1994

N-Ideals of a Lattice

Latif, Md. Abdul

University of Rajshahi

THE UNIVERSITY OF RAJSHAHI

BANGLADESH

n-IDEALS OF A LATTICE

A Thesis

Presented for the degree of Doctor of Philosophy

by

MD. ABDUL LATIF

B.Sc. Hons.(Rajshahi), M.Sc.(Rajshahi).

n-IDEALS OF A LATTICE

A Thesis

Presented for the degree of Doctor of Philosophy

by

MD. ABDUL LATIF

B.Sc. Hons.(Rajshahi), M.Sc.(Rajshahi)

in the

Department of MAthematics

University of Rajshahi,

Rajshahi Bangladesh.

January, 1994.

To my parents,

who have profoundly influenced my life.

TABLE OF CONTENTS

SUMMARY

This thesis studies the nature of n-ideals of a lattice. The topic arose out of a study on the kernels, around a particular element n, of a skeletal congruence on a distributive lattice. The idea of n-ideals in a lattice was first introduced by Cornish and Noor. For a fixed element n of a lattice L, a convex sublattice containing n is called an n-ideal. If L has a `0´, then replacing n by 0, an n-ideal becomes an ideal. Moreover if L has 1, an n-ideal bcomes a filter by replacing n by 1. Thus, the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. So any result involving n-ideals will give a generalization of the results on ideals and filters with 0 and 1 respectively in a lattice. In this thesis we give a series of results on n-ideals of a lattice which certainly extend and generalize many works in lattice theory.

Chapter I discusses n-ideals, finitely generated n-ideals and other results on n-ideals of a lattice which are basic to this thesis. We have shown that, a lattice L is modular (distributive) if and only if $I_n(L)$, the lattice of n-ideals is modular (distributive). We have also shown that the set of prime n-ideals of a distributive lattice L is unordered by

set inclusion if and only if $F_n(L)$, the lattice of finitely generated n-ideals is generalized boolean.

Chapter 2 discusses and generalize the concepts of the smallest and largest Congruences $\theta(I)$ and $R(I)$ respectively of a distributive lattice containing an n-ideal I as a class. Also we have given a characterization of distributivity of a lattice using $\theta(I)$. We have shown that in a distributive lattice L, the mapping $I \rightarrow \theta(I)$ is an imbedding from $I_n(L)$ to C(L), the lattice of congruences of L and there is an isomorphism if and only if $F_n(L)$ is generalized boolean. Also we have shown that there is an isomorphism between $C(F_n(L))$and C(L). Finally, we include a result on the permutability of the congruences $\theta(I)$ and $\theta(J)$ for n-ideals I and J of a distributive lattice L.

Chapter 3 studies the n-kernels of skeletal congruences on a distributive lattice. Previously, skeletal congruences have been studied by Cornish very extensively. This chapter generalizes several results of his works. Here we have given a description on $\theta(J)^*$ for an n-ideal J of a distributive lattice L. The Skeleton

$$SC(L) = \{\theta \in C(L) : \theta = \Phi^* \text{ for some } \Phi \in C(L)\}$$
$$= \{\theta \in C(L) : \theta = \theta^{**}\}.$$

We define $J^+ = \{x \in L : (x \wedge n) \vee (n \wedge j) \vee (x \wedge j) = n$ for all $j \in J\}$, which is of course an n-ideal. We also define $Ker_n\theta = \{ x \in L : x \equiv n\ \theta \}$ and $K_nSC(L) = \{ Ker_n\theta : \theta \in C(L)\}$.

This chapter establishes the following fundamental results :

(i) J^+ is the n-kernel of $\theta(J)^*$.

(ii) $\theta(J)$ is dense in C(L) if and only if the n-ideal J is both meet and join-dense and the n-kernels of each skeletal congruence is an annihilator n-ideal.

(iii) $F_n(L)$ is disjunctive if and only if each dense n-ideal J is both meet and join-dense.

(iv) $F_n(L)$ is generalized boolean if and only if $\theta(J^+) = \theta(J)^*$ for any n-ideal J.

(v) $F_n(L)$ is generalized boolean if and only if the map $\theta \rightarrow ker_n\theta$ is a lattice isomorphism of SC(L) onto $K_nSC(L)$ whose inverses the map $J \rightarrow \theta(J)$, where J is an n-ideal.

In chapter 4, we discuss on standard n-ideal of a lattice. Standard elements and ideals have been studied by many authors including Grätzer. From an open problem given by him,Fried and Schmidt have extended the idea to standard (convex) sublattices. In the light of their work we have developed the notion of standard n-ideals and showed that an n-ideal is standard if and only if it is a standard sublattice. We have also given a characterization of a standard n-ideal S interms of the congruence $\theta(S)$. Then we have proved the following results:-

(i) for an arbitrary n-ideal I and a standard n-ideal S of a lattice L, if $I \vee S$ and $I \cap S$ are principal n-ideals, then I itself a principal n-ideal.

(ii) For a neutral element n of a lattice with the proprerty that both (n] and [n) are relatively complemented, every homomorphism n-kernel of L is a standard n-ideal and every standard n-ideal is the n-kernel of precisely one congruence relation.

(iii) for a relatively complemented lattice L with 0 and 1, C(L) is a boolean algebra if and only if every standard n-ideal of L is a principal n-ideal.

Finally, we prove two isomorphism theorems on standard n-ideals which are extensions of the isomorphism theorems on standard ideals given by Grätzer and Schmidt [18].

STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain any material priviosly published or written by another person except where due reference is made in the text.

Md. Abdul Latif

Md. Abdul Latif.

ACKNOWLEDGEMENT

I wish to express my sincere gratitude to my superviser **Dr. A.S.A.Noor**, Professor, Department of Mathematics, Rajshahi University, for his constant guidance, suggestions, criticism and encouragement during the research for and preparation of this thesis.

I was fortunate in obtaining the expert services of **Md. Rashed Talukder** for the computer typing of this thesis. His patience, co-operation and careful attention to detail are deeply appreciated.

The acknowledgement will remain incomplete if I do not mention the inspiration that I received from my wife **Delowara Yasmin** and also deprived my daughters **Liz** and **Barsha** during the course of this work from my love and affection. I express my heartiest thanks to them as well.

The facilities that I obtained from the department of mathematics, Rajshahi University are greatefully acknowledged.

CHAPTER - I

"n -ideals of a Lattice"

Introduction :- The intention of this chapter is to outline and fix the notation for some of the concepts of n-ideals of a lattice which are basic to this thesis. The idea of n-ideals in a lattice was first introduced by Cornish and Noor in several papers [5], [34], [35]. Since then a little attention has been paid in these matters. For a fixed element n of a lattice L, a convex sublattice containing n is called an n-ideal. If L has a `0´, then replacing n by `0´ an n-ideal becomes an ideal. Moreover if L has 1, an n-ideal becomes a filter by replacing n by 1. Thus, the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. So any result involving n-ideals will give a generalization of the results on ideals and filters with 0 and 1 respectively in a lattice.

The set of all n-ideals of L is denoted by $I_n(L)$, which is an algebraic lattice under set-inclusion. Moreover, $\{n\}$ and L are respectively the smallest and largest elements of $I_n(L)$ while the set-theoretic intersection is the infimum.

For any two n-ideals I and J of L, it is easy to check that

$$I \cap J = \{x : x = m(i, n, j) \text{ for some } i \in I,\ j \in J\},$$

where $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$

and $I \vee J = \{x : i_1 \wedge j_1 \leq x \leq i_2 \vee j_2,$ for some $i_1, i_2 \in I$ and $j_1, j_2 \in J\}$.

The n-ideal generated by $a_1, a_2, \ldots a_m$ is denoted by $\langle a_1, a_2, \ldots a_m\rangle_n$.

Clearly $\langle a_1, a_2, \ldots a_m\rangle_n = \langle a_1\rangle_n \vee \ldots \vee \langle a_m\rangle_n$.

The n-ideal generated by a finite number of elements is called a finitely generated n-ideal. The set of all finitely generated n-ideals is denoted by $F_n(L)$. Of course $F_n(L)$ is a lattice. The n-ideal generated by a single element is called a principal n-ideal. The set of all principal n-ideals of L is denoted by $P_n(L)$. We have

$$\langle a\rangle_n = \{x \in L : a \wedge n \leq x \leq a \vee n\}.$$

The median operation
$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ is very well known in lattice theory. This has been used by several authors including Birkhoff and Kiss [03] for bounded distributive lattices, Jakubik and Kalibiar [26] for distributive lattices and Sholander [44] for median algebra.

An n-ideal P of a lattice L is called prime if $m(x, n, y) \in P$, $x, y \in L$ implies either $x \in P$ or $y \in P$.

Standard and neutral elements in a lattice were studied extensively in [18] and [16, chapter-3]. An element s of a lattice L is called standard if for all $x, y \in L$, $x \wedge (y \vee s) = (x \wedge y) \vee (x \wedge s)$. An element $n \in L$ is called neutral if it is standard and for all $x, y \in L$, $n \wedge (x \vee y) = (n \wedge x) \vee (n \wedge y)$. Of course 0 and 1 of a lattice are always neutral. An element $n \in L$ is called central if it is neutral and complemented in each interval containing n.
A lattice L with 0 is called sectionally complemented if $[0, x]$ is complemented for all $x \in L$.
A distributive lattice with 0, which is sectionally

complemented is called a generalized boolean lattice. For the background meterial we refer the reader to the texts of G. Grätzer [15], Birkhoff [04] and Rutherford [43].

In section 1, we have given some fundamental results on finitely generated n-ideals. We have shown that for a neutral element n of a lattice L, $P_n(L)$ is a lattice if and only if n is central. We have also shown that for a neutral element n, a lattice L is modular (distributive) if and only if $I_n(L)$ is modular (distributive). We proved that, in a distributive lattice L, if both supremum and infimum of two n-ideals are principal, then each of them is principal.

In section 2, we have studied the prime n-ideals of a lattice. Here we have generalized the seperation property for distributive lattices given by M.H. Stone [15, Th. 15, p-74] in terms of prime n-ideals. Then we showed that in a distributive lattice, every n-ideal is the intersection of prime n-ideals containing it. We have also shown that, in a distributive lattice L, the set of prime n-ideals is unordered by set inclusion if and only if $F_n(L)$ is generalized boolean, which generalizes a well known result of L.Nachbin [15, Th.22, p-76].

1. Finitely generated n-ideals.

1.1.1. We start this section with the following proposition which gives some simpler descriptions of $F_n(L)$.

1.1.2. Proposition : Let L be a lattice and $n \in L$. For $a_1, a_2, \ldots, a_m \in L$,

(i) $\langle a_1, a_2, \ldots, a_m\rangle_n \subseteq \{y \in L : (a_1] \cap \ldots \cap (a_m] \cap (n] \subseteq (y] \subseteq (a_1] \vee \ldots \vee (a_m] \vee (n]\}$.

(ii) $\langle a_1, a_2, \ldots, a_m\rangle_n = \{y \in L : a_1 \wedge a_2 \wedge \ldots a_m \wedge n \leq y \leq a_1 \vee \ldots \vee a_m \vee n\}$

(iii) $\langle a_1, a_2, \ldots, a_m\rangle_n = \{y \in L : a_1 \wedge \ldots \wedge a_m \wedge n \leq y = (y \wedge a_1) \vee \ldots \vee (y \wedge a_m) \vee (y \wedge n)$, when L is distributive $\}$

(iv) For any $a \in L$, $\langle a\rangle_n = \{y \in L : a \wedge n \leq y = (y \wedge a) \vee (y \wedge n)\}$

$= \{y \in L : y = (y \wedge a) \vee (y \wedge n) \vee (a \wedge n)\}$

whenever n is standard.

(v) Each finitely generated n-ideal is two generated.

Indeed $\langle a_1, \ldots, a_m\rangle_n = \langle a_1 \wedge \ldots \wedge a_m \wedge n, a_1 \vee \ldots \vee a_m \vee n\rangle_n$.

(vi) $F_n(L)$ is a lattice and its members are simply the intervals $[a,b]$ such that $a \leq n \leq b$ and for each intervals

$$[a,b] \vee [a_1, b_1] = [a \wedge a_1, b \vee b_1]$$

and $$[a,b] \cap [a_1, b_1] = [a \vee a_1, b \wedge b_1].$$

Proof: (i) Right hand side is clearly an n-ideal containing $a_1, a_2, \ldots, a_m$.

(ii) This clearly follows from (i) and by the convexity of n-ideals.

(iii) When L is distributive, then by (ii) $y \leq a_1 \vee a_2 \vee \ldots \vee a_m \vee n$ implies that $y = y \wedge [a_1 \vee a_2 \vee \ldots \vee a_m \vee n] = (y \wedge a_1) \vee (y \wedge a_2) \vee \ldots \vee (y \wedge a_m) \vee (y \wedge n)$, and (iii) follows.

(iv) By (ii) $\langle a\rangle_n = \{y \in L : a \wedge n \leq y \leq a \vee n\}$. Then $y = y \wedge (a \vee n) = (y \wedge a) \vee (y \wedge n)$, when n is standard. This proves (iv)

(v) This clearly follows from (ii)

(vi) First part is readily verifiable. For the second part, consider the intervals $[a,b]$ and $[a_1,b_1]$ where $a \leq n \leq b$, and $a_1 \leq n \leq b_1$.

Then using

(ii), $[a, b] \vee [a_1, b_1] = \langle a, a_1, b, b_1\rangle_n$

$$= [a \wedge a_1 \wedge b \wedge b_1 \wedge n,\ a \vee a_1 \vee b \vee b_1 \vee n]$$

$$= [a \wedge a_1,\ b \vee b_1], \text{ while}$$

$[a,b] \cap [a_1, b_1] = [a \vee a_1,\ b \wedge b_1]$ is trivial. ●

In general, the set of principal n-ideals $P_n(L)$ is not necessarily a lattice . The case is different when n is a central element . The following theorem also gives a characterization of central element of a lattice L.

1.1.3. Theorem: Let n be a neutral element of a lattice L. Then $P_n(L)$ is a lattice if and only if n is central.

Proof: Suppose n is central. Let $\langle a\rangle_n, \langle b\rangle_n \in P_n(L)$. Then using neutrality of n and proposition 1.1.2. (vi),

$$\langle a\rangle_n \cap \langle b\rangle_n = [a \wedge n,\ a \vee n] \cap [b \wedge n, b \vee n]$$

$$= [(a \vee b) \wedge n,\ (a \wedge b) \vee n]$$

and $\langle a\rangle_n \vee \langle b\rangle_n = [a \wedge b \wedge n,\ a \vee b \vee n]$.

Since n is central, there exist c and d such that

$$c \wedge n = (a \vee b) \wedge n, \quad c \vee n = (a \wedge b) \vee n$$

and $d \wedge n = a \wedge b \wedge n, \quad d \vee n = a \vee b \vee n.$

Which implies that $\langle a\rangle_n \cap \langle b\rangle_n = \langle c\rangle_n$ and $\langle a\rangle_n \vee \langle b\rangle_n = \langle d\rangle_n$ and so $P_n(L)$ is a lattice.

conversely, suppose that $P_n(L)$ is a lattice and $a \leq n \leq b$. Then $[a,b] = \langle a\rangle_n \vee \langle b\rangle_n$. Since $P_n(L)$ is a lattice, $\langle a\rangle_n \vee \langle b\rangle_n = \langle c\rangle_n$ for some $c \in L$. This implies that c is the relative complement of n in $[a,b]$. Therefore n is central. ●

Now, we like to discuss $F_n(L)$ when it is sectionally complemented.

1.1.4. Theorem : Let L be a lattice. Then $F_n(L)$ is sectionally complemented if and only if for each $a, b \in L$, with $a \leq n \leq b$, the intervals $[a, n]$ and $[n, b]$ are complemented.

Proof : Suppose $F_n(L)$ is sectionally complemented. Consider $a \leq c \leq n$ and $n \leq d \leq b$. Then $\langle n\rangle \subseteq [c, d] \subseteq [a, b]$. Since $F_n(L)$ is sectionally complemented, so there exists $[c', d']$ such that $[c, d] \cap [c', d'] = \langle n\rangle$ and $[c, d] \vee [c', d'] = [a, b]$. This implies $c \vee c' = n$, $c \wedge c' = a$ and $d \wedge d' = n$, $d \vee d' = b$. That is c' is the relative complement of c in $[a, n]$ and d' is the relative complement of d in $[n, b]$. Hence $[a, n]$ and $[n, b]$ are complemented for all $a, b \in L$ with $a \leq n \leq b$.

Conversely, suppose that $[a, n]$ and $[n, b]$ are complemented for all $a, b \in L$ with $a \leq n \leq b$. Consider $\langle n \rangle \subseteq [c, d] \subseteq [a, b]$. Then $a \leq c \leq n \leq d \leq b$. Since $[a, n]$ and $[n, b]$ are complemented so there exist c' and d' such that $c \wedge c' = a$, $c \vee c' = n$ and $d \wedge d' = n$, $d \vee d' = b$. Thus
$[c, d] \cap [c', d'] = [c \vee c', d \wedge d'] = [n, n] = \langle n \rangle$ and $[c, d] \vee [c', d'] = [c \wedge c', d \vee d'] = [a, b]$, which implies that $[c, d]$ has a relative complement $[c', d']$. Hence $F_n(L)$ is sectionally complemented.

●

The following corollaries follow immediately from above theorem.

1.1.5. Corollary : For a distributive lattice L, $F_n(L)$ is generalized boolean if and only if $[a, n]$ and $[n, b]$ are complemented for each $a, b \in L$ with $a \leq n \leq b$.

1.1.6. Corollary : For a distributive lattice L, $F_n(L)$ is generalized boolean if and only if both $(n]^d$ and $[n)$ are generalized boolean, where $(n]^d$ denotes the dual of the lattice $(n]$.

In lattice theory, it is well known that a lattice L is modular (distributive) if and only if the lattice of ideals $I(L)$ is modular (distributive). Our following theorems are nice generalizations of those results in terms of n-ideals when n is a neutral element. The following Lemma is needed for the next theorem, which is due to Grätzer [17].

1.1.7. Lemma: An element n of a lattice L is neutral if and only if

$$m(x, n, y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$$
$$= (x \vee y) \wedge (x \vee n) \wedge (y \vee n).$$

1.1.8. Theorem: Let L be a lattice with neutral element n. Then L is modular if and only if $I_n(L)$ is modular.

Proof: First assume that L is modular. Let $I, J, K \in I_n(L)$ with $K \subseteq I$. Obviously, $(I \wedge J) \vee K \subseteq I \wedge (J \vee K)$.

To prove the reverse inequality, let $x \in I \wedge (J \vee K)$. Then $x \in I$ and $x \in J \vee K$. Then $j_1 \wedge k_1 \leq x \leq j_2 \vee k_2$ for some $j_1, j_2 \in J$, $k_1, k_2 \in K$. since $I \supseteq K$ so $x \wedge k_1 \in I$ and $x \vee k_2 \in I$. Then by lemma 1.1.7.

$$m(x \wedge k_1, n, j_1) \wedge k_1$$
$$= k_1 \wedge [((x \wedge k_1) \vee n) \wedge (n \vee j_1) \wedge ((x \wedge k_1) \vee j_1)]$$
$$= [(x \wedge k_1) \vee n] \wedge (n \vee j_1) \wedge [(x \wedge k_1) \vee (k_1 \wedge j_1)]$$

as L is modular.

$\leq x$ as $j_1 \wedge k_1 \leq x$.

On the other hand

$$m(x \vee k_2, n, j_2) \vee k_2$$
$$=\{[(x \vee k_2) \wedge n] \vee (n \wedge j_2) \vee [(x \vee k_2) \wedge j_2]\} \vee k_2,$$
$$=[(x \vee k_2) \wedge n] \vee (n \wedge j_2) \vee [(x \vee k_2) \wedge (k_2 \vee j_2)],$$

as L is modular.

$\geq x$ as $j_2 \vee k_2 \geq x$

so we have

$$m(x \wedge k_1, n, j_1) \wedge k_1 \leq x \leq m(x \vee k_2, n, j_2) \vee k_2$$

Hence $x \in (I \wedge J) \vee K$.

Therefore

$I \wedge (J \vee K) = (I \wedge J) \vee K$ with $K \subseteq I$ and so $I_n(L)$ is modular.

Conversely, suppose that $I_n(L)$ is modular. Then for any $a, b, c \in L$ with $c \leq a$, consider the n-ideals $\langle a \vee n\rangle_n$, $\langle b \vee n\rangle_n$ and $\langle c \vee n)_n$. Then of course

$\langle c \vee n\rangle_n \subseteq \langle a \vee n\rangle_n$. Since $I_n(L)$ is modular,

so $\langle a \vee n\rangle_n \cap [\langle b \vee n\rangle_n \vee \langle c \vee n\rangle_n]$

$$= [\langle a \vee n\rangle_n \cap \langle b \vee n\rangle_n] \vee \langle c \vee n\rangle_n.$$

Then by proposition 1.1.2. (vi) and by neutrality of n, it is easy to show that

$$[a \wedge (b \vee c)] \vee n = [(a \wedge b) \vee c] \vee n \quad \ldots (A)$$

Again, consider the n-ideals $\langle a \wedge n\rangle_n$, $\langle b \wedge n\rangle_n$ and $\langle c \wedge n\rangle_n$, $c \le a$ implies $\langle a \wedge n\rangle_n \subseteq \langle c \wedge n\rangle_n$. Then using modularity of $I_n(L)$, we have

$$\langle a \wedge n\rangle_n \vee (\langle b \wedge n\rangle_n \wedge \langle c \wedge n\rangle_n)$$

$$= (\langle a \wedge n\rangle_n \vee \langle b \wedge n\rangle_n) \wedge \langle c \wedge n\rangle_n.$$

Then using proposition 1.1.2. (vi) again and the neutrality of n, it is easy to see that

$$[a \wedge (b \vee c)] \wedge n = [(a \wedge b) \vee c] \wedge n \quad \ldots (B)$$

From (A) & (B) we have $a \wedge (b \vee c) = (a \wedge b) \vee c$, with $c \le a$, as n is neutral. Therefore L is modular. ●

From the proof of above theorem, it can be easily seen that the following corollary holds which is an improvement of the above theorem.

1.1.9. Corollary : For a neutral element n of a lattice L, the following conditions are equivalent:-

(i) L is modular,

(ii) $I_n(L)$ is modular,

(iii) $F_n(L)$ is modular.

For the next theorem we omit the proof of only if part as it can be proved using the similar technique of the proof of above theorem.

1.1.10. Theorem: Let L be a lattice with neutral element n. Then L is distributive if and only if $I_n(L)$ is distributive.

Proof: First assume that L is distributive .Let I, J, $K \in I_n(L)$. Then obviously, $(I \wedge J) \vee (I \wedge K) \subseteq I \wedge (J \vee K)$. To prove the reverse inequality, let $x \in I \wedge (J \vee K)$ which implies $x \in I$ and $x \in J \vee K$. Then $j_1 \wedge k_1 \leq x \leq j_2 \vee k_2$ for some $j_1, j_2 \in J$, $k_1, k_2 \in K$. Since L is distributive,

$$\begin{aligned} m(x, n, j_1) \wedge m(x. n, k_1) &= [(x \wedge n) \vee (x \wedge j_1) \vee (n \wedge j_1)] \wedge \\ &\qquad [(x \wedge n) \vee (x \wedge k_1) \vee (n \wedge k_1)] \\ &= (x \wedge n) \vee (n \wedge j_1 \wedge k_1) \vee (x \wedge j_1 \wedge k_1) \\ &\leq x \vee (j_1 \wedge k_1) = x \end{aligned}$$

Also, $m(x, n, j_2) \vee m(x, n, k_2)$

$$= [(x \wedge n) \vee (x \wedge j_2) \vee (n \wedge j_2)] \vee [(x \wedge n) \vee (x \wedge k_2) \vee (n \wedge k_2)]$$

$$= (n \wedge (x \vee j_2 \vee k_2)) \vee (x \wedge (j_2 \vee k_2)),$$

$$= [n \wedge (j_2 \vee k_2)] \vee x \geq x$$

Then we have

$$m(x,n,j_1) \wedge m(x,n,k_1) \leq x \leq m(x,n,j_2) \vee m(x,n,k_2)$$

and so $x \in (I \wedge J) \vee (I \wedge K)$. Therefore $I \wedge (J \vee K) = (I \wedge J) \vee (I \wedge K)$, and so $I_n(L)$ is distributive.

Following corollary immediately follows from the above proof which is also an improvement of the above theorem.

1.1.11. Corollary : Let L be a lattice with a neutral element n. Then the following conditions are equivalent :

(i) L is distributive,

(ii) $I_n(L)$ is distributive,

(iii) $F_n(L)$ is distributive.

We conclude this section with a nice generalization of [15 : Lemma-5, p-71]. To prove this we need the following lemma:

1.1.12. Lemma: In a distributive lattice L, any finitely generated n-ideal which is contained in a principal n-ideal is principal.

Proof: Let $[b,c]$ be a finitely generated n-ideal such that $b \le n \le c$. Let $\langle a \rangle_n$ be a principal n-ideal such that $[b,c] \subseteq \langle a \rangle_n = [a \wedge n,\ a \vee n]$. Then $a \wedge n \le b \le n \le c \le a \vee n$. Suppose $t = (a \wedge c) \vee b$. Then

$$t \wedge n = [(a \wedge c) \vee b] \wedge n = (n \wedge a \wedge c) \vee (n \wedge b),$$

as L is distributive.

$$= b \wedge n = b$$

and $t \vee n = [(a \wedge c) \vee b] \vee n = (a \wedge c) \vee n$

$= (a \vee n) \wedge (c \vee n)$, as L is distributive.

$$= c \vee n = c$$

Hence $[b,\ c] = [t \wedge n,\ t \vee n] = \langle t \rangle_n$.

Therefore, $[b,\ c]$ is a principal n-ideal. ●

1.1.13. Theorem: Let I and J be n-ideals of a distributive lattice L. If $I \vee J$ and $I \wedge J$ are principal n-ideals, then I and J are also principal.

Proof: Let $I \vee J = \langle a \rangle_n$ and $I \wedge J = \langle b \rangle_n$. Then for all $i \in I$, $j \in J$, $i, j \le a \vee n$ and $i, j \ge a \wedge n$.

So there exist $i_1, i_2 \in I$ and $j_1, j_2 \in J$ such that $a \wedge n = i_1 \wedge j_1$ and $a \vee n = i_2 \vee j_2$.

Consider the n–ideal $[b \wedge i_1 \wedge n, b \vee i_2 \vee n]$.

Since $[b \wedge i_1 \wedge n, b \vee i_2 \vee n] \subseteq I \subseteq \langle a \rangle_n$,

$[b \wedge i_1 \wedge n, b \vee i_2 \vee n] = \langle t \rangle_n$, by lemma 1.1.12. for some $t \in L$. Then

$$\begin{aligned}\langle a \rangle_n = J \vee I &\supseteq J \vee [b \wedge i_1 \wedge n, b \vee i_2 \vee n] \\ &\supseteq [j_1 \wedge n, j_2 \vee n] \vee [b \wedge i_1 \wedge n, b \vee i_2 \vee n] \\ &= [j_1 \wedge n \wedge b \wedge i_1, j_2 \vee n \vee b \vee i_2] \\ &\supseteq [a \wedge n, a \vee n] = \langle a \rangle_n.\end{aligned}$$

This implies that

$$I \vee J = J \vee [b \wedge i_1 \wedge n, b \vee i_2 \vee n] = J \vee \langle t \rangle_n$$

Further,

$$\begin{aligned}\langle b \rangle_n = J \cap I &\supseteq J \cap [b \wedge i_1 \wedge n, b \vee i_2 \vee n] \\ &\supseteq J \cap [b \wedge n, b \vee n] = \langle b \rangle_n\end{aligned}$$

which implies that

$$\begin{aligned}J \cap I &= J \wedge [b \wedge i_1 \wedge n, b \vee i_2 \vee n] \\ &= J \cap \langle t \rangle_n\end{aligned}$$

Since L is distributive, $I_n(L)$ is also distributive by lemma 1.1.12., and using this distributivity we obtain that $I = \langle t \rangle_n$. Similarly we can show that J is also principal. ●

2. Prime n-ideals

1.2.1. Recall that an n-ideal P of a lattice L is prime if $m(x, n, y) \in P$, $x, y \in L$ implies either $x \in P$ or $y \in P$.

The set of all prime n-ideals of L is denoted by $P(L)$. The following seperation property for distributive lattices was given by M.H. Stone [15, Th. 15, p-74].

1.2.2. **Theorem :** Let L be a distributive lattice, let I be an ideal, let D be a dual ideal of L, and let $I \cap D = \Phi$. Then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \Phi$.

From the proof of above theorem given in [15], it can be easily seen that the following result also holds which is certainly an improvement of above.

1.2.3. **Theorem :** Let L be a distributive lattice, let I be an ideal, let D be a convex sublattice of L, and let $I \cap D = \Phi$. Then there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \Phi$.

Our next result gives a seperation property for distributive lattices interms of prime n-ideals which is of course an extension of the above results.

1.2.4. Theorem : In a distributive lattice L, suppose I is an n-ideal and D is a convex sublattice of L with $I \cap D = \Phi$. Then there exists a prime n-ideal P of L such that $P \supseteq I$ and $P \cap D = \Phi$.

Proof : Let χ be the set of all n-ideals of L that contains I and that are disjoint from D. Since $I \in \chi$, χ is non-empty. Let C be a chain in χ and let $T = \cup \{ X \mid X \in C \}$. If $a, b \in T$, then $a \in X$, $b \in Y$ for some $X, Y \in C$. Since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. Then $a, b \in Y$ and so $a \wedge b$, $a \vee b \in Y \subseteq T$, as Y is an n-ideal.

Thus, T is a sublattice.

If $a, b \in T$ and $a \leq r \leq b$, $r \in L$, then $a, b \in Y$ for some $Y \in C$, and so $r \in Y \subseteq T$ as Y is convex. Moreover $n \in T$. Therefore T is an n-ideal. Obviously $T \supseteq I$ and $T \cap D = \Phi$, which verifies that T is the maximum element of C. Hence by Zorn's lemma, χ has a maximal element, say P. We claim that P is a prime n-ideal.

Indeed, if P is not prime, then there exist $a, b \in L$ such that $a, b \notin P$ but $m(a, n, b) \in P$. Then by the maximality of P, $(P \vee \langle a\rangle_n) \cap D \neq \Phi$ and $(P \vee \langle b\rangle_n) \cap D \neq \Phi$. Then there exist $x, y \in D$ such that $p_1 \wedge a \wedge n \leq x \leq p_2 \vee a \vee n$ and $p_3 \wedge b \wedge n \leq y \leq p_4 \vee b \vee n$ for some $p_1, p_2, p_3, p_4 \in P$. Since $m(a, n, b) = (a \wedge n) \vee (b \wedge n) \vee (a \wedge b) \in P$, taking infimum with $p_1 \wedge p_3 \wedge n$, we have

$(p_1 \wedge p_3 \wedge a \wedge n) \vee (p_1 \wedge p_3 \wedge b \wedge n) \in P$.

Choosing $r = (p_1 \wedge p_3 \wedge a \wedge n) \vee (p_1 \wedge p_3 \wedge b \wedge n)$, we have $r \leq x \vee y$ with $r \in P$. Since $x \leq r \vee x \leq x \vee y$, $y \leq r \vee y \leq x \vee y$ and D is a convex sublattice, so $r \vee x, r \vee y \in D$.

Therefore $(r \vee x) \wedge (r \vee y) \in D$.

Again, $r \vee x \leq p_2 \vee a \vee n \leq p_2 \vee p_4 \vee a \vee n$ and $r \vee y \leq p_4 \vee b \vee n \leq p_2 \vee p_4 \vee b \vee n$ implies

$$(r \vee x) \wedge (r \vee y) \leq (p_2 \vee p_4 \vee a \vee n) \wedge (p_2 \vee p_4 \vee b \vee n) = s \text{ (say)}.$$

Since $m(a, n, b) = (a \vee n) \wedge (b \vee n) \wedge (a \vee b) \in P$, taking supremum with $p_2 \vee p_4 \vee n$, we have $s \in P$. Also, $r \leq (r \vee x) \wedge (r \vee y) \leq s$. Thus, again by convexity of P, $(r \vee x) \wedge (r \vee y) \in P$. This implies $P \cap D \neq \Phi$, which leads to a contradiction. Therefore, P is a prime n-ideal. ●

1.2.5. Corollary : Let I be an n-ideal of a distributive lattice L and let $a \notin I$, $a \in L$. Then there exists a prime n-ideal P of L such that $P \supseteq I$ and $a \notin P$.

1.2.6. Corollary : Every n-ideal I of a distributive lattice L is the intersection of all prime n-ideals containing it.

Proof : Let $I_1 = \cap \{ P : P \supseteq I,\ P$ is a prime n-ideal of $L \}$. If $I \neq I_1$, then there is an $a \in I_1 - I$. Then by above corollary, there is a prime n-ideal P with $P \supseteq I$, $a \notin P$. But $a \notin P \supseteq I_1$ gives a contradiction. ●

For any n-ideal J of a distributive lattice L, we define

$$J^+ = \{ x \in L : m(x, n, j) = n \text{ for all } j \in J \}.$$

Obviously, J^+ is an n-ideal and $J \cap J^+ = \{ n \}$. We will call J^+, the annihilator n-ideal of J.

It is well known from [15, Ch.2, Ex.27, P-79], that a distributive lattice with 0 is generalized boolean if and only if the set of prime ideals is unordered.

Our next theorem is a nice generalization of that result. To prove this we need following lemmas.

1.2.7. Lemma : [8,lemma 3.4] If L_1 is a sublattice of a distributive lattice L and P_1 is a prime ideal in L_1, then there exists a prime ideal P in L such that $P_1 = P \cap L_1$.

1.2.8. Lemma : In a distributive lattice L, a prime ideal containing n is also a prime n-ideal.

Proof : If P is a prime ideal containing n, then

$$m(x, n, y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n) \in P$$

implies $x \wedge y \in P$ and so either $x \in P$ or $y \in P$. Hence P is a prime n-ideal. ∎

1.2.9. Theorem : Let L be a distributive lattice and $n \in L$ be neutral. Then the following conditions are equivalent :

(i) $F_n(L)$ is generalized boolean.

(ii) For each principal n-ideal $\langle x \rangle_n$,

$$\langle x \rangle_n^+ = \{ y \in L : m(x, n, y) = n;\ x, y \in L \}$$

such that $\langle x \rangle_n^+ \vee \langle x \rangle_n = L$.

(iii) The set of prime n-ideals $P(L)$ is unordered by set inclusion.

Proof : (iii) $\Rightarrow$ (i). First suppose that $P(L)$ is unordered. Consider any interval $[n, b]$ in L. Let P_1, Q_1 be two prime ideals of $[n, b]$. Then by lemma 1.2.7., there exist prime ideals P and Q of L such that $P_1 = P \cap [n, b]$ and $Q_1 = Q \cap [n,b]$. Since P and Q contains n, they are also n-ideals. Then by lemma 1.2.8., they are also prime n-ideals. Since $P(L)$ is unordered, so P and Q are incomparable. This follows that P_1 and Q_1 are also incomparable. If not, let $P_1 \subset Q_1$. Then for any $z \in P$, by [8, lemma 3.4] $z \leq x$ for some $x \in P_1 \subset Q_1$. Which implies $z \in Q$. Thus, $P \subset Q$ which is a contradiction. Then by [15, Ch.2, Ex.27], $[n, b]$ is complemented.

Again consider the interval $[a, n]$ in L. Since the prime filters are the complements of prime ideals, so considering two prime filters of $[a, n]$ and using the same argument as above we see that $[a, n]$ is also complemented. Hence $F_n(L)$ is generalized boolean by 1.1.5. Which is (i).

(i) $\Rightarrow$ (iii). Suppose (i) holds, that is, $F_n(L)$ is generalized boolean. Then by 1.1.5. the intervals $[x, n]$ and $[n, y]$ are complemented for each $x, y \in L$ with $x \leq n \leq y$. If $P(L)$ is not unordered. Suppose there are prime n-ideals P, Q with $P \subset Q$. Let $b \in Q-P$. Now as Q is prime there exists $a \in L$ such that $a \notin Q$. Then either $a \wedge n \notin Q$ or $a \vee n \notin Q$. For otherwise $a \in Q$ by convexity of Q.

Suppose $a \vee n \notin Q$. Then $a \vee b \vee n \notin Q$. Since $[n, a \vee b \vee n]$ is complemented and $n \leq b \vee n \leq a \vee b \vee n$, so there exists $t \in [n, a \vee b \vee n]$ such that $t \wedge (b \vee n) = n$ and $t \vee b \vee n = a \vee b \vee n$.
So $t \wedge (b \vee n) = m(t, n, b \vee n) \in P$.
This implies either $t \in P$ or $b \vee n \in P$. If $t \in P$ then $a \vee b \vee n = t \vee b \vee n \in Q$, which is a contradiction to our assumption. Hence $b \vee n \in P$. So by convexity, $n \leq (a \wedge b) \vee n \leq b \vee n$ implies that $(a \wedge b) \vee n \in P$. But observe that $(a \wedge b) \vee n = m(a \vee n, n, b)$ and $a \vee n \notin P$, $b \notin P$. This is impossible as P is prime. Thus again we arrive at a contradiction. Therefore $a \vee n \in Q$.

Now, if $a \wedge n \notin Q$. Then $a \wedge b \wedge n \notin Q$. Since $b \wedge n$ has a relative complement in $[a \wedge b \wedge n, n]$, proceeding as above again we arrive at a contradiction. Thus $a \wedge n \in Q$. Since both $a \wedge n$ and $a \vee n$ belong to Q, so $a \in Q$ by convexity. Which gives a contradiction. Hence $P(L)$ must be unordered which is (iii).

Now, we shall prove (ii) $\Rightarrow$ (i). Suppose (ii) holds. Consider $\{n\} \subseteq [a, b] \subset [c, d]$. Then we have

$c \leq a \leq n \leq b \leq d$. Since

$\langle a\rangle_n \vee \langle a\rangle_n^+ = L$, so $c \in \langle a\rangle_n \vee \langle a\rangle_n^+$. Then

$i \wedge j \leq c \leq i_1 \vee j_1$ for some $i, i_1 \in \langle a\rangle_n$

and $j, j_1 \in \langle a\rangle_n^+$, which implies

$a \wedge n \wedge j \leq c$. That is, $a \wedge j \leq c$ and $c = c \vee (a \wedge j)$ $= (c \vee a) \wedge (c \vee j) = a \wedge (c \vee j)$, as L is distributive. Again $j \in \langle a\rangle_n^+$ implies $m(a, n, j) = n$, or $(a \wedge n) \vee (n \wedge j) \vee (a \wedge j) = n$, or $a \vee (n \wedge j) = n$.

Similarly, $d \in \langle b\rangle_n \vee \langle b\rangle_n^+$ implies that

$d = d \wedge (b \vee s)$ and

$b \wedge (n \vee s) = n$ for some $s \in \langle b\rangle_n^+$.

Now, consider an interval

$$[p, q] = [c \vee (n \wedge j),\ d \wedge (n \vee s)].$$ Then

$$\begin{aligned}[p, q] \cap [a, b] &= [c \vee (n \wedge j),\ d \wedge (n \vee s)] \cap [a, b]\\ &= [a \vee c \vee (n \wedge j),\ b \wedge d \wedge (n \vee s)]\\ &= [a \vee (n \wedge j),\ b \wedge (n \vee s)]\\ &= \{n\}.\end{aligned}$$

and

$$\begin{aligned}[p, q] \vee [a, b] &= [c \vee (n \wedge j),\ d \wedge (n \vee s)] \vee [a, b]\\ &= [a \wedge \{c \vee (n \wedge j)\},\ b \vee \{d \wedge (n \vee s)\}]\\ &= [(a \wedge c) \vee (a \wedge n \wedge j),\ (b \vee d) \wedge (b \vee n \vee s)]\\ &= [(a \wedge c) \vee (a \wedge j),\ (b \vee d) \wedge (b \vee s)]\\ &= [a \wedge (c \vee j),\ d \wedge (b \vee s)]\\ &= [c, d]\end{aligned}$$

Therefore, $[p, q]$ is the relative complement of $[a, b]$ in $\{n\} \subseteq [a, b] \subset [c.d]$.

Hence $F_n(L)$ is generalized boolean.

Now, we are to show that (i) → (ii). Suppose (i) holds, that is, $F_n(L)$ is generalized boolean. Suppose that $\langle x\rangle_n^+ \vee \langle x\rangle_n \neq L$. Then there exists $r \in L$ but $r \notin I = \langle x\rangle_n^+ \vee \langle x\rangle_n$.

This implies either $r \wedge x \wedge n \notin I$ or $r \vee x \vee n \notin I$. Suppose $r \vee x \vee n \notin I$. Now, $n \leq x \vee n \leq r \vee x \vee n$.

Since $F_n(L)$ is generalized boolean so by 1.1.5, we have $[n, r \vee x \vee n]$ is complemented. Then there exists $s \geq n \in L$ such that $s \wedge (x \vee n) = n$ and $s \vee (x \vee n) = r \vee x \vee n$.

Also, $n = s \wedge (x \vee n) = (n \vee s) \wedge (n \vee x)$

$= n \vee (s \wedge x)$, as L is distributive

$= (s \wedge n) \vee (s \wedge x) \vee (n \wedge x)$

$= m(s, n, x)$,

which implies that $s \in \langle x \rangle_n^+$. As $s \vee x \vee n = r \vee x \vee n$ so we have

$r \vee x \vee n \in \langle x \rangle_n \vee \langle x \rangle_n^+ = I$ which is a contradiction.

Similarly, for $r \wedge x \wedge n \notin I$, we arrive at a contradiction.

Hence $\langle x \rangle_n \vee \langle x \rangle_n^+ = L$. ●

CHAPTER - 2

"Congruences Corresponding to n-ideals in a Distributive Lattice"

Introduction : For any ideal I of a distributive lattice L, congruences $\Theta(I)$ and $R(I)$ represent the smallest and largest congruences of L containing I as a class respectively. These notations have been appeared in different instances in the literature; c.f. [15], [6], [7], [10]. $\Theta(I)$ is defined by $x \equiv y\ \Theta(I)$ if and only if $x \vee i = y \vee i$ for some $i \in I$. Again $R(I)$ is defined by $x \equiv y\ R(I)$ if and only if for any $r \in L$, $x \wedge r \in I$ if and only if $y \wedge r \in I$. For any $a \in L$, Θ_a denotes the congruence defined by $x \equiv y\ (\Theta_a)$, $(x, y \in L)$ if and only if $x \vee a = y \vee a$. Of course $\Theta_a = \Theta((a])$. Again Ψ_a denotes the congruence defined by $x \equiv y\ (\Psi_a)$, $(x, y \in L)$ if and only if $x \wedge a = y \wedge a$. Also $\Theta(a, b)$ denotes the smallest congruence which identifies a and b. Obviously Θ_a and Ψ_a are mutually complementary. Also for $a, b \in L$ with $a \leq b$, $\Theta(a,b)=\Psi_a \cap \Theta_b$, while its complement is $\Theta_a \vee \Psi_b$.

Of course $\theta_a = \theta(0, a) = \theta((a])$ if L has a 0 and $\Psi_a = \theta(a, 1)$, when L has a largest element 1.

In this chapter we generalize the concepts of $\theta(I)$ and $R(I)$ for n-ideals. Here we have shown that for a neutral element n of a lattice L, every n-ideal is a class of some congrunces if and only if L is distributive. Then we have shown that in a distributive lattice L, the mapping $I \rightarrow \theta(I)$ is an imbedding from the lattice of n-ideals to the lattice of congruences of L. Then we have generalized a well known result of J.Hashimoto [20] and showed that for a neutral element n of a lattice L, $I_n(L)$ is isomorphic to the congruence lattice $C(L)$ if and only if $F_n(L)$ is generalized boolean. We have also shown that there is an isomorphism between $C(F_n(L))$ and $C(L)$. Finally, we showed the permutability of congruences $\theta(I)$ and $\theta(J)$ for n-ideals I and J of a distributive lattice L. We showed that the above congruences permute for all I and J if and only if n is complemented in each interval containing it, (i,e. n is central as L is distributive).

1. Congruences Containing n-ideal as a class.

2.1.1. We start this chapter with the following theorem which gives a description of the smallest congruence relation of a distributive lattice L containing an n-ideal as a class where n is a fixed element of L.

2.1.2. Theorem: Let n be a fixed element of a distributive lattice L. Then for each n-ideal I of L the relation $\theta(I)$ on L defined by $x \equiv y\ \theta(I)$ if and only if $x \wedge i_1 = y \wedge i_1$ and $x \vee i_2 = y \vee i_2$ for some $i_1, i_2 \in I$, is the smallest congruence of L containing I as a class.

Proof: Clearly $\theta(I)$ is an equivalence relation. Now suppose $x \equiv y\ \theta(I)$. Then $x \wedge i_1 = y \wedge i_1$ and $x \vee i_2 = y \vee i_2$ for some $i_1, i_2 \in I$. So for any $m \in L$, $(x \vee m) \vee i_2 = (x \vee i_2) \vee m = (y \vee i_2) \vee m = (y \vee m) \vee i_2$

and
$$\begin{aligned}(x \vee m) \wedge i_1 &= (x \wedge i_1) \vee (m \wedge i_1)\\ &= (y \wedge i_1) \vee (m \wedge i_1)\\ &= (y \vee m) \wedge i_1,\end{aligned}$$

which shows that $x \vee m \equiv y \vee m\ \theta(I)$. Again clearly $(x \wedge m) \wedge i_1 = (y \wedge m) \wedge i_1$ and using distributivity of L, $(x \wedge m) \vee i_2 = (y \wedge m) \vee i_2$. This shows that $x \wedge m \equiv y \wedge m\ \theta(I)$.

Hence $\theta(I)$ is a congruence relation on L.

For any $i_1, i_2 \in I$, observe that

$$i_1 \vee (i_1 \vee i_2) = i_2 \vee (i_1 \vee i_2) = i_1 \vee i_2$$

and $$i_1 \wedge (i_1 \wedge i_2) = i_2 \wedge (i_1 \wedge i_2) = i_1 \wedge i_2.$$

This implies $i_1 \equiv i_2\ \theta(I)$. That is the elements of I belong to the same class of $\theta(I)$.

Now, suppose $m \in L$ and $m \equiv i\ \theta(I)$ for some $i \in I$. Then $m \wedge i_1 = i \wedge i_1$ and $m \vee i_2 = i \vee i_2$ for some $i_1, i_2 \in I$, which shows that $m \wedge i_1, m \vee i_2 \in I$ and so by convexity of I we get $m \in I$. Hence I is a congruence class of $\theta(I)$.

Finally, suppose that Φ is any congruence relation on L containing I as a class. Let $x \equiv y\ \theta(I)$. Then $x \wedge i_1 = y \wedge i_1$ and $x \vee i_2 = y \vee i_2$ for some $i_1, i_2 \in I$. Since L is distributive,

$$\begin{aligned}
x = x \wedge (x \vee i_2) &= x \wedge (y \vee i_2) \\
&= (x \wedge y) \vee (x \wedge i_2) \\
&\equiv (x \wedge y) \vee (x \wedge i_1)\ (\Phi) \\
&= (x \wedge y) \vee (y \wedge i_1) \\
&= y \wedge (x \vee i_1) \\
&\equiv y \wedge (x \vee i_2)\ (\Phi) \\
&= y \wedge (y \vee i_2) = y.
\end{aligned}$$

Thus, $x \equiv y\ (\Phi)$ and so $\Theta(I) \subseteq \Phi$. Therefore $\Theta(I)$ is the smallest congruence relation on L containing I as a class.●

Following theorem gives a characterization of distributivity of a lattice when the fixed element n is neutral in it. This is also a generalization of well known result.

2.1.3. Theorem: A lattice L with a neutral element n, is distributive if and only if for each n-ideal I of L, there exists a congruence on L, having I as a class.

Proof: If L is distributive, then by theorem 2.1.2, $\Theta(I)$ is the smallest congruence relation on L containing I as a class.

To prove the converse, suppose that every n-ideal I of L is a congruence class of some congruence relations on L. If L is not distributive, then it contains a sublattice isomorphic to N_5 or M_5 which are shown in figure 2.1.1. and figure 2.1.2. respectively.

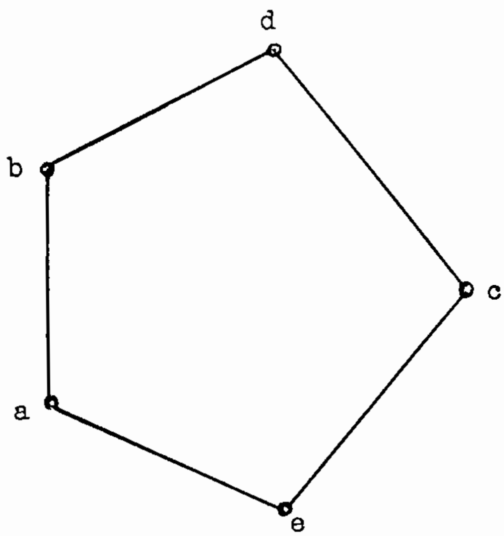

Figure 2.1.1

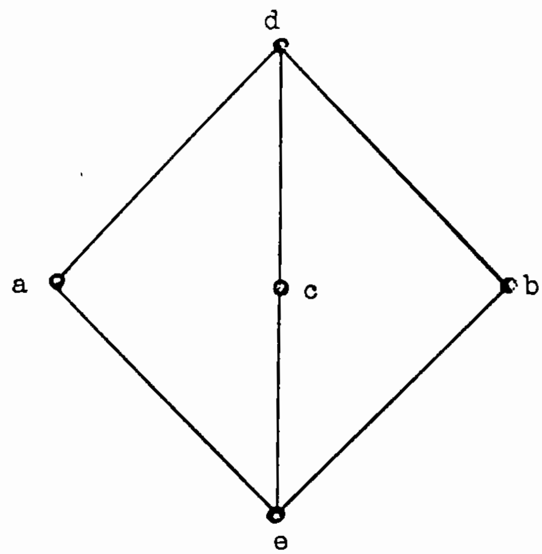

Figure 2.1.2

Here we have either $a \wedge n \neq b \wedge n$ or $a \vee n \neq b \vee n$. For, if $a \wedge n = b \wedge n$, $a \vee n = b \vee n$ then by neutrality of n, $a = b$, which is impossible. Without loss of generality, suppose $a \wedge n \neq b \wedge n$. Consider $I = \langle b \wedge n \rangle_n = [b \wedge n, n]$. Suppose θ is a congruence which contains I as a class. Since $b \wedge n \leq d \wedge n \leq n$, $d \wedge n \in I$.

Thus, $d \wedge n \equiv b \wedge n\ \theta(I)$

so $d \wedge n \wedge c \equiv b \wedge n \wedge c\ \theta(I)$. That is

$$c \wedge n \equiv e \wedge n\ \theta(I).\text{Then}$$

$$(c \wedge n) \vee (a \wedge n) \equiv (e \wedge n) \vee (a \wedge n)\ \theta(I),$$

and so

$$(c \vee a) \wedge n \equiv (e \vee a) \wedge n\ \theta(I) \text{ as,} n \text{ is neutral.}$$

This implies $d \wedge n \equiv a \wedge n\ \Theta(I)$, which shows that $a \wedge n \in I$. Then $b \wedge n \leq a \wedge n \leq n$.

Similarly, consider the n-ideal $\langle a \wedge n\rangle_n$, and proceeding as above we obtain $b \wedge n \in \langle a \wedge n\rangle_n$. Then $a \wedge n \leq b \wedge n \leq n$ and so $a \wedge n = b \wedge n$, which gives a contradiction to our assumption. Therefore L must be distributive.●

Following lemma is needed for our next theorem.

2.1.4. Lemma : Let L be a distributive Lattice. Then for any two n-ideals I & J of L,

(i) $\Theta(I \cap J) = \Theta(I) \cap \Theta(J)$

(ii) $\Theta(I \vee J) = \Theta(I) \vee \Theta(J)$

Proof : (i) Obviously, $\Theta(I \cap J) \subseteq \Theta(I) \cap \Theta(J)$. To prove the reverse inequality, let $x \equiv y\ \Theta(I) \cap \Theta(J)$. Then $x \wedge i_1 = y \wedge i_1$ and $x \vee i_2 = y \vee i_2$ for some $i_1, i_2 \in I$. Also $x \wedge j_1 = y \wedge j_1$ and $x \vee j_2 = y \vee j_2$ for some $j_1, j_2 \in J$. As $m(i_1, n, j_1), m(i_2, n, j_2) \in I \cap J$ and since L is distributive,

$$\begin{aligned} x \wedge m(i_1, n, j_1) &= x \wedge [(i_1 \wedge n) \vee (n \wedge j_1) \vee (i_1 \wedge j_1)] \\ &= (x \wedge i_1 \wedge n) \vee (x \wedge n \wedge j_1) \vee (x \wedge i_1 \wedge j_1) \end{aligned}$$

$= (y \wedge i_1 \wedge n) \vee (y \wedge j_1 \wedge n) \vee (y \wedge i_1 \wedge j_1)$

$= y \wedge m(i_1, n, j_1)$.

Similarly, using distributivity of L, we have
$x \vee m(i_2, n, j_2) = y \vee m(i_2, n, j_2)$, which shows that $x \equiv y\ \theta(I \cap J)$.

Hence $\theta(I \cap J) = \theta(I) \cap \theta(J)$.

(ii) Obviously, $\theta(I) \vee \theta(J) \subseteq \theta(I \vee J)$.
To prove the reverse inequality, let $x \equiv y\ \theta(I \vee J)$.
Then $x \vee p = y \vee p$ and $x \wedge q = y \wedge q$ for some $p, q \in I \vee J$. Then there exist $i_1, i_2, i_3, i_4 \in I$ and
$j_1, j_2, j_3, j_4 \in J$ such that $i_1 \wedge j_1 \leq p \leq i_2 \vee j_2$ and $i_3 \wedge j_3 \leq q \leq i_4 \vee j_4$. Thus, we have

$$x \vee i_2 \vee j_2 = y \vee i_2 \vee j_2$$

and

$$x \wedge i_3 \wedge j_3 = y \wedge i_3 \wedge j_3.$$

Observe that, $i_3 \wedge j_3 \equiv i_3 \wedge n\ \theta(J) \equiv i_2 \wedge n\ \theta(I)$

$\equiv i_2 \vee n\ \theta(I) \equiv i_2 \vee j_2\ \theta(J)$,

and so $i_3 \wedge j_3 \equiv i_2 \vee j_2\ \theta(I) \vee \theta(J)$. Then

$$x = x \wedge (x \vee i_2 \vee j_2)$$

$= x \wedge (y \vee i_2 \vee j_2)$

$\equiv x \wedge [y \vee (i_3 \wedge j_3)]\ (\theta(I) \vee \theta(J))$

$= (x \wedge y) \vee (x \wedge i_3 \wedge j_3)$

$= (x \wedge y) \vee (y \wedge i_3 \wedge j_3)$

$$= y \wedge [x \vee (i_3 \wedge j_3)]$$

$$\equiv y \wedge [x \vee i_2 \vee j_2] \ (\Theta(I) \vee \Theta(J))$$

$$= y \wedge (y \vee i_2 \vee j_2)$$

$$= y$$

Thus, $x \equiv y \ \Theta(I) \vee \Theta(J)$

Therefore $\Theta(I \vee J) = \Theta(I) \vee \Theta(J)$. ●

2.1.5. Theorem : For an element n of a distributive lattice L, the correspondence $I \to \Theta(I)$ is an imbedding from $I_n(L)$ to $C(L)$, where $I_n(L)$ is the lattice of n-ideals of L.

Proof: By above lemma , the mapping $I \to \Theta(I)$ is a homomorphism. So it is sufficient to show that the mapping is one-to-one. Suppose for n-ideals I and J, $\Theta(I) = \Theta(J)$. Let $i \in I$. Then for any $j \in J$, it is not hard to see that

$$m(i, n, j) \vee i \vee n = i \vee n$$

and $$m(i, n, j) \wedge i \wedge n = i \wedge n.$$

This implies $i \equiv m(i, n, j) \ \Theta(I) = \Theta(J)$. Then

$$i \wedge j_1 = m(i, n, j) \wedge j_1$$

and $i \vee j_2 = m(i, n, j) \vee j_2$ for some $j_1, j_2 \in J$.

Now, clearly $j \wedge j_1 \wedge n \leq m(i, n, j) \wedge j_1 \leq j \vee n$

and $j \wedge n \leq m(i, n, j) \vee j_2 \leq j_2 \vee j \vee n$.

Then by convexity of J, $m(i, n, j) \wedge j_1$ and $m(i, n, j) \vee j_2 \in J$ and so $i \wedge j_1$ and $i \vee j_2 \in J$. Since $i \wedge j_1 \leq i \leq i \vee j_2$, using convexity of J again, $i \in J$. Therefore $I \subseteq J$. Similarly $J \subseteq I$, and so $I = J$. Hence the mapping is one-to-one and so it is an imbedding. ●

We have already defined
$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$
for $x, y, z \in L$, we also define

$$m^d(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x).$$

In presence of distributivity of L, it is easy to show that $m(x, y, z) = m^d(x, y, z)$ for all $x,y,z \in L$.

Now, we give a describtion of the largest congruence of a distributive lattice containing an n-ideal as a class.

2.1.6. Theorem : Let n be a fixed element of a distributive lattice L. For each n-ideal I define the relation $R(I)$ on L by $x \equiv y\ R(I)$ if and only if for any $t \in L$, $m(x,n,t) \in I$ if and only if $m(y,n,t) \in I$. Then $R(I)$ is the largest congruence containing I as a class.

Proof : Obviously $R(I)$ is an equivalence relation. To prove the substitution property, Let $x \equiv y\ R(I)$. consider any $r \in L$. Suppose $m(x \wedge r, n, t) \in I$ for some $t \in L$. Then it is easy to check that

$$m(x \wedge r, n, t) \wedge n \leq m(x, n, (t \wedge r) \vee (t \wedge n))$$
$$\leq m(x \wedge r, n, t) \vee n.$$

Then by convexity of I,

$m(x, n, (t \wedge r) \vee (t \wedge n)) \in I$. Since $x \equiv y\ R(I)$ so $m(y, n, (t \wedge r) \vee (t \wedge n)) \in I$. Then using distributivity of L, a routine calculation shows that

$$m(x \wedge r, n, t) \wedge m(y, n, (t \wedge r) \vee (t \wedge n))$$
$$\leq m(y \wedge r, n, t)$$
$$\leq m(y, n, (t \wedge r) \vee (t \wedge n)).$$

Then by the convexity of I, $m(y \wedge r, n, t) \in I$.

Hence $x \wedge r \equiv y \wedge r\ R(I)$.

Since in a distributive lattice

$m(x, y, z) = m^d(x, y, z)$, a dual proof of above shows that $m(x \vee r, n, t) \in I$ for some $t \in L$ if and only if $m(y \vee r, n, t) \in I$. Therefore $x \vee r \equiv y \vee r\ R(I)$, and so $R(I)$ is a congruence.

Now, for any $i \in I$, and any $t \in L$,

$$i \wedge n \leq m(i, n, t) \leq i \vee n.$$

So by convexity $m(i, n, t) \in I$. Therefore, for any $i_1, i_2 \in I$, $i_1 \equiv i_2\ R(I)$.

Moreover, if $x \equiv i\ R(I)$ for $x \in L$ and $i \in I$, then $m(i, n, x) \in I$ implies that $x = m(x, n, x) \in I$. Therefore, $R(I)$ is a congruence containing I as a class.

Finally, let Φ be a congruence of L containing I as a class. Let $x \equiv y\ \Phi$. Suppose $m(x, n, t) \in I$ for some $t \in L$. Then $x \equiv y\ \Phi$ implies

$$\begin{aligned} m(x, n, t) &= (x \wedge n) \vee (t \wedge n) \vee (x \wedge t) \\ &\equiv (y \wedge n) \vee (t \wedge n) \vee (y \wedge t)\Phi \\ &= m(y, n, t). \end{aligned}$$

Since $m(x, n, t) \in I$ and I is a class of Φ, so $m(y,n,t) \in I$. Therefore, $x \equiv y\ R(I)$ and so $R(I)$ is the largest congruence containing I as a class.●

In lattice theory it is well Known that the lattice of ideals is isomorphic to the lattice of congruences if and only if the lattice is generalized boolean, c.f.[15.Th.8,p-91]. Our next theorem is a generalization to that result.

2.1.7. Theorem : For a neutral element n of a lattice L, $I_n(L) \cong C(L)$ if and only if $F_n(L)$ is generalized boolean.

Proof : First suppose that $F_n(L)$ is generalized boolean. Then by 1.2.8., L is distributive. Let us define a map $f : I_n(L) \to C(L)$ given by $f(J)=\Theta(J)$. Then by 2.1.5, f is a homomorphism and one-to-one. For ontoness, let $\Phi \in C(L)$. Consider $I = \{x \in L : x \equiv n\Phi\}$. Then clearly I is an n-ideal. Since $\Theta(I)$ is the smallest congruence containing I as a class, so $\Theta(I) \subseteq \Phi$. Now, let $x \equiv y\ (\Phi)$. Then $x \wedge y \equiv x \vee y\ (\Phi)$. Consider $[n, x \vee y \vee n] \in F_n(L)$. Here $n \le (x \wedge y) \vee n \le x \vee y \vee n$. As $F_n(L)$ is generalized boolean so by 1.1.5. there exists $t \in L$ such that $t \wedge [(x \wedge y) \vee n] = n$

and $t \vee [(x \wedge y) \vee n] = x \vee y \vee n$.

Now, $n = t \wedge [(x \wedge y) \vee n] \equiv t \wedge [x \vee y \vee n](\Phi) = t$. This implies $t \in I$. Also $t \vee [(x \wedge y) \vee n]$ $= x \vee y \vee n$. Then

$$(x \wedge y) \vee (t \vee n) = (x \vee y) \vee (t \vee n) \qquad \text{...(i)}$$

Again consider, $x \wedge y \wedge n \le (x \vee y) \wedge n \le n$. Since $[x \wedge y \wedge n, n]$ is complemented we can similarly show that there exists an $r \in I$ such that

$$(x \wedge y) \wedge (r \wedge n) = (x \vee y) \wedge (r \wedge n) \qquad \text{...(ii)}$$

combining (i) and (ii) we have $x \wedge y \equiv x \vee y\ \Theta(I)$, as $t \vee n, r \wedge n \in I$. This implies $\Phi \subseteq \Theta(I)$, and so $\Phi = \Theta(I)$. Thus f is onto. Therefore $I_n(L) \cong C(L)$.

Conversely, suppose that $I_n(L) \cong C(L)$. Then $I_n(L)$ is distributive and so by 1.1.10., both L and $F_n(L)$ are distributive. Consider the interval $[n, b]$ with $n \leq a < b$.

Let $I = \{ x \in L : x \equiv n\ \theta(a, b)\}$. Then I is an n-ideal. As $\theta(I)$ is the smallest congruence having I as a class and since $I_n(L) \cong C(L)$, so we have

$\theta(I) = \theta(a, b)$. Then $a \equiv b\ \theta(I)$ and $a \vee i_1 = b \vee i_1$ and $a \wedge i_2 = b \wedge i_2$ for some $i_1, i_2 \in I$. Then

$$i_1 \equiv n\ \theta(a, b) \text{ and } i_2 \equiv n\ \theta(a, b).$$

But $\theta(a, b) = \theta_b \cap \Psi_a$. Then $i_1 \vee b = n \vee b = b$ and $i_1 \wedge a = n \wedge a = n$. This implies i_1 is the relative complement of 'a' in $[n, b]$.

Again, considering any interval $[c, n]$ with $c < d \leq n$ and the principal congruence $\theta(c, d)$, we can similarly show that d has a relative complement in $[c, n]$. Therefore by (1.2.8) $F_n(L)$ is generalized boolean.●

Now, we describe an isomorphism between $C(F_n(L))$ and $C(L)$ in presence of distributivity. We prove this with the help of the following lemma.

2.1.8. Lemma : Let n be a neutral element of a lattice L. For each $\theta \in C(F_n(L))$, define a relation $\rho(\theta)$ on L given by $x \equiv y\ \rho(\theta)$ if and only if $\langle x\rangle_n \equiv \langle y\rangle_n \theta$. Then $\rho(\theta)$ is a congruence relation on L.

Moreover, for $\theta_i \in C(F_n(L))$, $i \in A$ where A is an indexed set

(i) $\rho(\cap\ \theta_i) = \cap\ \rho(\theta_i)$ and

(ii) $\rho(\vee\ \theta_i) = \vee\ \rho(\theta_i)$;

Proof : Clearly $\rho(\theta)$ is an equivalence relation. To prove the substitution property, suppose $x \equiv y\ \rho(\theta)$ and $t \in L$. Then $\langle x\rangle_n \equiv \langle y\rangle_n(\theta)$, and so

$\langle x\rangle_n \wedge [n, x \vee n] \equiv \langle y\rangle_n \wedge [n, x \vee n](\theta)$. Then by 1.1.2, $[n, x \vee n] \equiv [n, (y \vee n) \wedge (x \vee n)](\theta)$. Similarly, $[n, y \vee n] \equiv [n, (y \vee n) \wedge (x \vee n)](\theta)$. Thus, $[n, x \vee n] \equiv [n, y \vee n](\theta)$. Then $[n, x \vee n] \vee [n, t \vee n] \equiv [n, y \vee n] \vee [n, t \vee n](\theta)$. This implies

$$[n, x \vee t \vee n] \equiv [n, y \vee t \vee n](\theta) \qquad \ldots(i)$$

Again, $\langle x\rangle_n \cap [t \wedge n, n] \equiv \langle y\rangle_n \cap [t \wedge n, n](\theta)$. This implies

$$[(x \wedge n) \vee (t \wedge n), n] \equiv [(y \wedge n) \vee (t \wedge n), n](\theta) \qquad \ldots(ii)$$

Taking supremum of (i) and (ii), we have

$$[(x \wedge n) \vee (t \wedge n),\ x \vee t \vee n] \equiv [(y \wedge n) \vee (t \wedge n),\ y \vee t \vee n](\theta).$$

Thus, $[(x \vee t) \wedge n,\ x \vee t \vee n] \equiv [(y \vee t) \wedge n,\ y \vee t \vee n](\theta)$,

as n is neutral.

That is, $\langle x \vee t\rangle_n \equiv \langle y \vee t\rangle_n(\theta)$, and so $x \vee t \equiv y \vee t\ \rho(\theta)$. Similarly, a dual proof of above shows that $x \wedge t \equiv y \wedge t\ \rho(\theta)$, and so $\rho(\theta)$ is a congruence of L.

For the second part, the proof of (i) is trivial. For the proof of (ii), since ρ is order preserving, obviously $\vee\, \rho(\theta_i) \subseteq \rho(\vee\, \theta_i)$.

To prove the reverse inequality, assume that $x \equiv y\ \rho(\vee\, \theta_i)$. Then $\langle x\rangle_n \equiv \langle y\rangle_n\ (\vee\, \theta_i)$. Thus $\langle x\rangle_n \cap \langle y\rangle_n = \langle m(x, n, y)\rangle_n \equiv \langle x\rangle_n\ (\vee\, \theta_i)$ so by using 1.1.12. we have

$\langle m(x, n, y)\rangle_n = \langle z_0\rangle_n, \langle z_1\rangle_n, \ldots, \langle z_r\rangle_n = \langle y\rangle_n$,

with

$\langle z_{j-1}\rangle_n \equiv \langle z_j\rangle_n\ (\theta_{i_k})$; $i_k \in A$; $j = 1, 2, \ldots, r$;

$k = 1, 2, \ldots, r$.

This implies $z_{j-1} \equiv z_j \ \rho\ (\theta_{i_k})$, which shows that $m(x, n, y) \equiv x(\ \vee\ \rho\ (\theta_i))$. Similarly, $m(x, n, y) \equiv y\ (\ \vee\ \rho\ (\theta_i))$. Hence $x \equiv y\ (\vee\ \rho(\theta_i))$. So we have $\rho\ (\vee\ \theta_i) \subseteq \vee\ \rho(\theta_i)$.

Hence $\rho\ (\vee\ \theta_i) = \vee\ \rho(\theta_i)$.●

2.1.9. Theorem : Let L be a distributive lattice. The map $\rho : C(F_n(L)) \to C(L)$ is an isomorphism where for each $\theta \in C(F_n(L))$, $\rho(\theta)$ is defined by $x \equiv y\ \rho(\theta)$ if and only if $\langle x\rangle_n \equiv \langle y\rangle_n(\theta)$.

Proof : By above lemma, it is sufficient to prove that ρ is one - one and onto. Suppose $\rho(\theta) = \rho(\Phi)$. Let $[a, b] \equiv [c, d](\theta)$. Then

$[a, b] \cap \langle c\rangle_n \equiv [c, d] \cap \langle c\rangle_n\ (\theta)$. Thus by 1.1.2. we have $[a \vee c, n] \equiv [c, n](\theta)$. That is

$\langle a \vee c\rangle_n \equiv \langle c\rangle_n(\theta)$, and so

$a \vee c \equiv c\ \rho(\theta) = \rho(\Phi)$. Then

$\langle a \vee c\rangle_n \equiv \langle c\rangle_n(\Phi)$, and so

$[a \vee c, n] \equiv [c, n](\Phi)$

Similarly, considering

$[a, b] \cap \langle a\rangle_n \equiv [c, d] \cap \langle a\rangle_n(\theta)$,

we get $[a, n] \equiv [a \vee c, n](\Phi)$. Therefore

$[a, n] \equiv [c, n](\Phi)$.

Again considering

$$[a, b] \vee \langle b\rangle_n \equiv [c, d] \vee \langle b\rangle_n(\theta) \quad \text{and}$$

$$[a, b] \vee \langle d\rangle_n \equiv [c, d] \vee \langle d\rangle_n(\theta), \text{ we obtain}$$

$[n, b] \equiv [n, d]$ (Φ). Therefore $[a, b] \equiv [c, d]$ (Φ), and so $\theta \subseteq \Phi$. Similarly $\Phi \subseteq \theta$. Hence $\theta = \Phi$, and so ρ is one-to-one.

For ontoness, let $\Phi \in C(L)$. Define $\theta \in C(F_n(L))$ by

$$\theta = \vee \{ \theta (\langle a\rangle_n, \langle b\rangle_n) : a \equiv b \ \Phi \}.$$

If $x \equiv y$ (Φ), then $\langle x\rangle_n \equiv \langle y\rangle_n$ θ $(\langle x\rangle_n, \langle y\rangle_n)$, and so $\langle x\rangle_n \equiv \langle y\rangle_n$ (θ). This implies $x \equiv y$ $\rho(\theta)$ and so $\Phi \subseteq \rho(\theta)$

To prove the reverse inequality, let

$$x \equiv y \ \rho \ (\ \theta (\langle a\rangle_n, \langle b\rangle_n) : a \equiv b \ \Phi \ ). \text{ Then}$$

$$\langle x\rangle_n \equiv \langle y\rangle_n \ \theta \ (\langle a\rangle_n \cap \langle b\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n).$$

This implies $\langle x\rangle_n \cap \langle a\rangle_n \cap \langle b\rangle_n = \langle y\rangle_n \cap \langle a\rangle_n \cap \langle b\rangle_n$ and $\langle x\rangle_n \vee \langle a\rangle_n \vee \langle b\rangle_n = \langle y\rangle_n \vee \langle a\rangle_n \vee \langle b\rangle_n$. Then by some routine calculation, we get

$$(x \wedge n) \vee (a \wedge n) \vee (b \wedge n)$$
$$= (y \wedge n) \vee (a \wedge n) \vee (b \wedge n)$$

$$(x \vee n) \wedge (a \vee n) \wedge (b \vee n)$$
$$= (y \vee n) \wedge (a \vee n) \wedge (b \vee n)$$

and $\quad x \wedge a \wedge b \wedge n = y \wedge a \wedge b \wedge n$

$$x \vee a \vee b \vee n = y \vee a \vee b \vee n$$

Now, $x \wedge n = (x \wedge n) \wedge [(x \wedge n) \vee (a \wedge n) \vee (b \wedge n)]$

$= (x \wedge n) \wedge [(y \wedge n) \vee (a \wedge n) \vee (b \wedge n)]$

$\equiv (x \wedge n) \wedge [(y \wedge n) \vee (b \wedge n)]\ \theta(a, b)$

$= (x \wedge y \wedge n) \vee (x \wedge b \wedge n)$, as L is distributive

$\equiv (x \wedge y \wedge n) \vee (x \wedge a \wedge b \wedge n)\ \theta(a, b)$

$= (x \wedge y \wedge n) \vee (y \wedge a \wedge b \wedge n)$

$= (y \wedge n) \wedge [(x \wedge n) \vee (a \wedge b \wedge n)]$

$\equiv (y \wedge n) \wedge [(x \wedge n) \vee (a \wedge n) \vee (b \wedge n)]\ \theta(a, b)$

$= (y \wedge n) \wedge [(y \wedge n) \vee (a \wedge n) \vee (b \wedge n)] = y \wedge n.$

Thus, $x \wedge n \equiv y \wedge n\ \theta(a, b)$. Similarly, we can show that $x \vee n \equiv y \vee n\ \theta(a, b)$. Hence by distributivity $x \equiv y\ \theta(a, b)$. Also $\theta(a, b) \subseteq \Phi$. Thus $x \equiv y\ (\Phi)$. Therefore by lemma 2.1.8. (ii), $\rho(\theta) \subseteq \Phi$. Hence $\rho(\theta) = \Phi$ and so ρ is onto. ●

Since the lattice of ideals of a lattice L is isomorphic to the lattice of congruences if and only if L is generalized boolean, so using 2.1.7. and above theorem, we obtain the following corollary :

2.1.10. Corollary : For a fixed element n of a distributive lattice L, $I_n(L) \cong I(F_n(L))$ if $F_n(L)$ is generalized boolean.

We now turn our analogue to the permutability of the congruences $\Theta(I)$ and $\Theta(J)$ in a distributive lattice L, where I and J are n-ideals of L. In a lattice L, two congruences Θ and Φ permute if for $a, b, c \in L$ with $a \equiv b\ (\Theta)$ and $b \equiv c\ (\Phi)$ imply that there exists some $d \in L$ such that $a \equiv d\ (\Phi)$ and $d \equiv c\ (\Theta)$.

It is well known in lattice theory that for any two ideals I and J of a distributive lattice L, $\Theta(I)$ and $\Theta(J)$ always permute. But this is not true in general for n-ideals. For example, consider the 3—element chain $L = \{0, n, 1\}$.
Let $I = \{0, n\}$ and $J = \{n, 1\}$. Here $0 \equiv n\ \Theta(I)$ and $n \equiv 1\ \Theta(J)$. But there exists no $x \in L$ such that $0 \equiv x\ \Theta(J)$ and $x \equiv 1\ \Theta(I)$.

The following theorem shows that the permutability of those congruences hold when n is complemented in each interval containing it (i.e., n is cetral when L is distributive).

2.1.11. Theorem : Let L be a distributive lattice and $n \in L$. Then for $I, J \in I_n(L)$, the following conditions are equivalent :

(i) $\theta(I)$ and $\theta(J)$ permute ;

(ii) n is complemented in each interval containing it ;

(iii) $P_n(L)$ is a lattice.

Proof : (ii) $\leftrightarrow$ (iii) follows from 1.1.3.

(ii) $\rightarrow$ (i). Suppose (ii) holds. That is n is complemented in each interval containing it. Let $x, y, z \in L$ with $x \geq y \geq z$, and $x \equiv y\ \theta(I)$ and $y \equiv z\ \theta(J)$. Then

$$x \wedge i_1 = y \wedge i_1, \quad x \vee i_2 = y \vee i_2$$

and $$y \wedge j_1 = z \wedge j_1, \quad y \vee j_2 = z \vee j_2$$

for some $i_1, i_2 \in I$, $j_1, j_2 \in J$. Now consider an interval $[x \wedge (z \vee j_1) \wedge n,\ z \vee (x \wedge i_2) \vee n]$ and let t be the relative complement of n in this interval such that $t \wedge n = x \wedge (z \vee j_1) \wedge n$ and $t \vee n = z \vee (x \wedge i_2) \vee n$.

Now, $t \wedge n \wedge j_1 = x \wedge (z \vee j_1) \wedge n \wedge j_1 = x \wedge n \wedge j_1$,

and
$$\begin{aligned} t \vee n \vee j_2 &= z \vee (x \wedge i_2) \vee n \vee j_2 \\ &= y \vee j_2 \vee (x \wedge i_2) \vee n \\ &= j_2 \vee [(y \vee x) \wedge (y \vee i_2)] \vee n \\ &= j_2 \vee [x \wedge (x \vee i_2)] \vee n \\ &= x \vee n \vee j_2, \end{aligned}$$

which implies $x \equiv t\ \theta(J)$.

Again, $t \wedge n \wedge i_1 = x \wedge (z \vee j_1) \wedge n \wedge i_1$

$$= y \wedge i_1 \wedge (z \vee j_1) \wedge n$$

$$= i_1 \wedge [(y \wedge z) \vee (y \wedge j_1)] \wedge n$$

$$= i_1 \wedge [z \vee (z \wedge j_1)] \wedge n$$

$$= z \wedge n \wedge i_1,$$

and $t \vee n \vee i_2 = z \vee (x \wedge i_2) \vee n \vee i_2 = z \vee n \vee i_2$, which implies that $t \equiv z\ \theta(I)$.

Moreover, $t \wedge n \leq x \wedge n$ and $t \vee n \leq x \vee n$ implies $t \leq x$, and $t \wedge n \geq z \wedge n$ and $t \vee n \geq z \vee n$ implies $z \leq t$. Thus, $z \leq t \leq x$.

Now, for any $x, y, z \in L$, suppose $x \equiv y\ \theta(I)$ and $y \equiv z\ \theta(J)$. Then $x \equiv x \vee y\ \theta(I)$ and $x \vee y \equiv x \vee y \vee z\ \theta(J)$. Then by above there exists u with $x \leq u \leq x \vee y \vee z$ such that $x \equiv u\ \theta(J)$ and $u \equiv x \vee y \vee z\ \theta(I)$. Similarly, $z \equiv y \vee z\ \theta(J)$ and $y \vee z \equiv y \vee z \vee x\ \theta(I)$ implies there exists v with $z \leq v \leq y \vee z \vee x$ such that $z \equiv v\ \theta(I)$ and $v \equiv y \vee z \vee x\ \theta(J)$. Set $s = u \wedge v$. Then

$$s = u \wedge v \equiv u \wedge (y \vee z \vee x)\ \theta(J) = u\ \theta(J).$$

But $u \equiv x\ \theta(J)$. Thus, $s \equiv x\ \theta(J)$.

Again, $s = v \wedge u \equiv v \wedge (x \vee y \vee z)\ \theta(I) = v\ \theta(I)$. But $v \equiv z\ \theta(I)$. Thus, $s \equiv z\ \theta(I)$. Therefore $\theta(I)$ and $\theta(J)$ permute which is (i).

Now we are to show that (i) $\Rightarrow$ (ii). Suppose (i) holds, $\theta(I)$, $\theta(J)$ permute for all n-ideals I and J. Let $x \le n \le y$. Then $\theta(x, n)$, $\theta(n, y)$ permute. Now, $x \equiv n\ \theta(x, n)$ and $n \equiv y\ \theta(n, y)$, so there exists t with $x \le t \le y$ such that $x \equiv t\ \theta(n, y)$ and $t \equiv y\ \theta(x, n)$. This implies $x \wedge n = t \wedge n$ and $t \vee n = y \vee n$, and so t is the relative complement of n in $x \le n \le y$, which is (ii) ●

CHAPTER - 3

"The n-kernels of Skeletal Congruences on a Distributive Lattice"

Introduction : For any $\theta \in C(L)$, θ^* denotes the pseudocomplement of θ. By its very definition $\theta \cap \Phi = \omega$, (the smallest congruence) if and only if $\Phi \leq \theta^*$, $\Phi \in C(L)$. A subset T of a lattice L is called join-dense if each $z \in L$ is the join of its predecessors in T, while a meet-dense subset of L is defined dually. $\theta \in C(L)$ is called dense if $\theta^* = \omega$. A distributive lattice L with 0 is called disjunctive if $0 \leq a < b$ implies that there is an element $x \in L$ such that $x \wedge a = 0$ and $0 < x \leq b$.

For a distributive lattice L with 0, $I(L)$ is pseudocomplemented. The pseudocomplement J^* of an ideal J is the annihilator ideal $J^* = \{x \in L : x \wedge j = 0 \text{ for all } j \in J\}$. For any n-ideal J of a distributive lattice L, we already defined $J^+ = \{x \in L : m(x, n, j) = n \text{ for all } j \in J\}$.

Obviously J^+ is an n-ideal and $J \cap J^+ = \{n\}$. We call J^+, the annihilator n-ideal of J. We define the n-kernel of a congruence θ by $Ker_n\theta = \{x \in L : x \equiv n \ (\theta)\}$, which is clearly an n—ideal.

In [9], Cornish has studied the skeletal congruences extensively and gave several characterizations of disjunctive and generalized boolean lattices in terms of skeletal congruences. In this chapter we have extended several results of [9].

In section 1, we have studied the skeletal congruences θ^* of a distributive lattice L, where * represents the pseudocomplement. Then we have given a neat description of $\theta(J)^*$, where $\theta(J)$ is the smallest congruence of L containing n-ideal J as a class and showed that J^+ is the n-kernel of $\theta(J)^*$. We have also shown that the n-kernels of the skeletal congruences are precisely those n-ideals which are the intersection of relative annihilator ideals and dual relative annihilator ideals. Finally, we have shown that for any n-ideal J, $\theta(J)$ is dense in C(L) if and only if J is both meet and join-dense.

In section 2, we have shown that $F_n(L)$ is disjunctive if and only if each dense n-ideal J is both meet and join-dense. Moreover, the n-kernels of each skeletal congruence is an annihilator n-ideal. We have also shown that $F_n(L)$ is generalized boolean if and only if $\Theta(J^+) = \Theta(J)^*$ for any n-ideal J. Finally, we show that $F_n(L)$ is generalized boolean if and only if the map $\Theta \rightarrow Ker_n\Theta$ is a lattice isomorphism of $S\ C(L)$ onto $K_nSC(L)$ whose inverses the map $J \rightarrow \Theta(J)$ where J is an n-ideal.

1. Skeletal congruences.

For any $\theta \in C(L)$, the existence of θ^* is guaranted by the fact that $C(L)$ is a distributive algebraic lattice. The skeleton

$$SC(L) = \{ \theta \in C(L) : \theta = \Phi^* \text{ for some } \Phi \in C(L)\}$$
$$= \{ \theta \in C(L) : \theta = \theta^{**} \}.$$

The kernel of a congruence $\theta \in C(L)$ is

$\mathrm{Ker}\theta = \{ x \in L : x \equiv 0 \ (\theta) \}$. Of course,

$\mathrm{Ker}\ (\theta\ (J)) = J$. For $a, b \in L$, $\langle a, b\rangle$ denotes the relative annihilator. That is,

$\langle a, b\rangle = \{ x \in L : x \wedge a \leq b \}$. In the presence of distributivity, it is easy to show that each relative annihilator is an ideal. Also note that $\langle a, b\rangle = \langle a, a \wedge b\rangle$. Dual relative annihilator ideal $\langle a, b\rangle_d$ can be defined dually. For details on relative annihilator ideals, we refer the reader to consult [33].

The following theorem gives a neat description of the pseudocomplement θ^* of $\theta \in C(L)$, which is due to Cornish [9, Th.1.2., 1.3.]. This could also be deduced from Paperts description in [40, Th.2], also c.f.[2, 3.1. 3.2.].

3.1.1. Theorem : For a congruence Θ on a distributive lattice L, the following conditions are equivalent :

(i) For $x, y \in L$, $x \equiv y \ (\Theta^*)$;

(ii) For each $a, b \in L$ with $a \le b$ and $a \equiv b \ (\Theta)$,

$$(x \wedge b) \vee a = (y \wedge b) \vee a \ ;$$

(iii) $\Theta_x \cap \Theta = \Theta_y \cap \Theta$.

If L has a 0, then of course $\Theta_x = \Theta(0, x)$. Here our following theorem gives a nice generalization of a portion of the above result for a lattice L with 0.

3.1.2. Theorem : Let L be a distributive lattice and $n \in L$. Then for any $\Theta \in C(L)$, $x \equiv y \ (\Theta^*)$ if and only if $\Theta(n, x) \cap \Theta = \Theta(n, y) \cap \Theta$.

Proof : Define a relation Φ on L as $x \equiv y \ (\Phi)$ if and only if $\Theta(n, x) \cap \Theta = \Theta(n, y) \cap \Theta$. First we shall show that Φ is a congruence relation. Obviously, Φ is an equivalence relation. Let $x \equiv y \ \Phi$. As $\Theta(a, b) = \Theta(a \wedge b, a \vee b) = \Theta_{a \vee b} \cap \Psi_{a \wedge b}$.

So by definition of Φ we have

$$\Theta_{n \vee x} \cap \Psi_{n \wedge x} \cap \Theta = \Theta_{n \vee y} \cap \Psi_{n \wedge y} \cap \Theta.$$

Now, suppose $p \equiv q\ \Theta(n, x \wedge t) \cap \Theta$ for some $t \in L$.

Then $p \equiv q\ \Psi_{n \wedge x \wedge t}$ and so

$$p \wedge n \wedge x \wedge t = q \wedge n \wedge x \wedge t$$

This implies $p \wedge t \wedge n \equiv q \wedge t \wedge n\ \Theta\ (n, x) \cap \Theta$

$$= \Theta\ (n, y) \cap \Theta,$$

and so $p \wedge t \wedge n \wedge y = q \wedge t \wedge n \wedge y$. Thus

$$p \equiv q\ \Psi_{n \wedge y \wedge t} \qquad \ldots (i)$$

Again, $p \equiv q\ \Theta\ (n, x \wedge t) \cap \Theta$ implies

$p \vee n \vee (x \wedge t) = q \vee n \vee (x \wedge t)$, and so

$p \vee n \vee x = q \vee n \vee x$ and $p \vee n \vee t = q \vee n \vee t$.

Thus, $p \vee n \equiv q \vee n\ \Theta(n, x) \cap \Theta = \Theta(n, y) \cap \Theta$.

Therefore, $p \vee n \vee y = q \vee n \vee y$

and $p \vee n \vee t = q \vee n \vee t$, and so,

$(p \vee n \vee y) \wedge (p \vee n \vee t) = (q \vee n \vee y) \wedge (q \vee n \vee t)$

That is, $p \vee n \vee (y \wedge t) = q \vee n \vee (y \wedge t)$.

Thus, $$p \equiv q\ \Theta_{n \vee (y \wedge t)} \qquad \ldots\ldots (ii)$$

Combining (i) & (ii), $p \equiv q\ \Theta(n, y \wedge t)$.

Hence $\Theta(n, x \wedge t) \cap \Theta \subseteq \Theta(n, y \wedge t) \cap \Theta$.

Similarly, $\Theta(n, y \wedge t) \cap \Theta \subseteq \Theta(n, x \wedge t) \cap \Theta$,

and so $\Theta(n, x \wedge t) \cap \Theta = \Theta(n, y \wedge t) \cap \Theta$,

which implies $x \wedge t \equiv y \wedge t\ (\Phi)$.

A dual proof of above also gives

$x \vee t \equiv y \vee t\ (\Phi)$ for all $t \in L$. Therefore Φ is a congruence .

Now, suppose $x \equiv y\ (\theta \cap \Phi)$. Then $x \equiv y\ (\theta)$ and $\theta(n, x) \cap \theta = \theta(n, y) \cap \theta$. Observe that

$$(x \wedge y) \vee n \equiv x \vee n \quad \theta(n, x) \cap \theta = \theta(n, y) \cap \theta,$$

and so $y \vee n = x \vee y \vee n$. Thus, $x \vee n \leq y \vee n$. Similarly, we get $y \vee n \leq x \vee n$, and

hence $x \vee n = y \vee n$ ..... (iii)

Again, observe that

$$(x \vee y) \wedge n \equiv x \wedge n \quad \theta(n, x) \cap \theta = \theta(n, y) \cap \theta.$$

This implies $y \wedge n = x \wedge y \wedge n$ and so $y \wedge n \leq x \wedge n$. Similarly, we get $x \wedge n \leq y \wedge n$,

and so $x \wedge n = y \wedge n$ ..... (iv)

Combining (iii) & (iv) we obtain $x = y$, as L is distributive. Therefore, $\theta \cap \Phi = \omega$.

To show that $\Phi = \theta^*$, let Ψ be any other congruence such that $\theta \cap \Psi = \omega$. Suppose $x \equiv y\ (\Psi)$.

Let $a \equiv b\ \theta(n, x) \cap \theta$. This implies $a \vee n \vee x = b \vee n \vee x$ and $a \wedge n \wedge x = b \wedge n \wedge x$.

$$\begin{aligned} \text{Then} \quad a \wedge n \wedge y &\equiv a \wedge n \wedge x\ (\Psi) \\ &= b \wedge n \wedge x \\ &\equiv b \wedge n \wedge y\ (\Psi) \\ \text{and} \quad a \vee n \vee y &\equiv a \vee n \vee x\ (\Psi) \\ &= b \vee n \vee x \\ &\equiv b \vee n \vee y\ (\Psi). \end{aligned}$$

Also, we have $a \wedge n \wedge y \equiv b \wedge n \wedge y \ (\Theta)$

and $a \vee n \vee y \equiv b \vee n \vee y \ (\Theta)$

and so $a \wedge n \wedge y \equiv b \wedge n \wedge y \ (\Theta \cap \Psi) = \omega$, implies that $a \wedge n \wedge y = b \wedge n \wedge y$, which shows that $a \equiv b \ \Psi_{n \wedge y}$.

Similarly, $a \vee n \vee y \equiv b \vee n \vee y \ (\Theta \cap \Psi) = \omega$, implies $a \equiv b \ \Theta_{n \vee y}$. Thus, $a \equiv b \ \Theta(n, y) \cap \Theta$.

Therefore $\Theta(n, x) \cap \Theta \subseteq \Theta(n, y) \cap \Theta$.

Similarly, $\Theta(n, y) \cap \Theta \subseteq \Theta(n, x) \cap \Theta$, and so

$$\Theta(n, x) \cap \Theta = \Theta(n, y) \cap \Theta.$$

This implies $x \equiv y \ (\Phi)$. Therefore $\Psi \subseteq \Phi$, and so $\Phi = \Theta^*$. ●

The following theorem is due to Cornish [9].

3.1.3. Theorem : Let L be a distributive lattice with 0. Then the following conditions hold.

(i) For any ideal J, $x \equiv y \ (\Theta(J)^*)$, $(x, y \in L)$ if and only if $(x] \cap J = (y] \cap J$, i.e., if and only if $x \wedge j = y \wedge j$ for all $j \in J$.

(ii) For an ideal J, both $\Theta(J)^*$ and $\Theta(J^*)$ have J^* as their Kernel.

(iii) An ideal J is the kernel of a skeletal congruence if and only if it is the intersection of relative annihilator ideals.

(iv) Each principal ideal is an intersection of relative annihilator ideals.

The following theorem generalizes theorem 3.1.3.

3.1.4. Theorem : Let L be a distributive lattice and $n \in L$. Then the following conditions hold.

(i) For any n-ideal J of L, $x \equiv y$ $(\Theta(J)^*)$; $x, y \in L$ if and only if $\langle x\rangle_n \cap J = \langle y\rangle_n \cap J$, i.e., if and only if $m(x, n, j) = m(y, n, j)$ for all $j \in J$.

(ii) For any n-ideal j of L, both $\Theta(J)^*$ and $\Theta(J^+)$ have J^+ as their n-kernel.

(iii) The n-kernels of the skeletal congruences are precisely those n-ideals which are the intersection of relative annihilator ideals and dual relative annihilator ideals whose end points are of the form $x \vee n$ and $x \wedge n$ respectively.

(iv) Each principal n-ideal of L is the intersection of relative annihilator ideals and dual relative annihilator ideals whose end points are of the form $x \wedge n$ and $x \vee n$ respectively.

Proof : (i) For any two n-ideals I and J of L, we have $\Theta(I \cap J) = \Theta(I) \cap \Theta(J)$. Also,

$$\Theta(n, x) = \Theta(n \wedge x, n \vee x) = \Theta(\langle x \rangle_n).$$

Then by 3.1.2., $x \equiv y \ (\Theta(J)^*)$ if and only if

$\Theta(n, x) \cap \Theta(J) = \Theta(n, y) \cap \Theta(J)$ if and only if

$\Theta(\langle x \rangle_n) \cap \Theta(J) = \Theta(\langle y \rangle_n) \cap \Theta(J)$ if and only if

$\Theta(\langle x \rangle_n \cap J) = \Theta(\langle y \rangle_n \cap J)$ if and only if

$\langle x \rangle_n \cap J = \langle y \rangle_n \cap \Theta$, by 2.1.5. if and only if

$m(x, n, j) = m(y, n, j)$ for all $j \in J$.

Hence (i) holds.

(ii) If $x \in Ker_n (\Theta (J)^*)$, then $x \equiv n \ (\Theta(J)^*)$. Then by (i) above, $\langle x \rangle_n \cap J = \langle n \rangle_n \cap J$ if and only if $m(x, n, j) = m(n, n, j) = n$, for all $j \in J$ and so $x \in J^+$, and thus (ii) holds.

(iii) Consider $a, b \in L$ with $a \leq b$. Since $\Theta(a, b)^*$ $= \Theta(a, b)'$, so by [9. lemma 1.1.], $x \in Ker_n \Theta(a, b)^*$ if and only if $(x \wedge b) \vee a = (n \wedge b) \vee a$

Now, we shall show that $(x \wedge b) \vee a = (n \wedge b) \vee a$ is equivalent to $x \in \langle b \vee n, a \vee n \rangle \cap \langle a \wedge n, b \wedge n \rangle_a$.

Since $(x \wedge b) \vee a = (n \wedge b) \vee a$ implies $x \wedge b \leq a \vee n$, we have $x \wedge (b \vee n)$

$$= (x \wedge b) \vee (x \wedge n)$$

$$\leq a \vee n,$$

and so $x \in \langle b \vee n, a \vee n \rangle$.

Again from $(x \wedge b) \vee a = (n \wedge b) \vee a$, we have $b \wedge n \leq (x \wedge b) \vee a$. So $b \wedge n \leq (x \wedge b \wedge n) \vee (a \wedge n)$

$$\leq x \vee (a \wedge n),$$

which implies $x \in \langle a \wedge n, b \wedge n \rangle_d$.

Hence $x \in \langle b \vee n, a \vee n \rangle \cap \langle a \wedge n, b \wedge n \rangle_d$.

Conversely, let $x \in \langle b \vee n, a \vee n \rangle \cap \langle a \wedge n, b \wedge n \rangle_d$. Then, $x \wedge (b \vee n) \leq a \vee n$ and $x \vee (a \wedge n) \geq b \wedge n$. Now, $x \wedge (b \vee n) \leq a \vee n$ implies

$$\begin{aligned} x \wedge b &= x \wedge b \wedge (b \vee n) \\ &\leq (a \vee n) \wedge b \\ &= (a \wedge b) \vee (b \wedge n) \\ &= a \vee (b \wedge n), \end{aligned}$$

and so $(x \wedge b) \vee a \leq (b \wedge n) \vee a$. On the other hand, $b \wedge n \leq x \vee (a \wedge n)$ implies

$$\begin{aligned} b \wedge n &\leq b \wedge (x \vee (a \wedge n)) \\ &= (x \wedge b) \vee (a \wedge b \wedge n) \\ &= (x \wedge b) \vee (a \wedge n) \end{aligned}$$

and so, $(n \wedge b) \vee a \leq (x \wedge b) \vee a$. Combining both relations we have $(x \wedge b) \vee a = (n \wedge b) \vee a$. Since for any $\theta \in C(L)$,

$\theta^* = \cap \{ \theta(a,b)^* : a \equiv b \; \theta \}$, hence the result follows.

(iv) Since each principal n-ideal $\langle a\rangle_n = \mathrm{Ker}_n\Theta(\langle a\rangle_n) = \mathrm{Ker}_n\Theta(a \wedge n, a \vee n)$ and since $\Theta(a \wedge n, a \vee n)$ is skeletal so by (iii) the result follows. ●

3.1.5. A non-empty subset T of a lattice L is called large if $x \wedge t = y \wedge t$ for all $t \in T$, $x,y \in L$ implies $x = y$, while T is called join-dense if for each $z \in L$ is the join of its predecessors in T. Also T is called small if $x \vee t = y \vee t$ for all $t \in T$, $x,y \in L$ implies $x = y$, while T is called meet-dense if for each $z \in L$ is the meet of its successors in T. It can be easily shown that an ideal in a lattice is large if and only if it is join-dense. It is clear from 3.1.3 that an ideal J of a distributive lattice L is join-dense if and only if $\Theta(J)$ is dense in C(L), i.e., $\Theta(J)^* = \omega$, the smallest element of C(L).
Lemma. 3.1.6 and theorem 3.1.8 were suggested to the author by his supervisor Dr. Noor.

3.1.6. Lemma : A convex sublattice J of a distributive lattice L is large if and only if it is join-dense in L.

Proof : Obviously, every join-dense subset of L is large in L. Conversely, let J be large in L. Suppose

$x \in L$ and $\{j_i\}$ are its predecessors in J. Let t be an upper bound of $\{j_i\}$. Clearly, for any $j \in J$, $j_1 \wedge j \le x \wedge j \le j$ and so by convexity of J, $x \wedge j \in J$. Thus, $x \wedge j = j_k$ for some k.

Hence, $x \wedge j \le t$ for all $j \in J$ which shows that $x \wedge j = x \wedge j \wedge t$ for all $j \in J$. Since J is large, $x \wedge t = x$, i.e., $x \le t$. This implies that x is the supremum of $\{j_i\}$ ●

Similarly, a dual proof of above shows that a convex sublattice J of a lattice L is meet-dense if and only if $x \vee j = y \vee j$ for all $j \in J$ implies $x = y$. ●

Thus, we have the following corollary.

3.1.7. Corollary : An n-ideal of a distributive lattice L is large if and only if it is join dense in L.

3.1.8. Theorem : For any n-ideal J of a distributive lattice L, $\theta(J)$ is dense in C(L) if and only if J is both meet and join-dense.

Proof : Let $\theta(J)$ is dense in C(L), i.e., $\theta(J)^* = \omega$. Suppose $x \wedge j = y \wedge j$ for all $j \in J$. Then,

$m(x, n, j) = m(y, n, j)$ for all $j \in J$. Then by [3.1.4.(i)], we have $x \equiv y\ \theta(J)^* = \omega$. Hence $x = y$. Again, if $x \vee j = y \vee j$ for all $j \in J$, then

$$\begin{aligned} m(x, n, j) &= (x \vee n) \wedge (n \vee j) \wedge (x \vee j) \\ &= (y \vee n) \wedge (n \vee j) \wedge (y \vee j), \text{ as } n \in J \\ &= m(y, n, j) \text{ for all } j \in J. \end{aligned}$$

Thus, by [3.1.4.(i)], $x \equiv y\ \theta(J)^* = \omega$ and hence $x = y$, which shows that J is both meet and join-dense.

Conversely, let J be both meet and join-dense and $x \equiv y\ \theta(J)^*$. Then by 3.1.4., $m(x, n, j) = m(y, n, j)$ for all $j \in J$. Thus, $(x \wedge n) \vee j = m(x, n, j) \vee j$ $= m(y, n, j) \vee j = (y \wedge n) \vee j$ and $(x \vee n) \wedge j$ $= m(x, n, j) \wedge j = m(y, n, j) \wedge j = (y \vee n) \wedge j$ for all $j \in J$. These imply $x \wedge n = y \wedge n$ and $x \vee n = y \vee n$. Hence by the distributivity of L, $x = y$, i.e., $\theta(J)^* = \omega$, and so $\theta(J)$ is dense in $C(L)$. ●

2. 'Disjunctive and Generalized Boolean Lattices'

3.2.1. We recall that a distributive lattice L with 0 is disjunctive if $0 \leq a < b$ implies there is an element $x \in L$ such that $x \wedge a = 0$ and $0 < x \leq b$. We already know that for any n-ideal J of L, $R(J)$ denotes the largest congruence having J as its kernel, where $x \equiv y\ R(J)$ if and only if for each $r \in L$, $m(r, n, x) \in J$ if and only if $m(r, n, y) \in J$.

The following theorem gives a description of disjunctive lattices which is mentioned in section 2 of Cornish [9]. We omit the proof as it is very easy to show.

3.2.2. **Theorem** : For a distributive lattice L with 0, the following conditions are equivalent :

(i) L is disjunctive.

(ii) For each $a \in L$, $(a] = (a]^{**}$.

(iii) $R(\ (0]\ ) = \omega$. ●

We now extend the above result.

3.2.3. **Theorem** : Suppose L is a distributive lattice with an element n. Then the following conditions are equivalent :

(i) $F_n(L)$ is disjunctive.

(ii) For each $a \in L$, $\langle a\rangle_n = \langle a\rangle_n^{++}$.

(iii) $R(\{n\}) = \omega$.

Proof : (i) $\Rightarrow$ (ii) Suppose $F_n(L)$ is disjunctive and suppose that $\langle a\rangle_n \neq \langle a\rangle_n^{++}$ for some $a \in L$. Then there exists $t \in \langle a\rangle_n^{++}$ but $t \notin \langle a\rangle_n$, which implies either

$a \wedge n \nleq t$ or $t \nleq a \vee n$. In either case

$\langle a\rangle_n \subset [t \wedge a \wedge n, t \vee a \vee n]$. Since $F_n(L)$ is disjunctive , there exists $[b, c]$ with

$\{n\} \subset [b, c] \subseteq [t \wedge a \wedge n, t \vee a \vee n]$ such that

$\langle a\rangle_n \cap [b, c] = \{n\}$. This implies $[b, c] \in \langle a\rangle_n^{+}$ and

$(a \wedge n) \vee b = n = (a \vee n) \wedge c$. Then

$$
\begin{aligned}
[b, c] &= [b, c] \cap [t \wedge a \wedge n, t \vee a \vee n] \\
&= [(t \wedge a \wedge n) \vee b, (t \vee a \vee n) \wedge c] \\
&= [((t \wedge n) \vee b) \wedge ((a \wedge n) \vee b), \\
&\qquad ((t \vee n) \wedge c) \vee ((a \vee n) \wedge c)] \\
&= [((t \wedge n) \vee b) \wedge n, ((t \vee n) \wedge c) \vee n] \\
&= [(t \wedge n) \vee b, (t \vee n) \wedge c] \\
&= \langle t\rangle_n \cap [b, c] \\
&= \{n\}, \quad \text{as } t \in \langle a\rangle_n^{++} \text{and } [b, c] \subseteq \langle a\rangle_n^{+}.
\end{aligned}
$$

Thus, $[b, c] = \{n\}$, which is a contradiction.

Therefore, $\langle a\rangle_n = \langle a\rangle_n^{++}$ for all $a \in L$, which is (ii).

(ii) $\Rightarrow$ (i). Suppose that $\langle a\rangle_n = \langle a\rangle_n^{++}$ for all $a \in L$. Let $\{n\} \subseteq [a, b] \subset [c, d]$. Then either $c < a \leq n$ or $n \leq b < d$. Suppose $n \leq b < d$. Then $\{n\} \subseteq \langle b\rangle_n \subset \langle d\rangle_n$. Then $\langle b\rangle_n = \langle b\rangle_n^{++}$ and $\langle d\rangle_n = \langle d\rangle_n^{++}$, implies $\langle b\rangle_n^{+} \supset \langle d\rangle_n^{+}$. So there exists $r \in \langle b\rangle_n^{+}$ such that $r \notin \langle d\rangle_n^{+}$. This implies that $m(r, n, b) = n$ and $m(r, n, x) \neq n$ for some $x \in \langle d\rangle_n$. Since $b \geq n$ and $x \geq n$, We have $m(r, n, b) = (r \vee n) \wedge b = n$ and $m(r, n, x) = (r \vee n) \wedge x$. Then

$$\{n\} \subset \langle m(r, n, x)\rangle_n \subseteq \langle d\rangle_n,$$

and

$$\begin{aligned}
[a, b] \cap \langle m(r, n, x)\rangle_n &= [a, b] \cap [n, (r \vee n) \wedge x] \\
&= [n, (r \vee n) \wedge x \wedge b] \\
&= [n, x \wedge n] \\
&= \{n\}
\end{aligned}$$

which shows that $F_n(L)$ is disjunctive which is (i)

(i) $\Rightarrow$ (iii), suppose (i) holds. That is, $F_n(L)$ is disjunctive. Let $x \equiv y\ R(\{n\})$. If $x \neq y$, then either $x \wedge y < x$ or $x \wedge y < y$. suppose $x \wedge y < x$.

Since L is distributive, either $x \wedge y \wedge n < x \wedge n$ or $(x \wedge y) \vee n < x \vee n$. If $x \wedge y \wedge n < x \wedge n$, then $<x>_n \subset <x>_n \vee <y>_n$ and so $<x>_n \cap <y>_n \subset <y>_n$. If $(x \wedge y) \vee n < x \vee n$, then $<x>_n \cap <y>_n \subset <x>_n$. Thus $x \neq y$ implies either $<x>_n \cap <y>_n \subset <x>_n$ or $<x>_n \cap <y>_n \subset <y>_n$. Without loss of generality suppose $<x>_n \cap <y>_n \subset <x>_n$. Since $F_n(L)$ is disjunctive, there exists $\{n\} \subset [a, b] \subseteq <x>_n$ such that $[a, b] \cap <x>_n \cap <y>_n = \{n\}$. Now, by 1.1.12., $[a, b] = <t>_n$ for some $t \in L$.
Thus, $<t>_n \cap <x>_n \cap <y>_n = \{n\}$, and so $<t>_n \cap <y>_n = \{n\}$. That is $m(y, n, t) = n$. Since $x \equiv y\ R(\{n\})$, so $m(x, n, t) = n$, and so $<x>_n \cap <t>_n = \{n\}$. This implies $<t>_n = \{n\}$, which is a contradiction. Therefore, $x = y$ and so $R(\{n\}) = \omega$, which is (iii).

Finally, we show that (iii) $\Rightarrow$ (i). Let $R(\{n\}) = \omega$. If $F_n(L)$ is not disjunctive then for $\{n\} \subseteq [a, b] \subset [c, d]$, there exists no $[e, f] \neq \{n\}$ such that $[a, b] \cap [e, f] \neq \{n\}$. Since $[a, b] \subset [c, d]$ so either $c < a$ or $b < d$, Let $c < a$. Chose any $t \in L$. Then for all $[t \wedge n, b]$,

$$[t \wedge n, b] \cap [c, d] \neq \{n\} \text{ if and only if } [t \wedge n, b] \cap [a, b] \neq \{n\}$$

i.e., $[t \wedge n, b] \cap [c, d] = \{n\}$ if and only if

$$[t \wedge n, b] \cap [a, b] = \{n\}$$

or $[(t \wedge n) \vee c, b \wedge d] = \{n\}$ if and only if

$$[(t \wedge n) \vee a, b] = \{n\}$$

or $[(t \wedge n) \vee c, b] = \{n\}$ if and only if

$$[(t \wedge n) \vee a, b] = \{n\}$$

i.e., $(t \wedge n) \vee c = n$ if and only if $(t \wedge n) \vee a = n$

i.e., $m(c, n, t) = n$ if and only if $m(a, n, t) = n$

i.e., $c \equiv a\ R(\{n\}) = \omega$, and so $c = a$, which is a contradiction. So $F_n(L)$ must be disjunctive, which is (i). ●

An ideal J is called dense ideal if $J^* = (0]$. According to Cornish [9], we have the following result :

3.2.4. Theorem : In a distributive lattice L with 0, the following conditions are equivalent :

(i) L is disjunctive.

(ii) Each dense ideal J is join dense,

(iii) For each dense ideal J, $\Theta(J^*) = \Theta(J)^*$.

(iv) For each dense ideal J, $\Theta(J^{**}) = \Theta(J)^{**}$.

We call an n-ideal J of L is dense if $J^+ = \{n\}$. The following theorem is a generalization of above :

3.2.5. Theorem : Let L be a distributive lattice and $n \in L$, then the following conditions are equivalent:

(i) $F_n(L)$ is disjunctive.

(ii) Each dense n-ideal J is both join and meet-dense.

(iii) For each dense n-ideal J, $\Theta(J^+) = \Theta(J)^*$.

(iv) For each dense n-ideal J, $\Theta(J^{++}) = \Theta(J)^{**}$

Proof : (i) $\Rightarrow$ (ii). Suppose (i) holds. That is, $F_n(L)$ is disjunctive. Suppose J is a dense n-ideal. Then $J^+ = \{n\}$. Let $x \wedge j = y \wedge j$

for all $j \in J$, $x, y \in L$.

If $x \neq y$, then either $x \wedge y < x$ or $x \wedge y < y$. Without loss of generality suppose $x \wedge y < x$. Then either $x \wedge y \wedge n < x \wedge n$ or $(x \wedge y) \vee n < x \vee n$. Since $n \in J$, $x \wedge n = y \wedge n$. So $x \wedge y \wedge n = x \wedge n$. Thus, $(x \wedge y) \vee n < x \vee n$. Then $\{n\} \subseteq [n, (x \wedge y) \vee n] \subset [n, x \vee n]$. Since $F_n(L)$ is disjunctive, there exists

$$[n, b] \neq \{n\} \quad \text{and} \quad [n, b] \subseteq [n, x \vee n]$$

such that $[n, (x \wedge y) \vee n] \cap [n, b] = \{n\}$, which implies $[(x \wedge y) \vee n] \wedge b = n$. Then for all $j \in J$,

$$\begin{aligned} n &= n \wedge (j \vee n) \\ &= [(x \wedge y) \vee n] \wedge b \wedge (j \vee n) \\ &= b \wedge [(x \wedge y \wedge j) \vee n] \end{aligned}$$

$$= b \wedge [(x \wedge j) \vee n]$$
$$= b \wedge (x \vee n) \wedge (j \vee n)$$
$$= b \wedge (j \vee n)$$
$$= m(b, n, j)$$

which shows that $b \in J^{+} = \{n\}$ implies $b = n$ which is a contradiction. So, $x = y$, i.e., J is join-dense. Similarly we can show that J is also meet-dense. Hence (ii) holds.

(ii) $\Rightarrow$ (i). For any $a \in L$, $\langle a\rangle_n \vee \langle a\rangle_n^{+}$ is always a dense n-ideal. Since (ii) holds so we have $\langle a\rangle_n \vee \langle a\rangle_n^{+}$ is both meet and join-dense. Then by [3.1.8], $\omega = \theta(\langle a\rangle_n \vee \langle a\rangle_n^{+})^*$

$$= (\theta(\langle a\rangle_n) \vee \theta(\langle a\rangle_n^{+}))^*$$
$$= \theta(\langle a\rangle_n)^* \cap \theta(\langle a\rangle_n^{+})^*.$$

Thus $\theta(\langle a\rangle_n^{+})^* \subseteq \theta(\langle a\rangle_n)^{**} = \theta(\langle a\rangle_n)$.

Taking the n-kernels on both sides we have $\langle a\rangle_n^{++} \subseteq \langle a\rangle_n$ due to 3.1.4 (iii). It follows that $\langle a\rangle_n^{++} = \langle a\rangle_n$, which implies that $F_n(L)$ is disjunctive. Hence (i) holds.

Since J is dense n-ideal implies J is both meet and join-dense so we have $J^{+} = \{n\}$ if and only if $J^{++} = L$ and J is both meet and join-dense if and only if $\Theta(J)^{*} = \omega$, so obviously, (ii), (iii) and (iv) are equivalent. ●

The following theorem is a generalization of [9, Th. 2.2.].

3.2.6. Theorem : Let L be a distributive lattice and $n \in L$. Then the following conditions are equivalent:

(i) $F_n(L)$ is disjunctive.

(ii) For each congruence Φ, $\Phi^{*} = \Theta(Ker_n\Phi)^{*}$.

(iii) For each n-ideal J, $R(J)^{*} = \Theta(J)^{*}$

(iv) For each congruence Φ, $Ker_n(\Phi^{*}) = (Ker_n\Phi)^{+}$.

(v) For each congruence Φ, $Ker_n(\Phi^{**}) = (Ker_n\Phi)^{++}$.

(vi) The n-kernel of each skeletal congruence is an annihilator n-ideal.

Proof : (i) $\Rightarrow$ (ii). Since $\Theta(Ker_n\Phi) \subseteq \Phi$, so we have $\Theta^{*} \subseteq \Theta(Ker_n\Phi)^{*}$. So it is sufficient to prove that $\Phi \cap \Theta(Ker_n\Phi)^{*} = \omega$. Suppose (i) holds. That is, $F_n(L)$ is disjunctive. Suppose $x \leq y$ and $x \equiv y\ (\Phi \cap \Theta(Ker_n\Phi)^{*})$ implies $x \equiv y\ \Phi$ and $x \equiv y\ \Theta(Ker_n\Phi)^{*}$. If $x < y$, then either $x \wedge n < y \wedge n$

or $x \vee n < y \vee n$. Suppose that $x \vee n < y \vee n$. Then $\{n\} \subseteq [n, x \vee n] \subset [n, y \vee n]$. Since $F_n(L)$ is disjunctive so there exists $[n, a] \in F_n(L)$ with $a > n$ and $[n, a] \subseteq [n, y \vee n]$ such that $[n, a] \cap [n, x \vee n] = \{n\}$. This implies $a \wedge (x \vee n) = n$.

Now, $n = a \wedge (x \vee n) \equiv a \wedge (y \vee n) = a\ (\Phi)$ implies $a \in Ker_n\Phi$. Since $x \equiv y\ \Theta(Ker_n\Phi)^*$, so $x \vee n \equiv y \vee n\ \Theta(Ker_n\Phi)^*$ and since $a \in Ker_n\Phi$, so we have $m(x \vee n, n, a) = m(y \vee n, n, a)$. That is,

$$((x \vee n) \wedge n) \vee (a \wedge (x \vee n)) \vee (n \wedge a)$$
$$= ((y \vee n) \wedge n) \vee (a \wedge (y \vee n)) \vee (n \wedge a)$$

i.e., $n \vee (a \wedge (x \vee n)) \vee n = n \vee a \vee n$.

This implies, $n = a$, which is a contradiction.

Therefore, $x = y$ and so $\Phi \cap \Theta\ (Ker_n\Phi)^* = \omega$.

Hence (ii) holds.

(ii) $\Rightarrow$ (iii) holds since J is the n-kernel of $R(J)$ and $\Theta(J)$.

(iii) $\Rightarrow$ (i). Suppose (iii) holds. Since $\Theta(\{n\}) = \omega$ and since (iii) holds so $R(\{n\})^* = \Theta(\{n\})^* = \iota$ implies, $R(\{n\})^{**} = \omega$. Then by 3.2.3. we have $F_n(L)$ is disjunctive.

(ii) $\Rightarrow$ (iv) is clear since by 3.1.4.(ii) $\Theta(J)^*$ and $\Theta(J^+)$ have J^+ as their n-kernels.

(iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (vi) are obvious.

(vi) $\Rightarrow$ (i). Suppose (vi) holds. Let $\{n\} \subseteq [a, b] \subset [c, d]$, then either $c < a \leq n$ or $n \leq b < d$. Suppose $c < a \leq n$. Then by 3.1.4.(iii) $\langle c, a\rangle_d = \langle c \wedge n, a \wedge n\rangle_d$ is the n-kernel of a skeleton congruence. Since (vi) holds, so there is an annihilator n-ideal K such that $\langle c, a\rangle_d = K = K^{++}$. As $a \vee c \geq a$ implies $a \in \langle c, a\rangle_d = K = K^{++}$. Also, since $c < a$, $c \not\in \langle c, a\rangle_d = K = K^{++}$. So there exists $e \in K^+$ such that $m(c, n, e) \neq n$. But $m(a, n, e) = n$ implies $a \vee (n \wedge e) = n$. Now, consider the interval $[e \wedge n, n]$. Then

$$\begin{aligned}[e \wedge n, n] \cap [a, b] &= [(e \wedge n) \vee a, n \wedge b] \\ &= \{n\}\end{aligned}$$

Hence $F_n(L)$ is disjunctive, which is (i). ●

The following theorem is due to Cornish [9. Th. 2.3.], which characterizes generalized boolean lattice; Also c.f.[28, Th. 6].

3.2.7. Theorem : Let L be a distributive lattice with 0. Then the following conditions are equivalent :

(i) The lattice L is generalized boolean.

(ii) For each congruence Φ, $\Phi^* = \Theta(\mathrm{Ker}(\Phi^*))$.

(iii) For each ideal J, $\Theta(J)^* = \Theta(J^*)$.

(iv) For each ideal J, $\Theta(J)^{**} = \Theta(J^{**})$. ●

Now, we extend and generalize the above theorem.

3.2.8. Theorem : Let L be a distributive lattice and $n \in L$. Then the following conditions are equivalent:

(i) $F_n(L)$ is generalized boolean.

(ii) For each congruence Φ, $\Phi^* = \Theta(\mathrm{Ker}_n\Phi^*)$.

(iii) For each n-ideal J, $\Theta(J^+) = \Theta(J)^*$.

(iv) For each n-ideal J, $\Theta(J^{++}) = \Theta(J)^{**}$.

Proof : (i) $\Rightarrow$ (ii). Suppose (i) holds. Let Ψ be any congruence on L. Then by 2.1.7., $\Psi = \Theta\ (\mathrm{Ker}_n\Psi)$. Thus with $\Psi = \Phi^*$, we see that (i) implies (ii).

(ii) $\Rightarrow$ (iii) follows from [3.1.4.] and

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). Suppose (iv) holds. Put $J = \langle a\rangle_n \vee \langle a\rangle_n^+$. Since $J^{++} = L$, (iv) implies $\theta(\langle a\rangle_n \vee \langle a\rangle_n^+)^{**} = \iota$. It follows that $\theta(\langle a\rangle_n)^* \cap \theta(\langle a\rangle_n^+)^* = \omega$, and so $\theta(\langle a\rangle_n^+)^* \subseteq \theta(\langle a\rangle_n)^{**} = \theta(\langle a\rangle_n)$. Now by 3.1.4. $\langle a\rangle_n^+ = \mathrm{Ker}_n\theta(\langle a\rangle_n)^*$. Then, $\theta(\langle a\rangle_n^+) \subseteq \theta(\langle a\rangle_n)^*$, and so $\theta(\langle a\rangle_n) = \theta(\langle a\rangle_n)^{**} \subseteq \theta(\langle a\rangle_n^+)^*$. Therefore, $\theta(\langle a\rangle_n) = \theta(\langle a\rangle_n^+)^*$. But $\langle a\rangle_n^+ = \langle a\rangle_n^{+++}$, so by (iv) $\theta(\langle a\rangle_n)^* = \theta(\langle a\rangle_n^+)^{**} = \theta(\langle a\rangle_n^{+++}) = \theta(\langle a\rangle_n^+)$.

Now, let $n \le a \le b$. Then for all $j \in \langle a\rangle_n = [n, a]$, $m(a, n, j) = m(b, n, j) = j$.

Thus $a \equiv b\ \theta(\langle a\rangle_n)^* = \theta(\langle a\rangle_n^+)$. Then $a \vee r = b \vee r$ for some $r \in \langle a\rangle_n^+$. So $b = a \vee (b \wedge r)$. Again $r \in \langle a\rangle_n^+$ implies $(a \wedge r) \vee (a \wedge n) \vee (r \wedge n) = n$, and so $a \wedge r \le n$. Thus $a \wedge r = a \wedge r \wedge n = r \wedge n$. Now, put $p = (b \wedge r) \vee n$. Then $n \le p \le b$. Also

$$p \wedge a = (a \wedge b \wedge r) \vee (a \wedge n) = (a \wedge r) \vee (a \wedge n)$$
$$= (r \wedge n) \vee n = n,$$

and $p \vee a = (b \wedge r) \vee n \vee a = b \vee n = b$.

Hence $[n, b]$ is complemented for each $b \in L$, $(b \ge n)$.

On the other hand, let $b \le a \le n$. Then for all $j \in \langle a\rangle_n$, $m(a, n, j) = m(b, n, j) = j$. So, $a \equiv b\ \theta(\langle a\rangle_n)^* = \theta(\langle a\rangle_n^+)$. Then a dual proof of above shows that $[b, n]$ is also complemented for each $b \le n$. Hence by [1.1.5.], $F_n(L)$ is generalized boolean. ●

The skeleton $SC(L) = \{ \theta \in C(L) : \theta = \Phi^* \text{ for some } \Phi \in C(L) \} = \{ \theta \in C(L) : \theta = \theta^{**} \}$ is a complete boolean lattice. The meet of a set $\{ \theta_i \} \subseteq SC(L)$ is $\cap\, \theta_i$ as in $C(L)$, while the join is $\underline{\vee}\, \theta_i = (\vee\, \theta_i)^{**} = (\cap\, \theta_i)^*$ and the complement of $\theta \in SC(L)$ is θ^*. The fact that $SC(L)$ is complete follows from the fact that $SC(L)$ is precisely the set of closed elements associated with the closure operator $\theta \to \theta^{**}$ on the complete lattice $C(L)$ and $SC(L)$ is boolean because of Gliveanko´s theorem, c.f. Grätzer [15, Th.4. p.p.58]. The set $KSC(L) = \{ \text{Ker}\, \theta : \theta \in SC(L) \}$ is closed under arbitrary set theoretic intersection and hence is a complete lattice. Also, for any $n \in L$, $K_nSC(L) = \{ \text{Ker}_n\theta : \theta \in SC(L) \}$ is a complete lattice.

The following two theorems are due to Cornish [9, Th. 2.4. & 2.5.], Which are extensions of the classical theorem of Hashimoto [15, Th. 8, p-91] and also characterize generalized boolean lattices and give a one-to-one correspondence between ideals and congruence relations.

3.2.9. Theorem : Let L be a distributive lattice with 0. Then the following conditions are equivalent :

(i) The lattice L is disjunctive.

(ii) The map $\theta \to \mathrm{Ker}\theta$ of SC(L) onto KSC(L) is one-to-one and so is a one-to-one correspondence.

(iii) The map $\theta \to \mathrm{Ker}\theta$ of SC(L) onto KSC(L) preserves finite joins.

(iv) The map $\theta \to \mathrm{Ker}\theta$ is a lattice isomorphism of SC(L) onto KSC(L), whose inverse is the map $J \to \theta(J)^{**}$ ●

3.2.10. Theorem : Let L be a distributive lattice with 0. Then the lattice L is generalized boolean if and only if the map $\theta \to \mathrm{Ker}\theta$ is a lattice isomorphism of SC(L) onto KSC(L), whose inverse is the map $J \to \theta(J)$. ●

We conclude this section with the following generalizations of the above theorems.

3.2.11. Theorem : Let L be a distributive lattice with an element n. Then the following conditions are equivalent :

(i) $F_n(L)$ is disjunctive.

(ii) The map $\theta \to Ker_n\theta$ of SC(L) onto $K_nSC(L)$ is one-to-one and so is a one-to-one correspondence.

(iii) The map $\theta \to Ker_n\theta$ of SC(L) onto $K_nSC(L)$ preserves finite joins.

(iv) The map $\theta \to Ker_n\theta$ is a lattice isomorphism of SC(L) onto $K_nSC(L)$, whose inverse is the map $J \to \theta(J)^{**}$ for any n-ideal J in L ●

Proof : Firstly, we show that (i) → (iv). Suppose (i) holds, i.e., $F_n(L)$ is disjunctive. Then by 3.2.6.(iv) we have $K_nSC(L) = \{ J : J = J^{++} , J \text{ is n-ideal} \}$. Also, by 3.2.6.(ii) for any $\Phi \in SC(L)$, $\Phi = \Phi^{**} = \theta(Ker_n\Phi)^{**}$. Thus, the map $\theta \to Ker_n\theta$ of SC(L) onto $K_nSC(L)$ is one-to-one. Clearly this map preserves meets and it also preserves joins since for any $\theta, \Phi \in SC(L)$ $\theta \veebar \Phi = (\theta^* \cap \Phi^*)^*$ and $Ker_n(\theta \veebar \Phi) = Ker_n (\theta^* \cap \Phi^*)^*$

$= [\mathrm{Ker}_n\ (\theta^* \cap \Phi^*)]^+ = [(\mathrm{Ker}_n\theta)^+ \cap (\mathrm{Ker}_n\Phi)^+]^+$

$= (\mathrm{Ker}_n\theta)^{++} \vee (\mathrm{Ker}_n\Phi)^{++} = \mathrm{Ker}_n(\theta^{**}) \vee \mathrm{Ker}_n(\Phi^{**})$

$= \mathrm{Ker}_n\theta \vee \mathrm{Ker}_n\Phi$.

Thus, $\theta \to \mathrm{Ker}_n\theta$ is a lattice isomorphism. Also, note that, $\mathrm{Ker}_n(\theta(J)^{**}) = (\mathrm{Ker}_n\theta(J))^{++} = J^{++} = J$ for any n-ideal $J \in K_nSC(L)$, while $\theta(\mathrm{Ker}_n\Phi)^{**} = \Phi^{**} = \Phi$ for any $\Phi \in SC(L)$. Thus $J \to \theta(J)^{**}$ is the inverse of $\theta \to \mathrm{Ker}_n\theta$. Hence (iv) holds.

(iv) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). Suppose (ii) holds, i.e., $\theta \to \mathrm{Ker}_n\theta$ is one-to-one. Then it is a meet isomorphism of the lattice SC(L) onto the lattice $K_nSC(L)$. It follows that $\theta \to \mathrm{Ker}_n\theta$ is a lattice isomorphism and so (iii) holds.

Lastly, we shall show that (iii) $\Rightarrow$ (i). Suppose (iii) holds. Then $\theta \to \mathrm{Ker}_n\theta$ is a lattice isomorphism of SC(L) onto $K_nSC(L)$. Hence $K_nSC(L)$ must be boolean. It is not hard to see that $F_n(L)$ is a join-dense sublattice of $K_nSC(L)$. Since $K_nSC(L)$ is boolean, so $F_n(L)$ is disjunctive. Hence (i) holds.

3.2.12. Theorem : For a distributive lattice L with an element n, $F_n(L)$ is generalized boolean if and only if the map $\theta \to Ker_n\theta$ is a lattice isomorphism of SC(L) onto $K_nSC(L)$, whose inverse is the map $J \to \theta(J)$, J is an n-ideal of L.

Proof : Suppose $F_n(L)$ is generalized boolean. Then $F_n(L)$ is disjunctive and so by 3.2.11. the inverse of $\theta \to Ker_n\theta$ is $J \to \theta(J)^{**}$. But due to 3.2.8., $\theta(J)^{**} = \theta(J^{++})$ for any $J \in K_nSC(L)$. So due to 3.2.6., $J = J^{++}$. Hence $J \to \theta(J)$ is the inverse of $\theta \to Ker_n\theta$.

Conversely, let $J \to \theta(J)$ is the inverse of $\theta \to Ker_n\theta$. Then by 3.2.11., $F_n(L)$ is disjunctive and so by 3.2.6., $Ker_n(\theta(J)^{**}) = [Ker_n(\theta(J))]^{++} = J^{++}$ for any n-ideal J of L. Then by 3.1.4., we have $J^{++} \in K_nSC(L)$ We must also have, $\theta(J^{++}) = \theta(Ker_n(\theta(J))^{**}) = \theta(J)^{**}$. Then due to 3.2.8., $F_n(L)$ is generalized boolean.●

CHAPTER-4

Standard n-ideals

Introduction : Standard elements and ideals in a lattice were introduced by Grätzer and Schmidt [18]. Some additional work has been done by Janowitz [29]. While Fried and Schmidt [14] have extended the idea of standrad ideals to convex sublattices.

According to Grätzer and Schmidt [18], if a is an element of a lattice L, then

(i) a is called distributive if $a \vee (x \wedge y)$
$= (a \vee x) \wedge (a \vee y)$, for all $x, y \in L$.

(ii) a is called standard if $x \wedge (a \vee y)$
$= (x \wedge a) \vee (x \wedge y)$, for all $x, y \in L$.

(iii) a is called neutral if for all $x, y \in L$,

(a) $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$,
i,e, a is standard

and (b) $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$.

Grätzer [17] has shown that an element n in a lattice L is neutral if and only if

$$(n \wedge x) \vee (n \wedge y) \vee (x \wedge y)$$
$$= (n \vee x) \wedge (n \vee y) \wedge (x \vee y),$$
for all $x, y \in L$.

An ideal S of a lattice L is called standard if it is a standard element of the lattice of ideals $I(L)$.

Fried and Schmidt [14] have extended the idea of standard ideals to convex sublattices. Moreover, Nieminen in [37] has discussed on distributive and neutral (convex) sublattices. On the other hand, in a more recent paper Dixit and Paliwal [12], [13] have established some results on standard, neutral and distributive (convex) sublattices. But their technique is quite different from those of the above authors. We denote the set of all convex sublattices of L by Csub(L). According to [14] and [37], we define two operations $\mathbin{\dot\wedge}$ and $\mathbin{\dot\vee}$ (these notations have been used by Nieminen in [37] on Csub(L)) by

$$A \mathbin{\dot\wedge} B = \langle\{a \wedge b : a \in A,\ b \in B\}\rangle$$

and
$$A \mathbin{\dot\vee} B = \langle\{a \vee b : a \in A,\ b \in B\}\rangle$$

for all $A, B \in$ Csub (L), where $\langle H\rangle$ denotes the convex sublattice generated by a subset H of L.

If A and B are both ideals then $A \mathbin{\dot\vee} B$ and $A \mathbin{\dot\wedge} B$ are exactly the join and meet of A and B in the ideal lattice. However, in general case neither $A \subseteq A \vee B$ and $A \wedge B \subseteq A$ are valid. For example if $A = \{a\}$ and $B = \{b\}$, then both inequalities imply $A = B$.

According to [18], a convex sublattice S of a lattice L is called a standard convex sublattice (or simply a "standard sublattice") if

$$I \wedge \langle S,K\rangle = \langle I \wedge S,\ I \wedge K\rangle$$

and $I \vee \langle S,K\rangle = \langle I \vee S,\ I \vee K\rangle$ hold for any pair $\{I,K\}$ of Csub (L) whenever neither $S \cap K$ nor $I \cap \langle S,K\rangle$ are empty, where $\cap$ denotes the set theoretical intersection.

We call an n-ideal of a lattice L, a standard n-ideal if it is a standard element of the lattice of n-ideals $I_n(L)$.

In section 1, we give a characterization of standard n-ideals using the concept of standard sublattice when n is a neutral element. For a neutral element n of a lattice L, we prove the following :

(i) an n-ideal is standard if and only if it is a standard sublattice.

(ii) the intersection of a standard n-ideal and n-ideal I of a lattice L is a standard n-ideal in I.

(iii) the principal n-ideal $\langle a \rangle_n$ of a lattice L is a standard n-ideal if and only if $a \vee n$ is standard and $a \wedge n$ is dual standard.

(iv) for an arbitrary n-ideal I and a standard n-ideal S of a lattice L, if $I \vee S$ and $I \cap S$ are principal n-ideals, then I itself is a principal n—ideal.

In section 2, we have shown that if n is a neutral element and (n] and [n) are relatively complemented, then every homomorphism n-kernels of L is a standard n-ideal and every standard n-ideals is the n-kernel of precisely one congruence relation. We have also shown that for a relatively complemented lattice L with 0 and 1, C(L) is a boolean algebra if and only if every standard n-ideal of L is a principal n—ideal.

Finally, we prove two isomorphism theorems on standard n-ideals which are extensions of the isomorphism theorems on standard ideals given by Grätzer and Schmidt [18].

1. "Standard n-ideals"

According to Fried and Schmidt [14, Th.-1], we have a fundamental characterization theorem for standard convex sublattices :

4.1.1. Theorem : The following conditions are equivalent for each convex sublattice S of a lattice L :

(α) S is a standard sublattice,

(β) Let K be any convex sublattice of L such that $K \cap S \neq \Phi$. Then to each $x \in \langle S,K \rangle$, there exist $s_1, s_2 \in S$, $a_1, a_2 \in K$ such that

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \vee s_2) \wedge (x \vee a_2)$$

(β´) For any convex sublattice K of L and for each $s_2, s_1' \in S$, there are elements $s_1, s_2' \in S$, a_1, $a_2 \in K$ such that $x = (x \wedge s_1) \vee (x \wedge (a_1 \vee s_2))$

$$= (x \vee s_2') \wedge (x \vee (a_2 \wedge s_1')),$$

(γ) The relation $\Theta[S]$ on L defined by $x \equiv y \ (\Theta[S])$ if and only if $x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$ and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$ with suitable $t, s \in S$, is a congruence relation.

Following result which is due to [14] shows that the concept of standard sublattices and standard ideals coincides in case of ideals.

4.1.2. Proposition. **[14, Pro.2]** An ideal S of a lattice L is standard if and only if it is a standard sublattice.

Recall that an n-ideal I of a lattice L is called a standard n-ideal if it is a standard element of $I_n(L)$, the lattice of n-ideals.

The following theorem gives an extension of proposition 4.1.2. above.

4.1.3. Theorem : For a neutral element n of a lattice L, an n-ideal is standard if and only if it is a standard sublattice.

Proof : First assume that an n-ideal S of a lattice L is a standard sublattice. That is, for all convex sublattice I & K of L with $S \cap K \neq \Phi$ and $I \cap \langle S, K\rangle \neq \Phi$, we have, $I \wedge \langle S, K\rangle = \langle I \wedge S, I \wedge K\rangle$ and

$$I \vee \langle S, K\rangle = \langle I \vee S, I \vee K\rangle.$$

We are to show that S is a standard n-ideal in $I_n(L)$.

That is, for all n-ideals $I, K \in I_n(L)$,

$$I \cap (S \vee K) = (I \cap S) \vee (I \cap K).$$

Clearly, $(I \cap S) \vee (I \cap K) \subseteq I \cap (S \vee K)$.

So, let $x \in I \cap (S \vee K)$. Then $x \in I$ and $x \in S \vee K$, so by theorem 4.1.1., we have

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \vee s_2) \wedge (x \vee a_2),$$

for some $s_1, s_2 \in S$ and $a_1\ a_2 \in K$.

Now, $x = (x \wedge s_1) \vee (x \wedge a_1)$

$$\leq [(x \wedge s_1) \vee (x \wedge n) \vee (s_1 \wedge n)] \vee [(x \wedge a_1) \vee (x \wedge n) \vee (a_1 \wedge n)]$$

$$= m(x, n, s_1) \vee m(x, n, a_1),$$

that is, $x \leq m(x, n, s_1) \vee m(x, n, a_1)$

Again, $x = (x \vee s_2) \wedge (x \vee a_2)$

$$\geq [(x \vee s_2) \wedge (x \vee n) \wedge (s_2 \vee n)] \wedge [(x \vee a_2) \wedge (x \vee n) \wedge (a_2 \vee n)]$$

$$= m^d(x, n, s_2) \wedge m^d(x, n, a_2)$$

$$= m(x, n, s_2) \wedge m(x, n, a_2),$$ as n is neutral.

Hence $m(x, n, s_2) \wedge m(x, n, a_2) \leq x \leq m(x, n, s_1) \vee m(x, n, a_1)$. Which implies $x \in (I \cap S) \vee (I \cap K)$.

Thus, $I \cap (S \vee K) = (I \cap S) \vee (I \cap K)$ and so S is a standard n-ideal.

Conversely, suppose that n-ideal S of a lattice L is standard. Consider any convex sublattice K of L such that $S \cap K \neq \Phi$. Since S is an n-ideal, clearly $\langle S, K\rangle = \langle S, \langle K\rangle_n\rangle$. Let $x \in \langle S, K\rangle$. Then $x \in \langle S, \langle K\rangle_n\rangle = S \vee \langle K\rangle_n$. Then $x \in \langle x\rangle_n \cap (S \vee \langle K\rangle_n)$ $= (\langle x\rangle_n \cap S) \vee (\langle x\rangle_n \cap \langle K\rangle_n)$, as S is a standard n-ideal. This implies

$$\langle x\rangle_n = (\langle x\rangle_n \cap S) \vee (\langle x\rangle_n \cap \langle K\rangle_n) \quad \ldots\ldots\ldots \quad (1)$$

Since $x \vee n$ is the largest element of $\langle x\rangle_n$, so we have $x \vee n = m(x \vee n, n, s_1) \vee m(x \vee n, n, t)$

for some $s \in S$, $t \in \langle K\rangle_n$.

$$= ((x \vee n) \wedge s_1) \vee ((x \vee n) \wedge t) \vee n$$

$$= (x \wedge s_1) \vee (x \wedge t) \vee n, \text{ as } n \text{ is neutral.}$$

Now, $t \in \langle K\rangle_n$ implies $t \leq t_1 \vee n$ for some $t_1 \in K$.

Then $x \vee n \leq (x \wedge s_1) \vee (x \wedge (t_1 \vee n)) \vee n$

$$= (x \wedge s_1) \vee (x \wedge t_1) \vee n$$

$$\leq (x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n \leq x \vee n.$$

which implies that

$$x \vee n = (x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n.$$

Then $\quad x = x \wedge (x \vee n)$

$$= x \wedge [(x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee n]$$

$$= [x \wedge \{(x \wedge (s_1 \vee n)) \vee (x \wedge t_1)\}] \vee (x \wedge n),$$

as n is neutral.

$$= (x \wedge (s_1 \vee n)) \vee (x \wedge t_1) \vee (x \wedge n)$$

$$= (x \wedge (s_1 \vee n)) \vee (x \wedge t_1),$$

where $s_1 \vee n \in S$, $t_1 \in K$.

Since $x \wedge n$ is the smallest element of $\langle x\rangle_n$, using the relation (1) a dual proof of above shows that

$x = (x \vee (s_2 \wedge n)) \wedge (x \vee t_2)$ for some $s_2 \in S$, $t_2 \in K$. Hence from Th. 4.1.1. (β) we obtain that S is a standard sublattice. ●

Now, we give characterizations for standard n-ideals when n is a neutral element. We prefer to call it the " Fundamental Characterization Theorem" for standard n-ideals.

4.1.4. Theorem : If n is a neutral element of a lattice L. Then the following conditions are equivalent :

(a) S is a standard n-ideal;

(b) For any n-ideal K,

$$S \vee K = \{ x : x = (x \wedge s_1) \vee (x \wedge k_1) = (x \wedge s_1') \vee (x \wedge k_1') \vee (x \wedge n)$$

and $$x = (x \vee s_2) \wedge (x \vee k_2) = (x \vee s_2') \wedge (x \vee k_2') \wedge (x \vee n)$$

for some $s_1, s_2, s_1', s_2' \in S$; $k_1, k_2, k_1', k_2' \in K$ }.

(c) The relation $\theta(S)$ on L defined by $x \equiv y\ \theta(S)$ if and only if $x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$ and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$, for some $t, s \in S$, is a congruence relation.

Proof : (a) $\Rightarrow$ (b). Suppose S is a standard n-ideal and K be any n-ideal. Let $x \in S \vee K$. Since K is also a convex sublatice of L, we have from the proof of theorem 4.1.3., $x = (x \wedge (s_1 \vee n)) \vee (x \wedge t_1)$

$= (x \vee (s_2 \wedge n)) \wedge (x \vee t_2)$ for some s_1, $s_2 \in S$; $t_1, t_2 \in K$. Since n is neutral, from above we also have $x = (x \wedge s_1) \vee (x \wedge t_1) \vee (x \wedge n)$

$= (x \vee s_2) \wedge (x \vee t_2) \wedge (x \vee n)$.

Thus (b) holds.

(b) $\Rightarrow$ (c).

Let (b) holds. Let $\theta(S)$ be defined as $x \equiv y\ \theta(S)$ if and only if $x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$ and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$. For $x \geq y$, $y = (y \vee t) \wedge x$ and $x = (x \wedge s) \vee y$, for some $t, s \in S$, with $s \geq t$.

Obviously, $\theta(S)$ is reflexive and symmetric. Moreover, $x \equiv y\ \theta(S)$ if and only if $x \wedge y \equiv x \vee y\ \theta(S)$. Now suppose $x \geq y \geq z$ with $x \equiv y\ \theta(S)$ and $y \equiv z\ \theta(S)$.

Then $x = (x \wedge s_1) \vee y$, $y = (y \vee t_1) \wedge x$ and

$y = (y \wedge s_2) \vee z$, $z = (z \vee t_2) \wedge y$ for some

$s_1, s_2, t_1, t_2 \in S$.

Then
$$\begin{aligned} x = (x \wedge s_1) \vee y &= (x \wedge s_1) \vee (y \wedge s_2) \vee z \\ &\leq (x \wedge s_1) \vee (x \wedge s_2) \vee z \\ &\leq (x \wedge (s_1 \vee s_2)) \vee z \leq x, \end{aligned}$$

which implies $x = (x \wedge (s_1 \vee s_2)) \vee z$.

Similarly, we can show that $z = (z \vee (t_1 \wedge t_2)) \wedge x$.

This shows that $x \equiv z\ \Theta(S)$.

For the substitution property, suppose $x \geq y$ and $x \equiv y\ \Theta(S)$. Then $x = (x \wedge s) \vee y$ and $y = (y \vee t) \wedge x$, for some $s, t \in S$. From these relations it is easy to find $s, t \in S$ with $t \leq s$ satisfying the relations.

Then for every $z \in L$, $y \wedge z \leq x \wedge z$

and $y \wedge z \leq t \vee (y \wedge z)$.

Therefore,
$$\begin{aligned} y \wedge z &\leq (t \vee (y \wedge z)) \wedge (x \wedge z) \\ &\leq (t \vee y) \wedge (x \wedge z) \\ &= ((t \vee y) \wedge x) \wedge z \\ &= y \wedge z. \end{aligned}$$

This implies, $y \wedge z = (t \vee (y \wedge z)) \wedge (x \wedge z)$.

Let K be the n-ideal $\langle t \wedge y \wedge z, y \rangle_n$.

Since $s, t \wedge y \wedge z \in S \vee K$, so by the convexity of $S \vee K$, $t \wedge y \wedge z \leq t \wedge y \leq t \wedge x \leq s \wedge x \leq s$ as $t \leq s$.

This implies that

$s \wedge x \in S \vee K$. Hence $x = (s \wedge x) \vee y \in S \vee K$.

Also, by the convexity of $S \vee K$, $t \wedge y \wedge z \leq y \wedge z \leq x \wedge z \leq x$, implies $y \wedge z$, $x \wedge z \in S \vee K$. Then by (b) we have

$$x \wedge z = (x \wedge z \wedge s_1) \vee (x \wedge z \wedge k_1) \vee (x \wedge z \wedge n)$$
for some $s_1 \in S$, $k_1 \in K$.
$$= (x \wedge z \wedge s_1) \vee (x \wedge z \wedge (y \vee n)) \vee (x \wedge z \wedge n),$$
as $y \vee n$ is the largest element of K.
$$= (x \wedge z \wedge s_1) \vee (y \wedge z) \vee (x \wedge z \wedge n),$$
as n is neutral.
$$= ((x \wedge z) \wedge (s_1 \vee n)) \vee (y \wedge z),$$
where $s_1 \vee n \in S$. Therefore, $x \wedge z \equiv y \wedge z \ \theta(S)$.

Dually we can prove $x \vee z \equiv y \vee z \ \theta(S)$. Therefore using [15. Lemma 8.p-74], $\theta(S)$ is a congruence relation. Hence (c) holds.

Finally, we shall show that (c) $\rightarrow$ (a).
Let (c) holds. For any n-ideals I, K of L, obviously $(I \cap S) \vee (I \cap K) \subseteq I \cap (S \vee K)$. To prove the reverse inequality, suppose $x \in I \cap (S \vee K)$.

Then $x \in I$ and $x \in S \vee K$. Since $x \in S \vee K$, it is easy to find the elements $s_1, s_2 \in S$, $k_1, k_2 \in K$ with $s_1 \le n \le s_2$ and $k_1 \le n \le k_2$ such that

$$s_1 \wedge k_1 \le x \le s_2 \vee k_2.$$

Now, $s_1 \equiv s_2\ \Theta(S)$ implies $s_2 \vee k_2 \equiv s_1 \vee k_2 = k_2\ \Theta(S)$. Since $x \le s_2 \vee k_2$, we have $x = x \wedge (s_2 \vee k_2)$

$\equiv x \wedge k_2\ \Theta(S)$. Then by (c)

$x = (x \wedge s) \vee (x \wedge k_2)$ for some $s \in S$.

$\le m(x, n, s) \vee m(x, n, k_2)$.

Also, $s_1 \equiv s_2\ \Theta(S)$ implies $s_1 \wedge k_1 \equiv s_2 \wedge k_1$ $= k_1\ \Theta(S)$. So, $x = x \vee (s_1 \wedge k_1) \equiv x \vee k_1\ \Theta(S)$.

Applying (c) again we have

$x = (x \vee t) \wedge (x \vee k_1)$ for some $t \in S$.

$\ge m^d(x, n, t) \wedge m^d(x, n, k_1)$

$= m(x, n, t) \wedge m(x, n, k_1)$, as n is neutral.

Hence $x \in (I \cap S) \vee (I \cap K)$.

This implies $I \cap (S \vee K) = (I \cap S) \vee (I \cap K)$.

Therefore (a) holds.●

4.1.5. Corollary : Suppose n is a neutral element of a lattice L. Then for a standard n-ideal S of L, $\Theta(S)$ is the smallest congruence relation of L cotaining S as a class.

Proof : Clearly any two elements of S are related by

$\Theta(S)$. Now suppose $x \equiv y\ \Theta(S)$ with $x \geq y$. Then by theorem 4.1.4, we have $y = (y \vee t) \wedge x$ and $x = (x \wedge s) \vee y$ for some $s, t \in S$. Suppose $y \in S$. Then $y \leq x = (x \wedge s) \vee y \leq y \vee s$. Then, by the convexity of S, $x \in S$. On the other hand, if $x \in S$, then $x \geq y = (y \vee t) \wedge x \geq t \wedge x$ implies $y \in S$. Hence $\Theta(S)$ contains S as a class.

Let Φ be a congruence relation containing S as a class. We have $x \equiv y\ \Theta(S)$ with $x \geq y$, $x = (x \wedge s) \vee y$ and $y = (y \vee t) \wedge x$ for some $s, t \in S$.

Now, $x = (x \wedge s) \vee y \equiv (x \wedge n) \vee y\ \Phi$

$= (x \vee y) \wedge (n \vee y)$, as n is neutral.

$= x \wedge (n \vee y) \equiv x \wedge (y \vee t)\ \Phi = y\ \Phi$.

This implies $\Theta(S) \subseteq \Phi$. Hence $\Theta(S)$ is the smallest congruence containing S as a class. ●

4.1.6. Corollary : If n is a neutral element and S and T are two standard n-ideals of a lattice L, then $S \cap T$ is a standard n-ideal.

Proof : Clearly $S \cap T$ is an n-ideal. Suppose $x \equiv y\ (\Theta(S) \cap \Theta(T))$ with $x \geq y$. Since $x \equiv y\ \Theta(S)$, so we have $x = (x \wedge s_1) \vee y$ and $y = (y \vee s_2) \wedge x$, for some $s_1, s_2 \in S$. Here we can consider $s_2 \leq n \leq s_1$. Now $x \equiv y\ \Theta(T)$ implies $x \wedge s_1 \equiv y \wedge s_1\ \Theta(T)$, and so there exists $t_1 \in T$, $t_1 \geq n$ such that $x \wedge s_1 = ((x \wedge s_1) \wedge t_1) \vee (y \wedge s_1)$. Then $x = (x \wedge s_1) \vee y = [((x \wedge s_1) \wedge t_1) \vee (y \wedge s_1)] \vee y$

$= (x \wedge s_1 \wedge t_1) \vee y = (x \wedge (s_1 \wedge t_1)) \vee y$.

Again $x \equiv y\ \Theta(T)$ implies $x \vee s_2 \equiv y \vee s_2\ \Theta(T)$. Then we can find $t_2 \in T$ with $t_2 \leq n$ such that

$y \vee s_2 = ((y \vee s_2) \vee t_2) \wedge (x \vee s_2)$. Then

$$\begin{aligned} y = (y \vee s_2) \wedge x &= [((y \vee s_2) \vee t_2) \wedge (x \vee s_2)] \wedge x \\ &= (y \vee s_2 \vee t_2) \wedge (x \vee s_2) \wedge x \\ &= (y \vee (s_2 \vee t_2)) \wedge x. \end{aligned}$$

Now, $n \leq s_1 \wedge t_1 \leq s_1$ and $n \leq s_1 \wedge t_1 \leq t_1$ implies $s_1 \wedge t_1 \in S \cap T$. Also $s_2 \leq s_2 \vee t_2 \leq n$ and $t_2 \leq s_2 \vee t_2 \leq n$ implies $s_2 \vee t_2 \in S \cap T$. Hence $x \equiv y\ \Theta(S \cap T)$. Therefore $\Theta(S \cap T) = \Theta(S) \cap \Theta(T)$. Hence by 4.1.4. $S \cap T$ is also a standard n-ideal. ●

4.1.7. Corollary : Let n be a neutral element of a lattice L and S be a standard n-ideal. Then $x \equiv y\ \Theta(S)$ if and only if $\langle x\rangle_n \vee S = \langle y\rangle_n \vee S$.

Proof : Let $x \equiv y\ \theta(S)$. Then for $x \geq y$, we have $x = (x \wedge s_1) \vee y$ and $y = (y \vee s_2) \wedge x$ for some $s_1, s_2 \in S$. This implies $x \vee s_1 = y \vee s_1$, $x \wedge s_2 = y \wedge s_2$. Now, $y \leq x \leq x \vee s_1 = y \vee s_1$, which implies $x \in \langle y \rangle_n \vee S$. On the other hand, $x \wedge s_2 = y \wedge s_2 \leq y \leq x$ implies $y \in \langle x \rangle_n \vee S$. Hence $\langle x \rangle_n \vee S = \langle y \rangle_n \vee S$. Conversely, suppose that $\langle x \rangle_n \vee S = \langle y \rangle_n \vee S$. As $x \in \langle x \rangle_n \vee S = \langle y \rangle_n \vee S$, so by 4.1.4., $x = (x \wedge y_1) \vee (x \wedge s)$,

for some $y_1 \in \langle y \rangle_n$, $s \in S$.

$$= (x \wedge (y \vee n)) \vee (x \wedge s)$$
$$= (x \wedge y) \vee (x \wedge n) \vee (x \wedge s)$$
$$= y \vee [x \wedge (n \vee s)], \text{ as } n \text{ is neutral.}$$

Also, $y \in \langle y \rangle_n \vee S = \langle x \rangle_n \vee S$. Then applying 4.1.4. again we have $y = (y \vee x_1) \wedge (y \vee s')$,

for some $x \in \langle x \rangle_n$, $s' \in S$.

Then $y = (y \vee (x \wedge n)) \wedge (y \vee s')$
$$= (y \vee x) \wedge (y \vee n) \wedge (y \vee s')$$
$$= (x \wedge [y \vee (n \wedge s')],$$ as n is neutral. Since $n \vee s$, $n \wedge s' \in S$, so we have

$$x \equiv y\ \theta(S). \bullet$$

We know from [18] that the intersection of a standard ideal with an arbitrary ideal I of a lattice L is standard in I.

Following lemma is a generalization of this result.

4.1.8. Lemma : The intersection of a standard n-ideal and an n-ideal I of a lattice L is a standard n-ideal in I, where n is a neutral element.

Proof : Let S be a standard n-ideal of L. We are to show that $S \cap I$ is a standard n-ideal in I. Consider an n-ideal K of I, which is also an n-ideal of L. Now, let $x \in (S \cap I) \vee K \subseteq S \vee K$. Since S is standard, so we have by theorem 4.1.4.,
$x = (x \wedge s) \vee (x \wedge k)$, for some $s \in S$, $k \in K$. By the monotonity, we can choose both $s \geq n$, $k \geq n$.
Put $s' = (x \vee n) \wedge s$. Then $s' \leq s$
and $n = (x \vee n) \wedge n \leq (x \vee n) \wedge s = s' \leq x \vee n$.
Since $x \vee n \in I$, so by convexity of S and I,
$s' \in S \cap I$. Also $x \wedge s' = x \wedge s$. Thus
$x = (x \wedge s') \vee (x \wedge k)$, for some $s' \in S \cap I$, $k \in K$.

Also, by duality we get $x = (x \vee s'') \wedge (x \vee k')$ for some $s'' \in S \cap I$, $k' \in K$. Hence by theorem 4.1.4., we have $S \cap I$ is standard in I. ●

4.1.9. Lemma : Let n be a neutral element of a lattice L and Φ is a homomorphism of L onto a lattice L' such that $\Phi(n) = n'$, $n' \in L'$. Then for any standard n-ideal I of L, $\Phi(I)$ is a standard n'-ideal of L'.

Proof : Clearly $\Phi(I)$ is a sublattice of L'.
Let $p \leq t \leq q$, where $p, q \in \Phi(I)$, $t \in L'$. Then $p = \Phi(x)$ and $q = \Phi(y)$ for some $x, y \in I$. Since Φ is onto, $t = \Phi(r)$ for some $r \in L$.
Then $\Phi(r) = \Phi(r) \wedge \Phi(y) = \Phi(r \wedge y)$

and
$$\begin{aligned}\Phi(r) &= \Phi(r) \vee \Phi(x)\\ &= \Phi(x) \vee \Phi(r \wedge y)\\ &= \Phi(x \vee (r \wedge y)).\end{aligned}$$

Now, $x \leq x \vee (r \wedge y) \leq x \vee y$ and so by convexity we have $x \vee (r \wedge y) \in I$. Thus $t = \Phi(x \vee (r \wedge y)) \in \Phi(I)$.
Hence $\Phi(I)$ is a convex sublattice of L'.
Moreover $\Phi(n) = n'$ implies $\Phi(I)$ is an n'-ideal of L'.

For standardness, we shall prove (b) of theorem 4.1.4. for $\Phi(I)$. Let K' be any n'-ideal of L'. Then $K' = \Phi(K)$ for some n-ideal K of L.
Let $y \in \Phi(I) \vee \Phi(K) \subseteq \Phi(I \vee K)$. Then $y = \Phi(x)$ for some $x \in I \vee K$. Since I is a standard n-ideal of L, using (b) of Theorem 4.1.4.

we have $x = (x \wedge i_1) \vee (x \wedge k_1) \vee (x \wedge n)$,

for some $i_1 \in I$, $k_1 \in K$

$= (x \vee i_2) \wedge (x \vee k_2) \wedge (x \vee n)$,

for some $i_2 \in I$, $k_2 \in K$.

Then $y = \Phi(x)$

$= \Phi(x \wedge i_1) \vee \Phi(x \wedge k_1) \vee \Phi(x \wedge n)$

$= [\Phi(x) \wedge \Phi(i_1)] \vee [\Phi(x) \wedge \Phi(k_1)] \vee [\Phi(x) \wedge \Phi(n)]$

$= [y \wedge \Phi(i_1)] \vee [y \wedge \Phi(k_1)] \vee [y \wedge n']$.

Also, $y = \Phi(x)$

$= [y \vee \Phi(i_2)] \wedge [y \vee \Phi(k_2)] \wedge [y \vee n']$.

Then using (b) of theorem 4.1.4. again, $\Phi(I)$ is a standard n'-ideal of L'. ●

From Grätzer and Schmidt [18], we know that ideal (s] is standard if and only if s is standard in L. One may ask the question whether this is true for principal n-ideal when n is a neutral element. In fact this not even true when L is a complemented lattice. Figure 4.1.1. and Figure 4.1.2 exhibits the complemented lattice L, where n is neutral. There $\langle a\rangle_n$ is standard in $I_n(L)$ but a is not standard in L. Moreover b is standard in L but $\langle b\rangle_n$ is not standard.

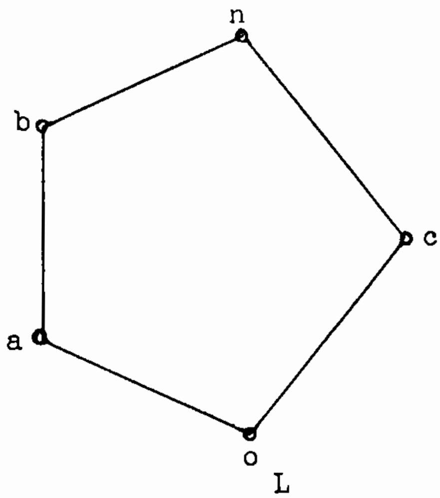

Figure 4.1.1.

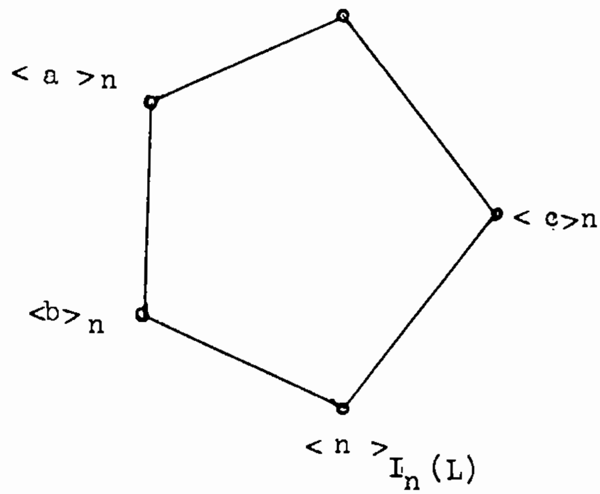

Figure 4.1.2

But we have the following result :

4.1.10. Lemma : For a neutral element n, the principal n-ideal $\langle a\rangle_n$ of a lattice L is a standard n-ideal if and only if $a \vee n$ is standard and $a \wedge n$ is dual standard.

Proof : First suppose that $a \vee n$ is standard and $a \wedge n$ is dual standard. We are to show that $\langle a\rangle_n$ is a standard n-ideal. Let us define a relation $\theta(\langle a\rangle_n)$ on L by $x \equiv y\ \theta(\langle a\rangle_n)$ if and only if

$$x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$$

and $$x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$$

for some $t, s \in \langle a\rangle_n$.

For $x \geq y$, we have

$x = (x \wedge s) \vee y$ and $y = (y \vee t) \wedge x$. Clearly $\theta(<a>_n)$ is reflexive and symmetric. Also $x \equiv y$ $\theta(<a>_n)$ if and only if $x \wedge y \equiv x \vee y$ $\theta(<a>_n)$. Now, let $x \geq y \geq z$ and $x \equiv y$ $\theta(<a>_n)$ and $y \equiv z$ $\theta(<a>_n)$. Then

$$x = (x \wedge s) \vee y, \; y = (y \vee t) \wedge x \text{ and}$$

$$y = (y \wedge p) \vee z, \; z = (z \vee q) \wedge y,$$

for some $s, t, p, q \in <a>_n$.

$$\begin{aligned}\text{Now, } x &= (x \wedge s) \vee y \\ &= (x \wedge s) \vee (y \wedge p) \vee z \\ &\leq (x \wedge s) \vee (x \wedge p) \vee z \\ &\leq [x \wedge (s \vee p)] \vee z \leq x,\end{aligned}$$

which implies $x = (x \wedge (s \vee p)) \vee z$.

$$\begin{aligned}\text{Also, } z &= (z \vee q) \wedge y \\ &= (z \vee q) \wedge (y \vee t) \wedge x \\ &\geq (z \vee q) \wedge (z \vee t) \wedge x \\ &\geq (z \vee (q \wedge t)) \wedge x \geq z,\end{aligned}$$

which implies $z = (z \vee (q \wedge t)) \wedge x$.

Hence $x \equiv z$ $\theta(<a>_n)$.

To prove the substitution property, let $x \equiv y$ $\theta(<a>_n)$, $x \geq y$ and $r \in L$. Then $x = (x \wedge s) \vee y$ and $y = (y \vee t) \wedge x$ for some $s, t \in <a>_n$. Since $s, t \in <a>_n$, $a \wedge n \leq s, t \leq a \vee n$. Set $s = a \vee n$, $t = a \wedge n$.

Then we have

$$x = (x \wedge s) \vee y = y \vee [x \wedge (a \vee n)]$$

$$= x \wedge (y \vee a \vee n), \text{ as } a \vee n \text{ is standard.}$$

Therefore, $x \wedge r = x \wedge r \wedge (y \vee a \vee n)$

$$= (x \wedge r \wedge y) \vee [(x \wedge r) \wedge (a \vee n)]$$

$$= [(x \wedge r) \wedge (a \vee n)] \vee (y \wedge r).$$

On the other hand, $y = (y \vee t) \wedge x$

$$= (y \vee (a \wedge n)) \wedge x$$

and so $y \wedge r = [(y \vee (a \wedge n)) \wedge x] \wedge r$

$$= (y \vee (a \wedge n)) \wedge (x \wedge r)$$

$$\geq [(y \wedge r) \vee (a \wedge n)] \wedge (x \wedge r)$$

$$\geq y \wedge r.$$

Thus, $y \wedge r = [(y \wedge r) \vee (a \wedge n)] \wedge (x \wedge r)$.

Therefore, $x \wedge r \equiv y \wedge r \; \Theta(\langle a \rangle_n)$.

Again, $y = (y \vee t) \wedge x = x \wedge (y \vee (a \wedge n))$

$$= y \vee (x \wedge (a \wedge n)), \text{ as } a \wedge n \text{ is dual}$$

standard.

Therefore, $y \vee r = y \vee r \vee (x \wedge (a \wedge n))$

$$= (y \vee r \vee x) \wedge ((y \vee r) \vee (a \wedge n)),$$

$$= (x \vee r) \wedge [(y \vee r) \vee (a \wedge n)].$$

On the other hand, $x = (x \wedge s) \vee y$

$$= (x \wedge (a \vee n)) \vee y$$

and so, $x \vee r = (x \wedge (a \vee n)) \vee y \vee r$

$$\leq [(x \vee r) \wedge (a \vee n)] \vee (y \vee r)$$

$$\leq x \vee r.$$

Thus $x \vee r = [(x \vee r) \wedge (a \vee n)] \vee (y \vee r)$. Therefore $x \vee r \equiv y \vee r\ \Theta(\langle a\rangle_n)$. Hence $\Theta(\langle a\rangle_n)$ is a congruence relation. Thus by theorem 4.1.4., $\langle a\rangle_n$ is a standard n-ideal.

Conversely, suppose that $\langle a\rangle_n$ is a standard n-ideal. We shall show that $a \vee n$ is standard and $a \wedge n$ is dual standard. Since $\langle a\rangle_n$ is standard so for any principal n-ideals $\langle x\rangle_n$, $\langle y\rangle_n$ we have

$\langle x\rangle_n \cap (\langle a\rangle_n \vee \langle y\rangle_n) = (\langle x\rangle_n \cap \langle a\rangle_n) \vee (\langle x\rangle_n \cap \langle y\rangle_n)$.

Then by some routine calculations, we get

$$[(x \wedge n) \vee \{(a \wedge n) \wedge (y \wedge n)\}, (x \vee n) \wedge \{(a \vee n) \vee (y \vee n)\}] = [\{(x \wedge n) \vee (a \wedge n)\} \wedge \{(x \wedge n) \vee (y \wedge n)\}, \{(x \vee n) \wedge (a \vee n)\} \vee \{(x \vee n) \wedge (y \vee n)\}] \quad \ldots\ldots \quad (1)$$

This implies, $(x \vee n) \wedge \{(a \vee n) \vee (y \vee n)\}$

$= \{(x \vee n) \wedge (a \vee n)\} \vee \{(x \vee n) \wedge (y \vee n)\}$.

Since n is neutral, so

$$\begin{aligned} \text{L.H.S.} &= (x \vee n) \wedge \{(a \vee n) \vee (y \vee n)\} \\ &= (x \vee n) \wedge (a \vee n \vee y) \\ &= [x \wedge (a \vee n \vee y)] \vee n, \end{aligned}$$

and

$$\begin{aligned} \text{R.H.S.} &= [(x \vee n) \wedge (a \vee n)] \vee [(x \vee n) \wedge (y \vee n)] \\ &= n \vee (x \wedge (a \vee n)) \vee (x \wedge y) \vee n, \\ &= (x \wedge y) \vee (x \wedge (a \vee n)) \vee n. \end{aligned}$$

Let $\quad A = x \wedge (y \vee (a \vee n))$

and $\quad B = (x \wedge y) \vee (x \wedge (a \vee n))$.

Now, $A \wedge n = x \wedge (y \vee (a \vee n)) \wedge n = x \wedge n$

and $\quad B \wedge n = [(x \wedge y) \vee (x \wedge (a \vee n)] \wedge n = x \wedge n$.

So by neutrality of n, $A = B$. That is,

$x \wedge (y \vee (a \vee n)) = (x \wedge y) \vee (x \wedge (a \vee n))$.

This implies $a \vee n$ is standard.

Also, from (1) we get

$$(x \wedge n) \vee \{(a \wedge n) \wedge (y \wedge n)\} = \{(x \wedge n) \vee (a \wedge n)\} \wedge \{(x \wedge n) \vee (y \wedge n)\}.$$

Then applying the similar technique we can show that

$$x \vee ((a \wedge n) \wedge y) = (x \vee (a \wedge n)) \wedge (x \vee y).$$

This implies $a \wedge n$ is dual standard. ●

In a distributive lattice, it is well known that if the infimum and supremum of two ideals are principal, then both of them are principal. In [18, lemma 8.], Grätzer and Schmidt have generalized that result for standard ideals. They showed that in an arbitrary lattice L, if I is an arbitrary ideal and S is standard ideal of L, and if $I \vee S$ and $I \wedge S$ are principal, then I itself is a principal ideal. The following theorem is a generalization of their result. To prove this we need the following Lemma :

4.1.11. Lemma : Let n be a neutral element of a lattice L. Then any finitely generated n-ideal which is contained in a principal n-ideal is principal.

Proof : Let $[b, c]$ be a finitely generated n-ideal such that $b \leq n \leq c$. Let $\langle a\rangle_n$ be a principal n-ideal which contains $[b,c]$. Then $a \wedge n \leq b \leq n \leq c \leq a \vee n$. Suppose $t = (a \vee b) \wedge c$. Since n is neutral, we have

$$n \wedge t = n \wedge [(a \vee b) \wedge c] = n \wedge (a \vee b)$$
$$= (n \wedge a) \vee (n \wedge b) = n \wedge b = b,$$

and
$$n \vee t = n \vee [(a \vee b) \wedge c]$$
$$= (n \vee a \vee b) \wedge (n \vee c)$$
$$= (n \vee a) \wedge c = c.$$

Hence $[b, c] = [n \wedge t, n \vee t] = \langle t\rangle_n$.

Therefore $[b, c]$ is a principal n-ideal. ●

4.1.12. Theorem : Let I be an arbitrary n-ideal and S be a standard n-ideal of a lattice L, where n is neutral. If $I \vee S$ and $I \cap S$ are principal n-ideals, then I itself is a principal n-ideal.

Proof : Let $I \vee S = \langle a\rangle_n = [a \wedge n, a \vee n]$ and $I \cap S = \langle b\rangle_n = [b \wedge n, b \vee n]$. Since S is a standard n-ideal, then by theorem 4.1.4.,

$a \vee n = [(a \vee n) \wedge s] \vee ((a \vee n) \wedge x)$

for some $s \in S$, $x \in I$.

$= s \vee x$.

Again, $a \wedge n \in S \vee I$. So by theorem 4.1.4. again there exist $s_1 \in S$ and $x_1 \in I$ such that

$a \wedge n = ((a \wedge n) \vee s_1) \wedge ((a \wedge n) \vee x_1) = s_1 \wedge x_1$.

Now, consider the n-ideal $[b \wedge x_1 \wedge n, b \vee x \vee n]$.

Obviously, $[b \wedge x_1 \wedge n, b \vee x \vee n] \subseteq I \subseteq \langle a \rangle_n$. So by above lemma, $[b \wedge x_1 \wedge n, b \vee x \vee n]$ is a principal n-ideal say $\langle t \rangle_n$ for some $t \in L$.

$$\begin{aligned}
\text{Then } \langle a \rangle_n &= I \vee S \supseteq S \vee [b \wedge x_1 \wedge n, b \vee x \vee n] \\
&\supseteq [s_1 \wedge n, s \vee n] \vee [b \wedge x_1 \wedge n, b \vee x \vee n] \\
&= [s_1 \wedge n \wedge b \wedge x_1 \wedge n, s \vee n \vee b \vee x \vee n] \\
&= [a \wedge n, a \vee n] = \langle a \rangle_n .
\end{aligned}$$

$$\begin{aligned}
\text{This implies } S \vee I &= S \vee [b \wedge x_1 \wedge n, b \vee x \vee n] \\
&= S \vee \langle t \rangle_n \qquad \dots\dots\dots\dots \qquad (A)
\end{aligned}$$

$$\begin{aligned}
\text{Further, } \langle b \rangle_n &= S \cap I \supseteq S \cap [b \wedge x_1 \wedge n, b \vee x \vee n] \\
&\supseteq S \cap [b \wedge n, b \vee n] = \langle b \rangle_n, \text{ as}
\end{aligned}$$

$b \wedge x_1 \wedge n \leq b \wedge n \leq b \vee n \leq b \vee x \vee n$. This implies

$S \cap I = S \cap [b \wedge x_1 \wedge n, b \vee x \vee n] = S \cap \langle t \rangle_n$ ...(B)

Since S is standard so we have from (A) & (B),

$I = \langle t \rangle_n$. Therefore I is a principal n-ideal. ●

In this section we shall deduce some important properties of standard elements and n-ideals from the fundamental characterization theorem. If S is a standard n-ideal, then we call the congruence relation $\Theta(S)$, generated by S, a standard n—congruence relation. If $S = \langle s\rangle_n$, then $\Theta(S) = \Theta(\langle s\rangle_n)$ and so $\Theta(\langle s\rangle_n)$ is a standard n-congruence relation which we call principal standard n—congruence. Firstly, we prove some results on the connection between standard n-ideals and standard n—congruence relations.

4.1.13. Theorem : Let n be a neutral element of a lattice L. Let S and T be two standard n-ideals of L. Then

(i) $\Theta(S \cap T) = \Theta(S) \cap \Theta(T)$

and (ii) $\Theta(S \vee T) = \Theta(S) \vee \Theta(T)$.

Proof : (i) This has already been proved in corollary 4.1.6.

(ii) Clearly, $\Theta(S) \vee \Theta(T) \subseteq \Theta(S \vee T)$. To prove the reverse inequality, let $x \equiv y\ \Theta(S \vee T)$ with $x \geq y$.

Then $y = (y \vee p) \wedge x$ and $x = (x \wedge q) \vee y$,

for some $p, q \in S \vee T$.

Then by theorem 4.1.4.,

$p = (p \wedge s_1) \vee (p \wedge t_1)$ and $p = (p \vee s_2) \wedge (p \vee t_2)$,

$q = (q \wedge s_3) \vee (q \wedge t_3)$ and $q = (q \vee s_4) \wedge (q \vee t_4)$

for some $s_1, s_2, s_3, s_4 \in S$ and $t_1, t_2, t_3, t_4 \in T$.

Now, $p = (p \wedge s_1) \vee (p \wedge t_1)$

$\equiv (p \wedge n) \vee (p \wedge t_1)\ \theta(S)$

$\equiv (p \wedge n) \vee (p \wedge n)\ \theta(T)$

$= p \wedge n$.

Thus, $p \equiv p \wedge n\ (\theta(S) \vee \theta(T))$

Again, $p = (p \vee s_2) \wedge (p \vee t_2)$

$\equiv (p \vee n) \wedge (p \vee t_2)\ \theta(S)$

$\equiv (p \vee n) \wedge (p \vee n)\ \theta(T)$

$= p \vee n$.

Thus, $p \equiv p \vee n\ (\theta(S) \vee \theta(T))$. This implies

$p \wedge n \equiv p \vee n\ (\theta(S) \vee \theta(T))$

and so $p \equiv n\ (\theta(S) \vee \theta(T))$.

Similarly, we have $q \equiv n\ (\theta(S) \vee \theta(T))$.

Now, $y = (y \vee p) \wedge x$

$\equiv (y \vee n) \wedge x\ (\theta(S) \vee \theta(T))$

$= (y \wedge x) \vee (n \wedge x)$, as n is neutral.

$= y \vee (x \wedge n)$

$\equiv y \vee (x \wedge q)\ (\theta(S) \vee \theta(T))$

$= x$.

This implies $x \equiv y\ (\theta(S) \vee \theta(T))$.

Therefore, $\theta(S \vee T) = \theta(S) \vee \theta(T)$,

which proves (ii). ●

4.1.14. Lemma : Let s be a standard element of a lattice L and `a´ be an arbitrary element of L. Then $m(a, n, s)$ is standard in $\langle a \rangle_n$, where n is neutral in L.

Proof : Let $p, q \in \langle a \rangle_n$. Then $a \wedge n \leq p, q \leq a \vee n$.

Also $p = p \wedge (a \vee n) = (p \wedge a) \vee (p \wedge n)$, and

$q = q \wedge (a \vee n) = (q \wedge a) \vee (q \wedge n)$, as n is neutral. Let $r = m(a, n, s)$.

Now, $p \wedge (q \vee r) = p \wedge [\{(q \wedge a) \vee (q \wedge n)\} \vee \{(a \wedge n) \vee (a \wedge s) \vee (n \wedge s)\}]$

$= p \wedge [\{(q \wedge a) \vee (q \wedge n)\} \vee \{(a \wedge s) \vee (n \wedge s)\}]$, as $q \wedge a \geq a \wedge n$.

$= p \wedge [\{q \wedge (a \vee n)\} \vee \{s \wedge (a \vee n)\}]$

$= p \wedge (a \vee n) \wedge (q \vee s)$, as s is standard.

$= p \wedge (q \vee s)$, as $p \leq a \vee n$,

$= (p \wedge q) \vee (p \wedge s)$, as s is standard.

$= (p \wedge q) \vee (p \wedge s) \vee (a \wedge n)$ ..... (A)

Also, $p \wedge r = p \wedge m(a, n, s)$

$= p \wedge [(a \wedge n) \vee (a \wedge s) \vee (n \wedge s)]$

$= [p \wedge \{(a \wedge n) \vee (a \wedge s)\}] \vee (p \wedge n \wedge s)$, as $n \wedge s$ is standard.

$= [p \wedge \{a \wedge (n \vee s)\}] \vee (p \wedge n \wedge s)$, as s is standard.

$= (p \wedge a \wedge n) \vee (p \wedge a \wedge s) \vee (p \wedge n \wedge s)$

$= (p \wedge a \wedge n) \vee [(p \wedge s) \wedge (a \vee n)]$, as n is neutral.

$= (a \wedge n) \vee (p \wedge s)$.

Hence from (A), $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and so $r = m(a, n, s)$ is standard in $\langle a \rangle_n$. ●

2. Homomorphisms and Standard n-ideals.

4.2.1. According to Grätzer and Schmidt [18], we know that a standard ideal of a lattice is a homomorphism kernel, but the converse is not true in general. For an example they consider the following figure. In this lattice, the principal ideal (a] is a homomorphism kernel because it is a prime ideal, but it is not standard for

$$x \wedge (a \vee t) = x \text{ but } (x \wedge a) \vee (x \wedge t) = y.$$

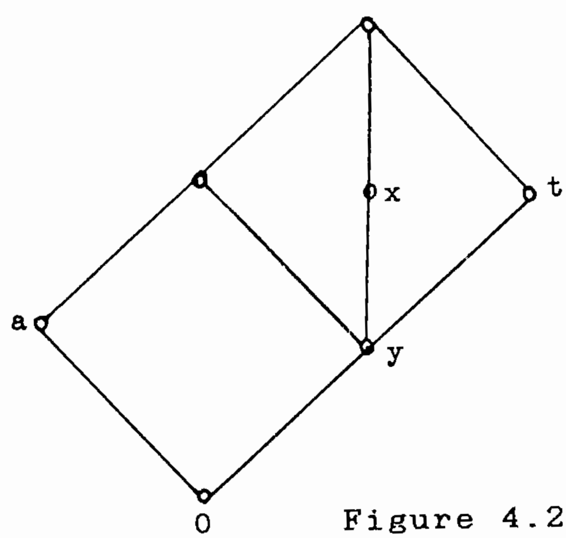

Figure 4.2.1.

In this section, we generalized their concepts to homomorphism n-kernels and standard n-ideals. Let Φ be a homomorphism of a lattice L, then

n-kernel $\Phi = \{x \in L : \Phi(x) = n\}$. Of course, if Φ is a homomorphism induced by the congruence relation Θ, then n-kernel $\Phi = \{x \in L : x \equiv n \ (\Theta)\}$.

It is already assured by corollary 4.1.5, that a standard n-ideal is a homomorphism n-kernel, where n is a neutral element of L. Considering n as the smallest element in figure 3, we find that the converse is not true in general. But the converse is true when L is a relatively complemented lattice. In this connection, we shall prove some of their theorems for standard n-ideals and finally we shall prove two isomorphism theorems for standard n-ideals.

4.2.2. Theorem : Let n be a neutral element of a lattice L with the property that both (n] and [n) are relatively complemented. Then every homomorphism n-kernel of L is a standard n-ideal and every standard n-ideal is the n-kernel of precisely one congruence relation.

Proof : Let I be the homomorphism n-kernel of L induced by the congruence relation θ. That is, $I = \{x \in L : x \equiv n\ \theta\}$. Clearly I is an n-ideal. We are to show that I is standard. Let $a \equiv b\ (\theta)$. Consider the interval $[n, a \vee b \vee n]$.
Now, $(a \wedge b) \vee n \in [n, a \vee b \vee n]$. Since [n) is relatively complemented so there exists

$r \in [n, a \vee b \vee n]$ such that $(a \wedge b) \vee n \vee r$ $= a \vee b \vee n$ and $((a \wedge b) \vee n) \wedge r = n$.

Since $a \equiv b\ (\theta)$ so we have $a \wedge b \equiv a \vee b\ (\theta)$. This implies $(a \wedge b) \vee n \equiv a \vee b \vee n\ (\theta)$. That is, $r \equiv n\ (\theta)$ and so $r \in I$.

$$\begin{aligned} \text{Now, } a \vee b \vee n &= (a \wedge b) \vee n \vee r = (a \wedge b) \vee r \\ &= (a \wedge b) \vee \{r \wedge (a \vee b \vee n)\} \\ &= (a \wedge b) \vee \{(r \wedge (a \vee b)) \vee (r \wedge n)\}, \\ &\qquad\qquad \text{as } n \text{ is neutral.} \\ &= (a \wedge b) \vee \{(a \vee b) \wedge r\} \vee n. \end{aligned}$$

$$\begin{aligned} \text{Also } a \vee b &= (a \vee b) \wedge (a \vee b \vee n) \\ &= (a \vee b) \wedge \{((a \vee b) \wedge r) \vee (a \wedge b) \vee n\} \\ &= [(a \vee b) \wedge \{((a \vee b) \wedge r) \vee (a \wedge b)\}] \\ &\qquad \vee ((a \vee b) \wedge n), \text{ as } n \text{ is neutral.} \\ &= ((a \vee b) \wedge r) \vee (a \wedge b) \vee ((a \vee b) \wedge n) \\ &= ((a \vee b) \wedge r) \vee (a \wedge b), \text{ where } r \in I. \end{aligned}$$

Again, consider the interval $[a \wedge b \wedge n, n]$.

Now, $(a \vee b) \wedge n \in [a \wedge b \wedge n, n]$. Since $(n]$ is relatively complemented, so there exists $s \in [a \wedge b \wedge n, n]$, such that $(a \vee b) \wedge n \wedge s$ $= a \wedge b \wedge n$ and $((a \vee b) \wedge n) \vee s = n$. Now, $a \wedge b \equiv$ $a \vee b\ (\theta)$ implies $a \wedge b \wedge n \equiv (a \vee b) \wedge n\ (\theta)$. Thus $s \equiv n\ (\theta)$ and so $s \in I$. Then by the dual proof of above it is not hard to show that

$a \wedge b = ((a \wedge b) \vee s) \wedge (a \vee b)$, where $s \in I$.

Thus, $a \equiv b\ (\theta)$ implies

$$a \vee b = ((a \vee b) \wedge r) \vee (a \wedge b)$$

and $a \wedge b = ((a \wedge b) \vee s) \wedge (a \vee b)$ for some $r, s \in I$. Hence by theorem 4.1.4. we have I is standard.

At the same time we have proved that if I is the homomorphism n-kernel of L induced by θ, then $\theta = \theta(I)$ which shows that every standard n-ideal is the homomorphism n-kernel of precisely one congruence relation. ●

4.2.3. Lemma : Let L be a relatively complemented lattice with 0 and 1 and n be neutral. Suppose $\langle s\rangle_n$ is a standard n-ideal, $s \in L$. If t is the complement of s, then $s \wedge n$, $t \wedge n$, $s \vee n$, $t \vee n$ are all neutral elements (and so they are central elements).

Proof : Since $\langle s\rangle_n$ is standard so by lemma 4.1.10. $s \vee n$ is standard and $s \wedge n$ is dual standard. Since L is relatively complemented so by [18 corollary 3, p-45] both $s \vee n$ and $s \wedge n$ are neutral and hence are central. Thus, $(s \vee n)' = s' \wedge n' = t \wedge n'$ and $(s \wedge n)' = s' \vee n' = t \vee n'$ are also central.

Since n is neutral, $t \vee n = (t \vee n) \wedge 1$

$$= (t \vee n) \wedge (n' \vee n)$$

$$= (t \wedge n') \vee n.$$

This implies $t \vee n$ is central.

Again as $s \wedge n$ is central, so $t \vee n' = s' \vee n'$ $= (s \wedge n)'$ is central. Therefore $t \wedge n = (t \wedge n) \vee 0$ $= (t \wedge n) \vee (n \wedge n') = n \wedge (t \vee n')$ is also central. Hence $s \wedge n$, $t \wedge n$, $s \vee n$, $t \vee n$ are all central . ●

In [18], authors proved that "In a relatively complemented lattice L with 0 and 1, C(L) is a boolean algebra if and only if every standard ideal of L is a principal ideal". The following theorem is a generalization of the above result :

4.2.4. Theorem : Let L be a relatively complemented lattice with 0 and 1. Then C(L) is a boolean algebra if and only if every standard n-ideal of L is a principal n-ideal.

Proof : Suppose every standard n-ideal of L is principal. Now, every congruence relation θ is of the form $\theta = \theta(S)$, where S is the n-kernel of the homomorphism induced by θ. Then by theorem 4.2.2.

S is a standard n-ideal. Since every standard n-ideal is principal, so $\Theta = \Theta(\langle s\rangle_n)$ for some $s \in L$. Then by Lemma-4.2.3., both $t \vee n$ and $t \wedge n$ are central, where t is the complemented of s. Thus by lemma 4.1.10, $\langle t\rangle_n$ is also standard. Hence by theorem 4.1.13., we have $\Theta(\langle s\rangle_n) \cap \Theta(\langle t\rangle_n) = \Theta(\langle s\rangle_n \cap \langle t\rangle_n)$

$$= \Theta((s \wedge n) \vee (t \wedge n), (s \vee n) \wedge (t \vee n))$$

$$= \Theta(n \wedge (s \vee t), n \vee (s \wedge t)),$$

as n is neutral.

$$= \Theta\ (n \wedge 1, n \vee 0)$$

$$= \Theta\ (n, n) = \omega.$$

Also, $\Theta(\langle s\rangle_n) \vee \Theta(\langle t\rangle_n) = \Theta(\langle s\rangle_n \vee \langle t\rangle_n)$

$$= \Theta(s \wedge t \wedge n, s \vee t \vee n)$$

$$= \Theta(0 \wedge n, 1 \vee n)$$

$$= \Theta(0, 1) = \iota,$$

which shows $\Theta(\langle t\rangle_n)$ is the complement of $\Theta(\langle s\rangle_n)$. Therefore, every congruence relation of C(L) has a complement. In other words C(L) is a Boolean algebra.

Conversely, suppose that C(L) is a Boolean algebra. By theorem 4.2.2, every congruence relation of L is of the form $\Theta(S)$, where S is a standard n-ideal. Suppose $\Theta(T)$ is the complement of $\Theta(S)$. Since C(L) is boolean, $\Theta(S)$ has a complement Φ. Then by 4.2.2. again, $\Phi = \Theta(T)$ for some standard n-ideal T.

Now, from theorem 4.1.13, we have

$$\theta(S \cap T) = \theta(S) \cap \theta(T) = \omega.$$

Also, $\theta(S \vee T) = \theta(S) \vee \theta(T) = \iota$.

Thus by theorem 4.2.2, $S \cap T = \{n\}$ and $S \vee T = L$. Since L has a unit element, so $L = \langle n' \rangle_n$, where n' is the complemented of n. So we have $S \cap T$ and $S \vee T$ are both principal n-ideals. Therefore S and T are principal n-ideals. This completes the proof. ●

In [18], Grätzer and Schmidt has proved two isomorphism theorems for standard ideals. In the next two theorems we give a generalization of their results in terms of standard n-ideals. For a standard n-ideal S of L, we denote the quotient lattice $L/\theta(S)$, simply by L/S.

4.2.5. Theorem : [First isomorphism theorem for standard n-ideals]. Let L be a lattice. Let S be a standard n-ideal and I be any n-ideal of L. Then $I \cap S$ is a standard n-ideal of I and

$$(I \vee S)/S \cong I/(I \cap S)$$

Proof : The first part has already been proved in lemma 4.1.8. For the second part, we use the first isomorphism theorem for Universal algebra. Then it remains to prove that every congruence class of $I \vee S$ may be represented by an element of I. So, let $x \in I \vee S$, then by theorem 4.1.4., we have

$$x = (x \wedge s_1) \vee (x \wedge a_1) = (x \vee s_2) \wedge (x \vee a_2),$$

for some $s_1, s_2 \in S$; $a_1, a_2 \in I$.

Without loss of generality we can chose $s_2 \leq n \leq s_1$ and $a_2 \leq n \leq a_1$.

Now, we have $s_1 \equiv s_2\ \theta(S)$, so $x \wedge s_1 \equiv x \wedge s_2\ \theta(S)$.

Then $x = (x \wedge s_1) \vee (x \wedge a_1)$

$\equiv (x \wedge s_2) \vee (x \wedge a_1)$

$= x \wedge a_1\ \theta(S)$.

Similarly, $x \equiv x \vee a_2\ \theta(S)$.

Let $y = (x \wedge a_1) \vee a_2$. Then $a_2 \leq y \leq a_1$, which implies $y \in I$ and $x \equiv x \vee a_2 \equiv (x \wedge a_1) \vee a_2$

$= y\ \theta(S)$.

That is, for any $x \in I \vee S$, there exists $y \in I$ such that $x \equiv y\ \theta\ (S)$. That is, $[x] = [y]\ \theta(S)$.

Therefore, $(I \vee S)/S \cong I/(I \cap S)$. ●

4.2.6. Theorem : [Second isomorphism theorem for standard n-ideals.] : Let L be a lattice. S be an n-ideal and T be a standard n-ideal of L such that $S \supseteq T$.

Then S is a standard n-ideal in L if and only if S/T is a standard [n]-ideal in L/T and in this case

$$L/S \cong \frac{L/T}{S/T} .$$

Proof : First suppose that S is a standard n-ideal of L. Let $\Phi : L \to L/\Theta(T)$ be the natural epimorphism. Then $x \to [x]\,\Theta(T)$ is homomorphism and onto. So by lemma 4.1.9, $\Phi(S)$ is a standard [n]-ideal of $L/\Theta(T)$. Now $\Phi(S) = S/\Theta(T) = S/T$. Hence S/T is a standard [n]-ideal of L/T.

Conversely, suppose that S/T is a standard [n]-ideal of L/T. We are to show that S is a standard n-ideal of L. Let us define a relation on S as follows :

$x \equiv y\ \Theta(S)$ defined by $x \wedge y = ((x \wedge y) \vee t) \wedge (x \vee y)$ and $x \vee y = ((x \vee y) \wedge s) \vee (x \wedge y)$,

for some $t, s \in S$.

We shall prove that $\Theta(S)$ is a congruence relation. Clearly $\Theta(S)$ is reflexive.

Now, let $x \geq y \geq z$ and $x \equiv y\ \Theta(S)$, $y \equiv z\ \Theta(S)$. Then from the proof of (b) $\Rightarrow$ (c) in theorem 4.1.4. we have $x \equiv z\ \Theta(S)$. For the substitution property, let $x \geq y$ with $x \equiv y\ \Theta(S)$ and $r \in L$.

Then $x = (x \wedge s) \vee y$ and $y = (y \vee t) \wedge x$

for some $s, t \in S$.

Now, $x \equiv y \;\Theta(S)$ implies $[x] \equiv [y] \;\Theta(S/T)$. Since S/T is standard, so $\Theta(S/T)$ is a congruence.

So, $[x] \wedge [r] \equiv [y] \wedge [r] \;\Theta(S/T)$. Since $[x] \wedge [r] \geq [y] \wedge [r]$ and S/T is standard in L/T, we have

$[y] \wedge [r] = (([y] \wedge [r]) \vee [s_1]) \wedge ([x] \wedge [r])$ ...(A)

and

$[x] \wedge [r] = (([x] \wedge [r]) \wedge [s_2]) \vee ([y] \wedge [r])$ ...(B)

for some $[s_1], [s_2] \in S/T$.

From (A) we get $y \wedge r \equiv ((y \wedge r) \vee s_1) \wedge (x \wedge r) \;\Theta(T)$.

Here $y \wedge r \leq ((y \wedge r) \vee s_1) \wedge (x \wedge r)$ and since T is standard in L, so we have

$y \wedge r = ((y \wedge r) \vee t) \wedge \{((y \wedge r) \vee s_1) \wedge (x \wedge r)\}$

for some $t \in T$.

$\geq ((y \wedge r) \vee (s_1 \wedge t)) \wedge x \wedge r \geq y \wedge r$.

This implies $y \wedge r = ((y \wedge r) \vee (s_1 \wedge t)) \wedge (x \wedge r)$.

Also from (B) we have

$[x] \wedge [r] = (([x] \wedge [r]) \wedge [s_2]) \vee ([y] \wedge [r])$

implies $x \wedge r \equiv ((x \wedge r) \wedge s_2) \vee (y \wedge r) \;\Theta(T)$.

Here $x \wedge r \geq ((x \wedge r) \wedge s_2) \vee (y \wedge r)$ and since T is standard in L, so we have

$x \wedge r = ((x \wedge r) \wedge t_1) \vee ((x \wedge r) \wedge s_2) \vee (y \wedge r)$

for some $t_1 \in T$.

$\leq \{(x \wedge r) \wedge (t_1 \vee s_2)\} \vee (y \wedge r) \leq (x \wedge r)$.

This implies $x \wedge r = ((x \wedge r) \wedge (t_1 \vee s_2)) \vee (y \wedge r)$. Hence $x \wedge r \equiv y \wedge r\ \theta(S)$, as $s_1 \wedge t \in S$ and $t_1 \vee s_2 \in S$. A dual proof will show that $x \vee r \equiv y \vee r\ \theta(S)$. Therefore $\theta(S)$ is a congruence relation and so by theorem 4.1.4. S is a standard n–ideal of L. ●

REFERENCES

[1] R. Balbaes and P. Dwinger, Distributive lattices, University of Missouripreea. Missouri (1974).

[2] R.A. Brooks and W.H. Cornish, Congruences on semilittices, Fund. Math.

[3] G. Birkhoff and S.A. Kiss, A ternary operation in a distributive lattices, Bull. Amer. Math. Soc. 53 ; 749-752 (1947).

[4] G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., 25, 3rd Edition, (1967).

[5] W.H. Cornish and A.S.A. Noor, Around a neutral element of a lattice. Comment. Math. Univ. Coroline, 28.2.(1987).

[6] W.H. Cornish, Pseudo-complemented modular semilattices, J. Austral. Math. Soc. 18(1974), 239-251.

[7] ————————, Characterizations of distributive and modular semilattices, Math. Japonica; 22(1977), 159-174.

[8] ————————, Normal lattices, J. Austral. Math. Soc. 14(1972), 200-215.

[9] ——————————, The kernels of skeletal congruences on a distributive lattice, Math. Nachr. 84(1978), 219-228.

[10] W.H.Cornish ;and R.C.Hicman,Weakly distributive semilattices, Acta Math.Acad.Sci. Hungar, 32(1978), 5-16.

[11] W.H.Cornish and A.S.A.Noor, standard elements in a nearlattice, Bull.Austral Math. Soc. 26(1982), 185-213.

[12] V.N. Dixit and D.K. Paliwal,On convex sublattices, Indian J.Pure appl. Math. 17(6)(1986), 769-777.

[13] ————————, Perspectivity and projectivity in NTC and W-modular lattices, Ind. J. Pure appl. Math.

[14] E.Fried and E.T. Schmidt, Standard sublattices, Alg.Univ. 5 (1975), 203-211.

[15] G.Grätzer, Lattice Theory. First concepts and distributive lattices, Freeman, san Francisco, 1971.

[16] ——————————, General lattice theory, Birkhauser verlag Basel (1978).

[17] ——————————, A characlerization of neutral elements in lattice (Notes on lattice theory 1), Magyar Tud. Akad.Math.Kutato Int. Kozl, 7(1962), 191-192.

[18] Grätzer and E.T. Sch midt Standard ideals in lattices,Acta Math. Acad.Sci.Hung. 12(1961), 17-86.

[19] ———— ,Ideals and congruence relations in lattices, Acta Math. Acad. Sci. Hung. 9(1958), 137-175.

[20] J.Hashimoto, A ternary operation in lattices, Math. Japonicae 2(1951), 49-52.

[21] ————, Ideal theory for lattices, Math. Japonicae 2(1952), 149-186.

[22] Iqbalunnisa, On neutral element in a lattice, J. Indian Math. Soc.(N.S) 28; (1964), 25-31.

[23] ————, On some problems of G. Gralzer and E.T. Schmidt, Fund. Math. 57(1965), 181-185.

[24] ————, Neutrality in weakly modular lattices, Acta Math. Acad. Sci. Hung. 16(1965a), 325-326.

[25] ————, Normal,Simple and neutral congruences on lattices, Illenois J. Math. 10(1966), 227-234.

[26] J.Jakubik and M.Kolibiar, On some properties of a pair of lattices (Russian), Czechosl. Math. J. 4(1954), 1-27.

[27] M.F. Janowitz, section semicomplemented lattices. Math. Zeit. 108(1968), 63-76.

[28] ——————, On a paper by Iqbalunnisa. Fund. Math. 78(1973), 177-182.

[29] ——————, A characterization of standard ideals, Acta Math. Acad. Sci. Hung. 16(1965), 289 -301.

[30] Latif and Noor, n-ideals of a lattice, submitted to the Rajshahi University studies.

[31] ——————, Two congruences corresponding to n-ideals in a distributive lattice, submitted to the Ganit, J. Bangladesh Mathematical Society.

[32] ——————, Finitily generated n-ideals of a lattice, submitted to the Indean Journal of Pure and Applied Mathematics.

[33] M.Mandelker, Relative annihilators in lattices, Duke Math. J.40 (1970), 377-386.

[34] A.S.A.Noor, Isotopes of nearlattices, Ganit (J. Bangladesh Math. Soc.) Vol.1, No.1 (1981), 63-72.

[35] ——————, A ternary operation in a medial nearlattice, The Rajshahi University studies, (part B)xiii : (1985), 89-96.

[36] A.S.A.Noor, Isotopes of Nearlattice Ph.D. Thesis Flinders University of South Australia,1980.

[37] J. Nieminen, Distributive, Standard and Neutral convex sublattices of a lattice, comment. Math. Univ. Sancti. pauli, Vol-33, No.1 (1984), 87-93.

[38] ——————, About generalized ideals in a distributive lattice, Manuscripta Math. Springer Verlag 7(1972), 13-21.

[39] ——————, On ideals and congruence relations on join-semilattices, Glasnik Math. Vol. 13(30) (1975), 223-230.

[40] D.Papert, Congruence relations in semilattices, J. London Math. Soc. 39(1964), 723-729.

[40]' Y.S.Pawar and N.K.Thakare, On prime semilattices, Canad. Math. Bull. Vol. 23(3) 1980, 291-298.

[41] P.V. Ramana Murty and V. Raman, Permutability of distributive congruence relations in a joint semilattice directed below, Math. slovaco 35. No.1. 43-49, 1985.

[42] J.B. Rhodes, modular and distributive semilattices, Trans. Amer. Math. Soc., 201(1975), 31-41.

[43] D.E. Rutherford- Introduction to lattice Theory, Oliver and Boyd, 1965.

[44] M. Sholander, Medians, lattices and trees, Proc. Amer. Math. Soc. 5(1954), 808-812.

[45] J.C. Varlet, Relative annihilators in semilattices, Bull. Austral. Math. Soc. 9(1973), 169-185.