gamma c$. and proves the lemma. ■

***Lemma* 4.06.** *Let M be a p-Γ-ring with identity* 1. *Then for $x, y \in M$, $x\gamma y - y\gamma x$ is in the intersection of the maximal ideals of M.*

Proof. We know that every ring has a maximal ideal. Let I be such a maximal ideal. Then the quotient ring M/I has an identity, and since I is a maximal right ideal of M, M/I has no maximal ideals other than *0* and M/I. Thus, M/I is a division ring. Since M is a p-Γ-ring, M/I is a p-Γ-ring (by *Lemma* 4.02). Then by *Lemma* 4.05, M/I is commutative. From this it follows that $x\gamma y - y\gamma x \in I$, for all $x, y \in M$. The conclusion of the lemma is now immediate. ■

***Lemma* 4.07.** *Let M be p-Γ-ring with identity* 1. *Then M is commutative.*

Proof. Suppose $x \neq 0$ is in every maximal ideal of M. Then $(x\gamma)^p x = x$, and $(x\gamma)^{p-1}x$ is an idempotent, say $(x\gamma)^{p-1}x = e \neq 0$ for all $p > 1$ and some $\gamma \in \Gamma$ and e must also be in every maximal ideal of M. Now, $1 - e$ can not be in any proper right ideal of M, for if it were, it would be in a maximal ideal K of M. Since $e \in K$, $1 = e + (1 - e)$ would be in K and hence $K = M$, a contradiction. Since $(1 - e)\gamma M \neq 0$ and since $(1 - e)\gamma M$ is a (right) ideal, it follows that $(1 - e)\gamma M = M$, whence $(1 - e)\gamma r = e$ for some $r \in M$. Thus, $0 = e\gamma(1 - e)\gamma r = e$, a contradiction. Thus, x can not be in every maximal ideal in M and the intersection of all the maximal ideals of M is 0. Thus, by *Lemma* 4.06, $x\gamma y - y\gamma x \in 0$, $x, y \in M$, that is, $x\gamma y = y\gamma x$, for all $x, y \in M$. ■

***Remarks*:** Since the intersection of all maximal ideals of a commutative Γ-ring with 1 is the Jacobson radical, so the Jacobson radical of p-Γ-ring with 1 is zero.

***Theorem* 4.08.** *If M is a p-Γ-ring, then M is commutative.*

Proof. Let e be an idempotent in M. Then, $e\gamma x = x\gamma e$ for all $x \in M$. Thus, $e\gamma M = M\gamma e = T$ is also a p-Γ-ring, but T has an identity, namely e. Hence by *Lemma* 4.07, T is commutative. Now, for all $x, y \in M$, $x\gamma y\gamma e = x\gamma y\gamma e\gamma e = (x\gamma e)\gamma(y\gamma e) = (y\gamma e)\gamma(x\gamma e) = y\gamma x\gamma e$, that is $(x\gamma y -$

$y\gamma x)\gamma e = 0$. Since $\{(x\gamma y - y\gamma x)\gamma\}^p(x\gamma y - y\gamma x) = (x\gamma y - y\gamma x)$ for some prime $p > 1$, so $\{(x\gamma y - y\gamma x)\gamma\}^{p-1}(x\gamma y - y\gamma x)$ is an idempotent, say e_1. Thus,

$$0 = (x\gamma y - y\gamma x)e_1 = \{(x\gamma y - y\gamma x)\gamma\}^p(x\gamma y - y\gamma x) = x\gamma y - y\gamma x,$$

that is, $x\gamma y = y\gamma x$. Hence, M is commutative ■

Lemma 4.09. *Let M be a commutative Γ-ring. Let I be an ideal of M such that I a p-Γ-ring. Then $e\gamma\{y - (y\gamma)^p y\} = 0$ for all $y\in M$ and some $\gamma\in\Gamma$ and $e\in I$ an idempotent.*

Proof. Let $x\in I$ and $y\in M$. Then $x\gamma y\in I$. Since I is a p-Γ-ring, $(x\gamma)^p x = x$. Also $\{(x\gamma y)\gamma\}^p(x\gamma y) = x\gamma y$ for some prime p and $\gamma\in\Gamma$.

Now, $\{(x\gamma y)\gamma\}^p(x\gamma y) = x\gamma y$,

i.e. $\{(x\gamma y)\gamma(x\gamma y)\gamma \ldots\ldots$ up to p times$\}(x\gamma y) = x\gamma y$,

i.e. $(y\gamma)^p(x\gamma)^p(x\gamma y) = x\gamma y$, since M is commutative,

i.e. $(y\gamma)^p x\gamma y = x\gamma y$.

i.e. $\{(y\gamma)^p y - y\}\gamma x = 0$,

so $(x\gamma)^{p-1}x\gamma\{y - (y\gamma)^p y\} = 0$ and hence $e\gamma\{y - (y\gamma)^p y\} = 0$, where $e = (x\gamma)^{p-1}x$ is an idempotent of I. ■

Lemma 4.10. *Let M be a Γ-ring and I an ideal of M. Then M is a p-Γ-ring if M/I and I are p-Γ-rings.*

Proof. Let M/I and I be p-Γ-rings. Let $x\in M$, then $x + I\in M/I$ and so

$\{(x + I)\gamma\}^p(x + I) = x + I$ for some prime p and $\gamma\in\Gamma$.

i.e. $(x\gamma + I)^p(x + I) = x + I$,

i.e. $\{(x\gamma)^p + I\}(x + I) = x + I$, i.e. $(x\gamma)^p x + I = x + I$.

Thus, $(x\gamma)^p x - x \in I$. Since I is a p-Γ-ring, $\{((x\gamma)^p x - x)\gamma\}^m \{(x\gamma)^p x - x\} = (x\gamma)^p x - x$ for some prime m. Let $e' = \{((x\gamma)^p x - x)\gamma\}^{m-1}\{(x\gamma)^p x - x\}$. Then e' is an idempotent of I. By *Lemma* 4.09, $e'\gamma\{(x\gamma)^p x - x\} = 0$ for every $x \in M$. Now, $0 = e'\gamma\{(x\gamma)^p x - x\} = \{((x\gamma)^p x - x)\gamma\}^{m-1}\{(x\gamma)^p x - x\}\gamma\{(x\gamma)^p x - x\} = \{((x\gamma)^p x - x)\gamma\}^m \{(x\gamma)^p x - x\} = (x\gamma)^p x - x$. Hence $(x\gamma)^p x = x$. Therefore M is a p-Γ-ring. ■

***Lemma* 4.11.** *If $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ .... is an ascending chain of ideals which are all p-Γ-rings, then $\cup_\alpha I_\alpha$ is a p-Γ-rings.*

Proof. Let $x \in \cup_\alpha I_\alpha$, then $x \in I_\alpha$ for some α. Since I_α is a p-Γ-ring, then $(x\gamma)^p x = x$ for some prime p and $\gamma \in \Gamma$. Hence $\cup_\alpha I_\alpha$ is a p-Γ-ring. ■

Thus, by *Lemma* 4.02, *Lemma* 4.10 and *Lemma* 4.11, we have the following theorem:

***Theorem* 4.12.** *The class of all p-Γ-rings is a radical class.*

Some Characterizations of *p*-Γ-rings

***Theorem* 4.13.** *Let M be a ring with* 1. *Let a, $x \in M$ such that $a = (x\gamma)^{p-2} x$. Then the following statements are equivalent:*

a) M is a p-Γ-ring.

b) Every principal ideal $M\gamma a$ is generated by an idempotent.

c) For every principal ideal $M\gamma a$ of M, there exists an element $b \in M$ such that $M = M\gamma a \oplus R\gamma b$.

d) Every principal ideal $M\gamma a$ is a direct summand of M.

Proof. ***a*** $\Rightarrow$ ***b***) Let $x \in M$. Then $(x\gamma)^p x = x$ for some prime p and $\gamma \in \Gamma$. Let $a \in M$ such that $a = (x\gamma)^{p-2}x$. Now, the principal ideal $M\gamma a$ is generated by the element $x\gamma a$ which is idempotent; for $(x\gamma a)\gamma(x\gamma a) = x\gamma\{(x\gamma)^{p-2}x\}\gamma\{x\gamma(x\gamma)^{p-2}x\} = (x\gamma)^p x\gamma(x\gamma)^{p-2}x = x\gamma a$.

b $\Rightarrow$ ***c***) Let $M\gamma a = M\gamma e$, where $e\gamma e = e$ and $a = (x\gamma)^{p-2}x$, $x \in M$. Since $1 = e + (1 - e)$, and if there exists $b \in M$ such that $a\gamma e = b\gamma(1 - e)$, then $a\gamma e = a\gamma e\gamma e = b\gamma(1 - e)\gamma e = 0$. So $M = M\gamma e \oplus M\gamma(1 - e)$.

c $\Rightarrow$ ***d***) Trivial.

d $\Rightarrow$ ***a***) Let $a \in M$. Then there exists an ideal I of M such that $M = M\gamma a \oplus I$. Hence $1 = x\gamma a + b$, where $b \in I$, so $x = x\gamma a\gamma x + b\gamma x$. Since $a = (x\gamma)^{p-2}x$, $b\gamma x = x - x\gamma a\gamma x \in M\gamma a \cap I = 0$, and therefore $x = x\gamma\{(x\gamma)^{p-2}x\}\gamma x = (x\gamma)^p x$. Hence M is a p-Γ-ring.

***Theorem* 4.14.** *Let M be a p-Γ-ring with* 1. *Then*

1) *Every finitely generated ideal is principal.*
2) *The intersection of any two principal ideals of M is principal.*

Proof. 1) It is enough to prove that if a, $b \in M$, then $M\gamma a + M\gamma b$ is principal. Since M is a p-Γ-ring, there exists elements x, $y \in M$ with $a = (x\gamma)^{p-2}x$ and $b = (y\gamma)^{p-2}y$ such that the elements $e_1 = x\gamma a$ and $e_2 = y\gamma b$ are the idempotent elements of $M\gamma a$ and $M\gamma b$ respectively and also $M\gamma a = M\gamma e_1$ and $M\gamma b = M\gamma e_2$ by *Theorem* 4.13(b). Now, $M\gamma a + M\gamma b = M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma(e_2 - e_2\gamma e_1)$ because $a_1\gamma e_1 + a_2\gamma e_2 =$

$(a_1 + a_2\gamma e_2)\gamma e_1 + a_2\gamma(e_2 - e_2\gamma e_1)$. If $s = \{(e_2 - e_2\gamma e_1)\gamma\}^{p-2}(e_2 - e_2\gamma e_1) \in M$, then

$(e_2 - e_2\gamma e_1)\gamma s\gamma(e_2 - e_2\gamma e_1) = \{(e_2 - e_2\gamma e_1)\gamma\}^p(e_2 - e_2\gamma e_1) = (e_2 - e_2\gamma e_1)$.

Then $e_2^{/} = s\gamma(e_2 - e_2\gamma e_1)$ is an idempotent of $M\gamma b$. Then $M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma e_2^{/}$ with $e_2^{/}\gamma e_1 = s\gamma(e_2 - e_2\gamma e_1)\gamma e_1 = 0$.

Finally, we have, $a_1\gamma e_1 + a_2\gamma e_2^{/} = (a_1\gamma e_1 + a_2\gamma e_2^{/})\gamma(e_1 + e_2^{/} - e_2^{/}\gamma e_1)$, $a_1, b_1 \in M$. Thus, $M\gamma e_1 + M\gamma e_2^{/} = M\gamma(e_1 + e_2^{/} - e_2^{/}\gamma e_1)$. Therefore $M\gamma a + M\gamma b = M\gamma(e_1 + e_2^{/} - e_2^{/}\gamma e_1)$. Thus, $M\gamma a + M\gamma b$ is a principal ideal.

2) Let $M\gamma a$ and $M\gamma b$ be two principal ideals. Since M is a p-Γ-ring, there exists elements $x, y \in M$ with $a = (x\gamma)^{p-2}x$ and $b = (y\gamma)^{p-2}y$ such that the elements $e_1 = x\gamma a$ and $e_2 = y\gamma b$ are the idempotents of $M\gamma a$ and $M\gamma b$ respectively and also $M\gamma a = M\gamma e_1$ and $M\gamma b = M\gamma e_2$ by *Theorem* 4.13(b). Hence $M = M\gamma e_1 \oplus M\gamma(1 - e_1) = M\gamma e_2 \oplus M\gamma(1 - e_2)$, and

$$M\gamma e_1 = Ann_M[(1 - e_1)\gamma M] = \{x \in M \mid x\gamma(1 - e_1)\gamma M = 0\},$$
$$M\gamma e_2 = Ann_M[(1 - e_2)\gamma M] = \{x \in M \mid x\gamma(1 - e_2)\gamma M = 0\}.$$

Indeed obviously $M\gamma e_1 \subseteq Ann_M[(1 - e_1)\gamma M]$.

Conversely, if $x \in M$ and $x\gamma(1 - e_1) = 0$, writing $x = a_1\gamma e_1 + b_1\gamma(1 - e_1)$, $a_1, b_1 \in M$, we have

$$a_1\gamma e_1\gamma(1 - e_1) + b_1\gamma(1 - e_1)\gamma(1 - e_1) = 0, \text{ and so}$$
$$b_1\gamma(1 - e_1) = 0, \text{ hence } x = a_1\gamma e_1 \in M\gamma e_1.$$

Thus, $M\gamma e_1 \cap M\gamma e_2 = Ann_M[(1 - e_1)\gamma M + (1 - e_2)\gamma M]$. Now, there exists $e_3 \in M$ such that $(1 - e_1)\gamma M + (1 - e_2)\gamma M = (1 - e_3)\gamma M$, and from

$M\gamma e_3 = Ann_M[(1 - e_3)\gamma M]$ we deduce that $M\gamma e_1 \cap M\gamma e_2 = M\gamma e_3$. Thus, $M\gamma e_1 \cap M\gamma e_2 = M\gamma a \cap M\gamma b$ is a principal ideal. ■

***Theorem* 4.15.** *Let M be a p-Γ-ring with unity* 1. *Then*

a) *The Jacobson radical* $\mathfrak{J}(M)$ *of M is zero.*
b) *M is a semisimple ring if and only if it is a Noetherian p-Γ-ring.*
c) *The centre of M is also a p-Γ-ring.*
d) *The p-Γ-ring M without zero divisors is a field.*
e) *Every ideal of M is nonsingular.*
f) *For any idempotent element e of M,* $(1 - e)\gamma M\gamma e = 0$.
g) *If* $(M_i)_{i\in I}$ *is a family of p-Γ-rings then* $\prod M_i$ *is a p-Γ-ring.*
h) *M is semihereditary.*

Proof. ***a***) Let $a \in \mathfrak{J}(M)$. Then $M\gamma a \subseteq \mathfrak{J}(M)$. Since $M\gamma a = M\gamma e$, where $e = x\gamma a$ is an idempotent with $a = (x\gamma)^{p-2}x$, so $e \in \mathfrak{J}(M)$. It follows that $(1 - e)$ is inevitable. So there exists $y \in M$ such that $1 = y\gamma(1 - e) = y - y\gamma e$. Hence $e = y\gamma e - y\gamma e\gamma e = y\gamma e - y\gamma e = 0$ and therefore $a = 0$. Thus, $\mathfrak{J}(M) = 0$.

b) First suppose that M is finitely generated. Then every ideal of M is finitely generated and hence a direct summand. So M is a semi-simple.

Conversely, let M be a semisimple ring. Then every principal ideal of M is a direct summand of M and hence M is a p-Γ-ring by *Theorem* 4.13(*d*). Since Jacobson radical $\mathfrak{J}(M)$ is the largest ideal of

M and since in a p-Γ-ring, $\mathfrak{J}(M) = 0$, so any ascending chain of ideals of M must be finite. Hence M is Noetherian.

c) Since p-Γ-ring is abelian, so centre of M is M itself, i.e. $C(M) = M$.

d) Let $a \in M$ with $a \neq 0$. Then $(a\gamma)^p a = a$ for some prime p. Then $(a\gamma)^p a - a = 0 \Rightarrow a\gamma\{(a\gamma)^{p-1} a - 1\} = 0$. Since $a \neq 0$, so $(a\gamma)^{p-1} a - 1 = 0$ and so $(a\gamma)^{p-2} a$ is the inverse of a. Since p-Γ-ring M is abelian, so M is a field.

e) Suppose that $x\gamma I = 0$ for some $x \in M$ and $I \subseteq M$ is an ideal of M. Let $M\gamma x$ be a principal ideal of M. Then there is an idempotent $e \in M$ such that $M\gamma x = M\gamma e$. Now, since $M\gamma e\gamma I = M\gamma x\gamma I = 0$, we see that $I \subseteq M\gamma(1 - e)$. Then $I \cap e\gamma M = 0$, whence $M\gamma e = 0$ and consequently $x = 0$. Thus, M is nonsingular.

f) Since $M\gamma e$ is a two-sided ideal, so $(1 - e)\gamma M\gamma e = M\gamma e - M\gamma e\gamma e = M\gamma e - M\gamma e = 0$.

g) Proof is obvious.

h) Since a finitely generated ideal of M is a direct summand of M and so is projective. Hence M is semihereditary. ■

Chapter - Five

J-Γ-Rings

The purpose of this chapter is to introduce J-Γ-rings and a few of their most basic properties. In this chapter we have proved that J-Γ-rings are commutative and also the class of all J-Γ-rings is a radical class. We also develop some characterizations of J-Γ-rings, which are analogues to the properties of J-rings.

Definition: *A Γ-ring M is said to be a J-Γ-ring if for every $x \in M$, there exists $\gamma \in \Gamma$ such that $(x\gamma)^n x = x$ for some $n = n(x, \gamma) > 1$.*

Example. Let $M = (Z_6, +, .)$ and $\Gamma = (Z_6, +)$. Then M is a J-Γ-ring.

***Lemma* 5.01.** *Let M be a J-Γ-ring. Then every right ideal I of M is a two-sided ideal of M.*

Proof. We first observe that M has no nonzero nilpotent elements. For if $x \neq 0$, then $(x\gamma)^n x = x$ implies that $(x\gamma)^m x \neq 0$ for all $m > 1$ and some $\gamma \in \Gamma$.

Next, let $a \in I$ and suppose $(a\gamma)^n a = a$ for some integer $n > 1$. Then

$$\{(a\gamma)^{n-1} a\}\gamma\{(a\gamma)^{n-1} a\} = \{(a\gamma)^{n-1} a\gamma\}\{(a\gamma)^{n-1} a\} = (a\gamma)^n (a\gamma)^{n-1} a$$
$$= (a\gamma)^n a\gamma(a\gamma)^{n-2} a = a\gamma(a\gamma)^{n-2} a = (a\gamma)^{n-1} a,$$

so $(a\gamma)^{n-1} a$ is an idempotent element.

Next, we show that an idempotent element commutes with every elements of M. To show this let e be an idempotent element of

M. Then for any $x \in M$, $(x\gamma e - e\gamma x\gamma e)\gamma(x\gamma e - e\gamma x\gamma e) = 0 = (e\gamma x - e\gamma x\gamma e)\gamma(e\gamma x - e\gamma x\gamma e)$. Thus, $x\gamma e - e\gamma x\gamma e = e\gamma x - e\gamma x\gamma e = 0$ and so $x\gamma e = e\gamma x$.

Now, for any $r \in M$ and $a \in I$ with $(a\gamma)^n a = a$,

$r\gamma a = r\gamma(a\gamma)^n a = r\gamma(a\gamma)^{n-1}(a\gamma)a = (a\gamma)^{n-1} a\gamma r\gamma a$

$= (a\gamma)(a\gamma)^{n-2} a\gamma r\gamma a = a\gamma(a\gamma)^{n-1} r\gamma a = a\gamma r'$, where $r' = (a\gamma)^{n-1} r\gamma a \in M$. Since $a\gamma r' \in I$, so does $r\gamma a$ and so I is a two-sided ideal. ■

***Lemma* 5.02.** *Let M be a J-Γ-ring and I an ideal of M. Then M/I is J-Γ-ring.*

Proof. Let $x \in M/I$, then $x = m + I$ for all $m \in M$ with $(m\gamma)^n m = m$, $n > 1$ and $\gamma \in \Gamma$. Now, $(x\gamma)^n x = \{(m + I)\gamma\}^n (m + I) = \{m\gamma + I\}^n (m + I) = \{(m\gamma)^n + I\}(m + I) = (m\gamma)^n m + I = m + I = x$. Thus, M/I is a J-Γ-ring. ■

***Lemma* 5.03.** *Let D be a division J-Γ-ring of characteristic $p \neq 0$ and let C be the center of D. Suppose that $a \in D$, $a \notin C$ is such that $(a\gamma)^{p^h} a = a$ for some $h > 0$. Then, there exists an element $x \in D$ such that $x\gamma a\gamma x^{-1} \neq a$.*

Proof. We define the mapping $f: D \to D$ by $f(x) = x\gamma a - a\gamma x$ for every $x \in D$. Now, $f^2(x) = ff(x) = f(x\gamma a - a\gamma x) = (x\gamma a - a\gamma x)\gamma a - a\gamma(x\gamma a - a\gamma x) = x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x$.

Again, $f^3(x) = f(x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x) = (x\gamma a\gamma a - 2a\gamma x\gamma a + a\gamma a\gamma x)\gamma a = (x\gamma a\gamma a\gamma a - 3a\gamma x\gamma a\gamma a - 3a\gamma a\gamma x\gamma a + a\gamma a\gamma a\gamma x)$. Thus, a simple computation yields that $f^{p}(x)=x\gamma(a\gamma)^{p}a-(a\gamma)^{p}a\gamma x$, where $\mathrm{char}D=p$, a prime.

Continuing we obtain that $f^{p^{h}}(x)=x\gamma(a\gamma)^{p^{k}}a-(a\gamma)^{p^{k}}a\gamma x$, for all $k\geq 0$. Let P denote the prime Γ-field of C; since a is algebraic over P, $P(a)$ must be a finite field having p^{m} elements, say. Hence $(a\gamma)^{p^{m}}a=a$ and so

$$f^{p^{m}}(x)=x\gamma(a\gamma)^{p^{m}}-(a\gamma)^{p^{m}}a\gamma x=x\gamma a-a\gamma x=f(x).$$

Thus, we see that the function $f^{p^{m}}=f$.

If $r\in P(a)$, then $f(r\gamma x)=(r\gamma x)\gamma a-a\gamma(r\gamma x)=r\gamma(x\gamma a-a\gamma x)=r\gamma f(x)$, since r commutes with a. If I denotes the identity map on D and rI denotes the map defined by $(rI)(x)=r\gamma x$, we have that $fo(rI)=(rI)of$ for all $x\in P$ $P(a)$. Since all elements of $P(a)$satisfy the polynomial $t^{p^{m}}-t$, we find that $t^{p^{m}}-t=\prod_{r\in P(a)}(t-r)$. Since rI commutes with f, we have that $0=f^{p^{m}}-f=\prod_{r\in P(a)}(f-r)$, where $(f-rI)(x)=f(x)-r\gamma x$. Now, Let $r_1=0$ (one of r's must be zero), and suppose for each $r_i\neq 0$, $(f-r_iI)\neq 0$, all $x\in D$, $x\neq 0$. Then $[(f-r_2I)o(f-r_3I)o$ .... $o(f-r_{p^{m}}I)](x)\neq 0$, for all $x\in D$, $x\neq 0$. But since $0=f^{p^{m}}=fo(f-r_2I)o(f-r_2I)o(f-r_2I)o$ ....
.... $o(f-r_{p^{m}}I)$, it follows that $f(x)=0$ for all $x\in D$. Thus, $0=f(x)=x\gamma a-a\gamma x$, whence $x\gamma a=a\gamma x$ for all $x\in D$. Thus, $a\in C$, contradicting the hypothesis. Thus, there is a $r_i\neq 0$, $r_i\in P(a)$ and $x\neq 0$ in D such that

$(f - r_i I)(x) = 0,$

i.e. $(f(x) - r_i I)(x) = 0$

i.e. $x\gamma a - a\gamma x - r_i \gamma x = 0$

i.e. $x\gamma a - a\gamma x = r_i \gamma x$

i.e. $x\gamma a\gamma x^{-1} - a\gamma x\gamma x^{-1} = r_i \gamma x\gamma x^{-1}$

i.e. $x\gamma a\gamma x^{-1} = r_i \gamma x\gamma x^{-1} + a\gamma x\gamma x^{-1} \neq a$, since $r_i \neq 0$.

This completes the proof. ■

***Lemma* 5.04.** *If D is a division Γ-ring of characteristic $p \neq 0$ and $G \subseteq D$ is a finite multiplicative subgroup of D, then G is commutative.*

Proof. let P be the prime field of D and let $A = \{r_i \gamma g_i / r_i \in D$ and $g_i \in G\}$. Clearly A is a finite subgroup of D under addition; moreover, since G is a group under multiplication, A is finite sub-Γ-ring of D. Therefore A is a finite division Γ-ring, hence is commutative. Since $G \subseteq A$, G is also commutative. ■

***Lemma* 5.05.** *Let D be a division Γ-ring such that for every $x \in D$ there exists an integer $n = n(x, \gamma) > 1$ such that $(x\gamma)^n x = x$. Then D is commutative.*

Proof. Suppose $a, b \in D$ are such that $c = a\gamma b - b\gamma a \neq 0$. By hypothesis $(c\gamma)^m c = c$ for some $m > 1$. If $r(\neq 0) \in C$, the center of D, then $r\gamma c = r\gamma(a\gamma b - b\gamma a) = (r\gamma a)\gamma b - b\gamma(r\gamma a)$, hence by hypothesis, $\{(r\gamma c)\gamma\}^n (r\gamma c) = r\gamma c$. Let $q = (m-1)(n-1) + 1$, $m > 1$, $n > 1$. Then $q > 1$. It follows that $(c\gamma)^q c = c$ and $\{(r\gamma c)\gamma\}^q (r\gamma c) = r\gamma c$, hence

$\{(r\gamma c)\gamma(r\gamma c)\gamma(r\gamma c)\gamma \ldots\ldots$ up to q times$\}(r\gamma c) = r\gamma c$

i.e. $(r\gamma)^q (c\gamma)^q (r\gamma c) = r\gamma c,$

i.e. $(r\gamma)^q (c\gamma)^q (c\gamma r) = r\gamma c$,

i.e. $(r\gamma)^q c\gamma r = r\gamma c$,

i.e. $(r\gamma)^q r\gamma c = r\gamma c$,

i.e. $\{(r\gamma)^q r - r\}\gamma c = 0$.

Since D is a division Γ-ring and $c \neq 0$, so $(r\gamma)^q r = r$ for every $r \in C$, $q > 1$ depending on r and γ. We know that C is of characteristic $p \neq 0$. Let P be the prime field of C.

We claim that if D is not commutative, we could have chosen our a, b such that not only is $c = a\gamma b - b\gamma a \neq 0$ but, in fact, c is not even in C. If not, all commutators are in C; hence $c \in C$ and C contains $a\gamma(a\gamma b) - (a\gamma b)\gamma a = a\gamma(a\gamma b) - a\gamma(b\gamma a) = a\gamma(a\gamma b - b\gamma a) = a\gamma c$. This would place $a \in C$ contrary to $c = a\gamma b - b\gamma a \neq 0$. Thus, we assume that $c = a\gamma b - b\gamma a \notin C$. Since $(c\gamma)^m c = c$, c is algebraic over P hence $(c\gamma)^{p^k} c = c$ for some $k > 0$. Thus, all the hypothesis of the *Lemma* 5.03 are satisfied for C. Hence we can find $x \in D$ such that $x\gamma c\gamma x^{-1} = c_1 \neq c$, that is $x\gamma c = c_1\gamma x \neq c\gamma x$. In particular, $d = x\gamma c - c\gamma x \neq 0$; but $d\gamma c = x\gamma c\gamma c - c\gamma x\gamma c = c_1\gamma x\gamma c - c\gamma c_1\gamma x = c_1\gamma x\gamma c - c_1\gamma c\gamma x$ (since $c_1 \in C$) $= c_1\gamma(x\gamma c - c\gamma x) = c_1\gamma d$. As a commutator, $(d\gamma)^t d = d$ for some $t > 1$ and $d\gamma c\gamma d = c_1$. Thus, the multiplicative subgroup of D generated by c and d is finite. Hence by *Lemma* 5.04, the multiplicative subgroup is abelian. This contradicts $c\gamma d \neq d\gamma c$ and proves the lemma. ■

***Lemma* 5.06.** *Let M be a J-Γ-ring with identity* 1. *Then for $x, y \in M$, $x\gamma y - y\gamma x$ is in the intersection of the maximal ideals of M*

Proof. Every ring has a maximal ideal. Let I be such a maximal ideal. Then the quotient ring M/I has an identity, and since I is a maximal right ideal of M, M/I has no maximal ideals other than 0 and M/I, Thus, M/I is a division ring. Since M is a J-Γ-ring, M/I is a J-Γ-ring (by *Lemma* 5.02). Then by *Lemma* 5.05, M/I is commutative. From this it follows that $x\gamma y - y\gamma x \in I$ for all $x, y \in M$. The conclusion of the lemma is now immediate. ■

***Lemma* 5.07.** *Let M be J-Γ-ring with identity* 1. *Then M is commutative.*

Proof. Suppose $x \neq 0$ is in every maximal ideal of M. Then $(x\gamma)^n x = x$, and $(x\gamma)^{n-1} x$ is an idempotent, say $(x\gamma)^{n-1} x = e \neq 0$ for all $n > 1$ and some $\gamma \in \Gamma$ and e must also be in every maximal ideal of M. Now, $1 - e$ can not be in any proper right ideal of M, for if it were, it would be in a maximal ideal K of M. Since $e \in K$, $1 = e + (1 - e)$ would be in K and hence $K = M$, a contradiction. Since $(1 - e)\gamma M \neq 0$ and since $(1 - e)\gamma M$ is a (right) ideal, it follows that $(1 - e)\gamma M = M$, whence $(1 - e)\gamma r = e$ for some $r \in M$. Thus, $0 = e\gamma(1 - e)\gamma r = e$, a contradiction. Thus, x can not be in every maximal ideal in M and the intersection of all the maximal ideals of M is 0. Thus, by *Lemma* 5.06, $x\gamma y - y\gamma x \in 0$, $x, y \in M$, that is, $x\gamma y = y\gamma x$ for all $x, y \in M$. ■

***Theorem* 5.08.** *If M is a J-Γ-ring, then M is commutative.*

Proof. Let e be an idempotent in M. Then, $e\gamma x = x\gamma e$ for all $x \in M$. Thus, $e\gamma M = M\gamma e = T$ is also a J-Γ-ring, but T has an identity, namely e. Hence by *Lemma* 5.07, T is commutative. Now, for all x, $y \in M$,

$x\gamma y\gamma e = x\gamma y\gamma e\gamma e = (x\gamma e)\gamma(y\gamma e) = (y\gamma e)\gamma(x\gamma e) = y\gamma x\gamma e$, that is $(x\gamma y - y\gamma x)\gamma e = 0$. Since $(x\gamma y - y\gamma x)\gamma\}^n(x\gamma y - y\gamma x) = (x\gamma y - y\gamma x)$ for some integer $n > 1$, so $\{(x\gamma y - y\gamma x)\gamma\}^{n-1}(x\gamma y - y\gamma x)$ is an idempotent, say e_1. Thus, $0 = (x\gamma y - y\gamma x)e_1 = \{(x\gamma y - y\gamma x)\gamma\}^n(x\gamma y - y\gamma x) = x\gamma y - y\gamma x$, that is, $x\gamma y = y\gamma x$. ■

***Lemma* 5.09.** *Let M be a commutative Γ-ring. Let I be an ideal of M such that I a J-Γ-ring. Then $e\gamma\{y - (y\gamma)^n y\} = 0$ for all $y \in M$ and some $\gamma \in \Gamma$ and e is an idempotent of I.*

Proof. Let $x \in I$ and $y \in M$. Then $x\gamma y \in I$. Since I is a J-Γ-ring, $(x\gamma)^n x = x$. Also $\{(x\gamma y)\gamma\}^n(x\gamma y) = x\gamma y$ for some $n > 1$and $\gamma \in \Gamma$.

Now, $\{(x\gamma y)\gamma\}^n(x\gamma y) = x\gamma y$,

i.e. $\{(x\gamma y)\gamma(x\gamma y)\gamma \ldots\ldots$up to n times$\}(x\gamma y) = x\gamma y$,

i.e. $(y\gamma)^n(x\gamma)^n(x\gamma y) = x\gamma y$, since M is commutative,

i.e. $(y\gamma)^n x\gamma y = x\gamma y$.

i.e. $\{(y\gamma)^n y - y\}\gamma x = 0$,

so $(x\gamma)^{n-1}x\gamma\{y - (y\gamma)^n y\} = 0$ and hence $e\gamma\{y - (y\gamma)^n y\} = 0$, where $e = (x\gamma)^{n-1}x$ is an idempotent of I. ■

***Lemma* 5.10.** *Let M be a Γ-ring and I an ideal of M. Then M is a J-Γ-ring if M/I and I are J-Γ-rings.*

Proof. Let M/I and I be J-Γ-rings. Let $x \in M$, then $x + I \in M/I$ and so $\{(x + I)\gamma\}^n(x + I) = x + I$ for some $n > 1$ and $\gamma \in \Gamma$.

i.e. $(x\gamma + I)^n(x + I) = x + I$,

i.e. $\{(x\gamma)^n + I\}(x + I) = x + I$,

i.e. $(x\gamma)^n x + I = x + I$.

Thus, $(x\gamma)^n x - x \in I$. Since I is a *J-Γ*-ring, $\{((x\gamma)^n x - x)\gamma\}^m \{(x\gamma)^n x - x\}$ $= (x\gamma)^n x - x$ for some $m > 1$. Let $e' = \{((x\gamma)^n x - x)\gamma\}^{m-1}\{(x\gamma)^n x - x\}$. Then e' is an idempotent of I. By *Lemma* 5.08, $e'\gamma\{(x\gamma)^n x - x\} = 0$ for every $x \in M$. Now, $0 = e'\gamma\{(x\gamma)^n x - x\} = \{((x\gamma)^n x - x)\gamma\}^{m-1}\{(x\gamma)^n x - x\}\gamma\{(x\gamma)^n x - x\} = \{((x\gamma)^n x - x)\gamma\}^m \{(x\gamma)^n x - x\} = (x\gamma)^n x - x)$. Hence $(x\gamma)^n x = x$. Therefore M is a *J-Γ*-ring. ■

***Lemma* 5.11.** *If $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ - - - - - - is an ascending chain of ideals which are all J-Γ-rings, then $\cup_\alpha I_\alpha$ is a J-Γ-rings.*

Proof. Let $x \in \cup_\alpha I_\alpha$, then $x \in I_\alpha$ for some α. Since I_α is a *J-Γ*-ring, then $(x\gamma)^n x = x$ for some $n > 1$ and $\gamma \in \Gamma$. Hence $\cup_\alpha I_\alpha$ is a *J-Γ*-ring. ■

Thus, by *Lemma* 5.02, *Lemma* 5.10 and *Lemma* 5.11, we have the following theorem:

***Theorem* 5.12.** *The class of all J-Γ-rings is a radical class.*

Some characterizations of *J-Γ*-rings.

***Theorem* 5.13.** *Let M be a Γ-ring with 1. Let a, $x \in M$ such that $a = (x\gamma)^{n-2}x$. Then the following statements are equivalent:*

a) M is a J-Γ-ring.

b) Every principal ideal $M\gamma a$ is generated by an idempotent.

c) For every principal ideal $M\gamma a$ of M, there exists an element $b \in M$ such that $M = M\gamma a \oplus M\gamma b$.

d) Every principal ideal $M\gamma a$ is a direct summand of M.

Proof. ***a)*** $\Rightarrow$ ***b)*** Let $x \in M$. Then $(x\gamma)^n x = x$ for some $n > 1$and $\gamma \in \Gamma$. Let $a \in M$ such that $a = (x\gamma)^{n-2}x$. Now, the principal ideal $M\gamma a$ is generated by the element $x\gamma a$ which is idempotent; for,

$$(x\gamma a)\gamma(x\gamma a) = x\gamma\{(x\gamma)^{n-2}x\}\gamma\{x\gamma(x\gamma)^{n-2}x\} = (x\gamma)^n x\gamma(x\gamma)^{n-2}x = x\gamma a.$$

b) $\Rightarrow$ ***c)*** Let $M\gamma a = M\gamma e$, where $e\gamma e = e$ and $a = (x\gamma)^{n-2}x$, $x \in M$. Since $1 = e + (1 - e)$, and if there exists $b \in M$ such that $a\gamma e = b\gamma(1 - e)$, then $a\gamma e = a\gamma e\gamma e = b\gamma(1- e)\gamma e = 0$. So $M = M\gamma e \oplus M\gamma(1 - e)$.

c) $\Rightarrow$ ***d)*** Trivial.

d) $\Rightarrow$ ***a)*** Let $a \in M$. Then there exists an ideal I of M such that $M = M\gamma a \oplus I$. Hence $1 = x\gamma a + b$, where $b \in I$, so $x = x\gamma a\gamma x + b\gamma x$. Since $a = (x\gamma)^{n-2}x$, $b\gamma x = x - x\gamma a\gamma x \in M\gamma a \cap I = 0$, and therefore $x = x\gamma\{(x\gamma)^{n-2}x\}\gamma x = (x\gamma)^n x$. Hence M is a J-Γ-ring. ■

***Theorem* 5.14.** *Let M be a J-Γ-ring with* 1. *Then*

1) *Every finitely generated ideal is principal.*
2) *The intersection of any two principal ideals of M is principal.*

Proof. 1) It is enough to prove that if a, $b \in M$, then $M\gamma a + M\gamma b$ is principal. Since M is a J-Γ-ring, there exists elements x, $y \in M$ with $a = (x\gamma)^{n-2}x$ and $b = (y\gamma)^{n-2}y$ such that the elements $e_1 = x\gamma a$ and $e_2 = y\gamma b$ are the idempotent elements of $M\gamma a$ and $M\gamma b$ respectively and also $M\gamma a = M\gamma e_1$ and $M\gamma b = M\gamma e_2$ by *Theorem* 5.13(*b*). Now, $M\gamma a + M\gamma b = M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma(e_2 - e_2\gamma e_1)$ because $a_1\gamma e_1 + a_2\gamma e_2 = (a_1 + a_2\gamma e_2)\gamma e_1 + a_2\gamma(e_2 - e_2\gamma e_1)$. If $s = \{(e_2 - e_2\gamma e_1)\gamma\}^{n-2}(e_2 -$

$e_2\gamma e_1)\in M$, then $(e_2 - e_2\gamma e_1)\gamma s\gamma(e_2 - e_2\gamma e_1) = \{(e_2 - e_2\gamma e_1)\gamma\}^n(e_2 - e_2\gamma e_1) = (e_2 - e_2\gamma e_1)$. Then $e_2' = s\gamma(e_2 - e_2\gamma e_1)$ is an idempotent of $M\gamma b$. Then $M\gamma e_1 + M\gamma e_2 = M\gamma e_1 + M\gamma e_2'$ with $e_2'\gamma e_1 = s\gamma(e_2 - e_2\gamma e_1)\gamma e_1 = 0$.

Finally, we have, $a_1\gamma e_1 + a_2\gamma e_2' = (a_1\gamma e_1 + a_2\gamma e_2')\gamma(e_1 + e_2' - e_2'\gamma e_1)$. Thus, $M\gamma e_1 + M\gamma e_2' = M\gamma(e_1 + e_2' - e_2'\gamma e_1)$. Therefore $M\gamma a + M\gamma b = M\gamma(e_1 + e_2' - e_2'\gamma e_1)$. Thus, $M\gamma a + M\gamma b$ is a principal ideal.

2) Let $M\gamma a$ and $M\gamma b$ be two principal ideals. Since M is a J-Γ-ring by *Theorem* 5.13(b), there exists elements $x, y\in M$ with $a = (x\gamma)^{n-2}x$ and $b = (y\gamma)^{n-2}y$ such that the elements $e_1 = x\gamma a$ and $e_2 = y\gamma b$ are the idempotents of $M\gamma a$ and $M\gamma b$ respectively and also $M\gamma a = M\gamma e_1$ and $M\gamma b = M\gamma e_2$. Hence $M = M\gamma e_1 \oplus M\gamma(1 - e_1) = M\gamma e_2 \oplus M\gamma(1 - e_2)$, and

$$M\gamma e_1 = Ann_M[(1 - e_1)\gamma M] = \{x\in M \mid x\gamma(1 - e_1)\gamma M = 0\},$$
$$M\gamma e_2 = Ann_M[(1 - e_2)\gamma M] = \{x\in M \mid x\gamma(1 - e_2)\gamma M = 0\}.$$

Indeed obviously $M\gamma e_1 \subseteq Ann_M[(1 - e_1)\gamma M]$.

Conversely, if $x\in M$ and $x\gamma(1 - e_1) = 0$, writing $x = a_1\gamma e_1 + b_1\gamma(1 - e_1)$, we have, $a_1\gamma e_1\gamma(1 - e_1) + b_1\gamma(1 - e_1)\gamma(1 - e_1) = 0$, and so $b_1\gamma(1 - e_1) = 0$, hence $x = a_1\gamma e_1\in M\gamma e_1$.

Thus, $M\gamma e_1 \cap M\gamma e_2 = Ann_M[(1 - e_1)\gamma M + (1 - e_2)\gamma M]$. Now, there exists $e_3\in M$ such that $(1 - e_1)\gamma M + (1 - e_2)\gamma M = (1 - e_3)\gamma M$, and from $M\gamma e_3 = Ann_M[(1 - e_3)\gamma M]$ we deduce that $M\gamma e_1 \cap M\gamma e_2 = M\gamma e_3$. Thus, $M\gamma e_1 \cap M\gamma e_2 = M\gamma a \cap M\gamma b$ is a principal ideal. ■

***Theorem* 5.15.** *Let M be a J-Γ-ring with unity* 1. *Then*

a) The Jacobson radical $\mathfrak{J}(M)$ *of M is zero.*

b) M is a semi-simple ring if and only if it is a Noetherian J-Γ-ring.

c) The centre of M is also a J-Γ-ring.

d) The J-Γ-ring M without zero-divisor is a field.

e) Every ideal of M is non-singular.

f) For any idempotent element e of M, $(1-e)\gamma M\gamma e = 0$.

g) If $(M_i)_{i\in I}$ *is a family of J-Γ-rings then* $\prod M_i$ *is a J-Γ-ring.*

h) M is semi-hereditary.

Proof. a) Let $a\in \mathfrak{J}(M)$. Then $M\gamma a \subseteq \mathfrak{J}(M)$. Since $M\gamma a = M\gamma e$, where $e = x\gamma a$ is an idempotent with $a = (x\gamma)^{n-2}x$, so $e\in \mathfrak{J}(M)$. It follows that $(1-e)$ is invertible. So there exists $y\in M$ such that $1 = y\gamma(1-e) = y - y\gamma e$. Hence $e = y\gamma e - y\gamma e\gamma e = y\gamma e - y\gamma e = 0$ and therefore $a = 0$. Thus, $\mathfrak{J}(M) = 0$.

b) First suppose that M is finitely generated. Then every ideal of M is finitely generated and hence a direct summand. So M is a semi-simple.

Conversely, let M be a semi-simple ring. Then every principal ideal of M is a direct summand of M and hence M is a *J-Γ*-ring by *Theorem* 5.13(*d*). Since Jacobson radical $\mathfrak{J}(M)$ is the largest ideal of M and since in a *J-Γ*-ring, $\mathfrak{J}(M) = 0$, so any ascending chain of ideals of M must be finite. Hence M is Noetherian.

c) Since *J-Γ*-ring is abelian, so centre of M is M itself, i.e. $C(M) = M$.

d) Let $a\in M$ with $a\neq 0$. Then $(a\gamma)^n a = a$ for some $n > 1$. Then $(a\gamma)^n a - a = 0 \Rightarrow a\gamma\{(a\gamma)^{n-1}a - 1\} = 0$. Since $a\neq 0$, so $(a\gamma)^{n-1}a - 1$

$= 0$ and so $(a\gamma)^{n-2}a$ is the inverse of a. Since J-Γ-ring M is abelian, so M is a field.

e) Suppose that $x\gamma I = 0$ for some $x \in M$ and $I \subseteq M$ is an ideal of M. Let $M\gamma x$ be a principal ideal of M. Then there is an idempotent $e \in M$ such that $M\gamma x = M\gamma e$. Now, since $M\gamma e\gamma I = M\gamma x\gamma I = 0$, we see that $I \subseteq M\gamma(1 - e)$. Then $I \cap e\gamma M = 0$, whence $M\gamma e = 0$ and consequently $x = 0$. Thus, M is non-singular.

f) Since $M\gamma e$ is a two-sided ideal, so $(1 - e)\gamma M\gamma e = M\gamma e - M\gamma e\gamma e = M\gamma e - M\gamma e = 0$.

g) Proof is obvious.

h) Since a finitely generated ideal of M is a direct summand of M and so is projective. Hence M is semi-hereditary. ■

Chapter-Six

Regular Gamma Rings

S. Kyuno, N. Nobusawa and B. Smith [14] defined a certain type of regular gamma rings and they developed their various properties. In this chapter, we have defined another type of regular gamma rings that are more significant and more general than that of S. Kyuno [14]. The main emphasis is on developing sufficient conditions for gamma rings to be regular. In 6.07 and 6.27, we derive the most fundamental and widely used properties of regular gamma rings, namely the large supply of idempotents of regular gamma rings, and the large supply of direct summands of projective modules over regular gamma rings.

Definition. *Let M be a Γ-ring. An element $a \in M$ is said to be regular in M if there exists γ, $\mu \in \Gamma$ and $x \in M$ such that $a = a\mu x\gamma a$. A Γ-ring M is said to be regular if all of its elements are regular.*

***Lemma* 6.01.** *If a is regular in M, then $[a, \mu]$ is regular in L, where L is the left operator ring in M.*

Proof. Since a is regular in M, $a = a\mu x\gamma a$ for some μ, $\gamma \in \Gamma$ and $x \in M$. This implies that $a\mu = a\mu x\gamma a\mu$, and hence $[a, \mu] = [a, \mu][x, \gamma][a, \mu]$ for some $\mu \in \Gamma$. Therefore $[a, \mu]$ is regular in L. ■

***Lemma* 6.02.** *If a is regular in M, then $[\gamma, a]$ is regular in R, where R is the right operator ring in M.*

Proof. Since a is regular in M, $a = a\mu x\gamma a$ for some μ, $\gamma\in\Gamma$ and $x\in M$. This implies that $\gamma a = \gamma a\mu x\gamma a$, and hence $[\gamma, a] = [\gamma, a][\mu, x][\gamma, a]$ for some $\gamma\in\Gamma$. Therefore $[\gamma, a]$ is regular in R. ■

***Lemma* 6.03.** *If $a, c\in M$, $a-c$ is regular in M and c is in $a\mu M\gamma a$, then a is regular.*

Proof. Since $a-c$ is regular, there exists μ, $\gamma\in\Gamma$ and $x\in M$ such that

$$a-c = (a-c)\mu x\gamma(a-c)$$
$$= a\mu x\gamma a - c\mu x\gamma a - a\mu x\gamma c + c\mu x\gamma c$$

This implies that $a = a\mu x\gamma a - c\mu x\gamma a - a\mu x\gamma c + c\mu x\gamma c + c$

Since $c\in a\mu M\gamma a$, then there exists $y\in M$ such that $c = a\mu y\gamma a$. Therefore

$$a = a\mu x\gamma a - a\mu y\gamma a\mu x\gamma a - a\mu x\gamma a\mu y\gamma a + a\mu y\gamma a\mu x\gamma a\mu y\gamma a + a\mu y\gamma a,$$

so that

$$a = a\mu(x - y\gamma a\mu x - x\gamma a\mu y + y\gamma a\mu x\gamma a\mu y + y)\gamma a$$
$$= a\mu x'\gamma a, \text{ where } x' = (x - y\gamma a\mu x - x\gamma a\mu y + y\gamma a\mu x\gamma a\mu y + y)\in M.$$

Therefore a is regular in M. ■

Definition. *Let I be an ideal of a Γ-ring M. If every element of I is regular, then I is regular.*

***Lemma* 6.04.** *If M is a regular Γ-ring and J is a two-sided ideal of M, then M/J is regular.*

Proof. Let $\bar{a}\in M/J$. Then $\bar{a} = a + J$, $a\in M$. Since M is regular, there exists μ, $\gamma\in\Gamma$ and $x\in M$ such that $a = a\mu x\gamma a$. Now, $\bar{a}\mu x\gamma\bar{a} = (a+J)\mu x\gamma(a+J) = a\mu x\gamma a + J = a + J = \bar{a}$. Therefore M/J is regular. ■

***Lemma* 6.05.** *If M/I and I are regular Γ-rings, then M is regular.*

Proof. Since M/I is regular for any $a \in M$, $a + I = (a + I)\mu x\gamma(a + I) = a\mu x\gamma a + I$ for some $\mu, \gamma \in \Gamma$ and $x \in M$. This implies that $a - a\mu x\gamma a \in I$. Since I is regular and $a\mu x\gamma a \in a\mu M\gamma a$, then by *Lemma* 6.03, a is regular. This implies that M is regular. ■

***Lemma* 6.06.** *Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ .... be the ascending chain of regular ideals. Then $\cup_\alpha I_\alpha$ is regular.*

Proof: It is obvious. ■

From the *Lemma* 6.04, *Lemma* 6.05 and *Lemma* 6.06, we have the following:

***Theorem* 6.07.** *The class of all regular Γ-rings is a radical class.*

***Theorem* 6.08(*a*).** *Let M be a Γ-ring with unity. Then the following statements are equivalent:*

i) *M is a regular Γ-ring.*
ii) *Every principal left ideal $M\Gamma a$ is generated by an idempotent.*
iii) *For every principal left ideal $M\Gamma a$ of M, there exists $b \in M$ such that $M = M\Gamma a \oplus M\Gamma b$.*
iv) *Every principal left ideal $M\Gamma a$ is a direct summand of M.*

Proof. **(*i*) $\Rightarrow$ (*ii*)** Given $a \in M$. Then there exists $\mu, \gamma \in \Gamma$ and $x \in M$ such that a $= a\mu x\gamma a$. Then $M\Gamma a$ is generated by $x\gamma a$ which is an idempotent element, for $(x\gamma a)\mu(x\gamma a) = x\gamma(a\mu x\gamma a) = x\gamma a$.

(*ii*) $\Rightarrow$ (*iii*) Let $M\gamma a = M\gamma e$ for some $\gamma \in \Gamma$, where $e = e\gamma e$. Since $1 = e + (1 - e)$, $M\gamma 1 = M\gamma e + M\gamma(1 - e)$ for some $\gamma \in \Gamma$. Therefore

$M = M\gamma e + M\gamma(1 - e)$. This implies that $M = M\Gamma e + M\Gamma(1 - e)$. If $a, b \in M$ such that $a\mu e = b\mu(1 - e)$ for some $\mu \in \Gamma$, then $a\mu e = a\mu e\gamma e = b\mu(1 - e)\gamma e = b\mu(1\gamma e - e\gamma e) = b\mu(e - e) = b\mu 0 = 0$. Hence $M = M\Gamma e \oplus M\Gamma(1 - e)$. This gives $M = M\Gamma a \oplus M\Gamma b$ with $b = (1 - e)$.

(*iii*) $\Rightarrow$ *iv*) It is trivial.

(*iv*) $\Rightarrow$ (*i*) Let $a \in M$. Then there exists an ideal I of M such that $M = M\Gamma a \oplus I$. Hence $1 = x\gamma a + b$, where $x \in M$, $b \in I$, so that $a = a\mu 1 = a\mu(x\gamma a + b) = a\mu x\gamma a + a\mu b$. This implies $a\mu b = a - a\mu x\gamma a$. Since $a\mu b \in M\Gamma a \cap I = 0$, then $a - a\mu x\gamma a = 0$ and so $a = a\mu x\gamma a$. Hence M is a regular Γ-ring. ■

***Theorem* 6.08(*b*).** *Let M be a Γ-ring with unity. The following statements are equivalent:*

i) M is a regular Γ-ring.

ii) Every principal right ideal $a\Gamma M$ is generated by an idempotent.

iii) For every principal right ideal $a\Gamma M$ of M, there exists $b \in M$ such that $M = a\Gamma M \oplus b\Gamma M$.

iv) Every principal right ideal $a\Gamma M$ is a direct summand of M.

Proof. The proof is similar to the proof of *Theorem* 6.08(a). ■

***Theorem* 6.09.** *If M is regular, then every finitely generated left (right) ideal is principal.*

Proof. (i) Let $a, b \in M$. We have to prove that then $M\Gamma a + M\Gamma b$ is principal. Since M is regular, every principal ideal is generated by some idempotents of M. So it is enough to prove that $M\gamma e_1 + M\mu e_2$ is principal (with e_1& e_2 idempotents) and $\gamma, \mu \in \Gamma$.

Now, $M\gamma e_1 + M\mu e_2 = M\gamma e_1 + M\mu(e_2 - e_2\gamma e_1)$, for

$a_1\gamma e_1 + a_2\mu e_2 = (a_1 + a_2\mu e_2)\gamma e_1 + a_2\mu(e_2 - e_2\gamma e_1)$.

If there exists an element $x \in M$ such that $(e_2 - e_2\gamma e_1)\xi x\delta(e_2 - e_2\gamma e_1) = (e_2 - e_2\gamma e_1)$ for some $\xi, \delta \in \Gamma$, then $x\delta(e_2 - e_2\gamma e_1) = e_2'$ is an idempotent of $M\Gamma b$ and so $M\gamma e_1 + M\mu e_2 = M\gamma e_1 + M\mu e_2'$ with

$$\begin{aligned} e_2'\gamma e_1 &= x\delta(e_2 - e_2\gamma e_1)\gamma e_1 \\ &= x\delta(e_2\gamma e_1 - e_2\gamma e_1\gamma e_1) \\ &= x\delta(e_2\gamma e_1 - e_2\gamma e_1) = 0. \end{aligned}$$

Finally, $M\gamma e_1 + M\mu e_2' = M\gamma(e_1 + e_2' - e_1\gamma e_2')$, because $a_1\gamma e_1 + a_2\mu e_2' = (a_1\gamma e_1 + a_2\mu e_2')\gamma(e_1 + e_2' - e_1\gamma e_2')$ for some $\gamma \in \Gamma$.

Similarly, we can prove that every finitely right ideal is principal. ■

***Theorem* 6.10.** *Let M be a regular Γ-ring. Then the intersection of any two principal left (right) ideals of M is principal.*

Proof. It is enough to prove that if $a, b \in M$ then $M\Gamma a \cap M\Gamma b$ is principal. To prove this we choose $e_1 = x\gamma a$ and $e_2 = y\delta b$, where $x, y \in M$ and $\gamma, \delta \in \Gamma$ are such that $a = a\mu x\gamma a$ and $b = b\mu y\delta b$. Then e_1 and e_2 are idempotents and $M\gamma a = M\gamma e_1$, $M\delta b = M\delta e_2$.

Hence $M = M\gamma e_1 \oplus M\mu(1 - e_1) = M\delta e_2 \oplus M\eta(1 - e_2)$, and

$$M\gamma e_1 = Ann_M[(1 - e_1)\mu M] = \{x \in M \mid x\gamma(1 - e_1)\mu M = 0\},$$

$$M\delta e_2 = Ann_M[(1 - e_2)\eta M] = \{x \in M \mid x\delta(1 - e_2)\eta M = 0\},$$

for some $\gamma, \delta, \eta \in \Gamma$.

Indeed obviously $M\gamma e_1 \subseteq Ann_M[(1 - e_1)\mu M]$.

Conversely, if $x \in M$ and $x\mu(1 - e_1) = 0$, writing $x = a_1\mu e_1 + b_1\mu(1 - e_1)$, we have

$$a_1\mu e_1\mu(1 - e_1) + b_1\mu(1 - e_1)\mu(1 - e_1) = 0,$$

and $b_1\mu(1 - e_1) = 0$, hence $x = a_1\mu e_1 \in M\mu e_1$.

Thus $M\gamma e_1 \cap M\delta e_2 = Ann_M[(1 - e_1)\gamma M + (1 - e_2)\delta M]$, as one may cheek easily. Now, there exists $e_3 \in M$ such that $(1 - e_1)\gamma M + (1 - e_2)\delta M = (1 - e_3)\xi M$ for some $\xi \in \Gamma$ and from $M\xi e_3 = Ann_M[(1 - e_3)\xi M]$, we deduce that $M\gamma e_1 \cap M\delta e_2 = M\xi e_3$.

Similarly, $a\Gamma M \cap b\Gamma M$ is a principal ideal. ■

***Theorem* 6.11.** *The Jacobson radical $\mathfrak{J}(M)$ of a regular Γ-ring M is equal to zero.*

Proof. Let $a \in \mathfrak{J}(M)$. Then $M\Gamma a \subseteq \mathfrak{J}(M)$. Since $M\gamma a$ is generated by an idempotent element e, $M\gamma a = M\gamma e$, and thus from $e \in \mathfrak{J}(M)$ it follows that $(1 - e)$ is invertible. So there exists $x \in M$ such that

$$1 = x\gamma(1 - e) = x\gamma 1 - x\gamma e = x - x\gamma e.$$

Hence $e = 1\gamma e = (x - x\gamma e)\gamma e = x\gamma e - x\gamma e\gamma e = x\gamma e - x\gamma e = 0$.

Therefore, $a = 0$. Hence Jacobson radical of a regular Γ-ring M is equal to zero. ■

***Theorem* 6.12.** *The centre of a regular Γ-ring M is also regular.*

Proof. Let $a \in C(M)$ (center of M). Let $x \in M$ and $\gamma \in \Gamma$ be such that $a = a\mu x\gamma a$.

Now, $a = a\mu x\gamma a = a\mu(x\gamma a) = (x\gamma a)\mu a = x\gamma(a\mu a)$.

Also $a = a\mu x\gamma a = (a\mu x)\gamma a = a\gamma(a\mu x) = (a\gamma a)\mu x$.

So, $a\mu x = a\gamma a\mu x\mu x$, or $a = a\gamma(a\mu x) = a\gamma(a\gamma a\mu x\mu x) = a\gamma(a\gamma x\mu a\mu x) = a\gamma(a\gamma x\mu x)\mu a$. Now, $a\mu x \in C(M)$ because if $y \in M$ then

$(a\mu x)\gamma y = (x\gamma y)\mu a = (x\gamma y)\mu(a\gamma a)\mu x = x\gamma y\mu a\gamma a\mu x = (x\gamma y\mu a\gamma x)\mu a = \mu(x\gamma y\mu a\gamma x) = a\mu x\gamma(y\mu x)\gamma a = a\mu x\gamma a\gamma(y\mu x) = a\gamma y\mu x = y\gamma(a\mu x)$.

Also $a\gamma x\mu x \in C(M)$ because

$(a\gamma x\mu x)\gamma y = (a\gamma x)\mu(x\gamma y) = (x\gamma y)\mu(a\gamma x) = x\gamma y\mu a\gamma x = x\gamma a\mu y\mu x = (a\gamma x)\mu(y\gamma x) = (y\gamma x)\mu(a\gamma x) = y\gamma(x\mu a\gamma x) = y\gamma(x\mu x)\gamma a = y\gamma a\gamma(x\mu x) = y\gamma(a\gamma x\mu x)$.

Hence the centre of M is a regular Γ-ring. ■

***Theorem* 6.13.** *Every regular Γ-ring without zero divisors is a skew Γ-field.*

Proof. Let $a \in M$, $a \neq 0$. Let $x \in M$ and μ, $\gamma \in \Gamma$ be such that $a = a\mu x\gamma a$. Then $a\mu(x\gamma a - 1) = 0$, $(a\mu x - 1)\gamma a = 0$, and hence $x\mu a = 1$, $a\mu x = 1$ for some μ, $\gamma \in \Gamma$ and so a is invertible. Hence M is a skew Γ- field. ■

***Theorem* 6.14.** *If M is a regular Γ-ring whose only nilpotent element is zero, then*

i) Every idempotent element of M is in the centre.

ii) If $a \in M$, $a \neq 0$, then there exists $b \in M$, $\gamma \in \Gamma$ such that $a\gamma b = b\gamma a = f$ is idempotent and $a\gamma f = f\gamma a = a$

iii) $M\Gamma a = a\Gamma M$ for all $a \in M$; hence every left (or right) ideal is a two-sided.

Proof. (*i*) Let $e \in M$ be idempotent. Let $a \in M$ be an arbitrary element and assume that zero is the only nilpotent element of M.

Since $[(1-e)\mu a\gamma e]\mu[(1-e)\mu a\gamma e]$

$= (1\mu a\gamma e - e\mu a\gamma e)\mu(1\mu a\gamma e - e\mu a\gamma e)$

$= (a\gamma e - e\mu a\gamma e)\mu(a\gamma e - e\mu a\gamma e)$

$= a\gamma e\mu a\gamma e - a\gamma e\mu e\mu a\gamma e - e\mu a\gamma e\mu a\gamma e + e\mu a\gamma e\mu e\mu a\gamma e$

$= a\gamma e\mu a\gamma e - a\gamma e\mu a\gamma e - e\mu a\gamma e\mu a\gamma e + e\mu a\gamma e\mu a\gamma e$

$= 0$

and

$[e\gamma a\mu(1 - e)]\mu[e\gamma a\mu(1 - e)]$

$= [e\gamma a\mu 1 - e\gamma a\mu e)\mu(e\gamma a\mu 1 - e\gamma a\mu e)]$

$= e\gamma a\mu e\gamma a - e\gamma a\mu e\mu e\gamma a - e\gamma a\mu e\gamma a\mu e + e\gamma a\mu e\mu e\gamma a\mu e$

$= e\gamma a\mu e\gamma a - e\gamma a\mu e\gamma a - e\gamma a\mu e\gamma a\mu e + e\gamma a\mu e\gamma a\mu e$

$= 0$ for some $\mu, \gamma \in \Gamma$.

Since the only nilpotent element is zero, so we have

$0 = (1 - e)\mu a\gamma e = 1\mu a\gamma e - e\mu a\gamma e = a\gamma e - e\mu a\gamma e,$

and $0 = e\gamma a\mu(1 - e) = e\gamma a\mu 1 - e\gamma a\mu e = e\gamma a - e\gamma a\mu e.$

Hence, $a\gamma e = e\mu a\gamma e = e\gamma a$ and so e is in the centre of M.

(ii) Let M be a regular Γ-ring having 0 as the only nilpotent element. Given $a \in M$, $a \neq 0$, let $x \in M$ be such that $a\mu x\gamma a = a$ for some $\mu, \gamma \in \Gamma$. Then $e = a\mu x$, $e' = x\delta a$ are idempotents elements of M; so e and e' belong to the centre, and $f = e\gamma e'$ is an idempotent.

It follows that $a\mu x\gamma x\delta a = (a\mu x)\gamma(x\delta a) = e\gamma e'$.

Also $(x\mu x\gamma a)\delta a = [x\mu(x\gamma a)]\delta a = [(x\gamma a)\mu x]\delta a = [x\gamma(a\mu x)]\delta a = [(a\mu x)\gamma x]\delta a = (a\mu x)\gamma(x\delta a) = e\gamma e'$.

Moreover, $a\mu f = a\mu e\gamma e' = e\mu a\gamma e' = a\mu x\mu a\gamma e' = a\gamma e' = a\gamma x\delta a = a$ and $f\mu a = e\gamma e'\mu a = e\gamma a\mu e' = e\gamma a\mu x\delta a = e\gamma a = a\mu x\gamma a = a.$

Thus $a\mu f = f\mu a = a$.

(iii) Given $y \in M$, we have $y\mu a = (y\mu a)\gamma e\gamma e' = e\gamma(y\mu a)\gamma e' = (a\mu x)\gamma(y\mu a)\gamma e' = a\mu x\gamma y\mu a\gamma e'$, and so there exists $z \in M$ such that $y\mu a = a\mu z$. This shows that $M\gamma a \subseteq a\gamma M$. The converse can be proved in a similar way.

Hence, since every left ideal J is the sum of the principal left ideals generated by its elements, J is also a right ideal and vice versa. ■

***Corollary* 6.15.** *Let M be a regular Γ-ring. Then*

(i) All one-sided ideals in M are idempotent.

(ii) All two-sided ideals in M are semiprime.

(iii) The Jacobson radical of M is zero.

(iv) M is right and left semihereditary.

(v) M is right and left nonsingular.

Proof. **(*i*)** Let J be a right ideal of M. Since M is regular, for each $a \in J$, $a = a\mu x\gamma a$ for some $\mu, \gamma \in \Gamma$, and $x \in M$. Consequently, $a = a\mu x\gamma a \in J\Gamma M\Gamma J$ and $J\Gamma M\Gamma J \subseteq J\Gamma J$. That is $J \subseteq J\Gamma J$. Also $J\Gamma J \subseteq J$. Hence $J = J\Gamma J$.

(*ii*) Let I be two-sided ideal of M. If A is a two-sided ideal of M such that $A\Gamma A \subseteq I$, then we have to show that $A \subseteq I$. Now, by (i) $A = A\Gamma A \subseteq I$.

(*iii*) Suppose that $e \in M$ is right quasi regular. Then $e = e\gamma x\mu e$ for some $\mu, \gamma \in \Gamma$ and $x \in M$. Let R be a right operator ring of M. Then there exist $r \in M$ such that $[\mu, e]\ o\ r = r + [\mu, e] - [\mu, e]r = 0$.

It follows that $[\mu, e] = [\mu, e]\ o\ 0$

$$= [\mu, e]\ o\ ([\mu, e]\ o\ r)$$

$$= ([\mu, e]\ o\ [\mu, e])\ o\ r$$

$$= [\mu , e] \; o \; r = 0$$

Thus $e = e\gamma x\mu e = e\gamma x[\mu , e] = e\gamma x0 = 0$.

Recall that $\mathfrak{I}(M) = \{e \in M \mid \langle e \rangle \text{ is right quasi regular}\}$. Since $e = 0$, $\langle e \rangle = 0$ and so $\mathfrak{I}(M) = 0$.

Note that in *Theorem* 6.11, this was proved by another method.

(*iv*) According to *Theorem* 6.08(*a*), every finitely generated one-sided ideal of M is a direct summand of M and so is projective.

(*v*) Suppose that $x\gamma J = 0$ for some $x \in M$ and some $J \subseteq M$ with $J \in \varphi(M)$, where $\varphi(M)$ be the class of all essential ideals in M. There is an idempotent $e \in M$ such that $M\mu e = M\mu x$ and since $M\mu e\gamma J = M\mu x\gamma J = 0$, we see that $J \subseteq (1 - e)\mu M$. Then $J \cap e\mu M = 0$ whence $e\mu M = 0$, and so $x = 0$. Thus M is nonsingular. ■

Subdirect sum. *By the direct product (or complete direct sum) of* Γ*-rings* M_r*, r is in some index set* I*, we mean the set* $\prod_{r \in I} M_r = \{f : I \to \cup_{r \in I} M_r \mid f(r) \in M_r \text{ all } r \in I\}$*. We give a ring structure to* $\prod_{r \in I} M_r$ *by defining* $(f + g)(r) = f(r) + g(r)$ *and* $(fg)(r) = f(r)g(r)$*.*

Let π_r *be the projection of* $\prod_{r \in I} M_r$ *onto* M_r*. A* Γ*-ring* M *is said to be a subdirect sum of* Γ*-rings* $\{M_r\}_{r \in I}$ *if there is a monomorphism* $\Phi : M \to \prod_{r \in I} M_r$ *such that* $M\Phi\pi_r = M_r$ *for each* $r \in I$*.*

***Theorem* 6.16.** *Any finite subdirect sum of regular* Γ*-rings is regular.*

Proof. It suffices to show that a subdirect sum of two regular Γ-rings is regular. Suppose that M has two ideals J and K such that $J \cap K = 0$. Now, $(J + K)/J$ is an ideal of M/J. Since $(J + K)/J \cong K/(J \cap K)$ and since $K/(J \cap K)$ is regular, then $(J + K)/J$ is regular. Since $(J + K)/J$ and J are regular, then $J + K$ is regular. Thus any finite subdirect sum of regular Γ-rings is regular. ■

Sub-directly irreducible. *A Γ-ring M is said to be a sub-directly irreducible if the intersection of all of its nonzero ideals is not zero.*

***Theorem* 6.17.** *In a regular Γ-ring M with no non-zero weakly nilpotent elements, every idempotent elements commutes with every elements of M.*

Proof. Let $e\delta e = e$, $\delta \in \Gamma$. Let $x \in M$. If $e = 0$, then $e\delta x = x\delta e$. Suppose $e \neq 0$. Then $\delta \neq 0$.

Now, $(e\delta x - e\delta x\delta e)\delta(e\delta x - e\delta x\delta e)$

$$= (e\delta x\delta e - e\delta x\delta e)([\delta, x] - [\delta, x\delta e])$$

$$= 0.$$

Therefore $(e\delta x - e)\delta(e\delta x - e) = 0$; and hence $e\delta x\delta e\delta x - e\delta e\delta x - e\delta x\delta e + e\delta e = 0$, or, $e\delta x - e\delta x - e\delta x\delta e + e\delta e = 0$. This implies $e\delta(e - x\delta e) = 0$. Since $e \neq 0$, therefore $e - x\delta e = 0$ and hence $e = x\delta e$.

Again, $(e\delta x\delta e - e\delta x)\delta(e\delta x\delta e - e\delta x)$

$$= (e\delta x\delta e - e\delta x\delta e)\delta([\delta, x\delta e] - [\delta, x]) = 0$$

Therefore, $(e - e\delta x)\delta(e - e\delta x) = 0$

$$\text{or } e\delta e - e\delta x\delta e - e\delta e\delta x + e\delta x\delta e\delta x = 0$$

$$\text{or } e\delta e - e\delta x\delta e - e\delta x - e\delta x = 0$$

$$\text{or } (e - e\delta x)\delta e = 0.$$

Since $e \neq 0$, therefore $e - e\delta x = 0$ and hence $e = e\delta x$. Therefore, $e\delta x = x\delta e$. Hence every idempotent elements commutes with every elements of M. ■

Theorem 6.18. *A non-zero subdirectly irreducible regular Γ-ring with no nonzero weakly nilpotent elements is a division Γ-ring.*

Proof. Theorem 6.18 shows that for any $x \in M$, $x\delta e = e\delta x$, where $e = e\delta e$. Let $a \in M$, $a \neq 0$. Let us consider two ideals $a\delta M$ and $A = \{x - a\delta M \mid x \in M\}$ whose intersection is zero. M is subdirectly reducible, so $a\delta M = 0$ or $A = 0$. But $a\delta M \neq 0$, hence $A = 0$, and thus $a\delta x = x$. So that we can write $x\delta e = e\delta x = x$. This means that $[e\,,\,\delta]$ and $[\delta\,,\,e]$ are the strong left and right unties respectively. Now, we have $a\gamma x = x = x\gamma a$ for $a, x \in M$ and so $a\gamma e = e = e\gamma a$, whence $(a\delta e)\gamma e = e = e\gamma(e\delta a)$, so that $a\delta(e\gamma e) = e = (e\gamma e)\delta e$. Therefore M is a division Γ-ring. ■

***Lemma* 6.19.** *If $x, y \in M$, $\gamma, \mu \in \Gamma$ and $x' = x - x\mu y\gamma x$, and if $x' = x'\mu a\gamma x'$ for some $a \in M$, then $x = x\mu b\gamma x$* for some $b \in M$.

Proof. We have,
$$\begin{aligned} x &= x' + x\mu y\gamma x \\ &= x'\mu a\gamma x' + x\mu y\gamma x \\ &= (x - x\mu y\gamma x)\mu a\gamma(x - x\mu y\gamma x) + x\mu y\gamma x \\ &= x\mu(a - a\gamma x\mu y - y\gamma x\mu a + y\gamma x\mu a\gamma x\mu y + y)\gamma x \\ &= x\mu b\gamma x \end{aligned}$$
where, $b = a - a\gamma x\mu y - y\gamma x\mu a + y\gamma x\mu a\gamma x\mu y + y \in M$. ■

***Lemma* 6.20.** *Let $J \subseteq K$ be two-sided ideals in a Γ-ring M, then K is regular if and only if K/J and J are both regular.*

Proof. If K is regular, then obviously K/J is regular. Given $x \in J$, $x\mu y\gamma x = x$ for some $\mu, \gamma \in \Gamma$ and $y \in K$. Then $z = y\delta x\xi y$ is an element of

J, δ, $\xi\in\Gamma$ and $x\mu z\gamma x = x\mu y\delta x\xi y\gamma x = x\xi y\gamma x = x$. Hence J is a regular Γ-*ring*.

Conversely, assume that K/J and J are both regular. Given $x\in K$, it follows from the regularity of K/J that $x - x\mu y\gamma x\in J$ for some $y\in K$. Consequently, $x - x\mu y\gamma x = (x - x\mu y\gamma x)\mu z\gamma(x - x\mu y\gamma x)$ for some $z\in J$ so that, $x - x\mu y\gamma x = x\mu z\gamma x - x\mu z\gamma x\mu y\gamma x - x\mu y\gamma x\mu z\gamma x + x\mu y\gamma x\mu z\gamma x\mu y\gamma x$.

Therefore $x = x\mu y\gamma x + x\mu(z - z\gamma x\mu y - y\gamma x\mu z + y\gamma x\mu z\gamma x\mu y)\gamma x$

$$= x\mu(y + z - z\gamma x\mu y - y\gamma x\mu z + y\gamma x\mu z\gamma x\mu y)\gamma x$$

$$= x\mu w\gamma x,$$

for some $w = y + z - z\gamma x\mu y\gamma - y\gamma x\mu z + y\gamma x\mu z\gamma x\mu y\in K$.

Therefore K is regular. ■

In particular, we can say that every two-sided ideal in a regular Γ-ring is regular. On the other hand, if J is two-sided ideal in a Γ-ring M such that M/J and J are both regular, then M is regular. This method of checking regularity is quite useful when constructing examples.

Subdirect product. *A Γ-ring M is said to be a subdirect product of the family $\{M_i\}_{i\in I}$ of Γ-rings if there is a natural projection $p_i : \Pi_{i\in I} M_i \to M_i$ such that $p_i(M) = M_i$ for each $i\in I$.*

Proposition 6.21. *Any finite sub-direct product of regular Γ-rings is regular.*

Proof. It suffices to consider the case of a Γ-ring M that is a sub-direct product of two regular Γ-rings. Then M has two-sided ideals J and K such that $J \cap K = 0$ and M/J and M/K are both regular. Since J is isomorphic to the two-sided ideal $(J + K)/J$ in the regular Γ-ring M/K, then from *Lemma* 6.20, we have, J is regular. So that M/J is regular and hence M is regular. ■

Note that a sub-direct product of infinitely many regular Γ-rings, such as $\mathbf{Z}$ (set of integers), need not be regular.

***Proposition* 6.22.** *Let M be a Γ-ring, and set $T = \{x \in M \mid M\Gamma x M\Gamma$ is a regular ideal$\}$. Then*

a) T is a regular two-sided ideal of M,

b) T contains all regular two-sided ideals of M,

c) M/T has no non-zero regular two-sided ideals.

Proof. ***a***) Given $x, y \in T$, we see that $M\Gamma y\Gamma M$ and $(M\Gamma x\Gamma M + M\Gamma y\Gamma M)/M\Gamma y\Gamma M$ are both regular, whence from *Lemma* 6.20, $M\Gamma x\Gamma M + M\Gamma y\Gamma M$ is regular. Thus $(M\Gamma x\Gamma M + M\Gamma y\Gamma M) \subseteq T$ for all $x, y \in T$. Hence T is two-sided ideal. It is clear that T is regular.

b) It is obvious and c) follows from *Lemma* 6.20. ■

In order to show that the Γ-ring of all m×n matrices over a regular Γ-ring is regular, we proceed via the following lemma, which is useful in other case as well.

***Lemma* 6.23:** *Let $e_1, e_2, e_3, \ldots\ldots, e_n$ be orthogonal idempotents in a Γ-ring M such that $e_1 + e_2 + e_3 + \ldots\ldots + e_n = 1$. Then M is regular if and only if for each $x \in e_i\mu M\gamma e_j$, there exists $y \in e_j\mu M\gamma e_i$ such that $x\mu y\gamma x = x$; $\mu, \gamma \in \Gamma$.*

Proof. First we assume that M is regular and let $x \in e_i \mu M \gamma e_j$. Then $x = x\mu y\gamma x$ for some $y \in M$. Now, $x\mu(e_i\mu z\gamma e_i)\gamma x = x\mu e_j\mu z\gamma e_i\gamma x = x\mu y\gamma x = x$, since $y \in e_j\mu M\gamma e_i$..

Conversely, assume that for any $x \in e_i\mu M\gamma e_j$, there exists $y \in e_j\mu M\gamma e_i$ such that $x\mu y\gamma x = x$. We proceed by induction on n. Since the case $n = 1$ is trivial we begin with the case $n = 2$. First consider an element $x \in M$ such that $e_1\mu x\gamma e_2 = 0$. There are elements $x \in e_1\mu M\gamma e_1$ and $z \in e_2\mu M\gamma e_2$ such that

$(e_1\mu x\gamma e_1)\mu y\gamma(e_1\mu x\gamma e_1) = e_1\mu x\gamma e_1$ and

$(e_2\mu x\gamma e_2)\mu z\gamma(e_2\mu x\gamma e_2) = e_2\mu x\gamma e_2$, then

$$\begin{aligned} x\mu(y+z)\gamma x &= (e_1\mu x\gamma e_1 + e_2\mu x\gamma e_1 + e_2\mu x\gamma e_2)\mu(y+z)\gamma(e_1\mu x\gamma e_1 + e_2\mu x\gamma e_1 \\ &\quad + e_2\mu x\gamma e_2) \\ &= e_1\mu x\gamma e_1\mu y\gamma e_1\mu x\gamma e_1 + e_2\mu x\gamma e_1\mu y\gamma e_1\mu x\gamma e_1 + \\ &\quad e_2\mu x\gamma e_2\mu z\gamma e_2\mu x\gamma e_1 + e_2\mu x\gamma e_2\mu z\gamma e_2\mu x\gamma e_2 \\ &= e_1\mu x\gamma e_1 + e_2\mu x\gamma e_2 + e_2\mu x\gamma(y+z)\mu x\gamma e_1. \end{aligned}$$

As a result, we see that the element $x' = x - x\mu(y+z)\gamma x$ lies in $e_2\mu M\gamma e_1$. Then $x'\mu w\gamma x' = x'$ for some $w \in e_1\mu M\gamma e_2$, whence $x\mu v\gamma x = x$ for some $v \in M$.

Now, consider a general element $x \in M$, and choose an element $y \in e_2\mu M\gamma e_1$ such that $(e_1\mu x\gamma e_2)\mu y\gamma(e_1\mu x\gamma e_2) = e_1\mu x\gamma e_2$. Since $y \in e_2\mu M\gamma e_1$, we see that $e_1\mu x\gamma y\mu x\gamma e_2 = e_1\mu x\gamma e_2$, whence $e_1\mu(x - x\gamma y\mu x)\gamma e_2 = 0$. By the case above there exists an element $z \in M$ such that $(x - x\mu y\gamma x)\mu z\gamma(x - x\mu y\gamma x) = x - x\mu y\gamma x$, hence $x\mu w\gamma x = x$ for some $w \in M$. Therefore, M is regular.

Finally, let $n > 2$, and assume that the lemma holds for $n - 1$ orthogonal idempotents. Setting $f = e_2 + \dots \dots \dots + e_n$ and $g = e_1 + e_2 + \dots \dots \dots + e_n$, we thus know that $f\mu M\gamma f$ and $g\mu M\gamma g$ are regular. Consider any element $x \in e_1\mu M\gamma f$. There exists $y \in e_2\mu M\gamma e_1$ such that $(x\delta e_2)\mu y\gamma(x\delta e_2) = x\delta e_2$, so that $(x - x\mu y\gamma x)\delta e_2 = 0$. Then $x - x\mu y\gamma x \in g\mu M\gamma g$, whence $(x - x\mu y\gamma x)\mu z\gamma(x - x\mu y\gamma x) = x - x\mu y\gamma x$ for some $z \in g\mu M\gamma g$. As a result, $x\mu w\gamma x = x$ for some $w \in M$. Hence we obtain $f\mu w\gamma e_1 \in f\mu M\gamma e_1$ such that $x\mu(f\mu w\gamma e_1)\gamma x = x$, likewise, for any $x \in f\mu M\gamma e_1$ there is some $t \in e_1\mu M\gamma f$ such that $x\mu t\gamma x = x$.
Applying the case $n = 2$ to the orthogonal idempotents e_1 and f, we conclude that M is regular. Therefore the induction works. ■

***Lemma* 6.24.** *A non-zero regular Γ-ring M is indecomposable (as a Γ-ring) if and only if its centre is a Γ-field.*

Proof. Assume that M is indecomposable. Let S denote the centre of M and let x be an element of S. Then by *Theorem* 6.12, $x\mu y\gamma x = x$ for some $y \in S$.

Now, $x\mu y x\delta y = x\delta y$, i .e., $x\delta y$ is a non-zero central idempotent in M. Since M is indecomposable, $x\delta y = 1$. Therefore, S is a Γ-field.

In particular, this lemma shows that the centre of any prime regular Γ-ring is a Γ-field. ■

Definition: *E is a projective left M-module when the following property holds: if f: M → N is any epimorphism, and g: E → N a homomorphism, there exits a homomorphism h: E → M such that g = foh.*

Definition: *Let E be an M-module. Then M is a free module whenever it has a basis. Thus every element of E can be written in one and only one way in the form* $x = \sum_{s \in S} a_s s$ *(where* $a_s \in M$*).*

Example. i) The zero module is free with empty basis.

ii) Every Γ-ring M is a free left (right) M-module; the set consisting only the unit elements is a basis.

***Theorem* 6.25.** *If E is finitely generated projective module over a regular* Γ*-ring M, then* $End_M(E)$ *is a regular* Γ*-ring.*

Proof. According to *Lemma* 6.23, $e\mu M_n(M)\gamma e$ is regular for any n and any idempotent $e \in M_n(M)$. ■

***Theorem* 6.26.** *If E is a projective right module over a regular* Γ*-ring M, then all finitely generated submodules of E are direct summand of E.*

Proof. Let E be a submodule of a free right M-module F. Given any finitely generated submodule $B \subseteq E$, we infer that F has a finitely generated free direct summand G which contains B. It suffices to prove that B is a direct summand of G, for then B is a direct summand of F and hence also of E.

Choose a positive integer n such that B can be generated by n elements, and embed G in a finitely generated free right M-module H which has a basis with at least n elements. Then there exists $f \in End_M(H)$ such that $f\gamma H = B$. According to *Theorem* 6.25, $End_M(H)$ is regular, hence there exists $g \in End_M(H)$ such that $f\mu g\gamma f = f$, consequently, $f\gamma g$ is an idempotent endomorphism of H such that

$f\mu g\gamma H = f\gamma H = B$, whence B is a direct summand of H. Therefore, B is a direct summand of G.

***Theorem* 6.27.** *A Γ-ring M is regular if and only if all right (left) M-modules are flat.*

Proof. First assume that M is regular. Let E be any free right M-module, and let K be any submodule of E. If F is any finitely generated submodule of K, then F is a direct summand of E by *Theorem* 6.26, whence E/F is projective. Now, E/K is the direct limit of the module E/F, where F ranges over all finitely generated submodules of K. Thus E/K is a direct limit of projective modules, whence E/K is flat.

Conversely, assume that all right M-modules are flat. Given $x \in M$, the flatness of $M/x\gamma M$ implies that the natural map $(M/x\gamma M) \oplus_M M\mu x \to M/(x\gamma M)$ must be injective, i.e., the map $M\mu x/x\mu M\mu x \to M/(x\gamma M)$ is injective. Thus $M\mu x \cap x\gamma M = x\mu M\gamma x$, and, consequently, $x \in x\mu M\gamma x$. Therefore, M is regular. ■

***Lemma* 6.28.** *For a commutative Γ-ring M, the following conditions are equivalent:*

a) M is regular.

b) M_M is a Γ-field for all maximal ideals M' of M.

c) M has no non-zero nilpotent elements and all prime ideals of M are maximal.

d) All simple M-modules are injective.

Proof. (*a*) $\Rightarrow$ (*d*) Let M' be a maximal ideal of M, let J be an ideal of M, and let f: $J \to M/M'$ be a non-zero homomorphism. Then $(M' \cap J)\Gamma(M' \cap J) = M' \cap J$.

Now, $M' \cap J = (M' \cap J)\Gamma(M' \cap J) \subseteq J\,M' \subseteq \ker f \subseteq J$.

Hence $J \not\subset M'$. Consequently, $x + y = 1$ for some $x \in M'$ and $y \in J$, and we set $w = f(y) \in M/M'$.

Given any $a \in J$, we have

$$(x + y)\gamma a = 1\gamma a = a$$

$$\Rightarrow x\gamma a + y\gamma a = a$$

$$\Rightarrow a - y\gamma a = x\gamma a \in M'\Gamma J \subseteq \ker f, \text{ whence } f(a - y\gamma a) = 0$$

$$\Rightarrow f(a) - f(y\gamma a) = 0$$

$$\Rightarrow f(a) = f(y\gamma a) = f(y)\gamma f(a) = w\gamma f(a).$$

Therefore, f extends to a map $M \to M/M'$.

(*d*) $\Rightarrow$ (*c*) We first claim that if M' is any maximal ideal of M, then $x \in x\Gamma M'$ for all $x \in M'$. If not, then $x\Gamma M/x\Gamma M' \neq 0$ for some $x \in M'$. Then $M/M' \cong x\Gamma M/x\Gamma M'$. Hence there exist an epimorphism f: $x\Gamma M \to M/M'$. Now, f extends to a map g: $M \to M/M'$, and so $f(x\Gamma M') \subseteq g(M) = 0$, which is false. Thus the claim holds.

Suppose that $x\gamma x = 0$ for some non-zero $x \in M$. The annihilator $J = \{m \in M \mid m\gamma x = 0\}$ is a proper ideal and so is contained in a maximal ideal M'. Since $x \in J \subseteq M'$, we have $x \in x\mu M'$ by the claim above. Then $x = x\gamma y$ for some $y \in M'$ and $(1 - y)\gamma x = 1\gamma x - y\gamma x = x - y\gamma x = 0$, $\Rightarrow (1 - y) \in J \subseteq M'$, which is impossible. Thus M cannot have any nonzero nilpotent elements.

Now, let P be a prime ideal of M, and let M' be a maximal ideal which contains P. Given any $x \in M'$, we have $x \in x\Gamma M'$ and so

$$x\gamma(1-y) = x\gamma 1 - x\gamma y = x - x\gamma y = 0 \text{ for some } y \in M'.$$

Since $1 - y \notin M'$, we also have $1 - y \notin P$, whence $x \in P$. Thus $M' = P$. So that P is maximal.

(*c*) $\Rightarrow$ (*b*) Since there are no prime ideals of M properly contained in M', we have seen that $M'\Gamma M_{M'}$ is the only prime ideal of $M_{M'}$, whence $M'\Gamma M_{M'}$ is nil. Given $x/s \in M'\Gamma M_{M'}$, we thus have $(x/s)^n = o$ for some n, hence $t\gamma x^n = 0$ for some $t \in M - M'$. Then $(t\gamma x)^n = 0$ and so $t\gamma x = 0$, whence $M'\Gamma M_{M'} = 0$. So that $M_{M'}$ is a Γ-field.

(*b*) $\Rightarrow$ (*a*) Let E be any M-module. For any maximal ideal M' of M, it follows from (*b*) that $E_{M'}$ is flat $M_{M'}$-module, and consequently E is a flat M-module. According to *Theorem* 6.27, M is regular. ■

***Theorem* 6.29.** *A Γ-ring M is regular if and only if*

a) M is semiprime.

b) The union of any chain of semiprime ideals of M is semiprime.

c) M/P is regular for all prime ideals P of M.

Proof. If M is regular then obviously (*c*) holds. In view of Corollary 6.15 (ii), all two-sided ideals of M are semiprime, whence (*a*) and (*b*) hold.

Conversely, assume that (*a*), (*b*), (*c*) hold. If M is not regular, then there is some elements $x \in M$ such that $x \notin x\mu M\gamma x$. Now, note that 0 ideal is a semiprime ideal of M such that $x \notin x\mu M\gamma x + 0$. From, (*b*) we see that there is a semiprime ideal J in M which is maximal with respect to the property $x \notin x\mu M\gamma x + J$.

Now, M/J is not regular, hence by (c), J is not prime. Thus there exist two-sided ideals A and B which properly contain J, such that $A\Gamma B \subseteq J$. Now, set $K = \{m \in M \mid m\Gamma B \subseteq J\}$ and $L = \{m \in M \mid K\Gamma M \subseteq J\}$. As J is semiprime, K and L are also semiprime. Since $(K \cap L)\Gamma(K \cap L) \subseteq K\Gamma L \subseteq J$, we have $K \cap L \subseteq J$. Clearly $A \subseteq K$ and $B \subseteq L$, hence K and L properly contain J.

Because of the maximality of J, there exist elements y, $z \in M$ such that $x - x\mu y\gamma x \in K$ and $x - x\mu z\gamma x \in L$.

$$\text{Now, } x - x\mu(y + z - y\gamma x\mu z)\gamma x = (x - x\mu y\gamma x) - (x - x\mu y\gamma x)\mu z\gamma x \in K.$$
$$= (x - x\mu z\gamma x) - x\mu y\gamma(x - x\mu z\gamma x) \in L.$$

We see that $x \in x\mu M\gamma x + (K \cap L) \subseteq x\mu M\gamma x + J$ which is a contradiction. Therefore M must be regular. ■

***Corollary* 6.30.** *A Γ-ring M is regular if and only if all two-sided ideals of M are idempotent and M/P is regular for all prime ideals P of M.*

Definition. *A completely prime ideal in a Γ-ring M is a proper two sided ideals P such that M/P is an integral domain (not necessarily commutative).*

***Lemma* 6.31.** *If M is a Γ-ring with no nonzero nilpotent elements, then every minimal prime ideal of M is completely prime.*

***Proof*:** We first claim that if $a_1, a_2, a_3, \ldots\ldots, a_n \in M$ and $a_1\gamma_1 a_2\gamma_2 a_3\gamma_3 \ldots\ldots\ldots \gamma_n a_n = 0$ for some $\gamma_1, \gamma_2, \gamma_3, \ldots\ldots\ldots, \gamma_n \in \Gamma$, then the product of $a_i\gamma_i a_j$ with $i \neq j$ in any order is zero. To prove this, it suffices to show that if $x\mu a\gamma b\delta y = 0$ in M for some μ, $\delta \in \Gamma$, then $x\mu b\gamma a\delta y = 0$. This is clear if $x = y = 1$, then

$x\mu a\gamma b\delta y = 0$

$\Rightarrow 1\mu a\gamma b\delta 1 = 0$

$\Rightarrow a\gamma b = 0$

and so $(b\gamma a)\gamma(b\gamma a) = b\gamma(a\gamma b)\gamma a = 0$, whence $b\gamma a = 0 \Rightarrow x\mu b\gamma a\delta y = 0$.

In case $x = 1$, then $(a\gamma b)\delta y = 0$

$$\Rightarrow y\delta(a\gamma b) = 0$$

$$\Rightarrow y\delta(a\gamma b\gamma a) = 0$$

$$\Rightarrow a\gamma b\gamma a\delta y = 0$$

$$\Rightarrow b\gamma a\gamma b\gamma a\delta y = 0$$

$$\Rightarrow (b\gamma a)\gamma(b\gamma a)\delta y = 0$$

$$\Rightarrow (b\gamma a)\gamma y\delta(b\gamma a) = 0$$

$$\Rightarrow (b\gamma a)\gamma y\delta(b\gamma a)\delta y = 0$$

$$\Rightarrow b\gamma a\delta y = 0.$$

For the general case,

$$x\mu(a\gamma b\delta y) = 0$$

$$\Rightarrow a\gamma b\delta y\mu x = 0$$

$$\Rightarrow (b\gamma a\delta y)\mu x = 0$$

$$\Rightarrow x\mu(b\gamma a\delta) = 0$$

This establishes the claim.

Now, let P be any minimal prime ideal of M. Recall that on m-system in M is a nonempty subset X such that $0 \notin X$ and whenever x, $y \in X$, there exists $n \in M$ such that $x\gamma n\gamma y \in X$. Then $M - P$ is an m-system and we may choose a maximal m-system $X \supseteq M - P$. If Q is two-sided ideal of M, maximal among all two-sided ideals disjoint from X, then Q is prime. Since Q is disjoint from $M - P$, we have $Q \subseteq P$ and thus $Q = P$, by minimality of P.

As a result, P is disjoint from X, whence $X = M - P$. Thus $M - P$ is maximal m-system.

Set $Y = \{x_1\gamma_1x_2\gamma_2x_3\gamma_3.........\gamma_nx_n \mid x_1, x_2, x_3, ..., x_n \in M - P$ and $\gamma_1, \gamma_2, \gamma_3,......,\gamma_n \in \Gamma\}$. If $0 \in Y$, then $x_1\gamma_1x_2\gamma_2x_3\gamma_3......\gamma_nx_n = 0$ for some $x_i \in M - P$ and $\gamma_1, \gamma_2, \gamma_3,...,\gamma_n \in \Gamma$. There exists $m_1, m_2, m_3,......, m_{n-1} \in M$ and $\mu_1, \mu_2, \mu_3,.........., \mu_n \in \Gamma$ such that $x_1\gamma_1m_1\mu_1x_2\gamma_2m_2\mu_2..................\mu_{n-2}x_{n-1}\gamma_{n-1}m_{n-1}\mu_{n-1}x_n \in X = M - P$. This implies $x_1\gamma_1m_1\mu_1x_2\gamma_2m_2\mu_2.........\mu_{n-2}x_{n-1}\gamma_{n-1}m_{n-1}\mu_{n-1}x_n \notin P$. Since $(x_1\gamma_1x_2\gamma_2x_3\gamma_3........\gamma_nx_n)\delta(m_1\mu_1m_2\mu_2 m_3\mu_3 ... ...\mu_nm_n) = 0$ for some $\delta \in \Gamma$, we see from the claim above that $x_1\gamma_1m_1\mu_1x_2\gamma_2m_2\mu_2 ... ... \mu_{n-2}x_{n-1}\gamma_{n-1}m_{n-1}\mu_{n-1}x_n = 0$ which is impossible. Thus $0 \notin Y$. Hence Y is an m-system. Clearly, $M - P \subseteq Y$. Hence by maximality of $M - P$, we obtain $M - P = Y$. Therefore $M - P$ is multiplicatively closed. So that M/P is an integral domain. ■

***Theorem* 6.32.** *Let M be a Γ-ring with no nonzero nilpotent elements. Then M is regular if and only if M/P is regular for all completely prime ideals P of M.*

Proof. Assume that M/P is regular for all completely prime ideals. If P is minimal prime ideal of M, then is completely prime by *Lemma* 6.31. Hence M/P is an integral domain and so is a division Γ-ring.

Consequently, we see that M/Q is a division Γ-ring for every prime ideal Q of M. Since every semiprime ideal of M is an intersection of prime ideals, we infer that the set of semiprime ideals of M coincides with the set of those two-sided ideals J such that M/J has no nonzero nilpotent elements.

As a result, we see that the union of chain of semiprime ideals of M must be semiprime. Therefore M is regular. ■

Chapter - Seven

Abelian Regular Γ-Rings

In this chapter, we have developed the basic properties of a class of regular Γ-rings of some "classical" type. Abelian regular Γ-rings are also known as strongly regular Γ-rings, which is, however, a more indirect concept. In that a nontrivial theorem is required to show that strongly regular Γ-rings are actually regular. For this reason, we view abelianness as the more general property. We first develop a number of equivalent characterizations of abelian regular Γ-rings before proving that "abelian regular" is equivalent to "strongly regular". We also develop a lattice theoretic characterization of the abelian regular Γ-rings.

Definition: *A regular Γ-ring M is abelian provided all idempotents in M are central.*

Obviously, any commutative regular Γ-ring is regular, as is any direct product of division Γ-rings.

***Lemma* 7.01.** *If e is idempotent in a semiprime Γ-ring M, then the following conditions are equivalent:*

(a) e *is central.*

(b) e *commutes with every idempotent in* M.

(c) $e\gamma M$ *is a two-sided ideal of* M.

(d) $M\gamma e$ *is a two-sided ideal of* M.

(e) $(1-e)\gamma M\gamma e = 0$, for some $\gamma\in\Gamma$.

(f) $e\gamma M\gamma(1-e) = 0$.

Proof. $(a) \Rightarrow (c)$ is trivial.

(e) $\Rightarrow$ **(a)** Since $(1 - e)\gamma M\gamma e = 0$, we see that $M\gamma e \subseteq e\gamma M$, whence $e\gamma M$ is a left ideal of M. Then $e\gamma M\gamma(1 - e)$ is a left ideal of M such that $[e\gamma M\gamma(1 - e)]\gamma[e\gamma M\gamma(1 - e) = 0$, hence $e\gamma M\gamma(1 - e) = 0$. Given $r\in M$, we have $e\mu r\gamma(1 - e) = 0$ as well as $(1 - e)\gamma r\mu e = 0$, whence $e\mu r = e\mu r\gamma e = e\gamma r\mu e = r\mu e$. Therefore e is central.

(a) $\Leftrightarrow$ **(d)** $\Leftrightarrow$ **(f)** by symmetry.

(a) $\Rightarrow$ **(b)** a priori.

(b) $\Rightarrow$ **(e)** Given any $x\in(1 - e)\gamma M\gamma e$, we see that $e + x$ is idempotent, hence $e\mu(e + x) = (e + x)\mu e$, i.e. $e\mu e + e\mu x = e\mu e + x\mu e$, i.e. $e\mu x = x\mu e$. Thus e commutes with x, so that $x = x\gamma e = e\gamma x = 0$. ■

***Theorem* 7.02.** *For regular Γ-ring M, the following conditions are equivalent*:

(a) M is abelian.

(b) M/P is a division Γ-ring for all prime ideals P of M.

(c) M has no nonzero nilpotent elements.

(d) All right (left) ideal of M are two-sided.

(e) Every nonzero right (left) ideal of M contains a nonzero central idempotent.

Proof. **(a)** $\Rightarrow$ **(b)** Since all the idempotents in the prime Γ-ring M/P come from idempotents in M, they are all central and hence we see that 0 and 1 are the only idempotents in M/P. As a result, $x\gamma(M/P) = (M/P)\gamma x = M/P$, for any nonzero $x\in M/P$ *and* for some $\gamma\in\Gamma$. Whence M/P is a division Γ-ring.

(b) $\Rightarrow$ ***(c)*** Since M is semiprime, it follows from (b) that M is isomorphic to a subdirect product of division Γ-rings, whence M has no nonzero nilpotent elements.

(c) $\Rightarrow$ ***(a)*** If $e \in M$ is an idempotent, then every element of $(1 - e)\gamma M \gamma e$ is nilpotent, whence $(1 - e)\gamma M \gamma e = 0$. By *Lemma* 7.01, e is central.

(a) $\Rightarrow$ ***(d)*** Each principal of M is generated by a central idempotent and so is two-sided, whence all right ideals of M are two-sided.

(d) $\Rightarrow$ ***(a)*** by *Lemma* 7.01.

(a) $\Rightarrow$ ***(e)*** is clear.

(e) $\Rightarrow$ ***(a)*** Let $e \in M$ be an idempotent, and let J be the right ideal of M generated by those central idempotents of M which lie in $e\gamma M$. Note that J is a two-sided ideal. In view of (e), we se that $J \leq e\gamma M$, whence $e\gamma(M/J)$ is singular. Given any $x \in M$, we have $x\gamma J \leq J \leq e\gamma M$ and so $(1 - e)\gamma x \gamma e \gamma M$ is homomorphic image of $e\gamma(M/J)$ and so is singular. Inasmuch as M_M is nonsingular, we obtain $(1 - e)\gamma x \gamma e \gamma M = 0$. Therefore $(1 - e)\gamma M \gamma e = 0$, hence e is central, by *Lemma* 7.01. ■

Proposition 7.03. *Let J be a two-sided ideal in a regular Γ-ring M, and let $f_1, f_2, \ldots\ldots$ be a finite or countably infinite sequence of orthogonal idempotents in M/J. Then there exist orthogonal idempotents $e_1, e_2, \ldots\ldots \ldots\ldots \in M$ such that $\overline{e_n} = f_n$ for all n.*

Proof. The proof is similar to the proof of [10, *Proposition* 2.18] ■

***Theorem* 7.04.** *Let M be a regular Γ-ring, and let N be the sum of all ideals of the form* $M\gamma e\gamma M\gamma(1 - e)\gamma M$, *where e is any idempotent in M. Then N equals to the sub Γ-ring (without identity) of M generated by the nilpotent elements, and M/N is abelian. Also N is contained in the subring of M generated by the idempotents.*

***Proof*.** Given any idempotent $f \in M/N$ there exists an idempotent $e \in M$ such that $\bar{e} = f$. Inasmuch as $e\gamma M\gamma(1 - e)\gamma N$, we obtain $f\gamma(M/N)\gamma(1 - f) = 0$, whence f is central (by *Lemma* 7.01). Thus, M/N is abelian.

Let S denote the sub Γ-ring (without identity) of M generated by the nilpotent elements. According to *Theorem* 7.02, M/N has no nonzero nilpotent elements, whence $S \subseteq N$. Given any idempotent $e \in M$, every element of $e\gamma M\gamma(1 - e)$ and $(1 - e)\gamma M\gamma e$ is nilpotent, hence we see that $e\gamma M\gamma(1 - e)\gamma M = [e\gamma M\gamma(1 - M)]\gamma[(1 - e)\gamma M\gamma e] + e\gamma M\gamma(1 - e)\gamma M\gamma(1 - e) \subseteq S$ and, similarly, $(1 - e)\gamma M\gamma e \subseteq S$. Consequently,

$$\begin{aligned} M\gamma e\gamma M\gamma(1 - e)\gamma M &= e\gamma M\gamma e\gamma M\gamma(1 - e)\gamma M + (1 - e)\gamma M\gamma e\gamma M(1 - e)\gamma M \\ &\subseteq e\gamma M\gamma(1 - e)\gamma M + (1 - e)\gamma M\gamma e\gamma M \subseteq S \end{aligned}$$

Therefore $N = S$.

Finally, let T denote the sub Γ-ring of M generated by the idempotents. If e is an idempotent in M and $x \in e\gamma M\gamma(1 - e)$, then $e + x$ is an idempotent as well. Then e and $e + x$ both lie in T, whence $x \in T$. Thus $e\gamma M\gamma(1 - e) \subseteq T$, and, likewise $(1 - e)\gamma M\gamma e \subseteq T$. Proceeding as above, we conclude that $M\gamma e\gamma M\gamma(1 - e)\gamma M \subseteq T$. Therefore $N \subseteq T$. ■

***Proposition* 7.05.** *If A and B are projective right Γ-modules over a regular Γ-ring R, then the following conditions are equivalent:*

(a) $Hom_M(A, B) \neq 0$.

(b) $Hom_M(B, A) \neq 0$.

(c) *There exist nonzero sub Γ-modules $A' \leq A$ and $B' \leq B$(which can be chosen to be cyclic if necessary) such that $A' \cong B'$.*

Proof. The proof is similar to the proof of [10, *proposition* 2.21] ■

***Theorem* 7.06.** *Let A be a finitely generated projective right Γ-module over a regular Γ-ring M, and set $T = End_M(A)$. Then the following conditions are equivalent:*

(a) *T is abelian.*

(b) *Isomorphic sub Γ-modules of A are equal.*

(c) *If B is any sub Γ-modules of A such that $2B \lesssim A$, then $B = 0$.*

(d) *If B and C are any sub Γ-modules of A such that $B \cap C = 0$, then $Hom_M(B, C) = 0$*

(e) *$L(A)$ is distributive.*

Proof. Obviously T is regular.

(***a***) $\Rightarrow$ (***b***) Let B and C be isomorphic Γ-modules of A. Given $x \in B$, there exists $y \in C$ such that $x\gamma M = y\gamma M$. There exist idempotents $e, f \in T$ such that $e\gamma A = x\gamma M$ and $f\gamma A = y\gamma M$ [by *Theorem* 6.08]. Since $e\gamma A = f\gamma A$, there exist elements $x \in e\gamma T\gamma f$ and $t \in f\gamma T\gamma e$ such that $s\gamma t = e$ and $t\gamma s = f$. Now, e and f are central in T, whence

$$e = s\gamma t = s\gamma f\gamma t = f\gamma s\gamma t = f\gamma e = t\gamma s\gamma e = t\gamma e\gamma s = t\gamma s = f,$$

and, consequently, $x\gamma M = e\gamma A = f\gamma A = y\gamma M \leq C$. Thus, $B \leq C$, and, by symmetry, $C \leq B$.

(***b***) $\Rightarrow$ (***c***) is clear.

$(c) \Rightarrow (d)$ If $\mathrm{Hom}_M(B, C) \neq 0$, then by *Proposition* 7.05, there exit nonzero sub Γ-modules $B' \leq B$ and $C' \leq C$ such that $B' \cong C'$. But then $2B' \lesssim B \oplus C \leq A$, which contradicts (c).

$(d) \Rightarrow (a)$ For any idempotent $e \in T$, we have $(1-e)\gamma T\gamma e \cong \mathrm{Hom}_M(e\gamma T, (1-e)\gamma T) = 0$ by (c). According to the *Lemma* 7.01, e must be central.

$(a) \Rightarrow (e)$ Let $B, C, D \in L(A)$, and choose idempotents $b, c, d \in T$ such that $b\gamma A = B$, $c\gamma A = C$ and $d\gamma A = D$. Since T is abelian, b, c, d are central in T. In particular, $c\gamma d = d\gamma c$, hence we compute that $e = c + d - c\gamma d$ is an idempotent. Clearly $e\gamma A \leq C + D$. Observing that $e\gamma c = c$ and $e\gamma d = d$, we see that $e\gamma A = C + D$. Consequently,

$$B \cap (C+D) = b\gamma A \cap e\gamma A = b\gamma e\gamma A \leq b\gamma c\gamma A + b\gamma d\gamma A$$
$$= (B \cap C) + (B \cap D).$$

The reverse induction is automatic.

$(e) \Rightarrow (a)$ For any idempotents $e, f \in T$, we have

$$e\gamma A = e\gamma A \cap [f\gamma A + (1-f)\gamma A] = [e\gamma A \cap f\gamma A] + [e\gamma A \cap (1-f\gamma A]$$

by (e). As a result, $f\gamma e\gamma A = e\gamma A \cap f\gamma A \leq e\gamma A$, whence $f\gamma e = e\gamma f\gamma e$. Likewise, $f\gamma(1-e) = (1-e)\gamma f\gamma(1-e)$, from which we obtain $e\gamma f\gamma(1-e) = 0$ and then $e\gamma f = e\gamma f\gamma e = f\gamma e$. Thus, all idempotents in T commutes with each other. By *Lemma* 7.01, all idempotents in T are central. ■

***Definition**: A Γ-ring M is said to be strongly regular if for each $x \in M$ there exits $y \in M$ such that* $(x\gamma)^2 y = x$, for some $\gamma \in \Gamma$

As the following theorem shows, this condition is left-right symmetric, and strongly regular Γ-rings are in fact regular.

***Theorem* 7.07.** *A Γ-ring M is strongly regular if and only if it is abelian regular.*

Proof. First assume that M is abelian regular. Given any $x \in M$, there exists $y \in M$ such that $x\gamma y\gamma x = x$. Since $x\gamma y$ is idempotent and, thus, is central in M, it follows that $x = (x\gamma y)\gamma x = (x\gamma x\gamma)y = (x\gamma)^2 y$. Hence M is strongly regular.

Conversely, assume that M is strongly regular. Obviously an element $x \in M$ can satisfy $(x\gamma)^2 x = 0$ only if $x = 0$, from which we infer that M has no nonzero nilpotent element. In particular, it follows that M is semiprime Γ-ring.

Next consider any prime ideal P of M, and note that M/P is strongly regular. If x, $y \in M/P$ are nonzero, then $y\gamma r\gamma x \neq 0$ for some $r \in M/P$ and so $\{(y\gamma r\gamma x)\gamma\}^2(y\gamma r\gamma x) \neq 0$, whence $x\gamma y \neq 0$. Thus, M/P is a domain. Given any nonzero $s \in M/P$, we have $(s\gamma)^2 t = s$ for some $t \in M/P$ and so $s\gamma(s\gamma t - 1) = 0$, whence $s\gamma t = 1$. Thus M/P is actually a division Γ-ring.

At this point, we could use *Theorem* 6.32 to conclude that M is regular. However, regularity is easy enough in this case to prove directly, as follows.

Now, let $x \in M$ and choose an element $y \in M$ such that $(x\gamma)^2 y = x$. Given any prime ideal P of M, we have $(\bar{x}\gamma)^2 \bar{y} = \bar{x}$ in the division Γ-ring M/P, from which we infer that $\overline{x\mu y\gamma x} = \bar{x}$, so that

$x\mu y\gamma x - x \in P$. Then $x\mu y\gamma x - x$ belongs to the intersection of all prime ideals of M, which is zero because M is semiprime. Thus $x\mu y\gamma x = x$.

Therefore M is regular. Since there are no nonzero nilpotent elements in M, *Theorem* 7.02 shows that M is abelian.■

***Proposition* 7.08.** *Every inverse limit of abelian regular Γ-rings is an abelian regular.*

Proof. We first claim that any abelian regular Γ-ring must satisfy the following property: (*) For each $x \in M$, there is a unique $y \in M$ such that $x\gamma y\mu x = x$ and $y\gamma x\mu y = y$.

First, there is some $z \in M$ such that $x\gamma z\mu x = x$. Setting $y = z\gamma x\gamma z$, we cheek that $x\gamma y\mu x = x$ and $y\gamma x\mu y = y$. Then $x\gamma y$ and $y\gamma x$ are idempotents in M and so are central, whence $x\gamma y = x\gamma(y\gamma x\mu y) = (x\gamma y)\gamma(x\mu y) = (y\gamma x)\gamma(y\mu x) = y\gamma(x\gamma y\mu x) = y\gamma x$. Now, we consider any $w \in M$ such that $x\gamma w\mu x = x$ and $w\gamma x\mu w = w$. As above, $x\gamma w = w\gamma x$ is central, hence $x\gamma w = (x\gamma y\mu x)\gamma w = (x\gamma y)\mu(x\gamma w) = (x\gamma w)\mu(x\gamma y) = (x\gamma w\mu x)\gamma y = x\gamma y$, and consequently,

$$w = w\gamma x\mu w = w\gamma(x\mu y) = w\gamma(y\mu x) = y\gamma(x\mu w) = y\gamma(x\mu y) = y.$$

Thus y is unique, proving (*).
It is clear (*) is preserved by unique limits. Thus, if M is an inverse limit of abelian regular Γ-rings M_i, we see that M is regular.

Inasmuch as r embeds in M_i, we conclude that all idempotents in M are central.

***Theorem* 7.09.** *If J is an ideal in an abelian regular Γ-ring M, then $End_M(J_M)$ is an abelian regular Γ-ring, and $End_M(J_M) \cong End_M({}_MJ)$*

***Proof*.** The set $X = \{x\gamma M \mid x \in J, \gamma \in \Gamma\}$ is a directed family of right ideals of M whose union is J. Since M is abelian, each $x\gamma M \in X$ is generated by a central idempotent, and so is a fully invariant sub Γ-module of any right ideal which contains it. Thus, we obtain restriction maps $End_M((y\gamma M)_M) \to End_M((x\gamma M)_M)$ whenever $x\gamma M \leq y\gamma M$ in X, and we infer that the inverse limit of this system of endomorphism rings and restriction maps is isomorphic to $End_M(J_M)$.

Given any $x\gamma M \in X$, we have $x\gamma M = e\gamma M$ for some central idempotent $e \in M$, whence $x\gamma M$ is an abelian regular Γ-subring (with unit e) of M, and, as a ring $x\gamma M$ is naturally isomorphic (via left multiplication) to $End_M((x\gamma M)_M)$. Consequently, we infer that $\varprojlim X \cong End_M(J_M)$ as rings. Since $x\gamma M$ is an abelian regular Γ-ring, *Theorem* 7.08 now says that $End_M(J_M)$ is an abelian regular Γ-ring.

Proceeding as above, we see that $X = \{M\gamma x \mid x \in J\}$, that each $M\gamma x \in X$ is an abelian regular Γ-ring which is naturally isomorphic (via right multiplication) to $End_M({}_M(M\mathrm{I}x))$, and that $\varprojlim X \cong End_M({}_MJ)$ as rings. Therefore $End_M({}_MJ) \cong End_M(J_M)$. ■

***Theorem* 7.10.** *Let M be an abelian regular Γ-ring, and let Q be the maximal right quotient Γ-ring of M. Then Q is abelian, and Q is also the maximal left quotient Γ-ring of M.*

Proof. We claim that any idempotent $e \in M$ is central in Q. Given $x \in Q$, we have $x\gamma J \leq M$ for some $J \leq_e M_M$. For all $r \in J$, note that e commutes with $x\gamma r$ as well as r, whence $e\gamma x\gamma r = x\gamma r\gamma e = x\gamma e\gamma r$. Thus $(e\gamma x - x\gamma e)\gamma J = 0$ and $e\gamma x = x\gamma e$, proving the claim.

Now, if K is any nonzero right ideal of Q, then $K \cap M \neq 0$ and so $K \cap M$ contains a nonzero idempotent, which must be central in Q by the claim above. Thus, every nonzero right ideal of Q contains a nonzero central idempotent, whence *Theorem* 7.02 shows that Q is abelian.

Given any nonzero element $i \in Q$ there exists $r \in M$ such that $x\gamma r \neq 0$ and $x\gamma r \in M$. Now, $r\gamma M = e\gamma M$ for some idempotent $e \in M$, and e is central in Q. Then $e\gamma x\gamma M = x\gamma e\gamma M = x\gamma r\gamma M$, whence $e\gamma x \neq 0$ and $e\gamma x \in M$. Thus, ${}_M M \leq_e {}_M Q$, so that Q is a left quotient Γ-ring of M.

As a result, Q is a sub Γ-ring of the maximal left quotient Γ-ring P of M. By symmetry, P is a right quotient Γ-ring of M, hence we conclude from the maximality of Q that $Q = P$. ■

Corollary 7.11. *Let M be an abelian regular Γ-ring. Then M is right self-injective if and only if M is self-injective.*

Proof. If M is right self-injective, then M is its own maximal right quotient Γ-ring. By *Theorem* 7.10, M is also its own maximal left quotient Γ-ring, whence M is left self-injective.■

Theorem 7.12. *Let $M \subseteq S$ be a regular Γ-ring such that M contains all the idempotents of S. Then*

(a) *S has a two-sided ideal N such that* $N \subseteq M$ *and the* Γ*-rings* M/N *and* S/N *are abelian.*

(b) *The rule* $\varphi(J) = JS$ *defines a lattice isomorphism* φ*:* $L(M_M) \to L(S_S)$*. For all* $K \in L(S_S)$*, we have* $\varphi^{-1}(K) = K \cap M$*.*

(c) *The rule* $\psi(J) = J \cap M$ *defines a lattice isomorphism* ψ*:* $L_2(S) \to L_2(K)$*. For all* $K \in L(M)$*, we have* $\psi^{-1}(K) = KS = SK$*.*

Proof. (*a*) Let N be the sum of all ideals of the form $S\gamma e\gamma S\gamma(1-e)\gamma S$, where e is any idempotent in S. According to *Theorem* 7.04, N is contained in the sub Γ-ring of S generated by the idempotents. Hence $N \subseteq M$. Also *Theorem* 7.04 says that S/N is abelian as well.

(*b*) Obviously φ is a monotone map of $L(M_M)$ into $L(S_S)$. Given any $K \in L(S_S)$, there is an idempotent $e \in S$ such that $e\gamma S = K$. Since $e \in M$ by assumption, we see that $K \cap M = e\gamma M \in L(M_M)$ and that $\varphi(K \cap M) = K$. Thus, the rule $\theta(K) = K \cap M$ defines a monotone map θ: $L(S_S) \to L(M_M)$ such that $\theta\varphi$ is the identity map on $L(S_S)$. Given any $J \in L(M_M)$, we have, $J = f\gamma M$ for some idempotent $f \in M$, whence $\theta\varphi(J) = f\gamma S \cap M = J$. Thus, $\theta\varphi$ is the identity map on $L(M_M)$. Therefore φ and θ are inverse order-isomorphisms, hence, also, lattice isomorphism.

(*c*) Obviously ψ is a monotone map of $L_2(S)$ into $L_2(M)$.

We next show that $K\Gamma S = S\Gamma K$ for any $K \in L_2(M)$. Given $x \in K$, there is an idempotent $e \in K$ such that $e\gamma M = x\gamma M$, and we note that $e\gamma S\gamma x \subseteq e\gamma S = x\gamma S \subseteq K\Gamma S$. Since $(1-e)\gamma S\gamma e \subseteq N \subseteq M$, we have $(1-e)\gamma S\gamma x \subseteq (1-e)\gamma S\gamma e\gamma M \subseteq M\gamma e\gamma M \subseteq K$, and, consequently, $S\gamma x =$

$e\gamma S\gamma x + (1 - e)\gamma S\gamma x \subseteq K\Gamma S$. Thus, $S\Gamma K \subseteq K\Gamma S$. By symmetry, $K\Gamma S \subseteq S\Gamma K$, so that $K\Gamma S = S\Gamma K$.

In particular, it follows that $K\Gamma S \in L_2(S)$. Now, the rule $\lambda(K) = K\Gamma S$ defines a monotone map λ: $L_2(M) \to L_2(S)$.Given $K \in L_2(M)$ and $x \in K\Gamma S \cap M$, we infer that $x \in e\gamma S$ for some idempotent $e \in K$, whence $x = e\gamma x \in K$. Thus, $\psi\lambda(K) = K\Gamma S \cap M = K$, so that $\psi\lambda$ is the identity map on $L_2(M)$.

Given $J \in L_2(S)$ and $y \in J$ we have $y \in f\gamma s$ for some idempotent $f \in J$. Since $f \in M$ by hypothesis, we obtain $y = f\gamma y \in (J \cap M)\Gamma S$. Thus, $\lambda\psi(J) = (J \cap M)S = J$, so that $\lambda\psi$ is the identity map on $L_2(S)$. Therefore ψ and λ are inverse order-isomorphisms, hence, also lattice isomorphisms. ■

Chapter-Eight

Unit-Regular Γ-rings

This chapter is concerned with unit-regular Γ-rings, which is equivalent to various cancellation properties for direct sums of finitely generated projective modules. We have developed a number of equivalent characterizations of the unit regularity of regular Γ-rings, mostly in the form of cancellation properties, either internal (within the lattice $L(M_R)$) or external(for finitely generated projective M-modules). These cancellation properties are then used to derive further properties of finitely generated projective modules over unit-regular Γ-rings. We also develop a lattice theoretic characterization of the unit-regularity of M, namely transitivity of the relation of perspectivity in the lattice $L(2M_R)$

Definition: *A Γ-ring M is said to be a **unit-regular** Γ-ring provided that, for each $x\in M$, there is a unit (i.e., an invariable element) $u\in M$ such that $x\mu u\gamma x = x$* for some μ, $\gamma\in\Gamma$. *For Example, any direct product of division Γ-rings is a **unit-regular** Γ-ring.*

Note that the class of **unit-regular** Γ-rings is closed under homomorphic images, direct products, and direct limits.

***Theorem* 8.01.** *The Jacobson radical of a unit-regular Γ-ring M is equal to zero.*

Proof. Let $a\in \mathfrak{J}(M)$(Jacobson radical). Then $M\Gamma a \subseteq \mathfrak{J}(M)$. Since $M\gamma a$ is generated by an idempotent element e (by *Theorem* 6.08(a)(ii)), $M\gamma a =$

$M\gamma e$, and thus, from $e \in \mathrm{J}(M)$ it follows that $(1 - e)$ is invertable. So there exists $x \in M$ such that

$$1 = x\gamma(1 - e) = x\gamma 1 - x\gamma e = x - x\gamma e.$$

Hence, $e = 1\gamma e = (x - x\gamma e)\gamma e = x\gamma e - x\gamma e\gamma e = x\gamma e - x\gamma e = 0.$

Therefore, $a = 0$. Hence, Jacobson radical of a regular Γ-ring M is zero. ■

***Theorem* 8.02.** *The centre of a unit-regular Γ-ring M is regular.*

Proof. Let $a \in C(M)$ (center of M). Then there is a unit $u \in M$ such that $a\mu u\gamma a = a$ for some $\mu, \gamma \in \Gamma$.

Now, $a = a\mu u\gamma a = a\mu(u\gamma a) = (u\gamma a)\mu a = u\gamma(a\mu a)$.

Also $a = a\mu u\gamma a = (a\mu u)\gamma a = a\gamma(a\mu u) = (a\gamma a)\mu u$.

So, $a\mu u = a\gamma a\mu u\mu u$, or $a = a\gamma(a\mu u) = a\gamma((a\gamma a\mu u)\mu u) = a\gamma((a\gamma u\mu a)\mu u) = a\gamma(a\gamma u\mu u)\mu a$. Now, $a\mu u \in C(M)$ because if $y \in M$ then $(a\mu u)\gamma y = (u\gamma y)\mu a = (u\gamma y)\mu(a\gamma a)\mu u = u\gamma y\mu a\gamma a\mu u = (u\gamma y\mu a\gamma u)\mu a = a\mu(u\gamma y\mu a\gamma u) = a\mu u\gamma(y\mu u)\gamma a = a\mu u\gamma a\gamma(y\mu u) = a\gamma y\mu u = y\gamma(a\mu u)$.

Also $a\gamma u\mu u \in C(M)$ because

$(a\gamma u\mu u)\gamma y = (a\gamma u)\mu(u\gamma y) = (u\gamma y)\mu(a\gamma u) = u\gamma y\mu a\gamma u = u\gamma a\mu y\gamma\mu u = (a\gamma u)\mu(y\gamma u) = (y\gamma u)\mu(a\gamma u) = y\gamma(u\mu a\gamma u) = y\gamma(u\mu u)\gamma a = y\gamma a\gamma(u\mu u) = y\gamma(a\gamma u\mu u)$.

Hence, the centre of M is a regular Γ-ring. ■

***Theorem* 8.03.** *Every unit-regular Γ-ring without zero divisors is a skew Γ-field.*

Proof. Let $a \in M$, $a \neq 0$. Then there is a unit $u \in M$ such that $a\mu u\gamma a = a$ for some $\mu, \gamma \in \Gamma$. Then $a\mu(u\gamma a - 1) = 0$ and $(a\mu u - 1)\gamma a = 0$, and

Hence, $u\mu a = 1$, $a\mu u = 1$ for some $\mu, \gamma \in \Gamma$ and so a is invariable. Hence, M is a skew Γ-field. ■

***Theorem* 8.04.** *If M is a unit-regular Γ-ring whose only nilpotent element is zero, then*

i) *Every idempotent element of M is in the centre.*

ii) *If $a \in M$, $a \neq 0$, then there exists $b \in M$, $\gamma \in \Gamma$ such that $a\gamma b = b\gamma a = f$ is idempotent and $a\gamma f = f\gamma a = a$.*

iii) *$M\Gamma a = a\Gamma M$ for all $a \in M$; Hence, every left (or right) ideal is a two sided ideal.*

Proof. i) Let $e \in M$ be idempotent. Let $a \in M$ be an arbitrary element and assume that zero is the only nilpotent element of M.

Since $[(1-e)\mu a\gamma e]\mu[(1-e)\mu a\gamma e]$

$= (1\mu a\gamma e - e\mu a\gamma e)\mu(1\mu a\gamma e - e\mu a\gamma e)$

$= (a\gamma e - e\mu a\gamma e)\mu(a\gamma e - e\mu a\gamma e)$

$= a\gamma e\mu a\gamma e - a\gamma e\mu e\mu a\gamma e - e\mu a\gamma e\mu a\gamma e + e\mu a\gamma e\mu e\mu a\gamma e$

$= a\gamma e\mu a\gamma e - a\gamma e\mu a\gamma e - e\mu a\gamma e\mu a\gamma e + e\mu a\gamma e\mu a\gamma e$

$= 0$

and

$[e\gamma a\mu(1-e)]\mu[e\gamma a\mu(1-e)]$

$= [e\gamma a\mu 1 - e\gamma a\mu e)\mu(e\gamma a\mu 1 - e\gamma a\mu e)]$

$= e\gamma a\mu e\gamma a - e\gamma a\mu e\mu e\gamma a - e\gamma a\mu e\gamma a\mu e + e\gamma a\mu e\mu e\gamma a\mu e$

$= e\gamma a\mu e\gamma a - e\gamma a\mu e\gamma a - e\gamma a\mu e\gamma a\mu e + e\gamma a\mu e\gamma a\mu e$

$= 0$ for some $\mu, \gamma \in \Gamma$.

Since the nilpotent element is zero, so we have

$0 = (1-e)\mu a\gamma e = 1\mu a\gamma e - e\mu a\gamma e = a\gamma e - e\mu a\gamma e,$

and $0 = e\gamma a\mu(1 - e) = e\gamma a\mu 1 - e\gamma a\mu e = e\gamma a - e\gamma a\mu e$.

Hence, $a\gamma e = e\mu a\gamma e = e\gamma a$ and so e is in the centre of M.

ii) Let M be a regular Γ-ring having 0 as the only nilpotent element. Given $a \in M$, $a \neq 0$. Then there is a unit $u \in M$ such that $a\mu u\gamma a = a$ for some μ, $\gamma \in \Gamma$. Then $e = a\mu u$, $e' = u\delta a$ are idempotent elements of M; so e and e' belong to the centre, and $f = e\gamma e'$ is an idempotent.

It follows that $a\mu u\gamma u\delta a = (a\mu u)\gamma(u\delta a) = e\gamma e'$.

Also $(u\mu u\gamma a)\delta a = [u\mu(u\gamma a)]\delta a = [(u\gamma a)\mu u]\delta a = [u\gamma(a\mu u)]\delta a = [(a\mu u)\gamma u]\delta a = (a\mu u)\gamma(u\delta a) = e\gamma e'$.

Moreover, $a\mu f = a\mu e\gamma e' = e\mu a\gamma e' = a\mu u\mu a\gamma e' = a\gamma e' = a\gamma u\delta a = a$,

and $f\mu a = e\gamma e'\mu a = e\gamma a\mu e' = e\gamma a\mu u\delta a = e\gamma a = a\mu u\gamma a = a$.

Thus, $a\mu f = f\mu a = a$.

iii) Given $y \in M$, we have $y\mu a = y\mu(a\gamma e\gamma e') = (y\mu a)\gamma e\gamma e' = e\gamma(y\mu a)\gamma e' = (a\mu u)\gamma(y\mu a)\gamma e' = a\mu u\gamma y\mu a\gamma e'$. So, there exists $z \in M$ such that $y\mu a = a\mu z$. This shows that $M\gamma a \subseteq a\gamma M$. The converse is proved in a similar way.

Hence, since every left ideal J is the sum of the principal left ideals generated by its elements, J is also a right ideal and vice versa. ■

The following fact is needed for the proof of *Theorem* 8.05.

***Fact* 1.** [3, *Theorem* 2]. *Suppose $Id(M) \subset C(M)$ and let $x \in M$. If x is regular, then x is unit regular.*

***Theorem* 8.05.** *If M is a unit-regular Γ-ring whose only nilpotent element is zero, then $C(M)$ is unit-regular.*

Proof. Since $C(M)$ of M is regular (*Theorem* 8.02), then for any $x \in C(M)$, x is regular and since $Id(M) \subset C(M)$, so by the *Theorem* 8.04 and by the *Fact* 1, x is unit-regular. ■

Definitions: *Let A be finitely generated projective module over a regular Γ-ring M. We use $L(A)$ to denote the set of all finitely generated submodules of A, partially ordered by inclusion. We also use $L(M_M)$ to denote the set of all principal right ideals of M. The partially ordered sets $L(A)$ are actually a lattice.*

***Theorem* 8.06.** *Suppose that A is a right M-module and $T = End_M(A)$ is a regular Γ-ring. Then the following conditions are equivalent:*

(a) T is a unit-regular Γ-ring.

(b) If $A = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then $B_1 \cong B_2$.

(c) $\ker x = \operatorname{coker} x$ for all $x \in T$.

(d) If $e, f \in T$ are idempotents such that $e\Gamma T = f\Gamma T$, then $(1 - e)\Gamma T \cong (1 - f)\Gamma T$.

Proof. (***a***) $\Rightarrow$ (***b***) Define $x \in T$ so that $x\Gamma B_1 = 0$ and x restricts to an isomorphism of A_1 onto A_2. There exists a unit $u \in T$ such that $x\mu u\gamma x = x$. Inasmuch as $x\mu(u\gamma x - 1) = 0$, we see that $A \leq u\gamma x\Gamma A + (\ker x) = u\Gamma A_2 + B_1$. In addition, since $x\mu u = 1$ on $x\Gamma A = A_2$, we see that $u\Gamma A_2 \cap B_1 = 0$, so that $A = u\Gamma A_2 + B_1$. Since u is an automorphism of A, we also have $A = u\Gamma A_2 \oplus u\Gamma B_2$, whence $B_1 \cong u\Gamma B_2 \cong B_2$.

(***b***) $\Rightarrow$ (***c***) There exists a unit $y \in T$ such that $x\mu y\gamma x = x$. Since $x\mu y$ and $y\gamma x$ are idempotents, we see that $A = y\gamma x\Gamma A \oplus (\ker x) = x\Gamma A \oplus (1 -$

$x\mu y)\Gamma A$. Observing that x restricts to an isomorphism of $y\gamma x\Gamma A$ onto $x\Gamma A$, we conclude from (b) that $\ker x \cong (1 - x\mu y)\Gamma A \cong \operatorname{coker} x$.

(c) $\Rightarrow$ **(a)** Given $x \in T$, there exists a unit $y \in T$ such that $x\mu y\gamma x = x$, and we note as above, that $A = y\gamma x\Gamma A \oplus (\ker x) = x\Gamma A \oplus (1 - x\mu y)\Gamma A$. By (c), $\ker x \cong \operatorname{coker} x \cong (1 - x\mu y)\Gamma A$. Also, x restricts to an isomorphism of $y\gamma x\Gamma A$ onto $x\Gamma A$. Define $u \in T$ so that u restricts to $x^{-1}: x\Gamma A \rightarrow y\gamma x\Gamma A$ and u restricts to an isomorphism of $(1 - x\mu y)\Gamma A$ onto $\ker x$. Then u is a unit in T such that $x\mu u\gamma x = x$.

(a) $\Leftrightarrow$ **(d)** is just the equivalence $(a) \Leftrightarrow (b)$ applied to the case $A = T_T$. ■

***Corollary* 8.07.** *Let M be a unit-regular Γ-ring, and let $\varphi : L(M_R) \rightarrow L(M_R)$ be the lattice isomorphism defined by the rule $\varphi(J) = \{x \in M / x\Gamma J = 0\}$. Let $J, K \in L(M_R)$. Then*

(a) $J \cong K$ if and only if $\varphi(J) \cong \varphi(K)$.

(b) $J \lesssim K$ if and only if $\varphi(K) \lesssim \varphi(J)$.

Proof. Choose idempotents $e, f \in M$ such that $e\Gamma M = J$ and $f\Gamma M = K$. Note that $\varphi(J) = M\Gamma(1 - e)$ and $\varphi(K) = M\Gamma(1 - f)$.

(a) If $J \cong K$, then there exists elements $x \in e\Gamma M\Gamma f$ and $y \in f\Gamma M\Gamma e$ such that $x\gamma y = e$ and $y\gamma x = f$, $\gamma \in \Gamma$. Then $M\Gamma e \cong M\Gamma f$; hence, *Theorem* 8.06 says that $M\Gamma(1 - e) \cong M\Gamma(1 - f)$. The converse is identical.

(b) If $J \lesssim K$, then there exist elements $x \in e\Gamma M\Gamma f$ and $y \in f\Gamma M\Gamma e$ such that $x\gamma y = e$, whence $M\Gamma e \lesssim M\Gamma f$. Then $M\Gamma f \cong M\Gamma e \oplus A$ for some A; hence, $M\Gamma e \oplus M\Gamma(1 - e) \cong (M\Gamma e \oplus A) \oplus M\Gamma(1 - f)$. By *Theorem* 8.06,

$M\Gamma(1-e) \cong A \oplus M\Gamma(1-f)$, so that $M\Gamma(1-f) \lesssim M\Gamma(1-e)$. The converse is identical. ■

***Corollary* 8.08.** *Let A be a finitely generated projective right module over a regular Γ-ring M, and set $T = End_M(A)$. Then T is unit-regular Γ-ring if and only if the following condition holds:*

(*) If $A_1, A_2 \in L(A)$ such that $A_1 \cong A_2$, then there exists $B \in L(A)$ such that $A = A_1 \oplus B = A_2 \oplus B$.

Proof. Clearly T is regular Γ-ring.

First assume that (*) holds. If $A = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then, by (*), there exists $B \in L(A)$ such that $A = A_1 \oplus B = A_2 \oplus B$, whence $B_1 \cong B \cong B_2$. Hence, by *Theorem* 8.06, T is unit-regular.

Conversely, assume that T is unit-regular Γ-ring. Given A_1, $A_2 \in L(A)$ with $A_1 \cong A_2$, choose decompositions $A_1 = (A_1 \cap A_2) \oplus C_1$ and $A_2 = (A_1 \cap A_2) \oplus C_2$, and $A = (A_1 + A_2) \oplus D$. Observing that $A = (D \oplus A_2) \oplus C_2$ and $A = (D \oplus A_2) \oplus C_1$ with $D \oplus A_1 \cong D \oplus A_2$ we say, by *Theorem* 8.06 that there exits an isomorphism $f: C_1 \to C_2$. Setting $C = \{x + f(x) \mid x \in C_1\}$, we infer that $A_1 + A_2 = A_1 \oplus C = A_2 \oplus C$, whence $A = A_1 \oplus (C \oplus D) = A_2 \oplus (C \oplus D)$. Therefore (*) holds. ■

***Theorem* 8.09.** *Let $A_1 \oplus \ldots\ldots\ldots \oplus A_n \cong B_1 \oplus \ldots\ldots\ldots \oplus B_n$ be finitely generated projective modules over a unit-regular Γ-ring. Then there exit decompositions $A_i = A_{i1} \oplus \ldots\ldots\ldots \oplus A_{ik}$ for $i = 1, \ldots\ldots, n$ such that $A_{1i} \oplus \ldots\ldots \oplus A_{ni} \cong B_i$ for $i = 1, \ldots\ldots, K$.*

Proof. The proof is similar to the proof of [10, *Theorem* 2.8]. ■

***Theorem* 8.10.** *Let M be a regular Γ-ring. Then M is unit-regular Γ-ring if and only if $A \oplus B \cong A \oplus C$ implies $B \cong C$, for all finitely generated projective right M-modules A, B, C.*

Proof. It is clear from *Theorem* 8.06 that this cancellation property implies unit-regularity.

Conversely, assume that M is unit-regular, and let A, B, C be finitely generated projective right M-modules such that $A \oplus B \cong A \oplus C$. Now, A is a direct summand of nM_R for some positive integer n, whence $nM_R \oplus B \cong nM_R \oplus C$. By induction, it suffices to prove the case $n = 1$.

Assume that $M_R \oplus B \cong M_R \oplus C$. Then by the *Theorem* 8.09, there exist decompositions $M_R = M_1 \oplus M_2$ and $B = B_1 \oplus B_2$ such that $M_1 \oplus B_1 \cong M_R$ and $M_2 \oplus B_2 \cong C$. Inasmuch as $M_1 \oplus B_1 \cong M_R = M_2 \oplus M_2$, we obtain $B_1 \cong M_2$ from *Theorem* 8.06, whence and $B = B_1 \oplus B_2 \cong M_2 \oplus B_2 \cong C$. ■

Defination: *Let L be a lattice with a least element 0 and greatest element 1. Two elements x, $y \in L$ are said to be perspective (in L) provided they have a common element, i.e., an element $z \in L$ such that $x \wedge z = y \wedge z = 0$ and $x \vee z = y \vee z = 1$.*

Note that the principal ideals J and K in a regular Γ-ring M are perspective in the lattice $L(M_R)$ if and only if there is some $H \in L(M_R)$ for which $M_R J \oplus H = K \oplus H$. Consequently, we see from *Theorem* 8.09 that M is unit-regular if and only if isomorphism implies perspectivity in $L(M_R)$.

***Proposition* 8.11.** *Let A be a finitely generated projective modules over regular Γ-ring, and let B, $C \in L(A)$. Then B and C are perspective in $L(A)$ if and only if $B/(B \cap C) \cong C/(B \cap C)$.*

***Proof*:** Choose decompositions $B = B' \oplus (B \cap C)$ and $C = C' \oplus (B \cap C)$. If B and C are perspective, then $A = B \oplus D = C \oplus D$ for some D. In this case, we have $A = B' \oplus [(B \cap C) \oplus D] = C' \oplus [(B \cap C) \oplus D]$, whence $B' \cong C'$.

Conversely, assume that there exists an isomorphism f: $B' \to C'$. Setting $D = \{x + f(x) \mid x \in B'\}$, we infer that $B + C = B \oplus D = C \oplus D$. Since $A = (B + C) \oplus E$ for some E, we conclude that $A = B \oplus (D \oplus E)$ $= C \oplus (D \oplus E)$. Therefore B and C are perspective. ■

***Corollary* 8.12.** *Let A be a finitely generated projective modules over a unit-regular Γ-ring, and let B, $C \in L(A)$. Then B and C are perspective in $L(A)$ if and only if $B \cong C$. Consequently, perspectivity is transitive in $L(A)$.*

***Proof*.** In view of *Theorem* 8.10, we have $B/(B \cap C) \cong C/(B \cap C)$ if and only if $B \cong C$. ■

***Theorem* 8.13.** *A regular Γ-ring is unit-regular Γ-ring if and if perspectivity is transitivity in $L(2M_R)$.*

***Proof*.** If M is unit-regular Γ-ring, then *Corollary* 8.12 shows that perspectivity is transitive in $L(2M_R)$.

Conversely, assume that perspectivity is transitive in $L(2M_R)$, and let $e, f \in M$ be idempotents such that $(1 - e)\Gamma M \cong (1 - f)\Gamma M$. Define A, B,

$C \in 2L(M_R)$ by setting $A = \{(x, y) \in 2M_R \mid x \in e\Gamma M\}$ and $B = \{(x, y) \in 2M_R \mid y \in e\Gamma M\}$, while $C = \{\{(x, y) \in 2M_R \mid x \in f\Gamma M\}$. Since $A/(A \cap B) \cong M/e\Gamma M \cong B/(A \cap B)$, we see by *Theorem* 8.10 that A and B are perspective in $L(2M_R)$. Observing that $B/(B \cap C) \cong (1-f)\Gamma M \cong (1-e)\Gamma M \cong C/(B \cap C)$, we also see from *Theorem* 8.10 that B and C are perspective. By transitivity, A and C are perspective, whence *Theorem* 8.11 shows that $A/(A \cap C) \cong C/(A \cap C)$.

Observe that $A/(A \cap C) \cong e\Gamma M/(e\Gamma M \cap f\Gamma M)$ and $C/(A \cap C) \cong f\Gamma M/(e\Gamma M \cap f\Gamma M)$, we conclude that $e\Gamma M \cong f\Gamma M$. Therefore M is a unit-regular Γ-ring. ■

Defination: *A Γ-ring M is said to be stable range* 1 *provided that whenever $a\Gamma M + b\Gamma M = M$, there exists $y \in M$ and $\gamma \in \Gamma$ such that $a + b\gamma y$ is a unit.*

***Proposition* 8.14.** *A regular Γ-ring M has stable range* 1 *if and only if it is unit-regular.*

Proof. First assume that M has a stable range 1. Given any $a \in M$, there exists $x \in M$ and $\mu, \gamma \in \Gamma$ such that $a\mu x\gamma a = a$. Now, $a\Gamma M + (1 - a\mu x)\Gamma M = M$; hence, there exists $y \in M$ such that $a + (1 - a\mu x)\gamma y$ is a unit. Then there is a unit $u \in M$ for which $[a + (1 - a\mu x)\gamma y]\delta u = 1$ for some $\delta \in \Gamma$, whence

$$a = a\mu x\gamma a = a\mu x\gamma[a + (1 - a\mu x)\gamma y]\delta u\gamma a = a\mu x\gamma a\delta u\gamma a = a\delta u\gamma a.$$

Therefore M is unit-regular.

Conversely, assume that M is unit-regular, and let $a\Gamma M + b\Gamma M = M$. Now, $b\Gamma M = (a\Gamma M \cap b\Gamma M) \oplus J$ for some J, and $a\Gamma M \oplus J = M_M$.

In addition, $M_M = K \oplus L$, where $K = \{r \in M \mid a\gamma r = 0\}$. Since $L \cong a\Gamma M$, we see from *Theorem* 3.06 that $K \cong J$; hence, there exists $c \in M$ such that $c\Gamma L = 0$ and left multiplication by c induces an isomorphism of K onto J. Note that $c\Gamma M = J \leq b\Gamma M$, whence $c = b\gamma y$ for some $y \in M$.

Now, left multiplication by a induces an isomorphism of L onto $a\Gamma M$, while left multiplication by c induces an isomorphism of K onto J. Inasmuch as $a\Gamma K = c\Gamma L = 0$, it follows that left multiplication by $a + c$ induces an isomorphism of $L \oplus K = M_M$ onto $a\Gamma R \oplus J = M_M$. Therefore $a + b\gamma y = a + c$ is a unit in M. ■

***Lemma* 8.15.** *If M is a unit-regular Γ-ring and J is a two-sided ideal of M, then M/J is unit-regular.*

Proof. Let $\bar{a} \in M/J$. Then $\bar{a} = a + J$, $a \in M$. Since M is regular, there exists μ, $\gamma \in \Gamma$ and unit $u \in M$ such that $a = a\mu u\gamma a$. Now, $\bar{a}\mu u\gamma\bar{a} = (a + J)\mu u\gamma(a + J) = a\mu u\gamma a + J = a + J = \bar{a}$. Therefore M/J is regular. ■

***Proposition* 8.16.** *Let J be a two-sided ideal of a unit-regular Γ-ring M, and let A_1, ..., A_n be finitely generated projective right M-modules such that the modules A_i/A_iJ are pairwise isomorphic. Then there exist decompositions $A_i = B_i \oplus C_i$ for each i such that the modules B_i are pairwise isomorphic and each $C_i = C_iJ$.*

Proof. The proof is similar to the proof of [10, *Theorem* 2.19]. ■

***Lemma* 8.17** *Let J be a two-sided ideal in a regular Γ-ring M. Then M is unit-regular if and only if*

(a) M/J is unit-regular.

(b) If e and f are idempotents in J such that $(1 - e)\Gamma M \cong (1 - f)\Gamma M$, *then* $e\Gamma M \cong f\Gamma M$.

Proof. If M is unit-regular, then (a) is clear by *Lemma* 8.15 and (b) follows from *Theorem* 8.05.

Conversely, assume that (a) and (b) hold. Given idempotents $g, h \in M$ such that $g\Gamma M \cong h\Gamma M$, we have $\bar{g}\Gamma(M/J) \cong \bar{h}(M/J)$ and so $(1-\bar{g})\Gamma(M/J) \cong (1-\bar{h})\Gamma(M/J)$, by (a). According to *Proposition* 8.16, there exist decompositions $(1 - g)\Gamma M = G_1 \oplus G_2$ and $(1 - h)\Gamma M = H_1 \oplus H_2$ such that $G_1 \cong H_1$, while $G_2 = G_2\Gamma J$ and $H_2 = H_2\Gamma J$. There exist idempotents $e, f \in M$ such that $e\Gamma M = G_2$, and $(1 - e)\Gamma M = g\Gamma M \oplus G_1$, While $f\Gamma M = H_2$ and $(1 - f)\Gamma M = h\Gamma M \oplus H_1$. Then $e, f \in J$ and $(1 - e)\Gamma M \cong (1 - f)\Gamma M$; hence, $e\Gamma M \cong f\Gamma M$ by (b), and consequently, $(1 - g)\Gamma M \cong (1 - h)\Gamma M$. Therefore M is unit-regular (by *Theorem* 8.06). ■

***Lemma* 8.18.** *Let J be a two-sided ideal in a unit-regular Γ-ring S and let M be a subring of S that contains J. If M/J is unit-regular, then so is M.*

Proof. Since J and M/J is regular, so is M. If e and f are idempotents in J such that $(1 - e)\Gamma M \cong (1 - f)\Gamma M$, then $(1 - e)\Gamma S \cong (1 - f)\Gamma S$, and consequently, $e\Gamma S \cong f\Gamma S$. Since $e, f \in J$ and $e\Gamma M = e\Gamma S$ and $f\Gamma M \cong f\Gamma S$, whence $e\Gamma M \cong f\Gamma M$. Therefore M is unit-regular, by *Lemma* 8.17. ■

***Lemma* 8.19.** *Let* $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ - - - - *be the ascending chain of unit-regular Γ- ideals. Then* $\cup_\alpha I_\alpha$ *is unit-regular.*

Proof. It is obvious. ■

From the *Lemma* 8.15, *Lemma* 8.17, *Lemma* 8.18 and *Lemma* 8.19, we get the following theorem:

***Theorem* 8.18.** *The class of all unit-regular Γ-rings is a radical class.*

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List of Special Symbols

R	–	Ring
Γ_N	–	Gamma rings in the sense of Nobusawa
M	–	Gamma ring
$\mathfrak{R}$	–	Radical class
$\boldsymbol{Z}$	–	Set of integers
$\boldsymbol{Z}_m$	–	Residue class modulo m
$C(M)$	–	Centre of M
$\mathfrak{J}(R)$	–	Jacobson radical of a ring R
Ann_M	–	Annihilator of M
$L(A)$	–	Set of all finitely generated submodules of A, partially ordered by inclusion.
f^{-1}	–	Inverse function
fog	–	Composite functions of f and g
i.e.	–	that is
Γ	–	Capital gamma
Π	–	Product of
Σ	–	Summation of
$\oplus$	–	Direct sum
$\cup$	–	Union
$\cap$	–	Intersection
$\propto$	–	Infinity
$\supset$	–	Strictly superset of

$\supseteq$ - Superset of

$\subset$ - Strictly subset of

$\subseteq$ - Subset of

$\in$ - Belongs to

$\notin$ - Not belongs to

Φ - Empty set

$\cong$ - Isomorphic to

$< >$ - Ideal generated by

$ID(M)$ - Set of all idempotent elements of M