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DEDICATED TO

MY PARENTS,

Who have profoundly influenced my life.

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Certified that the thesis entitled **"A study on finitely generated n-ideals of a lattice"** submitted by **Md. Ayub Ali** in fulfilment of the requirements for the degree of **Doctor of Philosophy** in Mathematics, University of Rajshahi, has been completed under my supervision. I believe that this research work is an original one and it has not been submitted elsewhere for any degree.

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STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my acknowledge and belief, does not contain material previously published or written by another person except where due reference is made in the text.

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ABSTRACT

This thesis studies extensively the finitely generated n-ideals of a lattice. The idea of n-ideals in a lattice was first introduced by Cornish and Noor in studying the kernels around a particular element n, of a skeletal congruence on a distributive lattice. Then Latif in his thesis "n-ideals of a lattice" studied thoroughly on the n-ideals and established many valuable results. For a fixed element n of a lattice L, a convex sublattice of L containing n is called an n-ideal. If L has a "0", then replacing n by 0, an n-ideal becomes an ideal and if L has a "1" then it becomes a filter by replacing n by 1. Thus, the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. The n-ideal generated by a finite number of elements of a lattice is called a finitely generated n-ideal, while the n-ideal generated by a single element is known as a principal n-ideal. Latif in his thesis has given a neat description on finitely generated n-ideals of a lattice and has provided a number of important results on them. According to Latif, for a lattice L, the lattice of all n-ideals of L and the lattice of all finitely generated n-ideals of L are denoted by $I_n(L)$ and $F_n(L)$ respectively, while $P_n(L)$ represents the set of principal n-ideals of L. In this thesis, we devote ourselves in studying several properties on $F_n(L)$ which will certainly enrich many

branches of lattice theory. Our results in this thesis generalize many results on Boolean, generalized Boolean, Stone, generalized Stone, and relatively Stone lattices. We also generalize several results on pseudocomplemented lattices satisfying the Lee's identity.

In this connection it should be mentioned that if L has a 0, then putting n=0 we find that $F_n(L)$ is the set of all principal ideals of L which is isomorphic to L. Thus, for every result on $F_n(L)$ in this thesis, we can obtain a result for the lattice L with 0 by substituting n=0. Hence the result in each chapter of the thesis regarding $F_n(L)$ are generalizations of the corresponding results in lattice theory.

In chapter 1, we discuss some fundamental properties of n-ideals which are basic to this thesis. Here we give an explicit description of $F_n(L)$ and $P_n(L)$ which are essential for the development of the thesis. Though $F_n(L)$ is always a lattice, $P_n(L)$ is not even a semilattice. But when n is a neutral element, $P_n(L)$ becomes a meet semilattice. Moreover, we show that $P_n(L)$ is a lattice if and only if n is a central element, and then in fact, $P_n(L)=F_n(L)$. We also show that, for a neutral element n, the lattice L is complemented if and only if $P_n(L)$ is so. In this chapter we also discuss on prime n-ideals. We give several properties and characterizations of prime n-ideals. We include a proof of the generalization of Stone's representation theorem. We also include a new proof of the result that for a distributive lattice L, $F_n(L)$ is generalized Boolean if and only if prime n-ideals are unorderd.

Chapter 2 discusses on minimal prime n-ideals of a lattice. We give some characterizations on minimal prime n-ideals which are essential for the further development of this chapter. Here we provide a number of results which are generalizations of the results on Stone and generalized Stone lattices. We prove that if $F_n(L)$ is a sectionally pseudocomplemented distributive lattice then $F_n(L)$ is generalized Stone if and only if each prime n-ideals of L contains a unique minimal prime n-ideal, which is also equivalent to $<x>_n^{+} \vee <x>_n^{++} = L$ for all $x \in L$.

In chapter 3 we introduce the notion of relative n-annihilators $<a, b>^n$. We characterize distributive and modular lattices in terms of relative n-annihilators. Then we generalize several results of Mandelker on annihilators. We use these results to characterize those $F_n(L)$ which are Stone lattices. Among many results we have shown that if $F_n(L)$ is a relatively pseudocomplemented distributive lattice, then $F_n(L)$ is relatively Stone if and only if any two incomparable prime n-ideals of L are comaximal. What is more, this is also equivalent to the condition

$$<<a>_n, <b>_n> \vee <<b>_n, <a>_n> = L \text{ for all } a, b \in L.$$

Pseudocomplemented distributive lattices satisfying Lee's identities form equational subclasses denoted by B_m, $-1 \leq m < \omega$. Cornish and Mandelker have studied distributive lattices analogues to B_1-lattices and relatively B_1-lattices. Moreover, Cornish, Beazer and Davey have each independently obtained several characterizations of (sectionally) B_m-lattices and relatively B_m-lattices. In chapter 4 we generalize their results by studying finitely gnerated n-ideals which form a (sectionally) B_m-lattice and a relatively B_m-lattice. We show that if $F_n(L)$ is (sectionally) pseudocomplemented and distributive, then $F_n(L)$ is (sectionally) in B_m if and only if for any $x_1, x_2, \text{--------}, x_m \in L$, $<x_0>_n^+ \vee \text{---------} \vee <x_m>_n^+ = L$, which is also equivalent to the condition that for any m+1 distinct minimal prime n-ideals $P_0, \text{--------}, P_m$ of L, $P_0 \vee \text{--------} \vee P_m = L$. In this chapter we also show that if $F_n(L)$ is relatively pseudocomplemented, then $F_n(L)$ is relatively in B_m if and only if any m+1 pairwise incomparable prime n-ideals are comaximal.

Chapter 5 introduces the concept of distributive and modular n-ideals of a lattice. Here we include several characterizations of those n-ideals. We prove some interesting results which generalize several results on distributive and modular ideals in lattices. Latif in his thesis has introduced the concept of standard n-ideals of a lattice. We conclude this thesis with some more properties of standard and neutral n-ideals.

Chapter-1

n-ideals of a lattice.

Introduction:

The intention of this chapter is to outline and fix the notation for some of the concepts of n-ideals of a lattice which are basic to this thesis. The idea of n-ideals was first introduced by Cornish and Noor in several papers [10] and [41]. The n-ideals have also been used in proving some results in [42].

The n-ideals of a lattice have been studied extensively by Noor and Latif in [31], [32], [33], [34], [35], [48], [49], [50], [51] and [52]. For a fixed element n of a lattice L, a convex sublattice containing n is called an n-*ideal*. If L has "0", then replacing n by "0" an n-ideal becomes an ideal. Moreover if L has 1, an n-ideal becomes a filter by replacing n by 1. Thus the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. So any result involving n-ideals of a lattice L will give a generalization of the results on ideals if $0 \in L$ and filters if $1 \in L$.

The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set inclusion. Moreover, $\{n\}$ and L are respectively the smallest and the largest elements of $I_n(L)$, while the set theoretic intersection is the infimum.

For any two n-ideals I and J of a lattice L, it is easy to check that

$I \cap J = \{x : x = m(i, n, j)$ for some $i \in I, j \in J\}$, where

$m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and

$I \vee J = \{x : i_1 \wedge j_1 \leq x \leq i_2 \vee j_2$, for some $i_1, i_2 \in I$ and $j_1, j_2 \in J\}$.

The n-ideal generated by $a_1, a_2, \ldots\ldots\ldots, a_m$ is denoted by $\langle a_1, a_2, \ldots\ldots, a_m \rangle_n$. Clearly $\langle a_1, a_2, \ldots\ldots, a_n \rangle_n = \langle a_1 \rangle_n \vee \langle a_2 \rangle_n \vee \ldots\ldots \vee \langle a_m \rangle_n$.

The n-ideal generated by a finite number of elements is called a finitely generated n-ideal. The set of all finitely generated n-ideals is denoted by $F_n(L)$. Of course, $F_n(L)$ is a lattice. The n-ideal generated by a single element is called a *principal* n-ideal. The set of all principal n-ideals of a lattice L is denoted by $P_n(L)$. We have $\langle a \rangle_n = \{x \in L : a \wedge n \leq x \leq a \vee n\}$.

The median operation $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ is very well known in lattice theory. This has been used by several authors including Birkhoff and Kiss [4] for bounded distributive lattices, Jakubik and Kalibiar [22] for distributive lattices and Sholander [57] for median algebras.

An n-ideal P of a lattice L is called *prime* if $m(x, n, y) \in P$ $(x, y \in L)$ implies $x \in P$ or $y \in P$.

Standard and neutral elements in a lattice were studied extensively in [14] and [18]. An element s of a lattice L is called *standard* if for all $x, y \in L$,

$$x \wedge (y \vee s) = (x \wedge y) \vee (x \wedge s).$$

An element $n \in L$ is called *neutral* if it is standard and for all $x, y \in L$,

$n \wedge (x \vee y) = (n \wedge x) \vee (n \wedge y)$. By [15], we know that $n \in L$ is neutral if and only if for all $x, y \in L$, $m(x, n, y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n) = (x \vee y) \wedge (x \vee n) \wedge (y \vee n)$. Of course 0 and 1 of a lattice are always neutral. In a distributive lattice clearly every element is standard and neutral.

Let L be a lattice with 0 and 1. For an element $a \in L$, a' is called the *complement* of a if $a \wedge a' = 0$ and $a \vee a' = 1$. A bounded lattice in which every element has a complement is called *complemented* lattice. In a distributive lattice it is easy to see that every element has at most one complement.

An element $n \in L$ is called *central* if it is neutral and complemented in each interval containing it.

A lattice L with 0 is called *sectionally complemented* if $[0, x]$ is complemented for all $x \in L$. A complemented distributive lattice is called a *Boolean lattice*, while a distributive lattice with 0, which is sectionally complemented is called a *generalized Boolean lattice*. For the background material on lattices we refer the reader to the texts of G. Grätzer [13], Birkhoff [3], Rutherford [55], Khanna [28] and Maeda and Maeda [37].

In this thesis we have studied the lattice $F_n(L)$ in different situations. If L has a 0, then putting n=0, we find that $\langle a_1, \text{--------}, a_m \rangle_n = (a_1 \vee \text{--------} \vee a_m]$. Hence for n=0, $F_n(L)$ is the set of all principal ideals of L which is isomorphic to L. Thus, for every result on $F_n(L)$ in this thesis, we can obtain a result for the lattice L by substituting n=0. Hence the result in each chapter of the thesis regarding $F_n(L)$ are generalizations of several results on Boolean, generalized Boolean, Stone, generalized Stone and relatively Stone lattices. Chapter 4 gives generalizations of several results on those lattices, which are in B_m, sectionally in B_m and relatively in B_m respectively.

In section 1 we have given an explicit description of $F_n(L)$ and $P_n(L)$ which will be needed for the development of the thesis. We have shown that $P_n(L)=F_n(L)$ if and only if n is central. We have proved that a lattice L is (modular) distributive if and only if $F_n(L)$ is so. We have also shown that for a neutral element n, lattice L is complemented if and only if $P_n(L)$ is complemented. Moreover, if a' is the complement of a in L, then $\langle a' \rangle_n$ is the complement of $\langle a \rangle_n$ in $P_n(L)$.

In section 2 we have discussed on prime n-ideals. We have given several properties of prime n-ideals. We have included a proof of generalization of Stone's representation theorem. Finally we include a new proof of the result that for a distributive lattice L, $F_n(L)$ is generalized Boolean if and only if prime n-ideals of L are unordered.

1. Finitely generated n-ideals.

We start this section with the following proposition which is due to [31], also see [33] and [48]. This gives some simpler description of $F_n(L)$.

Proposition 1.1.1. *Let* $F_n(L)$ *be a lattice and* $n \in L$. *For* $a_1, a_2, \ldots\ldots, a_m \in L$,

(i) $\langle a_1, a_2, \ldots\ldots, a_m \rangle_n \subseteq \{y \in L : (a_1] \cap \ldots\ldots \cap (a_m] \cap (n] \subseteq (y] \subseteq (a_1] \vee \ldots\ldots \vee (a_m] \vee (n]\}$;

(ii) $\langle a_1, a_2, \ldots\ldots, a_m \rangle_n = \{y \in L : a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_m \wedge n \leq y \leq a_1 \vee a_2 \vee \ldots\ldots \vee a_m \vee n\}$;

(iii) $\langle a_1, a_2, \ldots\ldots, a_m \rangle_n = \{y \in L : a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_m \wedge n \leq y = (y \wedge a_1) \vee \ldots\ldots \vee (y \wedge a_m) \vee (y \wedge n)\}$, *where* L *is distributive*;

(iv) *For any* $a \in L$, $\langle a \rangle_n = \{y \in L : a \wedge n \leq y = (y \wedge a) \vee (y \wedge n)\} = \{y \in L : y = (y \wedge a) \vee (y \wedge n) \vee (a \wedge n)\}$, *where* n *is standard* ;

(v) *Each finitely generated* n-*ideals is two generated. Indeed* $\langle a_1, a_2, \ldots\ldots, a_m \rangle_n = \langle a_1 \wedge a_2 \wedge \ldots\ldots \wedge a_m \wedge n, a_1 \vee a_2 \vee \ldots\ldots \vee a_m \vee n \rangle_n$;

(vi) $F_n(L)$ *is a lattice and its members are simply the intervals* $[a, b]$ *such that* $a \leq n \leq b$ *and for each intervals* $[a, b]$ *and* $[a_1, b_1]$,

$[a, b] \vee [a_1, b_1] = [a \wedge a_1, b \vee b_1]$ *and*

$[a, b] \cap [a_1, b_1] = [a \vee a_1, b \wedge b_1]$. □

For $n \in L$, suppose $(n]^d$ denotes the dual of the lattice $(n]$. Then for any $x, y \in (n]$, $x \vee^d y = x \wedge y$ and $x \wedge^d y = x \vee y$.

Theorem 1.1.2. *Let* L *be a lattice and* $n \in L$. *The maps* $\Phi: F_n(L) \to (n]^d \times [n)$ *and* $\Psi: (n]^d \times [n) \to F_n(L)$ *is given by* $\Phi([a, b]) = (a, b)$ *and* $\Psi((x, y)) = [x, y]$ *where* $[a, b] \in F_n(L)$ *and* $(x, y) \in (n]^d \times [n)$, *are mutually inverse lattice isomorphisms. In other words,* $F_n(L) \cong (n]^d \times [n)$.

Proof: Let $[a, b] \subseteq [a_1, b_1]$. Then $a_1 \le a \le n \le b \le b_1$, and so $a \le^d a_1$ in $(n]^d$ and $b \le b_1$ in $[n)$. Thus, $(a, b) \le (a_1, b_1)$ in $(n]^d \times [n)$. Hence Φ is order preserving. If $(a, b) \le (a_1, b_1)$ in $(n]^d \times [n)$, then $a \le^d a_1$ in $(n]^d$ and $b \le b_1$ in $[n)$. Thus $a_1 \le a \le n \le b \le b_1$ in L and so $[a, b] \subseteq [a_1, b_1]$. That is, Ψ is also order preserving. But Φ and Ψ are mutually inverse and so the theorem is established. $\square$

When n is a neutral element of a lattice L, then it is very easy to check that $P_n(L)$ is a meet semilattice. In fact, for any $a, b \in L$, $\langle a \rangle_n \cap \langle b \rangle_n = \langle m(a, n, b) \rangle_n$.

But $P_n(L)$ is not necessarily a lattice. The case is different when n is a central element. The following theorem also gives characterization of central elements of a lattice L.

Theorem 1.1.3. *Let* n *be neutral element of a lattice* L. *Then* $P_n(L)$ *is a lattice if and only if* n *is central. Then of course* $P_n(L)=F_n(L)$.

Moreover, for a central element $n \in L$, L *is bounded if and only if* $P_n(L)$ *is bounded.*

Also if L *is bounded and* n *is a central element of* L, *then for any* $x, y \in L$, $\langle x \rangle_n \vee \langle y \rangle_n = \langle m(x, n', y) \rangle_n$ *where* n' *is the complement of* n *in* L.

Proof: Suppose n is central. Since for all $a, b \in L$, $\langle a \rangle_n \cap \langle b \rangle_n = \langle m(a, n, b) \rangle_n$, we need only to check that $\langle a \rangle_n \vee \langle b \rangle_n \in P_n(L)$. Now, $\langle a \rangle_n \vee \langle b \rangle_n = [a \wedge b \wedge n, a \vee b \vee n]$. Since n is central, there exists $c \in L$ such that $c \wedge n = a \wedge b \wedge n$ and $c \vee n = a \vee b \vee n$ which implies that $\langle a \rangle_n \vee \langle b \rangle_n = \langle c \rangle_n$ and so $P_n(L)$ is a lattice.

Conversely, suppose that $P_n(L)$ is a lattice and $a \le n \le b$. Then $[a, b] = \langle a \rangle_n \vee \langle b \rangle_n$. Since $P_n(L)$ is a lattice, $\langle a \rangle_n \vee \langle b \rangle_n = \langle c \rangle_n$ for some $c \in L$. This implies that $c \wedge n = a$ and $c \vee n = b$. This implies c is the relative complement of n in [a, b]. Therefore n is central.

For the second part, if $L=[0, 1]$, then $\{n\}$ and $\langle n' \rangle_n$ are the smallest and the largest elements of $P_n(L)$, where

n' is the complement of n in L. Also if $P_n(L)$ is bounded, then there exists $n' \in L$ such that $<n'>_n$ is the largest element of $P_n(L)$. Therefore for any $x \in L$, $<x>_n \subseteq <n'>_n$. That is $n \wedge n' \leq x \wedge n \leq x \leq x \vee n \leq n \vee n'$. This implies $n \wedge n'$ and $n \vee n'$ are the smallest and the largest elements of L and so L is bounded. Last part is easily verifiable. □

Thus the following results are obvious from the Theorem 1.1.2.

Theorem 1.1.4. *Let* L *be a lattice. Then* $F_n(L)$ *is sectionally complemented if and only if for each* $a, b \in L$ *with* $a \leq n \leq b$, *the interval* $[a, n]$ *and* $[n, b]$ *are complemented.* □

Corollary 1.1.5. *For a distributive lattice* L, $F_n(L)$ *is generalized Boolean if and only if the interval* $[a, n]$ *and* $[n, b]$ *are complemented for each* $a, b \in L$ *with* $a \leq n \leq b$.□

Corollary 1.1.6. *For a distributive lattice* L, $F_n(L)$ *is generalized Boolean if and only if both* $(n]^d$ *and* $[n)$ *are generalized Boolean.* □

It is clear from the Corollary 1.1.4 that if L is relatively complemented, then $F_n(L)$ is sectionally complemented and in fact $F_n(L)=P_n(L)$. If L has 0 and 1, the largest element L of $I_n(L)$ is finitely generated. Then in fact, $L=[0, 1]$.

A lattice L with 0 is said to be *section-semi-complemented lattice* (*disjunctive*) *if* $0 \leq a < b$ ($a, b \in L$) implies there is an element $x \in L$ such that $x \wedge a = 0$ and $0 < x \leq b$, while a lattice satisfying the definition which is dual to that of a section-semi complemented lattice is called a *dual section-semi complemented lattice* (*dual disjunctive*).

A lattice L is called *implicative* (*relative pseudocomplemented*) if for any given elements a and b, the set of all $x \in L$ such that $a \wedge x \leq b$ contains a largest element which is denoted by $a \rightarrow b$. A dual implicative lattice is defined dually.

The following corollary holds because of Theorem 1.1.2.

Corollary 1.1.7. *Let* L *be a lattice and* $x \in L$. *Then*

(i) $F_n(L)$ *is section-semi complemented if and only if* $(n]$ *is dual section-semi complemented and* $[n)$ *is section-semi complemented* ;

(ii) $F_n(L)$ *is implicative if and only if* $(n]$ *is dual implicative and* $[n)$ *is implicative.* □

Theorem 1.1.8. *Let* n *be a neutral element of a bounded lattice* L. *Then* L *is complemented if and only if* $P_n(L)$ *is a complemented lattice.*

Moreover, a′ *is the complement of* a *in* L *if and only if* $<a'>_n$ *is the complement of* $<a>_n$ *in* $P_n(L)$.

Proof: Suppose L is complemented. Then by Theorem 1.1.3, $P_n(L)$ is a lattice with $\{n\}$ and $<n'>_n$ as the smallest and the largest elements. Moreover, $P_n(L)=F_n(L)$. Now let $<a>_n \in P_n(L)$. Suppose a′ is the complement of a in L. Then $<a>_n \cap <a'>_n=[a\wedge n,\ a\vee n]\cap[a'\wedge n,\ a'\vee n]=[(a\vee a')\wedge n,\ (a\wedge a')\vee n]$ $=[1\wedge n,\ 0\vee n]=\{n\}$. Also, $<a>_n \vee <a'>_n=[a\wedge a'\wedge n,\ a\vee a'\vee n]$ $=[0,\ 1]=<n'>_n$. This implies $P_n(L)$ is complemented, and $<a'>_n$ is the complement of $<a>_n$ for each $a\in L$.

Conversely, suppose $P_n(L)$ is complemented. Let $a\in L$, and let $<b>_n$ be the complement of $<a>_n$ in $P_n(L)$. Then $<a>_n \cap <b>_n=\{n\}$ and $<a>_n \vee <b>_n=[0,\ 1]$. Thus, $[(a\vee b)\wedge n,\ (a\wedge b)\vee n]=\{n\}$ and $[a\wedge b\wedge n,\ a\vee b\vee n]=[0,\ 1]$. Now, $[(a\vee b)\wedge n,\ (a\wedge b)\vee n]=\{n\}$ implies $a\wedge b\le n\le a\vee b$. Hence $[0,\ 1]$ $=[a\wedge b\wedge n,\ a\vee b\vee n]=[a\wedge b,\ a\vee b]$ and so $a\wedge b=0$ and $a\vee b=1$. This implies b is the complement of a in L. Therefore L is complemented. □

Thus we have the following corollary:

Corollary 1.1.9. *For a bounded distributive lattice* L *with* $n \in L$, L *is Boolean if and only if* $P_n(L)$ *is a Boolean lattice.* □

In lattice theory, it is well known that a lattice L is modular (distributive) if and only if the lattice of ideals I(L) is modular (distributive). Our following theorems are nice generalizations of those results in terms of n-ideals when n is a neutral element which is due to [31]. Also see [48].

Theorem 1.1.10. *For a neutral element* n *of a lattice* L, *the following conditions are equivalent*:

(i) L *is modular* ;

(ii) $I_n(L)$ *is modular* ;

(iii) $F_n(L)$ *is modular.* □

Following result is also due to [31].

Theorem 1.1.11. *Let* L *be a lattice with a neutral element* n. *Then the following conditions are equivalent*:

(i) L *is distributive* ;

(ii) $I_n(L)$ *is distributive* ;

(iii) $F_n(L)$ *is distributive.* □

For any two n-ideals I and J of a lattice we have already defined $I \vee J$ in the introduction. Now we include the following result, which will be used to prove several theorems in different chapters of the thesis.

Theorem 1.1.12. *Let* I *and* J *be two* n-*ideals of a distributive lattice. Then for any* $x\in I\vee J$, $x\vee n=i_1\vee j_1$ *and* $x\wedge n=i_2\wedge j_2$ *for some* $i_1, i_2\in I$, $j_1, j_2\in J$ *with* $i_1, j_1\geq n$ *and* $i_2, j_2\leq n$.

Proof: Let $x\in I\vee J$. Then $i\wedge j\leq x\leq i'\vee j'$ for some $i, i'\in I$, $j, j'\in J$. Now, $x\leq i'\vee j'$ implies $x\vee n\leq i'\vee j'\vee n$. Thus $x\vee n=(x\vee n)\wedge(i'\vee j'\vee n)=[(x\vee n)\wedge(i'\vee n)]\vee[(x\vee n)\wedge(j'\vee n)]$. But $n\leq(x\vee n)\wedge(i'\vee n)\leq i'\vee n$ implies by convexity that $(x\vee n)\wedge(i'\vee n)=i_1(\text{say})\in I$. Similarly, $(x\vee n)\wedge(j'\vee n)=j_1(\text{say})\in J$. Thus, $x\vee n=i_1\vee j_1$; $i_1\in I$, $j_1\in J$ and $i_1\geq n$, $j_1\geq n$. Similarly we can show that $x\wedge n=i_2\wedge j_2$ for some $i_2\in I$, $j_2\in J$ with $i_2, j_2\leq n$. □

We conclude this section with the following useful result which is due to [31]. This result will also be used in proving several results in different chapters of the thesis.

Theorem 1.1.13. *For a neutral element* n *of a lattice* L, *any finitely generated* n-*ideal of* L *which is contained in a principal* n-*ideal is a principal* n-*ideal.* □

2. Prime n-ideals.

Recall that an n-ideal P of a lattice L is prime if $m(x, n, y) \in P$, $x, y \in L$ implies either $x \in P$ or $y \in P$.

Since for any two n-ideals I and J of L, $I \cap J = \{m(i, n, j) : i \in I, j \in J\}$, so it is very easy to see that for any prime n-ideal P, $I \cap J \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$.

Theorem 1.2.1. *If* P *is a prime* n-*ideal of a lattice, then for any* $x \in L$, *at least one of* $x \wedge n$ *and* $x \vee n$ *is a member of* P.

Proof: Observe that $m(x \wedge n, n, x \vee n) = n \in P$. Thus, either $x \wedge n \in P$ or $x \vee n \in P$. □

Theorem 1.2.2. *If* P *is a prime* n-*ideal of a lattice, then* P *contains either* (n] *or* [n), *but not both.*

Proof: Suppose P is prime and $P \not\supseteq (n]$. Then there exists $r < n$ such that $r \notin P$. Now let $s \in [n)$. Then $m(r, n, s) = (r \wedge n) \vee (n \wedge s) \vee (s \wedge r) = r \vee n \vee r = n \in P$ implies that $s \in P$. That is, $P \supseteq [n)$. Similarly, if $P \not\supseteq [n)$, then we can show $P \supseteq (n]$.

Finally suppose that P contains both $(n]$ and $[n)$. Let $t \in L$. Then $t \wedge n \in P$ and $t \vee n \in P$. Then by convexity of n-ideals $t \in P$. This implies $P=L$, which is a contradiction to the primeness of P. □

Thus we have the following corollary:

Corollary 1.2.3. *If* P *is a prime* n*-ideal of a lattice* L, *then there exists at least one* $x \in L$ *such that both* $x \wedge n$ *and* $x \vee n$ *does not belong to* P. □

Theorem 1.2.4. *Let* n *be a neutral element of a lattice* L. *Then an* n*-ideal* P *is prime if and only if it is a prime ideal or a prime dual ideal (filter).*

Proof: Suppose the n-ideal P is prime. Then by Theorem 1.2.2, either $P \supseteq (n]$ or $P \supseteq [n)$. Suppose $P \supseteq (n]$. Let $x \in P$ and $t \leq x$, $t \in L$. Then $t \wedge n \in (n] \subseteq P$. Thus, by convexity of P, $t \wedge n \leq t \leq x$ implies that $t \in P$. This implies that P is an ideal. Also let $a \wedge b \in P$, $a, b \in L$. Then $(a \wedge b) \vee n \in P$ and $m(a, n, b) = (a \wedge n) \vee (n \wedge b) \vee (b \wedge a) \leq (a \wedge b) \vee n$ implies that $m(a, n, b) \in P$. Thus, either $a \in P$ or $b \in P$, and so P is a prime ideal.

On the other hand if $P \supseteq [n)$, we can similarly prove that P is a prime dual ideal. We omit the proof of the converse is trivial. □

Following lemma is due to [31, Lemma-1.2.8].

Lemma 1.2.5. *In a distributive lattice* L, *a prime ideal containing* n *is also a prime* n-*ideal.* □

Dually we can easily prove the following result.

Lemma 1.2.6. *In a distributive lattice* L, *a prime dual ideal (filter) containing* n *is also a prime* n-*ideal.* □

The set of all prime n-ideals of L is denoted by P(L). The following separation property for distributive lattices was given by M. H. Stone [13, Theorem-15, Page-74], which is known as Stone's representation theorem.

Theorem 1.2.7. *Let* L *be a distributive lattice, let* I *be an ideal, let* D *be a dual ideal of* L, *and let* $I \cap D = \emptyset$, *then there exists a prime ideal* P *of* L *such that* $P \supseteq I$ *and* $P \cap D = \emptyset$. □

Following result is an improvement of above theorem which is due to [31, Theorem-1.2.3].

Theorem 1.2.8. *Let* L *be a distributive lattice, let* I *be an ideal, let* D *be a convex sublattice of* L *and let* $I \cap D = \emptyset$, *then there exists a prime ideal* P *of* L *such that* $P \supseteq I$ *and* $P \cap D = \emptyset$. □

Now we give a separation property for distributive lattices in terms of prime n-ideals which is of course an extension of Stone's representation theorem. It should be mentioned that this result has also been obtained by Latif and Noor in [52]. Here we include a separate proof as it is much more simpler than that of [52].

Theorem 1.2.9. *In a distributive lattice* L, *suppose* I *is an* n-*ideal and* D *is a convex sublattice of* L *with* $I \cap D = \emptyset$. *Then there exists a prime* n-*ideal* P *of* L *such that* $P \supseteq I$ *and* $P \cap D = \emptyset$.

Proof: Since $I \cap D = \emptyset$, so either $(I] \cap D = \emptyset$ or $[I) \cap D = \emptyset$. If $(I] \cap D = \emptyset$, then by Theorem 1.2.8, there exists a prime ideal $P \supseteq I$ such that $P \cap D = \emptyset$. Similarly if $[I) \cap D = \emptyset$, then there exists a prime filter $Q \supseteq [I)$ such that $Q \cap D = \emptyset$. But by Lemma 1.2.5 and Lemma 1.2.6, both P and Q are prime n-ideals. □

Corollary 1.2.10. *Every* n*-ideal* I *of a distributive lattice* L *is the intersection of all prime* n*-ideals containing it.*

Proof: Let $I_1 = \cap\{P : P \supseteq I, P \text{ is a prime n-ideal of } L\}$. If $I \neq I_1$, then there is an element $a \in I_1 - I$. Then by above corollary, there is a prime n-ideal P with $P \supseteq I$, $a \notin P$. But $a \notin P \supseteq I$, gives a contradiction. □

For an n-ideal I of a distributive lattice L, the congruence $\Theta(I)$ has been studied in [53] and [31]. By [53], $x \equiv y\Theta(I)$ if and only if $x \wedge i_1 = y \wedge i_1$ and $x \vee i_2 = y \vee i_2$ for some $i_1, i_2 \in I$. Moreover $\Theta(I)$ is the smallest congruence of L containing I as a class. In chapter 2 of [31], Latif has proved the following result:

Theorem 1.2.11. *Let* L *be a distributive lattice. Then for any two* n*-ideals* I *and* J *of* L

(i) $\Theta(I \cap J) = \Theta(I) \cap \Theta(J)$;

(ii) $\Theta(I \vee J) = \Theta(I) \vee \Theta(J)$.

Moreover, the correspondence $I \to \Theta(I)$ *is an embedding from* $I_n(L)$ *to* $C(L)$. □

Theorem 1.2.12. *For a neutral element* n *of a lattice* L, $I_n(L) \cong C(L)$ *if and only if* $F_n(L)$ *is generalized Boolean.* □

For an n-ideal I of a distributive lattice L, Latif has also studied the congruence R(I) in [53]. By [53], the relation R(I) defined by "$x \equiv y R(I)$ if and only if for any $t \in L$, $m(x, n, t) \in I$ is equivalent to $m(y, n, t) \in I$" is the largest congruence of L containing I as a class. With the help of this congruence we will provide the following characterization of prime n-ideals of a distributive lattice.

Theorem 1.2.13. *Let* L *be a distributive lattice and* $n \in L$. *An* n-*ideal* P *is prime if and only if the quotient lattice* L/R(P) *is a two element chain.*

Proof: Suppose P is prime. Let $x, y \in L-P$. Then for any $t \in L$, $m(x, n, t) \in P$ implies $t \in P$. Since $t \wedge n \leq m(y, n, t) \leq t \vee n$, so by convexity of P, $m(y, n, t) \in P$. Therefore $x \equiv y R(P)$. Moreover, let $r \equiv x R(P)$ for some $x \in L-P$. Then $m(r, n, x) \notin P$ as $m(x, n, x) = x \notin P$. This implies $r \notin P$. For otherwise, $r \wedge n \leq m(r, n, x) \leq r \vee n$, would imply that $m(r, n, x) \in P$ by convexity of P and that is a contradiction. Thus L/R(P) is a two element chain $\{P, L-P\}$.

Conversely, suppose L/R(P) is a two element chain. Then L-P is a congruence class of the congruence R(P). If P is not prime, then there exists $x, y \in L-P$ such that $m(x, n, y) \in P$. Since L-P is a congruence class, so

$x \equiv y R(P)$. Thus $m(x, n, y) \in P$ implies $m(y, n, y) = y \in P$ which is a contradiction. Therefore P must be prime. □

For any n-ideal J of a distributive lattice L, we define

$J^+ = \{x \in L : m(x, n, j) = n \text{ for all } j \in J\}$. Obviously, J^+ is an n-ideal and $J \cap J^+ = \{n\}$. We will call J^+ as the annihilator n-ideal of J.

It is well known from [13, Theorem-22, Page-76] that a distributive lattice with 0 is generalized Boolean if and only if the set of prime ideals is unordered. We conclude the chapter with a nice generalization of that result which is due to [31, Theorem-1.2.9]; also see [48]. Here, we prefer to include a new proof of (i)⇒(iii), as it is much easier than that of [31].

Theorem 1.2.14. *Let* L *be a distributive lattice and* $n \in L$. *Then the following conditions are equivalent*:

(i) $F_n(L)$ *is generalized Boolean* ;

(ii) *For each principal* n-*ideal* $\langle x \rangle_n$, $\langle x \rangle_n \vee \langle x \rangle_n^{+} = L$, *where* $\langle x \rangle_n^{+} = \{y \in L : m(x, n, y) = n\}$;

(iii) *The set of prime* n-*ideals* P(L) *is unordered by set inclusion.*

Proof: (i)⇔(ii) and (iii)⇒(i) follows from [31, Theorem-1.2.9].

(i)$\Rightarrow$(iii). Suppose (i) holds. Then by Theorem 1.1.5, the intervals $[x, n]$ and $[n, y]$ are complemented for each $x, y \in L$ with $x \leq n \leq y$. Let P and Q be any two prime n-ideals of L. Then by Theorem 1.2.4, P and Q are either prime ideals or prime filters of L. If one of them is a prime ideal and the other is a prime filter, then of course they are unordered. If both P and Q are prime ideals, then $P \cap [n, y]$ and $Q \cap [n, y]$ are prime ideals of $[n, y]$. Since $[n, y]$ is a complemented lattice, so by [13, Theorem-22, Page-76], $P \cap [n, y]$ and $Q \cap [n, y]$ are unordered. Therefore P and Q are unordered. If P, Q are filters, then using the same argument we find that $P \cap [x, n]$ and $Q \cap [x, n]$ are unordered. Thus P and Q are unordered and this establishes (iii). $\square$

Chapter-2

Lattices whose finitely generated n-ideals form a Stone lattice.

Introduction:

Minimal prime ideals and Stone (generalized) lattices have been studied extensively by many authors including [1], [5], [6], [7], [19], [29], [58] and [61]. Chen and in Grätzer [5] and [6] studied the construction and structures of Stone lattices. Katrinak has given a new proof of construction theorem for Stone algebras in [25] and studied these algebras in [24], [26] and [27].

In this chapter we introduce the concept of minimal prime n-ideals and generalize some of the results on minimal prime ideals. Then we used these results to generalize several important results on Stone and generalized Stone lattices in terms of n-ideals.

A prime n-ideal P is said to be a *minimal prime* n-*ideal belonging to* n-*ideal* I *if*,

(i) $I \subseteq P$, and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

A prime n-ideal P of L is called a *minimal prime n-ideal* if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. Thus a minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

Let L be a lattice with 0 and 1. An element $a^* \in L$ is called a *pseudocomplement* of $a \in L$, if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. Of course $0^* = 1$ and $1^* = 0$. L is called *pseudocomplemented* if its every element has a *pseudocomplement*. Lattice L is called *relatively pseudocomplemented* if its every interval is pseudocomplemented. That is every element of each interval has a relative pseudocomplement in that interval.

A lattice L with 0 is called a *sectionally pseudocomplemented lattice* if the interval [0, x] is pseudocomplemented for each $x \in L$.

A distributive lattice L with 0 and 1 is called a *Stone lattice* if it is pseudocomplemented and for each $a \in L$, $a^* \vee a^{**} = 1$.

By [13, Theorem-3, Page-161], we also know that a distributive pseudocomplemented lattice is a Stone lattice if and only if for each $a, b \in L$, $(a \wedge b)^* = a^* \vee b^*$.

A distributive lattice L with 0 is called a *generalized Stone lattice* if $(x]^* \vee (x]^{**} = L$ for each $x \in L$. By [24] and [7], a distributive lattice L with 0 is called *generalized Stone* if and only if $[0, x]$ is Stone for each $x \in L$.

A distributive lattice L is called a *relatively Stone lattice* if every interval $[a, b]$, $a, b \in L$ is a Stone lattice.

For any n-ideal J of L, we have already defined in chapter 1 that

$J^+ = \{x \in L : m(x, n, j) = n \text{ for all } j \in J\}$.

Observe that J^+ is an n-ideal and $J \cap J^+ = \{n\}$. In fact, this is the largest n-ideal which annihilates J. Latif in [31] called this *an annihilator* n-*ideal of* J. We prefer to call this as the *pseudocomplement of* J *in* $I_n(L)$. Moreover, for a distributive lattice L, $I_n(L)$ is a distributive algebraic lattice and so it is pseudocomplemented. Observe that $F_n(L)$ has always the smallest element viz. $\{n\}$. But it does not necessarily contain the largest element. So in a general distributive lattice L with $n \in L$, we can not talk on pseudocomplementation in the lattice $F_n(L)$. But we can discuss on section pseudocomplementation in $F_n(L)$. Let $[a, b] \in F_n(L)$. By the interval $[\{n\}, [a, b]]$ in $F_n(L)$, we mean the set of all finitely generated n-ideals contained in $[a, b]$. $F_n(L)$ is called *sectionally pseudocomplemented* if for each $[a, b] \in F_n(L)$, the interval $[\{n\}, [a, b]]$ in $F_n(L)$ is pseudocomplemented. That is, each finitely generated n-ideal contained in $[a, b]$ has a relative pseudocomplement in $[\{n\}, [a, b]]$ which is also a member of $F_n(L)$.

We shall denote the relative pseudocomplement of $[c, d]$ by $[c, d]^0$, while $[c, d]^+$ denotes the pseudocomplement of $[c, d]$ in $I_n(L)$.

We shall call two prime n-ideals P and Q of L *comaximal* if $P \vee Q = L$.

In section 1, we have studied minimal prime n-ideals of L. There we have given some characterizations of minimal prime n-ideals, also see [43]. These results give nice generalizations of several results on minimal prime ideals which will be used to prove some important results in section 2.

In section 2, we have given several characterizations of those $F_n(L)$ which are Stone and generalized Stone lattices in terms of n-ideals. If $F_n(L)$ is sectionally pseudocomplemented, then we have proved that $F_n(L)$ is generalized Stone if and only if each prime n-ideal contains a unique minimal prime n-ideal.

1. Minimal prime n-ideals.

Recall that a prime n-ideal P is a minimal prime n-ideal belonging to an n-ideal I if

(i) $I \subseteq P$ and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

Following theorem is a generalization of [13, Lemma-4, Page-169].

Lemma 2.1.1. *Let* L *be a lattice with an element* n. *Then every prime* n-*ideal contains a minimal prime* n-*ideal.*

Proof: Let P be a prime n-ideal of L and let χ denotes the set of all prime n-ideals Q contained in P. Then χ is not void, since $P \in \chi$. If C is a chain in χ and $Q = \cap(X : X \in C)$, then Q is nonvoid because $n \in Q$ and Q is an n-ideal, in fact, Q is prime. Indeed, if $m(a, n, b) \in Q$ for some $a, b \in L$, then $m(a, n, b) \in X$ for all $X \in C$. Since X is prime, either $a \in X$ or $b \in X$. Thus, either $Q = \cap(X : a \in X)$ or $Q = \cap(X : b \in X)$, proving that $a \in Q$ or $b \in Q$. Therefore, we can apply to χ the dual form of Zorn's lemma to conclude the existence of a minimal member of χ. □

Now we give a characterization of minimal prime n-ideals of a distributive lattice L, when $F_n(L)$ is sectionally pseudocomplemented. In order to do this, we need the following lemmas:

Lemma 2.1.2. *Let* L *be a distributive lattice and* $n \in L$. *Then for any* $[a, b] \in F_n(L)$ *and for any* n-*ideal* I. $(I \cap [a, b])^+ \cap [a, b] = I^+ \cap [a, b]$.

Proof: Since $[a, b] \cap I \subseteq I$, so R.H.S$\subseteq$L.H.S. To prove the reverse inclusion, let $x \in$ L.H.S. Then $a \le x \le b$ and $m(x, n, t) = n$ for all $t \in [a, b] \cap I$. Since $x \in [a, b]$, so $m(x, n, i) \in [a, b] \cap I$ for all $i \in I$. Thus, $m(x, n, m(x, n, i)) = n$. But it can be easily seen that $m(x, n, m(x, n, i)) = m(x, n, i)$. This implies $m(x, n, i) = n$ for all $i \in I$. Hence, $x \in$ R.H.S. □

Lemma 2.1.3. *Suppose* $F_n(L)$ *is a sectionally pseudocomplented distributive lattice*, *and* $[c, d] \subseteq [a, b]$ *in* $F_n(L)$ *then*,

(i) $[c, d]^0 = [c, d]^+ \cap [a, b]$ *and*

(ii) $[c, d]^{00} = [c, d]^{++} \cap [a, b]$.

Proof: (i) is trivial. For (ii), using (i) we have $[c, d]^{00} = ([c, d]^0)^+ \cap [a, b] = ([c, d]^+ \cap [a, b])^+ \cap [a, b]$. Thus, by Lemma 2.1.2, $[c, d]^{00} = [c, d]^{++} \cap [a, b]$. □

Now we give the following characterizations of minimal prime n-ideals. (Also see [43]).

Theorem 2.1.4. *Let* $F_n(L)$ *be a sectionally pseudocomplemented distributive lattice, and* P *be a prime* n-*ideal of* L. *Then the following conditions are equivalent*:

(i) P *is minimal* ;

(ii) $x \in P$ *implies* $<x>_n^+ \not\subseteq P$;

(iii) $x \in P$ *implies* $<x>_n^{++} \subseteq P$;

(iv) $P \cap D(<t>_n) = \varnothing$ *for all* $t \in L-P$;

where $D(<t>_n) = \{x \in <t>_n : <x>_n^0 = \{n\}\}$.

Proof: (i)⇒(ii). Suppose P is minimal. If (ii) fails, then there exists $x \in P$ such that $<x>_n^+ \subseteq P$. Since P is a prime n-ideal, so by Theorem 1.2.4, P is a prime ideal or a prime dual ideal. Suppose P is a prime ideal. Let $D=(L-P) \vee [x)$. We claim that $n \notin D$. If $n \in D$, then $n = q \wedge x$ for some $q \in L-P$. Then $<q>_n \cap <x>_n = <(q \wedge x) \vee (q \wedge n) \vee (x \wedge n)>_n = \{n\}$ implies $<q>_n \subseteq <x>_n^+ \subseteq P$. Thus $q \in P$, which is a contradiction. Hence $n \notin D$. Then by Stone's representation theorem for n-ideals [52, Lemma-1.3], there exists a prime n-ideal Q with $Q \cap D = \varnothing$. Then $Q \subseteq P$ as $Q \cap (L-P) = \varnothing$ and $Q \neq P$ since $x \notin Q$. But this contradicts the minimality of P.

Hence, $<x>_n^+ \subseteq P$.

Similarly, we can prove that $<x>_n^+ \subseteq P$ if P is a prime dual ideal.

(ii)$\Rightarrow$(iii). Suppose (ii) holds and $x \in P$. Then $<x>_n^+ \not\subseteq P$. Since $<x>_n^+ \cap <x>_n^{++} = \{n\} \subseteq P$, and P is prime, so $<x>_n^{++} \subseteq P$.

(iii)$\Rightarrow$(iv). Suppose (iii) holds and $t \in L-P$. Let $x \in P \cap D(<t>_n)$. Then $x \in P$, $x \in D(<t>_n)$. Thus, $<x>_n^0 = \{n\}$ and so $<x>_n^{00} = <t>_n$. By (iii), $x \in P$ implies $<x>_n^{++} \subseteq P$. Also by Lemma 2.1.3, $<x>_n^{00} = <x>_n^{++} \cap <t>_n$. Hence $<x>_n^{++} \cap <t>_n = <t>_n$, and so $<t>_n \subseteq <x>_n^{++} \subseteq P$. That is, $t \in P$, which is a contradiction. Therefore, $P \cap D(<t>_n) = \varnothing$ for all $t \in L-P$.

(iv)$\Rightarrow$(i). Suppose P is not minimal. Then there exists a prime n-ideal $Q \subset P$. Let $x \in P-Q$. Since $<x>_n \cap <x>_n^+ = \{n\} \subseteq Q$, so $<x>_n^+ \subseteq Q \subset P$. Thus, $<x>_n \vee <x>_n^+ \subseteq P$. Choose any $t \in L-P$. Then $<t>_n \cap (<x>_n \vee <x>_n^+) \subseteq P$. Now $<t>_n \cap (<x>_n \vee <x>_n^+) = (<t>_n \cap <x>_n) \vee (<t>_n \cap <x>_n^+)$
$= <m(t, n, x)>_n \vee ((<t>_n \cap <x>_n)^+ \cap <t>_n)$ (by Lemma 2.1.2)
$= <m(t, n, x)>_n \vee (<m(t, n, x)>_n^+ \cap <t>_n)$
$= <m(t, n, x)>_n \vee <m(t, n, x)>_n^0$ [by Lemma 2.1.3] where $<m(t, n, x)>_n^0$ is the relative pseudocomplement of $<m(t, n, x)>_n$ in $<t>_n$.

Since $F_n(L)$ is sectionally pseudocomplemented, $<m(t, n, x)>_n^0$ is finitely generated and so $<m(t, n, x)>_n \vee <m(t, n, x)>_n^0$ is a finitely generated n-ideal contained in $<t>_n$. Therefore by Theorem 1.1.13, $<m(t, n, x)>_n \vee <m(t, n, x)>_n^0 = <r>_n$ for some $r \in <t>_n$. Moreover, $<r>_n^0 = <m(t, n, x)>_n^0 \cap <m(t, n, x)>_n^{00} = \{n\}$. Thus, $r \in P \cap D(<t>_n)$, which is a contradiction. Therefore P must be minimal. □

2. Lattices whose finitely generated n-ideals form generalized Stone lattices.

If $0, 1 \in L$, then of course, $[0, 1]=L$ which is the largest element of $F_n(L)$. Then we can talk on pseudocomplementation in $F_n(L)$. Since by Theorem 1.1.2, $F_n(L) \cong (n]^d \times [n)$. So we have the following result:

Theorem 2.2.1. *Let* L *be a lattice and* $n \in L$.

(i) $F_n(L)$ *is sectionally pseudocomplemented if and only if* $(n]$ *is sectionally dual pseudocomplemented and* $[n)$ *is sectionally pseudocomplemented.*

(ii) *If* $0, 1 \in L$, *then* $F_n(L)$ *is pseudocomplemented if and only if* $(n]$ *is dual pseudocomplemented and* $[n)$ *is pseudocomplemented.* □

For any $n \leq b \leq 1$, b^+ denotes the pseudocomplement of b in $[n, 1]$, while for $0 \leq a \leq n$, a^{+d} denotes the dual pseudocomplement of a in $[0, n]$.

Now we have the following result:

Corollary 2.2.2. *Let* $F_n(L)$ *be a distributive pseudocomplemented lattice (Then of course* $F_n(L)$ *has a largest element, and so* $0, 1 \in L$). *Then for* $[a, b] \in F_n(L)$, $[a, b]^+ = [a^{+d}, b^+]$.

Proof: Since $F_n(L)$ is pseudocomplemented. So by above theorem, $(n]$ is dual pseudocomplemented and $[n)$ is pseudocomplemented. Here $0 \leq a \leq n \leq b \leq 1$. Since a^{+d} is the dual pseudocomplement of a in $[0, n]$ and b^{+} is the pseudocomplement of b in $[n, 1]$.

So $[a, b] \cap [a^{+d}, b^{+}] = [a \vee a^{+d}, b \wedge b^{+}] = \{n\}$.

Now Let $x \in [a, b]^{+}$. Then $[x \wedge n, x \vee n] \subseteq [a, b]^{+}$. Thus $\{n\} = [x \wedge n, x \vee n] \cap [a, b] = [(x \wedge n) \vee a, b \wedge (x \vee n)]$ and so $(x \wedge n) \vee a = n = b \wedge (x \vee n)$. This implies $x \wedge n \geq a^{+d}$ and $x \vee n \leq b^{+}$.

Hence, $[x \wedge n, x \vee n] \subseteq [a^{+d}, b^{+}]$ and so $[a, b]^{+} \subseteq [a^{+d}, b^{+}]$. Therefore, $[a, b]^{+} = [a^{+d}, b^{+}]$. □

If $[a, b] \in [\{n\}, [c, d]]$. Then $\{n\} \subseteq [a, b] \subseteq [c, d]$. The relative pseudocomplement of $[a, b]$ in above interval is denoted by $[a, b]^{0}$. Here $c \leq a \leq n \leq b \leq d$. a^{0d} denotes the dual relative pseudocomplement of a in $[c, n]$ and b^{0} denotes the relative pesudocomplement of b in $[n, d]$. Since by Lemma 2.1.3, $[a, b]^{0} = [a, b]^{+} \cap [c, d]$. Using Corollary 2.2.2 above we have the following result:

Corollary 2.2.3. *Let* $F_n(L)$ *be a sectionally pseudocomplemented distributive lattice. Then for*

$\{n\} \subseteq [a, b] \subseteq [c, d]$, $[a, b]^{0} = [a^{0d}, b^{0}]$. □

A distributive lattice L with 0 is called a *generalized Stone lattice* if for each $x \in L$, $(x]^* \vee (x]^{**} = L$. By Katrinak [24], we know that a distributive lattice L with 0 is a generalized Stone lattice if and only if for each interval $[0, x]$, $x \in L$ is a Stone lattice. Thus if $F_n(L)$ is a distributive sectionally pseudocomplemented lattice, then $F_n(L)$ is a generalized Stone lattice if for each $[a, b] \in F_n(L)$, the interval $[\{n\}, [a, b]]$ in $F_n(L)$ is a Stone lattice.

Generalized Stone lattices have been studied by many authors including [7], [24] and [27]. Following result is a generalization of some of their work. This gives several characterizations of those $F_n(L)$ which are generalized Stone. To prove this result we need the following results. Lemma 2.2.4 and Corollary 2.2.5 are trivial from Theorem 1.1.2.

Lemma 2.2.4. *Suppose* $F_n(L)$ *is a sectionally pseudocomplemented distributive lattice. Then* $F_n(L)$ *is generalized Stone if and only if* $(n]$ *is dual generalized Stone and* $[n)$ *is generalized Stone.* □

Corollary 2.2.5. *Supposes* $F_n(L)$ *is a pseudocomplemented distributive lattice (Then of course,* $0, 1 \in L$*). Then* $F_n(L)$ *is Stone if and only if* $(n]$ *is a dual Stone lattice and* $[n)$ *is a Stone lattice.* □

Lemma 2.2.6. *Suppose* $F_n(L)$ *is a sectionally pseudocomplemented distributive lattice. Let* $x, y \in L$ *with* $\langle x\rangle_n \cap \langle y\rangle_n = \{n\}$. *Then the following conditions are equivalent:*

(i) $\langle x\rangle_n^+ \vee \langle y\rangle_n^+ = L$;

(ii) *For any* $t \in L$, $\langle m(x, n, t)\rangle_n^0 \vee \langle m(y, n, t)\rangle_n^0 = \langle t\rangle_n$, *where* $\langle m(x, n, t)\rangle_n^0$ *denotes the relative pseudocomplement of* $\langle m(x, n, t)\rangle_n$ *in* $[\{n\}, \langle t\rangle_n]$.

Proof: (i)$\Rightarrow$(ii). Suppose (i) holds. Then for any $t \in L$, using Lemma 2.1.3,

$\langle m(x, n, t)\rangle_n^0 \vee \langle m(y, n, t)\rangle_n^0$

$= (\langle x\rangle_n \cap \langle t\rangle_n)^0 \vee (\langle y\rangle_n \cap \langle t\rangle_n)^0$

$= ((\langle x\rangle_n \cap \langle t\rangle_n)^+ \cap \langle t\rangle_n) \vee ((\langle y\rangle_n \cap \langle t\rangle_n)^+ \cap \langle t\rangle_n)$

$= ((\langle x\rangle_n^+ \cap \langle t\rangle_n) \vee (\langle y\rangle_n^+ \cap \langle t\rangle_n)$ (by Lemma 2.1.2)

$= (\langle x\rangle_n^+ \vee \langle y\rangle_n^+) \cap \langle t\rangle_n = L \cap \langle t\rangle_n = \langle t\rangle_n$.

(ii)$\Rightarrow$(i). Suppose (ii) holds and $t \in L$. By (ii), $\langle m(x, n, t)\rangle_n^0 \vee \langle m(y, n, t)\rangle_n^0 = \langle t\rangle_n$. Then by calculation of (i)$\Rightarrow$(ii), we have $(\langle x\rangle_n^+ \vee \langle y\rangle_n^+) \cap \langle t\rangle_n = \langle t\rangle_n$. This implies $\langle t\rangle_n \subseteq \langle x\rangle_n^+ \vee \langle y\rangle_n^+$ and so $t \in \langle x\rangle_n^+ \vee \langle y\rangle_n^+$. Therefore, $\langle x\rangle_n^+ \vee \langle y\rangle_n^+ = L$. $\square$

Theorem 2.2.7. *Let* $F_n(L)$ *be a sectionally pseudocomplemented distributive lattice. Then the following conditions are equivalent*:

(i) $F_n(L)$ *is generalized Stone* ;

(ii) *For any* $x\in L$, $<x>_n^{+}\vee<x>_n^{++}=L$;

(iii) *For all* $x, y\in L$, $(<x>_n\cap<y>_n)^{+}=<x>_n^{+}\vee<y>_n^{+}$;

(iv) *For all* $x, y\in L$, $<x>_n\cap<y>_n=\{n\}$ *implies that* $<x>_n^{+}\vee<y>_n^{+}=L$.

Proof: (i)⇒(ii). Suppose (i) holds and $t\in L$. Then for any $x\in L$, $m(x, n, t)\in<t>_n$ and so $<m(t, n, x)>_n\in[\{n\}. <t>_n]$. Since $F_n(L)$ is generalized Stone, so

$<m(t, n, x)>_n^{0}\vee<m(t, n, x)>_n^{00}=<t>_n$. Then by Lemma 2.1.3, $<t>_n=(<m(t, n, x)>_n^{+}\cap<t>_n)\vee(<m(t, n, x)>_n^{++})\cap<t>_n$.

$=((<x>_n\cap<t>_n)^{+}\cap<t>_n)\vee((<x>_n\cap<t>_n)^{++}\cap<t>_n)$.

Thus by Lemma 2.1.2, $<t>_n=(<x>_n^{+}\cap<t>_n)\vee(<x>_n^{++}\cap<t>_n)$ $=(<x>_n^{+}\vee<x>_n^{++})\cap<t>_n$. This implies $<t>_n\subseteq<x>_n^{+}\vee<x>_n^{++}$ and so $t\in<x>_n^{+}\vee<x>_n^{++}$. Therefore, $<x>_n^{+}\vee<x>_n^{++}=L$.

(ii)⇒(iii). For any $x, y\in L$. $(<x>_n\cap<y>_n)\cap(<x>_n^{+}\vee<y>_n^{+})$

$=(<x>_n\cap<y>_n\cap<x>_n^{+})\vee(<x>_n\cap<y>_n\cap<y>_n^{+})$

$=\{n\}\vee\{n\}=\{n\}$. Now, let $<x>_n\cap<y>_n\cap I=\{n\}$ for some n-ideal I. Then $<y>_n\cap I\subseteq<x>_n^{+}$. Meeting $<x>^{++}$ with both sides, we have $<y>_n\cap I\cap<x>_n^{++}=\{n\}$. This implies n-ideal I. Then $<y>_n\cap I\subseteq<x>_n^{+}$. Meeting $<x>^{++}$ with both sides, we

have $<y>_n \cap I \cap <x>_n^{++}=\{n\}$. This implies $I \cap <x>_n^{++} \subseteq <y>_n^{+}$. Hence $I=I \cap L=I \cap (<x>_n^{+} \vee <x>_n^{++})$
$=(I \cap <x>_n^{+}) \vee (I \cap <x>_n^{++}) \subseteq <x>_n^{+} \vee <y>_n^{+}$. Therefore, $<x>_n^{+} \vee <y>_n^{+}=(<x>_n \cap <y>_n)^{+}$.

(iii)$\Rightarrow$(iv). Let $<x>_n \cap <y>_n=\{n\}$ for some $x,\ y \in L$. Then by (iii), $L=\{n\}^{+}=(<x>_n \cap <y>_n)^{+}=<x>_n^{+} \vee <y>_n^{+}$. Thus (iv) holds.

(iv)$\Rightarrow$(ii). Let $t \in L$. By Lemma 2.1.2, and by Lemma 2.1.3, for any $x \in L$, $(<x>_n^{+} \vee <x>_n^{++}) \cap <t>_n$

$$=(<x>_n^{+} \cap <t>_n) \vee (<x>_n^{++} \cap <t>_n)$$
$$=((<x>_n \cap <t>_n)^{+} \cap <t>_n) \vee ((<x>_n \cap <t>_n)^{++} \cap <t>_n)$$
$$=(<m(t,\ n,\ x)>_n^{+} \cap <t>_n) \vee (<m(t,\ n,\ x)>_n^{++} \cap <t>_n)$$
$$=<m(x,\ n,\ t)>_n^{0} \vee <m(x,\ n,\ t)>_n^{00}.$$

Here $<m(x,\ n,\ t)>_n^{0}$ is finitely generated n-ideal contained in $<t>_n$, as $F_n(L)$ is sectionally pseudocomplemented. Then by Theorem 1.1.13, $<m(x,\ n,\ t)>_n^{0}$ is a principal n-ideal, say $<r>_n$. Now $<m(x,\ n,\ t)>_n \cap <r>_n=\{n\}$. So by (iv) and Lemma 2.1.3, $<m(x,\ n,\ t)>_n^{0} \vee <r>_n^{0}=<t>_n$. Therefore, $(<x>_n^{+} \vee <x>_n^{++}) \cap <t>_n=<t>_n$ and so $t \in <x>_n^{+} \vee <x>_n^{++}$. This implies $<x>_n^{+} \vee <x>_n^{++}=L$. Thus (ii) holds.

To complete the proof we shall show that (iv)$\Rightarrow$(i). Since $F_n(L)$ is sectionally pseudocomplemented, so by

Theorem 2.2.1, (n] is sectionally dual pseudocomplemented and [n) is sectionally pseudocomplemented.

Suppose $n \le b \le d$. Let b^0 be the relative pseudocomplement of b in [n, d]. Now $b^0 \wedge b^{00} = n$. Thus $<b^0>_n \cap <b^{00}>_n = [n,\ b^0 \wedge b^{00}] = \{n\}$. Also, $<b^0>_n$, $<b^{00}>_n \subseteq <d>_n$. Then by equivalent condition of (iv) given in Lemma 2.2.6, we have $<m(b^0, n, d)>_n{}^0 \vee <m(b^{00}, n, d)>_n{}^0 = <d>_n$. But $m(b^0, n, d) = b^0$ and $m(b^{00}, n, d) = b^{00}$ as $n \le b^0$, $b^{00} \le d$. But by Corollary 2.2.3, $<b^0>_n{}^0 = <b^{00}>_n$ and $<b^{00}>_n{}^0 = <b^{000}>_n = <b^0>_n$. Therefore, $<d>_n = <b^{00}>_n \vee <b^0>_n = <b^0 \vee b^{00}>_n$ which gives $b^0 \vee b^{00} = d$. This implies [n, d] is a Stone lattice. That is [n) is generalized Stone.

A dual proof of above shows that (iv) also implies that (n] is a dual generalized Stone lattice. Therefore, by Lemma 2.2.4, $F_n(L)$ is generalized Stone. □

Following corollary is an immediate consequence of above result. This has also been proved in [44, Theorem-2.4].

Corollary 2.2.8. *Let* $F_n(L)$ *be a pseudocomplemented distributive lattice. Then the following conditions are equivalent*:

(i) $F_n(L)$ *is Stone* ;

(ii) *For all* $x \in L$, $<x>_n{}^+ \vee <x>_n{}^{++} = L$;

(iii) *For all* $x, y \in L$, $(\langle x\rangle_n \cap \langle y\rangle_n)^+ = \langle x\rangle_n^+ \vee \langle y\rangle_n^+$;

(iv) *For all* $x, y \in L$, $\langle x\rangle_n \cap \langle y\rangle_n = \{n\}$ *implies that* $\langle x\rangle_n^+ \vee \langle y\rangle_n^+ = L$. □

For a prime ideal P of a distributive lattice L with 0, Cornish in [7] has defined

$0(P)=\{x \in L : x \wedge y = 0$ for some $y \in L-P\}$. Clearly $0(P)$ is an ideal and $0(P) \subseteq P$. Cornish in [7] has shown that $0(P)$ is the intersection of all the minimal prime ideals of L which are contained in P.

For a prime n-ideal P of a distributive lattice L, we write $n(P)=\{y \in L : m(y, n, x)=n$ for some $x \in L-P\}$. Clearly, $n(P)$ is an n-ideal and $n(P) \subseteq P$.

Lemma 2.2.9. *Let* P *be a prime* n-*ideal in a distributive lattice* L. *Then each minimal prime* n-*ideal belonging to* n(P) *is contained in* P.

Proof: Let Q be a minimal prime n-ideal belonging to n(P). If $Q \not\subseteq P$, then choose $y \in Q-P$. By Theorem 1.2.4, we know that Q is either an ideal or a filter. Without loss of generality suppose Q is an ideal. Now let

$S=\{s \in L : m(y, n, s) \in n(P)\}$. We shall show that $S \not\subseteq Q$. If not, let $D=(L-Q) \vee [y)$. Then $n(P) \cap D = \varnothing$. For otherwise, $y \wedge r \in n(P)$ for some $r \in L-Q$. Then by convexity,

$y\wedge r\leq m(y, n, r)\leq(y\wedge r)\vee n$ implies $m(y, n, r)\in n(P)$. Hence $r\in S\subseteq Q$, which is a contradiction. Thus, by Stone's representation theorem for n-ideals, there exists a prime n-ideal R containing $n(P)$ disjoint to D. Then $R\subseteq Q$. Moreover, $R\neq Q$ as $y\notin R$, this shows that Q is not a minimal prime n-ideal belonging to $n(P)$, which is a contradiction. Therefore, $S\not\subseteq Q$. Hence there exists $z\notin Q$ such that $m(y, n, z)\in n(P)$. Thus $m(m(y, n, z), n, x)=n$ for some $x\in L-P$. It is easy to see that $m(m(y, n, z), n, x)=m(m(y, n, x), n, z)$. Hence, $m(m(y, n, x), n, z)=n$. Since P is prime and $y, x\notin P$, so $m(y, n, x)\notin P$. Therefore, $z\in n(P)\subseteq Q$, which is a contradiction. Hence $Q\subseteq P$. □

Proposition 2.2.10. *If* P *is a prime* n-*ideal in a distributive lattice* L, *then* n(P) *is the intersection of all minimal prime* n-*ideals contained in* P.

Proof: Clearly $n(P)$ is contained in any prime n-ideal which is contained in P. Hence $n(P)$ is contained in the intersection of all minimal prime n-ideals contained in P. Since L is distributive, so by Corollary 1.2.10, $n(P)$ is the intersection of all minimal prime n-ideals belonging to it. By Lemma 2.1.1, as each prime n-ideal contains a minimal prime n-ideal, above remarks and Lemma 2.2.9 establish the proposition. □

Theorem 2.2.11. *Let* $F_n(L)$ *be a sectionally pseudocomplemented distributive lattice. Then the following conditions are equivalent*:

(i) *For any* $x \in L$, $<x>_n^+ \vee <x>_n^{++} = L$,

equivalently, $F_n(L)$ *is generalized Stone* ;

(ii) *For any two minimal prime* n-*ideals* P *and* Q,

$P \vee Q = L$;

(iii) *Every prime* n-*ideal contains a unique minimal prime* n-*ideal* ;

(iv) *For each prime* n-*ideal* P, n(P) *is a prime* n-*ideal.*

Proof: (i)$\Rightarrow$(ii). Let $x \in P - Q$. Then $<x>_n \subseteq P - Q$. Now, $<x>_n \cap <x>_n^+ = \{n\} \subseteq Q$. So $<x>_n^+ \subseteq Q$ as Q is prime. Again $x \in P$ implies $<x>_n^{++} \subseteq P$ by Theorem 2.1.4. Hence by (i), $L = <x>_n^+ \vee <x>_n^{++} \subseteq Q \vee P$. Therefore, $P \vee Q = L$.

(ii)$\Leftrightarrow$(iii) is trivial.

(iii)$\Rightarrow$(iv) is direct consequence of Proposition 2.2.10.

(iv)$\Rightarrow$(i). Suppose (iv) holds. First we shall show that for all $x, y \in L$ with $<x>_n \cap <y>_n = \{n\}$ implies $<x>_n^+ \vee <y>_n^+ = L$. If it does not hold, then there exists $x, y \in L$ with $<x>_n \cap <y>_n = \{n\}$ such that $<x>_n^+ \vee <y>_n^+ \neq L$. As L is distributive, so by Theorem 1.2.9, there is a prime n-ideal P such that $<x>_n^+ \vee <y>_n^+ \subseteq P$. Then $<x>_n^+ \subseteq P$ and $<y>_n^+ \subseteq P$ imply $x \notin n(P)$ and $y \notin n(P)$. But n(P) is prime and so $m(x, n, y) = n \in n(P)$ is contradictory.

Thus for all x, $y \in L$ with $\langle x\rangle_n \cap \langle y\rangle_n = \{n\}$ implies that $\langle x\rangle_n^{+} \vee \langle y\rangle_n^{+} = L$. Hence by equivalent conditions of Theorem 2.2.7, (i) holds. □

Following result is an immediate consequence of above theorem, which has also been proved seperately in [44].

Corollary 2.2.12. *Let* $F_n(L)$ *be a pseudocomplemented distributive lattice. Then the following conditions are equivalent*:

(i) $F_n(L)$ *is Stone* ;

(ii) *For any two minimal prime* n-*ideals* P *and* Q, $P \vee Q = L$, *that is, they are comaximal* ;

(iii) *Every prime* n-*ideal contains a unique minimal prime* n-*ideal* ;

(iv) *For each prime* n-*ideal* P, n(P) *is a prime* n-*ideal*. □

Chapter-3

On finitely generated n-ideals, which form relatively Stone lattices.

Introduction:

Relative annihilators in lattices and semilattices have been studied by many authors including Mandelker [39] and Varlet [62]. Also Cornish [7] has used the annihilators in studying relative normal lattices. In this chapter we shall introduce the notion of relative annihilators around a fixed element $n \in L$ and then we will use it to generalize several results on relatively Stone lattices.

For a, $b \in L$, $<a, b> = \{x \in L : x \wedge a \leq b\}$ is known as annihilator of a relative to b, or simply a relative annihilator. It is very easy to see that in presence of distributivity, $<a, b>$ is an ideal of L.

Again for a, $b \in L$ we define $<a, b>_d = \{x : x \vee a \geq b\}$, which we call a *dual annihilator of* a *relative to* b, or simply a relative dual annihilator. In presence of distributivity of L, $<a, b>_d$ is a dual ideal (filter).

For a, $b \in L$ and a fixed element $n \in L$, we define

$<a, b>^n = \{x \in L : m(a, n, x) \in <b>_n\} = \{x \in L : b \wedge n \leq m(a, n, x) \leq b \vee n\}$. We call $<a, b>^n$ the annihilator of a relative to b around the element n or simply a relative n-annihilator. It is easy to see that for all $a, b \in L$, $<a, b>^n$ is always a convex subset containing n. In presence of distributivity, it can be easily seen that $<a, b>^n$ is an n-ideal.

For two n-ideals A and B of a lattice L, $<A, B>$ denotes $\{x \in L : m(a, n, x) \in B$ for all $a \in A\}$. In presence of distributivity, clearly $<A, B>$ is an n-ideal. Moreover, we can easily show that $<a, b>^n = <<a>_n, <b>_n>$.

Recall that a distributive lattice L is called a *Stone lattice* if it is pseudocomplemented and $x^* \vee x^{**} = 1$, for each $x \in L$. Also recall that a lattice L is relatively pseudocomplemented if its every interval [a, b] ($a, b \in L$, $a < b$) is pseudocomplemented. A distributive lattice L is called a *relatively Stone lattice* if its every interval [a, b] is Stone.

In section 1 of this chapter we shall give several characterizations of $<a, b>^n$. We will also give some characterizations of distributive and modular lattices in terms of relative n-annihilators. If $0 \in L$, then putting $n=0$, the n-ideals become ideals and $<a, b>^n = <a, b>$. So this section will generalize most of the results on annihilators in [39].

In section 2 we will characterize those $F_n(L)$ which are relatively Stone in terms of n-ideals and relative n-annihilators. These results are certainly generalizations of several results on relatively Stone lattices. At the end we will show that $F_n(L)$ is relatively Stone if and only if any two incomparable prime n-ideals of L are comaximal.

1. Relative annihilators around a neutral element of a lattice.

We start with the following characterization of $<a, b>^n$.

Theorem 3.1.1. *Let* L *be a lattice with a neutral element* n *in it. Then for all* a, b$\in$L, *the following conditions are equivalent*:

(i) $<a, b>^n$ *is an* n-*ideal* ;

(ii) $<a\wedge n, b\wedge n>_d$ *is a filter and*

$<a\vee n, b\vee n>$ *is an ideal.*

Proof: Suppose (i) holds. Let x, y$\in<a\vee n, b\vee n>$. Then $x\wedge(a\vee n)\leq b\vee n$. Thus $(x\wedge(a\vee n))\vee n\leq b\vee n$, then by the neutrality of n, $(x\vee n)\wedge(a\vee n)\leq b\vee n$. Also $m(x\vee n, n, a)$ $=(x\vee n\vee a)\wedge(x\vee n)\wedge(a\vee n)=(x\vee n)\wedge(a\vee n)\leq b\vee n$ implies that $x\vee n\in<a, b>^n$. Similarly, $y\vee n\in<a, b>^n$. Since $<a, b>^n$ is an n-ideal, so $x\vee y\vee n\in<a, b>^n$. This implies $m(x\vee y\vee n, n, a)\leq b\vee n$. That is, $(x\vee y\vee n)\wedge(a\vee n)\leq b\vee n$ and so $(x\vee y)\wedge(a\vee n)\leq b\vee n$. Therefore, $x\vee y\in<a\vee n, b\vee n>$. Moreover, for $x\in<a\vee n, b\vee n>$ and $t\leq x$ ($t\in L$) obviously $t\in<a\vee n, b\vee n>$. Hence $<a\vee n, b\vee n>$ is an ideal. A dual proof of above shows that $<a\wedge n, b\wedge n>_d$ is a filter.

(ii)$\Rightarrow$(i). Suppose (ii) holds and x, y$\in<a, b>^n$. Then

$m(x, n, a) \in \langle b \rangle_n$. Then using the neutrality of n, $b\wedge n \leq (x\wedge a)\vee(x\wedge n)\vee(a\wedge n) = (x\vee a)\wedge(x\vee n)\wedge(a\vee n) \leq b\vee n$. Similarly, $b\wedge n \leq (y\wedge a)\vee(y\wedge n)\vee(a\wedge n) = (y\vee a)\wedge(y\vee n)\wedge(a\vee n) \leq b\vee n$. So, $b\wedge n \leq [(x\wedge a)\vee(x\wedge n)\vee(a\wedge n)]\wedge n = (x\wedge n)\vee(a\wedge n)$. This implies $x\wedge n \in \langle a\wedge n, b\wedge n\rangle_d$. Similarly, $y\wedge n \in \langle a\wedge n, b\wedge n\rangle_d$. Since $\langle a\wedge n, b\wedge n\rangle_d$ is a filter, so we have $x\wedge y\wedge n \in \langle a\wedge n, b\wedge n\rangle_d$. Thus, $(x\wedge y\wedge n)\vee(a\wedge n) \geq (b\wedge n)$, and this implies $x\wedge y\wedge n \in \langle a, b\rangle^n$. Again, by the neutrality of n, $(x\vee n)\wedge(a\vee n)$ $= [(x\vee a)\wedge(x\vee n)\wedge(a\vee n)]\vee n \leq b\vee n$. Similarly, $(y\vee n)\wedge(a\vee n) \leq b\vee n$. Thus $((x\wedge y)\vee n)\wedge(a\vee n) \leq b\vee n$. But $((x\wedge y)\vee n)\wedge(a\vee n)$ $= m((x\wedge y)\vee n, n, a)$, as n is neutral. Therefore, $(x\wedge y)\vee n \in \langle a, b\rangle^n$, and so by the convexity of $\langle a, b\rangle^n$, $x\wedge y \in \langle a, b\rangle^n$.

A dual proof of above also shows that $x\vee y \in \langle a, b\rangle^n$. Clearly $\langle a, b\rangle^n$ contains n. Therefore, $\langle a, b\rangle^n$ is an n-ideal. $\square$

Proposition 3.1.2. *Let* L *be a lattice with a neutral element* n. *For all* a, $b\in L$ *the following hold*:

(i) $\langle a\vee n, b\vee n\rangle$ *is an ideal if and only if* [n) *is a distributive sublattice of* L ;

(ii) $\langle a\wedge n, b\wedge n\rangle_d$ *is a filter if and only if* (n] *is a distributive sublattice of* L.

Proof: (i). Suppose for all $a, b\in L$, $<a\vee n, b\vee n>$ is an ideal. Thus, for all $p, q\in[n)$, $<p, q>\cap[n)$ is an ideal in the sublattice $[n)$. Then by [39, Theorem-1], $[n)$ is distributive.

Conversely, suppose $[n)$ is distributive. Let $x, y\in<a\vee n, b\vee n>$. Then $x\wedge(a\vee n)\leq b\vee n$. Since n is neutral, so $(x\vee n)\wedge(a\vee n)$ $=[x\wedge(a\vee n)]\vee n\leq b\vee n$ implies that $x\vee n\in<a\vee n, b\vee n>$. Similarly, $y\vee n\in<a\vee n, b\vee n>$. Then $(x\vee y)\wedge(a\vee n)\leq(x\vee y\vee n)\wedge(a\vee n)$ $=[(x\vee n)\wedge(a\vee n)]\vee[(y\vee n)\wedge(a\vee n)]\leq b\vee n$, as $[n)$ is distributive. Therefore, $x\vee y\in<a\vee n, b\vee n>$. Since $<a\vee n, b\vee n>$ has always the hereditary property, so $<a\vee n, b\vee n>$ is an ideal.

(ii) can be proved dually. □

By Theorem 1.1.2, we know that $F_n(L)\cong(n]^d\times[n)$, where $(n]^d$ denotes the dual of the lattice $(n]$. Thus by Theorem 3.1.1 and above result we have the following result.

Theorem 3.1.3. *Let* L *be a lattice and* $n\in L$ *be neutral. Then for all* $a, b\in L$, $<a, b>^n$ *is an* n*-ideal if and only if* $F_n(L)$ *is distributive.* □

Now by [31], we know that L is distributive if and only if $F_n(L)$ is distributive. Therefore, we have the following corollary which is a generalization of [39, Theorem-1].

Corollary 3.1.4. *For all* a, $b \in L$ *and for a neutral element* $n \in L$, $\langle a, b\rangle^n$ *is an* n-*ideal if and only if* L *is distributive.* □

Following result also generalizes [39, Theorem-1]

Theorem 3.1.5. *Let* n *be a neutral element of a lattice* L. *Then the following conditions are equivalent*:

(i) L *is distributive* ;

(ii) $\langle a \vee n, b \vee n\rangle$ *is an ideal and* $\langle a \wedge n, b \wedge n\rangle_d$ *is a filter whenever* $\langle a\rangle_n \subseteq \langle b\rangle_n$.

Proof: (i)$\Rightarrow$(ii). Suppose (i) holds. Then by Corollary 3.1.4, $\langle a, b\rangle^n$ is an n-ideal for all $a, b \in L$. Thus (ii) holds by Theorem 3.1.1.

(ii)$\Rightarrow$(i). Suppose (ii) holds and $x, y, z \in [n)$. Clearly $(x \wedge y) \vee (x \wedge z) \leq x$. So $\langle x, (x \wedge y) \vee (x \wedge z)\rangle$ is an ideal as $\langle (x \wedge y) \vee (x \wedge z)\rangle_n \subseteq \langle x\rangle_n$. Since $x \wedge y \leq (x \wedge y) \vee (x \wedge z)$, so $y \in \langle x, (x \wedge y) \vee (x \wedge z)\rangle$. Similarly $z \in \langle x, (x \wedge y) \vee (x \wedge z)\rangle$. Hence $y \vee z \in \langle x, (x \wedge y) \vee (x \wedge z)\rangle$ and so $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. This implies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and so $[n)$ is distributive. Using the other part of (ii) we can similarly show that $(n]$ is also distributive. Thus, by Theorem 1.1.2, $F_n(L)$ is distributive, and so by [31], L is distributive. □

Theorem 3.1.6. *Let* n *be a neutral element of a lattice* L*. Then the following conditions are equivalent*:

(i) $F_n(L)$ *is modular* ;

(ii) *For* $a, b \in L$ *with* $<b>_n \subseteq <a>_n$, $x \in <b>_n$ *and* $y \in <a, b>^n$ *imply* $x \wedge y$, $x \vee y \in <a, b>^n$.

Proof: (i)$\Rightarrow$(ii). Suppose $F_n(L)$ is modular. Then by Theorem 1.1.2, $(n]$ and $[n)$ are modular. Here $<b>_n \subseteq <a>_n$. So $a \wedge n \leq b \wedge n \leq n \leq b \vee n \leq a \vee n$. Since $x \in <b>_n$, so $b \wedge n \leq x \leq b \vee n$. Hence, $a \wedge n \leq b \wedge n \leq x \wedge n \leq x \vee n \leq b \vee n \leq a \vee n$. Now, $y \in <a, b>^n$ implies $m(y, n, a) \in <b>_n$. Then by the neutrality of n, $(y \vee a) \wedge (y \vee n) \wedge (a \vee n) \leq b \vee n$, and so $((y \vee a) \wedge (y \vee n) \wedge (a \vee n)) \vee n$ $= (y \vee n) \wedge (a \vee n) \leq b \vee n$. Thus, using the modularity of $[n)$, $m(x \vee y \vee n, n, a) = (x \vee y \vee n) \wedge (a \vee n) = [(a \vee n) \wedge (y \vee n)] \vee (x \vee n)$, as $x \vee n \leq b \vee n \leq a \vee n$. This implies $m(x \vee y \vee n, n, a) \leq b \vee n$, and so $x \vee y \vee n \in <a, b>^n$. Since n is neutral, so $a \wedge n \leq b \wedge n \leq x \wedge n$ implies that $b \wedge n \leq (x \wedge n) \vee (y \wedge n) \vee (a \wedge n) = ((x \vee y) \wedge n) \vee (a \wedge n) =$ $m((x \vee y) \wedge n, n, a) \leq b \vee n$. Therefore, $(x \vee y) \wedge n \in <a, b>^n$. Hence by the convexity of $<a, b>^n$, $x \vee y \in <a, b>^n$. Again using the modularity of $(n]$, a dual proof of above shows that $x \wedge y \in <a, b>^n$.

Conversely, suppose (ii) holds. Let $x, y, z \in [n)$ with $x \leq z$. Then $x \vee (y \wedge z) \leq z$. This implies $\langle x \vee (y \wedge z)\rangle_n \subseteq \langle z \rangle_n$. Now $x \leq x \vee (y \wedge z)$ implies $x \in \langle x \vee (y \wedge z)\rangle_n$. Again $y \wedge z \leq x \vee (y \wedge z)$ implies $m(y, n, z) = y \wedge z \in \langle x \vee (y \wedge z)\rangle_n$. Hence $y \in \langle z, x \vee (y \wedge z)\rangle^n$. Thus by (ii), $x \vee y \in \langle z, x \vee (y \wedge z)\rangle^n$. That is, $(x \vee y) \wedge z \leq x \vee (y \wedge z)$ and so $(x \vee y) \wedge z = x \vee (y \wedge z)$. Therefore, $[n)$ is modular.

Similarly, using the condition (ii) we can easily show that $(n]$ is also modular. Hence by Theorem 1.1.2, $F_n(L)$ is modular. □

By [48, Theorem-3.2], we know that a lattice L is modular if and only if the lattice of all n-ideals $I_n(L)$ is modular. Following their proof it can be easily seen that L is modular if and only if $F_n(L)$ is modular. Hence we have the following result which generalizes [39, Theorem-2].

Corollary 3.1.7. *Let* n *be a neutral element of a lattice* L. *Then the following conditions are equivalent*:

(i) L *is modular* ;

(ii) *For* $a, b \in L$ *with* $\langle b \rangle_n \subseteq \langle a \rangle_n$, $x \in \langle b \rangle_n$ *and* $y \in \langle a, b \rangle^n$ *implies* $x \wedge y, x \vee y \in \langle a, b \rangle^n$. □

We conclude the section with the following characterization of minimal prime n-ideals belonging to an n-ideal. Since the proof of this is almost similar to Theorem 2.1.4, we omit the proof.

Theorem 3.1.8. *Let* L *be a distributive lattice and* P *be a prime* n-*ideal of* L *belonging to an* n-*ideal* J. *Then the following conditions are equivalent*:

(i) P *is minimal belonging to* J ;

(ii) $x \in P$ *implies* $<<x>_n, J> \not\subseteq P$. □

2. Some characterizations of those $F_n(L)$ which are relatively Stone lattices.

The following result is a generalization of [7, Lemma-3.6] which plays an important role in proving our main results in this section.

Theorem 3.2.1. *Let* L *be a distributive lattice. Then the following hold*:

(i) $<<x>_n \vee <y>_n, <x>_n> = <<y>_n, <x>_n>$;

(ii) $<<x>_n, J> = \bigvee_{y \in J} <<x>_n, <y>_n>$, *the supremum of* n-*ideals* $<<x>_n, <y>_n>$ *in the lattice of* n-*ideals of* L, *for any* $x \in L$ *and any* n-*ideal* J.

Proof: (i). L.H.S$\subseteq$R.H.S is obvious. Let $t \in$R.H.S, then $t \in <<y>_n, <x>_n>$. This implies $m(y, n, t) \subseteq <x>_n$. That is $<m(y, n, t)>_n \subseteq <x>_n$ and so $(<y>_n \cap <t>_n) \vee (<x>_n \cap <t>_n) \subseteq <x>_n$. That is, $<t>_n \cap [<x>_n \vee <y>_n] \subseteq <x>_n$ which implies $t \in <<x>_n \vee <y>_n, <x>_n>$. Thus, $t \in$L.H.S and so (i) holds.

(ii). R.H.S$\subseteq$L.H.S is obvious. Let $t \in$L.H.S, then $m(x, n, t) \in J$ that is $m(x, n, t) = j$ for some $j \in J$. This implies $t \in <<x>_n, <j>_n>$. Thus $t \in$R.H.S and so (ii) holds. □

Following lemma will be needed for further development of this chapter. This is in fact, the dual of [7, Lemma-3.6] and very easy to prove. So we prefer to omit the proof.

Lemma 3.2.2. *Let* L *be a distributive lattice. Then the following hold.*

(i) $<x\wedge y,\ x>_d=<y,\ x>_d$;

(ii) $<[x),\ F>_d=\bigvee_{y\in F}<x,\ y>_d$, *where* F *is a filter of* L ;

(iii) $\{<x,\ a>_d\vee<y,\ a>_d\}\cap[a,\ b]$

$=\{<x,\ a>_d\cap[a,\ b]\}\vee\{<y,\ a>_d\cap[a,\ b]\}$. □

Lemma 3.2.3 and Lemma 3.2.4 are essential for the proof of our main result of this section.

Lemma 3.2.3. *Let* L *be a distributive lattice with* $n\in L$. *Suppose* $a,\ b,\ c\in L$.

(i) *If* $a,\ b,\ c\geq n$, *then* $<<m(a,\ n,\ b)>_n,\ <c>_n>$

$=<<a>_n,\ <c>_n>\vee<<b>_n,\ <c>_n>$ *is equivalent to*

$<a\wedge b,\ c>=<a,\ c>\vee<b,\ c>$;

(ii) *If* $a,\ b,\ c\leq n$ *then*

$<<m(a,\ n,\ b)>_n,\ <c>_n>=<<a>_n,\ <c>_n>\vee<<b>_n,\ <c>_n>$

is equivalent to $<a\vee b,\ c>_d=<a,\ c>_d\vee<b,\ c>_d$.

Proof: (i). Suppose a, b, $c \geq n$ and

$<\langle a\rangle_n \cap \langle b\rangle_n, \langle c\rangle_n> = <\langle a\rangle_n, \langle c\rangle_n> \vee <\langle b\rangle_n, \langle c\rangle_n>$. Let

$x \in \langle a \wedge b, c\rangle$. Then $x \wedge a \wedge b \leq c$, $\langle x\rangle_n \cap \langle a \wedge b\rangle_n = \langle x\rangle_n \cap [n, a \wedge b]$

$= [n, (x \vee n) \wedge (a \wedge b)] = [n, (x \wedge a \wedge b) \vee n] \subseteq [n, c]$.

Hence $x \in <\langle a \wedge b\rangle_n, \langle c\rangle_n> = <\langle m(a, n, b)\rangle_n, \langle c\rangle_n>$

$= <\langle a\rangle_n, \langle c\rangle_n> \vee <\langle b\rangle_n, \langle c\rangle_n>$. Thus $x \leq p \vee q$, where

$p \in <\langle a\rangle_n, \langle c\rangle_n>$, $q \in <\langle b\rangle_n, \langle c\rangle_n>$. Then $\langle p\rangle_n \cap \langle a\rangle_n \subseteq \langle c\rangle_n$.

That is, $[p \wedge n, p \vee n] \cap [n, a] \subseteq [n, c]$. Thus, $[n, (p \vee n) \wedge a] \subseteq [n, c]$ which implies $p \wedge a \leq c$, and so $p \in \langle a, c\rangle$. Similarly, $q \in \langle b, c\rangle$ and so $x \in \langle a, c\rangle \vee \langle b, c\rangle$. Hence $\langle a \wedge b, c\rangle \subseteq \langle a, c\rangle \vee \langle b, c\rangle$. But $\langle a, c\rangle \vee \langle b, c\rangle \subseteq \langle a \wedge b, c\rangle$ is obvious. Therefore, $\langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$.

Conversely, suppose $\langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$. Let $x \in <\langle m(a, n, b)\rangle_n, \langle c\rangle_n>$. Then $\langle x\rangle_n \cap \langle m(a, n, b)\rangle_n = [x \wedge n, x \vee n] \cap [n, a \wedge b] \subseteq [n, c]$. That is, $[n, (x \vee n) \wedge (a \wedge b)] \subseteq [n, c]$. Thus, $[n, (x \wedge a \wedge b) \vee n] \subseteq [n, c]$ which implies $x \wedge a \wedge b \leq c$, and so $x \in \langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$. This implies $x = r \vee s$, where $r \in \langle a, c\rangle$ and $s \in \langle b, c\rangle$. Then $r \wedge a \leq c$ and $s \wedge b \leq c$. Now $\langle r\rangle_n \cap \langle a\rangle_n = [r \wedge n, r \vee n] \cap [n, a] = [n, (r \vee n) \wedge a]$

$= [n, (r \wedge a) \vee n] \subseteq [n, c] = \langle c\rangle_n$. Hence, $r \in <\langle a\rangle_n, \langle c\rangle_n>$.

Similarly, $s \in \langle\langle b\rangle_n, \langle c\rangle_n\rangle$. Thus $x \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$ and so $\langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle \subseteq \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$. Since $\langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle \subseteq \langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle$ is obvious, so $\langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle = \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$. A dual calculation of above proof proves (ii). □

Lemma 3.2.4. *Let* L *be a distributive lattice with* $n \in L$. *Suppose* $a, b, c \in L$

(i) *For* $a, b, c \geq n$,

$\langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle = \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$

is equivalent to $\langle c, a \vee b\rangle = \langle c, a\rangle \vee \langle c, b\rangle$;

(ii) *For* $a, b, c \leq n$, $\langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle$

$= \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$ *is equivalent to*

$\langle c, a \wedge b\rangle_d = \langle c, a\rangle_d \vee \langle c, b\rangle_d$.

Proof: Suppose $\langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle$ $= \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$. Let $x \in \langle c, a \vee b\rangle$. Then $x \wedge c \leq a \vee b$. Then $\langle x\rangle_n \cap \langle c\rangle_n = [x \wedge n, x \vee n] \cap [n, c] =$ $[n, (x \vee n) \wedge c] = [n, (x \wedge c) \vee n] \subseteq [n, a \vee b] = \langle a\rangle_n \vee \langle b\rangle_n$.

Thus, $x\in\langle\langle c\rangle_n, \langle a\rangle_n\vee\langle b\rangle_n\rangle=\langle\langle c\rangle_n, \langle a\rangle_n\rangle\vee\langle\langle c\rangle_n, \langle b\rangle_n\rangle$

so, $x\leq p\vee q$ where $p\in\langle\langle c\rangle_n, \langle a\rangle_n\rangle$ and $q\in\langle\langle c\rangle_n, \langle b\rangle_n\rangle$.

Then $[p\wedge n, p\vee n]\cap[n, c]\subseteq[n, a]$. Thus $[n, (p\vee n)\wedge c]\subseteq[n, a]$. That is, $[n, (p\wedge c)\vee n]\subseteq[n, a]$. This implies $p\wedge c\leq a$, and so $p\in\langle c, a\rangle$. Similarly, $q\in\langle c, b\rangle$. Hence $x\in\langle c, a\rangle\vee\langle c, b\rangle$ and so $\langle c, a\vee b\rangle\subseteq\langle c, a\rangle\vee\langle c, b\rangle$. Since the reverse inequality is trivial, so $\langle c, a\vee b\rangle=\langle c, a\rangle\vee\langle c, b\rangle$.

Conversely, suppose $\langle c, a\vee b\rangle=\langle c, a\rangle\vee\langle c, b\rangle$. Let $x\in\langle\langle c\rangle_n, \langle a\rangle_n\vee\langle b\rangle_n\rangle$. Then, $[x\wedge n, x\vee n]\cap[n, c]\subseteq[n, a\vee b]$, and so $[n, (x\vee n)\wedge c]\subseteq[n, a\vee b]$. That is, $[n, (x\wedge c)\vee n]\subseteq[n, a\vee b]$. This implies $x\wedge c\leq a\vee b$, and so $x\in\langle c, a\vee b\rangle=\langle c, a\rangle\vee\langle c, b\rangle$. Thus $x=r\vee s$, where $r\in\langle c, a\rangle$ and $s\in\langle c, b\rangle$. Now, $\langle r\rangle_n\cap\langle c\rangle_n=$ $[r\wedge n, r\vee n]\cap[n, c]=[n, (r\wedge c)\vee n]\subseteq[n, a]=\langle a\rangle_n$. So $r\in\langle\langle c\rangle_n, \langle a\rangle_n\rangle$.

Similarly, $s\in\langle\langle c\rangle_n, \langle b\rangle_n\rangle$. Hence

$x\in\langle\langle c\rangle_n, \langle a\rangle_n\rangle\vee\langle\langle c\rangle_n, \langle b\rangle_n\rangle$,

and so $\langle\langle c\rangle_n, \langle a\rangle_n\vee\langle b\rangle_n\rangle\subseteq\langle\langle c\rangle_n, \langle a\rangle_n\rangle\vee\langle\langle c\rangle_n, \langle b\rangle_n\rangle$.

Since the reverse inequality is trivial, so

$\langle\langle c\rangle_n, \langle a\rangle_n\vee\langle b\rangle_n\rangle=\langle\langle c\rangle_n, \langle a\rangle_n\rangle\vee\langle\langle c\rangle_n, \langle b\rangle_n\rangle$. By the dual calculation of above we can easily prove (ii). $\square$

Following result on Stone lattices is well known due to [13, Theorem-3, Page-161] and [7, Theorem-2.4].

Theorem 3.2.5. *Let* L *be a pseudocomplemented distributive lattice. Then the following conditions are equivalent*:

(i) L *is Stone* ;

(ii) *For each* $x, y \in L$, $(x \wedge y)^* = x^* \vee y^*$;

(iii) *If* $x \wedge y = 0$, $x, y \in L$, *then* $x^* \vee y^* = 1$. □

Similarly we can easily prove the following result which is dual to above theorem.

Theorem 3.2.6. *Let* L *be a dual pseudocomplemented distributive lattice. Then the following conditions are equivalent*:

(i) L *is dual Stone* ;

(ii) *For each* $x, y \in L$, $(x \vee y)^{*d} = x^{*d} \wedge y^{*d}$;

(iii) *If* $x \vee y = 1$, $x, y \in L$, *then* $x^{*d} \wedge y^{*d} = 0$,

where x^{*d} *denotes the dual pseudocomplement of* x. □

Now we prove the following result, which is dual to [7, Theorem-3.7]. This will be needed to prove the main result of this chapter.

Theorem 3.2.7. *Let* L *be a relatively dual pseudocomplemented distributive lattice. Let* $a, b, c \in L$ *be* *arbitrary elements, and* A, B *are arbitrary filters. Then the following are equivalent*:

(i) L *is relatively dual Stone* ;

(ii) $<a, b>_d \vee <b, a>_d = L$;

(iii) $<c, a\wedge b>_d = <c, a>_d \vee <c, b>_d$;

(iv) $<[c), A\vee B>_d = <[c), A>_d \vee <[c), B>_d$;

(v) $<a\vee b, c>_d = <a, c>_d \vee <b, c>_d$.

Proof: (i)$\Rightarrow$(ii). Let $z\in L$ be arbitrary. Consider the interval $I=[z, a\vee b\vee z]$. Then $a\vee b\vee z$ is the largest element of I. Since by (i), I is dual Stone, then by Theorem 3.2.6(iii), there exist $r, s\in I$ such that $a\vee s = a\vee b\vee z = b\vee z\vee r$ and $z = s\wedge r$. Now, $a\vee s\geq b$ implies $s\in <a, b>_d$ and $b\vee r = b\vee z\vee r = a\vee b\vee z\geq a$ implies $r\in <b, a>_d$. Hence (ii) holds.

(ii)$\Rightarrow$(iii). In (iii), R.H.S$\subseteq$L.H.S is obvious. Let $z\in <c, a\wedge b>_d$, then $z\vee c\geq a\wedge b$. Since (ii) holds, so $z = x\wedge y$, where $x\in <a, b>_d$ and $y\in <b, a>_d$. Then $x\vee a\geq b$ and $y\vee b\geq a$. Thus, $x\vee c = x\vee z\vee c\geq x\vee(a\wedge b) = (x\vee a)\wedge(x\vee b)\geq b$, which implies $x\in <c, b>_d$.

Similarly, $y\in <c, a>_d$. Hence $z = x\wedge y\in <c, a>_d \vee <c, b>_d$, and so $<c, a\wedge b>_d \subseteq <c, a>_d \vee <c, b>_d$. Since the reverse inclusion is obvious, so (iii) holds.

(iii)$\Rightarrow$(iv) follows from Lemma 3.2.2(ii).

(iv)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(ii) follows from Lemma 3.2.2(i) by putting $c=a\wedge b$.

(ii)$\Rightarrow$(v). Let $z\in\langle a\vee b, c\rangle_d$. Then by (ii), $z=x\wedge y$, where $x\vee a\geq b$ and $y\vee b\geq a$. Also $x\vee a=x\vee a\vee b\geq z\vee a\vee b\geq c$. This implies $x\in\langle a, c\rangle_d$.

Similarly, $y\in\langle b, c\rangle_d$. It follows that $\langle a\vee b, c\rangle_d\supseteq\langle a, c\rangle_d\vee\langle b, c\rangle_d$. Since the reverse inequality is obvious, so (v) holds.

(v)$\Rightarrow$(i). Let $x\in[a, b]$, $a<b$. Suppose x^{+d} denotes the relatively dual pseudocomplemented of x in [a, b]. Then clearly $[x^{+d})=[x)^{+d}=\{t\in[a, b] : t\vee x=b$, the largest element of $[a, b]\}$. It is easy to see that $[x)^{+d}=\langle a, b\rangle_d\cap[a, b]$.
Now Suppose $x, y\in[a, b]$ with $x\vee y=b$, then by (v),

$$\begin{aligned}
[x^{+d}\wedge y^{+d})&=[x^{+d})\vee[y^{+d})=[x)^{+d}\vee[y)^{+d}\\
&=(\langle x, b\rangle_d\cap[a, b])\vee(\langle y, b\rangle_d\cap[a, b])\\
&=(\langle x, b\rangle_d\vee\langle y, b\rangle_d)\cap[a, b] \quad \text{(by Lemma 3.2.2(iii))}\\
&=\langle x\vee y, b\rangle_d\cap[a, b]\\
&=\langle b, b\rangle_d\cap[a, b]=L\cap[a, b]\\
&=[a, b].
\end{aligned}$$

This implies $x^{+d}\wedge y^{+d}=a$. Hence by Theorem 3.2.6, [a, b] is dual Stone and so L is a relatively dual Stone lattice. □

Now we prove our main results of this chapter, which are generalizations of [7, Theorem-3.7] and [39, Theorem-5]. These give characterizations of those $F_n(L)$ which are relatively Stone in terms of n-ideals.

Theorem 3.2.8. *Let* $F_n(L)$ *be a relatively pseudocomplemented distributive lattice and* A *and* B *be two* n-*ideals of* L. *Then for all* a, b, c$\in$L, *the following conditions are equivalent*:

(i) $F_n(L)$ *is relatively Stone* ;

(ii) $<<a>_n, <b>_n> \vee <<b>_n, <a>_n> = L$;

(iii) $<<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n> \vee <<c>_n, <b>_n>$;

(iv) $<<c>_n, A \vee B> = <<c>_n, A> \vee <<c>_n, B>$;

(v) $<<m(a, n, b)>_n, <c>_n> = <<a>_n, <c>_n> \vee <<b>_n, <c>_n>$.

Proof: (i)$\Rightarrow$(ii). Let z$\in$L, consider the interval $I=[<a>_n \cap <b>_n \cap <z>_n, <z>_n]$ in $F_n(L)$. Then $<a>_n \cap <b>_n \cap <z>_n$ is the smallest element of the interval I. By (i), I is Stone. Then by Theorem 3.2.5, there exist finitely generated n-ideals [p, q], [r, s]$\in$I such that,

$<a>_n \cap <z>_n \cap [p, q]$

$= <a>_n \cap <b>_n \cap <z>_n$

$= <b>_n \cap <z>_n \cap [r, s]$ and

$<z>_n = [p, q] \vee [r, s]$.

Now, $\langle a\rangle_n\cap[p, q]=\langle a\rangle_n\cap[p, q]\cap\langle z\rangle_n=\langle a\rangle_n\cap\langle b\rangle_n\cap\langle z\rangle_n\subseteq\langle b\rangle_n$ implies $[p, q]\subseteq\langle\langle a\rangle_n, \langle b\rangle_n\rangle$. Also $\langle b\rangle_n\cap[r, s]=\langle b\rangle_n\cap\langle z\rangle_n\cap[r, s]$ $=\langle a\rangle_n\cap\langle b\rangle_n\cap\langle z\rangle_n\subseteq\langle a\rangle_n$ implies $[r, s]\subseteq\langle\langle b\rangle_n, \langle a\rangle_n\rangle$.

Thus, $\langle z\rangle_n\subseteq\langle\langle a\rangle_n, \langle b\rangle_n\rangle\vee\langle\langle b\rangle_n, \langle a\rangle_n\rangle$,

and so $z\in\langle\langle a\rangle_n, \langle b\rangle_n\rangle\vee\langle\langle b\rangle_n, \langle a\rangle_n\rangle$.

Hence $\langle\langle a\rangle_n, \langle b\rangle_n\rangle\vee\langle\langle b\rangle_n, \langle a\rangle_n\rangle=L$.

(ii)$\Rightarrow$(iii). Suppose (ii) holds. For (iii), R.H.S$\subseteq$L.H.S is obvious. Now, let $z\in\langle\langle c\rangle_n, \langle a\rangle_n\vee\langle b\rangle_n\rangle$. Then $z\vee n\in\langle\langle c\rangle_n, \langle a\rangle_n\vee\langle b\rangle_n\rangle$, and so $m(z\vee n, n, c)\in[a\wedge b\wedge n, a\vee b\vee n]$. That is, $(z\vee n)\wedge(c\vee n)\leq a\vee b\vee n$. Now by (ii), $z\vee n\in\langle\langle a\rangle_n, \langle b\rangle_n\rangle\vee\langle\langle b\rangle_n, \langle a\rangle_n\rangle$. So $z\vee n\leq(p\vee n)\vee(q\vee n)$ for some $p\vee n\in\langle\langle a\rangle_n, \langle b\rangle_n\rangle$ and $q\vee n\in\langle\langle b\rangle_n, \langle a\rangle_n\rangle$. Hence,

$z\vee n=((z\vee n)\wedge(p\vee n))\vee((z\vee n)\wedge(q\vee n))=r\vee s$ (say).

Now, $m(p\vee n, n, a)=(p\vee n)\wedge(a\vee n)\leq b\vee n$. So $(b\wedge n)\leq r\wedge(a\vee n)\leq b\vee n$. Hence, $r\wedge(c\vee n)=r\wedge(z\vee n)\wedge(c\vee n)$ $\leq r\wedge(a\vee b\vee n)=(r\wedge(a\vee n))\vee(r\wedge(b\vee n))\leq b\vee n$. This implies $r\in\langle\langle c\rangle_n, \langle b\rangle_n\rangle$. Similarly, $s\in\langle\langle c\rangle_n, \langle a\rangle_n\rangle$. Hence

$z\vee n\in\langle\langle c\rangle_n, \langle a\rangle_n\rangle\vee\langle\langle c\rangle_n, \langle b\rangle_n\rangle$.

Again $z \in \langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle$ implies $z \wedge n \in \langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle$. Then a dual calculation of above shows that $z \wedge n \in \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$. Thus by convexity, $z \in \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$ and so (iii) holds.

(iii)$\Rightarrow$(iv). Suppose (iii) holds. In (iv), R.H.S$\subseteq$L.H.S is obvious. Now let $x \in \langle\langle c\rangle_n, A \vee B\rangle$. Then $x \vee n \in \langle\langle c\rangle_n, A \vee B\rangle$. Thus $m(x \vee n, n, c) \in A \vee B$. Now $m(x \vee n, n, c) = (x \vee n) \wedge (n \vee c) \geq n$ implies $m(x \vee n, n, c) \in (A \vee B) \cap [n)$. Hence by Theorem 3.2.1(ii), $x \vee n \in \langle\langle c\rangle_n, (A \cap [n)) \vee (B \cap [n))\rangle$

$= \bigvee_{r \in (A \cap [n)) \vee (B \cap [n))} \langle\langle c\rangle_n, \langle r\rangle_n\rangle$. But by Theorem 1.1.12, $r \in (A \cap [n)) \vee (B \cap [n))$ implies $r = s \vee t$ for some $s \in A$, $t \in B$ and $s, t \geq n$. Then by (iii), $\langle\langle c\rangle_n, \langle r\rangle_n\rangle = \langle\langle c\rangle_n, \langle s \vee t\rangle_n\rangle$

$= \langle\langle c\rangle_n, \langle s\rangle_n \vee \langle t\rangle_n\rangle = \langle\langle c\rangle_n, \langle s\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle t\rangle_n\rangle$

$\subseteq \langle\langle c\rangle_n, A\rangle \vee \langle\langle c\rangle_n, B\rangle$. Hence $x \vee n \in \langle\langle c\rangle_n, A\rangle \vee \langle\langle c\rangle_n, B\rangle$. Also $x \in \langle\langle c\rangle_n, A \vee B\rangle$ implies $x \wedge n \in \langle\langle c\rangle_n, A \vee B\rangle$. Since $m(x \wedge n, n, c) = (x \wedge n) \vee (n \wedge c) \leq n$, so $x \wedge n \in \langle\langle c\rangle_n, (A \vee B) \cap (n]\rangle$. Then by Theorem 3.2.1(ii), $x \wedge n \in \langle\langle c\rangle_n, (A \cap (n]) \vee (B \cap (n])\rangle$

$=\bigvee_{t\in(A\cap(n])\vee(B\cap(n])}<\langle c\rangle_n, \langle t\rangle_n>$. Using Theorem 1.1.12 again, we see that $t=p\wedge q$ where $p\in A$, $q\in B$, $p, q\leq n$. Then by (iii), $<\langle c\rangle_n, \langle t\rangle_n>=<\langle c\rangle_n, \langle p\wedge q\rangle_n>=<\langle c\rangle_n, \langle p\rangle_n\vee\langle q\rangle_n>$ $=<\langle c\rangle_n, \langle p\rangle_n>\vee<\langle c\rangle_n, \langle q\rangle_n>\subseteq<\langle c\rangle_n, A>\vee<\langle c\rangle_n, B>$. Hence $x\wedge n\in<\langle c\rangle_n, A>\vee<\langle c\rangle_n, B>$. Therefore by convexity, $x\in<\langle c\rangle_n, A>\vee<\langle c\rangle_n, B>$, and so (iv) holds.

(iv)$\Rightarrow$(iii) is trivial.

(ii)$\Rightarrow$(v). In (v) R.H.S$\subseteq$L.H.S is obvious. Let $z\in$L.H.S. Then $z\in<\langle m(a, n, b)\rangle_n, \langle c\rangle_n>$, which implies $z\vee n\in<\langle m(a, n, b)\rangle_n, \langle c\rangle_n>$. By (ii), $z\vee n\in<\langle a\rangle_n, \langle b\rangle_n>\vee<\langle b\rangle_n, \langle a\rangle_n>$. Then by Theorem 1.1.12, $z\vee n=x\vee y$ for some $x\in<\langle a\rangle_n, \langle b\rangle_n>$ and $y\in<\langle b\rangle_n, \langle a\rangle_n>$ and $x, y\geq n$. Thus, $\langle x\rangle_n\cap\langle a\rangle_n\subseteq\langle b\rangle_n$, and so $\langle x\rangle_n\cap\langle a\rangle_n=\langle x\rangle_n\cap\langle a\rangle_n\cap\langle b\rangle_n\subseteq\langle z\vee n\rangle_n\cap\langle a\rangle_n\cap\langle b\rangle_n$. $=\langle z\vee n\rangle_n\cap\langle m(a, n, b)\rangle_n\subseteq\langle c\rangle_n$. This implies $x\in<\langle a\rangle_n, \langle c\rangle_n>$. Similarly $y\in<\langle b\rangle_n, \langle c\rangle_n>$, and so $z\vee n\in<\langle a\rangle_n, \langle c\rangle_n>\vee<\langle b\rangle_n, \langle c\rangle_n>$. Similarly, a dual calculation of above shows that $z\wedge n\in<\langle a\rangle_n, \langle c\rangle_n>\vee<\langle b\rangle_n, \langle c\rangle_n>$. Thus by convexity, $z \in \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$ and so (v) holds.

(v)$\Rightarrow$(i). Suppose (v) holds. Let a, b, $c \geq n$. By (v), $\langle\langle m(a, n, b)\rangle_n, \langle c\rangle_n\rangle = \langle\langle a\rangle_n, \langle c\rangle_n\rangle \vee \langle\langle b\rangle_n, \langle c\rangle_n\rangle$. But by Lemma 3.2.3(i), this is equivalent to $\langle a \wedge b, c\rangle = \langle a, c\rangle \vee \langle b, c\rangle$. Then by [7, Theorem-3.7], this shows that [n) is a relatively Stone Lattice. Similarly, for a, b, $c \leq n$, using the Lemma 3.2.3(ii) and Theorem 3.2.7, we find that (n] is relatively dual Stone. Therefore $F_n(L)$ is relatively Stone by Theorem 1.1.2.

Finally we need to prove (iii)$\Rightarrow$(i). Suppose (iii) holds. Let a, b, $c \in L \cap [n)$. By (iii), $\langle\langle c\rangle_n, \langle a\rangle_n \vee \langle b\rangle_n\rangle = \langle\langle c\rangle_n, \langle a\rangle_n\rangle \vee \langle\langle c\rangle_n, \langle b\rangle_n\rangle$. But by Lemma 3.2.4(i), this is equivalent to $\langle c, a \vee b\rangle = \langle c, a\rangle \vee \langle c, b\rangle$ which says by [7, Theorem-3.7] that [n) is relatively Stone. Similarly for a, b, $c \leq n$, using the Lemma 3.2.4(ii) and Theorem 3.2.7, we find that (n] is relatively dual Stone. Therefore by 1.1.2, $F_n(L)$ is relatively Stone. $\square$

We conclude the chapter by proving the following result, which is a generalizations of [7, Theorem-3.5].

To prove this we have used the following lemma which is due to [7, Lemma-3.4].

Lemma 3.2.9. *If* L_1 *is a sublattice of* L *and* P_1 *is a prime ideal in* L_1 *then there exists a prime ideal* P *in* L *such that* $P_1=L_1\cap P$. □

Theorem 3.2.10. *Let* $F_n(L)$ *be a relatively pseudocomplemented distributive lattice. Then the following conditions are equivalent*:

(i) $F_n(L)$ *is relatively Stone* ;

(ii) *Any two incomparable prime* n-*ideals* P *and* Q *are comaximal, that is* $P\vee Q=L$.

Proof: Suppose (i) holds. Let P, Q be two incomparable prime n-deals of L. Then there exist a, b$\in$L such that $a\in P-Q$ and $b\in Q-P$. Then $\langle a\rangle_n\subseteq P-Q$, $\langle b\rangle_n\subseteq Q-P$. Since $F_n(L)$ is relatively Stone, so by Theorem 3.2.8,

$\langle\langle a\rangle_n, \langle b\rangle_n\rangle\vee\langle\langle b\rangle_n, \langle a\rangle_n\rangle=L$. But as P, Q are prime, so it is easy to see that, $\langle\langle a\rangle_n, \langle b\rangle_n\rangle\subseteq Q$ and $\langle\langle b\rangle_n, \langle a\rangle_n\rangle\subseteq P$. Therefore $L\subseteq P\vee Q$ and so $P\vee Q=L$. That is, (ii) holds.

Conversely, suppose (ii) holds. Let P_1 and Q_1 be two incomparable prime ideals of [n). Then by Lemma 3.2.9, there exist incomparable prime ideals P and Q of L such that

$P_1=P\cap[n)$ and $Q_1=Q\cap[n)$. Since $n\in P_1$ and $n\in Q_1$, so by Lemma 1.2.5, P, Q are in fact two incomparable prime n-ideals of L. Then by (ii), $P\vee Q=L$. Therefore, $P_1\vee Q_1=(P\vee Q)\cap[n)=[n)$. Thus by [7, Theorem-3.5], [n) is relatively Stone. Similarly, considering two prime filters of (n] and proceeding as above and using the dual result of [7, Theorem-3.5] we find that (n] is relatively dual Stone. Therefore by Theorem 1.1.2, $F_n(L)$ is relatively Stone. □

Chapter-4

Characterization of finitely generated n-ideals which form sectionally and relatively B_m-lattices.

Introduction:

Lee in [36] also see Lakser [30] has determined the lattice of all equational subclasses of the class of all pseudocomplemented distributive lattices. They are given by $B_{-1} \subset B_0 \subset \text{-----} \subset B_m \subset \text{-----} \subset B_\omega$, where all the inclusions are proper and B_ω is the class of all pseudocomplemented distributive lattices, B_{-1} consists of all one element algebra, B_0 is the variety of Boolean algebras while B_m, for $-1 \le m < \omega$ consists of all algebras satisfying the equation $(x_1 \wedge x_2 \wedge \text{-----} \wedge x_m)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge x_2 \wedge \text{---------------} \wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \text{-----------} \wedge x_m)^* = 1$ where x^* denotes the pseudocomplement of x. Thus B_1 consists of all Stone algebras.

He also generalized Grätzer and Schmidt's theorem by proving that for $-1 \le m < \omega$ the mth variety consists of all lattices such that each prime ideal contains at most m minimal prime ideals.

Cornish in [7] and Mandelker in [39] have studied distributive lattices analogues to B_1-lattices and relatively B_1-lattices. Cornish [8], Beazer [2] and Davey [11] have each independently obtained several characterizations of (sectionally) B_m and relatively B_m-lattices. Moreover, Grätzer and Lakser in [16] and [17] have obtained some results on this topic.

A distributive lattice L with 0 is called *sectionally in* B_m, $-1 \le m < \omega$ if each interval $[0, x]$ $x \in L$ is in B_m. A distributive lattice L is called *relatively* in B_m if each interval $[0, x]$ $x \in L$ is in B_m.

Recall that a family of ideals of a lattice L is *comaximal* if their join is L. Similarly a family of n-ideals of a lattice L is *comiximal* if their join is L.

In section 1 we will study finitely generated n-ideals which form a (sectionally) B_m lattice. We will include several characterizations which generalize several results of [8], [11], [2] and [16]. We shall show that if $F_n(L)$ is (sectionally) pseudocomplemented and distributive then $F_n(L)$ is in (sectionally) B_m if and only if for any $x_1, x_2, \text{--------}, x_m \in L$, $\langle x_0\rangle_n^+ \vee \text{---------} \vee \langle x_m\rangle_n^+ = L$, which is also equivalent to the condition that for any m+1 distinct minimal prime n-ideals P_0,--------,P_m of L, $P_0 \vee$-------$\vee P_m$=L.

In section 2 we will study those $F_n(L)$ which are relatively in B_m. Here we will include a number of characterizations of those $F_n(L)$ which are generalizations of results on relatively B_m-lattices given in [8], [9] and [11]. We shall show that if $F_n(L)$ is relatively pseudocomplemented, then $F_n(L)$ is relatively in B_m if and only if any m+1 pairwise incomparable prime n-ideals are comaximal.

1. Lattices whose $F_n(L)$ form (sectionally) B_m-lattices.

The following result is due to [11, Lemma-2.2]. This follows from the corresponding result for commutative semigroups due to Kist [29].

Lemma 4.1.1. *Let* M *be a prime ideal containing an ideal* J. *Then* M *is a minimal prime ideal belonging to* J *if and only if for all* $x \in M$, *there exists* $x' \notin M$ *such that* $x \wedge x' \in J$. □

Now we generalize this result for n-ideals.

Lemma 4.1.2. *Let* M *be a prime* n-*ideal containing an* n-*ideal* J. *Then* M *is a minimal prime* n-*ideal belonging to* J *if and only if for all* $x \in M$ *there exists* $x' \notin M$ *such that* $m(x, n, x') \in J$.

Proof: Let M be a minimal prime n-ideal belonging to J and $x \in M$. Then by Theorem 3.1.8, $<<a>_n, J> \not\subseteq M$. So there exists x' with $m(x, n, x') \in J$ such that $x' \notin M$.

Conversely, suppose $x \in M$, then there exists $x' \notin M$ such that $m(x, n, x') \in J$. This implies $x' \notin M$, but $x' \in \langle\langle x\rangle_n, J\rangle$, that is $\langle\langle x\rangle_n, J\rangle \not\subseteq M$. Hence by Theorem 3.1.8, M is a prime n-ideal belonging to J. □

Davey in [11, Corollary-2.3] used the following result in proving several equivalent conditions on B_m-lattices. On the other hand, Cornish in [8] has used this result in studying n-normal lattices.

Proposition 4.1.3. *Let* M_0,--------,M_n *be* n+1 *distinct minimal prime ideals. Then there exist* a_0,----------,$a_n \in L$ *such that* $a_i \wedge a_j \in J$ $(i \neq j)$ *and* $a_j \notin M_j$ $j=0$,--------,n. □

The following result is a generalization of above result in terms of n-ideals.

Proposition 4.1.4. *Let* M_0,---------,M_n *be* n+1 *distinct minimal prime* n-*ideals. Then there exist* a_0,--------,$a_n \in L$ *such that* $m(a_i, n, a_j) \in J$ $(i \neq j)$ *and* $a_j \notin M_j$ ($j=0$,----------,n).

Proof: For n=1. Let $x_0 \in M_1 - M_0$ and $x_1 \in M_0 - M_1$. Then by Lemma 4.1.1, there exists $x_1' \notin M_0$ such that $m(x_1, n, x_1') \in J$. Hence $a_1 = x_1$, $a_0 = m(x_0, n, x_1')$ are the required elements. Observe that

$$m(a_0, n, a_1) = m(m(x_0, n, x_1'), n, x_1)$$

$$=(x_0 \wedge x_1 \wedge x_1') \vee (x_0 \wedge n) \vee (x_1 \wedge n) \vee (x_1' \wedge n)$$

$$=(x_0 \wedge m(x_1, n, x_1')) \vee (x_0 \wedge n) \vee (m(x_1, n, x_1') \wedge n)$$

$=m(x_0, n, m(x_1, n, x_1'))$.

Now, $m(x_1, n, x_1')\wedge n\leq m(x_0, n, m(x_1, n, x_1'))$ $\leq m(x_1, n, x_1')\vee n$ and $m(x_1, n, x_1')\in J$, so by convexity $m(a_0, n, a_1)\in J$.

Assume that the result is true for $n=m-1$, and let M_0,-----------,M_n be $n+1$ distinct minimal prime n-ideals. Let b_j ($j=0$,------,$m-1$) satisfy $m(b_i, n, b_j)\in J$ ($i\neq j$) and $b_j\notin M_j$. Now choose $b_m\in M_m-\bigcup_{j=0}^{m-1}M_j$ and by the Lemma 4.1.2, let b_m' satisfy $b_m'\notin M_m$ and $m(b_m, n, b_m')\in J$. Clearly, $a_j=m(b_j, n, b_m)$ ($j=0$,----------,$m-1$) and $a_m=b_m'$, establish the result. □

Let J be an n-ideal of a distributive lattice L. A set of elements x_0,------,$x_n\in L$ is said to be *pairwise* in J if $m(x_i, n, x_j)=n$ for all $i\neq j$.

The next result is due to [8, Lemma-2.3], which was suggested by Hindman in [21, Theorem-1.8].

Lemma 4.1.5. *Let* J *be an ideal in a lattice* L. *For a given positive integer* $n\geq 2$, *the following conditions are equivalent*:

(i) *For any* x_1,-------,$x_n\in L$ *which are "pairwise in* J*" that is,* $x_i\wedge x_j\in J$ *for any* $i\neq j$, *there exists* k *such that* $x_k\in J$;

(ii) *For any ideals* $J_1, \text{-------}, J_n$ *in* L *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$, *there exists* k *such that* $J_k \subseteq J$;

(iii) J *is the intersection of at most* n-1 *distinct prime ideals.* □

Our next result is a generalization of above result. This result will be needed in proving the next theorem which is the main result of this section. In fact, the following lemma is very useful in studying those $F_n(L)$ which are (sectionally) in B_m.

Lemma 4.1.6. *Let* J *be an* n-*ideal in a lattice* L. *For a given positive integer* $n \geq 2$, *the following conditions are equivalent*:

(i) *For any* $x_1, x_2, \text{--------}, x_m \in L$ *with* $m(x_i, n, x_j) \in J$ (*that is, they are pairwise in* J) *for any* $i \neq j$, *there exists* k *such that* $x_k \in J$;

(ii) *For any* n-*ideals* $J_1, \text{------}, J_m$ *in* L *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$, *there exists* k *such that* $J_k \subseteq J$;

(iii) J *is the intersection of at most* m-1 *distinct prime* n-*ideals.*

Proof: (i) and (ii) are easily seen to be equivalent.

(iii)⇒(i). Suppose $P_1, P_2, \text{---------}, P_k$ are k $(1 \leq k \leq m-1)$ distinct prime n-ideals such that $J = P_1 \cap \text{-----} \cap P_k$. Let $x_1, x_2, \text{------}, x_m \in L$ be such that $m(x_i, n, x_j) \in J$ for all $i \neq j$. Suppose no element x_i

is a member of J. Then for each r ($1 \leq r \leq k$) there is at most one i ($1 \leq i \leq m$) such that $x_i \in P_r$. Since $k<m$, there is some i such that $x_i \in P_1 \cap P_2 \cap \text{------------} \cap P_k$.

(i)$\Rightarrow$(iii). Suppose (i) holds for n=2, then it implies that J is a prime n-ideal. Then (iii) is trivially true. Thus we may assume that there is a largest integer $t<m$ such that the condition (i) does not hold for J (consequently condition (i) holds for $t+1$, $t+2$,------,m). For some $t<m$, we may suppose that there exist elements $a_1, a_2, \text{------}, a_t \in L$ such that $m(a_i, n, a_j) \in J$ for $i \neq j$, $i=1, 2, \text{------}, t$, $j=1, 2, \text{------}, t$, yet $a_1, a_2, \text{------}, a_t \notin J$.

As L is a distributive lattice, $<<a_i>_n, J>$ is an n-ideal for any $i \in \{1, 2, \text{------}, t\}$. Each $<<a_i>_n, J>$ is in fact a prime n-ideal. Firstly $<<a_i>_n, J> \neq L$, since $a_i \notin J$. Secondly, suppose that b and c are in L and $m(b, n, c) \in <<a_i>_n, J>$. Consider the set of t+1 elements $\{a_1, a_2, \text{-----------}, a_{i-1}, m(b, n, a_i), m(c, n, a_i), a_{i+1}, \text{---------}, a_t\}$. This set is pairwise in J and so, either $m(b, n, a_i) \in J$ or $m(c, n, a_i) \in J$ since condition (i) holds for t+1. That is, $b \in <<a_i>_n, J>$ or $c \in <<a_i>_n, J>$ and so $<<a_i>_n, J>$ is prime.

Clearly, $J \subseteq \bigcap_{1\le i\le t} \langle\langle a_i\rangle_n, J\rangle$. If $w \in \bigcap_{1\le i\le t} \langle\langle a_i\rangle_n, J\rangle$. Then $w, a_1, a_2, \ldots, a_t$ are pairwise in J and so $w \in J$. Hence $J = \bigcap_{1\le i\le t} \langle\langle a_i\rangle_n, J\rangle$ is the intersection of $t<m$ prime n-ideals. □

An ideal $J \neq L$ satisfying the equivalent conditions of Lemma 4.1.5 is called an m-*prime ideal.*

Similarly, an n-ideal $J \neq L$ satisfying the equivalent conditions of Lemma 4.1.6 is called an m-*prime* n-*ideal.*

Now we generalize a result of Davey in [11, Proposition-3.1].

Theorem 4.1.7. *Let* J *be an* n-*ideal of a distributive lattice* L. *Then the following conditions are equivalent*:

(i) *For any* m+1 *distinct prime* n-*ideals* $P_0, P_1, \ldots, P_m$ *belonging to* J, $P_0 \vee P_1 \vee \cdots \vee P_m = L$;

(ii) *Every prime* n-*ideal containing* J *contains at most* m *distinct minimal prime* n-*ideals belonging to* J ;

(iii) *If* $a_0, a_1, \ldots, a_m \in L$ *with* $m(a_i, n, a_j) \in J$ $(i \neq j)$ *then*

$$\bigvee_j \langle\langle a_j\rangle_n, J\rangle = L.$$

Proof: (i)⇒(ii) is obvious.

(ii)⇒(iii). Assume $a_0, a_1, \ldots, a_m \in L$ with $m(a_i, n, a_j) \in J$

and $\bigvee_j \langle\langle a_j\rangle_n, J\rangle \neq L$. It follows that $a_j \notin J$, for all j. Then by Theorem 1.2.9, there exists a prime n-ideal P such that $\bigvee_j \langle\langle a_j\rangle_n, J\rangle \subseteq P$. But by Theorem 1.2.4, we know that P is either a prime ideal or a prime filter. Suppose P is a prime ideal.

For each j, let $F_j = \{x\wedge y : x\geq a_j, x, y\geq n, y\notin P\}$.
Let $x_1\wedge y_1, x_2\wedge y_2 \in F_j$

$\therefore (x_1\wedge y_1)\wedge(x_2\wedge y_2)=(x_1\wedge x_2)\wedge(y_1\wedge y_2)$.

Now, $x_1\wedge x_2\geq a_j$ and $y_1\wedge y_2=m(y_1, n, y_2)$

so $t\geq x\wedge y$ implies $t=(t\vee x)\wedge(t\vee y)$.

Since $y\notin P$, so $t\vee y\notin P$. Hence $t\in F_j$, and so F_j is a dual ideal. We now show that $F_j\cap J=\varnothing$, for all $j=0, 1,$------------$,m$. If not, let $b\in F_j\cap J$, then $b=x\wedge y$, $x\geq a_j$, $x, y\geq n$, $y\notin P$. Hence $m(a_j, n, y)=(a_j\wedge n)\vee n\vee(a_j\wedge y)=(a_j\wedge y)\vee n=(a_j\vee n)\wedge(y\vee n)$. But $(a_j\vee n)\wedge(y\vee n)\in F_j$ and $n\leq(a_j\wedge y)\vee n\leq b$ implies $m(a_j, n, y)\in J$. Therefore, $m(a_j, n, y)\in F_j\cap J$. Again, $m(a_j, n, y)\in J$ with $y\notin P$ implies $\langle\langle a_j\rangle_n, J\rangle \not\subseteq P$, which is a contradiction. Hence $F_j\cap J=\varnothing$ for all j. For each j, let P_j be a minimal prime n-ideal belonging to J and $F_j\cap P_j=\varnothing$. Let $y\in P_j$. If $y\notin P$, then $y\vee n\notin P$. Then $m(a_j, n, y\vee n)=(a_j\vee n)\wedge(y\vee n)\in F_j$. But $m(a_j, n, y\vee n)\in\langle y\vee n\rangle_n\subseteq\langle y\rangle_n\subseteq P_j$, which is a contradiction. So $y\in P$.

Therefore $P_j \subseteq P$, and $a_j \notin P_j$. For if $a_j \in P_j$, then $a_j \vee n \in P_j$. Now, $a_j \vee n = (a_j \vee n) \wedge (a_j \vee n \vee y) \in F_j$ for any $y \notin P$. This implies $P_j \cap F_j \neq \emptyset$, which is a contradiction. So $a_j \notin P_j$.
But $m(a_i, n, a_j) \in J \subseteq P_j$ $(i \neq j)$ which implies $a_i \in P_j$ $(i \neq j)$ as P_j is prime. It follows that P_j form a set of $m+1$ distinct minimal prime n-ideals belonging to J and contained in P. This contradicts (ii). Therefore $\bigvee_j \langle\langle a_j \rangle_n, J\rangle = L$.

Similarly, if P is filter, then a dual proof of above also shows that $\bigvee_j \langle\langle a_j \rangle_n, J\rangle = L$, and hence (iii) holds.

(iii)⇒(i). Let $P_0, P_1, \text{--------}, P_m$ be $m+1$ distinct minimal prime n-ideals belonging to J. Then by Proposition 4.1.4, there exist $a_0, a_1, \text{-----------}, a_m \in L$ such that $m(a_i, n, a_j) \in J$ $(i \neq j)$ and $a_j \notin P_j$. This implies $\langle\langle a_j \rangle_n, J\rangle \subseteq P_j$ for all j. Then by (iii) $\langle\langle a_0 \rangle_n, J\rangle$

$\vee \langle\langle a_1 \rangle_n, J\rangle \vee \text{---------} \vee \langle\langle a_m \rangle_n, J\rangle \subseteq P_0 \vee P_1 \vee \text{--------} \vee P_m$,

which implies $P_0 \vee P_1 \vee \text{--------} \vee P_m = L$. □

We have already mentioned that Lee [36] and Lakser [30] have shown that the equational classes of pseudocomplemented distributive lattices form a chain $B_{-1} \subset B_0 \subset B_1 \subset \text{---------} \subset B_\omega$ where B_{-1} is the trivial class, B_0 is the class of Boolean algebras and B_1 is the class of

Stone lattices. Cornish in [7] and Mandelker in [39] considered distributive lattices analogues to B_1-lattices and relative B_1-lattices. In the following result characterizations are given for the distributive lattices analogues of B_n-lattices. This result is due to Cornish [8]. Beazer [2] and Davey [11] have each independently obtained a version of this result. Grätzer and Lakser in [16] (also see [13, Lemma-2 Page-169]) have shown that condition (iii) of the following theorem is equivalent to Lee's condition which characterizes the nth variety, for $0<n<\omega$, of distributive pseudocomplemented lattices. Thus, this theorem should be compared with Lee's Theorem 2 of [36].

Recall that for a prime ideal P of a distributive lattice L,

$0(P)=\{x : x\wedge y=0 \text{ for some } y\in L-P\}$, which is an ideal contained in P.

Theorem 4.1.8. *Let* L *be a distributive lattice. Then the following conditions are equivalent:*

(i) *For any* m+1 *distinct minimal prime ideals*

$P_0, P_1, \text{--------}, P_m$; $P_0\vee P_1\vee\text{---------}\vee P_m=L$;

(ii) *Every prime ideal contains at most* m *minimal prime ideals* ;

(iii) *For any* $x_0, x_1, \text{-------}, x_m\in L$ *such that* $x_i\wedge x_j=0$

for $(i\neq j)$, $i=0, 1, \text{--------}, m$, $j=0, 1, \text{---------}, m$

$(x_0]^* \vee (x_1]^* \vee \text{--------} \vee (x_m]^* = L$;

(iv) *For each prime ideal* P, 0(P) *is* m+1-*prime* ;

(v) *If* L *is (sectionally) pseudocomplemented, then* L *is (sectionally) in* B_m. □

Our next result is a nice extension of above result in terms of n-ideals.

Theorem 4.1.9. *Let* L *be a distributive lattice. Then the following conditions are equivalent*:

(i) *For any* m+1 *distinct minimal prime* n-*ideals* $P_0, P_1, \text{---------}, P_m$, $P_0 \vee P_1 \vee \text{---------} \vee P_m = L$;

(ii) *Every prime* n-*ideal contains at most* m-*minimal prime* n-*ideals* ;

(iii) *For any* $a_0, a_1, \text{--------}, a_m \in L$ *with* $m(a_i, n, a_j)=n$, $(i\neq j)$ $i=0, 1, \text{-------}, m$, $j=0, 1, \text{---------}, m$, $\langle a_0\rangle_n^+ \vee \langle a_1\rangle_n^+ \vee \text{---------} \vee \langle a_m\rangle_n^+ = L$;

(iv) *For each prime* n-*ideal* P, n(P) *is an* m+1-*prime* n-*ideal*.

Proof: (i)⇒(ii), (ii)⇒(iii) and (iii)⇒(i) easily hold by Theorem 4.1.7 replacing J by {n}. To complete the proof we need to show that (iv)⇒(iii) and (ii)⇒(iv).

(iv)⇒(iii). Suppose (iv) holds and $x_0, x_1, \text{--------}, x_m$ are m+1 elements of L such that $m(x_i, n, x_j)=n$ for $(i\neq j)$.

Suppose that $<x_0>_n^+ \vee <x_1>_n^+ \vee \text{---------} \vee <x_m>_n^+ \neq L$. Then by Theorem 1.2.9, there is a prime n-ideal P such that $<x_0>_n^+ \vee <x_1>_n^+ \vee \text{---------} \vee <x_m>_n^+ \subseteq P$.
Hence $x_0, x_1, \text{---------}, x_m \in L-n(P)$. This contradicts (iv) by Lemma 4.1.6, since $m(x_i, n, x_j)=n \in n(P)$ for all $i \neq j$. Thus, (iii) holds.

(ii)$\Rightarrow$(iv). This follows immediately from Proposition 2.2.10 and Lemma 4.1.6 above.

Following result is due to [8].

Proposition 4.1.10. *Let* L *be a distributive lattice with* 0. *If the equivalent conditions of Theorem* 4.1.8 *hold, then for any* m+1 *elements* $x_0, x_1, \text{--------}, x_m$,

$$(x_0 \wedge x_1 \wedge \text{----} \wedge x_m]^* = \bigvee_{0 \le i \le n} (x_0 \wedge x_1 \wedge \text{----} \wedge x_{i-1} \wedge x_{i+1} \wedge \text{----} \wedge x_m]^*. \quad \square$$

Proposition 4.1.11. *Let* L *be a distributive lattice and* $n \in L$. *If the equivalent conditions of Theorem* 4.1.9 *hold, then for any* m+1 *elements* $x_0, x_1, \text{---------}, x_m$;

$$(<x_0>_n \cap <x_1>_n \cap \text{--------} \cap <x_m>_n)^+$$

$$= \bigvee_{0 \le i \le n} (<x_0>_n \cap \text{--------} \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap \text{---------} \cap <x_m>_n)^+.$$

Proof: Let $<b_i>_n = <x_0>_n \cap \text{----------} \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap \text{------------} \cap <x_m>_n$ for each $0 \le i \le m$.

Suppose $x \in (\langle x_0 \rangle_n \cap \text{---------} \cap \langle x_m \rangle_n)^+$. Then

$\langle x \rangle_n \cap \langle x_0 \rangle_n \cap \text{---------} \cap \langle x_m \rangle_n = \{n\}$. For all $i \neq j$;

$(\langle x \rangle_n \cap \langle b_i \rangle_n) \cap (\langle x \rangle_n \cap \langle b_j \rangle_n) = \{n\}$.

So $(\langle x \rangle_n \cap \langle b_0 \rangle_n)^+ \vee \text{---------} \vee (\langle x \rangle_n \cap \langle b_m \rangle_n)^+ = L$.

Thus $x \in (\langle x \rangle_n \cap \langle b_0 \rangle_n)^+ \vee \text{---------} \vee (\langle x \rangle_n \cap \langle b_m \rangle_n)^+$.

Hence by Theorem 1.1.12, $x \vee n = a_0 \vee \text{---------} \vee a_m$ where $a_i \in (\langle x \rangle_n \cap \langle b_i \rangle_n)^+$ and $a_i \geq n$, for $i = 0, 1, \text{---------}, m$.

Then $x \vee n = (a_0 \wedge (x \vee n)) \vee \text{---------} \vee (a_m \wedge (x \vee n))$.

Now $a_i \in (\langle x \rangle_n \cap \langle b_i \rangle_n)^+$ implies $\langle a_i \rangle_n \cap \langle x \rangle_n \cap \langle b_i \rangle_n = \{n\}$.

Then by a routine calculation we find that $(a_i \wedge x \wedge b_i) \vee n = n$.

Thus, $\langle a_i \wedge (x \vee n) \rangle_n \cap \langle b_i \rangle_n = [n, (a_i \wedge x \wedge b_i) \vee n] = \{n\}$ implies that $a_i \wedge (x \vee n) \in \langle b_i \rangle_n^+$ and so $x \vee n \in \langle b_0 \rangle_n^+ \vee \text{---------} \vee \langle b_m \rangle_n^+$.

By a dual proof of above, we can easily show that $x \wedge n \in \langle b_0 \rangle_n^+ \vee \text{-------------} \vee \langle b_m \rangle_n^+$. Thus by convexity, $x \in \langle b_0 \rangle_n^+ \vee \text{---------} \vee \langle b_m \rangle_n^+$. This proves that L.H.S$\subseteq$R.H.S.

The reverse inclusion is trivial. $\square$

Theorem 4.1.12. *For a distributive lattice* L, *if* $F_n(L)$ *is sectionally pseudocomplemented then the following conditions are equivalent*:

(i) $F_n(L)$ *is sectionally in* B_m ;

(ii) *For* $a_0, \text{--------}, a_m$ *with* $m(a_i, n, a_j) = n$ $(i \neq j)$ *implies*

$\langle a_0 \rangle_n^+ \vee \text{---------} \vee \langle a_m \rangle_n^+ = L$.

Proof: (i)$\Rightarrow$(ii). Suppose $t \in L$, $\langle a_0 \rangle_n, \text{---------}, \langle a_m \rangle_n$ with $m(a_i, n, a_j) = n$, for all $i \neq j$. Consider the interval

$[\{n\}, <t>_n]$. Then

$\{n\}\subseteq<m(a_0, n, t)>_n$,-----------,$<m(a_m, n, t)>_n\subseteq<t>_n$, and

$<m(a_i, n, t)>_n\cap<m(a_j, n, t)>_n$

$=<a_i>_n\cap<t>_n\cap<a_j>_n\cap<t>_n$

$=\{n\}$.

Thus, $<m(a_i, n, t)>_n\subseteq<m(a_j, n, t)>_n^0$, for all $i\neq j$. Therefore,

$<m(a_0, n, t)>_n\subseteq<m(a_1, n, t)>_n^0\cap$--------$\cap<m(a_m, n, t)>_n^0$,

$<m(a_1, n, t)>_n\subseteq<m(a_1, n, t)>_n^{00}\cap<m(a_2, n, t)>_n^0\cap$-------$\cap<m(a_m, n, t)>_n^0$.

$<m(a_m, n, t)>_n\subseteq<m(a_1, n, t)>_n^0\cap$-------$\cap<m(a_m, n, t)>_n^{00}$.

Since $F_n(L)$ is sectionally in B_m, so applying Lee's identity to $<m(a_1, n, t)>_n^0$,------,$<m(a_m, n, t)>_n^0$ we obtain

$<m(a_0, n, t)>_n^0\vee$----------$\vee<m(a_m, n, t)>_n^0\supseteq(<m(a_1, n, t)>_n^0$

$\cap$-----------$\cap<m(a_m, n, t)>_n^0)^0\vee(<m(a_1, n, t)>_n^{00}\cap$-----------

$\cap<m(a_m, n, t)>_n^0)^0\vee$----------------$\vee(<m(a_1, n, t)>_n^0\cap$---------

----$\cap<m(a_m, n, t)>_n^{00})^0=<t>_n$.

Therefore $<t>_n=[<m(a_0, n, t)>_n^0\vee$--------$\vee<m(a_m, n, t)>_n^0]\cap<t>_n$

$=(<m(a_0, n, t)>_n^+\cap<t>_n)\vee$-----------$\vee(<m(a_m, n, t)>_n^+\cap<t>_n)$

(by Lemma 2.1.3).

$=((<a_0>_n\cap<t>_n)^+\cap<t>_n)\vee$--------$\vee((<a_m>_n\cap<t>_n)^+\cap<t>_n)$

$=(<a_0>_n^+\cap<t>_n)\vee$--------$\vee(<a_m>_n^+\cap<t>_n)$ (by Lemma 2.1.2)

$=(<a_0>_n^+\vee$------------$\vee<a_m>_n^+)\cap<t>_n$. This implies

$t\in<a_0>_n^+\vee$-------$\vee<a_m>_n^+$, and so $<a_0>_n^+\vee$-------$\vee<a_m>_n^+=L$.

(ii)$\Rightarrow$(i). Consider the interval $[n, d]$.

Let $x_1, \text{--------}, x_m \in [n, d]$. $x_1^0, x_2^0, \text{---------}, x_m^0$ denotes the relative pseudocomplements of $x_1, \text{---------}, x_m$ in $[n, d]$.

Let $b_0 = x_1 \wedge \text{---------------} \wedge x_m$

$b_1 = x_1^0 \wedge \text{--------------} \wedge x_m$

$b_2 = x_1 \wedge x_2^0 \wedge \text{-----------} \wedge x_m$

$b_m = x_1 \wedge x_2 \wedge \text{-----------} \wedge x_m^0$.

Then $b_i \wedge b_j = n$ for all $i \neq j$. That is $\langle b_i \rangle_n \cap \langle b_j \rangle_n = \{n\}$.

Hence by (ii), $\langle b_0 \rangle_n^+ \vee \text{------------------} \vee \langle b_m \rangle_n^+ = L$.

So $\langle d \rangle_n \cap L = (\langle b_0 \rangle_n^+ \cap \langle d \rangle_n) \vee \text{-------------} \vee (\langle b_m \rangle_n^+ \cap \langle d \rangle_n)$

Thus by Lemma 2.1.3 and Corollary 2.2.3,

$$[n, d] = \langle b_0 \rangle_n^0 \vee \text{---------------} \vee \langle b_m \rangle_n^0$$

$$= \langle x_1 \wedge \text{----------} \wedge x_m \rangle_n^0 \vee \langle x_1^0 \wedge \text{--------} \wedge x_m \rangle_n^0$$

$$\vee \text{-------------} \vee \langle x_1 \wedge x_2 \wedge \text{-------------} \wedge x_m^0 \rangle_n^0$$

$$= [n, (x_1 \wedge \text{---------} \wedge x_m)^0] \vee [n, (x_1^0 \wedge x_2 \wedge \text{----------} \wedge x_m)^0]$$

$$\vee \text{---------------} \vee [n, (x_1 \wedge x_2 \wedge \text{-------------} \wedge x_m^0)^0].$$

Thus, $[n, d] = [n, (x_1 \wedge x_2 \wedge \text{------} \wedge x_m)^0 \vee (x_1^0 \wedge x_2 \wedge \text{-------} \wedge x_m)^0$

$\vee \text{-----------} \vee (x_1 \wedge x_2 \wedge \text{--------} \wedge x_m^0)^0]$.

This implies $d = (x_1 \wedge x_2 \wedge \text{---------} \wedge x_m)^0 \vee (x_1^0 \wedge x_2 \wedge \text{-------------}$ $\text{---} \wedge x_m)^0 \vee \text{-------------} \vee (x_1 \wedge x_2 \wedge \text{------------} \wedge x_m^0)^0$, which is Lee's identity.

Therefore, [n) is sectionally in B_m. A dual proof of above shows that (n] is sectionally in dual B_m. Therefore by Theorem 1.1.2, $F_n(L)$ is sectionally in B_m. ∎

For a pseudocomplemented lattice L, we write $S(L)=\{a^* : a\in L\}$, which is known as the skeleton of L. We know that S(L) is a Boolean lattice, but it is not necessarily a sublattice of L. It is well known that S(L) is a subalgebra of L if and only if L is a Stone algebra.

We have already mentioned that if $0, 1\in L$, then $L=[0, 1]$ is the largest element of $F_n(L)$, and so $F_n(L)$ is a bounded lattice. Also we know that $F_n(L)$ is distributive if and only if L is distributive, so we have:

Theorem 4.1.13. *For a distributive lattice* L *with* 0 *and* 1, *if* $F_n(L)$ *is pseudocomplemented then the following are equivalent*:

(i) $F_n(L)$ *is in* B_m ;

(ii) *For* a_0,--------,a_m, *with* $m(a_i, n, a_j)=n$ $(i\neq j)$ *implies* $\langle a_0\rangle_n^+\vee$--------$\vee\langle a_m\rangle_n^+=L$;

(iii) $m(a_i, n, a_j)=n$, $(i\neq j)$ $i, j=0, 1$,---------,m *such that* $\langle a_0\rangle_n$,-----------,$\langle a_m\rangle_n\in S(F_n(L))$ *then* $\langle a_0\rangle_n^+\vee$-----------$\vee\langle a_m\rangle_n^+=L$.

Proof: (i)⇒(ii) is trivial by above theorem.

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i).

Let $<b_0>_n=<a_1>_n\cap$------------------$\cap<a_m>_n$

$<b_1>_n=<a_1>_n^+\cap$-----------------$\cap<a_m>_n$

$<b_2>_n=<a_1>_n\cap<a_2>_n^+\cap$---------$\cap<a_m>_n$

$<b_m>_n=<a_1>_n\cap$---------------$\cap<a_m>_n^+$.

These intersections are principal n-ideals as we know that any finitely generated n-ideal contained in a principal n-ideal is principal. Hence we also have $<b_i>_n\cap<b_j>_n=\{n\}$, for all $i\neq j$. So, $(<b_i>_n\cap<b_j>_n)^{++}=<b_i>_n^{++}\cap<b_j>_n^{++}=\{n\}$, for all $i\neq j$ and $<b_0>_n^{++}$,-----------,$<b_m>_n^{++}\in S(F_n(L))$.

Thus by (iii), $<b_0>_n^+\vee$-------------$\vee<b_m>_n^+=L$.

That is $(<a_1>_n\cap$--------------$\cap<a_m>_n)^+\vee$----------$\vee(<a_1>_n\cap$-------------$\cap<a_m>_n^+)^+=L$, which is Lee's identity. That is, $F_n(L)$ is in B_m. □

2. Generalizations of some results on relatively B_m-lattices.

Several characterizations on relative B_m-lattices have been given by Davey in [11]. Also Cornish have studied these lattices in [8] under the name of relatively n-normal lattices.

Recall that a lattice L is *relatively* in B_m if its every interval [a, b] (a, b $\in$ L a<b) is in B_m.

Following result gives some characterizations of $F_n(L)$ which are relatively in B_m, which is a generalization of [11, Theorem-3.4].

Theorem 4.2.1. *Let* L *be a distributive lattice with* n$\in$L. *Suppose* $F_n(L)$ *is relatively pseudocomplemented. Then the following conditions are equivalent*:

(i) $F_n(L)$ *is relatively in* B_m ;

(ii) *For all* $x_0, x_1, \ldots, x_m \in L$

$$<<x_1>_n \cap <x_2>_n \cap \cdots \cap <x_m>_n, <x_0>_n>$$

$$\vee <<x_0>_n \cap <x_2>_n \cap \cdots \cap <x_m>_n, <x_1>_n>$$

$$\vee \cdots \vee <<x_0>_n \cap <x_1>_n \cap \cdots \cap <x_{m-1}>_n, <x_m>_n> = L ;$$

(iii) *For all* $x_0, x_1, \ldots, x_m, z \in L$,

$<<x_0>_n \cap <x_1>_n \cap \text{-----------------} \cap <x_m>_n,\ <z>_n>$

$=<<x_1>_n \cap \text{-------------------------} \cap <x_m>_n,\ <z>_n>$

$\vee <<x_0>_n \cap <x_2>_n \cap \text{-----------------} \cap <x_m>_n,\ <z>_n>$

$\vee \text{----------} \vee <<x_0>_n \cap <x_1>_n \text{-------} \cap <x_{m-1}>_n,\ <z>_n>$.

(iv) *For any* m+1 *pairwise incomparable prime* n-*ideals* P_0, P_1,---------,P_m, $P_0 \vee \text{---------} \vee P_m = L$.

(v) *Any prime* n-*ideal contains at most* m *mutually incomparable prime* n-*ideals.*

Proof: (i)⇒(ii). Let $z \in L$, consider the interval $I=[<x_0>_n \cap <x_1>_n \cap \text{---------} \cap <x_m>_n \cap <z>_n,\ <z>_n]$ in $F_n(L)$. Then $<x_0>_n \cap <x_1>_n \cap \text{------------} \cap <x_m>_n \cap <z>_n$ is the smallest element of the interval I. For $0 \le i < m$, the set of elements $<t_i>_n = <x_0>_n \cap <x_1>_n \cap \text{------------} \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap \text{-------} \cap <x_m>_n \cap <z>_n$ are obviously pairwise disjoint in the interval I. Since I is in B_m. Then by Theorem 4.1.13, $<t_0>_n^{\,0} \vee \text{----------} \vee <t_m>_n^{\,0} = <z>_n$. So by Theorem 1.1.12,

$z \vee n = P_0 \vee \text{---------------------} \vee P_m$ where $P_i \ge n$.

Thus, $<P_0>_n \cap <t_0>_n = <P_1>_n \cap <t_1>_n = \text{------} = <P_m>_n \cap <t_m>_n$

= The smallest element of I

$= <x_0>_n \cap <x_1>_n \cap \text{-----------------} \cap <x_m>_n \cap <z>_n$.

Now, $<P_0>_n \cap <t_0>_n = <x_0>_n \cap <x_1>_n \cap \text{-----------} \cap <x_m>_n \cap <z>_n$ which implies $<P_0>_n \cap <t_0>_n \subseteq <x_0>_n$.

Again, $<P_0>_n \cap <t_0>_n = <P_0>_n \cap <x_1>_n \cap \text{---------} \cap <x_m>_n \cap <z>_n$

$=\langle P_0\rangle_n \cap \langle x_1\rangle_n \cap \text{------------------} \cap \langle x_m\rangle_n$, as $\langle P_0\rangle_n \subseteq \langle z\rangle_n$.

This implies, $\langle P_0\rangle_n \cap \langle x_1\rangle_n \cap \text{--------------} \cap \langle x_m\rangle_n \subseteq \langle x_0\rangle_n$

and so, $\langle P_0\rangle_n \in \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \text{---------} \cap \langle x_m\rangle_n,\ \langle x_0\rangle_n\rangle$

$\langle P_1\rangle_n \in \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \text{------------} \cap \langle x_m\rangle_n,\ \langle x_1\rangle_n\rangle$

$\langle P_m\rangle_n \in \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{---------} \cap \langle x_{m-1}\rangle_n,\ \langle x_m\rangle_n\rangle$.

Therefore, $z \vee n \subseteq \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \text{-------} \cap \langle x_m\rangle_n,\ \langle x_0\rangle_n\rangle$

$\vee \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \text{----------------------} \cap \langle x_m\rangle_n,\ \langle x_1\rangle_n\rangle$

$\vee \text{------------} \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{--------} \cap \langle x_{m-1}\rangle_n,\ \langle x_m\rangle_n\rangle$.

By a dual proof of above we can easily show that

$z \wedge n \subseteq \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \text{-------------------} \cap \langle x_m\rangle_n,\ \langle x_0\rangle_n\rangle$

$\vee \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \text{---------------------} \cap \langle x_m\rangle_n,\ \langle x_1\rangle_n\rangle$

$\vee \text{-----------} \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{-------} \cap \langle x_{m-1}\rangle_n,\ \langle x_m\rangle_n\rangle$.

Hence by convexity,

$z \in \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \text{-------------------} \cap \langle x_m\rangle_n,\ \langle x_0\rangle_n\rangle$

$\vee \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \text{--------------------} \cap \langle x_m\rangle_n,\ \langle x_1\rangle_n\rangle$

$\vee \text{-------} \vee \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{----------} \cap \langle x_{m-1}\rangle_n,\ \langle x_m\rangle_n\rangle$.

This implies (ii) holds.

(ii)$\Rightarrow$(iii). Suppose

$b \in \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{-------} \cap \langle x_m\rangle_n,\ \langle z\rangle_n\rangle$. Then by (ii) and Theorem 1.1.12, $b \vee n = s_0 \vee s_1 \vee \text{---------} \vee s_m$, for some

$s_0 \in \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \text{-------------} \cap \langle x_m\rangle_n,\ \langle x_0\rangle_n\rangle$

$s_1 \in \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \text{-------------} \cap \langle x_m\rangle_n,\ \langle x_1\rangle_n\rangle$

- -

- -

$s_m \in \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{-------------} \cap \langle x_{m-1}\rangle_n,\ \langle x_m\rangle_n\rangle$

and $s_i \geq n$, $i=0, 1, \text{---------------}, m$.

Thus, $\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \text{------------} \cap \langle x_m\rangle_n \cap \langle s_0\rangle_n \subseteq \langle x_0\rangle_n$

$\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \text{-------------} \cap \langle x_m\rangle_n \cap \langle s_1\rangle_n \subseteq \langle x_1\rangle_n$

$\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{----------} \cap \langle x_{m-1}\rangle_n \cap \langle s_m\rangle_n \subseteq \langle x_m\rangle_n$.

This implies $\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \text{-----------} \cap \langle x_m\rangle_n \cap \langle s_0\rangle_n$

$= \langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{------------------} \cap \langle x_m\rangle_n \cap \langle s_0\rangle_n$

$\subseteq \langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{----------} \cap \langle x_m\rangle_n \cap \langle b \vee n\rangle_n \subseteq \langle z\rangle_n$.

Hence, $s_0 \in \langle\langle x_1\rangle_n \cap \langle x_2\rangle_n \cap \text{-----------} \cap \langle x_m\rangle_n,\ \langle z\rangle_n\rangle$.

Similarly, $s_1 \in \langle\langle x_0\rangle_n \cap \langle x_2\rangle_n \cap \text{--------} \cap \langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\langle s_m\rangle_n \in \langle\langle x_0\rangle_n \cap \langle x_1\rangle_n \cap \text{---------} \cap \langle x_{m-1}\rangle_n,\ \langle z\rangle_n\rangle$.

Therefore, $b\vee n\in \langle\langle x_1\rangle_n\cap\langle x_2\rangle_n\cap\text{-------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\vee\langle\langle x_0\rangle_n\cap\langle x_2\rangle_n\cap\text{-------------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\vee\ \text{-----}\vee\langle\langle x_0\rangle_n\cap\langle x_1\rangle_n\cap\text{----}\cap\langle x_{m-1}\rangle_n,\ \langle z\rangle_n\rangle.$

The dual proof of above gives

$b\wedge n\in \langle\langle x_1\rangle_n\cap\langle x_2\rangle_n\cap\text{-------------------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\vee\langle\langle x_0\rangle_n\cap\langle x_2\rangle_n\cap\text{------------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle\vee\text{----}$

$\text{-------}\vee\langle\langle x_0\rangle_n\cap\langle x_1\rangle_n\cap\text{--------}\cap\langle x_{m-1}\rangle_n,\ \langle z\rangle_n\rangle.$

Thus by convexity,

$b\in \langle\langle x_1\rangle_n\cap\langle x_2\rangle_n\cap\text{--------------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\vee\langle\langle x_0\rangle_n\cap\langle x_2\rangle_n\cap\text{---------------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\vee\text{------}\vee\langle\langle x_0\rangle_n\cap\langle x_1\rangle_n\cap\text{-------}\cap\langle x_{m-1}\rangle_n,\ \langle z\rangle_n\rangle.$

Therefore, $\langle\langle x_0\rangle_n\cap\langle x_1\rangle_n\cap\text{------------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\subseteq\langle\langle x_1\rangle_n\cap\langle x_2\rangle_n\cap\text{----------------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\vee\langle\langle x_0\rangle_n\cap\langle x_2\rangle_n\cap\text{----------------}\cap\langle x_m\rangle_n,\ \langle z\rangle_n\rangle$

$\vee\text{-------}\vee\langle\langle x_0\rangle_n\cap\langle x_1\rangle_n\cap\text{------}\cap\langle x_{m-1}\rangle_n,\ \langle z\rangle_n\rangle.$

Since the reverse inequality always holds, so (iii) holds.

(iii)$\Rightarrow$(i). Suppose, $n\leq b\leq d$.

Let x_0, x_1,---------,$x_m\in[b, d]$ such that $x_i\wedge x_j=b$, for all $i\neq j$.

Let $t_0=x_1\vee x_2\vee\text{--------}\vee x_m$

$t_1=x_0\vee x_2\vee\text{--------}\vee x_m$

$t_m=x_0\vee x_1\vee\text{--------}\vee x_{m-1}$

clerly, $n\leq b\leq t_i\leq d$ and

$x_0=t_1\wedge t_2\wedge\text{--------}\wedge t_m$

$x_1=t_0\wedge t_2\wedge\text{--------}\wedge t_m$

$x_m=t_0\wedge t_1\wedge\text{--------}\wedge t_{m-1}$.

Then $[b, d]\cap\{<<x_0>_n, <b>_n>\vee\text{-------}\vee<<x_m>_n, <b>_n>\}$

$=[b, d]\cap\{<<t_1>_n\cap<t_2>_n\cap\text{------------}\cap<t_m>, <b>_n>$

$\vee<<t_0>_n\cap<t_2>_n\cap\text{--------------------}\cap<t_m>_n, <b>_n>$

$\vee\text{-----}\vee<<t_0>_n\cap<t_1>_n>\vee\text{-------}\vee<<t_{m-1}>_n, <b>_n>\}$

$=[b, d]\cap\{<<t_0>_n\cap<t_1>_n\cap\text{--------}\cap<t_m>_n, <b>_n>\}$

$=[b, d]\cap<<b>_n, <b>_n>=[b, d]\cap L=[b, d]$, that is,

$[b, d]$ is in B_m. Hence, $[n)$ is relatively in B_m.

A dual proof of above shows that $(n]$ is relatively in dual B_m. Since $F_n(L)\cong(n]^d\times[n)$ so, $F_n(L)$ is relatively in B_m.

(ii)⇒(iv). Suppose (ii) holds. Let $P_0, P_1, \text{---------}, P_m$ be m+1 pairwise incomparable prime n-ideals. Then, there exist $x_0, x_1, \text{----------------}, x_m \in L$ such that

$x_i \in P_j - \bigcup_{\substack{i=1 \\ i\neq j}}^{n} P_i$. Then by (ii),

$$<<x_1>_n \cap <x_2>_n \cap \text{-------------------} \cap <x_m>_n,\ <x_0>_n>$$

$$\vee <<x_0>_n \cap <x_2>_n \cap \text{-----------------} \cap <x_m>_n,\ <x_1>_n>$$

$$\vee \text{-----} \vee <<x_0>_n \cap <x_1>_n \cap \text{------} \cap <x_{m-1}>_n,\ <x_m>_n> = L.$$

Let $t_0 \in <<x_1>_n \cap <x_2>_n \cap \text{----------------} \cap <x_m>_n,\ <x_0>_n>$,

then, $<t_0>_n \cap <x_1>_n \cap <x_2>_n \cap \text{-----------} \cap <x_m>_n \subseteq <x_0>_n \subseteq P_0$.

Now, $x_i \notin P_0$, for $i=1, 2, \text{-------------------}, m$ implies that $<x_i>_n \not\subseteq P_0$ for $i=1, 2, \text{-----------------------------}, m$. Thus $<x_1>_n \cap <x_2>_n \cap \text{-----------------} \cap <x_m>_n \not\subseteq P_0$ as P_0 is prime. This implies $<t_0>_n \subseteq P_0$, and so $t_0 \in P_0$.

Therefore, $<<x_1>_n \cap <x_2>_n \cap \text{--------} \cap <x_m>_n,\ <x_0>_n> \subseteq P_0$.

Similarly, $<<x_0>_n \cap <x_2>_n \cap \text{---------} \cap <x_m>_n,\ <x_1>_n> \subseteq P_1$

$$<<x_0>_n \cap <x_1>_n \cap \text{---------} \cap <x_m>_n,\ <x_2>_n> \subseteq P_2$$

$$<<x_0>_n \cap <x_1>_n \cap \text{------} \cap <x_{m-1}>_n,\ <x_m>_n> \subseteq P_m.$$

Hence $P_0 \vee P_1 \vee \text{----------} \vee P_m = L$.

(iv)$\Leftrightarrow$(v) is trivial by Stone's representation theorem.

(iv)$\Rightarrow$(i). Let any m+1 pairwise incomparable prime n-ideals of L are comaximal. Consider the interval [b, d] in L with b, d$\geq$n, let P_0', P_1',--------,P_m' be m+1 distinct minimal prime ideals of [b, d]. Then by Lemma 3.2.9, there exist prime ideals P_0,--------,P_m of L such that $P_0'=P_0\cap[b, d]$-------$P_m'=P_m\cap[b, d]$. Since each P_i is an ideal, so $b\in P_i$. Moreover, $n\leq b$ implies that $n\in P_i$. Therefore each P_i is a prime n-ideal by Lemma 1.2.5. i=0, 1,-------,m. Since P_0',---------,P_m' are incomparable, so P_0,---------,P_m are also incomparable. Now by (iv), $P_0\vee$--------$\vee P_m=L$. Hence $P_0'\vee$--------$\vee P_m'$ $=(P_0\vee$--------$\vee P_m)\cap[b, d]=L\cap[b, d]=[b, d]$. Therefore by Theorem 4.1.8, [b, d] is in B_m. Hence [n) is relatively in B_m.

A dual proof of above shows that (n] is relatively in dual B_m. Since $F_n(L)\cong(n]^d\times[n)$, so $F_n(L)$ is relatively in B_m. □

We conclude this chapter with the following result which is also a generalization of [11, Theorem-3.4].

Theorem 4.2.2. *Let* L *be a distributive lattice with* $n\in L$. *Suppose* $F_n(L)$ *is relatively pseudocomplemented. Then the following conditions are equivalent*:

(i) $F_n(L)$ *is relatively in* B_m ;

(ii) *If* $b, a_0, a_1, \text{----------}, a_m \in L$ *with* $m(a_i, n, a_j) \in \langle b\rangle_n$ $(i \neq j)$, *then* $\langle\langle a_0\rangle_n, \langle b\rangle_n\rangle \vee \text{-------} \vee \langle\langle a_m\rangle_n, \langle b\rangle_n\rangle = L$.

Proof: (i)⇒(ii).

By Theorem 4.2.1(v), any prime n-ideal containing b contains at most m minimal prime n-ideals belonging to $\langle b\rangle_n$. Hence by Theorem 4.1.7 with $J=\langle b\rangle_n$, we have

$\langle\langle a_0\rangle_n, \langle b\rangle_n\rangle \vee \text{------} \vee \langle\langle a_m\rangle_n, \langle b\rangle_n\rangle = L$. Thus (ii) holds.

(ii)⇒(i). Consider $b, c \in [n)$ with $b \leq c$. Let $a_0, \text{-----}, a_m \in [b, c]$ with $a_i \wedge a_j = b$ $(i \neq j)$ then by $m(a_i, n, a_j) = b \in \langle b\rangle_n$. Then by (ii),

$\langle\langle a_0\rangle_n, \langle b\rangle_n\rangle \vee \text{-----------} \vee \langle\langle a_m\rangle_n, \langle b\rangle_n\rangle = L$. So,

$$[b, c] = (\langle\langle a_0\rangle_n, \langle b\rangle_n\rangle \cap [b, c]) \vee \text{-----} \vee (\langle\langle a_m\rangle_n, \langle b\rangle_n\rangle \cap [b, c])$$

$$= \langle a_0, b\rangle_{[b, c]} \vee \text{----------} \vee \langle a_m, b\rangle_{[b, c]}.$$

Hence by Theorem 4.1.8, $[b, c]$ is in B_m.

Therefore $[n)$ is relatively in B_m.

A dual proof of above shows that $(n]$ is relatively in dual B_m. Therefore, by Theorem 1.1.2, $F_n(L)$ is relatively in B_m. □

Chapter-5

Distributive and modular n-ideals of a lattice.

Introduction:

The notion of standard n-ideals of a lattice was introduced by Noor and Latif in [49]. Then they studied those n-ideals extensively and included several properties in [50] and [51]. Moreover, in [35] Latif has generalized isomorphism theorems for standard ideals in terms of n-ideals. In this section we give a notion of distributive and modular n-ideals of a lattice.

An n-ideal S of a lattice L is called a standard n-ideal if it is a standard element of the lattice $I_n(L)$. That is, S is called standard if for all I, $J \in I_n(L)$, $I \cap (S \vee J) = (I \cap S) \vee (I \cap J)$.

Distributive elements and ideals were studied extensively by Grätzer and Schmidt in [18]; also see [14]. On the other hand, [56] have studied the distributive elements and ideals in join semilattices which are directed below.

An element d of a lattice L is called *distributive* if for all x, $y \in L$, $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$. An ideal I is called *distributive* if it is a distributive element of the ideal lattice I(L).

In [59] and [60], Talukder and Noor have given the notion of a modular element and a modular ideal of a lattice. According to them, an element m of a lattice L is called *modular* if for all x, $y \in L$ with $y \leq x$, $x \wedge (m \vee y) = (x \wedge m) \vee y$. An ideal of L is called *modular* if it is a modular element of I(L). In [59], [60] authors have given several characterizations of modular elements and ideals of a lattice. On the other hand, Malliah and Bhatta in [38] have called an element m of a lattice modular, if for all x, $y \in L$ with $x \leq y$, $x \wedge m = y \wedge m$ and $x \vee m = y \vee m$ imply that $x = y$. It is very easy to see that both the definitions are equivalent. [59] have also shown that an element s is standard if and only if it is both distributive and modular.

Recall from chapter 1 that an element $s \in L$ is *standard* if for all x, $y \in L$, $x \wedge (s \vee y) = (x \wedge s) \vee (x \wedge y)$. An element $n \in L$ is called *neutral* if it is *standard* and for all x, $y \in L$, $n \wedge (x \vee y) = (n \wedge x) \vee (n \wedge y)$ that is, n is *dual distributive*.

In this connection it should be mentioned that Grätzer in [14] posed an open problem "generalize the concept of standard, distributive and neutral ideals to convex sublattices". Fried and Schmidt in [12] have given a neat description of standard convex sublattices. Neiminen in [40] have tried to give some descriptions on distributive and neutral convex sublattices. But some of his results are completely wrong which we do not wish to mention here, as it is beyond the scope of this thesis. On the other hand, Malliah and Bhatta [38] have given the concept of D-sublattices which is a nice generalization of distributive ideals to convex sublattices. They have also introduced the notion of M-sublattices which generalize the notion of modular ideals. Recently Noor and Rahman in [46], [47], have given new definitions of distributive and modular convex sublattices. Since the n-ideals are also convex sublattices, the notion of distributive and modular n-ideals easily follow from above notion as a particular case.

In section 1 of this chapter we introduced the concept of distributive n-ideals of a lattice. Then we have given several characterizations of it. For a distributive n-ideal I of a lattice L we have also given a definition of $\Theta(I)$, the congruence generated by I. We have shown that for a neutral element n of a lattice L, the principal n-ideal $\langle a \rangle_n$ is distributive if and only if $a \wedge n$ is dual distributive and $a \vee n$ is distributive.

Section 2 discusses the modular n-ideals with its several properties. Here we included several characterizations of modular n-ideals. We have proved some results similar to the results on standard n-ideals in

[49] and [50]. We have also proved that for a neutral element n, if for a modular n-ideal M and arbitrary n-ideal I, both $I \cap M$ and $I \vee M$ are principal, then I itself is principal.

Finally we have discussed some of the properties of standard and neutral n-ideals in section 3. We conclude the section by showing that for a neutral element n, the lattice of standard n-ideals is isomorphic to the lattice of standard n-congruences.

1. Distributive n-ideals of a lattice.

Recall that an n-ideal I of a lattice L is a distributive n-ideal if it is a distributive element of the lattice $I_n(L)$. That is, I is called *distributive* if for all J, $K \in I_n(L)$,

$I \vee (J \cap K) = (I \vee J) \cap (I \vee K)$.

We start this section with the following characterization of distributive n-ideal.

Theorem 5.1.1. *An* n-*ideal* I *of a lattice* L *is distributive if and only if*

$I \vee (<a>_n \cap <b>_n) = (I \vee <a>_n) \cap (I \vee <b>_n)$ *for all* a, $b \in L$.

Proof: If I is distributive, then the condition clearly holds from the definition. To prove the converse, suppose given equation holds for all a, $b \in L$. Let J and K be any two n-ideals of L. Obviously $I \vee (J \cap K) \subseteq (I \vee J) \cap (I \vee K)$. To prove the reverse inclusion, let $x \in (I \vee J) \cap (I \vee K)$. Then $x \in I \vee J$ and $x \in I \vee K$. Then $i_1 \wedge j_1 \leq x \leq i_2 \vee j_2$ and $i_3 \wedge k_3 \leq x \leq i_4 \vee k_4$ for some i_1, i_2, i_3, $i_4 \in I$, j_1, $j_2 \in J$ and k_3, $k_4 \in K$. Now $n \leq x \vee n \leq i_2 \vee j_2 \vee n$ implies that $x \vee n \in I \vee <j_2 \vee n>_n$. Similarly $n \leq x \vee n \leq i_4 \vee k_4 \vee n$ implies that $x \vee n \in I \vee <k_4 \vee n>_n$.

Thus, $x \vee n \in (I \vee <j_2 \vee n>_n) \cap (I \vee <k_4 \vee n>_n)$

$$= I \vee (<j_2 \vee n>_n \cap <k_4 \vee n>_n) \subseteq I \vee (J \cap k).$$

By a dual proof of above, we can show that $x\wedge n\in I\vee(J\cap K)$. Thus by convexity, $x\in I\vee(J\cap K)$. Therefore, $I\vee(J\cap K)=(I\vee J)\cap(I\vee K)$, and so I is distributive. □

Now we give another characterization of distributive n-ideal. To prove this we need the following lemma which is well known and is due to [14, Theorem-2, Page-139].

Lemma 5.1.2. *An element* a *of a lattice* L *is distributive if and only if the relation* θ_a *defined by* $x\equiv y\theta_a$ *if and only if* $x\vee a=y\vee a$ *is a congruence.* □

Theorem 5.1.3. *An* n-*ideal* I *of a lattice* L *is distributive if and only if the relation* $\Theta(I)$ *defined by* $x\equiv y\Theta(I)$ $(x, y\in L)$ *if and only if* $x\vee i_1=y\vee i_1$ *and* $x\wedge i_2=y\wedge i_2$ *for some* $i_1, i_2\in I$ *is the congruence generated by* I.

Proof: At first we shall show that $x\equiv y\Theta(I)$ if and only if $<x>_n\equiv<y>_n\Theta_I$ in $I_n(L)$. Let $x\equiv y\Theta(I)$. Then $x\vee i_1=y\vee i_1$ and $x\wedge i_2=y\wedge i_2$ for some $i_1, i_2\in I$. Now $x\wedge i_2=y\wedge i_2\leq y\leq y\vee i_1=x\vee i_1$ implies that $y\in<x>_n\vee I$. Similarly $x\in<y>_n\vee I$. Therefore, $<x>_n\vee I=<y>_n\vee I$, which implies that, $<x>_n\equiv<y>_n\Theta_I$ in $I_n(L)$. Conversely, if $<x>_n\equiv<y>_n\Theta_I$ in $I_n(L)$, then $<x>_n\vee I=<y>_n\vee I$. Then $x\in<y>_n\vee I$, and so $y\wedge n\wedge i_1\leq x\leq y\vee n\vee i_2$. Similarly $x\wedge n\wedge i_3\leq y\leq x\vee n\vee i_4$. Thus $x\leq y\vee n\vee i_2\leq x\vee n\vee i_2\vee i_4$ which implies $x\vee n\vee i_2\vee i_4=y\vee n\vee i_2\vee i_4$.

Similarly, $x\wedge n\wedge i_1\wedge i_3=y\wedge n\wedge i_1\wedge i_3$. That is, $x\vee i=y\vee i$ and $x\wedge i'=y\wedge i'$ where $i=n\vee i_2\vee i_4$ and $i'=n\wedge i_1\wedge i_3$. Therefore, $x\equiv y\Theta(I)$.

Above proof shows that $\Theta(I)$ is a congruence in L if and only if Θ_I is a congruence in $I_n(L)$. But by Lemma 5.1.2, Θ_I is a congruence if and only if I is distributive in $I_n(L)$, and this completes the proof. □

We know from [14] that an ideal generated by a set of distributive (standard) elements is distributive (standard). Now we generalize this result:

Theorem 5.1.4. *Let* n *be a neutral element of a lattice* L. *Then a finitely generated* n*-ideal* $<a_1,\ldots\ldots\ldots,a_m>_n$ *is distributive if* $a_1\wedge n,\ldots\ldots,a_m\wedge n$ *are dual distributive and* $a_1\vee n,\ldots\ldots\ldots,a_m\vee n$ *are distributive in* L.

Proof: Suppose $a_1\wedge n,\ldots\ldots\ldots,a_m\wedge n$ are dual distributive and $a_1\vee n,\ldots\ldots,a_m\vee n$ are distributive in L. Let J, $K\in I_n(L)$. Suppose $x\in(<a_1,\ldots,a_m>_n\vee J)\cap(<a_1,\ldots,a_m>_n\vee K)$. Then using distributivity of $a_1\vee n,\ldots\ldots\ldots,a_m\vee n$, we have

$$x\leq(a_1\vee\ldots\ldots\vee a_m\vee n\vee j)\wedge(a_1\vee\ldots\ldots\vee a_m\vee n\vee k)$$

$$=(a_1\vee n)\vee[(a_2\vee\ldots\ldots\ldots\vee a_m\vee n\vee j)\wedge(a_2\vee\ldots\ldots\ldots\vee a_m\vee n\vee k)]$$

for some $j\in J$, $k\in K$.

....

....

$=(a_1\vee n)\vee(a_2\vee n)\vee.........\vee(a_m\vee n)\vee(j\wedge k)$

$=(a_1\vee a_2\vee.........\vee a_m\vee n)\vee((j\vee n)\wedge(k\vee n))$.

But $(j\vee n)\wedge(k\vee n)=m(j\vee n,\ n,\ k\vee n)\in J\cap K$. Dually using the dual distributivity of $a_1\wedge n,......,a_m\wedge n$, it is easy to see that $a_1\wedge a_2\wedge.............\wedge a_m\wedge n\wedge((j_1\wedge n)\vee(k_1\wedge n))\le x$ for some $j_1\in J$, $k_1\in K$. Moreover, $(j_1\wedge n)\vee(k_1\wedge n)=m(j_1\wedge n,\ n,\ k_1\wedge n)\in J\cap K$. Thus by convexity $x\in\langle a_1,..........,a_m\rangle_n\vee(J\cap K)$. Since the reverse inclusion is trivial, so $\langle a_1,..........,a_m\rangle_n$ is distributive . □

It should be mentioned that the converse of above result is not necessarily true. For example consider the following lattice.

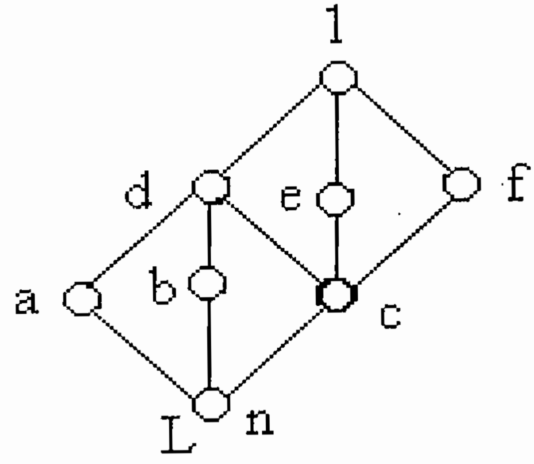

Figure 5.1

Here $\langle a,\ f\rangle_n=L$ which is of course distributive in $I_n(L)$. But neither $a\vee n$ nor $f\vee n$ is distributive in L.

But the converse holds for principal n-ideals.

Theorem 5.1.5. *Let* n *be a neutral element of a lattice* L. *Then* $\langle a\rangle_n$ *is distributive if and only if* $a\wedge n$ *is dual distributive and* $a\vee n$ *is distributive.*

Proof: If $a\wedge n$ is dual distributive and $a\vee n$ is distributive. Then by Theorem 5.1.4, $\langle a\rangle_n$ is distributive. To prove the converse, suppose $\langle a\rangle_n$ is distributive. Let $b, c\in L$. Then $\langle a\rangle_n\vee(\langle b\rangle_n\cap\langle c\rangle_n)=(\langle a\rangle_n\vee\langle b\rangle_n)\cap(\langle a\rangle_n\vee\langle c\rangle_n)$. Thus, $[a\wedge n, a\vee n]\vee([b\wedge n, b\vee n]\cap[c\wedge n, c\vee n])$
$=[a\wedge b\wedge n, a\vee b\vee n]\cap[a\wedge c\wedge n, a\vee c\vee n]$. This implies
$a\wedge n\wedge((b\wedge n)\vee(c\wedge n))=(a\wedge b\wedge n)\vee(a\wedge c\wedge n)$ and
$a\vee n\vee((b\vee n)\wedge(c\vee n))=(a\vee b\vee n)\wedge(a\vee c\vee n)$. That is,
$(a\wedge n)\wedge(b\vee c)=(a\wedge b\wedge n)\vee(a\wedge c\wedge n)$ and
$(a\vee n)\vee(b\wedge c)=(a\vee b\vee n)\wedge(a\vee c\vee n)$, as n is neutral. Therefore, $a\wedge n$ is dual distributive and $a\vee n$ is distributive in L. □

For a distributive n-ideal I of a lattice L, consider the lattice $\frac{L}{\Theta(I)}$. Suppose $I_n\left(\frac{L}{\Theta(I)}\right)$ represents the lattice of all convex sublattices of $\frac{L}{\Theta(I)}$ containing I as a class. We conclude the section by generalizing a result [14, Theorem-7, Page-148] by the following theorem.

Theorem 5.1.6. *Let* I *be a distributive* n*-ideal of a* *lattice* L. *Then* $I_n\left(\frac{L}{\Theta(I)}\right)$ *is isomorphic with the lattice of all* n-*ideals of* L *containing* I, *that is, with* [I, L] *in* $I_n(L)$.

Proof: Let φ be the homomorphism $x \to [x]\Theta(I)$ of L onto $\frac{L}{\Theta(I)}$. Then it is easy to see that the map $\psi: K \to K\varphi^{-1}$ maps $I_n\left(\frac{L}{\Theta(I)}\right)$ into [I, L]. To show that ψ is onto, it is sufficient to see that $[J]\Theta(I)=J$ for all $J \supseteq I$. Indeed, if $j \in J$ and $a \in L$ with $j \equiv a\Theta(I)$, then $j \vee i = a \vee i$ and $j \wedge i_1 = a \wedge i_1$ for some $i, i_1 \in I$. Thus $j \wedge i_1 \leq a \leq j \vee i$. Since $j \wedge i_1, j \vee i \in J$, so by convexity $a \in J$. Moreover, ψ is obviously an isotone and one-one. Therefore, it is an isomorphism. □

2. Modular n-ideals of a lattice.

An n-ideal M of a lattice L is called a *modular n-ideal* if it is a modular element of the lattice $I_n(L)$. In other words M is called modular if for all I, $J \in I_n(L)$ with $J \subseteq I$, $I \cap (M \vee J) = (I \cap M) \vee J$.

We know from [59] that a lattice L is modular if and only if its every element is modular. Also from [31], we know that for a neutral element n of a lattice L, L is modular if and only if $I_n(L)$ is so. Thus, for a neutral element n, the lattice L is modular if and only if its every n-ideal is modular.

Following result gives a characterization of modular n-ideals of a lattice.

Theorem 5.2.1. $M \in I_n(L)$ *is modular if and only if for any* J, $K \in P_n(L)$ *with* $K \subseteq J$, $J \cap (M \vee K) = (J \cap M) \vee K$.

Proof: Suppose M is modular. Then above relation obviously holds from the definition. Conversely, suppose $J \cap (M \vee K) = (J \cap M) \vee K$ for all J, $K \in P_n(L)$ with $K \subseteq J$. Let S, $T \in I_n(L)$ with $T \subseteq S$. We need to show that $S \cap (M \vee T) = (S \cap M) \vee T$. Clearly $(S \cap M) \vee T \subseteq S \cap (M \vee T)$. To prove the reverse inclusion let $x \in S \cap (M \vee T)$. Then $x \in S$ and

$x \in M \vee T$. Then $m \wedge t \leq x \leq m_1 \vee t_1$ for some $m, m_1 \in M$, $t, t_1 \in T$. Thus, $x \vee n \leq m_1 \vee t_1 \vee n$ which implies $x \vee n \in \langle m_1 \vee n \rangle_n \vee \langle t_1 \vee n \rangle_n \subseteq M \vee \langle t_1 \vee n \rangle_n$. Moreover, $x \vee n \in \langle x \vee t_1 \vee n \rangle_n$ and $\langle x \vee t_1 \vee n \rangle_n \supseteq \langle t_1 \vee n \rangle_n$. Hence by the given condition, $x \vee n \in \langle x \vee t_1 \vee n \rangle_n \cap (M \vee \langle t_1 \vee n \rangle_n) = (\langle x \vee t_1 \vee n \rangle_n \cap M) \vee \leq t_1 \vee n \rangle_n \subseteq (S \cap M) \vee T$. By a dual proof of above we can easily see that $x \wedge n \in (S \cap M) \vee T$. Thus by convexity $x \in (S \cap M) \vee T$. Therefore, $S \cap (M \vee T) = (S \cap M) \vee T$, and so M is modular. $\square$

Now we give another characterization of modular n-ideals when n is a neutral element in the lattice.

Theorem 5.2.2. *Suppose* n *is a neutral element of a lattice* L. *An* n-*ideal* M *is modular if and only if for any* $x \in M \vee \langle y \rangle_n$ *with* $\langle y \rangle_n \subseteq \langle x \rangle_n$, $x = (x \wedge m_1) \vee (x \wedge y) = (x \vee m_2) \wedge (x \vee y)$ *for some* $m_1, m_2 \in M$.

Proof: Suppose M is modular and $x \in M \vee \langle y \rangle_n$. Then $x \in \langle x \rangle_n \cap (M \vee \langle y \rangle_n) = (\langle x \rangle_n \cap M) \vee \langle y \rangle_n$. This implies $p \wedge y \wedge n \leq x \leq q \vee y \vee n$ for some $p, q \in \langle x \rangle_n \cap M$. By Proposition 1.1.1, $q \in \langle x \rangle_n \cap M$ implies that $q = (x \wedge q) \vee (x \wedge n) \vee (q \wedge n) = (x \wedge (q \vee n)) \vee (q \wedge n)$. Thus, $x \vee n \leq (x \wedge (q \vee n)) \vee y \vee n \leq x \vee n$, which implies $x \vee n = (x \wedge (q \vee n)) \vee y \vee n = (x \wedge (q \vee n)) \vee (y \wedge (x \vee n)) \vee n = (x \wedge (q \vee n)) \vee (x \wedge y) \vee n$, as n is neutral. Hence by the neutrality of n again, $x = x \wedge (x \vee n) = x \wedge [(x \wedge (q \vee n)) \vee (x \wedge y) \vee n]$

$=(x\wedge[(x\wedge(q\vee n))\vee(x\wedge y)])\vee(x\wedge n)=(x\wedge(q\vee n))\vee(x\wedge y)\vee(x\wedge n)$ $=(x\wedge(q\vee n))\vee(x\wedge y)$, which is the first relation where $m_1=q\vee n\in M$. A dual proof of above establishes the second relation.

Conversely, let $<y>_n\subseteq<x>_n$. By Theorem 5.2.1, we need to show that $<x>_n\cap(M\vee<y>_n)=(<x>_n\cap M)\vee<y>_n$. Clearly R.H.S$\subseteq$L.H.S. To prove the reverse inclusion let $t\in<x>_n\cap(M\vee<y>_n)$. Then $t\in<x>_n$ and $t\in M\vee<y>_n$. Then $m\wedge y\wedge n\leq t\leq m_1\vee y\vee n$ for some m, $m_1\in M$. Thus, $t\vee y\vee n\leq m_1\vee y\vee n$, and so $t\vee y\vee n\in M\vee<y\vee n>_n$ and $<y\vee n>_n\subseteq<t\vee y\vee n>_n$. So by the given condition $t\vee y\vee n=$ $((t\vee y\vee n)\wedge m')\vee(y\vee n)$ for some $m'\in M$. Since t, $y\in<x>_n$, so $t\vee y\vee n\in<x>_n$. Moreover, by the neutrality of n, $((t\vee y\vee n)\wedge m')\vee(y\vee n)=[(t\vee y\vee n)\wedge(m'\vee n)]\vee y$ $=m(t\vee y\vee n,\ n,\ m')\vee y\in(<x>_n\cap M)\vee<y>_n$. Therefore, $t\vee y\vee n\in(<x>_n\cap M)\vee<y>_n$. By a dual proof we can show that $t\wedge y\wedge n\in(<x>_n\cap M)\vee<y>_n$. Thus by the convexity, $t\in(<x>_n\cap M)\vee<y>_n$. Therefore, $<x>_n\cap(M\vee<y>_n)=(<x>_n\cap M)\vee<y>_n$ and so by Theorem 5.2.1, M is modular. □

In [38], it has been proved that for a modular ideal M and an arbitrary ideal I if $I\vee M$ and $I\cap M$ are principal, then I is itself principal. Now will generalize this result for modular n-ideals. It should be mentioned that similar result on standard n-ideals has been proved by Noor and Latif in [50].

Theorem 5.2.3. *Let* n *be a neutral element of a lattice* L. *Suppose* M *is a modular* n-*ideal and* I *is any* n-*ideal of* L. *If* $M \vee I = \langle a \rangle_n$ *and* $M \cap I = \langle b \rangle_n$, *then* I *is principal.*

Proof: Here $M \vee I = \langle a \rangle_n = [a \wedge n, a \vee n]$, then $a \vee n \leq m \vee i$ for some $m \in M$, $i \in I$. Since m, $i \leq a \vee n$, so $a \vee n = m \vee i$. Similarly $a \wedge n = m_1 \wedge i_1$ for some $m_1 \in M$ and $i_1 \in I$. Again, $M \cap I = \langle b \rangle_n$ implies $a \wedge n \leq b \leq a \vee n$. Thus,

$\langle a \rangle_n = M \vee I \supseteq M \vee [b \wedge i_1 \wedge n,\ b \vee i \vee n] \supseteq [m_1 \wedge n,\ m \vee n]$ $\vee [b \wedge i_1 \wedge n,\ b \vee i \vee n] = [a \wedge n,\ a \vee n] = \langle a \rangle_n$. This implies $M \vee I = M \vee [b \wedge i_1 \wedge n,\ b \vee i \vee n]$. On the other hand, $\langle b \rangle_n = M \cap I \supseteq M \cap [b \wedge i_1 \wedge n,\ b \vee i \vee n] \supseteq M \cap \langle b \rangle_n = \langle b \rangle_n$ implies that $M \cap I = M \cap [b \wedge i_1 \wedge n,\ b \vee i \vee n]$. Since $[b \wedge i_1 \wedge n,\ b \vee i \vee n] \subseteq I$, so by the modularity of M we have $I = [b \wedge i_1 \wedge n,\ b \vee i \vee n]$. Now by Theorem 1.1.13, we know that for a neutral element n, any finitely generated n-ideal contained in a principal n-ideal is principal. Since $[b \wedge i_1 \wedge n,\ b \vee i \vee n] \subseteq \langle a \rangle_n$, so I is principal. □

We conclude this section with the following result:

Theorem 5.2.4. *If* M *is a modular* n-*ideal and* n *is any* n-*ideal of a lattice* L, *then* $I \cap M$ *is also modular in the sublaltice* I.

Proof: Let J, K be any two n-ideals contained in I with $K \subseteq J$. Then $J \cap [(I \cap M) \vee K] = J \cap [I \cap (M \vee K)]$, as M is modular and $K \subseteq I$. Thus, $J \cap [(I \cap M) \vee K] = J \cap I \cap (M \vee K)$ $= J \cap (M \vee K) = (J \cap M) \vee K$ (using the modularity of M again) $= (J \cap (I \cap M)) \vee K$. This implies $I \cap M$ is a modular n-ideal in I. □

3. Some properties of standard and neutral n-ideals of a lattice.

Recall that an n-ideal S of a lattice L is standard if for any I, $J \in I_n(L)$, $I \cap (S \vee J) = (I \cap S) \vee (I \cap J)$. S is called neutral if

(i) it is standard and

(ii) for all I, $J \in I_n(L)$, $S \cap (I \vee J) = (S \cap I) \vee (S \cap J)$, that is, it is a dual distributive element of $I_n(L)$.

By [60], we know that any element of a lattice is standard if and only if it is distributive and modular. Thus, in a modular lattice every distributive element is standard. Not only that, in a modular lattice every standard element is also neutral. Therefore, an n-ideal is standard if and only if it is both distributive and modular. Since for a neutral element n of L, L is modular if and only if $I_n(L)$ is modular, so every distributive n-ideal of L is standard (also neutral) when L is modular and n is neutral.

Like Theorem 5.2.1, we can easily prove that the following result:

Theorem 5.3.1. *An* n-*ideal* S *is standard if and only if* $\langle a\rangle_n \cap (S \vee \langle b\rangle_n) = (\langle a\rangle_n \cap S) \vee (\langle a\rangle_n \cap \langle b\rangle_n)$ *for all* a, $b \in L$. □

Our next result is also very easy to prove as it is dual to the proof of Theorem 5.1.1. Thus we omit the proof.

Theorem 5.3.2. *An* n-*ideal* S *is dual distributive if and only if* $S\cap(\langle a\rangle_n\vee\langle b\rangle_n)=(S\cap\langle a\rangle_n)\vee(S\cap\langle b\rangle_n)$ *for all* $a, b\in L$.□

In [15] Grätzer have shown that an element n is neutral if and only if $m(x, n, y)=(x\wedge y)\vee(x\wedge n)\vee(y\wedge n)=(x\vee y)\wedge(x\vee n)\wedge(y\vee n)=m^d(x, n, y)$ for all $x, y\in L$. Combining this result with above theorems we obtain the following result which is also a generalization of [14, Theorem-6 Page-148].

Theorem 5.3.3. *An* n-*ideal* S *of a lattice* L *is neutral if and only if* $(S\cap\langle a\rangle_n)\vee(S\cap\langle b\rangle_n)\vee(\langle a\rangle_n\cap\langle b\rangle_n)$ $=(S\vee\langle a\rangle_n)\cap(S\vee\langle b\rangle_n)\cap(\langle a\rangle_n\vee\langle b\rangle_n)$ *for all* $a, b\in L$. □

In [50, Lemma-1.5], Noor and Latif have proved that for a neutral element n of a lattice L, $\langle a\rangle_n$ is standard if and only if $a\wedge n$ is dual standard and $a\vee n$ is standard. Moreover, for a finitely generated n-ideal we have the following result similar to Theorem 5.1.4.

Theorem 5.3.4. *Let* n *be a neutral element of a lattice* L. *Then* $\langle a_1,\text{----------------},a_m\rangle_n$ *is standard if*

$a_1\wedge n,\text{----------------------},a_m\wedge n$ *are dual standard and*

$a_1\vee n,\text{----------------------},a_m\vee n$ *are standard.*

Proof: Let I, $J\in I_n(L)$. Suppose $x\in I\cap(<a_1,---,a_m>_n\vee J)$. Then $x\in I$ and $x\in<a_1,-----,a_m>_n\vee J$. Then $a_1\wedge-------\wedge a_m\wedge n\wedge j \leq x\leq a_1\vee-------------\vee a_m\vee n\vee j_1$ for some j, $j_1\in J$. Thus, $x\vee n\leq a_1\vee-----------\vee a_m\vee n\vee j_1$ which implies $x\vee n=(x\vee n)\wedge(a_1\vee-------------------\vee a_m\vee n\vee j_1)$. Then using the standardness of $a_1\vee n,----------,a_m\vee n$, we have $x\vee n=((x\vee n)\wedge(a_1\vee n))\vee------\vee((x\vee n)\wedge(a_m\vee n))\vee((x\vee n)\wedge(j\vee n))$. But $(x\vee n)\wedge(a_i\vee n)=m(x\vee n, n, a_i\vee n)\in I\cap<a_i\vee n>_n \subseteq I\cap<a_1,-------------,a_m>_n$. Similarly, $(x\vee n)\wedge(j\vee n)\in I\cap J$. Therefore, $x\vee n\in(I\cap<a_1,------,a_m>_n)\vee(I\cap J)$. Dually, using the dual standardness of $a_1\wedge n,----------,a_m\wedge n$ we can show that $x\wedge n\in(I\cap<a_1,--------,a_m>_n)\vee(I\cap J)$, and so by convexity $x\in(I\cap<a_1,----------,a_m>_n)\vee(I\cap J)$. Therefore, $I\cap(<a_1,-------,a_m>_n\vee J)\subseteq(I\cap<a_1,---------,a_m>_n)\vee(I\cap J)$. Since the reverse inclusion is trivial, so $I\cap(<a_1,-------,a_m>_n\vee J)=(I\cap<a_1,---------,a_m>_n)\vee(I\cap J)$, and hence $<a_1,--------,a_m>_n$ is standard. □

Recall that by [15] an element $n\in L$ is neutral if and only if for all a, $b\in L$, $(a\wedge b)\vee(a\wedge n)\vee(b\wedge n)=(a\vee b)\wedge(a\vee n)\wedge(b\vee n)$. Since this relation is selfdual, so the dual condition of neutrality also implies the neutrality. Thus proceeding as above we can show that for a neutral element n of a lattice L, $<a>_n$ is neutral if and only if both $a\wedge n$ and $a\vee n$ are neutral.

Figure 5.1 again shows that the converse of above theorem is not true. There $<a, f>_n=L$ is standard in $I_n(L)$ but neither $a\vee n$ nor $f\vee n$ is standard in L.

In [50, Theorem-1.10], Noor and Latif have shown that in a relatively complemented lattice with 0 and 1, the congruence lattice C(L) is Boolean if and only if every standard n-ideal is a principal n-ideal, where n is a neutral element. Since in a modular lattice, every standard n-ideal is neutral, so we have the following result:

Theorem 5.3.5. *For a neutral element* n *of a complemented modular lattice* L, *the lattice of all congruence relations of* L *is a Boolean algebra if and only if every neutral* n-*ideal is principal.* □

By [49] we know that an n-ideal S of a lattice L is standard if and only if the relation $\Theta(S)$ defined by $x\equiv y\Theta(S)$ if and only if $x\wedge y=((x\wedge y)\vee t)\wedge(x\vee y)$ and $x\vee y=((x\vee y)\wedge s)\vee(x\wedge y)$ for some $s, t\in S$ is the smallest congruence containing S as a class. We also know by [50] that for two standard n-ideals S and T, both $S\cap T$ and $S\vee T$ are standard. Moreover,

$\Theta(S\cap T)=\Theta(S)\cap\Theta(T)$ and

$\Theta(S\vee T)=\Theta(S)\vee\Theta(T)$.

By [31], the congruence of the form $\Theta(S)$ where S is a standard n-ideal, are known as standard n-congruences. Above relations show that the standard n-congruences form a distributive lattice. We conclude the section with the following result which is a generalization of [14, Example-15, Page-150].

Theorem 5.3.6. *For a neutral element* n *of a lattice* L, *the lattice of all standard* n-*ideals is isomorphic to the lattice of all standard* n-*congruences.*

Proof: Between these two lattices consider the map $S\to\Theta(S)$. By above relations clearly this is a homomorphism and onto. So we need only to show that this is one–one. Suppose $\Theta(S)=\Theta(T)$ for two standard n-ideals S and T. Let $s\in S$. Then for any $t\in T$, $m(s, n, t)\in S$. Then $s\equiv m(s, n, t)\Theta(S)=\Theta(T)$. Since n is neutral, so $m(s, n, t)=(s\wedge t)\vee(s\wedge n)\vee(t\wedge n)=(s\vee t)\wedge(s\vee n)\wedge(t\vee n)$. Thus, $s\wedge m(s, n, t)=s\wedge(t\vee n)=(s\wedge t)\vee(s\wedge n)$, and $s\vee m(s, n, t)=s\vee(t\wedge n)$. Since $s\equiv m(s, n, t)\Theta(T)$, so $s\wedge m(s, n, t)$ $=((s\wedge m(s, n, t))\vee a)\wedge(s\vee m(s, n, t))$, and $s\vee m(s, n, t)=((s\vee m(s, n, t))\wedge b)\vee(s\wedge m(s, n, t))$ for some $a, b\in T$. Thus, $s\wedge(t\vee n)=((s\wedge(t\vee n))\vee a)\wedge(s\vee(t\wedge n))$ and $s\vee(t\wedge n)=((s\vee(t\wedge n))\wedge b)\vee(s\wedge(t\vee n))$. Hence, $a\wedge t\wedge n\leq s\wedge(t\vee n)\leq t\vee n$ which implies $s\wedge(t\vee n)\in T$. Then

$s\wedge(t\vee n)\leq s\leq s\vee(t\wedge n)\leq b\vee(s\wedge(t\vee n))$ implies by convexity that $s\in T$. This implies $S\subseteq T$. Similarly $T\subseteq S$, and so $S=T$. Therefore, above mapping is one-one and hence it is an isomorphism. □

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