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Spinning Particles in Curved Spacetime of General Relativity

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SPINNING PARTICLES IN CURVED SPACETIME OF GENERAL RELATIVITY

BY

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Certified that the thesis entitled "Spinning Particles in Curved Spacetime of General Relativity" submitted by Mr. M. Hossain Ali in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, University of Rajshahi, Rajshahi, has been completed under my supervision. I believe that this research work is an original one and it has not been submitted elsewhere for any degree.

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DEDICATED TO MY PARENTS

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NOTATION AND CONVENTION

Units are chosen such that the Newtonian gravitational constant G, the speed of light c, and $\hbar = h/2\pi$ with h the Planck's constant are equal to unity:

$$G = c = \hbar = 1.$$

Round brackets indicate full symmetrization, while square brackets denote full anti-symmetrization over the indices enclosed:

$$A_{(ij)} = \frac{1}{2!}\left(A_{ij} + A_{ji}\right)$$

$$A_{[ijk]} = \frac{1}{3!}\left(A_{ijk} - A_{ikj} + A_{jki} - A_{jik}\right)$$

A comma, a semicolon, and a dot respectively denote a directional derivative, a covariant derivative, and a differentiation with respect to the argument.

SUSY	stands for	Supersymmetry,
NUT	stands for	Newman-Unti-Tamburino,
RN	stands for	Reissner-Nordstrom,
NUT-RN	stands for	NUT-Reissner-Nordstrom,
NUT-KN	stands for	NUT-Kerr-Newman,
HNUTKNK	stands for	Hot NUT-Kerr-Newman Kasuya,
BRST	stands for	Becchi-Rouet Stora-Tyutin,
QFTs	stands for	Quantum Field Theories.

ABSTRACT

This thesis is organized as follows.

In Introduction we give a brief account of our work of studying spinning particles in curved spacetime of General Relativity.

In Chapter I we discuss the relevant equations for the motion of spinning particles in curved spacetime. We present the generalized Killing equations for spinning space and describe the constants of motion.

In Chapter II we derive the constants and the equations of motion for spinning particles moving in the Schwarzschild spacetime. We discuss various types of orbits and describe exact solutions in a plane.

In Chapter III we extend the work of Chapter II in the Reissner-Nordstrom spacetime, which is the Schwarzschild spacetime generalized with a charged parameter and then further extend this work in the Reissner-Nordstrom spacetime generalized with a NUT (or magnetic mass) parameter. In the Reissner-Nordstrom spacetime we investigate the motion of spinning particles on a plane for bound state orbits, while in the NUT-Reissner-Nordstrom spacetime we analyze the motion on a cone and on a plane.

In Chapter IV we study "nongeneric" supersymmetries of spinning particles in a curved spacetime. We present a general analysis of the conditions under which this type of supersymmetries appear, and describe Poisson-Dirac algebra of the resulting set of charges.

All members of the Kerr-Newman family of black-hole solutions admit a "nongeneric" supersymmetry. In Chapter V we describe this new supersymmetry along with the corresponding conserved quantity in the Kerr-Newman spacetime.

In Chapter VI we extend the work of Chapter V in the NUT-Kerr-Newman spacetime, which includes the Kerr-Newman black-Hole spacetime as well as NUT spacetime, which is sometimes considered as unphysical.

In Chapter VII we extend further the work of Chapter VI in the NUT-Kerr-Newman spacetime generalized with an extra magnetic monopole charge and a cosmological constant.

Finally, we present a discussion on our study at the end of this thesis.

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INTRODUCTION

In recent years there has been a renewed interest in the study of the motion of spinning particles in curved spacetime. The action of spin-*1/2* relativistic particle with spinning degrees of freedom characterized by anti-commuting Grassmann [1, 2] variables was first proposed by Berezin and Marinov [3, 4] and soon after that was discussed and investigated by many authors [5–14].

In spite of the fact that the anti–commuting Grassmann variables do not admit a direct classical interpretation, the Lagrangians for these models have a natural interpretation in the context of the path-integral description of the quantum dynamics. The pseudo-classical equations of motion acquire physical meaning when averaged over the inside of the functional integral [3, 4, 25]. In the semi-classical (classical metric with quantized matter) regime, neglecting higher order quantum correlations, it should be admissible to replace appropriate combinations of Grassmann spin–variables by real numbers, to obtain the corresponding quantum mechanical expectation values. These ideas have been used to study the motion of spinning particles in external fields in refs. [3, 4, 16–26].

In addition to such direct physical applications, generalizations of Riemannian Geometry based on anti–commuting variables have been found to be of wide mathematical interest. This interest is mainly raised by supersymmetry [27–44] and supergravity [38–53] –local version of supersymmetry – theories.

Supersymmetric point particle mechanics has found applications in the area of index theorem; for example, Witten's index [54], which exists in the context of $N \geq 1$ supersymmetric quantum field theories (QFTs) in any dimension, is an effective tool in addressing questions of supersymmetry breaking, while the supersymmetric index for $N=2$ supersymmetric QFTs in two dimensions is related to the geometry of the vacua [55,56]. In addition, the BRST (Bechhi–Rouet–Stora–Tyutin) [57, 58] methods, in which the gauge symmetry is replaced by a global fermionic symmetry in an extended phase space consisting of the original phase space together with the ghost variables and their momenta, are widely used in the study of topological invariants. For all of these reasons, the study of the geometry of graded pseudo–manifolds with both real number and Grassmann coordinates is well justified.

Recently, Rietdijk [59], Rietdijk *et al.* [60] and van Holten *et al.* [61] investigated the general relations between symmetries of graded pseudo-manifolds and constants of motion for spinning point particles in detail. These methods may be applied to any spacetime. More recently, Rietdijk and van Holten [62] studied spinning point particles in the Schwarzschild spacetime. Visinescu [63, 64], Vaman and Visinescu [65–67], van Holten [68] and Baleanu [69–72] investigated pseudo-classical spinning particles in the Taub–NUT spacetime.

In this thesis we would like to study spinning particles in the black–hole spacetime as well as in the spacetimes which are not black–hole spacetimes but have common feature with the black–hole spacetimes that they have horizons. We arrange our study in seven chapters as follows.

In Chapter I we would like to summarize the relevant equations for motion of pseudo–classical spinning point particles in curved spacetime along with their physical interpretation. We apply the generalized Killing equations derived in [60] for spinning space to describe constants of motion.

In Chapter II we would like to review the work of Rietdijk and van Holten [62] of investigating the motion of pseudo–classical spinning point particles in the Schwarzschild spacetime. We describe the full set of first integrals of motion and present an exact solution for planar orbits.

In Chapter III we extend the work of Chapter II to the Reissner–Nordstrom spacetime [73], which is the Schwarzschild spacetime generalized with a charged parameter, and then further extend this work to the Reissner–Nordstrom spacetime generalized with a NUT parameter, which has the interpretation of a gravitational magnetic monopole [74 – 79]. Our work may be interesting from the point of view of $N=2$ supergravity as well as string theory. Spacetime supersymmetry has previously been applied to charged black hole in the context of $N=2$ supergravity theory [80]. The string theory gives birth to general relativity at the linearized level in the low energy limit [81]. The NUT-Reissner-Nordstrom spacetime includes the NUT spacetime, which is sometimes considered as unphysical. According to Misner [82], the NUT spacetime is one which does not admit an interpretation without a periodic time coordinate, a spacetime without reasonable spacelike surfaces, and an asymptotically zero curvature spacetime, which apparently does not admit asymptotically rectangular coordinates. McGuire and Ruffini [83] suggested that the spacetimes endowed with NUT parameter should never be directly physically interpreted. So the study of pseudo-classical spinning particles in the NUT-Reissner-Nordstrom spacetime is interesting.

Spinning particles in curved spacetimes can have "nongeneric" supersymmetries (SUSYs) [80, 84–86]. The appearance of such SUSYs depends on the specific form of the metric tensor. In ref. [80] Gibbons *et al.* presented a systematic analysis of investigating "nongeneric" SUSYs of classical spacetime in terms of the motion of pseudo-classical spinning point particles in a curved Lorentzian manifold. This new SUSY is generated by the square root of bosonic constants of motion other than the Hamiltonian. We would like to review the work of Gibbons *et al.* [80] in Chapter IV. We summarize the formalism of pseudo-classical spinning point particles in an arbitrary background spacetime and describe "nongeneric" SUSY along with the other (universal) symmetries and give their algebras.

The existence of a "nongeneric" SUSY is closely related to the appearance of Killing–Yano tensors [87] of the bosonic manifold. Although there are not so many physically interpretable spacetimes in which Killing-Yano tensors exist [88–90], the Kerr–Newman [80, 91] and Taub-NUT [64, 68, 72] spacetimes admit

Killing–Yano tensors and hence they have "nongeneric" SUSYs. In Chapter V we would like to review the work of investigating "nongeneric" SUSY and the corresponding conserved quantity in the Kerr–Newman spacetime [80].

In Chapter VI we would like to extend the work of Chapter V to the Kerr–Newman spacetime generalized with a NUT (or magnetic mass) parameter [92]. This study is interesting in that the spacetime contains the unphysical NUT spacetime.

In Chapter VII we further extend the work of Chapter VI to the NUT-Kerr-Newman-Kasuya-de Sitter spacetime, which is the NUT-Kerr-Newman spacetime generalized with an extra magnetic monopole charge and a cosmological constant. The monopole hypothesis was propounded by Dirac relatively long ago. The ingenious suggestion by Dirac that magnetic monopole does exist was neglected due to the failure to detect such particle. However, in recent years the development of gauge theories has shed new light on it. On the other hand, in recent years there has been a renewed interest in cosmological constant as the cosmological constant is found to be present in the inflationary scenario of the early universe. In this scenario the universe undergoes a stage where it is geometrically similar to de Sitter spacetime [93]. Among other things inflation has led to the cold dark matter. According to cold dark matter theory, the bulk of the dark matter is in the form of slowly moving particles (axions or neutralinos). If the cold dark matter theory proves correct, it would shed light on the unification of forces [94, 95]. In view of these interests in the cosmological constant and because of the presence of magnetic monopole charge our study of this Chapter is interesting. Since the de Sitter spacetime has been interpreted as being hot [96], we shall call the spacetime the hot NUT-Kerr-Newman-Kasuya (H-NUT-KN-K) spacetime.

Finally, we present a discussion on our work at the end of this thesis.

CHAPTER I

MOTIONS OF SPINNING PARTICLES IN CURVED SPACETIME

1.1. INTRODUCTION

The configuration space of spinning particles (spinning space) is an extension of a (pseudo-) Euclidean manifold, described by local coordinates $\{x^{\mu}\}$, to a graded manifold described by local graded coordinates $\{x^{\mu}, \psi^{\mu}\}$ with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anti-commuting). Geodesic flow along time-like curves of such a graded manifold with Minkowskian signature $(+\,-\,-\,-)$ describes the classical limit of the motion of a relativistic point-like Dirac particle, which carries a spin $s=\hbar/2$ in quantum mechanics [4, 7–9, 14].

We arrange this chapter as follows. In section 1.2 we summarize the relevant equations for the motion of spinning particles in curved spacetime, and briefly discuss their physical interpretation. In section 1.3 we discuss the relation between symmetries of the spinning space and conservation laws, and describe the generalized Killing equations for spinning space. In section 1.4 we discuss the derivation of "generic" constants of motion in terms of the solutions of the generalized Killing equations.

1.2. SPINNING SPACE

Einstein's theory of gravity suggests that the world-lines of classical point particles in a curved spacetime are time-like geodesics. As geodesics are curves of extremal length, the equation for the world-line of a point particle can be derived from an action principle, with the action any smooth monotonic function of the spacetime interval along the curve

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = -d\tau^2. \qquad (1.2.1)$$

Here, $d\tau$ is the corresponding interval of proper time. The last equality holds only in the absence of external forces, like electromagnetic dipole forces [24].

In spinning space the additional fermionic dimensions are characterized by vectorial Grassmann coordinates ψ^μ. Since the number of bosonic and fermionic dimensions is the same, and the coordinates ψ^μ transform as 1-forms dx^μ, there can exist a supersymmetry in the geometry of the graded manifolds, which relates each x^μ with the corresponding ψ^μ, according to

$$\delta x^\mu = -i \in \psi^\mu, \qquad \delta\psi^\mu = \in \dot{x}^\mu , \qquad (1.2.2)$$

where the overdot denotes a derivative with respect to proper time, $d/d\tau$. A manifestly supersymmetric action that defines the extremal trajectories ("geodesics") of spinning space is given by

$$S = m \int_1^2 d\tau \left(\frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \frac{D\psi^\nu}{D\tau} \right) , \tag{1.2.3}$$

where the constant m has the dimension of mass. In the following we consider particles of unit mass: $m = 1$, but occasionally we re-instate the explicit mass dependence when this is physically relevant. The capital derivative denotes covariant derivative with respect to proper time and it is defined by

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma^\mu_{\lambda\,\nu} \psi^\nu . \tag{1.2.4}$$

Under a general variation of the co-ordinates (δx^μ, $\delta\,\psi^\mu$) the action S changes by

$$\delta S = \int_1^2 d\tau \left\{ - \delta x^\mu \left(g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} + \frac{i}{2} \psi^\kappa \psi^\lambda R_{\kappa\lambda\mu\nu} \dot{x}^\nu \right) \right.$$
$$\left. + i \Delta\psi^\mu g_{\mu\nu} \frac{D\psi^\nu}{D\tau} + \frac{d}{d\tau} \left(\delta x^\mu p_\mu - \frac{i}{2} \delta\psi^\mu g_{\mu\nu} \psi^\nu \right) \right\}, \tag{1.2.5}$$

where the canonical momentum is

$$p_\mu = g_{\mu\nu}\, \dot{x}^\nu - \frac{1}{2} i\, \Gamma_{\mu\kappa\lambda} \psi^\kappa \psi^\lambda ; \tag{1.2.6}$$

and $R_{\kappa\lambda\mu\nu}$ is the Riemann curvature tensor, while $\Delta\psi^\mu$ is the covariantized variation of ψ^μ:

$$\Delta\psi^\mu = \delta\psi^\mu + \delta x^\lambda\, \Gamma^\mu_{\lambda\,\nu} \psi^\nu . \tag{1.2.7}$$

The trajectories, which make the action S stationary under arbitrary variations δx^{μ} and $\delta\psi^{\mu}$ vanishing at the end points, are given by

$$\frac{D^2x^{\mu}}{D\tau^2} = \ddot{x}^{\mu} + \Gamma^{\mu}_{\lambda\,\nu}\,\dot{x}^{\lambda}\dot{x}^{\nu} = -\frac{1}{2}i\psi^{\kappa}\psi^{\lambda}R_{\kappa\lambda}{}^{\mu}{}_{\nu}\,\dot{x}^{\nu}\,, \tag{1.2.8}$$

$$\frac{D\psi^{\mu}}{D\tau} = 0. \tag{1.2.9}$$

The solutions of equations (1.2.8) and (1.2.9) for $x^{\mu}(\tau)$, with replacing ψ^{μ} by zero everywhere, give ordinary geodesics in the bosonic submanifold.

More interesting solutions are those for which one or more components ψ^{μ} are different from zero. We briefly discuss the physical interpretation of such solutions in the following. The anti-symmetric tensor

$$S^{\mu\nu} = -i\psi^{\mu}\psi^{\nu} \tag{1.2.10}$$

describes the relativistic spin of the particle [4, 7, 12, 16, 24, 25], and correspondingly equations (1.2.8) and (1.2.9) describe the classical motion of a Dirac particle. Equation (1.2.8) then becomes

$$\frac{D^2x^{\mu}}{D\tau^2} = \frac{1}{2}S^{\kappa\lambda}R_{\kappa\lambda}{}^{\mu}{}_{\nu}\,\dot{x}^{\nu}. \tag{1.2.11}$$

It implies the existence of a spin-dependent gravitational force [16, 17, 20, 24–26] analogous to the electromagnetic Lorentz force,

$$\ddot{x}^{\mu} = \frac{q}{m} F^{\mu}{}_{\nu} \dot{x}^{\nu}, \tag{1.2.12}$$

with the spin-polarization tensor replacing the scalar electric charge (here for unit mass). Equation (1.2.9) suggests that the spin is covariantly constant:

$$\frac{DS^{\mu\nu}}{D\tau} = 0. \tag{1.2.13}$$

The space-like components S^{ij} of the spin tensor are proportional to the particle's magnetic dipole moment, while the time-like components S^{i0} represent the electric dipole moment. For free Dirac particles like free electrons and quarks (the ultimate constituents of hadrons) the time-like spin components (electric dipole moment) vanish in the rest frame. This gives a covariant constraint

$$g_{\nu\lambda}(x) S^{\mu\nu} \dot{x}^{\lambda} = 0, \tag{1.2.14}$$

which, in terms of the Grassmann coordinates, is equivalent to

$$g_{\mu\nu}(x) \dot{x}^{\mu} \psi^{\nu} = 0. \tag{1.2.15}$$

This constraint has an elegant interpretation in the supersymmetry analysis. We shall return to this point in section 1.4 below.

An interesting consequence of equation (1.2.11) is that for real $S^{\mu\nu}$ one can measure the spin directly via its coupling to the gravitational field, instead of indirectly by determining its associated electromagnetic dipole moments.

1.3. SYMMETRIES AND GENERALIZED KILLING EQUATIONS

In classical mechanics there is a well-known relation between symmetries of the configuration space and conservation laws, as expressed by Noether's theorem. In case of a scalar point particle, moving in an arbitrary curved spacetime, this relation can be summarized as follows. The geodesic law of motion of the particle can be derived from an action principle. The simplest form of the action is

$$S = \int_1^2 d\tau \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu. \tag{1.3.1}$$

If this action be invariant under the transformations

$$\delta x^\mu = \mathscr{R}^\mu(x, \dot{x}) = R^\mu(x) + \dot{x}^\nu K^\mu{}_\nu(x) + \frac{1}{2}\dot{x}^\nu \dot{x}^\lambda L^\mu{}_{\nu\lambda}(x) + \ldots, \tag{1.3.2}$$

then the quantity

$$\mathscr{J}(x, \dot{x}) = J^{(0)}(x) + \dot{x}^\mu J^{(1)}_\mu(x) + \frac{1}{2}\dot{x}^\mu \dot{x}^\nu J^{(2)}_{\mu\nu}(x)$$

$$+ \frac{1}{3!}\dot{x}^\mu \dot{x}^\nu \dot{x}^\lambda J^{(3)}_{\mu\nu\lambda}(x) + \ldots \tag{1.3.3}$$

is a constant of motion [61]. The necessary and sufficient conditions for this are that the differential equations,

$$J^{(n)}_{(\mu_1 \ldots \mu_n ; \mu_{n+1})} = 0, \tag{1.3.4}$$

in which the parentheses denote full symmetrization over all indices enclosed. with total weight one, have to be satisfied for

$$J^{(1)}_{\mu}(x) = R_{\mu}(x),$$

$$J^{(2)}_{\mu\nu}(x) = K_{\mu\nu}(x),$$

$$J^{(3)}_{\mu\nu\lambda}(x) = L_{\mu\nu\lambda}(x), etc. \tag{1.3.5}$$

Explicit forms of the equations in (1.3.4) are

$$J^{(0)}_{,\mu} = 0, \tag{1.3.6}$$

$$R_{(\mu;\nu)} = 0, \tag{1.3.7}$$

$$K_{(\mu\nu;\lambda)} = 0, etc. \tag{1.3.8}$$

Equation (1.3.6) implies that $J^{(0)}$ is an irrelevant constant. Equation (1.3.7) is the standard equation for Killing vectors, while equation (1.3.8) and its higher-rank counterparts constitute tensorial generalizations of this equation. Because of this, one refers to $K_{\mu\nu}$ and higher-rank tensors satisfying (1.3.4) as Killing tensors.

We now generalize the above statements to the graded configuration space (spinning space) of spinning point particles. For this purpose it is necessary to consider specific variations δx^μ and $\Delta\psi^\mu$ which leave the action (1.2.3) invariant modulo boundary terms. Let us take the variations to be of the form

$$\delta x^\mu = \mathscr{R}^\mu(x,\dot{x},\psi) = R^{(1)\mu}(x,\psi) + \sum_{n=1}^{\infty} \frac{1}{n!}\dot{x}^{\nu_1}\ldots\dot{x}^{\nu_n} R^{(n+1)\mu}_{\nu_1\ldots\nu_n}(x,\psi),$$

$$\Delta\psi^\mu = \mathscr{S}^\mu(x,\dot{x},\psi) = S^{(0)\mu}(x,\psi) + \sum_{n=1}^{\infty} \frac{1}{n!}\dot{x}^{\nu_1}\ldots\dot{x}^{\nu_n} S^{(n)\mu}_{\nu_1\ldots\nu_n}(x,\psi) \qquad (1.3.9)$$

and the Lagrangian transform into a total derivative

$$\delta S = \int_1^2 d\tau \frac{d}{d\tau}\left(\delta x^\mu \wp_\mu - \frac{i}{2}\delta\psi^\mu g_{\mu\nu}\psi^\nu - \mathscr{J}(x,\dot{x},\psi)\right), \qquad (1.3.10)$$

where $\wp_\mu$ is the canonical momentum conjugate to x^μ, defined in (1.2.6). Then it follows that

$$\frac{d\mathscr{J}}{d\tau} = \mathscr{R}^\mu\left(g_{\mu\nu}\frac{D^2x^\nu}{D\tau^2} + \frac{i}{2}\psi^\kappa\psi^\lambda R_{\kappa\lambda\mu\nu}\dot{x}^\nu\right) + \mathscr{S}^\mu g_{\mu\nu}\frac{D\psi^\nu}{D\tau}. \qquad (1.3.11)$$

If the equations of motion (1.2.10), (1.2.11) are satisfied, the right-hand side of (1.3.11) vanishes. Hence the quantity $\mathscr{J}$ is conserved. This is Noether's theorem.

If we expand $\mathscr{J}(x, \dot{x}, \psi)$ in terms of the four-velocity,

$$\mathscr{J}(x, \dot{x}, \psi) = J^{(0)}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\mu_1} \ldots \dot{x}^{\mu_n} J^{(n)}_{\mu_1 \ldots \mu_n}(x, \psi), \tag{1.3.12}$$

and compare the left- and right-hand sides of equation (1.3.11) with the ansatz (1.3.9) for δx^{μ} and $\Delta \psi^{\mu}$, then the result gives the following identities:

$$J^{(n)}_{\mu_1 \ldots \mu_n}(x, \psi) = R^{(n)}_{\mu_1 \ldots \mu_n}(x, \psi), \, n \geq 1, \tag{1.3.13}$$

and

$$S^{(n)}_{\mu_1 \ldots \mu_n \nu}(x, \psi) = i \frac{\partial J^{(n)}_{\mu_1 \ldots \mu_n}}{\partial \psi^{\nu}}(x, \psi), \; n \geq 0. \tag{1.3.14}$$

These quantities satisfy a generalization of the Killing equations of the form [60]

$$J^{(n)}_{(\mu_1 \ldots \mu_n ; \mu_{n+1})} + \frac{\partial J^{(n)}_{(\mu_1 \ldots \mu_n}}{\partial \psi^{\sigma}} \Gamma_{\mu_{n+1})\kappa}{}^{\sigma} \psi^{\kappa} = \frac{i}{2} \psi^{\kappa} \psi^{\lambda} R_{\kappa \lambda \nu (\mu_{n+1}} J^{(n+1)}_{\mu_1 \ldots \mu_n)}{}^{\nu}. \tag{1.3.15}$$

If one substitutes as before $R^{(1)}_{\mu} = R_{\mu}, R^{(2)}_{\mu\nu} = K_{\mu\nu}, R^{(3)}_{\mu\nu\lambda} = L_{\mu\nu\lambda}, etc.$, and $J^{(0)} = B$, then this reduces for the lowest components to

$$B_{,\mu} + \frac{\partial B}{\partial \psi^{\sigma}} \Gamma^{\sigma}_{\mu\kappa} \psi^{\kappa} = \frac{i}{2} \psi^{\rho} \psi^{\sigma} R_{\rho\sigma\kappa\mu} R^{\kappa}, \tag{1.3.16}$$

$$R_{(\mu ; \nu)} + \frac{\partial R_{(\mu}}{\partial \psi^{\sigma}} \Gamma_{\nu)\kappa}{}^{\sigma} \psi^{\kappa} = \frac{i}{2} \psi^{\rho} \psi^{\sigma} R_{\rho\sigma\kappa(\mu} K_{\nu)}{}^{\kappa}, \tag{1.3.17}$$

$$K_{(\mu\nu;\lambda)} + \frac{\partial K_{(\mu\nu}}{\partial \psi^{\sigma}} \Gamma_{\lambda)\kappa}{}^{\sigma} \psi^{\kappa} = \frac{i}{2} \psi^{\rho} \psi^{\sigma} R_{\rho\sigma\kappa(\mu} L_{\nu\lambda)}{}^{\kappa}, etc. \tag{1.3.18}$$

These equations hold independently of the equations of motion. In the purely bosonic (ψ-independent) case, these equations reduce to equations (1.3.6)–(1.3.8). We note that, contrary to the bosonic case, the Killing scalar $B(x,\psi) = J^{(0)}(x,\psi)$ is not always an irrelevant constant, because it can depend non-trivially on x^{μ} and ψ^{μ}, as described in the equation (1.3.16).

1.4. GENERIC SOLUTIONS

In contrast to scalar particles, spinning particles admit several conserved quantities of motion in a curved spacetime with metric $g_{\mu\nu}(x)$. We can divide them into two classes. First, there are conserved quantities which exist in any theory, even though the specific functional form may depend on the metric $g_{\mu\nu}(x)$; these are called "generic" constants of motion. The second kind of conserved quantities result from the specific form of the metric $g_{\mu\nu}(x)$. These are model-dependent, and hence are called "nongeneric".

For spinning-particle models as defined by the action (1.2.3) there exist four independent "generic" constants of motion [60, 61]. We are going to describe them in the following:

1. Similar to the bosonic case the metric $g_{\mu\nu}(x)$ itself is a Killing tensor:

$$K_{\mu\nu} = g_{\mu\nu}, \tag{1.4.1}$$

with all other Killing vectors and tensors (bosonic as well as fermionic) equal to zero. The corresponding constant of motion is the world-line Hamiltonian

$$H(x,P) = \frac{1}{2} g^{\mu\nu} P_\mu P_\nu, \tag{1.4.2}$$

where P_μ is the covariant momentum defined by

$$P_\mu = p_\mu + \frac{1}{2} i\Gamma_{\mu\kappa\lambda} \psi^\kappa \psi^\lambda. \tag{1.4.3}$$

2. The Grassmann-odd Killing vectors

$$R^\mu = \psi^\mu, \qquad \mathrm{T}_\mu^{\ \nu} = i\delta_\mu^{\ \nu}, \tag{1.4.4}$$

provide another obvious solution. Here again all other Killing vectors and tensors are equal to zero. This solution gives the supercharge

$$Q = P_\mu \psi^\mu. \tag{1.4.5}$$

3. In addition, the spinning particle action has a second non-linear supersymmetry, generated by Killing vectors

$$R_\mu = \frac{-i^{[d/2]}}{(d-1)!} \sqrt{-g}\, \varepsilon_{\mu\nu_1 \ldots \nu_{d-1}} \psi^{\nu_1} \ldots \psi^{\nu_{d-1}},$$

$$T_{\mu\nu} = \frac{i^{[(d-2)/2]}}{(d-2)!}\sqrt{-g}\,\varepsilon_{\mu\nu\nu_1\ldots\nu_{d-2}}\psi^{\nu_1}\ldots\psi^{\nu_{d-2}}. \qquad (1.4.6)$$

Obviously, the Grassmann parities of $(R_\mu, T_{\mu\nu})$ depend on d, the number of spacetime dimensions. The corresponding constant of motion is the dual supercharge

$$Q^* = \frac{-i^{[d/2]}}{(d-1)!}\sqrt{-g}\,\varepsilon_{\mu_1\ldots\mu_d}P^{\mu_1}\psi^{\mu_2}\ldots\psi^{\mu_d}. \qquad (1.4.7)$$

4. Finally, there is a non-trivial Killing scalar

$$\Gamma_* \equiv J^{(0)} = \frac{-i^{[d/2]}}{d!}\sqrt{-g}\,\varepsilon_{\mu_1\ldots\mu_d}\psi^{\mu_1}\ldots\psi^{\mu_d}, \qquad (1.4.8)$$

which acts as the Hodge star duality operator on ψ^μ. In quantum mechanics it becomes the γ^{d+1} element of the Dirac algebra. Because of this reason Γ_* is referred to as the chiral charge.

From the fundamental Dirac brackets,

$$\{x^\mu, p_\nu\} = \delta^\mu{}_\nu,$$

$$\{\psi^\mu, \psi^\nu\} = -i\,g^{\mu\nu},$$

$$\{p_\mu, \psi^\nu\} = \frac{1}{2}g^{\kappa\nu}g_{\kappa\lambda,\mu}\psi^\lambda,$$

$$\{P_\mu, P_\nu\} = -\frac{1}{4} i g^{\kappa\nu} g_{\kappa\rho,\mu} g_{\lambda\sigma,\nu} \psi^\rho \psi^\sigma, \tag{1.4.9}$$

one can find the following non-trivial Dirac brackets between these universal constants of motion:

$$\{Q, Q\} = -2iH, \qquad \{Q, \Gamma_*\} = -iQ^*. \tag{1.4.10}$$

For $d=2$, Q^* becomes linear and acts as an ordinary supersymmetry:

$$\{Q^*, Q^*\} = -2iH, \qquad \{Q^*, \Gamma_*\} = -iQ. \tag{1.4.11}$$

This implies that the theory in two dimensions possesses an $N=2$ supersymmetry. In the case of $d\neq 2$, the right-hand side of equations (1.4.11) is to be replaced by zero.

The conservation of the supercharge Q is actually crucial for the consistency of the physical interpretation of the theory. In fact, the condition for the absence of an intrinsic electric dipole moment of physical fermions like leptons (the lighter particles such as electron muon, tau, *etc.*) and quarks as formulated in (1.2.15) becomes

$$Q=0. \tag{1.4.12}$$

Since Q is a conserved quantity, condition (1.4.12) can be satisfied at all times, irrespective of the presence of external fields, and at the same time it provides a clear physical interpretation of world-line supersymmetry.

CHAPTER II

SPINNING PARTICLES IN SCHWARZSCHILD SPACETIME

2.1. INTRODUCTION

The Schwarzschild spacetime is the simplest of all black-hole spacetimes. It describes a static, spherically symmetric gravitational field, which is asymptotically flat. Recently Rietdijk and van Holten [62] studied pseudo-classical spinning point particles in the Schwarzschild spacetime. We would like to review their work in the present chapter.

We arrange this chapter as follows. In section 2.2 we derive the equations of motion of a spinning particle moving in the Schwarzschild spacetime. In section 2.3 we discuss specific solutions, and derive an exact equation for the precession of the perihelion of planar orbits. In section 2.4 we present a discussion on the results.

2.2. SPINNING SCHWARZSCHILD SPACETIME

In this section we apply the results of Chapter I to the case of a spinning particle moving in the Schwarzschild spacetime which has the metric [97].

$$ds^2 = -\left(1-\frac{\alpha}{r}\right)dt^2 + \frac{1}{\left(1-\frac{\alpha}{r}\right)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \qquad (2.2.1)$$

where $\alpha=2M$ is the Schwarzschild radius, with M the total mass of the object which is the source of the field. The metric possesses four Killing vector fields of form

$$D^{(\beta)} \equiv R^{(\beta)\mu}(x)\partial_\mu\,, \quad \beta=0,\ldots,3, \qquad (2.2.2)$$

where

$$D^{(0)} = \frac{\partial}{\partial t}, \qquad (2.2.3a)$$

$$D^{(1)} = -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}, \qquad (2.2.3b)$$

$$D^{(2)} = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}, \qquad (2.2.3c)$$

$$D^{(3)} = \frac{\partial}{\partial\varphi}. \qquad (2.2.3d)$$

These Killing vector fields generate the Lie algebra $O(1, 1) \times SO(3)$:

$$\left[D^{(i)}, D^{(j)}\right] = -\varepsilon^{ijk}D^{(k)}, \qquad (i, j, k = 1, 2, 3). \qquad (2.2.4a)$$

$$\left[D^{(0)}, D^{(i)}\right] = 0, \qquad (2.2.4b)$$

and describe the time-translation invariance and the spatial rotation symmetry of the gravitational field.

The first generalized Killing equation (1.3.16) shows that for each Killing vector, $R_{\mu}^{(\beta)}(x)$, there is an associated Killing scalar, $B^{(\beta)}(x,\psi)$. Therefore, if we limit ourselves to variations (1.3.9) which terminate after the terms linear in $\dot{x}^{\mu}$, we obtain the constants of motion

$$J^{(\beta)} = B^{(\beta)} + m\,\dot{x}^{\mu}\,R_{\mu}^{(\beta)}\,, \tag{2.2.5}$$

which represent the total angular momentum, which is the sum of the orbital and the spin angular momentum. Equation (2.2.5) expresses the fact that the contribution of the Killing vector gives the orbital angular momentum, while the contribution of the spin is contained in the Killing scalars $B^{(\beta)}(x,\psi)$.

Inserting the expressions for the connection and the Riemann curvature components corresponding to the Schwarzschild spacetime in (1.3.16), we obtain for the Killing scalars

$$B^{(0)} = -i\,\frac{\alpha}{2r^2}\,\psi^t\,\psi^r\,, \tag{2.2.6a}$$

$$B^{(1)} = ir\,sin\varphi\,\psi^r\psi^\theta + ir\,sin\theta\,cos\theta\,cos\varphi\,\psi^r\psi^\varphi - ir^2 sin^2\theta\,cos\varphi\,\psi^\theta\psi^\varphi\,, \tag{2.2.6b}$$

$$B^{(2)} = -ir\,cos\varphi\,\psi^r\psi^\theta + ir\,sin\theta\,cos\theta\,sin\varphi\,\psi^r\psi^\varphi - ir^2 sin^2\theta\,sin\varphi\,\psi^\theta\psi^\varphi\,, \tag{2.2.6c}$$

$$B^{(3)} = -ir\sin^2\theta\,\psi^r\psi^\varphi - ir^2\sin\theta\cos\theta\,\psi^\theta\psi^\varphi. \tag{2.2.6d}$$

Substituting these in equation (2.2.5) and using the spin-tensor notation $S^{\mu\nu}$, introduced in equation (1.2.10), we obtain

$$J^{(0)} \equiv E = m\left(1-\frac{\alpha}{r}\right)\frac{dt}{d\tau} - \frac{\alpha}{2r^2}S^{rt}, \tag{2.2.7a}$$

$$J^{(1)} = -r\sin\varphi\left(m\,r\frac{d\theta}{d\tau} + S^{r\theta}\right) - \cos\varphi\left(\cot\theta\,J^{(3)} - r^2\,S^{\theta\varphi}\right), \tag{2.2.7b}$$

$$J^{(2)} = r\cos\varphi\left(m\,r\frac{d\theta}{d\tau} + S^{r\theta}\right) - \sin\varphi\left(\cot\theta\,J^{(3)} - r^2 S^{\theta\varphi}\right), \tag{2.2.7c}$$

$$J^{(3)} = r\sin^2\theta\left(m\,r\frac{d\varphi}{d\tau} + S^{r\varphi}\right) + r^2\sin\theta\cos\theta\,S^{\theta\varphi}. \tag{2.2.7d}$$

In addition to these constants of motion, the four generic conserved charges, described in the section (1.4) of Chapter I, also provide information about the allowed orbits of the particle. Finally the covariantly constant ψ^μ as formulated in (1.2.9) gives

$$\frac{d\psi^t}{d\tau} = -\frac{\alpha}{2r^2\left(1-\dfrac{\alpha}{r}\right)}\left(\frac{dr}{d\tau}\psi^t + \frac{dt}{d\tau}\psi^r\right), \tag{2.2.8a}$$

$$\frac{d\psi^r}{d\tau} = r\left(1-\frac{3\alpha}{2r}\right)\left(\frac{d\theta}{d\tau}\psi^\theta + \sin^2\theta\,\frac{d\varphi}{d\tau}\psi^\varphi\right), \tag{2.2.8b}$$

$$\frac{d\psi^\theta}{d\tau} = -\frac{1}{r}\left(\frac{dr}{d\tau}\psi^\theta + \frac{d\theta}{d\tau}\psi^r\right) + \sin\theta\cos\theta\frac{d\varphi}{d\tau}\psi^\varphi, \tag{2.2.8c}$$

$$\frac{d\psi^\varphi}{d\tau} = -\left(\frac{1}{r}\frac{dr}{d\tau} + \cot\theta\frac{d\theta}{d\tau}\right)\psi^\varphi - \frac{1}{r}\frac{d\varphi}{d\tau}\psi^r - \cot\theta\frac{d\varphi}{d\tau}\psi^\theta. \tag{2.2.8d}$$

Thus, we have twelve equations from which we want to solve four velocities and four components of ψ^μ. In order to construct a convenient set of equations incorporating physical boundary conditions, we consider motions for which

$$H = -\frac{1}{2}m, \tag{2.2.9}$$

or equivalently

$$g^{\mu\nu}P_\mu P_\nu + m^2 = 0. \tag{2.2.10}$$

This equation implies that the motion is geodesic, as described by (1.2.1). Combining equations (2.2.7) and (2.2.9), one can express the velocities as functions of the coordinates, the spin components and the constants of motion:

$$\frac{dt}{d\tau} = \frac{1}{\left(1-\frac{\alpha}{r}\right)}\left(\frac{E}{m} + \frac{\alpha}{2mr^2}S^{rt}\right), \tag{2.2.11a}$$

$$\frac{dr}{d\tau}=\left\{\left(1-\frac{\alpha}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2-\left(1-\frac{\alpha}{r}\right)\right.$$

$$\left.-r^2\left(1-\frac{\alpha}{r}\right)\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right]\right\}^{\frac{1}{2}}, \qquad (2.2.11b)$$

$$\frac{d\theta}{d\tau}=\frac{1}{mr^2}\left(-J^{(1)}\sin\varphi+J^{(2)}\cos\varphi-rS^{r\theta}\right), \qquad (2.2.11c)$$

$$\frac{d\varphi}{d\tau}=\frac{1}{mr^2\sin^2\theta}J^{(3)}-\frac{1}{mr}S^{r\varphi}-\frac{1}{m}\cot\theta\, S^{\theta\varphi}. \qquad (2.2.11d)$$

From equation (2.2.7) one can derive a useful identity

$$r^2\sin\theta\, S^{\theta\varphi}=J^{(1)}\sin\theta\cos\varphi+J^{(2)}\sin\theta\sin\varphi+J^{(3)}\cos\theta\,. \qquad (2.2.12)$$

In physical terms, this equation states that there is no orbital angular momentum in the radial direction: the total angular momentum in that direction is the spin angular momentum.

The supersymmetry constraint $Q=0$ (equation (1.4.12)) expresses the fact that spin represents only three independent degrees of freedom. Indeed, one can then solve for ψ^t in terms of the spatial components ψ^i:

$$\left(1-\frac{\alpha}{r}\right)\frac{dt}{d\tau}\psi^t=\frac{1}{\left(1-\frac{\alpha}{r}\right)}\frac{dr}{d\tau}\psi^r+r^2\left(\frac{d\theta}{d\tau}\psi^\theta+\sin^2\theta\frac{d\varphi}{d\tau}\psi^\varphi\right). \qquad (2.2.13)$$

As a result, the (classical) chiral charge Γ_* and the dual supercharge Q^* become zero:

$$\Gamma_* = Q^* = 0. \tag{2.2.14}$$

Equation (2.2.8a) is solved by the expression (2.2.13). The remaining equations, (2.2.8b)–(2.2.8d), can be rewritten in terms of the spin tensor components S^{ij}, $(i, j = r, \theta, \varphi)$, as follows:

$$\frac{dS^{r\theta}}{d\tau} = -\frac{1}{r}\frac{dr}{d\tau}S^{r\theta} + \sin\theta\cos\theta\frac{d\varphi}{d\tau}S^{r\varphi} - r\sin^2\theta\left(1 - \frac{3\alpha}{2r}\right)\frac{d\varphi}{d\tau}S^{\theta\varphi}, \tag{2.2.15a}$$

$$\frac{dS^{r\varphi}}{d\tau} = \cot\theta\frac{d\varphi}{d\tau}S^{r\theta} - \left(\frac{1}{r}\frac{dr}{d\tau} + \cot\theta\frac{d\theta}{d\tau}\right)S^{r\varphi} + r\left(1 - \frac{3\alpha}{2r}\right)\frac{d\theta}{d\tau}S^{\theta\varphi}, \tag{2.2.15b}$$

$$\frac{dS^{\theta\varphi}}{d\tau} = \frac{1}{r}\frac{d\varphi}{d\tau}S^{r\theta} - \frac{1}{r}\frac{d\theta}{d\tau}S^{r\varphi} - \left(\frac{2}{r}\frac{dr}{d\tau} + \cot\theta\frac{d\theta}{d\tau}\right)S^{\theta\varphi}. \tag{2.2.15c}$$

Equation (2.2.15c) is automatically solved by (2.2.12).

Equation (2.2.13) allows one to rewrite all time-like components S^{it} in terms of the space-like S^{ij}. In particular, using the anti-commuting character of the ψ-variables, and the expression for $dt/d\tau$ one can write

$$S^{rt} = \frac{mr^2}{E}\left(\frac{d\theta}{d\tau}S^{r\theta} + \sin^2\theta\frac{d\varphi}{d\tau}S^{r\varphi}\right). \tag{2.2.16}$$

Substitution of this result in (2.2.11a) gives

$$\frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{\alpha}{r}\right)} \left[\frac{E}{m} + \frac{\alpha}{2E} \left(\frac{d\theta}{d\tau} S^{r\theta} + \sin^2\theta \frac{d\varphi}{d\tau} S^{r\varphi} \right) \right]. \tag{2.2.17}$$

Equations (2.2.11), (2.2.15) and (2.2.17) can be integrated to give the full solution of the equations of motion for the coordinates and spins.

2.3. SPECIAL SOLUTIONS

In this section we apply the results obtained in the previous section to study the special case of motion in a plane, for which we choose $\theta=\pi/2$. For scalar particles, the orbital angular momentum is always conserved. Hence, any solution of scalar particles would actually describe a planar motion. But this is no longer true in general for spinning particles, for which only the total angular momentum is a conserved quantity.

For spinning particles, motion in a plane is strictly possible only in special cases, in which orbital and spin angular momentum are separately conserved. This may happen only if either the orbital angular momentum vanishes, or if spin and orbital angular momentum are parallel.

For $\theta=\pi/2$ the equations of motion (2.2.11) and (2.2.15) become

$$\frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{\alpha}{r}\right)} \left[\frac{E}{m} + \frac{\alpha}{2E} \frac{d\varphi}{d\tau} S^{r\varphi} \right], \tag{2.3.1a}$$

$$\frac{dr}{d\tau}=\left\{\left(1-\frac{\alpha}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2-\left(1-\frac{\alpha}{r}\right)-r^2\left(1-\frac{\alpha}{r}\right)\left(\frac{d\varphi}{d\tau}\right)^2\right\}^{\frac{1}{2}}, \tag{2.3.1b}$$

$$\frac{d\varphi}{d\tau}=\frac{1}{mr^2}J^{(3)}-\frac{1}{mr}S^{r\varphi}, \tag{2.3.1c}$$

$$\frac{d}{d\tau}\left(rS^{r\theta}\right)=-r^2\left(1-\frac{3\alpha}{2r}\right)\frac{d\varphi}{d\tau}S^{\theta\varphi}, \tag{2.3.1d}$$

$$\frac{d}{d\tau}\left(rS^{r\varphi}\right)=0. \tag{2.3.1e}$$

From equations (2.3.1c) and (2.3.1e) it follows that the orbital angular momentum and the component of the spin perpendicular to the plane in which the particle moves, are separately conserved:

$$rS^{r\varphi}\equiv\Sigma, \tag{2.3.2a}$$

$$m\,r^2\frac{d\varphi}{d\tau}=J^{(3)}-\Sigma\equiv L, \tag{2.3.2b}$$

where Σ and L are two constants. This result leads to the remark that the presence of spin-dependent forces modifies the gravitational red-shift. Indeed, equation (2.3.1a) becomes

$$dt=\frac{d\tau}{\left(1-\frac{\alpha}{r}\right)}\left[\frac{E}{m}+\frac{\alpha}{2mEr^3}L\Sigma\right]. \tag{2.3.3}$$

For a nonzero value of the orbital angular momentum L, it follows from equation (2.3.3) that there is an additional contribution to time-dilation from spin-orbit coupling. Thus the time-dilation is not a purely geometric effect; it also has a dynamical component [24, 25].

In the case of planar motion equation (2.2.11c) reduces to

$$rS^{r\theta} = -J^{(1)}\sin\varphi + J^{(2)}\cos\varphi, \tag{2.3.4}$$

and equation (2.2.12) to

$$r^2 S^{\theta\varphi} = J^{(1)}\cos\varphi + J^{(2)}\sin\varphi. \tag{2.3.5}$$

These two equations can be combined to obtain

$$\frac{d}{d\tau}\left(rS^{r\theta}\right) = -r^2 S^{\theta\varphi}\frac{d\varphi}{d\tau}. \tag{2.3.6}$$

There are indeed only two possibilities, if we compare (2.3.6) with (2.3.1d):

$$(I)\quad \frac{d\varphi}{d\tau} = 0, \qquad (II)\quad S^{\theta\varphi} = 0. \tag{2.3.7}$$

CASE I. Vanishing of $d\varphi/d\tau$ implies that there is no orbital angular momentum, and then the solution describes a particle moving along a fixed radius from or towards the source of the gravitational field. The motion of the particle for a distant observer is described by

$$\frac{dr}{dt}=\left(1-\frac{\alpha}{r}\right)\sqrt{1-\frac{m^2}{E^2}\left(1-\frac{\alpha}{r}\right)}, \tag{2.3.8}$$

as in the case of a scalar point particle. Since the orbital angular momentum vanishes, the (Cartesian) spin tensor components are all conserved, *e.g.*, if we choose $\varphi=0$ for the path of the particle, then

$$r^2 S^{\theta\varphi}=J^{(1)}, \qquad rS^{r\theta}=J^{(2)}, \qquad rS^{r\varphi}=J^{(3)}. \tag{2.3.9}$$

CASE II: If $\frac{d\varphi}{d\tau}\neq 0,$ then equation (2.3.7) gives

$$S^{\theta\varphi}=0, \tag{2.3.10a}$$

and hence

$$J^{(1)}=J^{(2)}=0. \tag{2.3.10b}$$

Then, equation (2.3.4) gives

$$S^{r\theta}=0, \tag{2.3.10c}$$

which means that the spin is parallel to the orbital angular momentum. The equation for the orbit of the particle is obtained from equations (2.3.1a)–(2.3.1c), which is

$$\frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2=\frac{\left(E^2-m^2\right)r^2}{L^2}-1+\frac{m\,\alpha}{L}\left(\frac{m\,r}{L}+\frac{J^{(3)}}{m\,r}\right). \tag{2.3.11}$$

Using the dimensionless variables

$$\in = \frac{E}{m}, \qquad x = \frac{r}{\alpha}, \qquad \ell = \frac{L}{m\alpha}, \qquad \Delta = \frac{\Sigma}{L}, \tag{2.3.12}$$

equation (2.3.11) can be put in the form

$$\in^2 = \alpha^2 \dot{x}^2 + \alpha^2 x^2 \left(\frac{d\varphi}{d\tau} \right)^2 + 1 - \frac{1}{x} - \frac{\ell^2 (1 + \Delta)}{x^3}. \tag{2.3.13}$$

The presence of Δ, which is a bilinear combination of anti-commuting variables ψ^{μ}, makes the equation rigorous. For the investigation of a possible motion, a numerical value needs to be assigned to Δ. As mentioned in the Introduction of this thesis, such a quantum mechanical expectation value is desirable. Then Δ can be used as a classical variable. In order to avoid any inconsistency that may result from this semiclassical approximation, the numerical value of Δ is supposed to be small: $\Delta << 1$.

Substituting the expression for $\frac{d\varphi}{d\tau}$ from (2.3.1c) on the right-hand side of equation (2.3.13), an effective potential $U_R(x, \ell^2)$ can be defined. Then

$$\frac{\ell^2}{x^4} \left(\frac{dx}{d\varphi} \right)^2 = \alpha^2 \left(\frac{dx}{d\tau} \right)^2 \equiv \in^2 - U_R\left(x, \ell^2\right), \tag{2.3.14a}$$

where

$$U_R\left(x,\ell^2\right)=1-\frac{1}{x}+\ell^2\,\frac{1}{x^2}-\left(1+\Delta\right)\ell^2\,\frac{1}{x^3}. \qquad (2.3.14b)$$

This equation is the same as one would obtain for a one-dimensional problem with a potential $U_R(x,\ell^2)$. In the one-dimensional problem, the particle is subject to a radial force

$$F\left(x,\ell^2\right)=-\frac{\partial}{\partial x}U_R\left(x,\ell^2\right) \qquad (2.3.15)$$

This is the effective force that the three-dimensional particle feels in the radial direction, including a contribution from the centripetal acceleration. As the radial kinetic energy is positive, the right-hand side of (2.3.14a) must be positive as well. Figure 1 shows a picture of a possible potential $U_R(x,\ell^2)$.

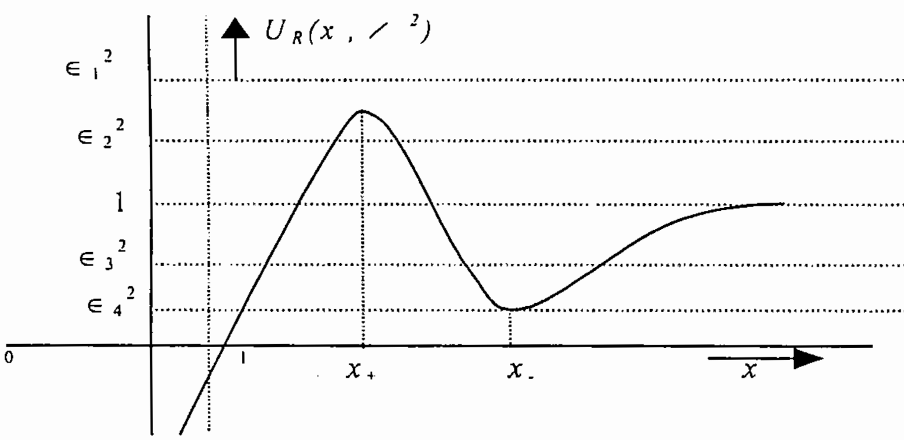

Figure 1. Effective potential of a particle in Schwarzschild spacetime for large values of orbital angular momentum.

The potential $U_R(x,\ell^2)$ is maximum at x_+ and minimum at x_-, where

$$x_{\mp} = \ell^2 \left(1 \pm \sqrt{1 - \frac{3(1+\Delta)}{\ell^2}} \right) \tag{2.3.16}$$

provided that $\ell^2 > 3\,(1 + \Delta)$. There are four possibilities:

(i) The approaching particle has the energy such that $\epsilon_1^2 > U_R\left(x_+, \ell^2\right)$. Far away an attractive effective force acts on the particle. After passing the minimum potential, $U_R(x_-, \ell^2)$, at x_- the effective force becomes repulsive. But the particle has enough kinetic energy to reach x_+, where it is again subject to an attractive effective force. The particle then being attracted hits the object, which is the source of the gravitational field, or in case of a black hole, it crosses the Schwarzschild radius at $x = 1$.

(ii) The energy of the approaching particle is such that $1 \leq \epsilon_2^2 < U_R\left(x_+, \ell^2\right)$ In this case the approaching particle from infinity is also subject to an attractive force, but it does not have enough energy to reach x_+. After passing x_- the particle faces a repulsive effective force and at the perihelion x_{ph}, $\partial x/\partial\varphi$ becomes zero. Hence, the particle disappears to infinity again. These orbits represent scattering states.

(iii) If the approaching particle has energy such that $U_R(x_-,\ell^2) < \epsilon_3^2 < 1$, then there are two values of x where $\partial x/\partial\varphi = 0$. These two values of x are x_{ph} and x_{ap}, which respectively represent the perihelion and the aphelion. The particle moves between these two extreme points in orbit, which is not necessarily closed. Also the particle cannot disappear to infinity. Hence, the orbit of the particle represents a bound state. For small relativistic effects these orbits approach precessing ellipses; hence, they are also referred to as quasi-elliptic orbits.

(iv) For $\epsilon_4^2 = U_R(x_-, \ell^2) < 1$, the energy equals the minimum of the potential and the perihelion coincides with the aphelion. There is no net effective force along the radial direction, *i.e.*, $\partial U_R/\partial x = 0$. Then the orbit becomes a circle. Figure 2 shows a picture of the effective potential $U_R(x, \ell^2)$ for several values of ℓ^2. For $\ell^2 < 3(1+\Delta)$(courve-1), there exists no bound state with $\epsilon < 1$, and all particles with energy such that $\epsilon \geq 1$ will finally hit the center.

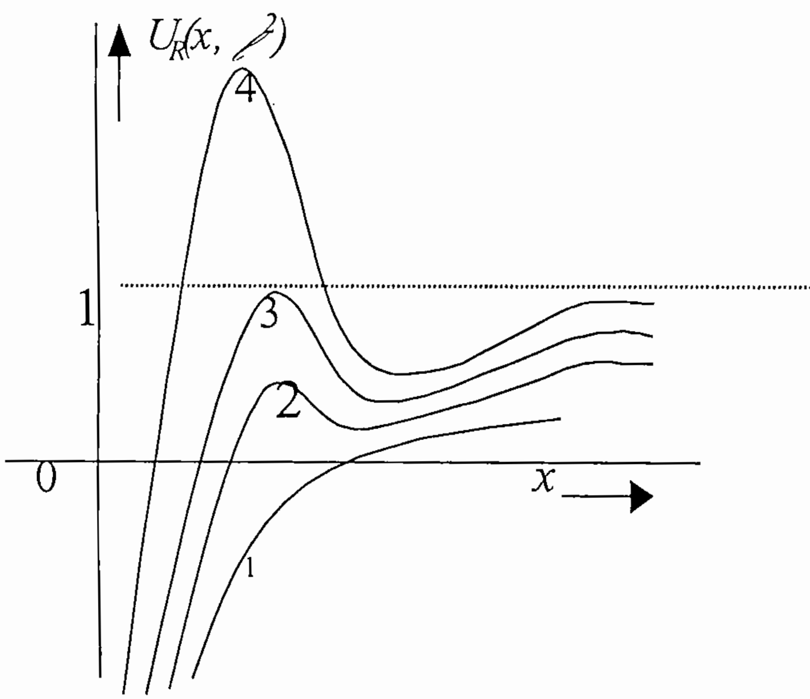

Figure 2. Effective potential for different values of angular momentum.

If $\ell^2 > 3(1+\Delta)$ (Courve-2–4), then bound states with $\epsilon < 1$ are possible. The bound states correspond to quasi-elliptic and circular orbits. The radius of the circular orbit is defined by $x = x_-$, which is minimum at the point of inflection of $U_R(x, \ell^2)$. The point of inflection occurs for $\ell^2 = 3(1+\Delta)$ and hence the radius of the smallest possible circular orbit is

$$x = \ell^2 = 3(1+\Delta). \tag{2.3.17}$$

The energy for this critical orbit is given by

$$\epsilon^2_{crit} = \frac{1}{9}(8+\Delta) \tag{2.3.18}$$

to the first order in Δ.

We now briefly discuss the quasi-elliptic orbits. In the classical case, the Kepler-type orbits, representing bound-state solutions, are circles and ellipses parameterized by

$$x = \frac{\kappa}{1 + \varepsilon \cos(\varphi - \varphi_0)}. \tag{2.3.19}$$

where $\kappa = k/\alpha$. The k is the semilatus rectum and ε is the eccentricity with $0 < \varepsilon < 1$ for ellipses and $\varepsilon = 0$ for a circle. The perihelion x_{ph} of the ellipse occurs at $\varphi = \varphi_0$. In the Schwarzschild spacetime, relativistic effects turn the perihelion during the motion of the particle. If the function $w(\varphi)$ describes this turning, then equation (2.3.19) becomes

$$x = \frac{\kappa}{1 + \varepsilon \cos\left[\varphi - w(\varphi)\right]}. \tag{2.3.20}$$

The perihelion and aphelion are now described by

$$\varphi_{ph}^{(n)} - w\left(\varphi_{ph}^{(n)}\right) = 2n\pi \quad , \tag{2.3.21a}$$

$$\varphi_{ah}^{(n)} - w\left(\varphi_{ah}^{(n)}\right) = (2n+1)\pi. \tag{2.3.21b}$$

The particle reaches its nth perihelion at the angle $\varphi_{ph}^{(n)}$ and the turning of the perihelion after n revolutions is given by the angle $w\left(\varphi_{\mathrm{ph}}^{(\mathrm{n})}\right)$. Hence, the precession of the perihelion after one revolution is

$$\Delta w \equiv w\left(\varphi_{ph}^{(1)}\right) - w\left(\varphi_{ph}^{(0)}\right) = \varphi_{ph}^{(1)} - \varphi_{ph}^{(0)} - 2\pi \equiv \Delta\varphi - 2\pi. \tag{2.3.22}$$

The energy at the perihelion / aphelion is given by

$$\epsilon^2 = 1 - \left(\frac{1 \pm \varepsilon}{\kappa}\right) + \ell^2 \left(\frac{1 \pm \varepsilon}{\kappa}\right)^2 - (1+\Delta)\ell^2 \left(\frac{1 \pm \varepsilon}{\kappa}\right)^3. \tag{2.3.23}$$

Since the energy ϵ is a constant of motion, it follows from comparison of both expressions for ϵ^2 that

$$\ell^2 = \frac{\kappa^2}{2\kappa - (1+\Delta)\left(3 + \varepsilon^2\right)}. \tag{2.3.24}$$

Using (2.3.20), (2.3.23) and (2.3.24) in equation (2.3.11) we obtain

$$\left(1-\frac{dw}{d\varphi}\right)^2 = 1-\frac{(1+\Delta)}{\kappa}\{3+\varepsilon\cos[\varphi-w(\varphi)]\}. \qquad (2.3.25)$$

Introducing

$$y=\varphi-w(\varphi), \qquad (2.3.26)$$

equation (2.3.25) can be put in the form

$$d\varphi=\frac{dy}{\sqrt{1-[(1+\Delta)/\kappa](3+\varepsilon\, cosy)}}. \qquad (2.3.27)$$

Then $\Delta\varphi$, as defined in (2.3.22), can be found by integrating (2.3.27) from one perihelion to the next one with $0\leq y\leq 2\pi$:

$$\Delta\varphi=\frac{1}{\sqrt{a}}\int_0^{2\pi}\frac{dy}{\sqrt{1-(b/a)cosy}}, \qquad (2.3.28a)$$

where

$$a=1-F, \qquad F=\frac{(1+\Delta)}{\kappa}, \qquad b=\varepsilon F. \qquad (2.3.28b)$$

The F contains the relativistic effects and Δ-dependence. So, we first expand the elliptic integral (2.3.28a) in a power series in F around the classical (Kepler) solution, and then integrate term by term. Using the expansion

$$(1-x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} A(m)x^m, \qquad A(m) = \frac{1}{2^{2m}}\binom{2m}{m}, \tag{2.3.29}$$

and the integrals

$$\int_0^{2\pi} dy \cos^{2m} y = 2\pi A(m), \tag{2.3.30a}$$

$$\int_0^{2\pi} dy \cos^{2m+1} y = 0, \tag{2.3.30b}$$

we obtain the following expression for $\Delta\varphi$:

$$\Delta\varphi = 2\pi\left\{1 + \frac{3F}{2} + \sum_{t=0}^{\infty} F^{t+2}\left[A(t+2)3^{t+2}\right.\right.$$

$$\left.\left. + \sum_{m=0}^{[t/2]}\sum_{n=0}^{t-2m} A(n)A(2m+2)A(m+1)3^{t-2m}\varepsilon^{2m+2}\binom{t-n+1}{t-n-2m}\right]\right\}$$

$$= 2\pi\left[1 + \frac{3\alpha}{2k}(1+\Delta) + \frac{3\alpha^2}{16k^2}\left(\varepsilon^2 + 18\right)(1+\Delta) + \ \dots\ \right], \tag{2.3.31}$$

where α=$2M$ is the Schwarzschild radius. For Δ=0 the lowest order contribution to the relativistic precession of the perihelion is given by the second term in the expansion. The spin of a particle contributes to this lowest order precession, which can be found by keeping terms of first order in Δ.

A similar effect is also expected for scattering states. The scattering orbits are possible for $\ell^2 \geq 4(1+\Delta)$, because in that case $U_R\,(x_+, \ell^2) \geq 1$. A particle coming in from infinity with energy

$$1 \leq \epsilon^2 \leq U_R(x_+, \ell^2) \qquad (2.3.32)$$

approaches the central mass until it reaches its perihelion and then disappears to infinity again.

The scattering angle is calculated from equation (2.3.14). Introducing

$$u = \frac{\ell}{x\sqrt{\epsilon^2 - 1}} \qquad (2.3.33)$$

it can be put in the form

$$\left(\frac{du}{d\varphi}\right)^2 = 1 + \sigma u - u^2 + \gamma u^3 \equiv -W(u, \sigma, \gamma), \qquad (2.3.34)$$

where

$$\sigma = \frac{1}{\ell\sqrt{\epsilon^2 - 1}}, \qquad \gamma = \frac{(1+\Delta)\sqrt{\epsilon^2 - 1}}{\ell} \qquad (2.3.35)$$

are constants of motion. Solution of this equation in the limit of vanishing γ gives the classical orbit. The exact scattering angle ϑ (figure 3) is given by

$$\vartheta = \pi - 2\int_0^{u_{ph}} \frac{du}{\sqrt{1 + \sigma u - u^2 + \gamma u^3}}. \qquad (2.3.36)$$

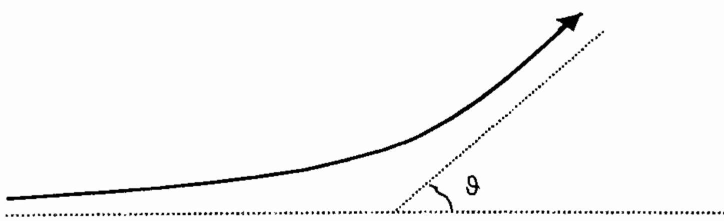

Figure 3. Scattering angle ϑ for quasi-hyperbolic orbits.

Since $dx/d\varphi=0$ in the perihelion, x_{ph} is one of the roots of (2.3.14):

$$\epsilon^2 - U_R(x, \ell^2) = 0. \qquad (2.3.37)$$

Equation (2.3.37) has three roots: x_1, x_2, and x_{ph}, say. One root is always negative and hence, not realistic. These roots correspond to the three roots u_1, u_2, u_{ph} of $W(u, \sigma, \gamma)$, defined by (2.3.34). The integral in (2.3.36) can be put in the form

$$\int_0^{u_{ph}} \frac{du}{\sqrt{\gamma(u_2 - u)(u_{ph} - u)(u - u_1)}}, \qquad (2.3.38)$$

where u_1, u_2, u_{ph} are given up to $\mathcal{O}(\gamma^2)$ by [62]

$$u_1 = \frac{\sigma - \sqrt{\sigma^2 + 4}}{2} - \frac{(\sigma^2 + 1)\left(\sigma - \sqrt{\sigma^2 + 4}\right) + 2\sigma}{2\sqrt{\sigma^2 + 4}}\gamma + \mathcal{O}(\gamma^2),$$

$$u_{ph} = \frac{\sigma + \sqrt{\sigma^2 + 4}}{2} + \frac{\left(\sigma^2 + 1\right)\left(\sigma + \sqrt{\sigma^2 + 4}\right) + 2\sigma}{2\sqrt{\sigma^2 + 4}}\gamma + \mathcal{O}\left(\gamma^2\right),$$

$$u_2 = \frac{1}{\gamma} - \sigma - \left(\sigma^2 + 1\right)\gamma + \mathcal{O}\left(\gamma^2\right) \qquad (2.3.39)$$

As the root u_2 is singular for $\gamma \to 0$, u_2 and thus x_2 do not exist in the classical case: in that case $W(u, \sigma, 0)$ is a parabola with roots u_1 and u_{ph}, given by (2.3.39) for $\gamma = 0$. Obviously, the factor $\gamma(u_2 - u) \to 1$ for $\gamma \to 0$. Using this the elliptical integral (2.3.38) can be solved as a series in γ. We write it in the form

$$\int_0^{u_{ph}} \left[\frac{du}{\sqrt{\gamma\left(u_2 - u\right)}} \frac{1}{\sqrt{\left(u_{ph} - u\right)\left(u - u_1\right)}} \right]. \qquad (2.3.40)$$

We note that the second factor in the integrand is the same as in the classical case, except for the γ-dependent terms in u_1 and u_{ph}. This integral can be performed by expanding the first factor to first order in γ. Expanding the resulting expression to first order in γ, one can derive the scattering angle, ϑ, in the form

$$\vartheta = \pi - 2\left(1 + \frac{3\sigma}{4}\gamma\right) \arccos\left(-\sqrt{\frac{\sigma^2}{\sigma^2 + 4}}\right) + \frac{\sigma^2}{\sigma^2 + 4}\gamma + \mathcal{O}\left(\gamma^2\right) \qquad (2.3.41)$$

This shows that the relativistic corrections to the scattering angle include contributions from spin.

2.4. DISCUSSION

The study of this chapter predicts spin dependence of the time dilation in a gravitational field, of the perihelion precession for bound state orbits, and of the scattering of particles by gravitational fields. This confirms the existence of a gravitational analogue of the Stern-Gerlach-type forces well known to appear in electromagnetic phenomena.

The equations of motion (1.2.8) and (1.2.9) remain valid if averaged inside a functional integral with the exponential of the action (1.2.3) in the integrand, *i.e.*, when $S^{\mu\nu}=-i\psi^{\mu}\psi^{\nu}$ is replaced by its quantum mechanical expectation value $<S^{\mu\nu}>$. This allows one to consider the results of this chapter as a semiclassical approximation to the Dirac theory, and provides a procedure for numerical evaluation of the components of the spin tensor, at least in principle. In general, $\langle S^{\mu\nu}\rangle^2 \neq \langle S^{\mu\nu^2}\rangle = 0$. Hence, this semiclassical approximation can only hold to first order in the spin.

Physically, in a macroscopic gravitational field like that of a star the effects of microscopic intrinsic spin of particles such as electrons can be completely omitted. Indeed, the ratio Δ (equation (2.3.12)) for an electron orbiting the sun is of the order of 10^{-17}. Therefore, effects of particle spins are appreciable only in strong gravitational interactions at short distances, near the Planck scale.

CHAPTER III

SPINNING PARTICLES IN REISSNER-NORDSTROM AND NUT-REISSNER-NORDSTROM SPACETIMES

3.1. INTRODUCTION

In ref. [62] Rietdijk and van Holten investigated the motion of pseudo-classical spinning point particles in the Schwarzschild spacetime. In this Chapter we like to extend their work to the Reissner-Nordstrom (RN) [73] and NUT-Reissner-Nordstrom (NUT-RN) spacetimes. The Reissner-Nordstrom spacetime [98, 99] is the Schwarzschild spacetime generalized with charged parameter. It is the unique [100], asymptotically flat, spherically symmetric solution of the Einstein-Maxwell equations that describes the geometry of a spherical star with charge q and mass M. This spacetime may be analytically extended to an electrovacuum solution representing a black hole for $0 < |q| < M$ [101, 102]. In the extremal case, *i.e.*, when $q = M$, the Reissner-Nordstrom black hole is distinguished by its coldness (vanishing Hawking temperature) and its supersymmetry. It admits supersymmetry in the context of $N = 2$ supergravity [103 – 108]. The extreme Reissner-Nordstrom black hole spacetime occupies a special position among the black hole solutions of the Einstein or Einstein-Maxwell equations because of its complete stability with respect to both classical and quantum process permitting its interpretation as a soliton [103,109]. So the study of spinning point particles in the Reissner-Nordstrom spacetime is interesting.

The NUT-RN spacetime is the Reissner-Nordstrom spacetime generalized with NUT parameter, which has the interpretation of magnetic mass [74–79]. This spacetime is stationary and asymptotically not flat. NUT parameter has peculiar properties [79, 82], its physical interpretation is not yet clear. Spacetimes endowed with NUT parameter are sometimes considered as unphysical [83]. Hence, our study of spinning point particles in the NUT-RN spacetime is interesting.

We arrange this chapter as follows. In section 3.2 we investigate the motion of pseudo-classical spinning particles in the Reissner-Nordstrom spacetime. We use the generalized Killing equations described in Chapter I for the symmetries of spinning particles in curved spacetime and derive the constants of motion in terms of the solutions of these equations. In section 3.3 we consider the motion in a plane and analyze the bound state solutions. The precession of the perihelion is also obtained. In section 3.4 we investigate the motion of pseudo-classical spinning particles in the NUT-Reissner-Nordstrom spacetime. In section 3.5 we solve the equations obtained in section 3.4 for special case of motion on a cone, and in a plane for which we choose $\theta = \pi/2$. In section 3.6 we present our remarks.

3.2. SPINNING REISSNER-NORDSTROM SPACETIME

In this section we apply the results of Chapter I to investigate the motion of a spinning point particle moving in the Reissner-Nordstrom spacetime, which has the metric [98, 99, 110]

$$ds^2 = -\left(1-\frac{\alpha}{r}+\frac{q^2}{r^2}\right)dt^2 + \frac{1}{\left(1-\frac{\alpha}{r}+\frac{q^2}{r^2}\right)}dr^2$$

$$+ r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \tag{3.2.1}$$

together with a vector potential, which has nonvanishing components: $A_t = \frac{q_e}{r}$, $A_\varphi = -q_m \cos\theta$. Here, $\alpha = 2M$ and $q^2 = q_e^2 + q_m^2$ with M the mass, q_e the electric charge, and q_m the magnetic charge of the object which is the source of the field. For $q^2 < M^2$, there are two zeros of g_{tt} at $r_\pm$ where

$$r_\pm = M \pm \sqrt{M^2 - q^2} \tag{3.2.2}$$

They correspond to two horizons: an event horizon at r_+ and an inner (Cauchy) horizon at r_- .The metric (3.2.1) describes the field outside the event horizon. The spacetime given by (3.2.1) describes the extreme Reissner-Nordstrom black hole spacetime for $q = M$, and the Schwarzschild spacetime for $q = 0$. The invariance of the metric (3.2.1) under time translation and spatial rotations is expressed by four ψ-independent solutions $R^{(\beta)}(x), (\beta = 0, \ldots, 3)$, of the generalized Killing equation (1.3.15). The corresponding vector fields have the form

$$D^{(\beta)} \equiv R^{(\beta)\mu}(x)\partial_\mu, \quad \beta = 0, \ldots, 3, \tag{3.2.3}$$

The explicit expressions for (3.2.3) are

$$D^{(0)} = \frac{\partial}{\partial t}, \tag{3.2.4a}$$

$$D^{(1)} = -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}, \tag{3.2.4b}$$

$$D^{(2)} = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}, \tag{3.2.4c}$$

$$D^{(3)} = \frac{\partial}{\partial\varphi}. \tag{3.2.4d}$$

These Killing vector fields generate the Lie algebra *O(1, 1) ×SO(3)*:

$$\left[D^{(a)}, D^{(b)}\right] = -\varepsilon^{abc}D^{(c)}, \quad (a, b, c = 1, 2, 3), \tag{3.2.5a}$$

$$\left[D^{(0)}, D^{(a)}\right] = 0. \tag{3.2.5b}$$

The first generalized Killing equation (1.3.16) suggests that with each Killing vector $R_\mu^{(\beta)}(x)$ there is associated a Killing scalar $B^{(\beta)}$. These Killing scalars are necessary to obtain the constants of motion

$$J^{(\beta)} = B^{(\beta)} + m\,\dot{x}^{\mu}\,R^{(\beta)}_{\mu}. \tag{3.2.6}$$

These constants of motion represent the total angular momentum, which is the sum of the orbital and the spin angular momentum. For scalar particles the orbital angular momentum is a constant of motion. However, this is no longer true for a particle with spin; therefore, the Killing vector itself does not give a conserved quantity of motion. The contribution of spin, which is contained in the Killing scalars, has to be added.

Inserting the $R^{(\beta)}(x)$, as given by (3.2.3) and (3.2.4), and the expressions for the connections and the Riemann curvature corresponding to the Reissner-Nordstrom spacetime in (1.3.16), we obtain for the Killing scalars the following expressions:

$$B^{(0)} = -i\left(\frac{\alpha}{2r^2} - \frac{q^2}{2r^3}\right)\psi^t\,\psi^r, \tag{3.2.7a}$$

$$B^{(1)} = ir\,sin\varphi\,\psi^r\psi^\theta + ir\,sin\theta\,cos\theta\,cos\varphi\,\psi^r\psi^\varphi - ir^2 sin^2\theta\,cos\varphi\,\psi^\theta\psi^\varphi, \tag{3.2.7b}$$

$$B^{(2)} = -ir\,cos\varphi\,\psi^r\psi^\theta + ir\,sin\theta\,cos\theta\,sin\varphi\,\psi^r\psi^\varphi - ir^2 sin^2\theta\,sin\varphi\,\psi^\theta\psi^\varphi, \tag{3.2.7c}$$

$$B^{(3)} = -ir\,sin^2\theta\,\psi^r\psi^\varphi - ir^2\,sin\theta\,cos\theta\,\psi^\theta\psi^\varphi. \tag{3.2.7d}$$

Substituting these in $J^{(\beta)}$ (equation (3.2.6)) and using the spin-tensor notation $S^{\mu\nu}$, introduced in equation (1.2.10), one can find

$$J^{(0)} \equiv E = m\left(1-\frac{\alpha}{r}+\frac{q^2}{r^2}\right)\frac{dt}{d\tau} - \left(\frac{\alpha}{2r^2}-\frac{q^2}{2r^3}\right)S^{tr}, \qquad (3.2.8a)$$

$$J^{(1)} = -r\sin\varphi\left(m\,r\frac{d\theta}{d\tau}+S^{r\theta}\right)-\cos\varphi\left(\cot\theta\, J^{(3)}-r^2 S^{\theta\varphi}\right), \qquad (3.2.8b)$$

$$J^{(2)} = r\cos\varphi\left(m\,r\frac{d\theta}{d\tau}+S^{r\theta}\right)-\sin\varphi\left(\cot\theta\, J^{(3)}-r^2 S^{\theta\varphi}\right), \qquad (3.2.8c)$$

$$J^{(3)} = r\sin^2\theta\left(m\,r\frac{d\varphi}{d\tau}+S^{r\varphi}\right)+r^2\sin\theta\cos\theta\, S^{\theta\varphi}. \qquad (3.2.8d)$$

In addition to these constants of motion, the four generic conserved charges, described in section 1.4 of Chapter I, provide information about the allowed orbits of the particle. Further, the covariantly constant ψ^{μ} as formulated in (1.2.9) gives

$$\frac{d\psi^t}{d\tau} = -\frac{\alpha r-2q^2}{2r^3\left(1-\frac{\alpha}{r}+\frac{q^2}{r^2}\right)}\left(\frac{dr}{d\tau}\psi^t+\frac{dt}{d\tau}\psi^r\right), \qquad (3.2.9a)$$

$$\frac{d\psi^r}{d\tau} = r\left(1-\frac{3\alpha}{2r}+\frac{2q^2}{r^2}\right)\left(\frac{d\theta}{d\tau}\psi^\theta+\sin^2\theta\,\frac{d\varphi}{d\tau}\psi^\varphi\right), \qquad (3.2.9b)$$

$$\frac{d\psi^\theta}{d\tau} = -\frac{1}{r}\left(\frac{dr}{d\tau}\psi^\theta+\frac{d\theta}{d\tau}\psi^r\right)+\sin\theta\cos\theta\frac{d\varphi}{d\tau}\psi^\varphi, \qquad (3.2.9c)$$

$$\frac{d\psi^{\varphi}}{d\tau} = -\left(\frac{1}{r}\frac{dr}{d\tau} + \cot\theta\,\frac{d\theta}{d\tau}\right)\psi^{\varphi} - \frac{1}{r}\frac{d\varphi}{d\tau}\psi^{r} - \cot\theta\,\frac{d\varphi}{d\tau}\psi^{\theta}. \tag{3.2.9d}$$

Altogether we have twelve equations for four velocities and four components of ψ^{μ}. Of course, there are only eight independent equations. In order to construct a convenient set of equations incorporating phycical boundary conditions, we consider motions for which

$$H = -\frac{1}{2}m, \tag{3.2.10}$$

or equivalently

$$g^{\mu\nu}P_{\mu}P_{\nu} + m^{2} = 0 \tag{3.2.11}$$

This equation implies that the motion is geodesic, as defined by (1.2.1). Combining equations (3.2.8) and (3.2.10) one can express the velocities as functions of the coordinates, the spin components and the constants of motion:

$$\frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{\alpha}{r} + \frac{q^{2}}{r^{2}}\right)}\left[\frac{E}{m} + \left(\frac{\alpha}{2mr^{2}} - \frac{q^{2}}{2mr^{3}}\right)S^{rt}\right], \tag{3.2.12a}$$

$$\frac{dr}{d\tau} = \left\{\left(1 - \frac{\alpha}{r} + \frac{q^{2}}{r^{2}}\right)^{2}\left(\frac{dt}{d\tau}\right)^{2} - \left(1 - \frac{\alpha}{r} + \frac{q^{2}}{r^{2}}\right)\right.$$

$$-r^2\left(1-\frac{\alpha}{r}+\frac{q^2}{r^2}\right)\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right]\Bigg\}^{\frac{1}{2}}, \qquad (3.2.12b)$$

$$\frac{d\theta}{d\tau}=\frac{1}{mr^2}\left(-J^{(1)}\sin\varphi+J^{(2)}\cos\varphi-rS^{r\theta}\right), \qquad (3.2.12c)$$

$$\frac{d\varphi}{d\tau}=\frac{1}{mr^2\sin^2\theta}J^{(3)}-\frac{1}{mr}S^{r\varphi}-\frac{1}{m}\cot\theta\, S^{\theta\varphi}. \qquad (3.2.12d)$$

From equations (3.2.8) one can derive another independent linear combination of $J^{(1)}$ and $J^{(2)}$:

$$r^2\sin\theta\, S^{\theta\varphi}=J^{(1)}\sin\theta\cos\varphi+J^{(2)}\sin\theta\sin\varphi+J^{(3)}\cos\theta. \qquad (3.2.13)$$

This equation states that there is no orbital angular momentum in the radial direction: the total angular momentum in that direction is the spin angular momentum.

The supersymmetry constraint $Q = 0$ [equation (1.4.12)] expresses the fact that spin represents only three independent degrees of freedom. Then one can solve for ψ^t in terms of the spatial components ψ^i:

$$\left(1-\frac{\alpha}{r}+\frac{q^2}{r^2}\right)\frac{dt}{d\tau}\psi^t=\frac{1}{\left(1-\frac{\alpha}{r}+\frac{q^2}{r^2}\right)}\frac{dr}{d\tau}\psi^r$$

$$+r^2\left(\frac{d\theta}{d\tau}\psi^\theta+\sin^2\theta\frac{d\varphi}{d\tau}\psi^\varphi\right). \qquad (3.2.14)$$

As a result, the chiral charge Γ_* and the dual supercharge Q^* become zero:

$$\Gamma_* = Q^* = 0. \tag{3.2.15}$$

Expression (3.2.14) satisfies equation (3.2.9a). Equations (3.2.9b)–((3.2.9d) can be rewritten in terms of the spin tensor components $S^{ij}, (i, j = r, \theta, \varphi)$, as follows:

$$\frac{dS^{r\theta}}{d\tau} = -\frac{1}{r}\frac{dr}{d\tau}S^{r\theta} + \sin\theta\cos\theta\frac{d\varphi}{d\tau}S^{r\varphi}$$

$$-r\sin^2\theta\left(1 - \frac{3\alpha}{2r} + \frac{2q^2}{r^2}\right)\frac{d\varphi}{d\tau}S^{\theta\varphi}, \tag{3.2.16a}$$

$$\frac{dS^{r\varphi}}{d\tau} = \cot\theta\frac{d\varphi}{d\tau}S^{r\theta} - \left(\frac{1}{r}\frac{dr}{d\tau} + \cot\theta\frac{d\theta}{d\tau}\right)S^{r\varphi}$$

$$+r\left(1 - \frac{3\alpha}{2r} + \frac{2q^2}{r^2}\right)\frac{d\theta}{d\tau}S^{\theta\varphi}, \tag{3.2.16b}$$

$$\frac{dS^{\theta\varphi}}{d\tau} = \frac{1}{r}\frac{d\varphi}{d\tau}S^{r\theta} - \frac{1}{r}\frac{d\theta}{d\tau}S^{r\varphi} - \left(\frac{2}{r}\frac{dr}{d\tau} + \cot\theta\frac{d\theta}{d\tau}\right)S^{\theta\varphi}. \tag{3.2.16c}$$

Equation (3.2.16c) is automatically solved by (3.2.13).

Using (3.2.14) all time-like components S^{it} can be rewritten in terms of the space-like $S^{ij}, (i, j = r, \theta, \varphi),$. In particular, the anti-commuting character of the ψ-variables and equation (3.2.12a) allow us to write

$$S^{rt} = \frac{mr^2}{E}\left(\frac{d\theta}{d\tau}S^{r\theta} + sin^2\theta\frac{d\varphi}{d\tau}S^{r\varphi}\right). \qquad (3.2.17)$$

Combining this result with (3.2.12a) we obtain

$$\frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)}\left[\frac{E}{m} + \frac{1}{2E}\left(\alpha - \frac{q^2}{r}\right)\right.$$

$$\left.\times\left(\frac{d\theta}{d\tau}S^{r\theta} + sin^2\theta\frac{d\varphi}{d\tau}S^{r\varphi}\right)\right]. \qquad (3.2.18)$$

Equations (3.2.12), (3.2.16) and (3.2.18) can be integrated to solve the equations of motion for the coordinates $\{x^\mu\}$ and spins $\{\psi^\mu\}$. For $q=0$ these equations reduce to those obtained in Chapter II for the Schwarzschild spacetime.

3.3. PLANAR MOTION IN REISSNER-NORDSTROM SPACETIME

In this section we solve the equations obtained in the previous section for the motion of spinning particles in a plane, for which we choose $\theta = \pi/2$. For scalar particles the orbital angular momentum is always conserved; hence, any solution of scalar particles would actually describe planar motion. But this is no longer true in general for spinning particles, because in this case only the total angular momentum is a constant of motion. Therefore, planar motion for spinning particles is strictly possible only in special cases, in which orbital and spin angular momentum are separately conserved. This may happen only if either the orbital angular momentum vanishes, or if spin and orbital angular momentum are parallel to each other.

The equations of motion (3.2.12) and (3.2.18) with $\theta = \pi/2$ and $d\theta/d\tau = 0$ become

$$\frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)}\left[\frac{E}{m} + \frac{1}{2E}\left(\alpha - \frac{q^2}{r}\right)\frac{d\varphi}{d\tau}S^{r\varphi}\right], \tag{3.3.1a}$$

$$\frac{dr}{d\tau} = \left\{\left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)^2\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)\right.$$

$$\left. - r^2\left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)\left(\frac{d\varphi}{d\tau}\right)^2\right\}^{\frac{1}{2}}, \tag{3.3.1b}$$

$$\frac{d\varphi}{d\tau} = \frac{1}{mr^2}J^{(3)} - \frac{1}{mr}S^{r\varphi}, \tag{3.3.1c}$$

$$\frac{d}{d\tau}\left(rS^{r\theta}\right) = -r^2\left(1 - \frac{3\alpha}{2r} + \frac{2q^2}{r^2}\right)\frac{d\varphi}{d\tau}S^{\theta\varphi}, \tag{3.3.1d}$$

$$\frac{d}{d\tau}\left(rS^{r\varphi}\right) = 0. \tag{3.3.1e}$$

Equations (3.3.1c) and (3.3.1e) imply that the orbital angular momentum and the component of the spin perpendicular to the plane of motion are separately conserved:

$$rS^{r\varphi} \equiv \Sigma \tag{3.3.2a}$$

$$m\,r^2 \frac{d\varphi}{d\tau} = J^{(3)} - \Sigma \equiv L \tag{3.3.2b}$$

where Σ and L are two constants. This result leads to the remark that the presence of spin-dependent forces modifies the gravitational red-shift. Indeed, equation (3.3.1a) becomes

$$dt = \frac{d\tau}{\left(1 - \dfrac{\alpha}{r} + \dfrac{q^2}{r^2}\right)} \left[\frac{E}{m} + \frac{\alpha - \dfrac{q^2}{r}}{2\,m\,E\,r^3} L\,\Sigma \right]. \tag{3.3.3}$$

For a nonzero value of the orbital angular momentum L, it follows from equation (3.3.3) that there is an additional contribution to time-dilation from spin-orbit coupling. This expresses the fact that the time-dilation is not a purely geometric effect, but also has a dynamical component [24, 25]. Equation (3.3.3) reduces to the Schwarzschild result for $q = 0$.

In the case of planar motion equation (3.2.13) simplifies to

$$r^2 S^{0\varphi} = J^{(1)} \cos\varphi + J^{(2)} \sin\varphi, \tag{3.3.4}$$

while equation (3.2.12c) to

$$rS^{r\theta} = -J^{(1)}\sin\varphi + J^{(2)}\cos\varphi. \tag{3.3.5}$$

Equations (3.3.4) and (3.3.5) can be combined to give

$$\frac{d}{d\tau}\left(rS^{r\theta}\right) = -r^2 S^{\theta\varphi}\frac{d\varphi}{d\tau}. \tag{3.3.6}$$

Comparing equation (3.3.6) with equation (3.3.1d) we find that there are only two possibilities:

$$\text{(I)}\quad \frac{d\varphi}{d\tau} = 0, \qquad \text{(II)}\quad S^{\theta\varphi} = 0. \tag{3.3.7}$$

CASE I: For $\frac{d\varphi}{d\tau} = 0$ the orbital angular momentum vanishes. Then the solution describes a particle moving along a fixed radius from or towards the source of the gravitational field. The motion of the particle for a distant observer is described by

$$\frac{dr}{dt} = \left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)\sqrt{1 - \frac{m^2}{E^2}\left(1 - \frac{\alpha}{r} + \frac{q^2}{r^2}\right)}, \tag{3.3.8}$$

as in the case of a spinless particle. Because of the vanishing of the orbital angular momentum the (Cartesian) spin tensor components are all conserved, *e.g.*, if we choose $\varphi = 0$ for the path of the particle, then

$$r^2 S^{0\varphi} = J^{(1)}, \qquad rS^{r\theta} = J^{(2)},$$

$$rS^{r\varphi} = J^{(3)}. \tag{3.3.9}$$

CASE II: If $\dfrac{d\varphi}{d\tau} \neq 0$, then equation (3.3.7) gives

$$S^{\theta\varphi} = 0, \tag{3.3.10a}$$

and hence

$$J^{(1)} = J^{(2)} = 0. \tag{3.3.10b}$$

Then equation (3.3.5) gives

$$S^{r0} = 0. \tag{3.3.10c}$$

This implies that the spin is parallel to the orbital angular momentum. From equations (3.3.1a)–(3.3.1c) we obtain

$$\frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2 = \frac{\left(E^2 - m^2\right)r^2}{L^2} - 1 + \frac{m}{L}\left(\alpha - \frac{q^2}{r^2}\right)\left(\frac{m\,r}{L} + \frac{J^{(3)}}{m\,r}\right) \tag{3.3.11}$$

for the orbit of the particle. Using the dimensionless variables

$$\epsilon = \frac{E}{m}, \qquad x = \frac{r}{\alpha}, \qquad \ell = \frac{L}{m\alpha},$$

$$\Delta = \frac{\Sigma}{L}, \qquad \delta = \frac{q}{\alpha}. \tag{3.3.12}$$

equation (3.3.11) can be put in the form

$$\in^2 = \alpha^2 \dot{x}^2 + \alpha^2 x^2 \left(\frac{d\varphi}{d\tau} \right)^2 + 1 - \left(\frac{1}{x} + \frac{\ell^2 (1+\Delta)}{x^3} \right) \left(1 - \frac{q^2}{\alpha^2 x} \right). \tag{3.3.13}$$

The presence of Δ, which is a bilinear combination of anti-commuting variables ψ^μ, makes the equation rigorous. For the investigation of a possible motion, a numerical value needs to be assigned to Δ. As mentioned in Introduction of this thesis, such a quantum mechanical expectation value is desirable. To avoid any inconsistency that may result from this semiclassical approximation, the numerical value of Δ is supposed to be small: $\Delta \ll 1$.

If the expression for $\frac{d\varphi}{d\tau}$ from (3.3.1c) is substituted on the right-hand side of equation (3.3.13), then it follows that

$$\frac{\ell^2}{x^4} \left(\frac{dx}{d\varphi} \right)^2 = \alpha^2 \left(\frac{dx}{d\tau} \right)^2 \equiv \in^2 - U_R \left(x, \ell^2, \delta^2 \right), \tag{3.3.14a}$$

where

$$U_R \left(x, \ell^2, \delta^2 \right) = 1 - \frac{1}{x} + \left(\ell^2 + \delta^2 \right) \frac{1}{x^2}$$

$$- (1+\Delta) \ell^2 \frac{1}{x^3} + (1+\Delta) \delta^2 \ell^2 \frac{1}{x^4} \tag{3.3.14b}$$

is defined as an effective potential. This equation is the same, as one would obtain for a one-dimensional problem with a potential $U_R\left(x, \ell^2, \delta^2\right)$ In the one-dimensional problem, the particle is subject to a radial force $F\left(x, \ell^2, \delta^2\right)$ given by

$$F\left(x, \ell^2, \delta^2\right) = -\frac{\partial}{\partial x} U_R\left(x, \ell^2, \delta^2\right) \tag{3.3.15}$$

This is the effective force that the three-dimensional particle feels in the radial direction, including a contribution from the centripetal acceleration. Because of the non-negativity of the kinetic energy, the right-hand side of (3.3.14a) must be positive.

Bound state orbits are possible for $\in < 1$. They correspond to quasi-elliptic and circular orbits. The radius of the possible circular orbit is minimum at the point of inflection of the function $U_R\left(x, \ell^2, \delta^2\right)$ This critical radius satisfies the equation

$$x^3 - 3\left(1+\Delta\right)x^2 + 9\left(1+\Delta\right)\delta^2 x - 8\left(1+\Delta\right)\delta^4 = 0, \tag{3.3.16}$$

and the corresponding energy and angular momentum are given by

$$\in^2_{crit} = \left(1 - \frac{X}{Y\,x}\right) \tag{3.3.17a}$$

and

$$\ell^2 = \frac{1}{Y}\left(x - 2\delta^2\right) \tag{3.3.17b}$$

respectively, provided that $Y>0$, where

$$X = 1 - 2(1+\Delta)\frac{1}{x}\left\{1 - \left(3 - \frac{2\delta^2}{x}\right)\frac{\delta^2}{x}\right\} - \frac{2\delta^2}{x}, \tag{3.3.17c}$$

$$Y = 2 - 3\ (1+\Delta)\frac{1}{x} + 4\ (1+\Delta)\delta^2\,\frac{1}{x^2}\ . \tag{3.3.17d}$$

For $q\ =\ 0$, equations (3.3.16) and (3.3.17) reduce to the Schwarzschild results described in Chapter II, if the terms first order in Δ are only considered.

We now discuss the quasi-elliptic orbits shortly. The classical Keplerian orbits are bound-state solutions, which are circles and ellipses parameterized by

$$x = \frac{\kappa}{1 + \varepsilon \cos\left(\varphi - \varphi_0\right)} \tag{3.3.18}$$

where $\kappa = k/\alpha$, k being the semilatus rectum and ε the eccentricity with $0<\varepsilon<1$ for ellipses and $\varepsilon=0$ for a circle. For an elliptic orbit the perihelion x_{ph} is reached for $\varphi=\varphi_0$. However, in Reissner-Nordstrom spacetime, relativistic effects turn the perihelion during the motion of the particle. Let the function $w(\varphi)$ describes this turning. Equation (3.3.18) then takes the form

$$x = \frac{\kappa}{1 + \varepsilon \cos\left[\varphi - w(\varphi)\right]}. \tag{3.3.19}$$

The perihelion and aphelion are now given by

$$\varphi_{ph}^{(n)} - w\left(\varphi_{ph}^{(n)}\right) = 2n\pi, \qquad (3.320a)$$

and

$$\varphi_{ah}^{(n)} - w\left(\varphi_{ah}^{(n)}\right) = (2n+1)\pi. \qquad (3.3.20b)$$

respectively. The particle reaches its nth perihelion at the angle $\varphi_{ph}^{(n)}$ and the turning of the perihelion after n revolutions is given by the angle $w\left(\varphi_{ph}^{(n)}\right)$. Hence, the precession of the perihelion after one revolution is

$$\Delta w \equiv w\left(\varphi_{ph}^{(1)}\right) - w\left(\varphi_{ph}^{(0)}\right) = \varphi_{ph}^{(1)} - \varphi_{ph}^{(0)} - 2\pi \equiv \Delta\varphi - 2\pi. \qquad (3.3.21)$$

The energy of the particle ϵ at the perihelion / aphelion is given by

$$\epsilon^2 = 1 - \left(\frac{1 \pm \varepsilon}{\kappa}\right) + \left(\ell^2 + \delta^2\right)\left(\frac{1 \pm \varepsilon}{\kappa}\right)^2 - (1+\Delta)\ell^2\left(\frac{1 \pm \varepsilon}{\kappa}\right)^3$$

$$+ (1+\Delta)\delta^2\ell^2\left(\frac{1 \pm \varepsilon}{\kappa}\right)^4. \qquad (3.3.22)$$

Since the energy ϵ is a constant of motion, comparing both expressions for ϵ^2 we obtain

$$\diagup^{2} = \frac{\kappa^2\left(\kappa - 2\delta^2\right)}{2\kappa^2 - (1+\Delta)\left[\left(3+\varepsilon^2\right)\kappa - 4\delta^2\left(1+\varepsilon^2\right)\right]}. \qquad (3.3.23)$$

Using (3.3.19), (3.3.22) and (3.3.23), equation (3.3.11) can be rewritten in terms of $w(\varphi)$ and φ. The result is

$$\left(1 - \frac{2\delta^2}{\kappa}\right)\left(1 - \frac{dw}{d\varphi}\right)^2 = 1 - \frac{(1+\Delta)}{\kappa^3}\left[\left\{3\kappa\left(\kappa - 3\delta^2\right) + 8\delta^4\right\}\right.$$

$$+ \varepsilon\left\{\kappa\left(\kappa - 6\delta^2\right) + 8\delta^4\right\}\cos(\varphi - w(\varphi))$$

$$\left. - \varepsilon^2\left\{\left(\kappa - 2\delta^2\right)\delta^2\cos^2(\varphi - w(\varphi)) + 2\delta^4\right\}\right]. \qquad (3.3.24)$$

Introducing

$$y = \varphi - w(\varphi), \qquad (3.3.25)$$

equation (3.3.24) can be put in the form

$$d\varphi = \frac{dy}{\sqrt{A\left(a - b\cos y + c\cos^2 y\right)}}, \qquad (3.3.26a)$$

where

$$A = \frac{\kappa}{\kappa - 2\delta^2}, \qquad F = (1+\Delta)\frac{1}{\kappa},$$

$$a = 1 - \exists F, \qquad b = D F, \qquad c = N F,$$

$$\exists = \frac{1}{\kappa^2}\left\{3\kappa\left(\kappa - 3\delta^2\right) + 8\delta^4 - 2\varepsilon^2\delta^4\right\},$$

$$D = \frac{\varepsilon}{\kappa^2}\left\{\kappa\left(\kappa - 6\delta^2\right) + 8\delta^4\right\},$$

$$N = \frac{\varepsilon^2}{\kappa^2}\delta^2\left(\kappa - 2\delta^2\right) \qquad (3.3.26b)$$

In order to obtain the precession of the perihelion after one revolution we need to integrate equation (3.3.26) from one perihelion to the next one with $0 \leq y \leq 2\pi$. The result gives $\Delta\varphi$, as defined in (3.3.21):

$$\Delta\varphi = \frac{1}{\sqrt{A\,a}} \int_0^{2\pi} \frac{dy}{\sqrt{1 - \left\{(b/a)\cos y - (c/a)\cos^2 y\right\}}} \,. \qquad (3.3.27)$$

The relativistic effects and all Δ-dependence are contained in F. So, we first expand the integral (3.3.27) in a power series in F and then integrate term by term. Using the expansion

$$(1 - x)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} A(m)\, x^m, \qquad A(m) = \frac{1}{2^{2m}}\binom{2m}{m}, \qquad (3.3.28)$$

and the integrals

$$\int_0^{2\pi} dy \cos^{2m} y = 2\pi\, A(m), \tag{3.3.29}$$

$$\int_0^{2\pi} dy \cos^{2m+1} y = 0, \tag{3.3.30}$$

we obtain the following expression for $\Delta\varphi$:

$$\Delta\varphi = 2\pi\,(1-\zeta)^{\frac{1}{2}}\left\{1 + 3\eta\left[1 - \frac{18+\varepsilon^2}{12}\zeta + \frac{8-\varepsilon^2}{12}\zeta^2\right](1+\Delta)\right.$$

$$+\frac{3}{4}\eta^2\left[18\left(1 - \frac{3}{2}\zeta + \frac{4-\varepsilon^2}{6}\zeta^2\right)^2 + \varepsilon^2\left(1 - 3\zeta + 2\zeta^2\right)^2\right.$$

$$-\frac{5}{3}\varepsilon^2\zeta\,(1-\zeta)\left(1 - \frac{72+9\varepsilon^2}{80}\zeta + \frac{32+\varepsilon^2}{80}\zeta^2\right)$$

$$\left.\left.+\frac{3}{16}\varepsilon^4\zeta^2(1-\zeta)^2\right](1+\Delta)^2 + \ldots\right\}, \tag{3.3.31a}$$

where

$$\zeta = \frac{q^2}{M^2}\eta, \qquad \eta = \frac{1}{2\kappa}. \tag{3.3.31b}$$

For $\Delta=0$ the lowest-order contribution to the relativistic precession of the perihelion is given by the second term in the expansion. Considering terms of first order in Δ, we find that in principle the spin of a particle contributes to this lowest-order precession. With $q=0$ the results in (3.3.22), (3.3.23) and (3.3.31) reduce to the Schwarzschild results described in the previous chapter.

3.4. SPINNING NUT-REISSNER-NORDSTROM SPACETIME

In this section we extend the work of section 3.2 in the Reissner-Nordstrom spacetime generalized with NUT parameter. This study gives the parallel results as we got in section 3.2. But it is interesting to note that a spacetime generalized with NUT parameter has not direct physical interpretation [83]. The NUT-Reissner-Nordstrom spacetime is described by the metric [111]

$$ds^2 = -U(r)\left[dt + 4n\sin^2\frac{\theta}{2}d\varphi\right]^2 + \frac{1}{U(r)}dr^2$$

$$+\left(r^2+n^2\right)\left[d\theta^2+\sin^2\theta\, d\varphi^2\right], \tag{3.4.1}$$

where

$$U(r) = 1 - \frac{2}{r^2+n^2}\left[M\, r + n^2 - \frac{q^2}{2}\right], \tag{3.4.2}$$

n is the NUT (or magnetic mass) parameter, q the charge and M the total mass of the gravitating body. The spacetime given by (3.4.1) and (3.4.2) gives

(i) for $q = 0$, the NUT spacetime [112],

(ii) for $n = 0$, the Reissner-Nordstrom spacetime [98, 99],

(iii) for $n = q = 0$, the Schwarzschild spacetime [97].

Spaces with a metric of the form given above have an isometry group $SU(2)\times U(1)$. The invariance of the metric (3.4.1) under time translations and spatial rotations is generated by the four ψ-independent solutions $R^{(\beta)\mu}(x)$ of the generalized Killing equation (1.3.15), ($\beta=0,\dots, 3$). The associated vector fields have the form

$$D^{(\beta)} \equiv R^{(\beta)\mu}(x)\partial_{\mu}, \quad \beta=0,\dots,3, \tag{3.4.3}$$

or explicitly

$$D^{(0)} = \frac{\partial}{\partial t}, \tag{3.4.4a}$$

$$D^{(1)} = -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi} - 2n\tan\frac{\theta}{2}\cos\varphi\frac{\partial}{\partial t}, \tag{3.4.4b}$$

$$D^{(2)} = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} - 2n\tan\frac{\theta}{2}\sin\varphi\frac{\partial}{\partial t}, \tag{3.4.4c}$$

$$D^{(3)} = \frac{\partial}{\partial \varphi} - 2n\frac{\partial}{\partial t}. \tag{3.4.4d}$$

$D^{(0)}$, which generates the $U(1)$ of t translations, commutes with the other Killing vectors. The remaining three vectors obey an $SU(2)$ algebra with

$$\left[D^{(a)}, D^{(b)}\right] = -\varepsilon^{abc} D^{(c)}, \qquad (a,b,c = 1,2,3). \tag{3.4.5}$$

This can be contrasted with the Reissner-Nordstrom spacetime, where the isometry group at spacelike infinity is $SO(3)\times U(1)$. This illustrates the essential topological character of the magnetic mass [113, 114].

In the purely bosonic case these invariances would correspond to conservation of the so-called "relative electric charge" and the angular momentum [79,115 –121]

$$q_r = -U\left[\frac{dt}{d\tau} + 4n\sin^2\frac{\theta}{2}\frac{d\varphi}{d\tau}\right], \tag{3.4.6}$$

$$\vec{j} = \vec{r}\times\vec{p} + 2nq_r\frac{\vec{r}}{r}. \tag{3.4.7}$$

The first generalized Killing equation (1.3.16) suggests that for each Killing vector, $R_\mu^{(\beta)}(x)$, there is an associated Killing scalar, $B^{(\beta)}$. Therefore, if we limit ourselves to variations (1.3.9) which terminate after the terms linear in $\dot{x}$, we obtain the constants of motion

$$J^{(\beta)} = B^{(\beta)} + \dot{x}^{\mu} R^{(\beta)}_{\mu}. \tag{3.4.8}$$

Equation (3.4.8) asserts that the Killing scalars contribute to the "relative electric charge" and the total angular momentum.

Inserting the expressions for the connection and the Riemann curvature components corresponding to the NUT-RN spacetime in (1.3.16), we obtain for the Killing scalars

$$B^{(0)} = VS^{tr} - 4nV\sin^2\frac{\theta}{2}S^{r\varphi} - 2nU\sin\theta\, S^{\theta\varphi}, \tag{3.4.9a}$$

$$B^{(1)} = -2nV\cos\varphi\tan\frac{\theta}{2}(1+\cos\theta)S^{tr}$$

$$-nU\cos\varphi\cos\theta\, S^{t\theta} - r\sin\varphi\, S^{r\theta}$$

$$+\cos\varphi\cot\theta\left[8n^2V\cos\theta\tan\frac{\theta}{2}\sin^2\frac{\theta}{2} - r\sin^2\theta\right]S^{r\varphi}$$

$$+\cos\varphi\left[\left(r^2+n^2\right)\sin^2\theta + 4n^2U - 8n^2U\tan^2\frac{\theta}{2}\right]S^{\theta\varphi}, \tag{3.4.9b}$$

$$B^{(2)} = -2nV\sin\varphi\tan\frac{\theta}{2}(1+\cos\theta)S^{tr}$$

$$-nU\sin\varphi\cos\theta\, S^{t\theta} + r\cos\varphi\, S^{r\theta}$$

$$+ \sin\varphi \cot\theta \left[8n^2 V \cos\theta \tan\frac{\theta}{2} \sin^2\frac{\theta}{2} - r \sin^2\theta \right] S^{r\varphi}$$

$$+ \sin\varphi \left[\left(r^2 + n^2\right)\sin^2\theta + 4n^2 U - 8n^2 U \tan^2\frac{\theta}{2} \right] S^{\theta\varphi}, \qquad (3.4.9c)$$

$$B^{(3)} = -2nV\cos\theta\, S^{tr} + 4nV \sin^2\frac{\theta}{2} S^{t\theta}$$

$$+ \left[r\sin^2\theta + 8n^2 V \sin^2\frac{\theta}{2} \cos^2\theta \right] S^{r\varphi}$$

$$+ \sin\theta \left[\left(r^2 + n^2\right)\cos\theta - 2n^2 U \left(1 + 4\sin^2\frac{\theta}{2} \right) \right] S^{\theta\varphi}, \qquad (3.4.9d)$$

where
$$V = \frac{1}{\left(r^2 + n^2\right)^2} \left[M\left(r^2 - n^2\right) + \left(2n^2 - q^2\right) r \right], \qquad (3.4.9e)$$

and U is given by (3.4.2).

Taking into accounts the contribution of the Killing scalars, one finds for the conserved quantities $J^{(\beta)}$,

$$J^{(0)} = B^{(0)} + q_r, \qquad (3.4.10a)$$

$$J^{(1)} = B^{(1)} - \left(r^2 + n^2\right)\sin\varphi \frac{d\theta}{d\tau} - \left(r^2 + n^2\right)\cos\theta \sin\theta \cos\varphi \frac{d\varphi}{d\tau}$$

$$+ 2nq_r \sin\theta \cos\varphi, \qquad (3.4.10b)$$

$$J^{(2)} = B^{(2)} + \left(r^2 + n^2\right)\cos\varphi \frac{d\theta}{d\tau} - \left(r^2 + n^2\right)\cos\theta \sin\theta \sin\varphi \frac{d\varphi}{d\tau}$$

$$+ 2nq_r \sin\theta \sin\varphi, \qquad (3.4.10c)$$

$$J^{(3)} = B^{(3)} + \left(r^2 + n^2\right)\sin^2\theta \frac{d\varphi}{d\tau} + 2nq_r \cos\theta, \qquad (3.4.10d)$$

where q_r has the expression from (3.4.6).

It is obvious that the "relative electric charge", q_r, is no longer conserved, contrasting with the purely bosonic case. On the other hand, the conserved total angular momentum is the sum of the orbital angular momentum, the Poincare contribution and the spin angular momentum:

$$\vec{J} = \vec{B} + \vec{j}, \qquad (3.4.11a)$$

where

$$\vec{J} = \left(J^{(1)}, J^{(2)}, J^{(3)}\right) \qquad \vec{B} = \left(B^{(1)}, B^{(2)}, B^{(3)}\right) \qquad (3.4.11b)$$

From (3.4.10) we can derive two very interesting relations:

$$J^{(1)} \sin\varphi - J^{(2)} \cos\varphi = -rS^{r\theta} - \left(r^2 + n^2\right)\frac{d\theta}{d\tau}. \qquad (3.4.12)$$

and

$$2nJ^{(0)} - \frac{\vec{J}\cdot\vec{r}}{r} = 4nVS^{tr} - \left(4V\sin^2\frac{\theta}{2} - U\sin\theta\right)n\cos\theta\, S^{t\theta}$$

$$-\,8n^2V\sin^2\frac{\theta}{2}\left[1+\cos^2\theta\left(\cos\theta+\tan\frac{\theta}{2}\right)\right]S^{r\varphi}$$

$$-\left[\left(r^2+n^2\right)+8n^2U\left(1-\tan^2\frac{\theta}{2}\right)\right.$$

$$\left.-\,2n^2U\left(1+4\sin^2\frac{\theta}{2}\right)\cos\theta\right]\sin\theta\, S^{\theta\varphi}. \qquad (3.4.13)$$

In the standard NUT-RN space, (3.4.13) reduces to

$$\frac{\vec{j}\cdot\vec{r}}{r} = \left|\,\vec{j}\,\right|\cos\theta = 2nq_r, \qquad (3.4.14)$$

which fixes the relative motion to lie on a cone whose vertex is at the origin and whose axis is $\vec{j}$. Equation (3.4.13) expresses the fact that the total angular momentum in the radial direction receives contributions from the spin angular momentum, the orbital angular momentum being absent in that direction.

In addition to these constants of motion there are four universal conserved charges described in Chapter I. Using the notation from this section they are

(i) The energy

$$E = \frac{1}{2U}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(r^2 + n^2\right)\left(\frac{d\theta}{d\tau} + \sin^2\theta\frac{d\varphi}{d\tau}\right)$$

$$-\frac{1}{2}U\left(\frac{dt}{d\tau} + 4n\sin^2\frac{\theta}{2}\frac{d\varphi}{d\tau}\right)^2, \tag{3.4.15}$$

(ii) The supercharge

$$Q = \frac{1}{U}\frac{dr}{d\tau}\psi^r + \left(r^2 + n^2\right)\frac{d\theta}{d\tau}\psi^\theta + q_r\psi^t$$

$$+\left[4n\sin^2\frac{\theta}{2}q_r + \left(r^2 + n^2\right)\sin^2\theta\frac{d\varphi}{d\tau}\right]\psi^\varphi, \tag{3.4.16}$$

(iii) The chiral charge

$$\Gamma_* = \left(r^2 + n^2\right)\sin\theta\,\psi^r\psi^\theta\psi^\varphi\psi^t, \tag{3.4.17}$$

(iv) The dual supercharge

$$Q^* = \left(r^2 + n^2\right)\sin\theta\left(\frac{dr}{d\tau}\psi^\theta\psi^\varphi\psi^t - \frac{d\theta}{d\tau}\psi^r\psi^\varphi\psi^t\right.$$

$$\left.+\frac{d\varphi}{d\tau}\psi^r\psi^\theta\psi^t - \frac{dt}{d\tau}\psi^r\psi^\theta\psi^\varphi\right), \tag{3.4.18}$$

Finally, having in mind that ψ^{μ} is covariantly constant as formulated in (1.2.9), the rate of change of spin is obtained as follows:

$$\frac{d\psi^{r}}{d\tau}=\left[rU-\left(r^{2}+n^{2}\right)V\right]\left(\frac{d\theta}{d\tau}\psi^{\theta}+sin^{2}\theta\frac{d\varphi}{d\tau}\psi^{\varphi}\right), \qquad (3.4.19a)$$

$$\frac{d\psi^{\theta}}{d\tau}=-\frac{r}{r^{2}+n^{2}}\frac{d\theta}{d\tau}\psi^{r}-\frac{r}{r^{2}+n^{2}}\frac{dr}{d\tau}\psi^{\theta}$$

$$+sin\theta\left[\left(cos\theta-\frac{4n^{2}U}{r^{2}+n^{2}}sin^{2}\frac{\theta}{2}\right)\frac{d\varphi}{d\tau}+\frac{nq_{r}}{r^{2}+n^{2}}\right]\psi^{\varphi}, \qquad (3.4.19b)$$

$$\frac{d\psi^{\varphi}}{d\tau}=\frac{nU}{r^{2}+n^{2}}cos\,ec\theta\frac{d\theta}{d\tau}\psi^{t}-\frac{r}{r^{2}+n^{2}}\frac{d\varphi}{d\tau}\psi^{r}$$

$$-\left(cot\theta\frac{d\varphi}{d\tau}+\frac{n\,cos\,ec\theta}{r^{2}+n^{2}}q_{r}\right)\psi^{\theta}$$

$$-\left[\frac{r}{r^{2}+n^{2}}\frac{dr}{d\tau}+\left(cot\theta-\frac{2n^{2}U}{r^{2}+n^{2}}tan\frac{\theta}{2}\right)\frac{d\theta}{d\tau}\right]\psi^{\varphi}, \qquad (3.4.19c)$$

$$\frac{d\psi^{t}}{d\tau}=\left(\frac{V}{U}\frac{dr}{d\tau}-\frac{2n^{2}U}{r^{2}+n^{2}}tan\frac{\theta}{2}\frac{d\theta}{d\tau}\right)\psi^{t}$$

$$+\left[4n\,sin^{2}\frac{\theta}{2}\left(\frac{r}{r^{2}+n^{2}}-\frac{2V}{U}\right)\frac{d\varphi}{d\tau}-\frac{V}{U^{2}}q_{r}\right]\psi^{r}$$

$$-\left(2n\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}\frac{d\varphi}{d\tau}-\frac{2n^2q_r}{r^2+n^2}\tan\frac{\theta}{2}\right)\psi^\theta$$

$$+\left[4n\sin^2\frac{\theta}{2}\left(\frac{r}{r^2+n^2}-\frac{V}{U}\right)\frac{dr}{d\tau}\right.$$

$$\left.-2n\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}\left(1+\frac{4n^2U}{r^2+n^2}\right)\frac{d\theta}{d\tau}\right]\psi^\varphi. \qquad (3.4.19d)$$

As a rule these complicated equations could be integrated to obtain the full solution of the equations of motion for the usual coordinates, $\{x^\mu\}$, and Grassmann coordinates, $\{\psi^\mu\}$. These equations are quite intricate and the general solution is by no means illuminating. Instead of the general solution, we shall discuss special solutions in the next section for the motion on a cone and in a plane.

We notice that the above equations reduce to those obtained in section 3.2 for the Reissner-Nordstrom spacetime when $n = 0$, and to those obtained in section 2.2 of Chapter II for the Schwarzschild spacetime when $n = q = 0$.

3.5. SPECIAL MOTION IN NUT-REISSNER-NORDSTROM SPACETIME

In this section we solve the equations derived in the previous section for the motion on a cone and in a plane. We first consider the motion on a cone.

Let us choose the z-axis along $\vec{j}$ so that the motion of the particle may be conveniently described in terms of polar coordinates

$$\vec{r} = r\,\vec{e}(\theta,\varphi) \tag{3.5.1}$$

with

$$\vec{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{3.5.2}$$

The equation of motion for the spin components, when $d\theta/d\tau = 0$, are

$$\frac{dS^{r\theta}}{d\tau} = -\frac{r}{r^2+n^2}\frac{dr}{d\tau}S^{r\theta} + \sin\theta\left[\left(\cos\theta - \frac{4n^2U}{r^2+n^2}\sin^2\frac{\theta}{2}\right)\frac{d\varphi}{d\tau} + \frac{nq_r}{r^2+n^2}\right]S^{r\varphi}$$

$$-\left[rU - \left(r^2+n^2\right)V\right]\sin^2\theta\frac{d\varphi}{d\tau}S^{\theta\varphi}, \tag{3.5.3a}$$

$$\frac{dS^{r\varphi}}{d\tau} = -\frac{r}{r^2+n^2}\frac{dr}{d\tau}S^{r\varphi} - \left(\cot\theta\frac{d\varphi}{d\tau} + \frac{n\,cosec\theta}{r^2+n^2}q_r\right)S^{r\theta}, \tag{3.5.3b}$$

$$\frac{dS^{\theta\varphi}}{d\tau} = -\frac{2r}{r^2+n^2}\frac{dr}{d\tau}S^{\theta\varphi} + \frac{r}{r^2+n^2}\frac{d\varphi}{d\tau}S^{r\theta}, \tag{3.5.3c}$$

$$\frac{dS^{\theta t}}{d\tau} = \left(\frac{V}{U} - \frac{r}{r^2+n^2}\right)\frac{dr}{d\tau}S^{\theta t}$$

$$+\sin\theta\left[\left(\cos\theta - \frac{4n^2U}{r^2+n^2}\sin^2\frac{\theta}{2}\right)\frac{d\varphi}{d\tau} + \frac{nq_r}{r^2+n^2}\right]S^{\varphi t}$$

$$-\left[4n\sin^2\frac{\theta}{2}\left(\frac{r}{r^2+n^2} - \frac{2V}{U}\right)\frac{d\varphi}{d\tau} - \frac{V}{U^2}q_r\right]S^{r\theta}$$

$$+ 4n \sin^2 \frac{\theta}{2}\left(\frac{r}{r^2+n^2} - \frac{V}{U}\right)\frac{dr}{d\tau} S^{\theta\varphi}, \tag{3.5.3d}$$

$$\frac{dS^{rt}}{d\tau} = \left[rU - \left(r^2+n^2\right)V\right]\sin^2\theta \frac{d\varphi}{d\tau} S^{\varphi t} + \frac{V}{U}\frac{dr}{d\tau} S^{rt}$$

$$- 2n \tan\frac{\theta}{2}\left(\sin^2\frac{\theta}{2}\frac{d\varphi}{d\tau} - \frac{nq_r}{r^2+n^2}\right) S^{r\theta}$$

$$+ 4n \sin^2 \frac{\theta}{2}\left(\frac{r}{r^2+n^2} - \frac{V}{U}\right)\frac{dr}{d\tau} S^{r\varphi}, \tag{3.5.3e}$$

$$\frac{dS^{\varphi t}}{d\tau} = -\frac{r}{r^2+n^2}\frac{d\varphi}{d\tau} S^{rt} - \left(\cot\theta \frac{d\varphi}{d\tau} + \frac{n \operatorname{cosec}\theta}{r^2+n^2} q_r\right) S^{\theta t}$$

$$+ \left(\frac{V}{U} - \frac{r}{r^2+n^2}\right)\frac{dr}{d\tau} S^{\varphi t}$$

$$- \left[4n \sin^2\frac{\theta}{2}\left(\frac{r}{r^2+n^2} - \frac{2V}{U}\right)\frac{d\varphi}{d\tau} - \frac{V}{U^2} q_r\right] S^{r\varphi}$$

$$+ 2n \tan\frac{\theta}{2}\left(\sin^2\frac{\theta}{2}\frac{d\varphi}{d\tau} - \frac{nq_r}{r^2+n^2}\right) S^{\theta\varphi}. \tag{3.5.3f}$$

Since we are looking for solutions with $d\theta/d\tau = 0$ and because $J^{(1)} = J^{(2)} = 0$, we have from (3.4.12),

$$S^{r\theta} = 0. \tag{3.5.4}$$

This relation implies that the special solutions investigated in this section are situated in the sector with

$$\Gamma_{*} = 0. \tag{3.5.5}$$

A particular solution may be obtained, if we choose $S^{\varphi t} = 0$, in the form

$$S^{r\varphi} = \frac{C^{r\varphi}}{\sqrt{(r^2 + n^2)}}, \tag{3.5.6a}$$

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 + n^2}, \tag{3.5.6b}$$

$$S^{\theta t} = \left(\frac{U}{r^2 + n^2}\right)^{\frac{1}{2}} C^{\theta t} - \frac{2n}{r^2 + n^2} \sin^2 \frac{\theta}{2} C^{\theta\varphi}, \tag{3.5.6c}$$

$$S^{rt} = \sqrt{U} C^{rt} - \frac{4n}{\sqrt{(r^2 + n^2)}} \sin^2 \frac{\theta}{2} C^{r\varphi}, \tag{3.5.6d}$$

where $C^{r\varphi}, C^{\theta\varphi}, C^{\theta t}, C^{rt}$ are Grassmann constants.

We investigate the case in which $Q = 0$ (equation (1.4.12)). As in section 3.2 (equation (3.2.15)) we have $\Gamma_{*} = Q^{*} = 0$. For the spin components we deduce the following relations:

$$\frac{1}{U}\frac{dr}{d\tau}S^{r\theta} = \left[\left(r^2+n^2\right)\sin^2\theta\frac{d\varphi}{d\tau}+4n\sin^2\frac{\theta}{2}q_r\right]S^{\theta\varphi}+q_rS^{\theta t}, \tag{3.5.7a}$$

$$\frac{1}{U}\frac{dr}{d\tau}S^{r\varphi} = q_rS^{\varphi t}, \tag{3.5.7b}$$

$$\frac{1}{U}\frac{dr}{d\tau}S^{rt} = -\left[\left(r^2+n^2\right)\sin^2\theta\frac{d\varphi}{d\tau}+4n\sin^2\frac{\theta}{2}q_r\right]S^{\varphi t}. \tag{3.5.7c}$$

The condition $Q = 0$ modifies drastically the form of the solutions.

In spite of the complexity of the equations, we have a simple exact solution for the components of the spin-tensor,

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}. \tag{3.5.8}$$

From (3.4.10) we can deduce that

$$q_r = J^{(0)} + 2nU\sin\theta\frac{C^{\theta\varphi}}{r^2+n^2}, \tag{3.5.9}$$

$$\frac{d\varphi}{d\tau} = \frac{1}{r^2+n^2}\left[\frac{2nJ^{(0)}}{\cos\theta}+\frac{\sin\theta}{\cos\theta}C^{\theta\varphi}\right.$$

$$\left.+\frac{4n^2U}{r^2+n^2}\cdot\frac{4-\cos\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)}C^{\theta\varphi}\right]. \tag{3.5.10}$$

These relations may be integrated to obtain the expressions for φ and t. We can deduce $dr/d\tau$ from the energy, given in equation (3.4.15).

We now study the special case of motion in a plane, for which we choose $\theta=\pi/2$. For scalar particles any solution would actually describe planar motion, because the orbital angular momentum is always conserved. But this is no longer true in general for spinning particles. As mentioned in section 3.3, motion in a plane for spinning particles occurs only in two kinds of situations: (I) the orbital angular momentum vanishes, and (II) spin and orbital angular momentum are parallel.

For $\theta=\pi/2$ the equations of motion are

$$\frac{dS^{r\theta}}{d\tau}=-\frac{r}{r^2+n^2}\frac{dr}{d\tau}S^{r\theta}-\frac{2n^2U}{r^2+n^2}\frac{d\varphi}{d\tau}S^{r\varphi}$$

$$-\left[rU-\left(r^2+n^2\right)V\right]\frac{d\varphi}{d\tau}S^{\theta\varphi}, \tag{3.5.11a}$$

$$\frac{dS^{r\varphi}}{d\tau}=-\frac{r}{r^2+n^2}\frac{dr}{d\tau}S^{r\varphi}, \tag{3.5.11b}$$

$$\frac{dS^{\theta\varphi}}{d\tau}=-\frac{2r}{r^2+n^2}\frac{dr}{d\tau}S^{\theta\varphi}+\frac{r}{r^2+n^2}\frac{d\varphi}{d\tau}S^{r\theta}, \tag{3.5.11c}$$

$$\frac{dS^{\theta t}}{d\tau}=\left(\frac{V}{U}-\frac{r}{r^2+n^2}\right)\frac{dr}{d\tau}S^{\theta t}-\frac{2n^2U}{r^2+n^2}\frac{d\varphi}{d\tau}S^{\varphi t}$$

$$-2n\left(\frac{r}{r^2+n^2}-\frac{2V}{U}\right)\frac{d\varphi}{d\tau}S^{r\theta}+2n\left(\frac{r}{r^2+n^2}-\frac{V}{U}\right)\frac{dr}{d\tau}S^{\theta\varphi}, \qquad (3.5.11d)$$

$$\frac{dS^{rt}}{d\tau}=\left[rU-\left(r^2+n^2\right)V\right]\frac{d\varphi}{d\tau}S^{\varphi t}+\frac{V}{U}\frac{dr}{d\tau}S^{rt}$$

$$-n\frac{d\varphi}{d\tau}S^{r\theta}+2n\left(\frac{r}{r^2+n^2}-\frac{V}{U}\right)\frac{dr}{d\tau}S^{r\varphi}, \qquad (3.5.11e)$$

$$\frac{dS^{\varphi t}}{d\tau}=-\frac{r}{r^2+n^2}\frac{d\varphi}{d\tau}S^{rt}+n\frac{d\varphi}{d\tau}S^{\theta\varphi}$$

$$+\left(\frac{V}{U}-\frac{r}{r^2+n^2}\right)\frac{dr}{d\tau}S^{\varphi t}-2n\left(\frac{r}{r^2+n^2}-\frac{2V}{U}\right)\frac{d\varphi}{d\tau}S^{r\varphi}. \qquad (3.5.11f)$$

CASE I. In this case the solution describes a particle moving along a fixed radius, for which $d\varphi/d\tau = 0$. We are able to obtain a simple exact solution,

$$S^{r\varphi}=\frac{1}{\sqrt{\left(r^2+n^2\right)}}C^{r\varphi}, \qquad (3.5.12a)$$

$$S^{\theta\varphi}=\frac{C^{\theta\varphi}}{r^2+n^2}, \qquad (3.5.12b)$$

$$S^{\theta t}=\left(\frac{U}{r^{2}+n^{2}}\right)^{\frac{1}{2}}C^{\theta t}-\frac{n}{r^{2}+n^{2}}C^{\theta\varphi},\qquad(3.5.12c)$$

$$S^{rt}=\sqrt{U}C^{rt}-\frac{2n}{\sqrt{\left(r^{2}+n^{2}\right)}}C^{r\varphi},\qquad(3.5.12d)$$

$$S^{\varphi t}=\left(\frac{U}{r^{2}+n^{2}}\right)^{\frac{1}{2}}C^{\varphi t}.\qquad(3.5.12e)$$

A special interest represents the case when the supersymmetry constraint $Q=0$. From this condition we obtain,

$$\frac{1}{U}\frac{dr}{d\tau}S^{r\theta}=\left(r^{2}+n^{2}\right)\frac{d\varphi}{d\tau}S^{\theta\varphi},\qquad(3.5.13a)$$

$$\frac{1}{U}\frac{dr}{d\tau}S^{rt}=-\left(r^{2}+n^{2}\right)\frac{d\varphi}{d\tau}S^{\varphi t}.\qquad(3.5.13b)$$

For $d\varphi/d\tau=0$ we have only a spin component nenule:

$$S^{\theta\varphi}=\frac{C^{\theta\varphi}}{r^{2}+n^{2}}.\qquad(3.5.14)$$

In this case $dr/d\tau$ and $dt/d\tau$ have a simple expression,

$$\frac{dr}{d\tau}=\sqrt{(2EU)},\qquad(3.5.15a)$$

$$\frac{dt}{d\tau}=\left(\frac{4n^2U}{r^2+n^2}-1\right)\frac{J^{(0)}}{U}. \qquad (3.5.15b)$$

CASE II. This possibility concerns motion for which $d\varphi/d\tau \neq 0$. From $Q=0$, we obtain the following relations:

$$\frac{1}{U}\frac{dr}{d\tau}S^{r\theta}=-J^{(3)}S^{\theta\varphi}, \qquad (3.5.16a)$$

$$\frac{1}{U}\frac{dr}{d\tau}S^{rt}=-J^{(3)}S^{t\varphi}. \qquad (3.5.16b)$$

It is very interesting that even in this case we have a spin component nenule:

$$S^{\theta\varphi}=\frac{1}{r^2+n^2}C^{\theta\varphi}. \qquad (3.5.17)$$

In this case the expressions for the $dt/d\tau$, $d\varphi/d\tau$ and $dr/d\tau$ can be integrated to give the full solution of the equations of motion for all coordinates and spins:

$$\frac{dt}{d\tau}=-\frac{J^{(0)}}{U}-2n\left(\frac{C^{\theta\varphi}}{r^2+n^2}+\frac{d\varphi}{d\tau}\right), \qquad (3.5.18a)$$

$$\frac{dr}{d\tau}=\left\{U\left[2E-\left(r^2+n^2\right)\frac{d\varphi}{d\tau}\right]\right\}^{\frac{1}{2}}, \qquad (3.5.18b)$$

$$\frac{d\varphi}{d\tau}=\frac{1}{r^2+n^2}\left(J^{(3)}+6n^2U\frac{C^{\theta\varphi}}{r^2+n^2}\right). \qquad (3.5.18c)$$

3.6. REMARKS

In this chapter we have studied the geodesic motion of pseudo-classical spinning particles in the Reissner-Nordstrom and NUT-Reissner-Nordstrom spacetimes. In this study we have restricted ourselves to the contribution of the spin contained in the Killing scalars $B^{(\beta)}(x, \psi)$, defined by (1.3.16). Despite the complexity of the equations, we are able to present special solutions for the motion in a plane in the Reissner-Nordstrom spacetime, and on a cone and in a plane in the NUT-Reissner-Nordstrom spacetime.

The result obtained in the Reissner-Nordstrom spacetime reduces to the Schwarzschild result [62] for $q = 0$, and to the result for the extreme Reissner-Nordstrom black hole spacetime when $q = M$.

The result obtained in the NUT-Reissner-Nordstrom spacetime reduces to the result for the Reissner-Nordstrom spacetime [73] when $n = 0$, and to the Schwarzschild result when $n = q = 0$. This study is interesting because of the fact that it not only encompasses the result obtained in the Schwarzschild and Reissner-Nordstrom black hole spacetimes but also provides similar result for the NUT spacetime, which is sometimes considered as unphysical [83].

CHAPTER IV

NONGENERIC SUSY IN CURVED SPACETIME

4.1. INTRODUCTION

One of the most remarkable properties of the Kerr black hole is that, in this background, particle motion is completely integrable. From the point of view of canonical analysis, this is a direct consequence of the existence of a nontrivial Stackel type Killing tensor $K_{\mu\nu}$ [122–125], which is the ψ-independent solution of the generalized Killing equation (1.3.15) with $n = 2$. This Killing tensor gives rise to the associated constant of motion

$$Z = \frac{1}{2} K^{\mu\nu} \wp_\mu \wp_\nu , \tag{4.1.1}$$

which is quadratic in the four-momentum $\wp_\mu$. That is, this constant of motion completes the maximal number of constants of motion in conjunction with the other three well-known constants of motion: the energy

$$E = -K^\mu \wp_\mu , \tag{4.1.2}$$

coming from the time translation invariance generated by the Killing field K^μ, the angular momentum

$$J = M^{\mu} p_{\mu}, \qquad (4.1.3)$$

coming from the axial symmetry generated by the Killing field M^{μ}, and the Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_{\mu} p_{\nu}. \qquad (4.1.4)$$

More surprisingly, various field equations, *e.g.*, the Dirac equation [126], separate in the Kerr geometry, and [91] this fact has a direct consequence of the existence of the Killing-Yano tensor $f_{\mu\nu}$, which is defined as an anti-symmetric second rank tensor satisfying the following Penrose-Floyd equation [127, 128]:

$$D_{(\mu} f_{\nu)\lambda} = 0. \qquad (4.1.5)$$

This Killing-Yano *2*-form $f_{\mu\nu}$ is a square root of the Stackel-Killing tensor $K^{\mu\nu}$:

$$K^{\mu}{}_{\nu} = f^{\mu}{}_{\lambda} f^{\lambda}{}_{\nu}. \qquad (4.1.6)$$

Here, indices are raised and lowered with the spacetime metric $g_{\mu\nu}$ and its inverse.

Recently, in ref. [80], Gibbons *etal.* have been able to show by considering supersymmetric particle mechanics that the Killing-Yano tensor can be understood as an object belonging to a larger class of possible structures which generate generalized supersymmetry algebras. This novel aspect has renewed people's interest in the Killing-Yano tensor, which has long been known for relativistic systems as a mysterious structure.

To describe "nongeneric" SUSYs, generated by the Killing-Yano tensors and the corresponding conserved quantities in an arbitrary background spacetime, we would like to review the work of Gibbons *etal.* [80] in this chapter.

The plan of this chapter is as follows. In section 4.2 we review the formalism of pseudo-classical spinning point particles in an arbitrary background spacetime, in which anti-commuting Grassmann variables describe the spin degrees of freedom. In section 4.3 we describe the general relation between symmetries, supersymmetries and constants of motion for these equations. In section 4.4 we discuss extra supersymmetries and their algebras. These "nongeneric" supersymmetries depend on the existence of a second-rank tensor field $f_{\mu\nu}$ which is referred to as f-symbols. In section 4.5 we describe the general properties of f-symbols and point out their relation to Killing-Yano tensors.

4.2. SPINNING PARTICLES IN CURVED SPACETIME

The pseudo-classical limit of the Dirac theory of a spin-1/2 fermion in curved spacetime is described by the supersymmetric extension of the ordinary relativistic point particle [4, 7–9, 12, 16, 20]. Local version of supersymmetry (supergravity) is described in terms of the vielbein (tetrad) [129] $e_\mu{}^a(x)$, which is the "square root" of the metric tensor $g_{\mu\nu}$ in some sense. In $e_\mu{}^a$ the Greek index, μ, is a "world" vector index in the curved spacetime; it transforms like a vector under coordinate transformations and is raised (or lowered) with $g^{\mu\nu}$ (or $g_{\mu\nu}$). The Latin index, a, is a tangent space (flat-space) index; it transforms under (local) Lorentz transformations as a Lorentz vector and is raised (or lowered) with the Minkowski space metric η^{ab} (or η_{ab}). The $e_\mu{}^a$ is the "square root" of $g_{\mu\nu}$ in the sense that

$$g_{\mu\nu} = e_\mu{}^a \, e_{a\nu} = \eta_{ab} \, e_\mu{}^a \, e_\nu{}^b. \tag{4.2.1}$$

This allows one to translate any formula involving the metric tensor into a corresponding one, which involves vielbeins.

The configuration space of the theory is spanned by the real position coordinates $x^\mu(\tau)$ and the Grassmann-valued spin coordinates $\psi^a(\tau)$, where μ and a both run from $1,\ldots,\ d,$ with d the dimension of the spacetime. The world and tangent vector indices (*i.e.*, a and μ) can be converted into each other by the vielbein $e_\mu{}^a(x)$ and its inverse $e^\mu{}_a(x)$; for example, it is sometimes convenient to introduce the object

$$\psi^\mu(x) = e^\mu{}_a(x)\psi^a, \tag{4.2.2}$$

transforming under general coordinate and local Lorentz transformation as a world vector rather than a local Lorentz vector. The world-line parameter τ is the invariant proper time,

$$c^2 d\tau^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu . \tag{4.2.3}$$

We choose units such that $c = 1$.

The equations of motion of the pseudo-classical Dirac particle can be obtained from the Lagrangian

$$\mathscr{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{1}{2} i \eta_{ab} \psi^a \frac{d\psi^b}{d\tau} . \tag{4.2.4}$$

The overdot, here and in the following, represents a derivative with respect to τ. The covariant derivative of the spin variable is

$$\frac{D\psi^a}{D\tau} = \dot{\psi}^a - \dot{x}^\mu \omega_\mu{}^a{}_b \psi^b , \tag{4.2.5}$$

where $\omega_\mu{}^a{}_b$ is the spin connection.

To fix the dynamics completely one has to add the condition expressed by equation (4.2.3), which is equivalent to the mass-shell condition, together with others necessary to select the physical solutions of the equations of motion; for example, the restriction that spin be space-like, as expressed by (1.4.12), reads

$$\mathscr{Q} \equiv e_{\mu a} \dot{x}^\mu \psi^a = 0 , \tag{4.2.6}$$

which implies that ψ has no time-component in the rest frame. These supplementary conditions have to be compatible with the equations of motion derived from the Lagrangian $\mathscr{L}$[59, 61, 62]. However, in the formulation of spinning particle dynamics these additional conditions are only to be imposed after solving the equations of motion of the theory.

The configuration space of spinning particles spanned by $\left(x^{\mu}, \psi^{a}\right)$ is sometimes referred to as spinning space. The solutions of the Euler-Lagrange equations derived from the Lagrangian (4.2.4) may be considered as generalizations of the concept of geodesics to spinning space. The supplementary conditions then select those geodesics that correspond to the world lines of the physical spinning particles.

Under arbitrary variations $\left(\delta x^{\mu}, \delta\psi^{a}\right)$, the variation of the Lagrangian (4.2.4) is given by

$$\delta\mathscr{L} = \delta x^{\mu}\left(-g_{\mu\nu}\frac{D^2 x^{\nu}}{D\tau^2} - \frac{1}{2} i\psi^{a}\psi^{b}R_{ab\mu\nu}\dot{x}^{\nu}\right)$$

$$+\Delta\psi^{a}\eta_{ab}\frac{D\psi^{b}}{D\tau} + total\ derivative, \tag{4.2.7}$$

where

$$\Delta\psi^{a} = \delta\psi^{a} - \delta x^{\mu}\,\omega_{\mu}{}^{a}{}_{b}\,\psi^{b} \tag{4.2.8}$$

is the covariantized variation of ψ^{a} [60]. The equations of motion can immediately be cast in the following form:

$$\frac{D^2 x^\mu}{D\tau^2} = \ddot{x}^\mu - \Gamma_{\lambda\nu}{}^\mu \dot{x}^\lambda \dot{x}^\nu = -\frac{1}{2} i \psi^a \psi^b \mathrm{R}_{ab}{}^\mu{}_\nu \dot{x}^\nu , \tag{4.2.9a}$$

$$\frac{d\psi^a}{d\tau} = 0 . \tag{4.2.9b}$$

The canonical momenta conjugates to x^μ and ψ^a are

$$\wp_\mu = \frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu - \frac{1}{2} i \omega_{\mu ab} \psi^a \psi^b , \tag{4.2.10a}$$

and

$$\pi_a = \frac{\partial \mathscr{L}}{\partial \dot{\psi}^a} = -\frac{1}{2} i \psi_a , \tag{4.2.10b}$$

respectively. This gives a second-class constraint. Eliminating the constraint by Dirac's procedure one can obtain the canonical Poisson-Dirac brackets

$$\{x^\mu , \wp_\nu\} = \delta^\mu_\nu , \qquad \{\psi^a , \psi^b\} = -\eta^{ab} . \tag{4.2.11}$$

Accordingly, the Poisson-Dirac brackets for general functions F and G of the canonical phase-space variables (x, $\wp$, ψ) read

$$\{F,G\} = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial \wp_\mu} - \frac{\partial F}{\partial \wp_\mu}\frac{\partial G}{\partial x^\mu} + i(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a} , \tag{4.2.12}$$

where a_F is the Grassmann parity of F with $a_F=(0, 1)$ for F = *(even, odd)*. The canonical Hamiltonian of the theory has the form

$$H = \frac{1}{2} g^{\mu\nu} \left(p_\mu + \omega_\mu \right)\left(p_\nu + \omega_\nu \right), \tag{4.2.13a}$$

where

$$\omega_\mu = \frac{1}{2} i \omega_{\mu a b} \psi^a \psi^b. \tag{4.2.13b}$$

The time-evolution of any function $F(x, p, \psi)$ is generated by its Poisson-Dirac brackets with this Hamiltonian:

$$\frac{dF}{d\tau} = \{F, H\}. \tag{4.2.14}$$

Equations (4.2.11)–(4.2.14) describe the canonical formulation of the theory. The disadvantage of this formulation is, that one loses manifest covariance. For this reason it is often convenient to formulate the theory in terms of covariant phase-space variables x^μ, Π_μ, ψ^a where

$$\Pi_\mu \equiv p_\mu + \omega_\mu = g_{\mu\nu} \dot{x}^\nu \tag{4.2.15}$$

is the covariant momentum. The Poisson-Dirac brackets for functions of the covariant phase-space variables (x, Π, ψ) then become

$$\{F,G\} = (\mathscr{D}_\mu F)\frac{\partial G}{\partial \Pi_\mu} - \frac{\partial G}{\partial \Pi_\mu}(\mathscr{D}_\mu G)$$

$$- \mathscr{R}_{\mu\nu}\frac{\partial F}{\partial \Pi_\mu}\frac{\partial G}{\partial \Pi_\nu} + i(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a}, \qquad (4.2.16)$$

where

$$\mathscr{D}_\mu F \equiv \partial_\mu F + \Gamma_{\mu\nu}{}^{\lambda}\Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} + \omega_\mu{}^a{}_b\,\psi^b \frac{\partial F}{\partial \psi^a} \qquad (4.2.17)$$

is the phase-space covariant derivative and

$$\mathscr{R}_{\mu\nu} \equiv \frac{1}{2} i \psi^a \psi^b R_{ab\mu\nu} \qquad (4.2.18)$$

is the spin-valued Riemann tensor. It follows from (4.2.16) that

$$\{\Pi_\mu, \Pi_\nu\} = -\mathscr{R}_{\mu\nu}, \qquad (4.2.19)$$

which is the classical analogue of the Ricci identity when there is no torsion. In terms of the covariant phase-space variables the Hamiltonian becomes

$$H = \frac{1}{2} g^{\mu\nu}\,\Pi_\mu \Pi_\nu. \qquad (4.2.20)$$

The dynamical equation (4.2.14) remains unaltered, while the constraints in (4.2.3) and (4.2.6) become

$$2H = g^{\mu\nu}\, \Pi_\mu \Pi_\nu = -1, \tag{4.2.21}$$

$$\mathscr{Q} = \Pi \cdot \psi = 0. \tag{4.2.22}$$

Since these are not compatible with the Poisson-Dirac brackets in general, they are to be imposed only after solving the theory. However, one easily finds that

$$\{\mathscr{Q},\, H\} = 0. \tag{4.2.23}$$

As the Hamiltonian itself is trivially conserved, equation (4.2.23) implies the conservation of $\mathscr{Q}$. Hence, the values of H and $\mathscr{Q}$ given by (4.2.21), (4.2.22), are preserved in time, and the physical conditions imposed on the theory are consistent with the equations of motion [59].

4.3. SYMMETRIES AND CONSTANTS OF MOTION

The theory of a pseudo-classical spinning particle model possesses a number of symmetries, which are very useful in solving the equations of motion explicitly [62] because of their connection with constants of motion via Noether's theorem. As mentioned in Chapter I, these symmetries can be divided into two classes, *generic* and *nongeneric* symmetries. The generic kind exists for any spacetime metric $g_{\mu\nu}(x)$, while the latter type depends on the explicit form of the metric. The theory described by the Lagrangian (4.2.4) admits four generic symmetries [59–61], two of which are proper-time translations generated by the

Hamiltonian H, and supersymmetry generated by the supercharge $\mathscr{Q}$, equation (4.2.22). The other two are chiral symmetry generated by the chiral charge

$$\Gamma_* = -\frac{i^{[d/2]}}{d!}\varepsilon_{a_1 \ldots a_d}\psi^{a_1}\ldots\psi^{a_d}, \tag{4.3.1}$$

and dual supersymmetry, generated by the dual supercharge

$$\mathscr{Q}^* = i\{\mathscr{Q},\Gamma_*\}$$

$$= -\frac{i^{[d/2]}}{(d-1)!}\varepsilon_{a_1 \ldots a_d}e^{\mu a_1}\Pi_\mu \psi^{a_2}\ldots\psi^{a_d}. \tag{4.3.2}$$

It can be checked that $\{H, \Gamma_* \}=0$. Then the Jacobi identity with (4.2.23) confirms that all the above quantities are constants of motion.

In order to obtain all the symmetries, including the nongeneric ones, we now find all functions $\mathscr{J}(x, \Pi, \psi)$ such that

$$\{H, \mathscr{J}\}=0. \tag{4.3.3}$$

Using the covariant form (4.2.16) of the brackets, we simplify (4.3.3) to

$$\Pi^\mu \left(\mathscr{D}_\mu \mathscr{J} + \mathscr{R}_{\mu\nu}\frac{\partial \mathscr{J}}{\partial \Pi_\nu} \right) = 0. \tag{4.3.4}$$

The second term in (4.3.4) vanishes if $\mathscr{J}$ depends on the covariant momentum only via the Hamiltonian: $\mathscr{J}(x,\Pi,\psi)=\mathscr{J}(x,H,\psi)$. Then the equation (4.3.4) simplifies to

$$\Pi \cdot \mathscr{D}\mathscr{J} = 0. \tag{4.3.5}$$

In all other cases we need the full equation (4.3.4). This equation is satisfied for arbitrary Π_μ if and only if the components of $\mathscr{J}$ in the expansion

$$\mathscr{J} = \sum_{n=0}^{\infty} \frac{1}{n!} J^{(n)\mu_1 \dots \mu_n}(x,\psi)\Pi_{\mu_1} \dots \Pi_{\mu_n} \tag{4.3.6}$$

satisfy the generalized Killing equations

$$D_{(\mu_{n+1}} J^{(n)}_{\mu_1 \dots \mu_n)} + \omega_{(\mu_{n+1}}{}^{a}{}_{b}\, \psi^b \frac{\partial J^{(n)}_{\mu_1 \dots \mu_n)}}{\partial \psi^a} = \mathscr{R}_{\nu(\mu_{n+1}} J^{(n+1)}_{\mu_1 \dots \mu_n)}{}^{\nu}, \tag{4.3.7}$$

where D_μ is an ordinary covariant derivative, and the parentheses denote full symmetrization over the indices enclosed.

Further, any constant of motion $\mathscr{J}$ satisfies

$$\{\mathscr{O},\mathscr{J}\} = -\psi^\mu \left(\mathscr{D}_\mu \mathscr{J} + \mathscr{R}_{\mu\nu} \frac{\partial \mathscr{J}}{\partial \Pi_\nu} \right) - ie^{\mu a}\Pi_\mu \frac{\partial \mathscr{J}}{\partial \psi^a}. \tag{4.3.8}$$

If the curvature term undergoes three contractions with the anticommuting spin variables, then with the Bianchi identity $R_{[\mu\nu\lambda]\kappa} = 0$, equation (4.3.8) can be written as

$$\{\mathscr{Q}, \mathscr{J}\} = -\left(\psi \cdot \mathscr{D}\mathscr{J} + i\Pi \cdot \frac{\partial \mathscr{J}}{\partial \psi}\right). \tag{4.3.9}$$

In particular, for $\mathscr{J} = \mathscr{Q}$ we obtain the usual supersymmetry algebra:

$$\{\mathscr{Q}, \mathscr{Q}\} = -2iH. \tag{4.3.10}$$

Then, the Jacobi identity for two $\mathscr{Q}$'s and any constant of motion $\mathscr{J}$ confirms that the quantity

$$\Theta \equiv \{\mathscr{Q}, \mathscr{J}\} \tag{4.3.11}$$

is a superinvariant and hence a constant of motion as well:

$$\{\mathscr{Q}, \Theta\} = 0, \qquad \{H, \Theta\} = 0. \tag{4.3.12}$$

This result implies that constants of motion generally come in supermultiplets $\{\mathscr{J}, \Theta\}$, of which the prime example is the multiplet ($\mathscr{Q}$, H) itself. The only exceptions to this result are the constants of motion for which $\Theta = 0$, but which are not themselves obtained from the bracket of $\mathscr{Q}$ with another constant of motion.

It follows from (4.3.9) that a superinvariant is a solution of the equation

$$\{\mathscr{Q},\mathscr{J}\}=-\left(\psi\cdot\mathscr{D}\mathscr{J}+i\Pi\cdot\frac{\partial\mathscr{J}}{\partial\psi}\right)=0. \tag{4.3.13}$$

Expanding $J^{(n)\mu_1\ldots\mu_n}(x,\psi)$ of (4.3.6) in powers of ψ^a and letting the coefficients be $f^{(m,n)\mu_1\ldots\mu_n}_{a_1\ldots a_m}(x)$, the series expansion of $\mathscr{J}$ can be put in the form

$$\mathscr{J}(x,\Pi,\psi)=\sum_{m,n=0}^{\infty}\frac{i^{[d/2]}}{m!\,n!}\psi^{a_1}\ldots\psi^{a_m}f^{(m,n)\mu_1\ldots\mu_n}_{a_1\ldots a_m}(x)\Pi_{\mu_1}\ldots\Pi_{\mu_n}, \tag{4.3.14}$$

where $f^{(m,n)}$ is completely symmetric in the n upper indices $\{\mu_k\}$ and completely antisymmetric in the m lower indices $\{a_i\}$. Equation (4.3.13) then gives the component equation

$$nf^{(m+1,n-1)(\mu_1\ldots\mu_{n-1}}_{a_0a_1\ldots a_m}e^{\mu_n)a_0}=mD_{[a_1}f^{(m-1,n)\mu_1\ldots\mu_n}_{a_2\ldots a_m]}, \tag{4.3.15}$$

where $D_a=e^{\mu}{}_aD_\mu$, and square brackets denote full antisymmetrization, while parentheses denote full symmetrization over the indices enclosed, all with unit weight. In particular for $m=0$,

$$f^{(1,n)(\mu_1\ldots\mu_n}_{a}e^{\mu_{n+1})a}=0. \tag{4.3.16}$$

In a certain sense these equations represent a square root of the generalized Killing equations (4.3.7). They only provide sufficient, but not necessary conditions for obtaining solutions. However, at least one component of each supermultiplet (singlet or non-singlet) is a solution of equation (4.3.13). Having found Θ one can then proceed to reconstruct the corresponding $\mathscr{J}$ by solving (4.3.11).

The content of equations (4.3.15) is twofold. On the one hand they partly solve $f^{(m+1,n-1)}$, which is symmetrized in one flat index and all $(n-1)$ curved indices, in terms of $f^{(m-1,n)}$. On the other hand Equations (4.3.15) do not automatically mean that $f^{(m+1,n-1)}$ is completely anti-symmetric in the first $(m+1)$ indices. If that condition is imposed on equations (4.3.15), one can find a new set of equations which are precisely the generalized Killing equations for that part of $f^{(m+1,n-1)}$ which was not given in terms of $f^{(m-1,n)}$, and which should still be solved for. This is the anti-symmetrized part of $f^{(m+1,n-1)}$ in one curved index and all $(m+1)$ flat indices.

Thus equations (4.3.15) clearly have advantages over the generalized Killing equations (4.3.7). To obtain the constant of motion corresponding to a Killing tensor of rank n,

$$\mathscr{D}_{(\mu_{n+1}} J^{(n)}_{\mu_1 \cdots \mu_n)} = 0, \tag{4.3.17}$$

we have to solve the complicated hierarchy of partial differential equations (4.3.7) for $\left(J^{(n-1)}, \ldots, J^{(0)}\right)$ and add the terms, as in expression (4.3.6). However, if one has a solution $f^{(m,n)\mu_1 \ldots \mu_n}_{a_1 \ldots a_m}$ of the equation

$$f^{(m,n)\,(\mu_1\ldots\mu_n}_{a_1\ldots a_m}\, e^{\mu_{n+1})a_1} = 0, \qquad (4.3.18)$$

then one can generate at least part of the components $f^{(m+2\alpha,n-\alpha)\mu_1\ldots\mu_{n-\alpha}}_{a_1\ldots a_{m+2\alpha}}$ for $\alpha=1,\ldots,n$ by mere differentiation. Then, equation (4.3.14) gives the corresponding constant of motion. We consider an example in section 4.4 in which these advantages become clear.

Finally we note that equations (43.11) and (4.3.12) imply that the Poisson-Dirac bracket with $\mathcal{Q}$ defines a nilpotent operation in the space of constants of motion. Thus, the supersinglets span the cohomology of the supercharge, while the supermultiplets $(\mathcal{J},\Theta)$ form pairs of $\mathcal{Q}$-exact and $\mathcal{Q}$-coexact forms. Then, the solutions of equation (4.3.13) correspond to the $\mathcal{Q}$-closed forms.

4.4. NONGENERIC SUPERSYMMETRIES

For any constant of motion $\mathcal{J}$ there exist infinitesimal transformations of the coordinates which leave the equations of motion invariant:

$$\delta x^\mu = \delta\alpha\{x^\mu, \mathcal{J}\},$$

$$\delta\psi^a = \delta\alpha\{\psi^a, \mathcal{J}\}, \qquad (4.4.1)$$

where $\delta\alpha$ is the infinitesimal parameter of the transformation. For example, the action as defined by the Lagrangian in (4.2.4) remains invariant under the generic symmetries, such as supersymmetry:

$$\delta x^{\mu} = i \in \left\{\mathcal{Q}, x^{\mu}\right\} = -i \in e^{\mu}{}_{a}\, \psi^{a},$$

$$\delta\psi^{a} = i \in \left\{\mathcal{Q}, \psi^{a}\right\}$$

$$= \in e_{\mu}{}^{a}\, \dot{x}^{\mu} + \delta x^{\mu} \omega_{\mu}{}^{a}{}_{b}\, \psi^{b}, \tag{4.4.2}$$

where the infinitesimal parameter $\in$ of the transformation is Grassmann-odd.

We now investigate whether the theory admits other (nongeneric) supersymmetries of the type

$$\delta x^{\mu} = -i \in f^{\mu}{}_{a}\, \psi^{a} \equiv -i \in J^{(1)\mu}. \tag{4.4.3}$$

Such a transformation is generated by a phase-space function $\mathcal{Q}_f$,

$$\mathcal{Q}_f \equiv J^{(1)\mu}\, \Pi_{\mu} + J^{(0)}, \tag{4.4.4}$$

where $J^{(1)}(x, \psi)$ and $J^{(0)}(x, \psi)$ are independent of Π. If this ansatz is inserted into the generalized Killing equations (4.3.7), it follows that

$$J^{(0)}(x, \psi) = \frac{i}{3!} C_{abc}(x)\, \psi^{a} \psi^{b} \psi^{c} \tag{4.4.5}$$

with the tensors $f^{\mu}{}_{a}$ and C_{abc} satisfying the conditions

$$D_{\mu} f_{\nu a} + D_{\nu} f_{\mu a} = 0, \tag{4.4.6}$$

and

$$D_{\mu} C_{abc} = -\left(R_{\mu\nu ab}\, f^{\nu}{}_{c} + R_{\mu\nu bc}\, f^{\nu}{}_{a} + R_{\mu\nu ca}\, f^{\nu}{}_{b}\right). \tag{4.4.7}$$

Let there be N such symmetries specified by N sets of tensors $\left(f_{i}^{\mu}{}_{a}, C_{abc}\right), i = 1, \ldots, N$. Then the corresponding generators will be

$$\mathscr{Q}_{i} = f_{i}^{\mu}{}_{a}\, \Pi_{\mu} \psi^{a} + \frac{i}{3!} C_{iabc}\, \psi^{a} \psi^{b} \psi^{c}. \tag{4.4.8}$$

Obviously, for $f^{\mu}{}_{a} = e^{\mu}{}_{a}$ and $C_{abc} = 0$, the supercharge in (4.2.22) is precisely of this form. It is therefore convenient to assign the index $i = 0 : \mathscr{Q} = \mathscr{Q}_{0}$, $e^{\mu}{}_{a} = f_{0a}^{\mu}$, *etc.*, when we refer to the quantities defining the standard supersymmetry.

The covariant form (4.2.16) of Poisson-Dirac brackets gives the following algebra for the conserved charges $\mathscr{Q}_{i}$:

$$\left\{\mathscr{Q}_{i}, \mathscr{Q}_{j}\right\} = -2i\, Z_{ij}, \tag{4.4.9}$$

where

$$Z_{ij} = \frac{1}{2} K_{ij}^{\mu\nu} \Pi_\mu \Pi_\nu + I_{ij}^\mu \Pi_\mu + G_{ij} \ , \tag{4.4.10}$$

and

$$K_{ij}^{\mu\nu} = \frac{1}{2}\left(f_i^\mu{}_a f_j^{\nu a} + f_i^\nu{}_a f_j^{\mu a} \right), \tag{4.4.11}$$

$$I_{ij}^\mu = \frac{1}{2} i \psi^a \psi^b I_{ijab}^\mu$$

$$= \frac{1}{2} i \psi^a \psi^b \Big(f_i^\nu{}_b D_\nu f_j^\mu{}_a + f_j^\nu{}_b D_\nu f_i^\mu{}_a$$

$$+ \frac{1}{2} f_i^{\mu c} C_{jabc} + \frac{1}{2} f_j^{\mu c} C_{iabc} \Big), \tag{4.4.12}$$

$$G_{ij} = -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d G_{ijabcd}$$

$$= -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d \left(R_{\mu\nu ab} f_i^\mu{}_c f_j^\nu{}_d + \frac{1}{2} C_{iab}{}^e C_{jcde} \right). \tag{4.4.13}$$

From an explicit calculation one can infer that $K_{ij\mu\nu}$ is a symmetric Killing tensor of second rank:

$$D_{(\lambda} K_{ij\mu\nu)} = 0, \tag{4.4.14}$$

while I_{ij}^μ is the corresponding Killing vector:

$$\mathscr{D}_{(\mu} I_{i j \nu)} = \frac{1}{2} i \psi^a \psi^b D_{(\mu} I_{i j \nu) a b}$$

$$= \frac{1}{2} i \psi^a \psi^b R_{a b \lambda (\mu} K_{i j \nu)}{}^{\lambda}, \tag{4.4.15}$$

and G_{ij} the corresponding Killing scalar:

$$\mathscr{D}_{\mu} G_{i j} = -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d D_{\mu} G_{i j abcd}$$

$$= \frac{1}{2} i \psi^a \psi^b R_{a b \lambda \mu} I^{\lambda}_{i j}. \tag{4.4.16}$$

Thus we see that the Grassmann-even Phase-space functions Z_{ij} satisfy the generalized Killing equations. Hence, their bracket with the Hamiltonian vanishes and they are constants of motion:

$$\frac{d Z_{i j}}{d\tau} = 0. \tag{4.4.17}$$

For $i = j = 0$, (4.4.9) reduces to the usual supersymmetry algebra (4.3.10). If i or j is not equal to zero, the Z_{ij} correspond to new bosonic symmetries, unless $K^{\mu\nu}_{i j} = \lambda_{(i j)} g^{\mu\nu}$, with $\lambda_{(i j)}$ a constant (may be zero). Then, the corresponding Killing vector $I^{\mu}_{i j}$ and Killing scalar G_{ij} vanish identically. Further, if $\lambda_{(i j)} \neq 0$, the corresponding supercharges close on the Hamiltonian. This proves the existence of a second supersymmetry of the standard type. We then have an N-extended supersymmetry with $N \geq 2$. On the other hand, if we have a second independent

Killing tensor $K^{\mu\nu}$ not proportional to $g^{\mu\nu}$, there exists a genuine new type of supersymmetry.

Following (4.3.12) we obtain

$$\{\mathcal{Q}_i, \mathcal{Q}\} = 0, \tag{4.4.18}$$

and hence $\mathcal{Q}_i$ is a superinvariant, if and only if

$$K_{oi}^{\mu\nu} = f^{\mu}{}_{a}\, e^{\nu a} + f^{\nu}{}_{a}\, e^{\mu a} = 0. \tag{4.4.19}$$

In the language of $\mathcal{Q}$-cohomology, $\mathcal{Q}_i$ is $\mathcal{Q}$-closed. Using the discussion given at the end of section 4.3 one can then construct the full constant of motion Z_{ij} directly by repeated differentiation of $f^{\mu}{}_{a}$.

Since the Z_{ij} are symmetric in $(i\ j)$ by construction, we can diagonalize them. Thus we obtain an algebra

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = -2i\delta_{ij}\, Z_i, \tag{4.4.20}$$

with $N+1$ conserved bosonic charges Z_i. If all $\mathcal{Q}_i$ satisfy condition (4.4.19), the first of these diagonal charges (with $i=0$) is the Hamiltonian: $Z_0 = H$.

4.5. PROPERTIES OF *f*-SYMBOLS

In this section we turn our attention to the quantities $f^{\mu}{}_{a}$ to study the properties of the new supersymmetries. For convenience we introduce the second rank tensor

$$f_{\mu\nu} = f_{\mu a}\, e_{\nu}{}^{a}, \tag{4.5.1}$$

which will be referred to as the f-symbol. Condition (4.4.6) then gives

$$D_\nu f_{\lambda\mu} + D_\lambda f_{\mu\nu} = 0. \tag{4.5.2}$$

This implies that the divergence on the first index of the f-symbol vanishes:

$$D_\nu f^{\nu}{}_{\mu} = 0. \tag{4.5.3}$$

On contraction, equation (4.5.2) gives

$$D_\nu f_\mu{}^\nu = -\partial_\mu f_\nu{}^\nu, \tag{4.5.4}$$

and hence, the f-symbol will also be divergenceless on the second index if and only if its trace is constant:

$$D_\nu f_\mu{}^\nu = 0 \Leftrightarrow f_\mu{}^\mu = constant. \tag{4.5.5}$$

If the trace is constant, then, since the metric tensor $g_{\mu\nu}$ is a trivial solution of equation (4.5.2), it may be subtracted from the f-symbol without spoiling condition (4.5.2). In this case one may without loss of generality always take the constant equal to zero and then, f is traceless.

From equation (4.4.11) with $f^\mu_{oa} = e^\mu{}_a$, the symmetric part of the i-th f-symbol is the tensor

$$S_{\mu\nu} \equiv K_{io\mu\nu} = \frac{1}{2}\left(f_{\mu\nu} + f_{\nu\mu}\right). \tag{4.5.6}$$

which satisfies the generalized Killing equation

$$D_{(\mu} S_{\nu\lambda)} = 0. \tag{4.5.7}$$

Also, the antisymmetric part can be constructed as

$$B_{\mu\nu} = -B_{\nu\mu} = \frac{1}{2}\left(f_{\mu\nu} - f_{\nu\mu}\right), \tag{4.5.8}$$

satisfying the condition

$$D_{\nu} B_{\lambda\mu} + D_{\lambda} B_{\nu\mu} = D_{\mu} S_{\nu\lambda}. \tag{4.5.9}$$

Therefore, if the symmetric part does not vanish and is not covariantly constant, the antisymmetric part $B_{\mu\nu}$ is not a solution of equation (4.5.2). Also, then the antisymmetric part of f can not vanish either. Hence, f can be completely symmetric only if it is covariantly constant.

The interesting case is that in which the f-symbol is completely antisymmetric: $f_{\mu\nu}=B_{\mu\nu}$. This is precisely the condition (4.4.19) for the supercharge, $\mathcal{Q}_f$, to anticommute with ordinary supersymmetry in the sense of Poisson-Dirac brackets. Also, equation (4.5.5) is trivially satisfied in this case.

The antisymmetric $f_{\mu\nu}$ leads to say much more about the explicit form of the quantities introduced above. If the symmetric part of a certain $f_{i\mu\nu}$ vanishes:

$$S_i^{\mu\nu} = K_{io}^{\mu\nu} = 0, \tag{4.5.10}$$

then the corresponding Killing vector I^{μ}_{io} and the Killing scalar G_{io} vanish as well. Thus for this particular i, $Z_{io}=0$ and then

$$\{\mathscr{O}_i, \mathscr{O}\}=0, \tag{4.5.11}$$

which shows that $\mathscr{O}_i$ is superinvariant. To prove this assertion, we first note that equation (4.5.2) for antisymmetric $f_{\mu\nu}$ becomes

$$D_\nu B_{\lambda\mu} = -D_\lambda B_{\nu\mu}. \tag{4.5.12}$$

Then, it follows that the gradient is completely antisymmetric:

$$D_\mu B_{\nu\lambda} = -D_{[\mu} B_{\nu\lambda]} \equiv H_{\mu\nu\lambda}. \tag{4.5.13}$$

Taking the second covariant derivative of $f_{\mu\nu}$, and then commuting the derivatives and using equation (4.5.2) we obtain the identity

$$D_\mu D_\nu f_{\lambda\kappa} = R_{\nu\lambda\mu}{}^{\sigma} f_{\sigma\kappa} + \frac{1}{2}\left(R_{\nu\lambda\kappa}{}^{\sigma} f_{\mu\sigma} + R_{\mu\lambda\kappa}{}^{\sigma} f_{\nu\sigma} - R_{\mu\nu\kappa}{}^{\sigma} f_{\lambda\sigma}\right) \tag{4.5.14}$$

For antisymmetric $f_{\mu\nu}$, equation (4.5.14) implies

$$D_\mu H_{\nu\lambda\kappa} = \frac{1}{2}\left(R_{\nu\lambda\mu}{}^{\sigma} f_{\sigma\kappa} + R_{\lambda\kappa\mu}{}^{\sigma} f_{\sigma\nu} + R_{\kappa\nu\mu}{}^{\sigma} f_{\sigma\lambda}\right) \tag{4.5.15}$$

Comparing equation (4.5.15) with (4.4.7) we find that

$$-\frac{1}{2} C_{abc} = H_{abc} = e^{\mu}{}_{a}\, e^{\nu}{}_{b}\, e^{\lambda}{}_{c}\, H_{\mu\nu\lambda}, \tag{4.5.16}$$

modulo a covariantly constant term. This result is an instance of equation (4.3.15) with $n=1$, $m=2$.

The covariantly constant three-index tensor C_{abc} provides another independent symmetry corresponding to the Killing vector

$$I_\mu = \frac{1}{2} i \psi^a \psi^b e_\mu{}^c C_{abc} . \tag{4.5.17}$$

More precisely, if we choose

$$D_\mu C_{abc} = 0, \tag{4.5.18}$$

then

$$\mathscr{D}_\mu I_\nu = 0, \tag{4.5.19}$$

and automatically I_μ satisfies the generalized Killing equation.

We observe that according to equation (4.5.10), $K_{0i}^{\mu\nu} = 0$. Furthermore, since $C_{0abc} = 0$ identically, the right-hand side of equation (4.4.12) becomes

$$I_{i0\mu\nu\lambda} \equiv I_{i0\mu ab}\, e_\nu{}^a\, e_\lambda{}^b = D_\lambda B_{i\mu\nu} + \frac{1}{2} C_{i\mu\nu\lambda} = 0, \tag{4.5.20}$$

where the last equality follows from equation (4.5.16). Finally, the Killing scalar G_{i0} becomes zero because of the cyclic Bianchi identity for the Riemann tensor $R_{\mu\nu\lambda\kappa}$ and the vanishing of C_{0abc}. This proves equation (4.5.11).

CHAPTER V

NONGENERIC SUSY IN KERR-NEWMAN SPACETIME

5.1. INTRODUCTION

The Kerr-Newman spacetime [130] is an axisymmetric asymptotically flat stationary solution of the Einstein-Maxwell equations that describes the geometry of a charged rotating black hole.

Recently, Gibbons *et al.* in ref. [80] investigated nongeneric supersymmetry in the Kerr-Newman spacetime in terms of the motion of pseudo-classical spinning point particles. In view of extending this work in a more general spacetime in the subsequent chapter we would like to review it in this chapter.

We arrange this chapter as follows. In section 5.2 we derive first the Killing-Yano tensor in the Kerr-Newman spacetime and then describe the corresponding Killing tensor, Killing vector and Killing scalar, which generate the nongeneric supersymmetry. In section 5.3 we present a discussion.

5.2. SPINNING KERR-NEWMAN SPACETIME

In this section, the results of Chapter IV have been applied to show that a new kind of supersymmetry exists in Kerr-Newman spacetime

The gravitational and electromagnetic field of a rotating body with mass M and charge q is described by the Kerr-Newman metric, which is

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta\, d\varphi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$+\frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\varphi - a\,dt\right]^2, \qquad (5.2.1)$$

and the electromagnetic field tensor

$$F = \frac{q}{\rho^4}\left(r^2 - a^2\cos^2\theta\right)dr \wedge \left(dt - a\sin^2\theta\, d\varphi\right)$$

$$+\frac{2\,q\,a\,r\cos\theta\sin\theta}{\rho^4}d\theta \wedge \left[-a\,dt + \left(r^2+a^2\right)d\varphi\right], \qquad (5.2.2)$$

where

$$\Delta = r^2 + a^2 - 2Mr + q^2,$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad (5.2.3)$$

and the total angular momentum is $J=Ma$. The Kerr-Newman spacetime has two horizons which are the event horizon located at r_+ and the (Cauchy) inner horizon located at r_- where

$$r_\pm = M \pm \sqrt{M^2 - q^2 - a^2} \qquad (5.2.4)$$

Expressions in (5.2.1) and (5.2.2) describe fields only outside the event horizon.

As was found by Carter [123], the Kerr-Newman spacetime admits two independent second-rank Killing tensors. One is the metric tensor $g_{\mu\nu}$, here defined by (5.2.1), which is a Stackel-Killing tensor for any geometry and the corresponding conserved quantity is the Hamiltonian H given by (4.1.4):

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu .$$

The other one is the Stackel-Killing tensor $K_{\mu\nu}$, which corresponds to the conserved quantity Z, given by (4.1.1):

$$Z = \frac{1}{2} K^{\mu\nu} p_\mu p_\nu .$$

In order to apply to spinning particles a supersymmetric extension of this result is required. Such a type of extension is based on the antisymmetric Killing-Yano tensor $f_{\mu\nu}$ found by Penrose and Floyd [128, 129], which satisfies equation (4.5.2):

$$D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0.$$

The Stackel-Killing tensor $K_{\mu\nu}$ is exactly the covariant square of this tensor. Then the new supersymmetry in the Kerr-Newman spacetime is generated by a supercharge of the form given in equation (4.4.8), with the Killing-Yano tensor as the f-symbol of the double vector $f_\mu{}^a$,

$$f_\mu{}^a = f_{\mu\nu}\, e^{\nu a} \ ,$$

and a corresponding three-index tensor C_{abc} obtained as in equation (4.5.16).

We now derive the explicit expression for the new supercharge and use this to obtain the Killing vector I_μ and the Killing scalar G which correspond to the Stackel-Killing tensor $K_{\mu\nu}$ in the Kerr-Newman spacetime and which define the corresponding conserved charge Z.

The Killing-Yano tensor in the Kerr-Newman spacetime is defined by [127,128]

$$\frac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\nu = a \cos\theta \, dr \wedge \left(dt - a \sin^2\theta \, d\varphi\right)$$

$$+ r \sin\theta \, d\theta \wedge \left[-a\, dt + \left(r^2 + a^2\right) d\varphi\right]. \qquad (5.2.5)$$

The vielbein $e_\mu{}^a(x)$ corresponding to the metric (5.2.1) has the following expressions:

$$e_\mu{}^0 \, dx^\mu = -\frac{\sqrt{\Delta}}{\rho}\left(dt - a \sin^2\theta \, d\varphi\right),$$

$$e_\mu{}^1 \, dx^\mu = \frac{\rho}{\sqrt{\Delta}}\, dr,$$

$$e_\mu{}^2\, dx^\mu = \rho\, d\theta,$$

$$e_\mu{}^3\, dx^\mu = \frac{\sin\theta}{\rho}\left[-a\, dt + \left(r^2 + a^2\right) d\varphi\right]. \qquad (5.2.6)$$

Then, one finds the following components of $f_\mu{}^a\,(x)$:

$$f_\mu{}^0\, dx^\mu = \frac{\rho}{\sqrt{\Delta}}\, a\cos\theta\, dr,$$

$$f_\mu{}^1\, dx^\mu = -\frac{\sqrt{\Delta}}{\rho}\, a\cos\theta\left(dt - a\sin^2\theta\, d\varphi\right),$$

$$f_\mu{}^2\, dx^\mu = -\frac{r\sin\theta}{\rho}\left[-a\, dt + \left(r^2 + a^2\right) d\varphi\right],$$

$$f_\mu{}^3\, dx^\mu = \rho\, r d\theta. \qquad (5.2.7)$$

It can be checked that this $f_\mu{}^a(x)$ indeed satisfies equation (4.4.6). In order to find a conserved quantity we now need to calculate $C_{abc}(x)$ from (4.5.16). Its components are

$$C_{012} = \frac{2a\sin\theta}{\rho}, \quad C_{013} = 0, \quad C_{023} = 0, \quad C_{123} = -\frac{2\sqrt{\Delta}}{\rho}. \qquad (5.2.8)$$

Using the quantities derived in equations (5.2.7), (5.2.8) we obtain from (4.4.8) the new supersymmetry generator $\mathscr{Q}'_f$ for the Kerr-Newman spacetime. From equations (4.4.11)–(4.4.13), the Killing tensor $K_{\mu\nu}$, Killing vector I_μ and Killing scalar G can be constructed as follows:

$$K_{\mu\nu}(x)\,dx^\mu dx^\mu = -\frac{\rho^2 a^2 \cos^2\theta}{\Delta}dr^2 + \frac{\Delta\, a^2 \cos^2\theta}{\rho^2}\left(dt - a\sin^2\theta\, d\varphi\right)^2$$

$$+\frac{r^2\sin^2\theta}{\rho^2}\left[-a\,dt + \left(r^2+a^2\right)d\varphi\right]^2 + \rho^2 r^2 d\theta^2, \tag{5.2.9}$$

$$I_\mu(x)dx^\mu = \frac{2i}{\rho^2}\left(r\sin\theta\,\psi^1 + \sqrt{\Delta}\cos\theta\,\psi^2\right)\left(a\sin\theta\,\psi^0 - \sqrt{\Delta}\,\psi^3\right)$$

$$\times\left[-a\,dt + \left(r^2+a^2\right)d\varphi\right]$$

$$-i\sqrt{\Delta}\cos\theta\,\psi^2\left(a\sin\theta\;\psi^0 - \sqrt{\Delta}\,\psi^3\right)d\varphi$$

$$+i\sqrt{\Delta}\left(r\sin\theta\;\psi^1 + \sqrt{\Delta}\cos\theta\;\psi^2\right)\psi^3\,d\varphi$$

$$+\frac{i\,a\sin\theta}{\sqrt{\Delta}}\left(r\,\psi^0\psi^3 + a\cos\theta\,\psi^1\psi^2\right)dr$$

$$+i\sqrt{\Delta}\;\left(a\cos\theta\,\psi^0\psi^3 - r\,\psi^1\psi^2\right)d\theta, \tag{5.2.10}$$

$$G = -\frac{2q\;a\cos\theta}{\rho^2}\psi^0\psi^1\psi^2\psi^3. \tag{5.2.11}$$

The above expressions and $\mathcal{Q}_f$ then define the conserved charge

$$Z = \frac{1}{2}\,i\left\{\mathcal{Q}_f,\;\mathcal{Q}_f\right\}.$$

5.3. DISCUSSION

As expressed in (1.2.10) the anticommuting spin variables are related to the standard antisymmetric spin tensor S^{ab}, which appears in the definition of the generators of local Lorentz transformations, by $S^{ab}=-i\psi^a\psi^b$. This relation makes the physical interpretation of the equations (5.2.9)–(5.2.11) more clear. Indeed, using the Dirac-Poisson brackets (4.2.16), it can be verified straightforwardly that these equations satisfy the *SO(3, 1)* algebra. The full Lorentz transformations are then generated by $M^{ab}=L^{ab}+S^{ab}$, with L^{ab} the orbital part. Likewise, the generators of other symmetries such as Z also receive a spin-dependent part. The Killing tensor $K_{\mu\nu}$ given in (5.2.9) is the one, which was found in ref. [12]. For spinless point particles in Kerr-Newman spacetime it defines a constant of motion directly, but for spinning particles it now requires the nontrivial contributions from spin. This spin dependent part contains the Killing vector and Killing scalar computed in (5.2.10) and (5.2.11).

CHAPTER VI

NONGENERIC SUSY IN NUT-KERR-NEWMAN SPACETIME

6.1. INTRODUCTION

In the previous chapter we have reviewed the work of Gibbons *et al.* [80] of investigating nongeneric supersymmetry in the Kerr-Newman spacetime. In this Chapter we would like to extend that work in the Kerr-Newman spacetime generalized with NUT (magnetic mass) parameter [92]. This type of extension is interesting in that the spacetime endowed with NUT parameter should never be directly physically interpreted [83].

We arrange this chapter as follows. In section 6.2 we derive the Killing-Yano tensor in the NUT-Kerr-Newman spacetime and calculate the corresponding Killing tensor, Killing vector and Killing scalar, which generate the nongeneric supersymmetry. In section 6.3 we present our remarks.

6.2. SPINNING NUT-KERR-NEWMAN SPACETIME

In this section we apply the results obtained in Chapter IV to the motion of pseudo-classical spinning point particles moving in a stationary axisymmetric spacetime of general relativity described by the NUT-Kerr-Newman metric, which has the form

$$ds^2 = -\frac{\Delta}{\rho^2}\left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right)d\varphi\right]^2$$

$$+\frac{\sin^2\theta}{\rho^2}\left[a\,dt - \left(r^2 + a^2\right)d\varphi\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \quad (6.2.1)$$

where

$$\Delta = r^2 + a^2 - n^2 - 2Mr + q^2,$$

$$\rho^2 = r^2 + (n - a\cos\theta)^2, \quad (6.2.2)$$

M is the mass, q the charge, $a=(J/M)$ the specific angular momentum of the gravitating body, and n the NUT (or magnetic mass) parameter. The electromagnetic field tensor associated with the spacetime (6.2.1) is expressed by

$$F = \frac{q}{\rho^4}\left[r^2 - (n - a\cos\theta)^2\right] dr \wedge \left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right)d\varphi\right]$$

$$+\frac{2q}{\rho^4}(n - a\cos\theta)r\sin\theta\, d\theta \wedge \left[a\,dt - \left(r^2 + a^2\right)d\varphi\right] \quad (6.2.3)$$

The NUT-Kerr-Newman spacetime has two horizons, which are respectively the event horizon located at r_+ and the inner (Cauchy) horizon at r_-, where

$$r_\pm = M \pm \sqrt{M^2 - q^2 - a^2 + n^2}. \quad (6.2.4)$$

Expressions in (6.2.1) and (6.2.3) describe fields only outside the event horizon. The NUT-KN spacetime has two independent second-rank Killing tensors. The metric tensor $g_{\mu\nu}$, here given by equation (6.2.1), is a Stackel-Killing tensor, which exists for any geometry, and the corresponding conserved quantity is the Hamiltonian H given by (4.1.4):

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu .$$

The other Killing tensor is the Stackel-Killing tensor $K_{\mu\nu}$, which is the ψ-independent solution of the generalized Killing equation (4.3.7) with $n=2$. The corresponding conserved quantity Z is given by (4.1.1):

$$Z = \frac{1}{2} K^{\mu\nu} p_\mu p_\nu .$$

We need a supersymmetric extension of the above result to apply to spinning particles. This type of extension is based on the antisymmetric Killing-Yano tensor, $f_{\mu\nu}$, found by Penrose and Floyd [127, 128], which satisfies equation (4.5.2):

$$D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0.$$

The Stackel-Killing tensor $K_{\mu\nu}$ is exactly the covariant square of this tensor. Then the new supersymmetry in the NUT–KN spacetime is obtained from a supercharge given in equation (4.4.8), with the Killing–Yano tensor as the f–symbol of the double vector $f_\mu{}^a$,

$$f_\mu{}^a = f_{\mu\nu}\, e^{\nu a},$$

and a corresponding three-index tensor C_{abc} as obtain in equation (4.5.16).

We first derive the explicit expression for the new supercharge. Using this we then obtain the Killing vector I_μ and the Killing scalar G which correspond to the Stackel-Killing tensor $K_{\mu\nu}$ in the NUT-Kerr-Newman spacetime and define the conserved charge Z.

As was defined in [127, 128], the Killing-Yano tensor in the NUT-KN spacetime is given by

$$\frac{1}{2}\, f_{\mu\nu}\, dx^\mu \wedge dx^\nu = -(n - a\cos\theta)\, d\, r \wedge \left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a} \right) d\varphi \right]$$

$$- r\sin\theta\, d\theta \wedge \left[a\, dt - \left(r^2 + a^2 \right) d\varphi \right] \qquad (6.2.5)$$

The vielbein $e_\mu{}^a(x)$ corresponding to the metric (6.2.1) has the following expressions:

$$e_\mu{}^0\, dx^\mu = -\frac{\sqrt{\Delta}}{\rho} \left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a} \right) d\varphi \right],$$

$$e_\mu{}^1\, dx^\mu = \frac{\rho}{\sqrt{\Delta}}\, dr,$$

$$e_\mu{}^2\, dx^\mu = \rho\, d\theta,$$

$$e_\mu{}^3\, dx^\mu = -\frac{\sin\theta}{\rho}\left[a\, dt - \left(r^2 + a^2\right) d\varphi\right]. \tag{6.2.6}$$

Using the vielbein one then finds the following components of $f_\mu{}^a(x)$:

$$f_\mu{}^0\, dx^\mu = -\frac{\rho}{\sqrt{\Delta}}\left(n - a\cos\theta\right) dr,$$

$$f_\mu{}^1\, dx^\mu = \frac{\sqrt{\Delta}}{\rho}\left(n - a\cos\theta\right)\left[dt - \left(a - \frac{\left(n - a\cos\theta\right)^2}{a}\right) d\varphi\right],$$

$$f_\mu{}^2\, dx^\mu = \frac{r\sin\theta}{\rho}\left[a\, dt - \left(r^2 + a^2\right) d\varphi\right],$$

$$f_\mu{}^3\, dx^\mu = \rho\, r\, d\theta. \tag{6.2.7}$$

It can be checked that this $f_\mu{}^a(x)$ indeed satisfies equation (4.4.6). Finally, to find a conserved quantity we need to calculate $C_{abc}(x)$. Using equation (4.5.16) its components are given as follows:

$$C_{012} = \frac{2a\sin\theta}{\rho}, \quad C_{013} = 0, \quad C_{023} = 0, \quad C_{123} = -\frac{2\sqrt{\Delta}}{\rho}. \tag{6.2.8}$$

Inserting the quantities derived in equations (6.2.7), (6.2.8) into equation (4.4.8) we obtain the new supersymmetry generator $\mathscr{Q}_f$ for the NUT-KN spacetime. From equations (4.4.11)–(4.4.13) we construct the Killing tensor, vector and scalar as follows:

$$
\begin{aligned}
K_{\mu\nu}(x)\,dx^{\mu}dx^{\mu} = &-\frac{\rho^2 (n - a\cos\theta)^2}{\Delta}\,dr^2 + \frac{\Delta (n - a\cos\theta)^2}{\rho^2} \\
&\times\left[dt - \left(a - \frac{(n - a\cos\theta)^2}{a}\right)d\varphi\right]^2 \\
&+ \frac{r^2\sin^2\theta}{\rho^2}\left[a\,dt - \left(r^2 + a^2\right)d\varphi\right]^2 + \rho^2 r^2 d\theta^2,
\end{aligned}
\tag{6.2.9}
$$

$$
\begin{aligned}
I_{\mu}(x)\,dx^{\mu} = &-\frac{2i}{\rho^2}\left(r\sin\theta\,\psi^1 + \sqrt{\Delta}\cos\theta\,\psi^2\right)\left(a\sin\theta\,\psi^0 - \sqrt{\Delta}\,\psi^3\right) \\
&\times\left[a\,dt - \left(r^2 + a^2\right)d\varphi\right] \\
&- i\sqrt{\Delta}\cos\theta\,\psi^2\left(a\sin\theta\,\psi^0 - \sqrt{\Delta}\,\psi^3\right)d\varphi \\
&+ i\sqrt{\Delta}\left(r\sin\theta\,\psi^1 + \sqrt{\Delta}\cos\theta\,\psi^2\right)\psi^3\,d\varphi \\
&+ \frac{i\,a\sin\theta}{\sqrt{\Delta}}\left[r\psi^0\psi^3 - (n - a\cos\theta)\psi^1\psi^2\right]dr \\
&- i\sqrt{\Delta}\left[(n - a\cos\theta)\,\psi^0\psi^3 + r\,\psi^1\psi^2\right]d\theta,
\end{aligned}
\tag{6.2.10}
$$

$$
G = \frac{2q\,(n - a\cos\theta)}{\rho^2}\,\psi^0\psi^1\psi^2\psi^3. \tag{6.2.11}
$$

The expressions for $\mathscr{Q}_f$ and (6.2.9)–(6.2.11) then define the conserved charge

$$Z = \frac{1}{2}\, i \left\{ Q_f ,\ Q_f \right\}.$$

We note that for $n=0$ the above results reduce to those obtained for the Kerr-Newman spacetime [80] (described in Chapter V).

6.3. REMARKS

The supersymmetric extension of the NUT-Kerr-Newman spacetime admits nongeneric supersymmetries.

The Killing tensor $K_{\mu\nu}$ given in (6.2.9) defines a constant of motion directly for spinless particles in the NUT-Kerr-Newman spacetime, whereas for spinning particles it now requires the nontrivial contributions from spin which involve the Killing vector and Killing scalar computed in (6.2.10) and (6.2.11).

The results obtained in this Chapter for the NUT-Kerr-Newman spacetime [92] go for the NUT spacetime when $a=q=0$, and for the Kerr-Newman spacetime [80] when $n=0$. The study thus not only encompasses the results of Gibbons *et al.* [80], but also provides similar results for the NUT spacetime, which is sometimes considered as unphysical [82].

CHAPTER VII

NONGENERIC SUSY IN HOT NUT-KERR-NEWMAN-KASUYA SPACETIME

7.1. INTRODUCTION

In Chapter VI we have investigated a new kind of supersymmetry and the corresponding conserved quantity in the NUT-Kerr-Newman spacetime [92] in terms of the motion of pseudo-classical spinning point particles. In this chapter we would like to extend that work in a more general spacetime called the combined NUT-Kerr-Newman-Kasuya-de Sitter spacetime. This is the NUT-Kerr-Newman spacetime generalized with an extra magnetic monopole charge and a cosmological constant. This spacetime is asymptotically de Sitter and since de Sitter spacetime has been interpreted as being hot [96], we call it the hot NUT-Kerr-Newman-Kasuya (H-NUT-KN-K) spacetime.

In recent years there has been a renewed interest in the study of magnetic monopole [113, 117–120, 131–134]. Our work is also interesting in that regard. Besides, because of the presence of cosmological constant, this work is interesting from the point of view of inflationary scenario of early universe.

We arrange this chapter as follows. In section 7.2 we derive the Killing-Yano tensor in the H-NUT-KN-K spacetime and calculate the corresponding Killing tensor, Killing vector and Killing scalar, which generate the nongeneric supersymmetry. In section 7.3 we present our remarks.

7.2. SPINNING HOT NUT-KERR-NEWMAN-KASUYA SPACETIME

In this section we apply the results of Chapter IV to the motion of pseudo-classical spinning particles moving in a more general spacetime in general relativity described by the hot NUT-Kerr-Newman-Kasuya metric, which has the form

$$ds^2 = \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\exists^{-2} \Delta_\theta \sin^2\theta}{\Sigma} \left(-a\, dt + \rho\, d\varphi\right)^2 \cdot$$

$$- \frac{\exists^{-2} \Delta_r}{\Sigma} \left(dt - A\, d\varphi\right)^2, \tag{7.2.1}$$

where

$$\Sigma = r^2 + \left(n + a\cos\theta\right)^2,$$

$$\Delta_\theta = 1 + \frac{1}{3} \Lambda a^2 \cos^2\theta,$$

$$\Delta_r = \left(r^2 + a^2 + n^2\right)\left[1 - \frac{1}{3}\Lambda\left(r^2 + 5n^2\right)\right] - 2\left(M r + n^2\right) + q_e^2 + q_m^2 \ ,$$

$$\exists = 1 + \frac{1}{3} \Lambda a^2 \quad ,$$

$$\rho = r^2 + a^2 + n^2 \quad ,$$

$$A = a \sin^2\theta - 2n\cos\theta \, . \qquad (7.2.2)$$

Besides the cosmological constant Λ, the metric possesses the mass parameter M, the NUT (or magnetic mass) parameter n, the specific angular momentum parameter a $(=J/M)$, the electric charge parameter q_e, and the magnetic monopole charge parameter q_m. The surface $\Delta_r=0$ gives the horizons of the spacetime. The electromagnetic field tensor associated with this spacetime is expressed by

$$F = \frac{\exists^{-1}(q_e + iq_m)}{\Sigma^4}\left[r^2 - (n + a\cos\theta)^2\right] dr \wedge (dt - A\, d\varphi)$$

$$- \frac{2(q_e + iq_m)\exists^{-1} r\,(n + a\cos\theta)\sin\theta}{\Sigma^4} d\theta \wedge (-a\, dt + \rho\, d\varphi). \qquad (7.2.3)$$

The spacetime given by (7.2.1) and (7.2.2) includes:

(i) NUT-Kerr-Newman-Kasuya (NUT-KN-K) spacetime for $\Lambda=0$;

(ii) hot Kerr-Newman-Kasuya (H-KN-K) spacetime for $n=0$;

(iii) hot NUT-Kerr-Newman (H-NUT-KN) spacetime for $q_m=0$;

(iv) hot Kerr-Newman (H-KN) spacetime [135, 136] for $n=q_m=0$;

(v) hot Kerr spacetime [136, 137] for $n=q_m=q_e=0$;

(vi) hot Reissner-Nordstrom spacetime for $n=q_m=a=0$;

(vii) hot Schwarzschild spacetime [136, 138] for $n=q_m=a=q_e=0$;

(viii) hot NUT spacetime [139] for $a=q_e=q_m=0$;

(ix) de Sitter spacetime [140] for $M=n=a=q_e=q_m=0$.

Thus we observe that the H-NUT-KN-K spacetime includes the NUT-KN-K, H-KN-K, H-NUT-KN, hot NUT, de Sitter spacetimes as well as all the black hole spacetimes (iv)–(vii) which are asymptotically de Sitter. Further, if we put $\Lambda=0$ in the cases (ii)–(vii), we get the Kerr-Newman-Kasuya, NUT-Kerr-Newman spacetimes and all the black hole spacetimes which are asymptotically flat. In the limit $\Lambda=0$, the case (viii) reduces to the NUT spacetime, which is sometimes considered as unphysical.

The H-NUT-KN-K spacetime has two independent second-rank Stackel-Killing tensors. One is the metric tensor $g_{\mu\nu}$, here defined by equation (7.2.1), which exists for any geometry and the corresponding conserved quantity is the Hamiltonian H, given by (4.1.4):

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu .$$

The other Stackel-Killing tensor is the tensor $K_{\mu\nu}$ and the corresponding conserved quantity Z is given by (4.1.1):

$$Z = \frac{1}{2} K^{\mu\nu} p_\mu p_\nu .$$

To apply to spinning particles we need a supersymmetric extension of this result and such an extension is based on the antisymmetric Killing-Yano tensor $f_{\mu\nu}$ found by Penrose and Floyd [127, 128], which satisfies equation (4.5.2):

$$D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0.$$

The Stackel-Killing tensor $K_{\mu\nu}$ is exactly the covariant square of this tensor. Then the new supersymmetry in the H-NUT-KN-K spacetime is obtained from a supercharge given in equation (4.4.8), with the Killing-Yano tensor as the f-symbol of the double vector $f_\mu{}^a$,

$$f_\mu{}^a = f_{\mu\nu} \, e^{\nu a} .$$

and a corresponding three-index tensor C_{abc} as obtained in equation (4.5.16).

We first derive the explicit expression for the new supercharge. Using this we then obtain the Killing vector I_μ and the Killing scalar G, which correspond to the Stackel-Killing tensor $K_{\mu\nu}$ in the H-NUT-KN-K spacetime and define the conserved charge Z.

As was defined in [127, 128], the Killing-Yano tensor in the H-NUT-KN-K spacetime is given by

$$\frac{1}{2} f_{\mu\nu}\, dx^{\mu} dx^{\nu} = \frac{(n + a\cos\theta)}{\exists} dr \wedge (dt - A\, d\varphi)$$

$$+ \frac{r\sin\theta}{\exists} dr \wedge (-a\, dt + \rho\, d\varphi). \qquad (7.2.4)$$

The vielbein $e_{\mu}{}^{a}(x)$ corresponding to the metric (7.2.1) has the following expressions:

$$e_{\mu}{}^{0}\, dx^{\mu} = -\frac{\sqrt{\Delta_r}}{\exists\sqrt{\Sigma}} (dt - A\, d\varphi),$$

$$e_{\mu}{}^{1}\, dx^{\mu} = \frac{\sqrt{\Sigma}}{\sqrt{\Delta_r}} dr,$$

$$e_{\mu}{}^{2}\, dx^{\mu} = \frac{\sqrt{\Sigma}}{\sqrt{\Delta_\theta}} d\theta,$$

$$e_{\mu}{}^{3}\, dx^{\mu} = \frac{\sqrt{\Delta_\theta}}{\exists\sqrt{\Sigma}} (-a\, dt + \rho\, d\varphi). \qquad (7.2.5)$$

Using the vielbein one then finds the following components of $f_{\mu}{}^{a}$ (x):

$$f_{\mu}{}^{0}\, dx^{\mu} = \frac{\sqrt{\Sigma}}{\sqrt{\Delta_r}} (n + a\cos\theta)\, dr,$$

$$f_{\mu}{}^{1}\, dx^{\mu} = -\frac{\sqrt{\Delta_r}}{\exists\sqrt{\Sigma}} (n + a\cos\theta)(dt - A\, d\varphi),$$

$$f_\mu{}^2\, dx^\mu = -\frac{\sqrt{\Delta_\theta}\; r \sin\theta}{\exists\sqrt{\Sigma}}\left(-a\, dt + \rho\, d\varphi\right),$$

$$f_\mu{}^3\, dx^\mu = \frac{\sqrt{\Sigma}}{\sqrt{\Delta_\theta}}\, r\, d\theta. \tag{7.2.6}$$

It can be checked that this $f_\mu{}^a(\mathrm{x})$ indeed satisfies equation (4.4.6). Finally, to find a conserved quantity we need to calculate $C_{abc}(x)$. Using equation (4.5.16) its components are given as follows:

$$C_{012} = \frac{2a\sqrt{\Delta_\theta}\,\sin\theta}{\exists\sqrt{\Sigma}}, \quad C_{013} = 0, \quad C_{023} = 0, \quad C_{123} = -\frac{2\sqrt{\Delta_r}}{\exists\sqrt{\Sigma}}. \tag{7.2.7}$$

Inserting the quantities derived in equations (7.2.6), (7.2.7) into equation (4.4.8) we obtain the new supersymmetry generator $\mathscr{Q}_f$ for the H-NUT-KN-K spacetime. From equations (4.4.11)–(4.4.13) we construct the Killing tensor, vector and scalar as follows:

$$K_{\mu\nu}(x)\, dx^\mu dx^\mu = -\frac{(n + a\cos\theta)^2\,\Sigma}{\Delta_r}\, dr^2 + \frac{\Delta_r (n + a\cos\theta)^2}{\exists^2\Sigma}\left(dt - A\, d\varphi\right)^2$$

$$+\frac{\Delta_\theta r^2 \sin^2\theta}{\exists^2\Sigma}\left(-a\, dt + \rho\, d\varphi\right)^2 + \frac{\Sigma}{\Delta_\theta} r^2\, d\theta^2, \tag{7.2.8}$$

$$I_\mu(x) dx^\mu = \frac{2i}{\exists^2\Sigma}\left(r \sin\theta\, \psi^1 + \sqrt{\Delta_r}\, \cos\theta\, \psi^2\right)$$

$$\times\left(a\sqrt{\Delta_\theta}\,\sin\theta\,\psi^0-\sqrt{\Delta_r}\,\psi^3\right)\left[-a\,dt+\rho\,d\varphi\right]$$

$$-i\sqrt{\Delta_r}\,\cos\theta\,\psi^2\left(a\sqrt{\Delta_\theta}\,\sin\theta\,\psi^0-\sqrt{\Delta_r}\,\psi^3\right)d\varphi$$

$$+i\sqrt{\Delta_r}\left(r\sin\theta\,\psi^1+\sqrt{\Delta_r}\,\cos\theta\,\psi^2\right)\psi^3\,d\varphi$$

$$+\frac{i\,a\sqrt{\Delta_\theta}\,\sin\theta}{\sqrt{\Delta_r}}\left(r\,\psi^0\psi^3+(n+a\cos\theta)\,\psi^1\psi^2\right)dr$$

$$+\frac{i\sqrt{\Delta_r}}{\sqrt{\Delta_\theta}}\left[(n+a\cos\theta)\psi^0\psi^3-r\psi^1\psi^2\right]d\theta, \tag{7.2.9}$$

$$G=-\frac{2(q_e+iq_m)}{\Sigma}(n+a\cos\theta)\psi^0\psi^1\psi^2\psi^3. \tag{7.2.10}$$

The expressions for $\mathcal{Q}_f$ and (7.2.8)–(7.2.10) then define the conserved charge

$$Z=\frac{1}{2}\,i\left\{\mathcal{Q}_f,\ \mathcal{Q}_f\right\}.$$

We note that the above results reduce to the results of the NUT-Kerr-Newman spacetime, described in Chapter VI, for $\Lambda=q_m=0$, and of the Kerr-Newman spacetime, described in Chapter V, for $n=\Lambda=q_m=0$.

7.3. REMARKS

The supersymmetric extension of the hot NUT-Kerr-Newman-Kasuya spacetime admits nongeneric supersymmetries.

The Killing tensor $K_{\mu\nu}$ given in (7.2.8) defines a constant of motion directly for spinless particles in the H-NUT-KN-K spacetime, whereas for spinning particles it now requires the nontrivial contributions from spin which involve the Killing vector and Killing scalar computed in (7.2.9) and (7.2.10).

The result obtained in this Chapter for the H-NUT-KN-K spacetime goes for the NUT-KN spacetime [92] when $\Lambda=q_m=0$, and for the Kerr-Newman spacetime [80] when $n=\Lambda=q_m=0$.

This study not only encompasses the result of Gibbons *et al.* in the context of Kerr-Newman black hole spacetime and of our work in the context of NUT-Kerr-Newman spacetime, but also provides similar result if the Kerr-Newman spacetime is involved with magnetic monopole and/or cosmological constant. So, it is interesting to note that the physical result remains the same whether or not the magnetic monopole does exist in nature.

DISCUSSION

Our main concern has been the geodesic motion of pseudo-classical spinning particles in the Schwarzschild spacetime [Chapter II] generalized with a charge parameter [Chapter III] along with a NUT parameter [Chapter III] and nongeneric supersymmetry in the Kerr-Newman spacetime [Chapter V] generalized with a NUT parameter [Chapter VI] along with an extra magnetic monopole charge and a cosmological constant [Chapter VII]. From this work, it appears that the mathematical treatment for studying the spinning particles in the non-black hole spacetimes having horizons is the same as in the black hole spacetimes. Not only the mathematical treatment for studying the spinning particles but also for other cases this assertion holds true. For example, we would like to mention different works of Ahmed [141–146], Ahmed *etal.* [147–152] and of ours [153]. Ahmed extensively studied different problems such as superradiance phenomena, Hawking radiation in the spacetimes, which are not black hole spacetimes but the spacetimes having horizons. Ahmed observed in his different works that the physical results in superradiance phenomena and Hawking radiation are not only true for the black hole spacetimes but also true for the spacetimes having horizons. The mathematical treatment followed by Ahmed in all of these cases is analogous to those used for the study of radiation for the black hole spacetime.

The supersymmetric extension of curved spacetime admits "nongeneric" supersymmetries along with "generic" ones. Spacetime supersymmetry has previously been applied to charged black holes in the context of $N=2$ supergravity theory. The application of world-line supersymmetry in Chapters IV–VII seems at first sight to be unrelated to that work. The results concerning a 'hidden' supersymmetry related to the motion of spinning point particles are applicable to all members of the Kerr-Newman family of black-hole spacetimes and to the spacetimes which are not black-hole spacetimes but have horizons such as hot NUT-Kerr-Newman-Kasuya spacetime. On the other hand, the Killing spinors giving rise to symmetries of the spacetimes of charged black-holes in the context of $N=2$ supergravity theory, arise only in the extreme cases (or indeed naked singularities) in which mass and charge in suitable units are equal.

Supersymmetry and its local version –supergravity–are relevant in the fundamental theory of particle interactions. In modern particle theory, SUSY is the most general symmetry of the S-matrix consistent with relativistic quantum field theory [154]. So it is not inconceivable that nature might make some use of it. Indeed, superstrings [155, 156] are the present best candidates for a consistent quantum theory unifying gravity with all other fundamental interactions, and SUSY appears to play a very important role for the quantum stability of superstring solutions in four-dimensional spacetime.

For all of the above reasons, the study of spinning particles in curved spacetime is well motivated.

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