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Certificate From The Supervisor

This is to certify that the thesis entitled "**Studies of Spinning Particles in Curved Spacetimes**" submitted by **Akhtara Banu,** who got her name registered in July/2005 M.Phil/Ph.D. batch for the award of the degree of **Doctor of Philosophy** in Science (Mathematics) of **Rajshahi University,** is absolutely based upon her own work under my supervision and that neither this thesis nor any part of it has been submitted for any degree or any other academic award anywhere before.

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Declaration

I do hereby declare that the thesis entitled "Studies of Spinning Particles in Curved Spacetimes" submitted by me to Rajshahi University for the award of Ph.D. degree in Science (Mathematics) has not been submitted to any Institute or University for any degree or award.

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Dedicated To My Parents

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The Author

Abstract

We investigate the motion of spinning particles, such as Dirac fermions, in curved spacetimes by pseudo-classical mechanics models in which the spin degrees of freedom are characterized in terms of anti-commuting Grassmann variables. The work of this thesis has been organized in **7 chapters** along with an introduction at the very outset and a discussion at the end of the thesis.

In the **Introduction** we present a short account of the work of investigating pseudo-classical spinning point particles in the four-dimensional curved spacetimes of general relativity.

We review the relevant formulations and the equations of motion for the theory in **chapters 1** and **2**. We generalize the Killing equations in the spinning space, i.e., the configuration space of the spinning particles, spanned by the usual position coordinates and spin variables. We describe the symmetries and the corresponding constants of motion in terms of the solutions of the generalized Killing equations. Spinning space can have fermionic symmetries along with the standard kind of symmetries. The standard or *generic* symmetries exist for any spacetime metric $g_{\mu\nu}(x)$ (chapter 1), while the new *nongeneric* symmetries depend on the specific form of spacetime and are generated by the square root of bosonic constants of motion other than the Hamiltonian (chapter 2). These formalisms have been exploited in **chapters 4–7** to study the pseudo-classical spinning point particles in particular types of black hole spacetimes.

In **chapter 3** we investigate geodesic motion of pseudo-classical spin one half point particles in the purely de Sitter spacetime and asymptotically de

Sitter Schwarzschild spacetime. We apply the formalism of chapter 1 to solve the equations of motion of the pseudo-classical spinning particles in a plane.

In **chapter 4** we investigate geodesic motions of pseudo-classical spinning point particles in the Euclidean Taub-NUT space. Applying the formalism of chapters 1 and 2, we describe the symmetries and derive the corresponding constants of motion. The spinning space is found to admit hidden supersymmetries, generated by the mysterious Killing-Yano tensors. We use these formalisms to analyze the motion of the pseudo-classical spinning particles on a cone and plane.

In **chapter 5** we analyze the geodesic motion of pseudo-classical spinning particles in the NUT-Taub spacetime. We find that spinning spacetime admits new fermionic supersymmetries along with generic symmetries. The corresponding constants of motion have been obtained and the motion has been described on a cone and on a plane.

In **chapter 6** we study geodesic motion of pseudo-classical spinning particles in the Taub-NUT-de Sitter spacetime, which is the Taub-NUT spacetime generalized with cosmological constant. We obtain the conserved quantities for spinning space and describe the motion of the pseudo-classical Dirac fermion on a cone and plane.

Geodesic motions of pseudo-classical spinning point particles are analyzed for the generalized NUT spacetime in **chapter 7**. Both types of symmetries, *generic* and *nongeneric*, are found to exist in the spinning space. We obtain the corresponding conserved quantities and investigate the motion of the pseudo-classical spinning particles on a cone and plane.

Finally, we present the concluding remarks of this work in the **discussion**, at the end of the thesis.

Table of Contents

Notations, Conventions and Acronyms

- We use natural units in which the gravitational constant G, the vacuum speed of light c, and the reduced Planck's constant $\hbar = h/2\pi$ are set to unity: $G = c = \hbar = 1$.
- The signature of the metric is $(-+++)$.
- Greek indices $\mu, \nu, \cdots$ and Latin indices $a, b, c, \cdots$ at tensors cycle the numbers 0 to 3. The temporal index is denoted by t and number 0.
- *Einstein's summation convention*: Sum on any index that is repeated in a product.
- $e_\mu{}^a(x)$ is the vielbein (tetrad) where μ labels components of world vectors and a is a tangent space (flat-space) index; and $g_{\mu\nu}(x) = e_\mu{}^a e_{a\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b$.
- Spinning space is spanned by $\{x^\mu, \psi^\mu\}$, where x^μ denotes the usual position coordinates and ψ^μ the anti-commuting Grassmann spin-variables.
- $A_{(\mu\nu)} = \frac{1}{2!}\left(A_{\mu\nu} + A_{\nu\mu}\right)$, i.e. parentheses denote full symmetrization over the indices enclosed.
- $A_{[\mu\nu\lambda]} = \frac{1}{3!}\left(A_{\mu\nu\lambda} - A_{\mu\lambda\nu} + A_{\nu\lambda\mu} - A_{\nu\mu\lambda}\right)$, i.e. square brackets denote full anti-symmetrization over the indices enclosed.
- A comma, a semicolon, and a dot respectively denote a directional derivative, a covariant derivative, and a differentiation with respect to the argument.
- SUSY $\rightarrow$ Supersymmetry
- BRST $\rightarrow$ Becchi-Rouet-Stora-Tyutin
- KY $\rightarrow$ Killing-Yano
- dS $\rightarrow$ de Sitter
- SdS $\rightarrow$ Schwarzschild-de Sitter
- NUT $\rightarrow$ Newman-Unti-Tamburino
- TN $\rightarrow$ Taub-NUT
- ETN $\rightarrow$ Euclidean Taub-NUT
- AdS $\rightarrow$ Anti-de Sitter
- CFT $\rightarrow$ Conformal Field Theory

Citations to Previously Published Works

Part of the contents of this thesis has already appeared in the following papers:

Refereed Journals:

- *Geodesic Motions in Euclidean Taub-NUT Spinning Spaces,* **A. Banu** and M.A. Ansary, International Journal of Theoretical Physics **48,** (2009), 2987 – 3000.

- *Symmetries and Motions in NUT-Taub Spinning Space*, **A. Banu** and M.A. Ansary, International Journal of Theoretical Physics **46,** (2007) 3072 – 3087.

In Preparation:

- *Motion of Spinning Particles in de Sitter Spacetime*, [Chapter 3].

- *Motions in Taub-NUT-de Sitter Spinning Spacetime*, [Chapter 6].

- *Geodesic Motions of Spinning Particles in Generalized NUT Spacetime*, [Chapter 7].

Introduction

Spinning point particles, such as Dirac fermions, in curved spacetimes are described by pseudo-classical mechanics models in which the spin degrees of freedom are characterized in terms of Grassmann anti-commuting variables [1, 2, 3, 4, 5].

The relativistic spinning particle models have been proposed for a long time. The pioneer work concerning the Lagrangian description of the relativistic spinning particles was done by Frenkel [6] in 1926 and literatures on it grew vast [7] afterward. Characterizing spinning degrees of freedom by Grassmann (odd) variables [8, 9], the action of spin-$\frac{1}{2}$ relativistic particles was first proposed by Berezin and Marinov [1, 10] and soon after that was discussed and investigated by many authors [11, 12, 13, 14, 15, 16]. Although the anticommuting Grassmann variables do not admit a direct classical interpretation, the Lagrangians for these models have a natural interpretation in the context of the path-integral description of the quantum dynamics. The pseudo-classical equations get physical meaning when averaged over the inside of the functional integral [1, 10, 17]. In this semi-classical regime, neglecting higher order quantum correlations, appropriate combinations of Grassmann spin-variables should be admissible to replace by real numbers. These ideas have been exploited in analyzing the motion of spinning particles in external fields [1, 10, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

In addition to such direct physical applications, it has been proved that generalizations of Riemannian geometry based on anticommuting variables have wide mathematical interests. The supersymmetric point particle mechanics has found applications in the area of index theorem [31, 32, 33, 34], while the BRST (Bechhi-Rouet-Stora-Tyutin) [35, 36] methods are widely used in the study of topological invariants [37, 38]. Thus the study of the geometry of graded pseudo-manifold with both real number and anticommuting variables is well motivated.

In refs. [39, 40, 41] the relations between symmetries of graded pseudo-manifolds and constants of motion for spinning point particles have been described in detail. These relations are more complicated than in the case of scalar particles moving in a Riemannian manifold. This is because of two reasons; firstly, the presence of anti-commuting variables modifies the Killing equations themselves. Secondly, Killing vectors alone no longer construct the constants of motion, there correspond in principle associated Killing scalars which must be added to the expressions involving the Killing vectors to obtain the conserved quantities of motion. These formulations may be applied to any spacetime, and some works in this regard can be found in refs. [42, 43, 44, 45, 46, 47, 48, 49, 50, 51].

The symmetries of spacetime are very important to investigate the motions of spin-$\frac{1}{2}$ particles in the curved spacetime. The configuration space (spinning space) for the pseudo-classical spinning point particles is spanned by local graded coordinates $\{x^\mu, \psi^\mu\}$ with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anti-commuting). This graded space can have two types of symmetries: (i) "generic" symmetries which exist in any spacetime, and (ii) "nongeneric" supersymmetries (SUSYs) [52, 53, 54, 55, 56, 57, 58], appearance of which depends on the specific form of the metric $g_{\mu\nu}(x)$. This fermionic symmetries are generated by the square root of bosonic constants of motion other than the Hamiltonian. In ref. [53] Gibbons et.al. presented a general analysis of the conditions under which such new type of SUSYs appear, and discussed the Poisson-Dirac algebra of the resulting set of charges, including the conditions of closure of the new algebra.

In this thesis we investigate the motion of pseudo-classical relativistic spinning point particles in some particular curved spacetimes and arrange the work in **seven chapters**.

Following Rietdijk and van Holten [40, 41, 42] we present a review work in **chapter 1** on the "generic" symmetries and the corresponding conserved quantities of spinning space. We summarize the relevant equations for the motion of spinning point particles in curved spacetime with briefly discussing their physical interpretation. We describe the generalized Killing equations for spinning space and give the derivation of constants of motion in terms of the solutions of these equations.

The generic symmetries discussed in chapter 1 may be applied to any spacetime, but since our study also includes the motion of spinning particles in axially symmetric spacetime such as the Taub-NUT solution of the Einstein vacuum equations in four-dimensions, which admits nongeneric symmetries as well as generic ones, we describe nongeneric symmetries and their corresponding conserved quantities in **chapter 2**.

The axially symmetric spacetimes are invariant under two continuous symmetries: time translations and rotations about an axis of symmetry, which are generated by Killing fields K^μ and M^μ, respectively. These symmetries generate two constants of motion: energy E and angular momentum J, for particles orbiting in these backgrounds. Both constants of motion are linear in the particles 4-momentum p_μ:

$$E = -K^\mu p_\mu, \tag{1}$$

and

$$J = M^\mu p_\mu. \tag{2}$$

The complete integrability of particle motions in the four-dimensional Kerr-Newman and Taub-NUT spacetimes demands the existence of a non-trivial Stäckel-Killing tensor $K_{\mu\nu}$ [59, 60, 61], associated with which is the constant of motion:

$$Z = \frac{1}{2} K^{\mu\nu} p_\mu p_\nu, \tag{3}$$

quadratic in the 4-momentum p_μ [62]. This constant of motion commutes with the covariant Hamiltonian

$$H = \frac{1}{2} \mathrm{g}^{\mu\nu} p_\mu p_\nu \tag{4}$$

in the sense of Poisson brackets, because $K^{\mu\nu}$ is a symmetric second-rank contravariant tensor field satisfying a generalized Killing equation (chapter 1). The four constants of motion (E, J, Z, H) form a mutually Poisson-commuting set of functions on the cotangent bundle.

More surprisingly, the separability of various field equations in Kerr background spacetimes, e.g., the Dirac equation [63, 64], has the direct consequence of the existence of a Killing-Yano tensor $f_{\mu\nu}$ [65], which satisfies the Penrose-Floyd equation [66, 67]:

$$D_{(\mu}f_{\nu)\lambda} = 0, \tag{5}$$

and is the square-root of the Stäckel-Killing tensor:

$$K^{\mu}{}_{\nu} = f^{\mu}{}_{\lambda}f^{\lambda}{}_{\nu} \tag{6}$$

with the properties $f_{\mu\nu} = -f_{\nu\mu}$. These useful and mysterious discoveries made possible a whole range of calculations, both classical and quantum mechanical, and can be applied to various physical processes near black holes. This novel aspect has renewed peoples interest in the Killing-Yano tensor which has long been known to relativists as a mysterious thing. Although there are not so many physically interpretable spacetimes in which Killing-Yano tensors exist [68, 69, 70], Kerr-Newman [53, 71, 72] or Taub-NUT [73] background spacetimes admit Killing-Yano tensors and hence pseudo-classical spinning point particles' motion in these spacetimes can have nongeneric SUSYs.

Using supersymmetric particle mechanics involving classically anticommuting Grassmann variables, Gibbons et. al. [53] demonstrated the Killing-Yano tensor $f_{\mu\nu}$ as an object belonging to a larger class of possible structures which generate generalized SUSY algebras and addressed the classical counter part of Carter and McLenaghan's work on the Dirac equation. These useful and mysterious discoveries made possible a whole range of calculations, both classical and quantum mechanical, and can be applied to various physical processes near black holes.

In **chapter 2** we review the work of Gibbons et. al. [53] for investigating nongeneric SUSYs. We summarize the formalism of pseudo-classical spinning point particles in an arbitrary background spacetime and describe the nongeneric SUSYs along with the other (universal) symmetries and their corresponding conserved quantities.

In **chapter 3** we apply the formalisms of chapters 1 and 2 to investigate the motion of pseudo-classical spinning particles in the purely de Sitter (dS) spacetime as well as asymptotically dS black hole spacetime such as Schwarzschild-de Sitter (SdS) spacetime. The purely dS or SdS spacetime is vacuum solution of Einstein's eld equations with a positive cosmological constant $\Lambda > 0$. The dS spacetime metric was found by de Sitter in 1917 [74, 75, 76], the year of Λ and of Einstein's static universe. It too was put forward as a static universe (up to the horizon), although one of vanishing density. However, it was soon realized that free particles would stream away from the center, driven by the force, and eventually dS spacetime became a model for an expanding universe. It still plays a role in modern cosmology as the limit of a whole family of non-empty models. Indeed, recent astronomical observations suggest the existence of a positive cosmological constant in our universe and that the universe will asymptotically approach a dS spacetime [77]. This has sparked a sense of urgency in resolving the quantum-gravitational mysteries of dS spacetime [78, 79]. Physicists have thus growing interest in the dS and asymptotically dS spacetimes. The work of this chapter may be interesting in view of the inflationary scenario of the universe.

In **chapter 4** we present our work [80] of investigating the geodesic motion of the pseudo-classical spin-$\frac{1}{2}$ point particle in the geometry of Euclidean Taub-NUT, which is a $D = 4$ self-dual space. The Euclidean Taub-NUT space has attracted much attention in physics. It gives rise to the gravitational analog of the Yang-Mills instanton [81]. The metric of this space is the space part of the line element of the celebrated Kaluza-Klein monopole of Gross and Perry [82] and Sorkin [83]. Moreover, the motion of well-separated monopole-monopole interactions is described approximately by the geodesics of this space [84, 85, 86, 87]. The Euclidean Taub-NUT background contains also interesting specific features of the quantum theory in the case of the scalar fields [88] as well as for Dirac fields of spin-$\frac{1}{2}$ fermions [89, 90, 91]. There exist large algebras of conserved observables in both cases [92]. The Taub-NUT family of metrics has also attracted physicists in studying many other modern studies like strings, membranes, etc. This chapter, therefore, contains our work in the interesting Euclidean Taub-NUT spacetime.

We describe our work [93] in **chapter 5** of investigating the motion of a pseudo-classical spin-$\frac{1}{2}$ particle in the NUT-Taub space which is a stationary and axisymmetric solution of Einstein's empty space equation.

The similarity between the group structure for the NUT symmetry and the group structure for spherical symmetry leads to the interpretation that the NUT-Taub metric corresponds to a vacuum cosmological-like solution with periodic time. According to this identification, the NUT-Taub space admits peculiar properties [94, 95]: it does not admit an interpretation without a periodic time coordinate, has no reasonable space-like surface, and is an asymptotically zero curvature space which apparently does not admit asymptotically rectangular coordinates.

The NUT-Taub spacetime is the Schwarzschild spacetime generalized with the NUT parameter n which has the identification of the gravitational magnetic mass or magnetic monopole [96, 97, 98, 99, 100]. The monopole hypothesis was propounded by Dirac relatively long ago. The ingenious suggestion by Dirac that magnetic monopole does exist in nature was neglected because of the failure to identify such thing. In recent years, however, the development of gauge theories [101, 102] has shed new light on it. The string theory [103] predicts the existence of this type of objects. In a recent work [104], it was shown that the NUT charge generates a "rotational effect", so that the spacetime must be assigned a "specific angular momentum" due to the NUT charge. Thus the study of this chapter gives result for the interesting NUT-Taub spacetime.

In **chapter 6** we investigate the motion of a pseudo-classical spin-$\frac{1}{2}$ particle in the Taub-NUT-de Sitter (TN-dS) spacetime, which is a stationary and axisymmetric solution of the vacuum Einstein equations with a cosmological constant. The TN-dS metric is a generalization of the Schwarzschild metric with the NUT parameter n and cosmological constant Λ. Instead of being asymptotically flat, it is an asymptotically de Sitter TN spacetime. The spacetime is interesting in that it contains the TN spacetime, which has played an important role in the conceptional development of general relativity and in the construction of brane solutions in string theory and M-theory [105, 106, 107]. The NUT charge induces a topology in the Euclidean section at infinity that is a Hopf fibration of a circle over a 2-sphere and plays the role of a magnetic mass. The curious properties of the Taub-NUT spaces induced Misner [95] to consider it as "a counter example to almost anything". This spacetime plays a significant role in exhibiting the type of effects that can arise in strong gravitational fields. The TN spacetime has been of particular interest in recent years because of the role it plays in furthering our understanding of the AdS/CFT correspondence [108, 109, 110]. On the other hand, there has been a renewed interest in cosmological constant as it is found to be present in the inflationary scenario of the early universe. In this scenario the universe undergoes a phase where it is geometrically similar to the de Sitter space [111]. Among other things inflation has led to the cold dark matter in the form of slowly moving particles (axions or neutralinos). If the cold dark matter theory proves correct, it would shed light on the unification of forces [112, 113]. Comprehensive reviews of the cosmological constant or dark energy, including the observational evidence for it and the problems associated with it, have been done by many authors [114, 115, 116, 117, 118, 119, 120, 121, 122]. The TN-dS spacetime is therefore interesting in the inflationary scenario of the early universe and the study of this chapter provides result for this interesting spacetime.

In **chapter 7** we investigate the motion of a pseudo-classical spin-$\frac{1}{2}$ particle in the black hole spacetime described by the generalized NUT metric with six parameters: the mass, the electric and magnetic charges, the NUT charge, the cosmological constant, and a continuous parameter. The spacetime represents a stationary axisymmetric solution of the Einstein-Maxwell field equations with cosmological constant. The work of this chapter is interesting in that it gives in special cases the results obtained in chapters 1–6 along with the results for some other interesting spacetimes, e.g., the Reissner-Nordström [43] and the Reissner-Nordström-de Sitter [44] spacetimes.

Finally, we present our remarks in the **discussion** at the end of this thesis.

Chapter 1

Spinning Space, Its Generic Symmetries and Conserved Charges

1.1 Introduction

In this chapter we briefly discuss the relevant equations for the motion of pseudo-classical spinning point particles in curved spacetimes. We describe the symmetries of spinning spaces, the generalized Killing equations, and derive the constants of motion. The analysis of this chapter is a generic one and can be applied to any spacetime with the metric $g_{\mu\nu}$.

We arrange this chapter as follows. In section 1.2 we describe the starting point of the work by summarizing the relevant equations for the motion of spinning point particles in curved spacetime and briefly discuss their physical interpretation. In section 1.3 we recall the generalized Killing equations for a scalar particle in curved spacetime and extend the theory to the case of spinning particles in section 1.4. In section 1.5 we describe the derivation of the constants of motion, which exist in any spacetime, in terms of the solutions of the generalized Killing equations.

1.2 Spinning Space

According to Einstein's theory of gravity, the world-line of a classical point particles in curved spacetime is a time-like geodesic. Since geodesics are curves of extremal length, the equation for the world-line of a point particle can be obtained from an action principle. With this action any smooth monotonic function of the spacetime interval ds along the curve is given by

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = -d\tau^2, \tag{1.1}$$

where $d\tau$ is the corresponding interval of proper time. The last equality holds only in the absence of external forces such as electromagnetic dipole forces [28, 29].

Spinning spaces are the configuration spaces of spinning particles and are extensions of ordinary Riemannian manifolds described by local coordinates $\{x^\mu\}$ to graded manifolds described by local graded coordinates $\{x^\mu, \psi^\mu\}$ with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anticommuting). Geodesic flow along time-like curves of such a graded manifold with Minkowskian signature $(+, -, -, -)$ describes the classical limit of the motion of a relativistic point-like Dirac fermion with spin-$\frac{1}{2}$ in quantum mechanics [1, 3, 4, 5, 16].

The number of bosonic and fermionic dimension dimensions is the same and the Grassmann variables ψ^μ transform as 1-forms dx^μ. So, one can realize a supersymmetry in the geometry of the graded manifolds, which acts on the coordinates as

$$\delta x^\mu = -\mathrm{i}\epsilon\psi^\mu, \qquad \delta\psi^\mu = \epsilon\dot{x}^\mu, \tag{1.2}$$

where the overdot denotes a derivative with respect to proper time, $\frac{d}{d\tau}$. The supersymmetric action that defines the extremal trajectories in the graded manifold is given by

$$S = m\int_1^2 d\tau \left(\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{\mathrm{i}}{2}g_{\mu\nu}(x)\psi^\mu\frac{D\psi^\nu}{D\tau}\right), \tag{1.3}$$

where the constant m of proportionality has the dimension of mass. In the following we consider particles of unit mass: $m = 1$, but occasionally we reinstate the explicit mass dependence when it is physically relevant. The covariant derivative of ψ^μ is defined by

$$\frac{D\psi^\nu}{D\tau} := \dot{\psi}^\mu + \dot{x}^\lambda\Gamma^\mu_{\lambda\nu}\psi^\nu. \tag{1.4}$$

Under arbitrary variations of the coordinates $(\delta x^\mu, \delta\psi^\mu)$ the action S changes by

$$\delta S = \int_1^2 d\tau \left\{ -\delta x^\mu \left(g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} + \frac{i}{2} \psi^\kappa \psi^\lambda R_{\kappa\lambda\mu\nu} \dot{x}^\nu \right) \right.$$
$$\left. + i \Delta \psi^\mu g_{\mu\nu} \frac{D\psi^\nu}{D\tau} + \frac{d}{d\tau} \left(\delta x^\mu p_\mu - \frac{i}{2} \delta \psi^\mu g_{\mu\nu} \psi^\nu \right) \right\}, \tag{1.5}$$

where p_μ is the canonical momentum:

$$p_\mu = g_{\mu\nu} \dot{x}^\nu - \frac{1}{2} i \Gamma_{\mu\kappa\lambda} \psi^\kappa \psi^\lambda, \tag{1.6}$$

$R_{\kappa\lambda\mu\nu}$ is the Riemann curvature tensor and $\Delta\psi^\mu$ is the covariantized variation of ψ^μ:

$$\Delta \psi^\mu = \delta \psi^\mu + \delta x^\lambda \Gamma^\mu_{\lambda\nu} \psi^\nu. \tag{1.7}$$

The action S is stationary under variations δx^μ and $\delta\psi^\mu$ vanishing at the end points, if the following equations of motion are satisfied:

$$\frac{D^2 x^\nu}{D\tau^2} = -\frac{i}{2} R^\mu{}_{\nu\kappa\lambda} \dot{x}^\nu \psi^\kappa \psi^\lambda, \tag{1.8}$$

$$\frac{D\psi^\mu}{D\tau} = 0, \tag{1.9}$$

where $\frac{D^2 x^\mu}{D\tau^2} := \ddot{x}^\mu + \Gamma^\mu_{\lambda\nu} \dot{x}^\lambda \dot{x}^\nu$ and $R^\mu{}_{\nu\kappa\lambda} := \partial_\kappa \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\kappa} + \Gamma^\mu_{\kappa\rho} \Gamma^\rho_{\nu\lambda} - \Gamma^\mu_{\lambda\rho} \Gamma^\rho_{\nu\kappa}$ is the Riemann curvature tensor. Obviously, the solutions for $x^\mu(\tau)$ in the case of replacing ψ^μ by zero everywhere are ordinary geodesics in the bosonic submanifold.

The interesting solutions are those for which one or more components of ψ^μ do not vanish. In order to interpret physically such solutions, the following antisymmetric tensor has been defined:

$$S^{\mu\nu} = -\mathrm{i}\psi^\mu\psi^\nu, \tag{1.10}$$

which describes the relativistic spin of the particle [1, 2, 3, 5, 16, 18, 19, 28, 29]. Accordingly, equations (1.8) and (1.9) describe the classical motion of a Dirac fermion.

Equation (1.8) with the spin tensor (1.10) implies the existence of spin-dependent gravitational forces [18, 19, 20, 24, 28, 29, 30]

$$\frac{D^2x^\nu}{D\tau^2} = \frac{1}{2}S^{\kappa\lambda}R^\mu{}_{\nu\kappa\lambda}\dot{x}^\nu. \tag{1.11}$$

This is analogous to the electromagnetic Lorentz force

$$\ddot{x}^\mu = \frac{q}{m}F^\mu{}_\nu\dot{x}^\nu, \tag{1.12}$$

with spin tensor replacing the scalar electric charge [28, 29, 30] (here for unit mass). The second equation, (1.9), states that the spin is covariantly constant:

$$\frac{DS^{\mu\nu}}{D\tau} = 0, \tag{1.13}$$

where $\frac{DS^{\mu\nu}}{D\tau} := \dot{S}^{\mu\nu} + \Gamma^{\mu}_{\kappa\lambda}\dot{x}^{\kappa}S^{\lambda\nu} + \Gamma^{\nu}_{\kappa\lambda}\dot{x}^{\kappa}S^{\mu\lambda}$. The spin interpretation of $S^{\mu\nu}$ is supported by the study of electromagnetic interactions of the particle [1, 5, 10, 20, 28, 29]. It follows that spacelike components S^{ij} represent the particle's magnetic dipole moment and the time-like components S^{i0} correspond to the electric dipole moment. It is, of course, required for free Dirac fermions (like free electrons and quarks) to vanish the electric dipole moment in the rest frame. This can be expressed as a covariant constraint [30]

$$g_{\nu\lambda}(x)S^{\mu\nu}\dot{x}^{\lambda} = 0, \tag{1.14}$$

which is equivalent to

$$g_{\mu\nu}(x)\dot{x}^{\mu}\psi^{\nu} = 0, \tag{1.15}$$

using the Grassmann coordinates. It has an elegant interpretation in terms of supersymmetry.

1.3 Scalar Particles in Curved Spacetime

We start our discussion with the well-known relation that exists in classical mechanics between symmetries of the conguration space and conservation laws, as expressed by Noether's theorem. For a scalar point particle moving in an arbitrary curved spacetime, this relation can be described as follows. Let us consider the action

$$S = \int_1^2 d\tau \frac{1}{2} g_{\mu\nu}(x)\dot{x}^\mu \psi^\nu, \tag{1.16}$$

the stationary points of which are precisely given by $\frac{D^2 x^\mu}{D\tau^2} = 0$, the equation (1.8) with vanishing ψ^μ. The general variation of S is

$$\delta S = \int_1^2 d\tau \left\{ -\delta x^\mu g_{\mu\nu} \frac{D^2 x^\mu}{D\tau^2} + \frac{d}{d\tau}(\delta x^\mu p_\mu) \right\}, \tag{1.17}$$

where $p_\mu = g_{\mu\nu}\dot{x}^\nu$ is the canonical momentum. Then $\delta S = 0$ for any arbitrary variation of x^μ with fixed end points if and only if $\frac{D^2 x^\mu}{D\tau^2} = 0$ is satisfied.

Now the question arises, whether there exist variations x^μ for which $\delta S = 0$ modulo boundary terms even when the equations of motion are not satisfied, i.e. $\frac{D^2 x^\mu}{D\tau^2} \neq 0$.

We consider variations δx^μ of the type

$$\delta x^\mu = \mathcal{R}^\mu(x, \dot{x}) = R^\mu(x) + \dot{x}^\nu K_\nu{}^\mu(x) + \frac{1}{2}\dot{x}^\nu \dot{x}^\lambda L_{\nu\lambda}{}^\mu(x) + \cdots; \tag{1.18}$$

then the variation of the action (1.16) is

$$\delta S = \int_1^2 d\tau \left\{ \delta x^\mu p_\mu - \mathcal{J}(x, \dot{x}) \right\}. \tag{1.19}$$

We restrict variations (1.18) to depend only on the first derivative $\dot{x}^\mu$, because the second and higher derivatives can always be expressed in terms of these modulo of the equations of motion: $\frac{D^2 x^\mu}{D\tau^2} = 0$. Comparing (1.19) with (1.18) one immediately obtains

$$\frac{d\mathcal{J}}{d\tau} = \mathcal{R}^{\mu} g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} \approx 0, \tag{1.20}$$

where last equality holds only when the equations of motion: $\frac{D^2 x^\nu}{D\tau^2} = 0$ is satisfied. Thus the quantities $\mathcal{J}(x, \dot{x})$ are conserved for physical motions. This is Noether's theorem.

Assuming that $\mathcal{J}(x, \dot{x})$ can be expanded in the four-velocity as

$$\begin{aligned} \mathcal{J}(x, \dot{x}) &= J^{(0)}(x) + \dot{x}^\mu J^{(1)}_\mu(x) + \frac{1}{2}\dot{x}^\mu \dot{x}^\nu J^{(2)}_{\mu\nu}(x) \\ &+ \frac{1}{3!}\dot{x}^\mu \dot{x}^\nu \dot{x}^\lambda J^{(3)}_{\mu\nu\lambda}(x) + \cdots, \end{aligned} \tag{1.21}$$

one can find from (1.20) with the ansatz (1.18) the following identification of the coefficients:

$$\begin{aligned} J^{(1)}_\mu(x) &= R_\mu(x), \\ J^{(2)}_{\mu\nu}(x) &= K_{\mu\nu}(x), \\ J^{(3)}_{\mu\nu\lambda}(x) &= L_{\mu\nu\lambda}(x), \quad \textit{etc.} \end{aligned} \tag{1.22}$$

These relations indicate that all covariant tensors on the right hand side of

(1.18) should be taken to be completely symmetric. Moreover, it follows that the following differential equations have to be satisfied

$$J^{(n)}_{(\mu_1\cdots\mu_n;\mu_{n+1})} = 0, \tag{1.23}$$

in which the parentheses denote full symmetrization over all indices enclosed, with total weight one. Equations (1.23) constitute a straightforward generalization of the Killing equation and give in explicit forms

$$J^{(0)}_{,\mu} \;=\; 0, \tag{1.24}$$

$$R_{(\mu;\nu)} \;=\; 0, \tag{1.25}$$

$$K_{(\mu\nu;\lambda)} \;=\; 0, \quad etc. \tag{1.26}$$

These equations hold independently of the equations of motion: $\frac{D^2x^\nu}{D\tau^2} = 0$. The first equation (1.24) gives $J^{(0)}$ as an irrelevant constant. The second equation (1.25) represents the standard equation for Killing vectors, while (1.26) and its higher-rank counterparts constitute tensorial generalizations of this equation.

1.4 Spinning Particles in Curved Spacetime

We now generalize the above theory to the graded configuration space (spinning space). The supersymmetric action in this space is given by (1.3). We look for specific variations δx^μ and $\Delta\psi^\mu$ which leave the action off-shell invariant modulo boundary terms. Let the variations be of the form

$$
\begin{aligned}
\delta x^\mu &= \mathcal{R}^\mu(x, \dot{x}, \psi) \\
&= R^{(1)\mu}(x,\psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} R^{(n+1)\mu}_{\nu_1 \cdots \nu_n}(x,\psi), \\
\Delta\psi^\mu &= \mathcal{S}^\mu(x, \dot{x}, \psi) \\
&= S^{(0)\mu}(x,\psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\nu_1} \cdots \dot{x}^{\nu_n} S^{(n)\ \ \mu}_{\nu_1 \cdots \nu_n}(x,\psi).
\end{aligned} \tag{1.27}
$$

If the Lagrangian transforms under the variations (1.27) into a total derivative

$$
\delta S = \int_1^2 d\tau \frac{d}{d\tau} \left(\delta x^\mu p_\mu - \frac{\mathrm{i}}{2} \delta\psi^\mu \mathrm{g}_{\mu\nu} \psi^\nu - \mathcal{J}(x, \dot{x}, \psi) \right), \tag{1.28}
$$

it then follows that

$$
\frac{d\mathcal{J}}{d\tau} = \mathcal{R}^\mu \left(\mathrm{g}_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} + \frac{\mathrm{i}}{2} \psi^\kappa \psi^\lambda R_{\kappa\lambda\mu\nu} \dot{x}^\nu \right) + \mathrm{i}\mathcal{S}^\mu \mathrm{g}_{\mu\nu} \frac{D\psi^\nu}{D\tau}, \tag{1.29}
$$

When the equations of motions (1.8) and (1.9) are satisfied, it then follows from (1.29) that $\frac{d\mathcal{J}}{d\tau} = 0$, i.e. $\mathcal{J}$ is conserved. This is Noether's theorem, again.

If we expand $\mathcal{J}(x, \dot{x}, \psi)$ in terms of the four-velocity,

$$\mathcal{J}(x, \dot{x}, \psi) = J^{(0)}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \dot{x}^{\mu_1} \cdots \dot{x}^{\mu_n} J^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi), \tag{1.30}$$

and compare the left- and right-hand sides of (1.29) with the ansatz (1.27) for δx^μ and $\Delta\psi^\mu$, then we get the following identities:

$$J^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi) = R^{(n)}_{\mu_1 \cdots \mu_n}(x, \psi), \qquad n \geq 1, \tag{1.31}$$

$$S^{(n)}_{\mu_1 \cdots \mu_n \nu}(x, \psi) = \mathrm{i} \frac{\partial J^{(n)}_{\mu_1 \cdots \mu_n}}{\partial \psi^\nu}(x, \psi), \qquad n \geq 0. \tag{1.32}$$

These equations satisfy a generalization of the Killing equations of the form [39, 40, 53]

$$J^{(n)}_{(\mu_1 \cdots \mu_n ; \mu_{n+1})} + \frac{\partial J^{(n)}_{(\mu_1 \cdots \mu_n}}{\partial \psi^\sigma} \Gamma_{\mu_{n+1})\kappa}{}^{\sigma} \psi^\kappa = \frac{\mathrm{i}}{2} \psi^\kappa \psi^\lambda R_{\kappa\lambda\nu(\mu_{n+1}} J^{(n+1)\ \nu}_{\mu_1 \cdots \mu_n)}, \tag{1.33}$$

If we write, as before, $R^{(1)}_\mu = R_\mu$, $R^{(2)}_{\mu\nu} = K_{\mu\nu}$, $R^{(3)}_{\mu\nu\lambda} = K_{\mu\nu\lambda}$, *etc.*, and $J^{(0)} = B$, this reduces for the lowest components to

$$B_{,\mu} + \frac{\partial B}{\partial \psi^\sigma}\Gamma_{\mu\kappa}{}^{\sigma}\psi^\kappa = \frac{\mathrm{i}}{2}\psi^\rho\psi^\sigma R_{\rho\sigma\kappa\mu}R^\kappa, \tag{1.34}$$

$$R_{(\mu;\nu)} + \frac{\partial R_{(\mu}}{\partial \psi^\sigma}\Gamma_{\nu)\kappa}{}^{\sigma}\psi^\kappa = \frac{\mathrm{i}}{2}\psi^\rho\psi^\sigma R_{\rho\sigma\kappa(\mu}K_{\nu)}{}^{\kappa}, \tag{1.35}$$

$$K_{(\mu\nu;\lambda)} + \frac{\partial K_{(\mu\nu}}{\partial \psi^\sigma}\Gamma_{\nu)\kappa}{}^{\sigma}\psi^\kappa = \frac{\mathrm{i}}{2}\psi^\rho\psi^\sigma R_{\rho\sigma\kappa(\mu}L_{\nu\lambda)}{}^{\kappa}, \quad etc. \tag{1.36}$$

These equations hold independently of the equations of motion (1.8) and (1.9). For purely bosonic case the $\{\psi^\mu\}$-parts vanish and then these equations reduce to those obtained for the scalar particle, equations (1.24)–(1.26). We note that, in contrast to the bosonic case, the Killing scalar $B(x,\psi) = J^{(0)}(x,\psi)$ is not always an irrelevant constant, since it can depend non-trivially on x^μ and ψ^μ, as follows from the equation (1.34).

1.5 Generic Solutions for Spinning Space

For spinning particle models as defined by the action (1.3), there exist four types of generic (i.e. existing in any spacetime) constants of motion [40, 41]. They are the following:

1. Like in the bosonic case $g_{\mu\nu}$ itself is a Killing tensor:

$$K_{\mu\nu} = g_{\mu\nu}, \tag{1.37}$$

with vanishing all other Killing vectors and tensors (bosonic as well as fermionic). The associated constant of motion is the world-line

Hamiltonian,

$$H = \frac{1}{2} g_{\mu\nu}(x) P_\mu P_\nu, \tag{1.38}$$

where P_μ is the covariant momentum defined by

$$P_\mu = p_\mu + \frac{\mathrm{i}}{2} \Gamma_{\mu\kappa\lambda} \psi^\kappa \psi^\lambda. \tag{1.39}$$

2. The Grassmann-odd Killing vectors

$$R^\mu = \psi^\mu, \qquad T^\nu_\mu = \mathrm{i}\delta^\nu_\mu \tag{1.40}$$

provide another obvious solution. Again, all other Killing vectors and tensors equal to zero. The supercharge corresponding to this solution is

$$Q = P_\mu \psi^\mu. \tag{1.41}$$

3. In addition to ordinary supersymmetry, the spinning particle action admits a second non-linear supersymmetry that is generated by Killing vectors

$$\begin{aligned} R_\mu &= \frac{-\mathrm{i}^{[\frac{d}{2}]}}{(d-1)!}\sqrt{-\mathrm{g}}\,\varepsilon_{\mu\nu_1\cdots\nu_{d-1}}\psi^{\nu_1}\cdots\psi^{\nu_{d-1}}, \\ T_{\mu\nu} &= \frac{-\mathrm{i}^{[\frac{d-2}{2}]}}{(d-2)!}\sqrt{-\mathrm{g}}\,\varepsilon_{\mu\nu\nu_1\cdots\nu_{d-1}}\psi^{\nu_1}\cdots\psi^{\nu_{d-1}}. \end{aligned} \tag{1.42}$$

The Grassmann parities of $(R_\mu, T_{\mu\nu})$ evidently depend on d, the number of spacetime dimensions. The associated constant of motion is the dual supercharge

$$Q^* = \frac{-\mathrm{i}^{[\frac{d}{2}]}}{(d-1)!}\sqrt{-\mathrm{g}}\,\varepsilon_{\mu_1\cdots\mu_d}P^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_d}. \tag{1.43}$$

4. Finally, there is found to exist a non-trivial Killing scalar

$$\Gamma_* \equiv J^{(0)} = \frac{-\mathrm{i}^{[\frac{d}{2}]}}{d!}\sqrt{-\mathrm{g}}\,\varepsilon_{\mu_1\cdots\mu_d}\psi^{\mu_1}\cdots\psi^{\mu_d}, \tag{1.44}$$

which acts as the Hodge star duality operator on ψ^μ. In quantum mechanics it becomes the γ^{d+1} element of the Dirac algebra and for this reason Γ_* is referred to as the chiral charge.

The fundamental Dirac brackets in the spinning space are as follows:

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu,$$

$$
\begin{aligned}
\{\psi^\mu, \psi^\nu\} &= -\mathrm{i}\mathrm{g}^{\mu\nu}, \\
\{p_\mu, \psi^\nu\} &= \frac{1}{2}\mathrm{g}^{\kappa\nu}\mathrm{g}_{\kappa\lambda,\mu}\psi^\lambda, \\
\{p_\mu, p_\nu\} &= -\frac{\mathrm{i}}{4}\mathrm{g}^{\kappa\lambda}\mathrm{g}_{\kappa\rho,\mu}\mathrm{g}_{\lambda\sigma,\nu}\psi^\rho\psi^\sigma.
\end{aligned} \tag{1.45}
$$

Using these results the non-trivial Dirac-brackets between the generic constants of motion are obtained as follows:

$$
\{Q, Q\} = -2\mathrm{i}H, \qquad \{Q, \Gamma_*\} = -\mathrm{i}Q^*. \tag{1.46}
$$

Clearly, $d = 2$ is an exceptional case, because Q^* then is linear and acts as an ordinary supersymmetry:

$$
\{Q^*, Q^*\} = -2\mathrm{i}H, \qquad \{Q^*, \Gamma_*\} = -\mathrm{i}Q. \tag{1.47}
$$

Hence, the theory actually possesses an $N = 2$ supersymmetry in the two dimensions. If $d \neq 2$, the right-hand side of the equations in (1.47) vanishes.

The supercharge Q is actually crucial for the consistency of the physical interpretation of the theory. Indeed, the absence of an intrinsic electric dipole moment of physical fermions like leptons and quarks gives the constraint (1.15) which implies that

$$Q = 0. \tag{1.48}$$

Since Q is a conserved quantity, the physical condition (1.48) can be satisfied at all times, irrespective of the presence of external fields, and at the same time this provides a clear physical interpretation of world-line supersymmetry.

Chapter 2

New Supersymmetry in Curved Spacetime

2.1 Introduction

In this chapter we present a brief analysis on a new type of supersymmetries for the pseudo-classical spinning point particles in curved spacetime. As described in the Introduction of this thesis, these fermionic symmetries, called nongeneric SUSYs, are generated by the square root of bosonic constants of motion other than the Hamiltonian. In particular, a spacetime can have nongeneric SUSYs, if it admits the existence of Killing-Yano tensor $f_{\mu\nu}$, which is square root of the Stäckel-Killing tensor: $K_{\mu\nu} = f_{\mu\lambda}f^{\lambda}{}_{\nu}$ with the properties $f_{\mu\nu} = -f_{\nu\mu}$. Using supersymmetric particle mechanics involving classically anticommuting Grassmann variables, Gibbons et.al.

[53] demonstrated the Killing-Yano tensor $f_{\mu\nu}$ as an object belonging to a larger class of possible structures which generate generalized SUSY algebras and play an important role in solving the Dirac equation in curved spacetimes.

This chapter is organized as follows. In section 2.2 we give the canonical description of the configuration space of spinning particles and then describe it in terms of the covariant phase-space variables in 2.3. In section 2.4 we review the general relation between symmetries, supersymmetries and constants of motion for these equations. In section 2.5 we address the question of the existence of extra supersymmetries and their algebras. Supersymmetries are dependent on the existence of a second rank tensor field $f_{\mu\nu}$ which we call f-symbols. We describe the general properties of f-symbols and their relation to Killing-Yano tensors in section 2.6.

2.2 Canonical Structures in Spinning Space

The pseudo-classical limit of the Dirac theory of fermions in a curved spacetime is described by supersymmetric extension of the ordinary relativistic point particle [2, 3, 4, 5, 10, 16, 18, 24].

The vielbein (tetrad) $e_\mu{}^a(x)$ [123], which is the "square root" of the metric tensor $g_{\mu\nu}$ in some sense, is used to describe the local version of supersymmetry (supergravity). The Greek index μ labels components of vectors in spacetime (world vectors), transforms like a vector under coordinate transformations and is raised (or lowered) with $g^{\mu\nu}$ (or $g_{\mu\nu}$). The Latin index a is a tangent space (flat-space) index, transforms under (local) Lorentz transformations as a Lorentz vector and is raised (or lowered) with the Minkowski space metric η^{ab} (or η_{ab}). The $e_\mu{}^a$ is the "square root" of the metric tensor $\mathrm{g}_{\mu\nu}$ in the sense that

$$\mathrm{g}_{\mu\nu} = e_\mu{}^a e_{a\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b. \tag{2.1}$$

The configuration or spinning space of the theory is spanned by the real position coordinates $x^\mu(\tau)$ and the Grassmann-valued spin coordinates $\psi^a(\tau)$ where $\mu, a = 1, \cdots d$ with d the dimension of the spacetime. The world and tangent vectors indices (i.e. μ and a) can be converted into each other by the vielbein $e_\mu{}^a(x)$ and its inverse $e^\mu{}_a(x)$. It is sometimes convenient to introduce the object

$$\psi^\mu(x) = e^\mu{}_a(x)\psi^a, \tag{2.2}$$

which transforms a local Lorentz vector into a world vector. The world-line parameter τ is the invariant proper time,

$$c^2 d\tau^2 = -\mathrm{g}_{\mu\nu}(x) dx^\mu dx^\nu. \tag{2.3}$$

We choose units such that $c = 1$.

The Lagrangian of the theory is given by

$$L = \frac{1}{2}\mathrm{g}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{\mathrm{i}}{2}\eta_{ab}\psi^a \frac{D\psi^b}{D\tau}, \tag{2.4}$$

where the overdot denotes an ordinary proper-time derivative $\frac{d}{d\tau}$ and

$$\frac{D\psi^a}{D\tau} := \dot{\psi}^a - \dot{x}^\mu \omega_{\mu\ b}^{\ a} \psi^b \tag{2.5}$$

is the covariant derivative of the spin variable, transforming as a local Lorentz vector with $\omega_{\mu\ b}^{\ a}$ the spin connection. The restriction for the spin to be space-like is

$$\mathcal{Q} \equiv e_{\mu a} \dot{x}^\mu \psi^a = 0, \tag{2.6}$$

which expresses that ψ has no time-component in the rest frame. These supplementary conditions are only to be imposed after solving the equations of motion derived from the Lagrangian L. Indeed, the solutions of the Euler-Lagrange equations may be considered as generalizations of the concept of geodesics to spinning space. The supplementary conditions then select those geodesics which correspond to the world lines of the physical spinning particles.

Under arbitrary variations $(\delta x^\mu, \delta\psi^a)$ the Lagrangian changes by

$$\begin{aligned} \delta L &= \delta x^\mu \left(-g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} - \frac{i}{2} \psi^a \psi^b R_{ab\mu\nu} \dot{x}^\nu \right) \\ &\quad + \Delta\psi^a \eta_{ab} \frac{D\psi^b}{D\tau} + \text{total derivative}, \end{aligned} \tag{2.7}$$

where

$$\Delta\psi^a := \delta\psi^a - \delta x^\mu \omega_{\mu\ b}^{\ a}\psi^b \tag{2.8}$$

is the covariantized variation of ψ^a [40]. The equations of motion can be cast in the form

$$\begin{aligned} \frac{D^2 x^\mu}{D\tau^2} &= -\frac{\mathrm{i}}{2}\psi^a\psi^b R^\mu{}_{\nu ab}\dot{x}^\nu, \\ \frac{D\psi^a}{D\tau} &= 0. \end{aligned} \tag{2.9}$$

The canonical momenta conjugates to x^μ and ψ^a are, respectively,

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = \mathrm{g}_{\mu\nu}\dot{x}^\nu - \frac{\mathrm{i}}{2}\omega_{\mu ab}\psi^a\psi^b \tag{2.10}$$

and

$$\pi_a = \frac{\partial L}{\partial \dot{\psi}^a} = -\frac{\mathrm{i}}{2}\psi_a. \tag{2.11}$$

When this second-class constraint is eliminated by Dirac's procedure, one obtains the canonical Poisson-Dirac brackets

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{\psi^a, \psi^b\} = -\mathrm{i}\,\eta^{ab}. \tag{2.12}$$

Consequently, the Poisson-Dirac brackets for general functions F and G of the canonical phase-space variables (x, p, ψ) is given by

$$\{F, G\} = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial x^\mu} + \mathrm{i}(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a}, \tag{2.13}$$

where a_F is the Grassmann parity of F: $a_F = (0, 1)$ for $F =$(even, odd). The canonical Hamiltonian of the theory is expressed by

$$H = \frac{1}{2}\mathrm{g}^{\mu\nu}(p_\mu + \omega_\mu)(p_\nu + \omega_\nu), \tag{2.14}$$

where

$$\omega_\mu = \frac{\mathrm{i}}{2}\omega_{\mu ab}\psi^a\psi^b. \tag{2.15}$$

Indeed, the time-evolution of any function $F(x, p, \psi)$ is generated by

$$\frac{dF}{d\tau} = \{F, G\}. \tag{2.16}$$

Equations (2.12)–(2.15) summarize the canonical structure of the theory of pseudo-classical Dirac's fermions in spinning space. The inconvenience of the canonical formulation is that one looses manifest covariance.

2.3 Covariant Formulation in Spinning Space

It is often convenience to describe the theory of pseudo-classical spin one half point particles in terms of a set of covariant phase-space variables, defined by x^μ, ψ^a and the covariant momentum

$$\Pi_\mu \equiv p_\mu + \omega_\mu = g_{\mu\nu}\dot{x}^\nu. \tag{2.17}$$

For functions of the covariant phase-space variables $F(x, \Pi, \psi)$ the Poisson-Dirac brackets become

$$\{F, G\} = \mathcal{D}_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu}\mathcal{D}_\mu G - \mathcal{R}_{\mu\nu}\frac{\partial F}{\partial \Pi_\mu}\frac{\partial G}{\partial \Pi_\nu} + \mathrm{i}(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a}, \tag{2.18}$$

where the phase-space covariant derivative is defined by

$$\mathcal{D}_\mu F := \partial_\mu F + \Gamma_{\mu\nu}{}^{\lambda}\Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} + \omega_\mu{}^a{}_b \psi^b \frac{\partial F}{\partial \psi_a}, \tag{2.19}$$

and the spin-valued curvature tensor is given by

$$\mathcal{R}_{\mu\nu} = \frac{\mathrm{i}}{2}\psi^a\psi^b R_{ab\mu\nu}. \tag{2.20}$$

It results that

$$\{\Pi_\mu, \Pi_\nu\} = -\mathcal{R}_{\mu\nu}, \tag{2.21}$$

which is the classical analogue of the Ricci-identity in the absence of torsion. The Hamiltonian in the covariant phase-space becomes

$$H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu. \tag{2.22}$$

The dynamical equation (2.16) remains unaltered, but the constraints (2.3) and (2.9) become

$$2H = g^{\mu\nu} \Pi_\mu \Pi_\nu = -1, \quad \mathcal{Q} = \Pi \cdot \psi = 0. \tag{2.23}$$

In general, these are not compatible with the Poisson-Dirac brackets. Hence, they are to be imposed only after solving the theory. However, it clearly follows that

$$\{\mathcal{Q}, H\} = 0, \tag{2.24}$$

which shows the conservation of $\mathcal{Q}$. Since the Hamiltonian itself is trivially conserved, the values of H and $\mathcal{Q}$ as chosen in (2.23) are preserved in time. Therefore, the imposed physical conditions are consistent with the equations of motion of the theory.

2.4 Symmetries and Constants of Motion

The theory of a spinning-particle model described by the Lagrangian (2.4), or equivalently the Hamiltonian (2.23), possesses a number of symmetries which are very useful in obtaining explicit solutions of the equations of motion. These symmetries are of two kinds: *generic* symmetries, which exist in any spacetime, and *nongeneric* symmetries, which depend on the explicit form of the metric $g_{\mu\nu}$. As described in chapter 1, the Lagrangian (2.4) possesses four generic symmetries: proper-time translations, generated by the Hamiltonian H; supersymmetry generated by the supercharge $\mathcal{Q}$, equation (2.23); chiral symmetry, generated by the chiral charge

$$\Gamma_* = -\frac{\mathrm{i}^{[\frac{d}{2}]}}{d!}\varepsilon_{a_1\cdots a_d}\psi^{a_1}\cdots\psi^{a_d}, \tag{2.25}$$

and dual supersymmetry, generated by dual supercharge

$$Q^* = \mathrm{i}\{\mathcal{Q},\Gamma_*\} = \frac{-\mathrm{i}^{[\frac{d}{2}]}}{(d-1)!}\varepsilon_{a_1\cdots a_d}e^{\mu a_1}\Pi_\mu\psi^{a_2}\cdots\psi^{a_d}. \tag{2.26}$$

It can be checked that $\{H,\Gamma_*\}=0$, and then with (2.24) it follows from Jacobi identity that all these quantities have vanishing Poisson-Dirac brackets with the Hamiltonian, and therefore are constants of motion.

To find all symmetries, including the nongeneric ones, we search for all functions $\mathcal{J}(x,\Pi,\psi)$ which commute with the Hamiltonian:

$$\{H,\mathcal{J}\}=0, \tag{2.27}$$

With the Poisson-Dirac brackets (2.18) it gives

$$\Pi^{\mu}\left(\mathcal{D}_{\mu}\mathcal{J}+\mathcal{R}_{\mu\nu}\frac{\partial\mathcal{J}}{\partial\Pi_{\nu}}\right)=0. \tag{2.28}$$

Obviously, the second term vanishes identically when $\mathcal{J}$ depends on the covariant momentum only via the Hamiltonian: $\mathcal{J}(x,\Pi,\psi)=\mathcal{J}(x,H,\psi)$. Then (2.28) simplifies to

$$\Pi\cdot\mathcal{D}\mathcal{J}=0, \tag{2.29}$$

The power series of $\mathcal{J}(x,\Pi,\psi)$ in the covariant momentum:

$$\mathcal{J}=\sum_{n=0}^{\infty}\frac{1}{n!}J^{(n)\mu_1\cdots\mu_n}(x,\psi)\Pi_{\mu_1}\cdots\Pi_{\mu_n}, \tag{2.30}$$

satisfies (2.28) for arbitrary Π_{μ} if and only if the components of $\mathcal{J}$ satisfy the generalized Killing equations [40, 41]:

$$D_{(\mu_{n+1}}J^{(n)}_{\mu_1\cdots\mu_n)}+\omega_{(\mu_{n+1}b}^{\quad\ a}\psi^{b}\frac{\partial J^{(n)}_{\mu_1\cdots\mu_n)}}{\partial\psi^{a}}=\mathcal{R}_{\nu(\mu_{n+1}}J^{(n+1)\ \nu}_{\mu_1\cdots\mu_n)}, \tag{2.31}$$

where the parentheses denote full symmetrization with norm one over the indices enclosed.

Furthermore, any constant of motion $\mathcal{J}$ satisfies

$$\{\mathcal{Q}, \mathcal{J}\} = -\psi^\mu \left(\mathcal{D}_\mu \mathcal{J} + \mathcal{R}_{\mu\nu} \frac{\partial \mathcal{J}}{\partial \Pi_\nu} \right) - \mathrm{i}\, e^{\mu a} \Pi_\mu \frac{\partial \mathcal{J}}{\partial \psi_a}, \tag{2.32}$$

in which the curvature term contains three contractions with the anticommuting spin variables. Then, in combination with the Bianchi identity: $R_{[\mu\nu\lambda]\kappa} = 0$, (2.32) reduces to

$$\{\mathcal{Q}, \mathcal{J}\} = -\left(\psi \cdot \mathcal{D}\mathcal{J} + \mathrm{i}\, \Pi \cdot \frac{\partial \mathcal{J}}{\partial \psi} \right), \tag{2.33}$$

In particular, with $\mathcal{J} = \mathcal{Q}$ it results the conventional SUSY algebra

$$\{\mathcal{Q}, \mathcal{Q}\} = -2\mathrm{i}H. \tag{2.34}$$

It follows, from this result, with the Jacobi identity for 2 $\mathcal{Q}$'s and any constant of motion $\mathcal{J}$, that

$$\Theta \equiv \{\mathcal{Q}, \mathcal{J}\} \tag{2.35}$$

is a superinvariant and hence a constant of motion as well:

$$\{\mathcal{Q}, \Theta\} = 0, \qquad \{H, \Theta\} = 0. \tag{2.36}$$

This shows that constants of motion generally come in supermultiplets $(\mathcal{J}, \Theta)$, the prime example of which is the multiplet $(\mathcal{Q}, H)$ itself. However, there is an exception to this result, namely, the constants of motion whose bracket with the supercharge vanishes ($\Theta = 0$), but which are not themselves obtained from the bracket of $\mathcal{Q}$ with another constant of motion.

Equation (2.33) states that a superinvariant is a solution of the equation

$$\{\mathcal{Q}, \mathcal{J}\} = -\left(\psi \cdot \mathcal{D}\mathcal{J} + \mathrm{i}\Pi \cdot \frac{\partial \mathcal{J}}{\partial \psi}\right) = 0. \tag{2.37}$$

Let us expand the coefficient $J^{(n)\mu_1\cdots\mu_n}(x, \psi)$ of the series (2.30) in powers of ψ^a:

$$J^{(n)\mu_1\cdots\mu_n} = \sum_{n=0}^{\infty} \frac{\mathrm{i}^{[\frac{m}{2}]}}{m!} f^{(m,n)\mu_1\cdots\mu_n}_{a_1\cdots a_m}(x)\psi^{a_1}\cdots\psi^{a_m}, \tag{2.38}$$

where $f^{(m,n)}$ is completely anti-symmetric in the m lower indices $\{a_i\}$, while it is completely symmetric in the n upper indices $\{\mu_k\}$. Then the power series (2.30) for $\mathcal{J}$ becomes

$$\mathcal{J}(x, \Pi, \psi) = \sum_{m,n=0}^{\infty} \frac{\mathrm{i}^{[\frac{m}{2}]}}{m!n!}\psi^{a_1}\cdots\psi^{a_m} f^{(m,n)\mu_1\cdots\mu_n}_{a_1\cdots a_m}(x)\Pi_{\mu_1}\cdots\Pi_{\mu_n}, \tag{2.39}$$

and the component equation takes the form

$$n f^{(m+1,n-1)(\mu_1\cdots\mu_{n-1}}_{a_0a_1\cdots a_m}e^{\mu_n)a_0} = mD_{[a_1} f^{(m-1,n)\mu_1\cdots\mu_n}_{a_2\cdots a_m]}, \tag{2.40}$$

where $D_a = e^{\mu}{}_a D_\mu$ are ordinary covariant derivatives. Square brackets denote full antisymmetrization, parentheses denote full symmetrization over the indices enclosed, all with unit weight. In particular, with $m = 0$ (2.40) gives

$$f_a^{(1,n)(\mu_1 \cdots \mu_n} e^{\mu_{n+1})a} = 0. \tag{2.41}$$

These equations represent a square root of the generalized Killing equations, in a certain sense, and only provide sufficient, not necessary conditions for obtaining solutions. Nevertheless, at least one component of each supermultiplet (singlet or non-singlet) is a solution of equation (2.37). First finding Θ one can then try to reconstruct the corresponding $\mathcal{J}$ by solving (2.35).

Equations (2.40) partly solve only that part of $f^{(m+1,n-1)}$ (in terms of $f^{(m-1,n)}$) which is symmetrized in one flat index and all $(n-1)$ curved indices. On the contrary, (2.40) do not automatically imply that $f^{(m+1,n-1)}$ is completely antisymmetric in the first $(m+1)$ indices. If that condition is imposed on (2.40), one finds the part of $f^{(m+1,n-1)}$ which is antisymmetrized in one curved index and all $(m+1)$ flat indices. This set of equations are precisely the generalized Killing equations for that part of $f^{(m+1,n-1)}$ which was not given in terms of $f^{(m-1,n)}$), and which should still be solved for.

Thus equations (2.40) evidently have advantages over the generalized Killing equations (2.31). The constant of motion corresponding to a Killing tensor of rank n:

$$\mathcal{D}_{(\mu_{n+1}} \mathcal{J}^{(n)}_{\mu_1 \cdots \mu_n)} = 0, \tag{2.42}$$

can be obtained by solving the complicated hierarchy of partial differential equations (2.31) for $(\mathcal{J}^{(n-1)}, \cdots, \mathcal{J}^{(0)})$ and adding the terms, as in expression (2.30). If $f^{(m,n)\mu_1\cdots\mu_n}_{a_1\cdots a_m}$ be a solution of

$$f^{(m,n)(\mu_1\cdots\mu_n}_{a_1\cdots a_m} e^{\mu_{n+1})a_1} = 0, \tag{2.43}$$

one can then generate at least part of the components $f^{(m+2\alpha,\, n-\alpha)\mu_1\cdots\mu_{n-\alpha}}_{a_1\cdots a_{m+2\alpha}}$ for $\alpha = 1, \cdots, n$ by mere differentiation. The corresponding constant of motion is obtained from (2.39).

It follows from (2.35) and (2.36) that the Poisson-Dirac bracket with $\mathcal{Q}$ defines a nilpotent operation in the space of constants of motion. So, the supersinglets span the cohomology of the supercharge and the supermultiplets $(\mathcal{J}, \Theta)$ form pairs of $\mathcal{Q}$-exact and $\mathcal{Q}$-coexact forms. Then the solutions of (2.37) correspond to the $\mathcal{Q}$-closed forms.

2.5 Nongeneric Supersymmetries

The constants of motion $\mathcal{J}$ generate infinitesimal transformations of the coordinates:

$$\delta x^\mu = \delta\alpha\{x^\mu, \mathcal{J}\}, \qquad \delta\psi^a = \delta\alpha\{\psi^a, \mathcal{J}\}, \tag{2.44}$$

which leave the equations of motion invariant. Here $\delta\alpha$ is the infinitesimal parameter of the transformation. For example, the action as defined by L in (2.4) remains invariant under the generic symmetries, such as supersymmetry:

$$\delta x^\mu = \mathrm{i}\epsilon\{\mathcal{Q}, x^\mu\} = -\mathrm{i}\epsilon e^\mu{}_a \psi^a,$$

$$\delta\psi^a = \mathrm{i}\epsilon\{\mathcal{Q}, \psi^a\} = \epsilon e_\mu{}^a \dot{x}^\mu + \delta x^\mu \omega_\mu{}^a{}_b \psi^b, \tag{2.45}$$

the infinitesimal parameter ϵ being Grassmann-odd of the transformation.

We now look for other (non-generic) SUSYs that the theory might admit. Such SUSYs are of the type

$$\delta x^\mu = -\mathrm{i}\epsilon f^\mu{}_a \psi^a \equiv -\mathrm{i}\epsilon J^{(1)\mu}, \tag{2.46}$$

generated by a phase-space function $\mathcal{Q}_\mathrm{f}$:

$$\mathcal{Q}_\mathrm{f} = J^{(1)\mu}\Pi_\mu + J^{(0)}, \tag{2.47}$$

where $J^{(0,1)}(x,\psi)$ are independent of Π. If this ansatz is inserted into the generalized Killing equations (2.31), it then follows that

$$J^{(0)}(x,\psi) = \frac{\mathrm{i}}{3!} c_{abc}(x)\psi^a\psi^b\psi^c, \tag{2.48}$$

where the tensors $f^\mu{}_a$ and c_{abc} satisfy the conditions

$$D_\mu f_{\nu a} + D_\nu f_{\mu a} = 0, \tag{2.49}$$

$$D_\mu c_{abc} = -\left(R_{\mu\nu ab} f^\nu{}_c + R_{\mu\nu bc} f^\nu{}_a + R_{\mu\nu ca} f^\nu{}_b\right), \tag{2.50}$$

If there exist N such symmetries specified by N sets of tensors $(f_i^\mu{}_a, c_{iabc})$, $i = 1, \cdots, N$, the corresponding generators will be

$$\mathcal{Q}_i = f_i^\mu{}_a \Pi_\mu \psi^a + \frac{\mathrm{i}}{3!} c_{iabc}(x) \psi^a \psi^b \psi^c. \tag{2.51}$$

Evidently, if $f^\mu{}_a = e^\mu{}_a$ and $c_{abc} = 0$, the supercharge (2.23) is precisely of this form. It is, therefore, convenient to refer to the quantities defining the standard SUSY by assigning them the index $i = 0$: $\mathcal{Q} = \mathcal{Q}_0$, $e^\mu{}_a = f_0^\mu{}_a$, etc.

The covariant form (2.18) of the Poisson-Dirac brackets gives the following algebra for the conserved charges $\mathcal{Q}_i$:

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} = -2\mathrm{i} Z_{ij}, \tag{2.52}$$

where

$$Z_{ij} = \frac{1}{2} K_{ij}^{\mu\nu} \Pi_\mu \Pi_\nu + I_{ij}^\mu \Pi_\mu + G_{ij}, \tag{2.53}$$

and

$$K_{ij}^{\mu\nu} = \frac{1}{2}\left(f_i^{\mu}{}_a f_j^{\nu a} + f_i^{\nu}{}_a f_j^{\mu a}\right), \tag{2.54}$$

$$\begin{aligned} I_{ij}^{\mu} &= \frac{1}{2}\mathrm{i}\psi^a\psi^b I_{ijab}^{\mu} \\ &= \frac{\mathrm{i}}{2}\psi^a\psi^b\left(f_i^{\nu}{}_b D_\nu f_j^{\mu}{}_a + f_j^{\nu}{}_b D_\nu f_i^{\mu}{}_a\right. \\ &\quad \left. + \frac{1}{2}f_i^{\mu c}c_{jabc} + \frac{1}{2}f_j^{\mu c}c_{iabc}\right), \end{aligned} \tag{2.55}$$

$$\begin{aligned} G_{ij} &= -\frac{1}{4}\psi^a\psi^b\psi^c\psi^d G_{ijabcd} \\ &= -\frac{1}{4}\psi^a\psi^b\psi^c\psi^d\left(R_{\mu\nu ab}f_i^{\mu}{}_c f_j^{\nu}{}_d + \frac{1}{2}c_{iab}{}^e c_{jcde}\right). \end{aligned} \tag{2.56}$$

Here $K_{ij\mu\nu}$ is a symmetric Killing tensor of second rank:

$$D_{(\lambda}K_{ij\mu\nu)} = 0, \tag{2.57}$$

while I_{ij}^{μ} is the corresponding Killing vector:

$$\mathcal{D}_{(\mu}I_{ij\nu)} = \frac{\mathrm{i}}{2}\psi^a\psi^b D_{(\mu}I_{ij\nu)ab} = \frac{\mathrm{i}}{2}\psi^a\psi^b R_{ab\lambda(\mu}K_{ij\nu)}{}^{\lambda}, \tag{2.58}$$

and G_{ij} the corresponding Killing scalar:

$$\mathcal{D}_\mu G_{ij} = -\frac{1}{4}\psi^a\psi^b\psi^c\psi^d D_\mu G_{ijabcd} = \frac{1}{2}\mathrm{i}\psi^a\psi^b R_{ab\lambda\mu}I_{ij}^{\lambda}. \tag{2.59}$$

Since the functions Z_{ij} satisfy the generalized Killing equations, their brackets with the Hamiltonian vanish and they are constants of motion:

$$\frac{dZ_{ij}}{d\tau} = 0. \tag{2.60}$$

In particular, when $i = j = 0$, (2.52) reduces to the usual SUSY algebra

$$\{Q, Q\} = -2\mathrm{i}H, \tag{2.61}$$

with H the Hamiltonian. If i or j is not equal to zero, Z_{ij} correspond to new bosonic symmetries, unless $K^{\mu\nu}_{ij} = \lambda_{(ij)}\mathrm{g}^{\mu\nu}$, with $\lambda_{(ij)}$ a constant (may be zero). In such case the corresponding Killing vector I^{μ}_{ij} and Killing scalar G_{ij} vanish identically. Moreover, the supercharges for $\lambda_{(ij)} \neq 0$ close on the Hamiltonian. This proves the existence of a second SUSY of the standard type. Then the theory admits an N-extended SUSY with $N \geq 2$. Further, if there exists a second independent Killing tensor $K^{\mu\nu}$ not proportional to $\mathrm{g}^{\mu\nu}$, there exists a new type of SUSY.

The quantity $\mathcal{Q}_i$ is a superinvariant, $\{\mathcal{Q}_i, \mathcal{Q}\} = 0$, for the bracket (2.18), if and only if,

$$K^{\mu\nu}_{0i} = f^{\mu}{}_{a}e^{\nu a} + f^{\nu}{}_{a}e^{\mu a}. \tag{2.62}$$

Then the full constants of motion Z_{ij} can be constructed directly by repeated differentiation of $f^{\mu}{}_{a}$. By construction the Z_{ij} are symmetric in

(ij) and hence, we can diagonalize them and obtain the algebra

$$\{Q_i, Q_j\} = -2i\delta_{ij} Z_i, \tag{2.63}$$

where Z_i are $N+1$ conserved bosonic charges of which the first one is the Hamiltonian: $Z_0 = H$.

2.6 f-symbols and their Properties

In this section we describe the properties of the quantities $f^{\mu}{}_{a}$. For convenience we introduce the second rank tensor

$$f_{\mu\nu} = f_{\mu a} e_{\nu}{}^{a}, \tag{2.64}$$

and refer it as the f-symbol. In refs.[62, 64] the antisymmetric f-symbols and their corresponding Killing-tensors: $K_{\mu\nu}$ have extensively been studied in the related context of finding solutions of the Dirac-equation in nontrivial curved space-time.

Equation (2.49) with (2.64) gives

$$D_{\nu} f_{\lambda\mu} + D_{\lambda} f_{\nu\mu} = 0. \tag{2.65}$$

The f-symbol is divergence-less on its first index:

$$D_{\nu} f^{\nu}{}_{\mu} = 0. \tag{2.66}$$

Equation (2.65) gives on contraction

$$D_\nu f_\mu{}^\nu = -\partial_\mu f_\nu{}^\nu. \tag{2.67}$$

Then the divergence on the second index vanishes if and only if the trace of the f-symbol is constant:

$$D_\nu f_\mu{}^\nu = 0 \quad \Leftrightarrow \quad f_\mu{}^\mu = \text{const.} \tag{2.68}$$

The metric tensor $g_{\mu\nu}$ is a trivial solution of (2.65); so, if the trace is constant, it maybe subtracted from the f-symbol without destroying condition (2.65). Then the constant may always be taken equal to zero without loss of generality and hence f is traceless.

From (2.54), with $f_0{}^\mu{}_a = e^\nu{}_a$, the symmetric part of the i-th f-symbol is the tensor

$$S_{\mu\nu} \equiv K_{i0\mu\nu} = \frac{1}{2}(f_{\mu\nu} + f_{\nu\mu}), \tag{2.69}$$

satisfying the generalized Killing equation

$$D_{(\mu}S_{\nu\lambda)} = 0. \tag{2.70}$$

The anti-symmetric part can also be constructed as

$$B_{\mu\nu} = -B_{\nu\mu} = \frac{1}{2}(f_{\mu\nu} - f_{\nu\mu}), \tag{2.71}$$

which satisfies the condition

$$D_\nu B_{\lambda\mu} + D_\lambda B_{\nu\mu} = D_\mu S_{\nu\lambda}. \tag{2.72}$$

If the symmetric part does not vanish and is not covariantly constant, it then follows that the anti-symmetric part $B_{\mu\nu}$ by itself is not a solution of (2.65). However, the same token does not lead to vanish the antisymmetric part of f. Therefore, f is completely symmetric only if it is covariantly constant.

The considerably interesting case is that in which the f-symbol is completely antisymmetric: $f_{\mu\nu} = B_{\mu\nu}$. The condition (2.62) is precisely this case for the supercharge $\mathcal{Q}_{\mathrm{f}}$ to anti-commute with ordinary supersymmetry in the sense of Poisson-Dirac brackets. Equation (2.68) is also satisfied automatically in this case.

If the symmetric part of a certain $f_{i\mu\nu}$ vanishes:

$$S_i^{\mu\nu} = K_{i0}^{\mu\nu} = 0, \tag{2.73}$$

then the corresponding Killing vector I_{i0}^{μ} and the Killing scalar G_{i0} vanish as well. As a result, the complete $Z_{i0} = 0$ for this particular i and $\mathcal{Q}_i$ is superinvariant, since then

$$\{\mathcal{Q}_i, \mathcal{Q}\} = 0. \tag{2.74}$$

These assertions can be proved as follows. Equation (2.65) for antisymmetric $f^{\mu\nu}$ gives

$$D_\nu B_{\lambda\mu} = -D_\lambda B_{\nu\mu}. \tag{2.75}$$

Since $B_{\mu\nu}$ is antisymmetric, it follows that the gradient is completely antisymmetric:

$$D_\mu B_{\nu\lambda} = D_{[\mu} B_{\nu\lambda]} \equiv H_{\mu\nu\lambda}. \tag{2.76}$$

Commuting the second covariant derivative of $f_{\mu\nu}$ and applying (2.65), one can derive the identity

$$D_\mu D_\nu f_{\lambda\kappa} = R_{\nu\lambda\mu}{}^{\sigma} f_{\sigma\kappa} + \frac{1}{2}(R_{\nu\lambda\kappa}{}^{\sigma} f_{\mu\sigma} + R_{\mu\lambda\kappa}{}^{\sigma} f_{\nu\sigma} - R_{\mu\nu\kappa}{}^{\sigma} f_{\lambda\sigma}). \tag{2.77}$$

For antisymmetric $f_{\mu\nu}$ this gives

$$D_\mu H_{\nu\lambda\kappa} = \frac{1}{2}(R_{\nu\lambda\mu}{}^{\sigma} f_{\sigma\kappa} + R_{\lambda\kappa\mu}{}^{\sigma} f_{\sigma\nu} + R_{\kappa\nu\mu}{}^{\sigma} f_{\sigma\lambda}). \tag{2.78}$$

Comparing with (2.50) one can find

$$-\frac{1}{2}c_{abc} = H_{abc} = e^{\mu}{}_{a}e^{\nu}{}_{b}e^{\lambda}{}_{c}H_{\mu\nu\lambda}, \tag{2.79}$$

modulo a covariantly constant term. This result with $n = 1$, $m = 2$ is an example of (2.40). Since one only needs a particular solution of (2.50) to construct a constant of motion, the covariantly constant term can always be chosen to vanish.

If a covariantly constant three-index tensor c_{abc} exists, then it always gives another symmetry corresponding to the Killing vector

$$I_{\mu} = \frac{\mathrm{i}}{2}\psi^{a}\psi^{b}e_{\mu}{}^{c}c_{abc}. \tag{2.80}$$

More precisely, $D_{\mu}c_{abc} = 0$ implies that

$$\mathcal{D}_{\mu}I_{\nu} = 0, \tag{2.81}$$

and the generalized Killing equation is automatically satisfied for I_{μ}. Hence we are free to add the term with c_{abc} to the supercharge. However, it is not required, since both terms are conserved separately.

According to (2.73), $K_{0i}^{\mu\nu} = 0$, and since $c_{0abc} = 0$ identically, the right-hand side of (2.55) becomes

$$I_{i0\mu\nu\lambda} \equiv I_{i0\mu ab}e_{\nu}{}^{a}e_{\lambda}{}^{b} = D_{\lambda}B_{i\mu\nu} + \frac{1}{2}c_{i\mu\nu\lambda} = 0, \tag{2.82}$$

where the last equality follows from (2.79). Using the cyclic Bianchi identity for the Riemann tensor $R_{\mu\nu\lambda\kappa}$ and the vanishing of at least one of the three-index tensors: $c_{0abc} = 0$, it results that the Killing scalar G_{i0} vanishes. Thus assertion (2.74) is proved.

The analysis presented in this chapter shows, that Killing-Yano tensors belong to a larger class of possible structures which generate generalized supersymmetry algebras.

Chapter 3

Motion of Spinning Particles in de Sitter Spacetime

3.1 Introduction

In this chapter we investigate the motion of pseudo-classical spinning particles in the purely de Sitter (dS) spacetime and asymptotically dS black hole spacetime such as the Schwarzschild-de Sitter (SdS) spacetime. These are the maximally symmetric solutions of the Einstein's field equations with positive cosmological constant, having line elements

$$ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3.1}$$

with $V(r) \mapsto V_{\mathrm{dS}}(r) = 1 - \frac{r^2}{\beta^2}$ for dS spacetime and $V(r) \mapsto V_{\mathrm{SdS}}(r) =$

$1 - \frac{\alpha}{r} - \frac{r^2}{\beta^2}$ for SdS spacetime, where $\beta = \sqrt{3/\Lambda}$ is the curvature radius of dS spacetime (for an observer located at $r = 0$) and $\alpha = 2M$, M being the total mass of SdS spacetime. The coordinate variables have the usual ranges: $-\infty < t < +\infty$, $r \geq 0$, $0 \leq \theta < \pi$, and $0 \leq \varphi < 2\pi$. At large r, the SdS spacetime tends to the dS space limit. The explicit dS case is obtained from SdS spacetime by setting $M = 0$ while the explicit Schwarzschild case is obtained by setting $\Lambda = 0$. Just like the Schwarzschild metric *inside* its horizon, the dS metric *outside* its horizon is non-static.

For $0 < M < M_n = \beta/\sqrt{27}$, there exist two positive zeros of $V_{\text{SdS}}(r)$ at $r_h = (-2\beta/\sqrt{3})\cos[(A+\pi)/3]$ and $r_c = (2\beta/\sqrt{3})\cos(A/3) > r_h$, where $\pi/2 < A < \pi$, $A \equiv \arccos[-(27M^2/\beta^2)^{\frac{1}{2}}]$. They are associated with the black hole event and cosmological horizons, respectively. For $M = M_n$ both horizons coalesce and it then results the Nariai solution [124], which represents the largest black hole one can have in dS spacetime. For $M < 0$ the black hole disappears, and the spacetime describe a naked singularity in $r = 0$ surrounded by a cosmological horizon. Finally, there is no static region for $M > M_n$ and the solution in this case is asymptotically dS only in the far past. Hence, one has to discard the solutions with $M > M_n$, if one wants to consider only spacetimes that approach dS in both past and future.

Our work of analyzing the motion of pseudo-classical spinning point particles in this chapter may be interesting in view of the inflationary scenario of the universe.

The plane of this chapter is as follows. In section 3.2 we find the vectors and Killing scalars for the dS/SdS spacetime. The constants of motion are derived. In section 3.3 we consider the motion in a plane and analyze specific solutions. The perihelion precession is also discussed. In section 3.4 we present our concluding remarks.

3.2 Spinning Particles in dS/SdS Spacetime

In this section we apply the formalisms described in chapters 1 and 2 to investigate the geodesic motion of spinning particles in the spacetimes described by the metric (3.1). This metric possesses four Killing vector fields of the form

$$D^{(\alpha)} \equiv R^{(\alpha)\mu}\partial_\mu, \qquad \alpha = 0, \cdots, 3, \tag{3.2}$$

where

$$D^{(0)} = \frac{\partial}{\partial t}, \qquad D^{(1)} = -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi},$$
$$D^{(2)} = \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}, \qquad D^{(3)} = \frac{\partial}{\partial\varphi}. \tag{3.3}$$

These Killing vector fields describe the time-translation invariance and the spatial rotation symmetry of the field and generate the Lie algebra $O(1,1)\times SO(3)$:

$$[D^{(i)}, D^{(j)}] = -\varepsilon^{ijk}D^{(k)}, \quad [D^{(0)}, D^{(i)}] = 0, \quad (i,j,k = 1,2,3). \tag{3.4}$$

The first generalized Killing equation of (1.33) shows that for each Killing vector $R_\mu^{(\alpha)}$ there is an associated Killing scalar $B^{(\alpha)}$, which is necessary to obtain the constants of motion:

$$J^{(\alpha)} = B^{(\alpha)} + m\dot{x}^\mu R_\mu^{(\alpha)}. \tag{3.5}$$

This asserts that the contribution of spin is contained in the Killing scalars $B^{(\alpha)}$ and the Killing vector itself does not give a conserved quantity of motion without the Killing scalars. For the spacetime of (3.1) we obtain the the Killing scalars

$$\begin{aligned}
B^{(0)} &= \frac{1}{2}\frac{dV}{dr}S^{tr}, \\
B^{(1)} &= -r\sin\varphi S^{r\theta} - r\sin\theta\cos\theta\cos\varphi S^{r\varphi} + r^2\sin^2\theta\cos\varphi S^{\theta\varphi}, \\
B^{(2)} &= r\cos\varphi S^{r\theta} - r\sin\theta\cos\theta\sin\varphi S^{r\varphi} + r^2\sin^2\theta\sin\varphi S^{\theta\varphi}, \\
B^{(3)} &= r\sin^2\theta S^{r\varphi} + r^2\sin\theta\cos\theta S^{\theta\varphi}.
\end{aligned} \tag{3.6}$$

The four conserved quantities $J^{(\alpha)}$ are found as follows:

$$J^{(0)} \equiv E = mV\frac{dt}{d\tau} + B^{(0)},$$

$$J^{(1)} = B^{(1)} - mr^2 \left(\sin\varphi \frac{d\theta}{d\tau} + \sin\theta\cos\theta\cos\varphi \frac{d\varphi}{d\tau} \right),$$

$$J^{(2)} = B^{(2)} + mr^2 \left(\cos\varphi \frac{d\theta}{d\tau} - \sin\theta\cos\theta\sin\varphi \frac{d\varphi}{d\tau} \right),$$

$$J^{(3)} = B^{(3)} + mr^2 \sin^2\theta \frac{d\varphi}{d\tau}. \tag{3.7}$$

In addition to these conserved quantities, there are four generic constants of motion as described in chapter 1, given by equations (1.38), (1.41), (1.43) and (1.44). We consider motion for $H = -m^2/2$, which yields geodesic motion: $g_{\mu\nu} dx^\mu dx^\nu = -d\tau^2$. The condition for the absence of an intrinsic electric dipole moment of physical fermions (leptons and quarks) as formulated in (1.15) gives the supersymmetric constraint $Q = 0$, which gives ψ^t in terms of the spatial components ψ^i:

$$V \frac{dt}{d\tau} \psi^t = \frac{1}{V} \frac{dr}{d\tau} \psi^r + r^2 \left(\frac{d\theta}{d\tau} \psi^\theta + \sin^2\theta \frac{d\varphi}{d\tau} \psi^\varphi \right) \tag{3.8}$$

and as a result, the dual supercharge Q^* and the chiral charge Γ^* vanish as well: $Q^* = \Gamma^* = 0$. The independent linear combination of $J^{(1)}$ and $J^{(2)}$ given by

$$r^2 \sin\theta S^{\theta\varphi} = J^{(1)} \sin\theta\cos\varphi + J^{(2)} \sin\theta\sin\varphi + J^{(3)} \cos\theta, \tag{3.9}$$

implies that there is only the spin angular momentum in the radial direction.

One can now obtain a complete set of first integrals of motion for physical fermions, expressing the velocities as functions of the co-ordinates, the spatial spin components and the constants of motion:

$$\begin{aligned}
\frac{dt}{d\tau} &= \frac{1}{V}\left(\frac{E}{m} + \frac{1}{2m}\frac{dV}{dr}S^{rt}\right), \\
\frac{dr}{d\tau} &= \left\{V^2\left(\frac{dt}{d\tau}\right)^2 - V - r^2V\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2\right]\right\}^{\frac{1}{2}}, \\
\frac{d\theta}{d\tau} &= \frac{1}{mr^2}\left(-J^{(1)}\sin\varphi + J^{(2)}\cos\varphi - rS^{r\theta}\right), \\
\frac{d\varphi}{d\tau} &= \frac{1}{mr^2\sin^2\theta}J^{(3)} - \frac{1}{mr}S^{r\varphi} - \frac{1}{m}\cot\theta S^{\theta\varphi},
\end{aligned} \tag{3.10}$$

where

$$S^{rt} = \frac{mr^2}{E}\left(\frac{d\theta}{d\tau}S^{r\theta} + \sin^2\theta\frac{d\varphi}{d\tau}S^{r\varphi}\right). \tag{3.11}$$

Finally, (1.13) gives equations for the rate of change of the spins. The equations which are left for solution are

$$\frac{dS^{r\theta}}{d\tau} = -\frac{1}{r}\frac{dr}{d\tau}S^{r\theta} + \sin\theta\cos\theta\frac{d\varphi}{d\tau}S^{r\varphi} - r\sin^2\theta\frac{d\theta}{d\tau}S^{\theta\varphi},$$

$$\frac{dS^{r\varphi}}{d\tau} = \cot\theta \frac{d\varphi}{d\tau} S^{r\theta} - \left(\frac{1}{r}\frac{dr}{d\tau} + \cot\theta\frac{d\theta}{d\tau}\right) S^{r\varphi} + r\frac{d\theta}{d\tau} S^{\theta\varphi}, \quad (3.12)$$

where $S^{\theta\varphi}$ is given by (3.9). Equations (3.10)–(3.12) have to be integrated for the full solution of the equations of motion for all coordinates and spins.

3.3 Special Solutions

We solve the equations obtained in section 3.2 for the special case of motion in a plane with $\theta = \pi/2$. In contrast to scalar point particles, this is not the generic case. Because orbital angular momentum is not separately conserved in general. Planar motion for spinning particles occurs only in two kinds of situations. One possibility is radial motion, for which $\dot{\varphi} = 0$. This case indicates that there is no orbital angular momentum and spin is conserved independently. The other possibility concerns motion for which $\dot{\varphi} \neq 0$. This case happens if spin and orbital angular momentum are parallel.

Equations (3.10)–(3.12), with $\theta = \pi/2$ and $\dot{\theta} = 0$, are written as

$$\frac{dt}{d\tau} = \frac{1}{V}\left(\frac{E}{m} + \frac{r^2}{2E}\frac{dV}{dr}\frac{d\varphi}{d\tau}S^{rt}\right),$$

$$\frac{dr}{d\tau} = \left\{V^2\left(\frac{dt}{d\tau}\right)^2 - V - r^2 V\left(\frac{d\varphi}{d\tau}\right)^2\right\}^{\frac{1}{2}},$$

$$\begin{aligned}
\frac{d\varphi}{d\tau} &= \frac{1}{mr^2}J^{(3)} - \frac{1}{mr}S^{r\varphi}, \\
\frac{d}{d\tau}\left(rS^{r\theta}\right) &= -r^2\left(V - \frac{r}{2}\frac{dV}{dr}\right)S^{\theta\varphi}\frac{d\varphi}{d\tau}, \\
\frac{d}{d\tau}\left(rS^{r\varphi}\right) &= 0.
\end{aligned} \tag{3.13}$$

The third and the last parts of (3.13) describe that the orbital angular momentum and the component of the spin perpendicular to the plane in which the particle moves, are separately conserved:

$$mr^2\dot{\varphi} = J^{(3)} - \Sigma \equiv L, \qquad rS^{r\varphi} \equiv \Sigma, \tag{3.14}$$

where Σ and L are two constants. The first of (3.13) gives a formula for the gravitational redshift in the form

$$dt = \frac{d\tau}{V}\left(\frac{E}{m} + \frac{1}{2mEr}\frac{dV}{dr}L\Sigma\right), \tag{3.15}$$

which shows that the time-dilation receives a contribution from spin-orbit coupling for nonzero orbital angular momentum L. Hence, time-dilation is not a purely geometric effect, but also has a dynamical component.

Equations (3.9), third of (3.10), and fourth of (3.13) with $\theta = \pi/2$ yield indeed only two possibilities for the planar motion:

$$(i)\ \dot{\varphi} = 0, \qquad (ii)\ S^{\theta\varphi} = 0. \tag{3.16}$$

The first possibility $\dot{\varphi} = 0$ implies that $L = 0$, that is, the particle moves along a fixed radius. The motion of the particle for a distant observer is described by

$$\frac{dr}{dt} = \frac{V}{E}\sqrt{(E^2 - m^2 V)} \tag{3.17}$$

as in the case of a spinless particle. Along the path of the particle with $\varphi = 0$, the spin tensor components are all conserved:

$$r^2 S^{\theta\varphi} = J^{(1)}, \quad r S^{r\theta} = J^{(2)}, \quad r S^{r\varphi} = J^{(3)}. \tag{3.18}$$

The second possibility $\dot{\varphi} \neq 0$ gives

$$S^{\theta\varphi} = 0, \quad S^{r\theta} = 0, \quad J^{(1)} = J^{(2)} = 0, \tag{3.19}$$

which states that the spin is parallel to the orbital angular momentum. Equation (3.13) for $\dot{r}$ and $\dot{\varphi}$ give the following equation for the orbit of the particle:

$$\begin{aligned} \frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2 &= \frac{E^2 - m^2}{L^2} r^2 - 1 - \frac{m}{L} r^2 \frac{dV}{dr}\left(\frac{mr}{L} + \frac{J^{(3)}}{mr}\right) \\ &+ \left(1 - V - r\frac{dV}{dr}\right)\left(1 + \frac{m^2 r^2}{L^2}\right). \end{aligned} \tag{3.20}$$

3.3.1 Spinning Particles in dS Spacetime

We now consider the particle in the dS spacetime. Taking $V(r) \mapsto V_{\mathrm{dS}}(r) = 1 - \frac{r^2}{\beta^2}$, the equation (3.20) for the orbit of the particle becomes

$$\begin{aligned} \frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2 &= \frac{E^2 - m^2}{L^2} r^2 - 1 - \frac{2mr^3}{\beta^2 L} \\ &\quad \times \left(\frac{mr}{L} + \frac{J^{(3)}}{mr}\right) + \frac{3r^2}{\beta^2}\left(1 + \frac{m^2 r^2}{L^2}\right). \end{aligned} \tag{3.21}$$

In terms of the dimensionless quantities

$$\epsilon = \frac{E}{m}, \quad x = \frac{r}{\beta}, \quad l = \frac{L}{m\beta}, \quad \Delta = \frac{\Sigma}{L}, \tag{3.22}$$

(3.21) can be written as

$$\frac{l^2}{x^4}\left(\frac{dx}{d\varphi}\right)^2 = \beta^2 \dot{x}^2 = \epsilon^2 - U_R(x, l^2), \tag{3.23}$$

where

$$U_R(x, l^2) = 1 - l^2(1 - 2\Delta) - x^2 + \frac{l^2}{x^2} \tag{3.24}$$

defines an effective potential.

All of the manipulations presented here are rather formal, as Δ is not a pure number but a bilinear combination of anti-commuting variables

ψ^μ. In order to analyze possible motions of the spinning particle based on (3.23) one needs to replace Δ in certain limiting cases by a real number. As mentioned in the Introduction of this thesis, such a limit might arise in the semi-classical regime of the quantum theory, as implied by the correspondence principle. Henceforth we assume that such a numerical value of Δ has been derived and leads to valid results, at least in expansions to first order in Δ with the fact that $\Delta^2 = 0$ plays no role.

Equation (3.23) is the same as one would find for a one-dimensional problem with a potential $U_R(x, l^2)$ which produces a radial force

$$F(x, l^2) = -\frac{\partial}{\partial x} U_R(x, l^2), \tag{3.25}$$

This is the effective force that the three-dimensional particle feels in the radial direction, including a contribution from the centripetal acceleration. The radial kinetic energy is non-negative, hence the right-hand side of (3.23) must be non-negative as well.

Since $\partial U_R/\partial x < 0$ the approaching particle is subject to an repulsive effective force. Particles with energy exceeding $[U_R]_{x=1} = 2l^2\Delta$ pass the horizon, but they stream away from the center instead of reaching it. We note that this is different from the spinless particle case. The orbit of the particle is given by

$$(\varphi - \varphi_0) = \pm\frac{1}{2l}\sin^{-1}\frac{\gamma - 2l^2u^2}{\sqrt{\gamma^2+4}}, \tag{3.26}$$

where

$$\gamma = 1 - 2\Delta - \frac{1-\epsilon^2}{l^2}, \qquad u = \frac{1}{x},$$

and obviously, it contains terms contributed by particle's spin.

The scattering angle of the particle can be calculated from (3.23) [42]. Considering l^2 is small and $\epsilon^2 - 1$ is large, such that

$$1 \leq \epsilon^2 \leq U_R(x, l^2), \tag{3.27}$$

and defining

$$u = \frac{l}{x\sqrt{\epsilon^2 - 1}}, \tag{3.28}$$

we obtain

$$\left(\frac{du}{d\varphi}\right)^2 = 1 + \gamma + \frac{\delta^2}{u^2} - u^2 \equiv -W(u, \delta, \gamma), \tag{3.29}$$

where

$$\delta = \frac{l}{\epsilon^2 - 1}, \qquad \gamma = \frac{l^2(1-2\Delta)}{\epsilon^2 - 1} \tag{3.30}$$

are constants of motion. In the limit of vanishing γ, the solution of (3.29) gives the classical orbit. By expanding to first order in γ, small relativistic correction can be calculated. The scattering angle Θ is given by

$$\Theta = \pi - 2\int_0^{u_{\rm ph}} \frac{du}{\sqrt{(1+\gamma)+(\delta^2/u^2)-u^2}}. \tag{3.31}$$

In the perihelion $dx/d\varphi = 0$, and hence $x_{\rm ph}$ is one of the roots of

$$\epsilon^2 - U_R(x, l^2) = 0, \tag{3.32}$$

which has four roots, x_1, x_2, x_3, $x_{\rm ph}$, two of them are imaginary, one is negative and the remaining one is positive. The negative or imaginary roots are not realistic. These roots correspond to the four zeros u_1, u_2, u_3, $u_{\rm ph}$ of $W(u, \delta, \gamma)$, defined by (3.29).

The integral in (3.31) can be written in the form

$$\int_0^{u_{\rm ph}} \frac{u\,du}{\sqrt{(u_3-u)(u_2-u)(u_{\rm ph}-u)(u-u_1)}} \tag{3.33}$$

with u_1, u_2, u_3 and $u_{\rm ph}$ given by

$$u_{\rm ph} = \left(\frac{1}{2}(1+\gamma)+\frac{1}{2}\sqrt{(1+\gamma)^2+4\delta^2}\right)^{\frac{1}{2}},$$

$$u_1 = -\left(\frac{1}{2}(1+\gamma)+\frac{1}{2}\sqrt{(1+\gamma)^2+4\delta^2}\right)^{\frac{1}{2}},$$

$$u_2 = \mathrm{i}\left(\frac{1}{2}\sqrt{(1+\gamma)^2+4\delta^2}-\frac{1}{2}(1+\gamma)\right)^{\frac{1}{2}},$$

$$u_3 = -\mathrm{i}\left(\frac{1}{2}\sqrt{(1+\gamma)^2+4\delta^2}-\frac{1}{2}(1+\gamma)\right)^{\frac{1}{2}}. \tag{3.34}$$

Expanding the factors to first order in γ and performing integration, we obtain

$$\begin{aligned}\Theta &= \pi - 2\arcsin\left[\frac{2+5\delta^2}{2+4\delta^2}\left(1+\frac{3-4\delta^2}{2+10\delta^2}\gamma\right)\right] \\ &+\frac{2\delta}{1+4\delta^2}(1-\gamma)+O(\gamma^2).\end{aligned} \tag{3.35}$$

The limit $\gamma \to 0$ gives the result for the classical particle. The spin of particle contributes to this value a small relativistic corrections only if the coefficients of terms of $O(\gamma^s)$ for $s > 1$ do not blow-up due to their dependence on δ. One can check that terms of the expansion (3.35) converges under the condition (3.27).

3.3.2 Spinning Particles in SdS Spacetime

The pseudo-classical spinning particles' motion in the SdS spacetime was investigated in [125], which contains a mistake in the expression for the effective potential U_R. In this section we review this work with the correct form of U_R.

For $V(r) \mapsto V_{\mathrm{SdS}}(r) = 1 - \frac{\alpha}{r} - \frac{r^2}{\beta^2}$, the equation (3.20) for the orbit of the spinning particle in SdS spacetime takes the form

$$\frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2 = \frac{E^2-m^2}{L^2}r^2 - 1 + \frac{m}{L}\left(\alpha - \frac{2r^3}{\beta^2}\right) \times \left(\frac{mr}{L} + \frac{J^{(3)}}{mr}\right) + \frac{3r^2}{\beta^2}\left(1 + \frac{m^2r^2}{L^2}\right). \tag{3.36}$$

With the dimensionless quantities

$$\epsilon = \frac{E}{m}, \quad x = \frac{r}{r_h}, \quad l = \frac{L}{mr_h},$$
$$\Delta = \frac{\Sigma}{L}, \quad \lambda = \frac{\beta}{r_h}, \quad \eta = \frac{\alpha}{r_h}, \tag{3.37}$$

(3.36) can be written as

$$\frac{l^2}{x^4}\left(\frac{dx}{d\varphi}\right)^2 = r_h^2\dot{x}^2 = \epsilon^2 - U_R(x, l^2, \lambda^2, \eta), \tag{3.38}$$

where

$$U_R(x, l^2, \lambda^2, \eta) = 1 - \frac{1}{\lambda^2}[x^2 + (1-2\Delta)l^2] - \frac{\eta}{x} + \frac{l^2}{x^2} - \eta l^2(1+\Delta)\frac{1}{x^3} \tag{3.39}$$

defines an effective potential. In [125] the dimensional variables were defined by using α instead of r_h and there were two extra terms (x-term and

x^3-term) inserted in the expression of U_R. As $\eta \to 0$, $\lambda \to 1$ and r_h is replaced by $r_c = \beta$, (3.38) and (3.39) reduce to the dS case (3.23) and (3.24).

When $\epsilon \geq 1$ there exist open orbits for which there is at most one point of closest approach, the perihelion. The particle can cross into the central region of the potential ($x < 1$), if the energy exceeds some critical value $\epsilon_{\text{crit}} = \epsilon(x_m)$ for fixed l, x_m being a minimum point of U_R.

For $\epsilon < 1$ there are bound states that correspond to quasi-elliptic and circular orbits. With

$$l^2 = \frac{\lambda^2 \eta x^2 - 2x^5}{\lambda^2[2x - 3\eta(1+\Delta)]}, \tag{3.40}$$

there exists a circular orbit at the point of infection of $U_R(x, l^2, \lambda^2, \eta)$ with minimum radius x_{crit} given by

$$8x_{\text{crit}}^4 - 15\eta(1+\Delta)x_{\text{crit}}^3 - \lambda^2\eta x_{\text{crit}} + 3\lambda^2\eta^2(1+\Delta) = 0. \tag{3.41}$$

The energy for this critical orbit is given by

$$\epsilon_{\text{crit}}^2 = 1 - \left(\frac{\eta}{x_{\text{crit}}} + \frac{x_{\text{crit}}^2}{\lambda^2}\right) - l^2 \left[\frac{1}{\lambda^2}(1 - 2\Delta) - \frac{1}{x_{\text{crit}}^2} + \eta(1+\Delta)\frac{1}{x_{\text{crit}}^3}\right], \tag{3.42}$$

while the time-dilation factor is expressed by

$$\left(\frac{dt}{d\tau}\right)_{\text{crit}} = \frac{1}{(1-\eta/x_{\text{crit}} - x_{\text{crit}}^2/\lambda^2)}\left[\epsilon_{\text{crit}} + \frac{l^2\Delta}{2\epsilon_{\text{crit}}}\left(\frac{\eta}{x_{\text{crit}}^3} - \frac{2}{\lambda^2}\right)\right]. \tag{3.43}$$

If one imposes the limit $\lambda \to \infty$, the results exactly reduce to the Schwarzschild spacetime case [42] in which the radius of the minimal circular orbit is

$$x_{\text{crit}} = l^2 = 3(1+\Delta), \tag{3.44}$$

and the energy and the time-dilation for this orbit are respectively given by

$$\epsilon_{\text{crit}} = \frac{1}{9}(8+\Delta), \qquad \left(\frac{dt}{d\tau}\right)_{\text{crit}} = \sqrt{2}\left(1-\frac{3}{8}\Delta\right) \tag{3.45}$$

to first order in Δ.

We now discuss the quasi-elliptic orbits. In the case of non-circular motion the perihelion of the orbit precesses as for a spinless particle, but at a different rate.

The orbits approaching the precessing ellipses are described by

$$x = \frac{\kappa}{1+e\cos[\varphi - w(\varphi)]}, \tag{3.46}$$

where $\kappa = k/r_h$, k being the semilatus rectum and e the eccentricity with $0 < e < 1$. The perihelion and aphelion are defined by

$$\varphi_{\rm ph}^{(n)} - w(\varphi_{\rm ph}^{(n)}) = 2n\pi, \qquad \varphi_{\rm ah}^{(n)} - w(\varphi_{\rm ah}^{(n)}) = (2n+1)\pi. \tag{3.47}$$

The particle reaches its n-th perihelion at the angle $\varphi_{\rm ph}^{(n)}$ and the amount of precession of the perihelion after n revolutions is given by $w(\varphi_{\rm ph}^{(n_o)})$. For example, the precession of the perihelion after one revolution is found to be

$$\Delta w \equiv w(\varphi_{\rm ph}^{(1)}) - w(\varphi_{\rm ph}^{(0)}) = \varphi_{\rm ah}^{(1)} - \varphi_{\rm ah}^{(0)} - 2\pi \equiv \Delta\varphi - 2\pi. \tag{3.48}$$

The energy ϵ is a constant of motion and its value at the perihelion/aphelion is given by

$$\begin{aligned}\epsilon^2 = 1 - \frac{1}{\lambda^2}\left[\left(\frac{\kappa}{1\pm e}\right)^2 + l^2(1-2\Delta)\right] - \eta\left(\frac{1\pm e}{\kappa}\right) \\ +l^2\left(\frac{1\pm e}{\kappa}\right)^2 - \eta l^2(1+\Delta)\left(\frac{1\pm e}{\kappa}\right)^3.\end{aligned} \tag{3.49}$$

From comparison of both expressions for ϵ^2, it follows that

$$l^2 = \frac{\kappa^2}{2\kappa - \eta(3+e^2)(1+\Delta)}\left[\eta - \frac{2\kappa^3}{\lambda^2(1-e^2)^2}\right]. \tag{3.50}$$

Defining

$$y = \varphi - w(\varphi), \tag{3.51}$$

substituting (3.46) for x, using (3.49) and (3.50), (3.38) can be put in the form

$$d\varphi = \left(\frac{e}{\kappa}\right) \frac{\sin y(1 + e\cos y)dy}{\sqrt{\sum\limits_{r=0}^{5} c_r(e\cos y)^r}}, \tag{3.52}$$

where

$$\begin{aligned}
c_o &= a_1 + \frac{1}{\lambda^2\kappa^2}\left\{(2a_2\kappa + 3)\kappa^2 - \lambda^2(1 - a_2\eta)\right\} \\
&\quad + a_3F + \frac{1}{\lambda^2\kappa^2}\left\{\lambda^2\eta - 2\kappa^3 - (3 + e^2)a_2\eta(\lambda^2\eta + \kappa^3)\right\}F, \\
c_1 &= 2a_1 + \frac{2}{\lambda^2\kappa^2}\left\{3\kappa^2 - \lambda^2(2 - 3a_2\eta)\right\} \\
&\quad + 2a_3F + \frac{1}{\lambda^2\kappa^2}\left\{5\lambda^2\eta - 4\kappa^3 - 3(3 + e^2)a_2\lambda^2\eta^2\right\}F, \\
c_2 &= a_1 + \frac{3}{\lambda^2\kappa^2}\left\{\kappa^2 - 2\lambda^2(1 - a_2\eta)\right\} \\
&\quad + a_3F + \frac{1}{\lambda^2\kappa^2}\left\{10\lambda^2\eta - 2\kappa^3 - 3(3 + e^2)a_2\lambda^2\eta^2\right\}F,
\end{aligned}$$

$$
\begin{aligned}
c_3 &= -\frac{2}{\kappa^2}(2 - a_2\eta) + \frac{\eta}{\kappa^2}\left\{10 - (3+e^2)a_2\eta\right\}F, \\
c_4 &= -\frac{1}{\kappa^2}(1 - 5\eta F), \\
c_5 &= \kappa\eta F, \qquad F = \frac{1+\Delta}{\kappa}, \\
a_1 &= -\frac{2a_2a_4}{\kappa} + \left(\frac{1+e}{\kappa}\right)^2 - \frac{3}{\lambda^2}, \\
a_2 &= \frac{\lambda^2(1-e^2)^2}{\lambda^2\eta(1-e^2)^2 - 2\kappa^3}, \\
a_3 &= \frac{1}{\kappa}a_2a_4\eta(3+e^2) - \frac{1}{\kappa^2}\eta(1+e)^3 + \frac{2\kappa}{\lambda^2}, \\
a_4 &= \frac{1}{\lambda^2}\left(\frac{\kappa}{1+e}\right)^2 + \eta\left(\frac{1+e}{\kappa}\right).
\end{aligned}
$$

In the limit $\lambda \to \infty$ this equation gives

$$
d\varphi = \frac{dy}{\sqrt{1 - [(1+\Delta)/\kappa](3 + e\cos y)}}, \tag{3.53}
$$

the Schwarzschild result [42], which gives for $\Delta\varphi$ as defined in (3.48) on integration from one perihelion to the next one with $0 \leq y \leq 2\pi$ the result

$$
\Delta\varphi = 2\pi\left[1 + \frac{3}{2}\left(\frac{1+\Delta}{\kappa}\right) + \frac{3}{16}(e^2+18)\left(\frac{1+\Delta}{\kappa}\right)^2 + \cdots\right]. \tag{3.54}
$$

For equation (3.52) we find

$$
\begin{aligned}
\Delta\varphi &= 2\pi\frac{e/\kappa}{\sqrt{\xi_1}}\left(1-\frac{1}{2}\xi_2F+\frac{3}{8}(\xi_2F)^2+\cdots\right)\\
&\quad\times\left[1-\frac{525}{4096}\left\{\frac{1}{\xi_1}[(\xi_3+\xi_5)-(\xi_3(\xi_2-\xi_4)+\xi_5(\xi_2-\xi_6))\right.\right.\\
&\quad\quad\left.\left.\times(F-\xi_2F^2)]e^2+\frac{432}{175}\right\}+\cdots\right], \qquad (3.55)
\end{aligned}
$$

where ξ's are such that $c_o = \xi_1(1+\xi_2F)$, $c_1 = \xi_3(1+\xi_4F)$ and $c_2 = \xi_5(1+\xi_6F)$. As $\lambda \to \infty$, (3.55) reduces to

$$
\Delta\varphi = 2\pi\left[1+\frac{3}{2}\left(\frac{1+\Delta}{\kappa}\right)+\frac{27}{256}(32+5e^2)\left(\frac{1+\Delta}{\kappa}\right)^2+\cdots\right], \qquad (3.56)
$$

which is similar with the result in (3.54) and gives exactly the same result if we discard terms second and higher orders in $\left(\frac{1+\Delta}{\kappa}\right)$. The second term in the expansion with $\Delta = 0$ is the well-known contribution to the relativistic precession of the perihelion. We observe that in principle the spin of a particle contributes to this lowest-order precession, if terms of first order in Δ are retained.

3.4 Concluding Remarks

This study mainly concerns the investigation of spinning point particle's motion in the purely de Sitter and asymptotically de Sitter Schwarzschild spacetimes by using pseudo-classical mechanics models. In these models spinning spaces are graded extensions of ordinary Riemannian manifolds, with additional fermionic dimensions characterized by vectorial Grassmann coordinates ψ^{μ}. The spin of the particle is described by the antisymmetric spin-tensor $S^{\mu\nu} = -\mathrm{i}\psi^{\mu}\psi^{\nu}$.

The model we have used is very simple and describes a classical limit of the Dirac equation. Even though there is no satisfactory quantum theory for gravitational interaction, this study is justified and not at all trivial. The results of this chapter can be used to investigate the aspects of the motion of fermions such as electrons or, possibly, massive neutrinos (or photinos, gravitinos, etc.) in an empty de Sitter and asymptotically de Sitter Schwarzschild spacetimes. These aspects include the spin-orbit coupling and the corresponding fine splitting, resulting from dependence of the energy on the values and relative orientation of the orbital and spin angular momentum. Our study shows that the time dilation, perihelion precession for bound-state orbits, and the scattering of particles in the de Sitter/asymptotically de Sitter spacetime receive contributions from spin. This leads to the existence of a gravitational analogue of the Stern-Gerlach-type forces well known to appear in electromagnetic phenomena. This spin-dependent contribution can be larger or smaller than for a spinless particle according as the sign of Δ (defined in (3.22)), i.e. the relative orientation of L and Σ. This is interpreted as a classical analogue of fine splitting. Although an *a priori* numerical value for the ratio Δ cannot be assigned, its appearance still allows the pseudo-classical theory to make quantitative predictions by comparing different physical processes in the regime where the semi-classical limit applies.

The equations of motion (1.8) and (1.9) remain valid if averaged inside a functional integral with the exponential of the action (1.3) in the integrand, that is, when $S^{\mu\nu} = -i\psi^{\mu}\psi^{\nu}$ is replaced by its quantum mechanical expectation value $\langle S^{\mu\nu}\rangle$. This permits to consider our results as a semiclassical approximation to the quantum Dirac theory, and provides a procedure to evaluate numerically the components of the spin tensor (at least in principle). However, since $\langle S^{\mu\nu}\rangle^2 \neq \langle (S^{\mu\nu})^2\rangle$ in general, this approximation can only hold to first order in the spin.

The results obtained for the Schwarzschild-de Sitter spacetime reduce to that of the Schwarzschild spacetime [42] for the vanishing of the cosmological constant, $\Lambda = 0$. As the gravitating mass $M \to 0$ the Schwarzschild-de Sitter spacetime becomes the pure de Sitter spacetime and the results then correspond to that of the resulting de Sitter spacetime.

In recent years some important progresses have been achieved in the astronomical observations [126, 127], which have led to the surprising conclusion that the recent universe is dominated by a "dark" exotic form of energy density that acts repulsively at large scales. The simplest and best known candidate for the "dark energy" is the cosmological constant. In this scenario the de Sitter geometry [74, 75, 76] appears to take the double role of reference geometry of the universe, namely the geometry of spacetime deprived of its matter and radiation content and of geometry that the universe approaches asymptotically. This spacetime is of great theoretical as well as cosmological interest. In this view the study of the de Sitter geometry of the graded pseudo-manifolds with both real number $\{x^\mu\}$ and anticommuting variables $\{\psi^\mu\}$ is well motivated.

Chapter 4

Geodesic Motions in Euclidean Taub-NUT Spinning Spaces

4.1 Introduction

In this chapter we investigate the geodesic motion of the pseudo-classical spin-$\frac{1}{2}$ point particle in the geometry of Euclidean Taub-NUT (ETN) [80]. The ETN manifold M_4 is a 4-dimensional Kaluza-Klein space which has static charts with the Cartesian coordinates x^μ ($\mu = 1, 2, 3, 4$). Here, x^i ($i = 1, 2, 3$) are the physical Cartesian space coordinates and x^4 is the Cartesian extra-coordinate. Using the usual three-dimensional vector notations, $\mathbf{x} = (x^1, x^2, x^3)$, $r = |\mathbf{x}|$ and $dl^2 = d\mathbf{x} \cdot d\mathbf{x}$, the line element can be put in the form

$$ds^2 = f(r)dl^2 + \frac{1}{f(r)}\left[dx^4 + A_i(\mathbf{x})dx^i\right]^2, \tag{4.1}$$

where

$$f(r) = 1 + \frac{\mu}{r}, \quad A_1 = -\frac{\mu}{r}\frac{x^2}{r+x^3}, \quad A_2 = \frac{\mu}{r}\frac{x^1}{r+x^3}, \quad A_3 = 0, \tag{4.2}$$

μ being a real parameter. If $\mathbf{A}$ is interpreted as the gauge field of a monopole, it results the magnetic field with central symmetry

$$\mathbf{B} = \mu\frac{\mathbf{x}}{r^3}. \tag{4.3}$$

In the chart of spherical coordinates $(r, \theta, \varphi, \psi)$ where r, θ, φ are commonly related to the physical Cartesian ones x^i, the apparent singularity at the origin is unphysical if x^4 is periodic with period $4\pi\mu$ [128, 129, 130, 131, 132]. As a result, the fourth coordinate ψ is defined such that

$$x^4 = -\mu(\psi + \varphi). \tag{4.4}$$

This chart covers the domain where $r > 0$ for $\mu > 0$ or $r > |\mu|$ for $\mu < 0$, the angular coordinates θ, φ cover the sphere S^2 and $0 \leq \psi < 4\pi$. Since $A_r = A_\theta = 0$, $A_\varphi = \mu(1 - \cos\theta)$, with $\mu = 2m$ the line element can be put in the form

$$ds^2 = f(r)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) + g(r)(d\psi + \cos\theta d\varphi)^2, \tag{4.5}$$

where

$$f(r) = 1 + \frac{2m}{r}, \qquad g(r) = 1 + \frac{4m^2}{f(r)}. \tag{4.6}$$

The organization of the chapter is as follows. In section 4.2 we investigate the motion of pseudo-classical spinning particles in the Euclidean Taub-NUT space. We examine the generalized Killing equations for this spinning space and derive the constants of motion in terms of the Killing-Yano tensors. In section 4.3 we solve the equations derived in section 4.2 for special case of motion on a cone and on a plane. Finally, in section 4.4 we present our concluding remarks.

4.2 Motion in Euclidean Taub-NUT Spinning Space

We use the formalisms described in chapters 1 and 2 for the pseudo-classical spinning point particles in curved spacetime and investigate the motion in the ETN space with the metric (4.5). The invariance of the metric under spatial rotations and translations is generated by four Killing vectors [128, 130]

$$D^{(\alpha)} \equiv R^{(\alpha)\mu}\partial_\mu, \qquad \alpha = 0, \cdots, 3 \qquad \mu = (r, \theta, \varphi, \psi), \tag{4.7}$$

where

$$R^{(0)} = (0,0,0,1), \quad R^{(1)} = (0,\, -\sin\varphi,\, -\cot\theta\cos\varphi,\, \csc\theta\cos\varphi),$$
$$R^{(2)} = (0,\, \cos\varphi,\, -\cot\theta\sin\varphi,\, \csc\theta\sin\varphi), \quad R^{(3)} = (0,0,1,0). \tag{4.8}$$

$D^{(0)}$, which generates the $U(1)$ of ψ translations, commutes with the other Killing vectors. The remaining three vectors, corresponding to the invariance of the metric (4.5) under spatial rotations ($\alpha = 1, 2, 3$), obey an $SU(2)$ algebra with

$$[D^{(i)}, D^{(j)}] = -\varepsilon^{ijk} D^{(k)}, \quad (i,j,k = 1,2,3). \tag{4.9}$$

This is contrasted with the Schwarzschild space, where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This demonstrates the essential topological character of the magnetic monopole mass [133].

These invariances, in the bosonic case, would correspond to the conservation of the so-called "relative electric charge" and the angular momentum [128, 129, 130, 131]:

$$q = g(r)(\dot\psi + \cos\theta\dot\varphi), \tag{4.10}$$

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} + q\frac{\mathbf{r}}{r}, \tag{4.11}$$

where $\mathbf{p} = f(r)\dot{\mathbf{r}}$ is the *mechanical momentum* canonically conjugate to $\mathbf{r}$.

From the first generalized Killing equation of (1.33), it follows that to each Killing vector there is associated a Killing scalar. If we limit ourselves to variations (1.27) that terminate after the terms linear in $\dot{x}^\mu$, the corresponding constants of motion would be of the form

$$\mathcal{J}^{(\alpha)} = B^{(\alpha)} + m\dot{x}^\mu R_\mu^{(\alpha)}, \tag{4.12}$$

which asserts that the Killing scalars $B^{(\alpha)}$ contribute to the "relative electric charge" and the total angular momentum.

For the Euclidean Taub-NUT metric (4.5), we obtain

$$\begin{aligned}
B^{(0)} &= \frac{g'}{2}S^{r\psi} + \frac{g'}{2}\cos\theta S^{r\varphi} - \frac{g}{2}\sin\theta S^{\theta\varphi}, \\
B^{(1)} &= -\frac{g'}{2}\sin\theta\cos\varphi S^{r\psi} - \frac{g}{2}\cos\theta\cos\varphi S^{\theta\psi} + \frac{g}{2}\sin\theta\sin\varphi S^{\varphi\psi}, \\
&+\frac{1}{2}(2rf + r^2 f')\sin\varphi S^{r\theta} + (2rf + r^2 f' - g')\sin\theta\cos\theta\cos\varphi S^{r\varphi} \\
&+(g + 2fr^2\cos^2\theta)\cos\varphi S^{\theta\varphi}, \\
B^{(2)} &= -\frac{\partial}{\partial\varphi}B^{(1)},
\end{aligned}$$

$$\begin{aligned} B^{(3)} &= -g'\cos\theta S^{r\psi} + \frac{g}{2}\sin\theta S^{\theta\psi} - \frac{1}{2}(r^2 f - g)\sin 2\theta S^{\theta\varphi} \\ &+ \left[\left(fr + \frac{1}{2}f'r^2\right)\sin^2\theta - g'\cos^2\theta\right] S^{r\varphi}, \end{aligned} \tag{4.13}$$

where $f' = df/dr$ and $g' = dg/dr$. Then the conserved total angular momentum in the spinning space is given by

$$\boldsymbol{\mathcal{J}} = \mathbf{B} - \mathbf{j}, \qquad \mathcal{J}^{(0)} = B^{(0)} + q \tag{4.14}$$

with $\boldsymbol{\mathcal{J}} = (\mathcal{J}^{(1)}, \mathcal{J}^{(2)}, \mathcal{J}^{(3)})$ and $\mathbf{B} = (B^{(1)}, B^{(2)}, B^{(3)})$. The components of $\boldsymbol{\mathcal{J}}$ are as follows:

$$\begin{aligned} \mathcal{J}^{(1)} &= B^{(1)} - r^2 f \sin\varphi\dot{\theta} - r^2 f\cos\theta\sin\theta\cos\varphi\dot{\varphi} - q\sin\theta\cos\varphi, \\ \mathcal{J}^{(2)} &= B^{(2)} + r^2 f\cos\varphi\dot{\theta} - r^2 f\cos\theta\sin\theta\sin\varphi\dot{\varphi} - q\sin\theta\sin\varphi, \\ \mathcal{J}^{(3)} &= B^{(3)} + r^2 f\sin^2\theta\dot{\varphi} - q\cos\theta. \end{aligned} \tag{4.15}$$

We obtain, from (4.15), two interesting relations:

$$\begin{aligned} \mathcal{J}^{(1)}\sin\varphi - \mathcal{J}^{(2)}\cos\varphi &= \frac{1}{2}(2rf + r^2 f')S^{r\theta} \\ &+ \frac{g}{2}\sin\theta S^{\varphi\psi} - r^2 f\dot{\theta}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} \mathcal{J}^{(0)} + \frac{\mathcal{J}\cdot\mathbf{r}}{r} &= -\frac{g'}{2}\cos^2\theta S^{r\psi} \\ &+ \left[3\left(rf + \frac{1}{2}r^2 f'\right)\sin^2\theta - \frac{g'}{2}\right]\cos\theta S^{r\varphi} \\ &+ \left[(r^2 f + g)\cos^2\theta + \frac{g}{2}\right]\sin\theta S^{\theta\varphi}. \end{aligned} \tag{4.17}$$

In addition to the above constants of motion, there are the four universal conserved charges described by equations (1.38), (1.41), (1.43) and (1.44) in chapter 1. In terms of the notation of this section they are given by

(i) The energy

$$E = \frac{1}{2}f\dot{r}^2 + \frac{1}{2}fr^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + \frac{1}{2g}q^2; \tag{4.18}$$

(ii) The supercharge

$$Q = f\dot{r}\psi^r + r^2 f(\dot{\theta}\psi^\theta + \sin^2\theta\dot{\varphi}\psi^\varphi) + q(\psi^\psi + \cos\theta\psi^\varphi); \tag{4.19}$$

(iii) The chiral charge

$$\Gamma_* = r^2 f\sqrt{gf}\,\sin\theta\psi^r\psi^\theta\psi^\varphi\psi^\psi; \tag{4.20}$$

(iv) The dual supercharge

$$
\begin{aligned}
Q^* &= r^2 f\sqrt{gf}\,\sin\theta\Big(\dot{r}\psi^\theta\psi^\varphi\psi^\psi - \dot{\theta}\psi^r\psi^\varphi\psi^\psi \\
&\qquad + \dot{\varphi}\psi^r\psi^\theta\psi^\psi - \dot{\psi}\psi^r\psi^\theta\psi^\varphi\Big). \qquad (4.21)
\end{aligned}
$$

The equation of motion formulated in (1.9) shows that ψ^μ is covariantly constant, from which we obtain

$$
\begin{aligned}
\dot{\psi}^\psi &= \left(\frac{g'}{2g}\dot{r} + \frac{g\cot\theta}{2r^2 f}\dot{\theta}\right)\psi^\psi + \left(\frac{2f+rf'}{2rf}\cos\theta\dot{\varphi} - \frac{g'}{2g^2}q\right)\psi^r \\
&+ \left[\cos\theta\left(\cot\theta + \frac{1}{2}\tan\theta\right)\dot{\varphi} - \frac{g}{2r^2 f}\cot\theta\right]\psi^\theta \\
&- \cos\theta\left(\frac{g'}{2g} - \frac{2f+rf'}{2rf}\right)\dot{r}\psi^\varphi \\
&- \cos\theta\left(\frac{g\cot\theta}{2r^2 f} - \cot\theta - \frac{1}{2}\tan\theta\right)\dot{\theta}\psi^\varphi, \\
\dot{\psi}^r &= -\frac{f'}{2f}\dot{r}\psi^r + \frac{g'}{2gf}q\left(\psi^\psi + \cos\theta\,\psi^\theta\right) \\
&+\frac{r^2 f' + 2rf}{2f}\left(\dot{\theta}\psi^\theta + \sin^2\theta\,\dot{\varphi}\psi^\varphi\right),
\end{aligned}
$$

$$\begin{aligned}
\dot{\psi}^\theta &= \frac{2f + rf'}{2rf}\left(\dot{\theta}\psi^r + \dot{r}\psi^\theta\right) + \sin\theta\cos\theta\,\dot{\varphi}\psi^\varphi \\
&\quad -\frac{\sin\theta}{2r^2 f}\left[\left(q + g\cos\theta\,\dot{\varphi}\right)\psi^\varphi + g\dot{\varphi}\psi^\psi\right], \\
\dot{\psi}^\varphi &= \frac{g}{2r^2 f}\csc\theta\,\dot{\theta}\psi^\psi - \frac{2f + rf'}{2rf}\dot{\varphi}\psi^r \\
&\quad + \left(\frac{q}{2r^2 f}\csc\theta - \cot\theta\,\dot{\varphi}\right)\psi^\theta \\
&\quad - \left[\frac{2f + rf'}{2rf}\dot{r} + \left(1 - \frac{g}{2r^2 f}\right)\cot\theta\,\dot{\theta}\right]\psi^\varphi. \qquad (4.22)
\end{aligned}$$

The Taub-NUT geometry admits a conserved vector, analogous to the Runge-Lenz vector of the Kepler-type problem [130, 131, 132]:

$$\mathbf{K} = \frac{1}{2}\mathbf{K}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \mathbf{p}\times\mathbf{j} + \left(\frac{q^2}{2m} - 2mE\right)\frac{\mathbf{r}}{r}, \qquad (4.23)$$

where E is the conserved energy given by (4.18) and $K = (K^{(1)}, K^{(2)}, K^{(3)})$. The three Stäckel-Killing tensors $K^{(\alpha)}_{\mu\nu}$ $(\alpha = 1,\, 2,\, 3)$ are such that $D_{(\lambda}K^{(\alpha)}_{\mu\nu)} = 0$.

Four Killing-Yano tensors are found to exist in the Taub-NUT space, three of which, denoted by f_i $(i = 1,\, 2,\, 3)$, are special because they are covariantly constant [73]. In the 2-form notation the explicit expressions for the f_i are given by

$$\begin{aligned} f_i &= 4m\,(d\psi + \cos\theta d\varphi) \wedge dx_i \\ &-\varepsilon_{ijk}\left(1+\frac{2m}{r}\right) dx_j \wedge dx_k, \end{aligned} \tag{4.24}$$

which obey the quaternion algebra

$$\begin{aligned} f_i f_j + f_j f_i = -2\delta_{ij}, \\ f_i f_j - f_j f_i = 2\varepsilon_{ijk} f_k \end{aligned} \tag{4.25}$$

and the corresponding supercharges are

$$Q_i = f_i^{\mu}{}_a \Pi_\mu \psi^a. \tag{4.26}$$

The fourth Killing-Yano tensor, which is not trivial and leads to new constants of motion, is given in the 2-form notation by

$$\begin{aligned} f_Y &= 4m\,(d\psi + \cos\theta d\varphi) \wedge dr \\ &+4r(r+m)\left(1+\frac{2m}{r}\right) \sin\theta d\theta \wedge d\varphi. \end{aligned} \tag{4.27}$$

The field strength contains one independent non-vanishing component, given by

$$H_{r\theta\varphi} = 2\left(1 + \frac{r}{2m}\right) r \sin\theta, \tag{4.28}$$

and the corresponding supercharges have the simple form

$$Q_Y = f^{\mu}_{Y\ a} \Pi_\mu \psi^a - \frac{\mathrm{i}}{3} H_{abc} \psi^a \psi^b \psi^c. \tag{4.29}$$

It follows that

$$\{Q_Y, Q_Y\} = -2\mathrm{i}\left(H + \frac{\mathcal{J}^2 - \mathcal{J}^{(0)^2}}{m^2}\right). \tag{4.30}$$

In terms of (4.7), (4.24) and (4.27), the components of the Runge-Lenz vector (4.23) can be written as follows:

$$\begin{aligned} K_{i\mu\nu} &= \frac{1}{2} m \left(f_{Y\mu\lambda} f_i^{\lambda}{}_{\nu} + f_{Y\nu\lambda} f_i^{\lambda}{}_{\mu} \right) \\ &+ \frac{1}{4m}\left(R^{(0)}_\mu R^{(i)}_\nu + R^{(0)}_\nu R^{(i)}_\mu \right). \end{aligned} \tag{4.31}$$

A detail expression for the components of the Runge-Lenz vector $\mathcal{K}$ in the spinning case is given by [135]

$$\begin{aligned}
\mathcal{K}_i &= \left((f_Y f_i)_{\mu\nu} + \frac{1}{4m^2} R^{(i)}_{(\mu} R^{(0)}_{\nu)} \right) \Pi^\mu \Pi^\nu \\
&+ m \left[f_i^{\lambda}{}_a D_\lambda f_{Y\mu a} + f_i^{\lambda}{}_\mu D_\lambda f_{Yab} \right. \\
&\left. - \frac{1}{4m^2} \left(D_b R^{(i)}_a R^{(0)}_\mu + D_b R^{(0)}_a R^{(i)}_\mu \right) \right] S^{ab} \Pi^\mu \\
&+ \frac{1}{8m} S^{ab} S^{cd} D_b R^{(i)}_a D_d R^{(0)}_c , \qquad (4.32)
\end{aligned}$$

which satisfies the following Dirac brackets:

$$\{\mathcal{K}_i, Q\} = 0, \qquad \{\mathcal{K}_i, \mathcal{J}_j\} = \varepsilon_{ijk} \mathcal{K}_k$$

$$\{\mathcal{K}_i, \mathcal{K}_j\} = \varepsilon_{ijk} \mathcal{J}_k \left[\frac{\mathcal{J}^{(0)^2}}{4m^2} - 2H \right] . \qquad (4.33)$$

The geometrical origin of the nongeneric symmetries generated by the Killing-Yano tensors and the Runge-Lenz vector in the Taub-NUT space is traced and their algebraic structure is described in [73, 134, 135].

4.3 Special Solutions

In this section we solve the equations derived in the preceding section to obtain the full solution of the equations of motion for the usual coordinates

x^μ and Grassmann coordinates ψ^μ. These equations are quite intricate and the general solution is by no means illuminating. Instead of the general solution, we investigate special solutions for the motion on a cone and a plane.

4.3.1 Motion on a Cone

We choose the z-axis along $\mathbf{j}$ so that the motion of the particle may conveniently be described in terms of polar coordinates

$$\mathbf{r} = r\mathbf{e}(\theta, \varphi), \qquad \mathbf{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \tag{4.34}$$

For this choice of axis, we have

$$\dot{\theta} = \frac{1}{2r^2 f}(2rf + r^2 f')S^{r\theta} + \frac{g\sin\theta}{2r^2 f}S^{\varphi\psi}, \tag{4.35}$$

$$\begin{aligned}\dot{\varphi} \;=\; & -\frac{q}{r^2 f\cos\theta} - \frac{g'}{2r^2 f\cos\theta}S^{r\psi} - \frac{g}{2r^2 f\sin\theta}S^{\theta\psi} \\ & + \frac{1}{r^2 f}(2rf + r^2 f' - g')S^{r\varphi} \\ & + \frac{1}{r^2 f\sin\theta\cos\theta}(2r^2 f\cos^2\theta + g)S^{\theta\varphi}.\end{aligned} \tag{4.36}$$

In what follows we consider $\dot{\theta} = 0$, which solves $S^{r\theta}$ in terms of $S^{\varphi\psi}$. As a result, it follows that $\Gamma_* = Q^* = 0$. Then, using (4.22), (4.35) and

(4.36), the equations of motion for the spin components other than $S^{r\theta}$ or $S^{\varphi\psi}$ are given by

$$
\begin{aligned}
\dot{S}^{r\varphi}+\frac{f+rf'}{rf}\dot{r}S^{r\varphi} &= -\frac{1}{2}\left[3+\frac{r^2g'}{g^2}\left(2rf+r^2f'\right)\right]\cot\theta\dot{\varphi}S^{r\theta} \\
&\quad +\frac{g'}{2g}r^2\cos\theta\dot{\varphi}S^{\theta\varphi}, \\
\dot{S}^{\theta\varphi}+\frac{2f+rf'}{rf}\dot{r}S^{\theta\varphi} &= 0, \\
\dot{S}^{\theta\psi} &= \left(\frac{g'}{2g}-\frac{2f+rf'}{2rf}\right)\dot{r}S^{\theta\psi} \\
&\quad -\frac{1}{2g}\left[3\left(2rf+r^2f'\right)+\frac{r^2fg'}{g}\right]\cos\theta\dot{\varphi}S^{r\theta} \\
&\quad -\left(\frac{g'}{2g}-\frac{2f+rf'}{2rf}\right)\cos\theta\dot{r}S^{\theta\varphi} \\
\dot{S}^{r\psi}-\left(\frac{g'}{2g}-\frac{f'}{2f}\right)\dot{r}S^{r\psi} &= \frac{1}{2}(3\cot\theta+\tan\theta)\cos\theta\dot{\varphi}S^{r\theta} \\
&\quad -\left(\frac{g'}{2g}-\frac{2f+rf'}{2rf}\right)\cos\theta\dot{r}S^{r\varphi} \\
&\quad -\frac{r^2g'}{2g}\cos^2\theta\dot{\varphi}S^{\theta\psi}, \\
&\quad +\frac{2rf+r^2f'}{2f}\sin^2\theta\dot{\varphi}S^{\varphi\psi}. \qquad (4.37)
\end{aligned}
$$

A particular solution may be found, if one chooses $S^{r\varphi} = S^{r\psi} = S^{r\theta} = 0$, in the form

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f} \qquad S^{\theta\psi} = \frac{\sqrt{g}}{r^2 f} C^{\theta\psi} + \cos\theta \frac{C^{\theta\varphi}}{r^2 f}, \tag{4.38}$$

where $C^{\theta\varphi}$ and $C^{\theta\psi}$ are Grassmann constants.

The constraint $Q = 0$ with (4.19) yields that $\Gamma_* = Q^* = 0$. For the spin components, one then can obtain

$$\begin{aligned} p_r S^{r\varphi} &= q S^{\varphi\psi}, \\ p_r S^{r\psi} &= -p_\varphi S^{\varphi\psi}, \\ p_r S^{r\theta} &= p_\varphi S^{\theta\varphi} + q S^{\theta\psi}, \end{aligned} \tag{4.39}$$

where $\mathbf{p} = f\dot{\mathbf{r}}$. The condition $Q = 0$ modifies drastically the form of the solutions. In spite of the complexity of the equations, we have a simple exact solution for the components of the spin-tensor:

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f} \qquad S^{\theta\psi} = \frac{\sqrt{g}}{r^2 f} C^{\theta\psi}. \tag{4.40}$$

For the equations of motion, we obtain

$$\left[\frac{1}{u^2}\left(\frac{du}{d\varphi}\right)^2 + \sin^2\theta\right]\dot{\varphi}^2 fg = \left(2gE - q^2\right)u^2,$$

$$q = \mathcal{J}^{(0)} + \frac{gu^2}{2f}\sin\theta\, C^{\theta\varphi},$$

$$\dot{\varphi} = \frac{u^2}{f}\left[-\frac{q}{\cos\theta} - \frac{g\sqrt{g}u^2}{2f\sin\theta}C^{\theta\psi} + \frac{2f\cos^2\theta + gu^2}{f\sin\theta\cos\theta}C^{\theta\varphi}\right], \qquad (4.41)$$

where $u = 1/r$.

4.3.2 Motion on a Plane

As we know, the orbital angular momentum for scalar particles is always conserved, but this is not true for spinning particles. For the latter case only the total angular momentum is a constant of motion. Hence, planar motion for spinning particles happens only in two kinds of situations: (i) the orbital angular momentum vanishes, or (ii) spin and orbital angular momentum are parallel. We consider the plane $\theta = \pi/2$ and discuss the cases separately. From (4.35) and (4.36) we obtain

$$S^{r\theta} = -\frac{g}{2rf + r^2 f'}S^{\varphi\psi},$$

$$q = -\frac{g'}{2}S^{r\psi} + gS^{\theta\varphi}. \qquad (4.42)$$

Then the equations of motion for the spin components take the following form:

$$
\begin{aligned}
\dot{S}^{r\varphi} + \frac{f + rf'}{rf}\dot{r}S^{r\varphi} &= 0, \\
\dot{S}^{\theta\varphi} + \frac{2f + rf'}{rf}\dot{r}S^{\theta\varphi} &= 0, \\
\dot{S}^{\theta\psi} - \left(\frac{g'}{2g} - \frac{2f + rf'}{2rf}\right)\dot{r}S^{\theta\psi} &= 0, \\
\dot{S}^{r\psi} - \left(\frac{g'}{2g} - \frac{f'}{2f}\right)\dot{r}S^{r\psi} &= \frac{2rf + r^2 f'}{2f}\dot{\varphi}S^{\varphi\psi}.
\end{aligned}
\tag{4.43}
$$

Case (i). The solution describes a particle moving along a fixed radius, for which $\dot{\varphi} = 0$. We obtain the solution

$$
\begin{aligned}
S^{r\varphi} &= \frac{C^{r\varphi}}{rf}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f}, \\
S^{\theta\psi} &= \frac{\sqrt{g}}{r^2 f}C^{\theta\psi}, \qquad S^{r\psi} = \sqrt{\frac{g}{f}}\,C^{r\psi}.
\end{aligned}
\tag{4.44}
$$

The SUSY constraint $Q = 0$ gives a nenule spin component

$$
S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f}, \tag{4.45}
$$

and consequently, the orbit of the particle is described by

$$\dot{r} = \frac{1}{\sqrt{gf}}\left(2gE - q^2\right)^{\frac{1}{2}},$$

$$q = \mathcal{J}^{(0)} + \frac{g\,C^{\theta\varphi}}{2\,r^2 f}. \tag{4.46}$$

Case (ii). The concerned motion is for $\dot{\varphi} \neq 0$, and if one chooses $S^{\varphi\psi} = 0$, the solution to the equations of motion for the spin components, (4.43), is just as given in (4.44). Interestingly, $Q = 0$ implies even in this case a spin component nenule: $S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 f}$. Subsequently, for the orbit of the particle, we obtain

$$\frac{1}{u^2}\left(\frac{du}{d\varphi}\right)^2 \dot{\varphi}^2 f g = \left(2gE - q^2\right) u^2,$$

$$\dot{\varphi} = \frac{u^2}{f}\mathcal{J}^{(3)}\left(1 - \frac{gu^2}{2qf}C^{\theta\varphi}\right)^{-1},$$

$$q = \mathcal{J}^{(0)} + \frac{gu^2}{2f}C^{\theta\varphi}, \tag{4.47}$$

where $u = 1/r$.

4.4 Concluding Remarks

Our main concern of this study has been the geodesic motion of pseudo-classical Dirac fermions in the four-dimensional Euclidean Taub-NUT space. The supersymmetric extension of the Taub-NUT geometry admits fermionic symmetries along with four standard SUSYs. The appearance of these *nongeneric* SUSYs are closely related to the existence of four Killing-Yano tensors, three of which are complex structures recognizing the quaternion algebra and the Taub-NUT manifold is hyper-Kähler [130]. Beside these three vector-like Killing-Yano tensors, there is a scalar one which has a nonvanishing field strength and which exists by virtue of the metric being type D. With a plentiful symmetries the family of Taub-NUT metrics provides an excellent background to analyze the classical and quantum conserved quantities on curved spaces.

We have described the conserved quantities of the Euclidean Taub-NUT spinning-space spanned by $\{x^\mu, \psi^\mu\}$ and obtained the geodesic equations for the motion of pseudo-classical spin one half particles with the spin characterized by the anticommuting spin-polarization tensor $S^{\mu\nu} = -\mathrm{i}\psi^\mu\psi^\nu$, ψ^μ being anticommuting Grassmann coordinates. The conserved quantities admit contributions from the spin variables. In spite of the complexity of the equations, we are able to present special solutions for the motion on a cone and on a plane. The supersymmetric constraint $Q = 0$ with (4.19) plays an important role for the forms of solutions.

The results show spin dependence of the orbits of the particles in a gravitational field. This leads to the existence of a gravitational analogue of the Stern-Gerlach-type forces well known to appear in electromagnetic phenomena.

It is thus well motivated to study the geometry of graded pseudo-manifold with the coordinates $\{x^\mu, \psi^\mu\}$.

Chapter 5

Symmetries and Motions in NUT-Taub Spinning Space

5.1 Introduction

In this chapter we investigate pseudo-classical spinning point particles in the NUT-Taub (NT) space [93]. The linearized Einstein equations for the NT metric are analogous to the case in electromagnetism of a semiinfinite magnetic solenoid or a magnetic monopole. This means, the NT metric is a particle-like solution whose spherically symmetric source has both ordinary mass and "magnetic-like" mass. The NT space has the metric [96, 142]

$$\begin{aligned} ds^2 &= -U(dt - 2n\cos\theta d\varphi)^2 + \frac{1}{U}dr^2 \\ &\quad +(r^2+n^2)(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \tag{5.1}$$

where

$$U(r) = 1 - \frac{2}{r^2 + n^2}(Mr + n^2). \tag{5.2}$$

M is the ordinary mass of the source and n is the NUT parameter, which has the identification of the gravitational "magnetic" mass or magnetic monopole [96, 97, 98, 99, 100]. In the limit that $n = 0$ the NT metric reduces to the Schwarzschild metric.

For a nonzero n the vanishing of dt.$g_{\mu\nu}$ in (5.1) identifies the singularities at $\theta = 0$ and $\theta = \pi$. Because of this axial singularity the metric admits different physical interpretations. Misner [94, 95] introduced a periodic time coordinate to remove the singularity; but this makes the metric an uninteresting particle-like solution. To avoid a periodic time coordinate, Bonnor [136] removed the singularity at $\theta = 0$ and related the singularity at $\theta = \pi$ to a semiinfinite massless source of angular momentum along the axis of symmetry. This is analogous to representing the magnetic monopole in electromagnetic theory by semiinfinite solenoid [137]. The singularity along z-axis is analogous to the Dirac string.

The NT metric possesses properties similar to both the Schwarzschild and the Kerr metrics. Like the Kerr and Schwarzschild, the NT space is Petrov-type D and has a Killing horizon at $r_o = M + \sqrt{(M^2 + n^2)}$. Like the Schwarzschild metric, the single nonvanishing Riemann curvature scalar is spherically symmetric. Also the NT space, like the Schwarzschild space, admits a four parameter group of motion with three space-like generators having the same commutator algebra as do the generators for angular momentum.

The NT metric (5.1) is Kerr-like in regard to that it has a crossed space-time metric component $g_{t\varphi}$ which generates gravimagnetic effects. In the Kerr metric the cross term breaks spherical symmetry and produces an ergosphere and frame dragging. On the contrary, the cross term in the metric (5.1) does not generate ergosphere, but it does produce an effect similar to the dragging of inertial frames. Moreover, although the cross term in (5.1) singles out the z-axis and appears to break spherical symmetry, the space components of the geodesics as a function of proper time are spherically symmetric. However, the geodesic coordinate time component is not spherically symmetric. Since the time component is dependent on the orientation of the "Dirac string", we say that the geodesics are only "almost" spherically symmetric. This suggests that the energy of the "Dirac string" makes contribution to the solution.

The NT space, as was suggested by McGuire and Ruffini [138], admits no direct physical interpretation. It is sometimes considered as unphysical. Our study of pseudo-classical spin-$\frac{1}{2}$ particles in such a peculiar space is interesting.

The organization of this chapter is as follows. In section 5.2 we investigate the motion of pseudo-classical spinning particles in the NUT-Taub space. We examine the generalized Killing equations for this spinning space and derive the constants of motion in terms of the Killing-Yano tensors. In section 5.3 we solve the equations derived in the previous section for special case of motion on a cone and on a plane. Finally, we present our concluding remarks in section 5.4.

5.2 Motion in NUT-Taub Spinning Space

In this section we analysis the motion of pseudo-classical spinning point particles in the NUT-Taub space by exploiting the formalisms described in chapters 1 and 2. The NT space, described by the metric (5.1) has an isometry group $SU(2) \times U(1)$. The four Killing vectors associated with this metric are given by

$$D^{(\alpha)} \equiv R^{(\alpha)\mu}\partial_{\mu}, \qquad \alpha = 0, \cdots, 3, \tag{5.3}$$

or explicitly

$$\begin{aligned}
D^{(0)} &= \frac{\partial}{\partial t}, \\
D^{(1)} &= -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi} + 2n\cot\theta\cos\varphi\frac{\partial}{\partial t}, \\
D^{(2)} &= \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} + 2n\cot\theta\cos\varphi\frac{\partial}{\partial t}, \\
D^{(3)} &= \frac{\partial}{\partial\varphi} + 2n\frac{\partial}{\partial t}.
\end{aligned} \tag{5.4}$$

$D^{(0)}$, which generates the $U(1)$ of t translation, commutes with the other Killing vectors. The remaining three vectors, corresponding to the invariance of the metric (5.1) under spatial rotations ($\alpha = 1, 2, 3$), obey an $SU(2)$ algebra with

$$[D^{(i)}, D^{(j)}] = -\varepsilon^{ijk} D^{(k)}, \qquad (i,j,k=1,2,3). \tag{5.5}$$

This is contrasted with the Schwarzschild space, where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This illustrates the essential topological character of the magnetic mass [83, 133].

These invariances, in purely bosonic case, would correspond to conservation of the so-called "relative electric charge" and the angular momentum [128, 129, 130, 131, 132, 139]:

$$q = -U(\dot{t} - 2n\cos\theta\dot{\varphi}), \tag{5.6}$$

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} + 2nq\frac{\mathbf{r}}{r}. \tag{5.7}$$

The first generalized Killing equation of (1.33) shows that for each Killing vector there is an associated Killing scalar, and if we limit ourselves to variations (1.27) that terminate after the terms linear in $\dot{x}^\mu$, the corresponding constants of motion would be of the form

$$\mathcal{J}^{(\alpha)} = B^{(\alpha)} + m\dot{x}^\mu R_\mu^{(\alpha)}, \tag{5.8}$$

which asserts that the Killing scalars $B^{(\alpha)}$ contribute to the "relative electric charge" and the total angular momentum.

For the NUT-Taub space, we obtain

$$
\begin{aligned}
B^{(0)} &= VS^{tr} + 2nV\cos\theta S^{r\varphi} - nU\sin\theta S^{\theta\varphi}, \\
B^{(1)} &= -2nV\cos\varphi\cot\theta(1+\cos\theta)S^{tr} \\
&\quad + \frac{1}{2}nU\cos\varphi\cos\theta S^{t\theta} - \frac{1}{2}nU\sin\varphi\sin\theta S^{t\varphi} - r\sin\varphi S^{r\theta} \\
&\quad - \cos\varphi\cot\theta\left[(9n^2V + r)\sin^2\theta + 4n^2V\cos\theta(1+\cos\theta)\right]S^{r\varphi} \\
&\quad + \cos\varphi\left[(r^2+n^2)\sin^2\theta + 3n^2U(1-2\cos^2\theta) - 2n^2U\cos\theta\right]S^{\theta\varphi}, \\
B^{(2)} &= -\frac{\partial}{\partial\varphi}B^{(1)}, \\
B^{(3)} &= -2nV(1-2\cos\theta)S^{tr} - \frac{1}{2}nU\sin\theta S^{t\theta} \\
&\quad + \left[r\sin^2\theta - 7n^2V\cos^2\theta - \frac{3}{2}n^2V(1+4\cos\theta)\right]S^{r\varphi} \\
&\quad + \sin\theta\left[(r^2+n^2)\cos\theta - 2n^2U(1-3\cos\theta)\right]S^{\theta\varphi}, \qquad (5.9)
\end{aligned}
$$

where $2V = dU/dr$. The conserved total angular momentum in the spinning case is given by

$$\mathbf{J} = \mathbf{B} + \mathbf{j}, \qquad J^{(0)} = B^{(0)} - q \tag{5.10}$$

with $\mathbf{J} = (J^{(1)}, J^{(2)}, J^{(3)})$ and $\mathbf{B} = (B^{(1)}, B^{(2)}, B^{(3)})$. For the components of $\mathbf{J}$, we obtain

$$\begin{aligned}
J^{(1)} &= B^{(1)} - (r^2 + n^2)\sin\varphi\,\dot\theta \\
&\quad -(r^2+n^2)\cos\theta\sin\theta\cos\varphi\,\dot\varphi + 2nq\sin\theta\cos\varphi, \\
J^{(2)} &= B^{(2)} + (r^2 + n^2)\cos\varphi\,\dot\theta \\
&\quad -(r^2+n^2)\cos\theta\sin\theta\sin\varphi\,\dot\varphi + 2nq\sin\theta\sin\varphi, \\
J^{(3)} &= B^{(3)} + (r^2+n^2)\sin^2\theta\,\dot\varphi + 2nq\cos\theta.
\end{aligned} \tag{5.11}$$

We obtain two interesting relations:

$$J^{(1)}\sin\varphi - J^{(2)}\cos\varphi = -rS^{r\theta} - (r^2+n^2)\dot\theta - \frac{1}{2}nU\sin\theta S^{t\varphi}, \tag{5.12}$$

$$J^{(1)}\sin\theta\cos\varphi + J^{(2)}\sin\theta\sin\varphi + J^{(3)}\cos\theta$$

$$= -2nJ^{(0)} + 2nV(2\cos\theta + \sin^2\theta)S^{tr}$$

$$+\sin\theta\left[(r^2+n^2) - n^2U\{1+2\cos\theta(3\cos\theta-2)\}\right]S^{r\varphi}$$

$$-\frac{1}{2}n^2V\cos\theta\left[4\cos\theta(\cos\theta+5)+13\right]S^{r\varphi}. \tag{5.13}$$

The four universal conserved charges, described in equations (1.38), (1.41), (1.43) and (1.44), are given by

(i) The energy

$$E = \frac{1}{2U}\dot{r}^2 + \frac{1}{2}(r^2+n^2)(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - \frac{1}{2U}q^2; \tag{5.14}$$

(ii) The supercharge

$$Q = \frac{1}{U}\dot{r}\psi^r + (r^2+n^2)\dot{\theta}\psi^\theta + q\psi^t$$

$$-\left[2nq\cos\theta - (r^2+n^2)\sin^2\theta\dot{\varphi}\right]\psi^\varphi; \tag{5.15}$$

(iii) The chiral charge

$$\Gamma_* = (r^2+n^2)\sin\theta\psi^r\psi^\theta\psi^\varphi\psi^t; \tag{5.16}$$

(iv) The dual supercharge

$$\begin{aligned} Q^* &= (r^2+n^2)\sin\theta\left(\dot{r}\psi^\theta\psi^\varphi\psi^t - \dot{\theta}\psi^r\psi^\varphi\psi^t\right. \\ &\qquad \left. + \dot{\varphi}\psi^r\psi^\theta\psi^t - \dot{t}\psi^r\psi^\theta\psi^\varphi\right). \end{aligned} \tag{5.17}$$

Keeping in mind that ψ^μ is covariantly constant as formulated in (1.9), we obtain

$$\begin{aligned} \dot{\psi}^t &= \left(\frac{2n^2U}{r^2+n^2}\cot\theta\,\dot{\theta} - \frac{V}{U}\dot{r}\right)\psi^t - \left(\frac{2nr}{r^2+n^2}\cos\theta\,\dot{\varphi} + \frac{V}{U^2}q\right)\psi^r \\ &\quad - \left[n\cos\theta\,(\tan\theta + 2\cot\theta)\,\dot{\varphi} + \frac{2n^2q}{r^2+n^2}\cot\theta\right]\psi^\theta \\ &\quad -2n\cos\theta\left(\frac{r}{r^2+n^2} - \frac{V}{U}\right)\dot{r}\psi^\varphi \\ &\quad +n\cos\theta\left(\tan\theta + 2\cot\theta + \frac{4n^2U}{r^2+n^2}\cot\theta\right)\dot{\theta}\psi^\varphi, \end{aligned}$$

$$
\begin{aligned}
\dot{\psi}^r &= \left[rU - (r^2+n^2)V\right]\left(\dot{\theta}\psi^\theta + \sin^2\theta\,\dot{\varphi}\psi^\varphi\right),\\
\dot{\psi}^\theta &= -\frac{r\dot{\theta}}{r^2+n^2}\psi^r - \frac{r\dot{r}}{r^2+n^2}\psi^\theta + \sin\theta\cos\theta\,\dot{\varphi}\psi^\varphi\\
&\quad + \frac{n\sin\theta}{r^2+n^2}\left[\left(q + 2nU\cos\theta\,\dot{\varphi}\right)\psi^\varphi - U\dot{\varphi}\psi^t\right],\\
\dot{\psi}^\varphi &= \frac{nU}{r^2+n^2}\csc\theta\,\dot{\theta}\psi^t - \frac{r\dot{\varphi}}{r^2+n^2}\psi^r\\
&\quad - \left(\cot\theta\,\dot{\varphi} + \frac{nq}{r^2+n^2}\csc\theta\right)\psi^\theta\\
&\quad - \left[\frac{r\dot{r}}{r^2+n^2} + \left(1 + \frac{2n^2U}{r^2+n^2}\right)\cot\theta\,\dot{\theta}\right]\psi^\varphi.
\end{aligned}
\tag{5.18}
$$

We now turn to nongeneric SUSYs generated by the functions Q_i of (2.51). The Killing-Yano tensor $f_{\mu\nu}$ for the metric (5.1) is given by

$$
\begin{aligned}
\frac{1}{2}f_{\mu\nu}dx^\mu \wedge dx^\nu &= ndr \wedge (dt - 2n\cos\theta d\varphi)\\
&\quad + r\sin\theta d\theta \wedge (r^2+n^2)d\varphi.
\end{aligned}
\tag{5.19}
$$

For the vierbein $e_\mu{}^a(x)$ we have the following expressions:

$$
e_\mu{}^0 dx^\mu = -\sqrt{U}(dt - 2n\cos\theta d\varphi),
$$

$$
\begin{aligned}
e_\mu{}^1 dx^\mu &= \frac{1}{\sqrt{U}} dr, \\
e_\mu{}^2 dx^\mu &= \sqrt{(r^2+n^2)}\, d\theta, \\
e_\mu{}^3 dx^\mu &= \sqrt{(r^2+n^2)}\, \sin\theta d\varphi.
\end{aligned} \tag{5.20}
$$

The components of $f_\mu{}^a(x)$ are given by

$$
\begin{aligned}
f_\mu{}^0 dx^\mu &= \frac{1}{\sqrt{U}} n dr, \\
f_\mu{}^1 dx^\mu &= -n\sqrt{U}(dt - 2n\cos\theta d\varphi) \\
f_\mu{}^2 dx^\mu &= -r\sqrt{(r^2+n^2)}\, \sin\theta d\varphi, \\
f_\mu{}^3 dx^\mu &= r\sqrt{(r^2+n^2)}\, d\theta.
\end{aligned} \tag{5.21}
$$

From (2.79) the components of c_{abc} are obtained follows:

$$
c_{012} = 0, \qquad c_{013} = 0, \qquad c_{023} = 0, \qquad c_{123} = -2\sqrt{U}. \tag{5.22}
$$

The new SUSY generator Q_{f} given by (2.51) takes the following form for the NT space:

$$\begin{aligned} Q_{\mathrm{f}} &= -r\sqrt{(r^2+n^2)}\,[2n\,q\cos\theta - (r^2+n^2)\sin^2\theta\,\dot{\varphi}]\psi^\theta \\ &+ \left[\frac{2n^2\cos\theta}{\sqrt{U}}\dot{r} - r(r^2+n^2)^{\frac{3}{2}}\sin\theta\,\dot{\theta}\right]\psi^\varphi \\ &- \frac{n}{\sqrt{U}}\left[\dot{r}\psi^t - q\psi^r\right] - 2\mathrm{i}\sqrt{U}\,\psi^r\psi^\theta\psi^\varphi. \end{aligned} \tag{5.23}$$

The Killing tensor, vector, and scalar are constructed from (2.54)–(2.56) and are given by

$$K_{\mu\nu}\Pi^\mu\Pi^\nu = -\frac{n^2}{U}\left(\dot{r}^2 - q^2\right) + r^2\left(r^2+n^2\right)\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right), \tag{5.24}$$

$$I_\mu\Pi^\mu = \sqrt{(r^2+n^2)U}\left[r\sin\theta S^{r\varphi}\dot{\varphi} + \left(nS^{t\varphi} + rS^{r\theta}\right)\dot{\theta}\right], \tag{5.25}$$

$$G = -\frac{2n^2}{r^2+n^2}S^{tr}S^{\theta\varphi}. \tag{5.26}$$

The new conserved charge is then given by

$$Z = \frac{1}{2}K_{\mu\nu}\Pi^\mu\Pi^\nu + I_\mu\Pi^\mu + G. \tag{5.27}$$

The equations of this section are to be integrated for the trajectories of the particle in terms of the usual coordinates $\{x^\mu\}$ and Grassmann coordinates $\{\psi^\mu\}$. These equations are quite intricate and the general solution is by no means illuminating. We therefore discuss special solutions in the next section for the motion on a cone and on a plane.

5.3 Special solutions

We solve the equations derived in the previous section for special kind of motions of the spin-$\frac{1}{2}$ particle in NUT-Taub spinning space.

5.3.1 Motion on a Cone

We choose the z-axis along $\mathbf{J}$. Then the motion of the particle may conveniently be described in terms of polar coordinates

$$\mathbf{r} = r\mathbf{e}(\theta, \varphi), \qquad \mathbf{e} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,, \tag{5.28}$$

and for this choice of axis, we have

$$(r^2 + n^2)\dot{\theta} = -rS^{r\theta} - \frac{1}{2}nU\sin\theta S^{t\varphi}, \tag{5.29}$$

$$\begin{aligned}
\dot{\varphi} &= \frac{2nq}{(r^2+n^2)\cos\theta} - \frac{2nV(1+\cos\theta)}{(r^2+n^2)\sin^2\theta}S^{tr} + \frac{nU}{2(r^2+n^2)\sin\theta}S^{t\theta} \\
&\quad - \frac{1}{(r^2+n^2)\sin^2\theta}\left[(9n^2V+r)\sin^2\theta + 4n^2V\cos\theta(1+\cos\theta)\right]S^{r\varphi} \\
&\quad + \left(\tan\theta + \frac{3n^2U}{(r^2+n^2)\sin\theta\cos\theta}\right)S^{\theta\varphi} \\
&\quad - \frac{2n^2U(3\cos\theta+1)}{(r^2+n^2)\sin\theta}.
\end{aligned} \tag{5.30}$$

The equations of motion for the spin components when $\dot{\theta}=0$ are given by

$$\begin{aligned}
\dot{S}^{r\theta} &= -\frac{r\dot{r}}{r^2+n^2}S^{r\theta} + \sin\theta\left[\cos\theta\left(1+\frac{2n^2U}{r^2+n^2}\right)\dot{\varphi} + \frac{nq}{r^2+n^2}\right]S^{r\varphi} \\
&\quad - \frac{nU\sin\theta}{r^2+n^2}\dot{\varphi}S^{rt} - \left[rU-(r^2+n^2)V\right]\sin^2\theta S^{\theta\varphi}, \\
\dot{S}^{r\varphi} &= -\frac{r\dot{r}}{r^2+n^2}S^{r\varphi} - \left(\cot\theta\,\dot{\varphi} + \frac{nq}{r^2+n^2}\csc\theta\right)S^{r\theta}, \\
\dot{S}^{\theta\varphi} &= -\frac{2r\dot{r}}{r^2+n^2}S^{\theta\varphi} + \frac{r\dot{\varphi}}{r^2+n^2}S^{r\theta} - \frac{nU\sin\theta}{r^2+n^2}\dot{\varphi}S^{t\varphi},
\end{aligned}$$

$$
\begin{aligned}
\dot{S}^{\theta t} &= -\left(\frac{r}{r^2+n^2}+\frac{V}{U}\right)\dot{r}S^{\theta t} - 2n\cos\theta\left(\frac{r}{r^2+n^2}-\frac{V}{U}\right)\dot{r}S^{\theta\varphi} \\
&\quad + \sin\theta\left[\cos\theta\left(1+\frac{2n^2U}{r^2+n^2}\right)\dot{\varphi}+\frac{nq}{r^2+n^2}\right]S^{\varphi t} \\
&\quad + \left(\frac{2nr}{r^2+n^2}\cos\theta\,\dot{\varphi}+\frac{qV}{U^2}\right)S^{r\theta}, \\
\dot{S}^{rt} &= \left[rU-(r^2+n^2)V\right]\sin^2\theta\,\dot{\varphi}S^{\varphi t} - \frac{V}{U}\dot{r}S^{rt} \\
&\quad - n\cos\theta\left[(\tan\theta+2\cot\theta)\,\dot{\varphi}+\frac{2nq}{r^2+n^2}\csc\theta\right]S^{r\theta} \\
&\quad - 2n\cos\theta\left(\frac{r}{r^2+n^2}-\frac{V}{U}\right)\dot{r}S^{r\varphi}, \\
\dot{S}^{\varphi t} &= -\frac{r\dot{\varphi}}{r^2+n^2}S^{rt} - \left(\cot\theta\,\dot{\varphi}+\frac{nq}{r^2+n^2}\csc\theta\right)S^{\theta t} \\
&\quad - \left(\frac{r}{r^2+n^2}+\frac{V}{U}\right)\dot{r}S^{\varphi t} + \left(\frac{2n\cos\theta}{r^2+n^2}r\dot{\varphi}+\frac{qV}{U^2}\right)S^{r\varphi} \\
&\quad + n\cos\theta\left[(\tan\theta+2\cot\theta)\,\dot{\varphi}+\frac{2nq}{r^2+n^2}\csc\theta\right]S^{\theta\varphi}. \qquad (5.31)
\end{aligned}
$$

Since we are looking for solutions with $\dot{\theta}=0$, we have from (5.29),

$$S^{r\theta} + \frac{nU\sin\theta}{2r} S^{t\varphi} = 0. \tag{5.32}$$

This relation implies $\Gamma_* = 0$. Using (5.32) we can express $S^{r\theta}$ through $S^{t\varphi}$. The system of equations (5.31) reduces to a more tractable form

$$\dot{S}^{r\varphi} + \frac{r\dot{r}}{r^2+n^2} S^{r\varphi} = \frac{3n^2 qU}{2r(r^2+n^2)} S^{t\varphi},$$

$$\dot{S}^{\theta\varphi} + \frac{2r\dot{r}}{r^2+n^2} S^{\theta\varphi} = -\frac{3n^2 qU\tan\theta}{(r^2+n^2)^2} S^{t\varphi},$$

$$\dot{S}^{\theta t} + \left(\frac{r}{r^2+n^2} + \frac{V}{U}\right) \dot{r} S^{\theta t}$$

$$= -2n\cos\theta \left(\frac{r}{r^2+n^2} - \frac{V}{U}\right) \dot{r} S^{\theta\varphi}$$

$$-\sin\theta \left(\frac{nV}{rU} + \frac{3n}{r^2+n^2} + \frac{6n^3 U}{(r^2+n^2)^2}\right) q S^{t\varphi},$$

$$\dot{S}^{rt} + \frac{V}{U}\dot{r} S^{rt} = -\frac{2nq\sin\theta\tan\theta}{r^2+n^2}\left[rU - (r^2+n^2)V - \frac{n^2U}{2r}\right] S^{t\varphi}$$

$$+\frac{3n^3 q\cos\theta\, U}{r(r^2+n^2)} S^{t\varphi}$$

$$-2n\cos\theta \left(\frac{r}{r^2+n^2} - \frac{V}{U}\right) \dot{r} S^{r\varphi}. \tag{5.33}$$

A particular solution may be found, if one chooses $q = 0$, in the form

$$
\begin{aligned}
S^{r\varphi} &= \frac{C^{r\varphi}}{\sqrt{(r^2+n^2)}}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{(r^2+n^2)}, \\
S^{\theta t} &= \frac{C^{\theta t}}{\sqrt{U(r^2+n^2)}} + n\cos\theta \frac{C^{\theta\varphi}}{(r^2+n^2)}, \\
S^{rt} &= \frac{C^{rt}}{\sqrt{U}} + n\cos\theta \frac{C^{r\varphi}}{\sqrt{(r^2+n^2)}}, \\
S^{r\theta} &= \frac{C^{r\theta}}{\sqrt{(r^2+n^2)}}, \qquad S^{t\varphi} = \frac{C^{t\varphi}}{\sqrt{U(r^2+n^2)}}.
\end{aligned} \tag{5.34}
$$

where $C^{\mu\nu}$ are Grassmann constants. The case of choice $S^{t\varphi} = 0$ is included into the case $q = 0$.

The constraint $Q = 0$ (see (1.48)) enables one to solve for ψ^t in terms of the spatial components ψ^i. As a result, it gives $\Gamma_* = Q^* = 0$. For the spin components we have

$$
\dot{r} S^{r\varphi} = -qUS^{t\varphi}, \qquad qS^{rt} = \left[2nq\cos\theta - (r^2+n^2)\sin^2\theta\,\dot{\varphi}\right] S^{r\varphi},
$$

$$
\dot{r} S^{r\theta} = -U\left[2nq\cos\theta - (r^2+n^2)\sin^2\theta\,\dot{\varphi}\right] S^{\theta\varphi} + qUS^{\theta t}. \tag{5.35}
$$

The condition $Q = 0$ modifies drastically the form of the solutions.

In spite of the complexity of the equations, we have a simple exact solution for the components of the spin-tensor:

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}, \tag{5.36}$$

For the equations of motion, we obtain

$$\begin{aligned}
\dot{t} &= 2n\cos\theta\,\dot{\varphi} - \frac{q}{U}, \\
q &= -J^{(0)} + nU\sin\theta\frac{C^{\theta\varphi}}{r^2+n^2}, \\
\dot{\varphi} &= \frac{1}{r^2+n^2}\left[\frac{2nq}{\cos\theta} + \frac{2n^2U(2-\cos\theta-4\cos^2\theta)}{(r^2+n^2)\sin\theta\cos\theta}C^{\theta\varphi} + \frac{\sin\theta}{\cos\theta}C^{\theta\varphi}\right], \\
\dot{r} &= \left\{U\left[2E-(r^2+n^2)\sin^2\theta\,\dot{\varphi}^2\right]+q^2\right\}^{\frac{1}{2}}.
\end{aligned} \tag{5.37}$$

5.3.2 Motion on a Plane

Planar motion for spinning particles happen only in two kinds of situations: (i) the orbital angular momentum vanishes, or (ii) spin and orbital angular momentum are parallel.

For the plane we consider $\theta = \pi/2$. Then the equations of motion for the spin components become

$$
\begin{aligned}
\dot{S}^{r\theta} &= -\frac{r\dot{r}}{r^2+n^2}S^{r\theta} + \frac{nq}{r^2+n^2}S^{r\varphi} \\
&\quad -\frac{nU}{r^2+n^2}\dot{\varphi}S^{rt} - \left[rU - (r^2+n^2)V\right]\dot{\varphi}S^{\theta\varphi}, \\
\dot{S}^{r\varphi} &= -\frac{r\dot{r}}{r^2+n^2}S^{r\varphi} - \frac{nq}{r^2+n^2}S^{r\theta}, \\
\dot{S}^{\theta\varphi} &= -\frac{2r\dot{r}}{r^2+n^2}S^{\theta\varphi} + \frac{r\dot{\varphi}}{r^2+n^2}S^{r\theta} - \frac{nU}{r^2+n^2}\dot{\varphi}S^{t\varphi}, \\
\dot{S}^{\theta t} &= -\left(\frac{r}{r^2+n^2} + \frac{V}{U}\right)\dot{r}S^{\theta t} + \frac{nq}{r^2+n^2}S^{\varphi t} + \frac{qV}{U^2}S^{r\theta}, \\
\dot{S}^{rt} &= \left[rU - (r^2+n^2)V\right]\dot{\varphi}S^{\varphi t} - \frac{V}{U}\dot{r}S^{rt}, \\
\dot{S}^{\varphi t} &= -\left(\frac{r}{r^2+n^2} + \frac{V}{U}\right)\dot{r}S^{\varphi t} + \frac{qV}{U^2}S^{r\varphi} \\
&\quad -\frac{r\dot{\varphi}}{r^2+n^2}S^{rt} - \frac{nq}{r^2+n^2}S^{\theta t}.
\end{aligned}
\tag{5.38}
$$

From (5.12) and (5.13) we obtain

$$
\begin{aligned}
rS^{r\theta} &= -\frac{1}{2}nUS^{t\varphi}, \\
q &= -\frac{1}{2n}\left(r^2+n^2+n^2U\right)S^{\theta\varphi}.
\end{aligned}
\tag{5.39}
$$

Case (i). The solution describes a particle moving along a fixed radius, for which $\dot{\varphi} = 0$. We obtain

$$S^{r\varphi} = \frac{C^{r\varphi}}{\sqrt{(r^2+n^2)}}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{(r^2+n^2)},$$

$$S^{\theta t} = \frac{C^{\theta t}}{\sqrt{U(r^2+n^2)}}, \qquad S^{rt} = \frac{C^{rt}}{\sqrt{U}},$$

$$S^{\varphi t} = \frac{C^{\varphi t}}{\sqrt{U(r^2+n^2)}}. \tag{5.40}$$

The SUSY constraint $Q = 0$ gives a nenule spin component

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}. \tag{5.41}$$

In this case the orbit of the particle is described by

$$\dot{r} = \left(2UE + q^2\right)^{\frac{1}{2}},$$

$$q = -\frac{1}{2n}\left(r^2 + n^2 + n^2 U\right)\frac{C^{\theta\varphi}}{r^2+n^2},$$

$$\dot{t} = \frac{1}{U}\left(J^{(0)} + nU\frac{C^{\theta\varphi}}{r^2+n^2}\right). \tag{5.42}$$

Case (ii). The concerned motion is for $\dot{\varphi} \neq 0$. From $Q = 0$ we obtain

$$\frac{\dot{r}}{U} S^{r\theta} = -J^{(3)} S^{\theta\varphi}, \qquad \frac{\dot{r}}{U} S^{rt} = -J^{(3)} S^{t\varphi}. \tag{5.43}$$

Interestingly, even in this case a spin component is nenule: $S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}$. For the orbit of the particle we obtain

$$\begin{aligned}
\dot{r} &= \left\{2UE - U(r^2+n^2)\dot{\varphi}^2 + q^2\right\}^{\frac{1}{2}}, \\
q &= -\frac{1}{2n}\left(r^2+n^2+n^2U\right)\frac{C^{\theta\varphi}}{r^2+n^2}, \\
\dot{\varphi} &= \frac{1}{r^2+n^2}\left(J^{(3)} + 2n^2U\frac{C^{\theta\varphi}}{r^2+n^2}\right), \\
\dot{t} &= \frac{1}{U}\left(J^{(0)} + nU\frac{C^{\theta\varphi}}{r^2+n^2}\right).
\end{aligned} \tag{5.44}$$

5.4 Concluding Remarks

The spinning particle model is a worldline supersymmetric extension of the theory of a scalar particle. It describes a relativistic particle with spin-$\frac{1}{2}$.

The main concern of our study has been the motion of pseudo-classical spinning particles in the NUT-Taub space. The supersymmetric extension of this space admits fermionic symmetries along with four standard SUSYs. The appearance of these *nongeneric* SUSYs are closely related to the existence of Killing-Yano tensors [53].

In spite of the complexity of the equations, we are able to present special solutions for the motion on a cone and on a plane. The supersymmetric constraint $Q = 0$ (1.48) plays a very important role for the forms of solutions.

The results we obtain show spin dependence of the time dilation and of the orbits of the particles in a gravitational field. This leads to the existence of a gravitational analogue of the Stern-Gerlach-type forces well known to appear in electromagnetic phenomena.

Although the Killing tensor $K_{\mu\nu}$ given in (5.24) defines a constant of motion (directly) for spinless particles in the NUT-Taub space, it requires for spinning particles the nontrivial contributions from spin which involve Killing vector and Killing scalar computed in (5.25) and (5.26).

The NUT-Taub space is the Schwarzschild space generalized with NUT or magnetic monopole parameter n. The monopole hypothesis was propounded by Dirac relatively long ago. The ingenious suggestion by Dirac that magnetic monopole does exist in nature was neglected because of the failure to identify such thing. In recent years, however, the development of gauge theories has shed new light on it. The result of this chapter is interesting in view of the presence of n parameter. With $n = 0$ our result reduces to that of the Schwarzschild space [42].

Supersymmetry/supergravity is relevant in the fundamental theory of particle interactions. It is thus not inconceivable that nature might make some use of it. In regard to this, the study of the geometry of graded pseudo-manifolds with both real number and anticommuting variables is well justified.

Chapter 6

Motions in Taub-NUT-de Sitter Spinning Spacetime

6.1 Introduction

In this chapter we investigate motion of pseudo-classical spinning point particles in the Taub-NUT-de Sitter spacetime. The metric of the spacetime can be written in the form [140]

$$\begin{aligned} ds^2 &= -F(r)(dt + 2n\cos\theta d\varphi)^2 + \frac{1}{F(r)}dr^2 \\ &\quad +(r^2+n^2)(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \tag{6.1}$$

where

$$F(r) = \frac{1}{r^2+n^2}\left[(r^2+n^2) - 2Mr - \frac{1}{\ell^2}\left\{(r^2+3n^2)^2 - 12n^4\right\}\right], \qquad (6.2)$$

and $\ell^2 = 3/\Lambda$, with Λ the positive cosmological constant. Here M is a (generalized) mass parameter and n is the NUT parameter, which has the identification of the gravitational "magnetic" mass or magnetic monopole [96, 97, 98, 99, 100] and generates a "rotational effect" [104]. Because of the presence of the cosmological constant the TN-dS spacetime is interesting in the inflationary scenario of the early universe.

The metric (6.1) reduces to (i) the Taub-NUT spacetime for $\Lambda = 0$, (ii) the Schwarzschild-de Sitter spacetime for $n = 0$, (iii) the Schwarzschild spacetime for $\Lambda = 0$, $n = 0$, and (iv) the pure de Sitter spacetime for $M = 0$, $n = 0$.

This chapter is organized as follows. In section 6.2 we investigate the motion of pseudo-classical spinning particles in the TN-dS spacetime. We examine the generalized Killing equations for this spinning spacetime and derive the constants of motion in terms of the Killing-Yano tensors. In section 6.3 we solve the equations derived in section 6.2 for special case of motion on a cone and on a plane. Finally, we present our concluding remarks in section 6.4.

6.2 Spinning Point Particles in TN-dS Spacetime

In this section we use the results of chapter 1 and 2 to investigate the motion of a pseudo-classical spinning particle in the Taub-NUT-de Sitter spacetime described by the metric (6.1). The TN-dS spacetime has an isometry group $SU(2) \times U(1)$. The invariance of the metric under time translations and spatial rotations is generated by four ψ-independent solutions $R^{(\alpha)\mu}$ of (1.35), ($\alpha = 0, \cdots, 3$). The corresponding vector fields have the form

$$D^{(\alpha)} \equiv R^{(\alpha)\mu}\partial_{\mu}, \tag{6.3}$$

or equivalently

$$\begin{aligned} D^{(0)} &= \frac{\partial}{\partial t}, \\ D^{(1)} &= -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi} - 2n\cot\theta\cos\varphi\frac{\partial}{\partial t}, \\ D^{(2)} &= \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} - 2n\cot\theta\cos\varphi\frac{\partial}{\partial t}, \\ D^{(3)} &= \frac{\partial}{\partial\varphi} - 2n\frac{\partial}{\partial t}. \end{aligned} \tag{6.4}$$

$D^{(0)}$ generates the $U(1)$ of t translation and commutes with the remaining three vectors $D^{(i)}$, which obey an $SU(2)$ algebra:

$$[D^{(0)}, D^{(i)}] = 0, \quad [D^{(i)}, D^{(j)}] = -\varepsilon^{ijk} D^{(k)}, \quad (i, j, k = 1, 2, 3). \tag{6.5}$$

This is contrasted with the Schwarzschild spacetime, where the isometry group at spacelike infinity is $SO(3) \times U(1)$. This illustrates the essential topological character of the magnetic mass [83, 133].

In purely bosonic case, these invariances would correspond to conservation of the so-called "relative electric charge" and the angular momentum [128, 129, 130, 131]:

$$q = -F(\dot{t} + 2n \cos\theta\, \dot{\varphi}),$$

$$\mathbf{j} = \mathbf{r} \times \mathbf{p} + 2nq\frac{\mathbf{r}}{r}. \tag{6.6}$$

However, for a spinning particle, only the sum of the orbital and the spin angular momentum is conserved. Indeed, the first generalized Killing equation of (1.33) shows that to each Killing vector there is associated a Killing scalar. If we limit ourselves to variations (1.27) that terminate after the terms linear in $\dot{x}^\mu$, the corresponding constants of motion, with $J^{(0)} = B^{(\alpha)}$, $J^{(1)}_\mu = R^{(\alpha)}_\mu$, would be of the form

$$J^{(\alpha)} = B^{(\alpha)} + m\dot{x}^\mu R^{(\alpha)}_\mu, \tag{6.7}$$

which asserts that the Killing scalars $B^{(\alpha)}$ contribute to the "relative electric charge" q and the total angular momentum $\mathbf{j}$.

For the TN-dS spacetime, we obtain

$$
\begin{aligned}
B^{(0)} &= NS^{tr} - 2nN\cos\theta S^{r\varphi} + nF\sin\theta S^{\theta\varphi}, \\
B^{(1)} &= 2nN\cos\varphi\cot\theta(1+\cos\theta)S^{tr} \\
&\quad -\frac{1}{2}nF\cos\varphi\cos\theta S^{t\theta} + \frac{1}{2}nF\sin\varphi\sin\theta S^{t\varphi} - r\sin\varphi S^{r\theta} \\
&\quad -\cos\varphi\cot\theta\left[(9n^2N+r)\sin^2\theta + 4n^2N\cos\theta(1+\cos\theta)\right]S^{r\varphi} \\
&\quad +\cos\varphi\left[(r^2+n^2)\sin^2\theta + 3n^2F(1-2\cos^2\theta) - 2n^2F\cos\theta\right]S^{\theta\varphi}, \\
B^{(2)} &= -\frac{\partial}{\partial\varphi}B^{(1)}, \\
B^{(3)} &= 2nN(1-2\cos\theta)S^{tr} + \frac{1}{2}nF\sin\theta S^{t\theta} \\
&\quad +\left[r\sin^2\theta - 7n^2N\cos^2\theta - \frac{3}{2}n^2N(1+4\cos\theta)\right]S^{r\varphi} \\
&\quad +\sin\theta\left[(r^2+n^2)\cos\theta - 2n^2F(1-3\cos\theta)\right]S^{\theta\varphi}, \qquad (6.8)
\end{aligned}
$$

where

$$
N = \frac{1}{(r^2+n^2)^2}\left[M(r^2-n^2) + 2n^2r - \frac{1}{\ell^2}r\left\{(r^2+n^2)^2 + 8n^4\right\}\right], \qquad (6.9)
$$

and F is given by (6.2). The conserved total angular momentum in the spinning case is given by

$$\mathbf{J} = \mathbf{B} + \mathbf{j}, \qquad J^{(0)} = B^{(0)} - q \tag{6.10}$$

with $\mathbf{J} = (J^{(1)}, J^{(2)}, J^{(3)})$ and $\mathbf{B} = (B^{(1)}, B^{(2)}, B^{(3)})$. For the components of $\mathbf{J}$, we obtain

$$\begin{aligned} J^{(1)} &= B^{(1)} - (r^2 + n^2)\sin\varphi\,\dot{\theta} \\ &\quad -(r^2+n^2)\cos\theta\sin\theta\cos\varphi\,\dot{\varphi} + 2nq\sin\theta\cos\varphi, \\ J^{(2)} &= B^{(2)} + (r^2+n^2)\cos\varphi\,\dot{\theta} \\ &\quad -(r^2+n^2)\cos\theta\sin\theta\sin\varphi\,\dot{\varphi} + 2nq\sin\theta\sin\varphi, \\ J^{(3)} &= B^{(3)} + (r^2+n^2)\sin^2\theta\,\dot{\varphi} + 2nq\cos\theta. \end{aligned} \tag{6.11}$$

We obtain two interesting relations:

$$J^{(1)}\sin\varphi - J^{(2)}\cos\varphi = -rS^{r\theta} - (r^2+n^2)\dot{\theta} + \frac{1}{2}nF\sin\theta S^{t\varphi}, \tag{6.12}$$

$$\begin{aligned}
2nJ^{(0)} + \frac{\mathbf{J}\cdot\mathbf{r}}{r} &= -\frac{1}{2}n^2 N\cos\theta\left[4\cos\theta(\cos\theta+5)+29\right]S^{r\varphi} \\
&\quad + \sin\theta\left[(r^2+n^2)+n^2F(3-4\cos\theta)\right]S^{\theta\varphi} \\
&\quad + 2nN(2\cos\theta+\sin^2\theta)S^{tr}. \qquad (6.13)
\end{aligned}$$

In addition to the above constants of motion, there are the four universal conserved charges described by equations (1.38), (1.41), (1.43) and (1.44) in chapter 1. In terms of the notation of this section they are as follows:

(i) The energy

$$E = \frac{1}{2F}\dot{r}^2 + \frac{1}{2}(r^2+n^2)(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2) - \frac{1}{2F}q^2; \qquad (6.14)$$

(ii) The supercharge

$$\begin{aligned}
Q &= \frac{1}{F}\dot{r}\psi^r + (r^2+n^2)\dot{\theta}\psi^\theta + q\psi^t \\
&\quad + \left[2nq\cos\theta + (r^2+n^2)\sin^2\theta\,\dot{\varphi}\right]\psi^\varphi; \qquad (6.15)
\end{aligned}$$

(iii) The chiral charge

$$\Gamma_* = (r^2+n^2)\sin\theta\psi^r\psi^\theta\psi^\varphi\psi^t; \tag{6.16}$$

(iv) The dual supercharge

$$\begin{aligned} Q^* &= (r^2+n^2)\sin\theta\left(\dot{r}\psi^\theta\psi^\varphi\psi^t - \dot{\theta}\psi^r\psi^\varphi\psi^t\right. \\ &\qquad \left. + \dot{\varphi}\psi^r\psi^\theta\psi^t - \dot{t}\psi^r\psi^\theta\psi^\varphi\right). \end{aligned} \tag{6.17}$$

The covariantly constant ψ^μ, as formulated in (1.9), gives

$$\begin{aligned} \dot{\psi}^t &= \left(\frac{N}{F}\dot{r} - \frac{2n^2F}{r^2+n^2}\cot\theta\,\dot{\theta}\right)\psi^t + \left(\frac{2nr}{r^2+n^2}\cos\theta\,\dot{\varphi} - \frac{qN}{F^2}\right)\psi^r \\ &\quad - \left[n\cos\theta\,(\tan\theta + 2\cot\theta)\,\dot{\varphi} - \frac{2n^2q}{r^2+n^2}\cot\theta\right]\psi^\theta \\ &\quad + 2n\cos\theta\left(\frac{r}{r^2+n^2} - \frac{N}{F}\right)\dot{r}\psi^\varphi \\ &\quad - n\cos\theta\left(\tan\theta + 2\cot\theta + \frac{4n^2F}{r^2+n^2}\cot\theta\right)\dot{\theta}\psi^\varphi, \\ \dot{\psi}^r &= \left[rF - (r^2+n^2)N\right]\left(\dot{\theta}\psi^\theta + \sin^2\theta\,\dot{\varphi}\psi^\varphi\right), \end{aligned}$$

$$
\begin{aligned}
\dot{\psi}^{\theta} &= -\frac{r\dot{\theta}}{r^2+n^2}\psi^r - \frac{r\dot{r}}{r^2+n^2}\psi^{\theta} + \sin\theta\cos\theta\,\dot{\varphi}\psi^{\varphi} \\
&\quad + \frac{n\sin\theta}{r^2+n^2}\left[(q-2nF\cos\theta\,\dot{\varphi})\,\psi^{\varphi} - F\dot{\varphi}\psi^t\right], \\
\dot{\psi}^{\varphi} &= \frac{nF}{r^2+n^2}\csc\theta\,\dot{\theta}\psi^t - \frac{r\dot{\varphi}}{r^2+n^2}\psi^r \\
&\quad - \left(\cot\theta\,\dot{\varphi} + \frac{nq}{r^2+n^2}\csc\theta\right)\psi^{\theta} \\
&\quad - \left[\frac{r\dot{r}}{r^2+n^2} + \left(1 - \frac{2n^2F}{r^2+n^2}\right)\cot\theta\,\dot{\theta}\right]\psi^{\varphi}. \qquad (6.18)
\end{aligned}
$$

We now turn to *nongeneric* SUSYs generated by the functions Q_i defined in (2.51). The Killing-Yano tensor $f_{\mu\nu}$ for the metric (6.1) is given by

$$
\begin{aligned}
\frac{1}{2}f_{\mu\nu}dx^{\mu}\wedge dx^{\nu} &= ndr\wedge(dt+2n\cos\theta d\varphi) \\
&\quad + r\sin\theta d\theta\wedge(r^2+n^2)d\varphi. \qquad (6.19)
\end{aligned}
$$

For the vierbein $e_{\mu}{}^{a}(x)$ we obtain

$$
e_{\mu}{}^{0}dx^{\mu} = -\sqrt{F}(dt+2n\cos\theta d\varphi),
$$

$$
\begin{aligned}
e_\mu{}^1 dx^\mu &= \frac{1}{\sqrt{F}} dr, \\
e_\mu{}^2 dx^\mu &= \sqrt{(r^2+n^2)}\, d\theta, \\
e_\mu{}^3 dx^\mu &= \sqrt{(r^2+n^2)}\, \sin\theta d\varphi,
\end{aligned} \tag{6.20}
$$

while for $f_\mu{}^a(x)$ we have

$$
\begin{aligned}
f_\mu{}^0 dx^\mu &= \frac{1}{\sqrt{F}} n dr, \\
f_\mu{}^1 dx^\mu &= -n\sqrt{F}(dt + 2n\cos\theta d\varphi) \\
f_\mu{}^2 dx^\mu &= -r\sqrt{(r^2+n^2)}\, \sin\theta d\varphi, \\
f_\mu{}^3 dx^\mu &= r\sqrt{(r^2+n^2)}\, d\theta.
\end{aligned} \tag{6.21}
$$

From (2.79) the components of c_{abc} are obtained as follows:

$$
c_{012} = 0, \qquad c_{013} = 0, \qquad c_{023} = 0, \qquad c_{123} = -2\sqrt{F}. \tag{6.22}
$$

Then the new SUSY generator Q_{f} given by (2.51) has the following form for the Taub-NUT-de Sitter spacetime:

$$\begin{aligned} Q_{\mathrm{f}} &= r\sqrt{(r^2+n^2)}\left[2n\,q\cos\theta+(r^2+n^2)\sin^2\theta\,\dot{\varphi}\right]\psi^{\theta} \\ &\quad -\left[\frac{2n^2\cos\theta}{\sqrt{F}}\dot{r}+r(r^2+n^2)^{\frac{3}{2}}\sin\theta\,\dot{\theta}\right]\psi^{\varphi} \\ &\quad -\frac{n}{\sqrt{F}}\left[\dot{r}\psi^t-q\psi^r\right]-2\mathrm{i}\sqrt{F}\,\psi^r\psi^{\theta}\psi^{\varphi}. \end{aligned} \tag{6.23}$$

From (2.54)–(2.56) the Killing tensor, vector, and scalar are constructed as follows:

$$K_{\mu\nu}\Pi^{\mu}\Pi^{\nu} = -\frac{n^2}{F}\left(\dot{r}^2-q^2\right)+r^2\left(r^2+n^2\right)\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right), \tag{6.24}$$

$$I_{\mu}\Pi^{\mu} = \sqrt{(r^2+n^2)F}\left[r\sin\theta S^{r\varphi}\dot{\varphi}+\left(nS^{t\varphi}+rS^{r\theta}\right)\dot{\theta}\right], \tag{6.25}$$

$$G = -\frac{2n^2}{r^2+n^2}S^{tr}S^{\theta\varphi}, \tag{6.26}$$

where S^{ab} is the spin tensor defined in (1.10).The new conserved charge is then given by

$$Z=\frac{1}{2}K_{\mu\nu}\Pi^{\mu}\Pi^{\nu}+I_{\mu}\Pi^{\mu}+G. \tag{6.27}$$

For the trajectories of the pseudo-classical particle, one needs to solve the equations of this section. These are quite intricate and the general solution is by no means illuminating. We therefore analyze special solutions in the next section for the motion on a cone and on a plane.

6.3 Special solutions

In order to solve the equations derived in the preceding section we choose the z-axis along $\mathbf{J}$ so that the motion of the particle may conveniently be described in terms of polar coordinates

$$\mathbf{r} = r\mathbf{e}(\theta, \varphi), \qquad \mathbf{e} = (\sin\theta\cos\varphi,\, \sin\theta\sin\varphi,\, \cos\theta)\,, \tag{6.28}$$

For this choice of axis, we have

$$(r^2 + n^2)\dot{\theta} = -rS^{r\theta} + \frac{1}{2}nF\sin\theta S^{t\varphi}; \tag{6.29}$$

$$\begin{aligned}
\dot{\varphi} \;=\;& \frac{2nq}{(r^2+n^2)\cos\theta} + \frac{2nN(1+\cos\theta)}{(r^2+n^2)\sin^2\theta}S^{tr} - \frac{nF}{2(r^2+n^2)\sin\theta}S^{t\theta} \\
&+\frac{n^2N}{(r^2+n^2)\sin^2\theta}\left[\cos\theta(5\cos\theta-4) - r\sin^2\theta - 9\right]S^{r\varphi} \\
&+\left[\tan\theta + \frac{n^2F\{1-2\cos\theta(1+3\cos\theta)\}}{(r^2+n^2)\sin\theta\cos\theta}\right]S^{\theta\varphi}.
\end{aligned} \tag{6.30}$$

6.3.1 Motion on a Cone

We consider the case $\dot{\theta} = 0$. This solves $S^{r\theta}$ in terms of $S^{t\varphi}$ and as a result, it implies that $\Gamma_* = Q^* = 0$. We then the equations of motion for the spin components as follows:

$$\dot{S}^{r\varphi} + \frac{r\dot{r}}{r^2+n^2}S^{r\varphi} = -\frac{3nq}{r^2+n^2}\csc\theta S^{r\theta},$$

$$\dot{S}^{\theta\varphi} + \frac{2r\dot{r}}{r^2+n^2}S^{\theta\varphi} = -\frac{r\dot{\varphi}}{r^2+n^2}S^{r\theta},$$

$$\begin{aligned}\dot{S}^{r\theta} &+ \frac{r\dot{r}}{r^2+n^2}S^{r\theta} \\ &= -\left[rF - (r^2+n^2)N\right]\sin^2\theta\,\dot{\varphi}S^{\theta\varphi} - \frac{nF\sin\theta}{r^2+n^2}\dot{\varphi}S^{rt} \\ &+ \sin\theta\left[\cos\theta\,\dot{\varphi} + \frac{n}{r^2+n^2}(q - 2nF\cos\theta\,\dot{\varphi})\right]S^{r\varphi},\end{aligned}$$

$$\begin{aligned}\dot{S}^{rt} &+ \frac{N}{F}\dot{r}S^{rt} \\ &= -\frac{2r}{nF}\left[rF - (r^2+n^2)N\right]\sin\theta\,\dot{\varphi}S^{r\theta} \\ &-n\cos\theta\left[(\tan\theta + 2\cot\theta)\,\dot{\varphi} - \frac{2nq}{r^2+n^2}\csc\theta\right]S^{r\theta} \\ &+2n\cos\theta\left(\frac{r}{r^2+n^2} - \frac{N}{F}\right)\dot{r}S^{r\varphi}.\end{aligned} \tag{6.31}$$

The choice $S^{r\theta} = 0$ yields the following particular solution:

$$S^{r\varphi} = \frac{C^{r\varphi}}{\sqrt{(r^2+n^2)}}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{(r^2+n^2)},$$

$$S^{rt} = \frac{C^{rt}}{\sqrt{F}} - n\cos\theta\frac{C^{r\varphi}}{\sqrt{(r^2+n^2)}}. \tag{6.32}$$

where $C^{\mu\nu}$ are Grassmann constants.

The constraint $Q = 0$ (see (1.48)) yields $\Gamma_* = Q^* = 0$ and modifies drastically the form of the solutions. It gives, for the spin components, the followings:

$$\dot{r}S^{r\theta} = -F\left[2nq\cos\theta + (r^2+n^2)\sin^2\theta\,\dot{\varphi}\right]S^{\theta\varphi} + qFS^{\theta t},$$

$$qS^{rt} = \left[2nq\cos\theta + (r^2+n^2)\sin^2\theta\,\dot{\varphi}\right]S^{r\varphi},$$

$$\dot{r}S^{r\varphi} = -qFS^{t\varphi}. \tag{6.33}$$

We have a simple exact solution for the components of the spin-tensor:

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}, \tag{6.34}$$

despite the complexity of the equations. For the equations of motion, we obtain

$$
\begin{aligned}
\dot{r} &= \left\{F\left[2E-(r^2+n^2)\sin^2\theta\,\dot{\varphi}^2\right]+q^2\right\}^{\frac{1}{2}}, \\
\dot{\varphi} &= \frac{2nq}{(r^2+n^2)\cos\theta} \\
&\quad +\frac{C^{\theta\varphi}}{r^2+n^2}\left[\frac{4n^2F\{1-\cos\theta(1+3\cos\theta)\}}{(r^2+n^2)\sin 2\theta}+\tan\theta\right], \\
\dot{t} &= -\frac{1}{F}(2nF\cos\theta\,\dot{\varphi}+q), \\
q &= -J^{(0)}+nF\sin\theta\frac{C^{\theta\varphi}}{r^2+n^2}.
\end{aligned}
\tag{6.35}
$$

The third of (6.35) defines the gravitational redshift, and shows that the time-dilation receives a contribution from spin-orbit coupling. This demonstrates that time-dilation is not a purely geometric effect, but also has a dynamical component.

6.3.2 Motion on a Plane

Although the orbital angular momentum for scalar particles is always conserved, this is not true for spinning particles in general. Hence, planar motion for spinning particles occurs only in two kinds of situations: (i) the orbital angular momentum vanishes, or (ii) spin and orbital angular momentum are parallel.

For the plane we choose $\theta = \pi/2$. Then from (6.29) and (6.30) we obtain

$$
\begin{aligned}
rS^{r\theta} &= -\frac{1}{2}nFS^{\varphi t}, \\
q &= -\frac{1}{2n}\left(r^2+n^2+n^2F\right)S^{\theta\varphi},
\end{aligned}
\tag{6.36}
$$

and the equations of motion for the spin components become

$$
\begin{aligned}
&\dot{S}^{r\varphi} + \frac{r\dot{r}}{r^2+n^2}S^{r\varphi} = 0, \\
&\dot{S}^{\theta\varphi} + \frac{2r\dot{r}}{r^2+n^2}S^{\theta\varphi} = -\frac{r\dot{\varphi}}{r^2+n^2}S^{r\theta}, \\
&\dot{S}^{r\theta} + \frac{r\dot{r}}{r^2+n^2}S^{r\theta} \\
&\qquad = -\left[rF-(r^2+n^2)N\right]\dot{\varphi}S^{\theta\varphi} - \frac{nF}{r^2+n^2}\dot{\varphi}S^{rt} \\
&\dot{S}^{rt} + \frac{N}{F}\dot{r}S^{rt} \\
&\qquad = -\frac{2r}{nF}\left[rF-(r^2+n^2)N\right]\dot{\varphi}S^{r\theta} - n\dot{\varphi}S^{r\theta}.
\end{aligned}
\tag{6.37}
$$

Case (i). For $\dot{\varphi} = 0$, the solution describes a particle moving along a fixed radius. We are then led to the exact solution:

$$S^{r\varphi} = \frac{C^{r\varphi}}{\sqrt{(r^2+n^2)}}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{(r^2+n^2)},$$
$$S^{r\theta} = \frac{C^{r\theta}}{\sqrt{(r^2+n^2)}}, \qquad S^{rt} = \frac{C^{rt}}{\sqrt{F}}. \tag{6.38}$$

The SUSY constraint $Q = 0$ provides the existence of a nenule spin component

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2+n^2}, \tag{6.39}$$

and consequently, the orbit of the particle is described by

$$\dot{r} = \left(2FE + q^2\right)^{\frac{1}{2}},$$
$$\dot{t} = -\frac{q}{F} = \frac{1}{F}\left(J^{(0)} - nF\frac{C^{\theta\varphi}}{r^2+n^2}\right). \tag{6.40}$$

Case (ii). The concerned motion is for $\dot{\varphi} \neq 0$, and if one chooses $S^{r\theta} = 0$, the solution to (6.37) is given by

$$S^{r\varphi} = \frac{C^{r\varphi}}{\sqrt{(r^2+n^2)}}, \qquad S^{\theta\varphi} = \frac{C^{\theta\varphi}}{(r^2+n^2)}, \qquad S^{rt} = \frac{C^{rt}}{\sqrt{F}}. \tag{6.41}$$

Interestingly, even in this case $Q = 0$ yields $S^{\theta\varphi}$ as the only nenule spin component.

Introducing two constants of motion:

$$L = (r^2 + n^2)\dot{\varphi}, \qquad \Sigma = J^{(3)} - L = -2n^2 F S^{\theta\varphi}, \tag{6.42}$$

one could obtain for the orbit of the particle the followings:

$$\begin{aligned} &\frac{1}{r^2+n^2}\frac{dr}{d\varphi} \\ &\quad = \sqrt{\left(2E - \frac{1}{r^2+n^2}\right)\frac{F}{L^2} + \frac{1}{4n^2}\left[1 - \frac{1}{L}\left(2nJ^{(0)} + J^{(3)}\right)\right]^2}, \\ &dt = \frac{d\tau}{F}\left[J^{(0)} + \frac{\Sigma}{2n}\right]. \end{aligned} \tag{6.43}$$

The gravitational redshift formula in (6.40) or (6.43) shows that the time-dilation receives contribution from the spin. The spin degrees of freedom modify the particle's orbit.

6.4 Concluding Remarks

The aim of this chapter has been to investigate by pseudo-classical mechanics models the quantum objects, namely spin one half particles, in the

NUT-Taub-de Sitter spacetime. Having in mind the lack of a satisfactory quantum theory for gravitational interaction, this study is justified and not at all trivial.

The supersymmetric extension of the NUT-Taub-de Sitter spacetime admits fermionic symmetries (generated by Q_{f} in (6.23)) along with four standard SUSYs (given in (6.14)–(6.17). The appearance of these *non-generic* SUSYs are closely related to the existence of Killing-Yano tensors, obtained in (6.21). The new conserved charges then receive contributions from the spin-polarization tensor $S^{\mu\nu}$ (defined in (1.10)) and are given by (6.27) with (6.24)–(6.26).

Although the equations of motion of the pseudo-classical Dirac fermions are complex enough and exact solutions are not illuminating, we are able to present special solutions for the motion on a cone and on a plane. The supersymmetric constraint for physical fermions $Q = 0$ (1.48) plays an important role for the forms of solutions.

The results show spin dependence of the time-dilation and of the orbits of the particles in a gravitational field. This leads to the existence of a gravitational analogue of the Stern-Gerlach-type forces well known to appear in electromagnetic phenomena.

The results of this study may be interesting in the study of fermion modes in gravitational instantons as well as in the long-range monopole dynamics. Our results reduce to the case of (i) the Taub-NUT spacetime for $\Lambda = 0$ (chapter 5), (ii) the Schwarzschild-de Sitter spacetime for $n = 0$ (chapter 3), (iii) the Schwarzschild spacetime for $\Lambda = 0$, $n = 0$, [42] and (iv) the pure de Sitter spacetime for $M = 0$, $n = 0$ (chapter 3).

In recent years there is a renewed interest in cosmological constant, since it is found to be present in the inflationary scenario of the early universe. In this scenario the universe undergoes a phase where it is geometrically analogous to the de Sitter space [111]. Moreover, the Taub-NUT space has a peculiar character – a counter example to almost everything [95] – and it is sometimes considered as unphysical [138]. The study of this chapter in such an interesting spacetime is well motivated.

Chapter 7

Geodesic Motions of Spinning Particles in Generalized NUT Spacetime

7.1 Introduction

In this chapter we investigate the motion of a pseudo-classical spin-$\frac{1}{2}$ particle in the black hole spacetime described by the generalized NUT metric [141]

$$\begin{aligned} ds^2 &= -V(r)(dt - 2p_{\rm n}\cos\theta d\varphi)^2 + \frac{1}{V(r)}dr^2 \\ &\quad +(r^2+p_{\rm n}^2)(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \tag{7.1}$$

where

$$V(r) = 1 - \frac{\Lambda}{3}(r^2 + 5p_{\rm n}^2) - 2\frac{Mr + np_{\rm n}}{r^2 + p_{\rm n}^2} + \frac{q^2}{r^2 + p_{\rm n}^2},$$

$$n = p_{\rm n} - \frac{4\Lambda}{3}p_{\rm n}^3, \qquad q^2 = q_{\rm e}^2 + q_{\rm m}^2, \tag{7.2}$$

where $p_{\rm n}$ is a continuous parameter, M the gravitating mass, $q_{\rm e}$ the electric charge, $q_{\rm m}$ the magnetic monopole charge, n the NUT charge (magnetic mass [96, 97, 98, 99, 100]), and Λ the cosmological constant.

The metric (7.1) represents a stationary axisymmetric solution of the Einstein-Maxwell field equations with cosmological constant. It solves the field equations with an electromagnetic vector potential

$$A = -\frac{q_{\rm e} r}{(r^2 + p_{\rm n}^2)\sqrt{V}}\, e^0 - \frac{q_{\rm m} r \cos\theta}{\sqrt{(r^2 + p_{\rm n}^2)}\, \sin\theta}\, e^3, \tag{7.3}$$

and an associated field strength tensor which can be expressed in terms of the 2-form

$$F = -\frac{1}{(r^2 + p_{\rm n}^2)^2}\left[q_{\rm e}(r^2 - p_{\rm n}^2) + 2q_{\rm m}p_{\rm n}r\right] e^0 \wedge e^1,$$

$$+\frac{1}{(r^2 + p_{\rm n}^2)^2}\left[q_{\rm m}(r^2 - p_{\rm n}^2) - 2q_{\rm e}p_{\rm n}r\right] e^2 \wedge e^3, \tag{7.4}$$

where we have defined the vierbein field

$$e^0 = \sqrt{V}(dt - 2p_\mathrm{n}\cos\theta d\varphi), \qquad e^1 = \frac{1}{\sqrt{V}}dr,$$

$$e^2 = \sqrt{(r^2+p_\mathrm{n}^2)}\,d\theta, \qquad e^3 = -\sqrt{(r^2+p_\mathrm{n}^2)}\,\sin\theta d\varphi, \tag{7.5}$$

In the case of $p_\mathrm{n} \to 0$, the metric (7.1) represents the generic Reissner-Nordström solution with the cosmological constant for $q_\mathrm{m} = 0$ and the cosmological Schwarzschild solution for $q_\mathrm{e} = q_\mathrm{m} = 0$, if additionally $\Lambda = 0$, we obtain the basic Schwarzschild solution. For $q_\mathrm{m} \neq 0$, we get a slight generalization of the cosmological Reissner-Nordström solution due to the presence of the electromagnetic field of a magnetic monopole. The monopole hypothesis was propounded by Dirac relatively long ago. It was ingeniously suggested by Dirac that magnetic monopoles do exist in nature, but this prediction was neglected due to the failure to identify such things. In recent years, however, the development of gauge theories [101, 102] has shed new light on it. The string theory [103] predicts the existence of this type of objects.

With $p_\mathrm{n} \neq 0$ and switching off the electromagnetic field: $q_\mathrm{e} = q_\mathrm{m} = 0$, the metric (7.1) gives the NUT solution generalized by the presence of the cosmological constant. If $\Lambda = 0$, the parameter n coincides with p_n and the resulting NUT solution [94, 142] plays an important role in the conceptional development of general relativity and in the construction of brane solutions in string theory and M-theory [105, 106, 107]. In recent years a particular interest has been paid to the NUT solution because of the role it plays in furthering our understanding of the AdS/CFT correspondence [108, 109, 110]. The NUT solution also has become renowned for being "a counter example to almost anything" [95]. It represents a nontrivial generalization of the Schwarzschild solution [143]. It has the usual interpretation of describing a gravitational dyon with both ordinary and magnetic masses. Similar as that electric and magnetic charges are dual within Maxwell theory, the NUT charge n plays a role dual to that of the ordinary mass M [97, 137]. In a recent work [104], it was shown that the NUT charge generates a "rotational effect", so that the spacetime must be assigned a "specific angular momentum" due to the NUT charge. We note that n vanishes with $p_{\rm n} = 0$ or with $\Lambda = \frac{3}{4}p_{\rm n}^{-2}$. Thus if $M = n = 0$ in addition to $p_{\rm n} \neq 0$ and $q_{\rm e} = q_{\rm m} = 0$, the metric (7.1) just describes the de Sitter space, which has properties similar to a black hole [126, 127, 144, 145, 146, 147]. There has been a renewed interest in cosmological constant as it is found to be present in the inflationary scenario of the early universe. In this scenario the universe undergoes a phase where it is geometrically similar to the de Sitter space [111].

Supersymmetric extension of the generalized NUT spacetime admits Killing-Yano tensors. We investigate the symmetries of the generalized NUT spacetime and find the existence of four standard supersymmetries (SUSYs) (which exist in any spacetime) plus several nonstandard ones generated by the Killing-Yano tensors. We calculate these SUSYs and analyze the geodesic motions of spin-$\frac{1}{2}$ point particles.

The chapter is organized as follows. In section 7.2 we investigate the motion of pseudo-classical spinning particles in the generalized NUT spacetime. We examine the generalized Killing equations for this spinning spacetime and derive the constants of motion in terms of the Killing-Yano tensors. In section 7.3 we analyze the equations describing the pseudo-classical spinning point particles, derived in section 7.2, for special cases of motion on a cone and on a plane. Finally, we present our concluding remarks in section 7.4.

7.2 Motion in Generalized NUT Spinning Spacetime

In this section, the formalisms of chapters 1 and 2 have been exploited to investigate the motion of a pseudo-classical spinning point particle in the generalized NUT spacetime described by the metric (7.1), which has an isometry group $SU(2) \times U(1)$ and possesses four Killing vector fields of the form

$$D^{(\alpha)} \equiv R^{(\alpha)\mu}\partial_{\mu}, \qquad (\alpha = 0, \cdots, 3), \tag{7.6}$$

or equivalently

$$D^{(0)} = \frac{\partial}{\partial t},$$

$$
\begin{aligned}
D^{(1)} &= -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi} - 2p_{\rm n}\cot\theta\cos\varphi\frac{\partial}{\partial t}, \\
D^{(2)} &= \cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi} - 2p_{\rm n}\cot\theta\cos\varphi\frac{\partial}{\partial t}, \\
D^{(3)} &= \frac{\partial}{\partial\varphi} - 2p_{\rm n}\frac{\partial}{\partial t}.
\end{aligned}
\tag{7.7}
$$

$D^{(0)}$, which generates the $U(1)$ of t translation, commutes with the Killing vectors. The remaining three vectors obey an $SU(2)$ algebra with

$$
\begin{aligned}
&[D^{(0)}, D^{(i)}] = 0, \\
&[D^{(i)}, D^{(j)}] = -\varepsilon^{ijk}D^{(k)}, \qquad (i,j,k = 1,2,3).
\end{aligned}
\tag{7.8}
$$

This is contrasted with the Schwarzschild geometry, where the isometry group at spacelike infinity is $SO(3) \times U(1)$, and illustrates the essential topological character of the parameter $p_{\rm n}$ or magnetic mass [83, 133].

These invariances would correspond to conservation of the so-called "relative electric charge" and the angular momentum [128, 130, 131, 132, 139] in purely bosonic case:

$$q_{\rm p} = -V(\dot{t} - 2p_{\rm n}\cos\theta\,\dot{\varphi}),$$

$$\mathbf{j} = \mathbf{r}\times\mathbf{p} + 2p_{\rm n}q_{\rm p}\frac{\mathbf{r}}{r}. \tag{7.9}$$

It follows, from the first generalized Killing equation of (1.33) with $J^{(0)} = B^{(\alpha)}$, $J^{(1)}_{\mu} = R^{(\alpha)}_{\mu}$, that the constants of motion would be of the form

$$J^{(\alpha)} = B^{(\alpha)} + m\dot{x}^{\mu}R^{(\alpha)}_{\mu}, \tag{7.10}$$

which asserts that the Killing scalars $B^{(\alpha)}$ contribute to the "relative electric charge" $q_{\rm p}$ and the total angular momentum $\mathbf{j}$.

For the metric (7.1), we obtain

$$\begin{aligned}
B^{(0)} &= NS^{tr} - 2p_{\rm n}N\cos\theta S^{r\varphi} + p_{\rm n}V\sin\theta S^{\theta\varphi},\\
B^{(1)} &= 2p_{\rm n}N\cos\varphi\cot\theta(1+\cos\theta)S^{tr}\\
&\quad -\frac{1}{2}p_{\rm n}V\cos\varphi\cos\theta S^{t\theta} + \frac{1}{2}p_{\rm n}V\sin\varphi\sin\theta S^{t\varphi} - r\sin\varphi S^{r\theta}\\
&\quad -\cos\varphi\cot\theta\left[(9p_{\rm n}^2N + r)\sin^2\theta + 4p_{\rm n}^2N\cos\theta(1+\cos\theta)\right]S^{r\varphi}\\
&\quad +\cos\varphi\left[(r^2+p_{\rm n}^2)\sin^2\theta + 3p_{\rm n}^2V(1-2\cos^2\theta) - 2p_{\rm n}^2V\cos\theta\right]S^{\theta\varphi},
\end{aligned}$$

$$
\begin{aligned}
B^{(2)} &= -\frac{\partial}{\partial\varphi}B^{(1)}, \\
B^{(3)} &= 2p_{\rm n}N(1-2\cos\theta)S^{tr}+\frac{1}{2}p_{\rm n}V\sin\theta S^{t\theta} \\
&+\left[r\sin^2\theta-7p_{\rm n}^2N\cos^2\theta-\frac{3}{2}p_{\rm n}^2N(1+4\cos\theta)\right]S^{r\varphi} \\
&+\sin\theta\left[(r^2+p_{\rm n}^2)\cos\theta-2p_{\rm n}^2V(1-3\cos\theta)\right]S^{\theta\varphi}, \qquad (7.11)
\end{aligned}
$$

where

$$
N=\frac{1}{(r^2+p_{\rm n}^2)^2}\left[M(r^2-p_{\rm n}^2)+2np_{\rm n}r-rq^2-\frac{\Lambda}{3}r(r^2+p_{\rm n}^2)^2\right]. \qquad (7.12)
$$

The conserved total angular momentum in the spinning case is expressed by

$$
\mathbf{J}=\mathbf{B}+\mathbf{j}, \qquad J^{(0)}=B^{(0)}-q_{\rm p} \qquad (7.13)
$$

where $\mathbf{J}=(J^{(1)},J^{(2)},J^{(3)})$ and $\mathbf{B}=(B^{(1)},B^{(2)},B^{(3)})$. The components of $\mathbf{J}$ are as follows:

$$
\begin{aligned}
J^{(1)} &= B^{(1)}-(r^2+p_{\rm n}^2)\sin\varphi\,\dot\theta \\
&-(r^2+p_{\rm n}^2)\cos\theta\sin\theta\cos\varphi\,\dot\varphi+2p_{\rm n}q_{\rm p}\sin\theta\cos\varphi,
\end{aligned}
$$

$$\begin{aligned}
J^{(2)} &= B^{(2)} + (r^2 + p_{\rm n}^2)\cos\varphi\,\dot\theta \\
&\qquad -(r^2 + p_{\rm n}^2)\cos\theta\sin\theta\sin\varphi\,\dot\varphi + 2p_{\rm n}q_{\rm p}\sin\theta\sin\varphi, \\
J^{(3)} &= B^{(3)} + (r^2 + p_{\rm n}^2)\sin^2\theta\,\dot\varphi + 2p_{\rm n}q_{\rm p}\cos\theta.
\end{aligned} \tag{7.14}$$

From (7.14) we obtain two interesting relations:

$$J^{(1)}\sin\varphi - J^{(2)}\cos\varphi = -rS^{r\theta} - (r^2 + p_{\rm n}^2)\dot\theta + \frac{1}{2}p_{\rm n}V\sin\theta S^{t\varphi}, \tag{7.15}$$

$$\begin{aligned}
2p_{\rm n}J^{(0)} + \frac{\mathbf{J}\cdot\mathbf{r}}{r} &= -\frac{1}{2}p_{\rm n}^2 N\cos\theta\left[4\cos\theta(\cos\theta + 5) + 29\right]S^{r\varphi} \\
&\quad + \sin\theta\left[(r^2 + p_{\rm n}^2) + p_{\rm n}^2 V(3 - 4\cos\theta)\right]S^{\theta\varphi} \\
&\quad + 2p_{\rm n}N(2\cos\theta + \sin^2\theta)S^{tr}.
\end{aligned} \tag{7.16}$$

In addition, the four universal conserved charges, described in (1.38), (1.41), (1.43) and (1.44), become

(i) The energy

$$E = \frac{1}{2V}\dot r^2 + \frac{1}{2}(r^2 + p_{\rm n}^2)(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2) - \frac{1}{2V}q_{\rm p}^2; \tag{7.17}$$

(ii) The supercharge

$$Q = \frac{1}{V}\dot{r}\psi^r + (r^2+p_{\rm n}^2)\dot{\theta}\psi^\theta + q_{\rm p}\psi^t$$
$$+ \left[2p_{\rm n}q_{\rm p}\cos\theta + (r^2+p_{\rm n}^2)\sin^2\theta\,\dot{\varphi}\right]\psi^\varphi; \qquad (7.18)$$

(iii) The chiral charge

$$\Gamma_* = (r^2+p_{\rm n}^2)\sin\theta\psi^r\psi^\theta\psi^\varphi\psi^t; \qquad (7.19)$$

(iv) The dual supercharge

$$Q^* = (r^2+p_{\rm n}^2)\sin\theta\left(\dot{r}\psi^\theta\psi^\varphi\psi^t - \dot{\theta}\psi^r\psi^\varphi\psi^t\right.$$
$$\left.+ \dot{\varphi}\psi^r\psi^\theta\psi^t - \dot{t}\psi^r\psi^\theta\psi^\varphi\right). \qquad (7.20)$$

From the covariantly constant ψ^μ, formulated in (1.9), we obtain the following equations for the spin coordinates:

$$
\begin{aligned}
\dot{\psi}^t &= \left(\frac{N}{V}\dot{r} - \frac{2p_\mathrm{n}^2 V}{r^2+p_\mathrm{n}^2}\cot\theta\,\dot{\theta}\right)\psi^t + \left(\frac{2p_\mathrm{n} r}{r^2+p_\mathrm{n}^2}\cos\theta\,\dot{\varphi} - \frac{q_\mathrm{p} N}{V^2}\right)\psi^r \\
&\quad - \left[p_\mathrm{n}\cos\theta\,(\tan\theta + 2\cot\theta)\,\dot{\varphi} - \frac{2p_\mathrm{n}^2 q_\mathrm{p}}{r^2+p_\mathrm{n}^2}\cot\theta\right]\psi^\theta \\
&\quad + 2p_\mathrm{n}\cos\theta\left(\frac{r}{r^2+p_\mathrm{n}^2} - \frac{N}{V}\right)\dot{r}\psi^\varphi \\
&\quad - p_\mathrm{n}\cos\theta\left(\tan\theta + 2\cot\theta + \frac{4p_\mathrm{n}^2 V}{r^2+p_\mathrm{n}^2}\cot\theta\right)\dot{\theta}\psi^\varphi, \\
\dot{\psi}^r &= \left[rV - (r^2+p_\mathrm{n}^2)N\right]\left(\dot{\theta}\psi^\theta + \sin^2\theta\,\dot{\varphi}\psi^\varphi\right), \\
\dot{\psi}^\theta &= -\frac{r\dot{\theta}}{r^2+p_\mathrm{n}^2}\psi^r - \frac{r\dot{r}}{r^2+p_\mathrm{n}^2}\psi^\theta + \sin\theta\cos\theta\,\dot{\varphi}\psi^\varphi \\
&\quad + \frac{p_\mathrm{n}\sin\theta}{r^2+p_\mathrm{n}^2}\left[(q_\mathrm{p} - 2p_\mathrm{n}V\cos\theta\,\dot{\varphi})\,\psi^\varphi - V\dot{\varphi}\psi^t\right], \\
\dot{\psi}^\varphi &= \frac{p_\mathrm{n}V}{r^2+p_\mathrm{n}^2}\csc\theta\,\dot{\theta}\psi^t - \frac{r\dot{\varphi}}{r^2+p_\mathrm{n}^2}\psi^r \\
&\quad - \left(\cot\theta\,\dot{\varphi} + \frac{p_\mathrm{n} q_\mathrm{p}}{r^2+p_\mathrm{n}^2}\csc\theta\right)\psi^\theta \\
&\quad - \left[\frac{r\dot{r}}{r^2+p_\mathrm{n}^2} + \left(1 - \frac{2p_\mathrm{n}^2 V}{r^2+p_\mathrm{n}^2}\right)\cot\theta\,\dot{\theta}\right]\psi^\varphi.
\end{aligned}
\tag{7.21}
$$

The metric (7.1) admits Killing-Yano tensor $f_{\mu\nu}$ given by

$$\begin{aligned} \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu &= p_{\mathrm{n}} dr \wedge (dt + 2p_{\mathrm{n}} \cos\theta d\varphi) \\ &+ r \sin\theta d\theta \wedge (r^2 + p_{\mathrm{n}}^2) d\varphi. \end{aligned} \tag{7.22}$$

For $f_\mu{}^a(x)$ we have

$$\begin{aligned} f_\mu{}^0 dx^\mu &= \frac{1}{\sqrt{V}} p_{\mathrm{n}} dr, \\ f_\mu{}^1 dx^\mu &= -p_{\mathrm{n}} \sqrt{V} (dt + 2p_{\mathrm{n}} \cos\theta d\varphi) \\ f_\mu{}^2 dx^\mu &= -r \sqrt{(r^2 + p_{\mathrm{n}}^2)} \sin\theta d\varphi, \\ f_\mu{}^3 dx^\mu &= r \sqrt{(r^2 + p_{\mathrm{n}}^2)} \, d\theta. \end{aligned} \tag{7.23}$$

The components of c_{abc} are derived from (2.79) as follows:

$$c_{012} = 0, \qquad c_{013} = 0, \qquad c_{023} = 0, \qquad c_{123} = -2\sqrt{V}. \tag{7.24}$$

When the quantities from (7.23) and (7.24) are inserted into (2.51), the result gives the new SUSY generator Q_{f} for the metric (7.1) in the following form:

$$
\begin{aligned}
Q_{\mathrm{f}} &= r\sqrt{(r^2+p_{\mathrm{n}}^2)}\,\left[2p_{\mathrm{n}}\,q_{\mathrm{p}}\cos\theta+(r^2+p_{\mathrm{n}}^2)\sin^2\theta\,\dot{\varphi}\right]\psi^{\theta} \\
&\quad -\left[\frac{2p_{\mathrm{n}}^2\cos\theta}{\sqrt{V}}\dot{r}+r(r^2+p_{\mathrm{n}}^2)^{\frac{3}{2}}\sin\theta\,\dot{\theta}\right]\psi^{\varphi} \\
&\quad -\frac{p_{\mathrm{n}}}{\sqrt{V}}\left[\dot{r}\psi^{t}-q_{\mathrm{p}}\psi^{r}\right]-2\mathrm{i}\sqrt{V}\,\psi^{r}\psi^{\theta}\psi^{\varphi}. \qquad (7.25)
\end{aligned}
$$

Using (2.54)–(2.56) the Killing tensor, vector, and scalar, which define the conserved charge Z in (2.52), are constructed as follows:

$$
\begin{aligned}
K_{\mu\nu}\Pi^{\mu}\Pi^{\nu} &= -\frac{p_{\mathrm{n}}^2}{V}dr^2+p_{\mathrm{n}}^2V(dt-2p_{\mathrm{n}}\cos\theta\,d\varphi)^2 \\
&\quad +r^2(r^2+p_{\mathrm{n}}^2)\left(d\theta^2+\sin^2\theta\,d\varphi^2\right), \qquad (7.26)
\end{aligned}
$$

$$
\begin{aligned}
I_{\mu}\Pi^{\mu} &= \sqrt{(r^2+p_{\mathrm{n}}^2)V} \\
&\quad \times\left[r\sin\theta S^{r\varphi}d\varphi+\left(p_{\mathrm{n}}S^{t\varphi}+rS^{r\theta}\right)d\theta\right], \qquad (7.27)
\end{aligned}
$$

$$
G = -2p_{\mathrm{n}}\frac{\sqrt{(q^2+np_{\mathrm{n}})}}{r^2+p_{\mathrm{n}}^2}S^{tr}S^{\theta\varphi}, \qquad (7.28)
$$

where $S^{\mu\nu}$ is the spin-tensor defined in (1.10).

The equations and conserved quantities derived in this section are applied to obtained the trajectories of the pseudo-classical spinning particles in terms of the usual coordinates $\{x^\mu\}$ and Grassmann coordinates $\{\psi^\mu\}$. Since these equations are quite intricate and the general solution is by no means illuminating, we discuss special solutions in the subsequent section for the motion on a cone and on a plane.

7.3 Special solutions

We solve the equations derived in the preceding section for special kind of motions of pseudo-classical spin-$\frac{1}{2}$ point particles in the generalized NUT spacetime.

7.3.1 Motion on a Cone

Let us choose the z-axis along $\mathbf{J}$ so that the motion of the particle may conveniently be described in terms of polar coordinates

$$\mathbf{r} = r\mathbf{e}(\theta, \varphi), \qquad \mathbf{e} = (\sin\theta\cos\varphi, \, \sin\theta\sin\varphi, \, \cos\theta)\,. \tag{7.29}$$

We consider the case $\dot{\theta} = 0$. Equation (7.15) then solves $S^{r\theta}$ in terms of $S^{t\varphi}$. For this choice of axis, we have

$$rS^{r\theta} = \frac{1}{2} p_{\mathrm{n}} V \sin\theta S^{t\varphi}, \tag{7.30}$$

and as a result, it implies that $\Gamma_* = Q^* = 0$. The equations of motion for the spin components are obtained as follows:

$$\dot{S}^{r\varphi} + \frac{r\dot{r}}{r^2+p_{\rm n}^2} S^{r\varphi} = -\frac{3p_{\rm n}q_{\rm p}}{r^2+p_{\rm n}^2} \csc\theta S^{r\theta},$$

$$\dot{S}^{\theta\varphi} + \frac{2r\dot{r}}{r^2+p_{\rm n}^2} S^{\theta\varphi} = -\frac{r\dot{\varphi}}{r^2+p_{\rm n}^2} S^{r\theta},$$

$$\begin{aligned}
&\dot{S}^{r\theta} + \frac{r\dot{r}}{r^2+p_{\rm n}^2} S^{r\theta} \\
&\quad = -\left[rV - (r^2+p_{\rm n}^2)N\right] \sin^2\theta\, \dot{\varphi} S^{\theta\varphi} - \frac{p_{\rm n} V \sin\theta}{r^2+p_{\rm n}^2} \dot{\varphi} S^{rt} \\
&\quad + \sin\theta \left[\cos\theta\,\dot{\varphi} + \frac{p_{\rm n}}{r^2+p_{\rm n}^2}\left(q_{\rm p} - 2p_{\rm n} V \cos\theta\,\dot{\varphi}\right)\right] S^{r\varphi},
\end{aligned}$$

$$\begin{aligned}
&\dot{S}^{rt} + \frac{N}{V}\dot{r} S^{rt} \\
&\quad = -\frac{2r}{p_{\rm n}V}\left[rV - (r^2+p_{\rm n}^2)N\right] \sin\theta\, \dot{\varphi} S^{r\theta} \\
&\quad -p_{\rm n}\cos\theta \left[(\tan\theta + 2\cot\theta)\,\dot{\varphi} - \frac{2p_{\rm n}q_{\rm p}}{r^2+p_{\rm n}^2}\csc\theta\right] S^{r\theta} \\
&\quad +2p_{\rm n}\cos\theta \left(\frac{r}{r^2+p_{\rm n}^2} - \frac{N}{V}\right) \dot{r} S^{r\varphi}.
\end{aligned} \tag{7.31}$$

The constraint $Q = 0$ (see (1.48)) yields for the spin components the following relations:

$$\dot{r}S^{r\varphi} = -q_{\mathrm{p}}VS^{t\varphi},$$

$$\dot{r}S^{r\theta} = -V\left[2p_{\mathrm{n}}q_{\mathrm{p}}\cos\theta + (r^2+p_{\mathrm{n}}^2)\sin^2\theta\,\dot{\varphi}\right]S^{\theta\varphi} + q_{\mathrm{p}}VS^{\theta t},$$

$$q_{\mathrm{p}}S^{rt} = \left[2p_{\mathrm{n}}q_{\mathrm{p}}\cos\theta + (r^2+p_{\mathrm{n}}^2)\sin^2\theta\,\dot{\varphi}\right]S^{r\varphi}. \tag{7.32}$$

Setting $S^{r\varphi}=0$ and using (7.30)–(7.32), we get a single equation

$$\dot{S}^{\theta\varphi} + \frac{2r\dot{r}}{r^2+p_{\mathrm{n}}^2}S^{\theta\varphi} = 0, \tag{7.33}$$

and then (7.16) gives

$$\begin{aligned}
&\left(1-\frac{p_{\mathrm{n}}V}{2q_{\mathrm{p}}}\sin\theta S^{\theta\varphi}\right)\dot{\varphi}\\
&\quad= \frac{2p_{\mathrm{n}}q_{\mathrm{p}}}{(r^2+p_{\mathrm{n}}^2)\cos\theta} + \tan\theta S^{\theta\varphi} + \frac{2p_{\mathrm{n}}V}{(r^2+p_{\mathrm{n}}^2)\sin 2\theta}\\
&\qquad\times\left[\cos^2\theta + p_{\mathrm{n}}(1-2\cos\theta-6\cos^2\theta)\right]S^{\theta\varphi}.
\end{aligned} \tag{7.34}$$

Equation (7.33) is solved by

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{(r^2+p_{\mathrm{n}}^2)}. \tag{7.35}$$

where $C^{\theta\varphi}$ is a Grassmann constant.

The orbit of the particle is spin dependent and is described by the following equations:

$$
\begin{aligned}
\dot{t} &= -\frac{1}{V}(2p_{\mathrm{n}} V \cos\theta\,\dot{\varphi} + q_{\mathrm{p}}), \\
\dot{r} &= \left\{V\left[2E - (r^2 + p_{\mathrm{n}}^2)\sin^2\theta\,\dot{\varphi}^2\right] + q_{\mathrm{p}}^2\right\}^{\frac{1}{2}}, \\
&\left(1 - \frac{p_{\mathrm{n}} V \sin\theta\, C^{\theta\varphi}}{2q_{\mathrm{p}}(r^2 + n^2)}\right)\dot{\varphi} \\
&= \frac{2p_{\mathrm{n}} q_{\mathrm{p}}}{(r^2 + p_{\mathrm{n}}^2)\cos\theta} + \left[\tan\theta + \frac{2p_{\mathrm{n}} V}{(r^2 + p_{\mathrm{n}}^2)\sin 2\theta}\right. \\
&\quad \times \left.\left(\cos^2\theta + p_{\mathrm{n}}(1 - 2\cos\theta - 6\cos^2\theta)\right)\right]\frac{C^{\theta\varphi}}{r^2 + p_{\mathrm{n}}^2}, \\
q_{\mathrm{p}} &= -J^{(0)} + p_{\mathrm{n}} V \sin\theta \frac{C^{\theta\varphi}}{r^2 + p_{\mathrm{n}}^2}.
\end{aligned}
\tag{7.36}
$$

The first of (7.36) defines the gravitational redshift, and demonstrates that the time-dilation receives a contribution from spin-orbit coupling. This displays that time-dilation is not a purely geometric effect, but also has a dynamical component.

7.3.2 Motion on a Plane

We now turn to the special case of motion in a plane with $\theta = \pi/2$. Since orbital angular momentum is not separately conserved in general, planar motion for spinning particles occurs only in two kinds of situations: (i) the orbital angular momentum vanishes, or (ii) spin and orbital angular momentum are parallel.

With $\theta = \pi/2$, (7.30) and (7.16) give

$$
\begin{aligned}
rS^{r\theta} &= -\frac{1}{2} p_{\mathrm{n}} V S^{\varphi t}, \\
q_{\mathrm{p}} &= -\frac{1}{2p_{\mathrm{n}}} \left(r^2 + p_{\mathrm{n}}^2 + p_{\mathrm{n}}^2 V\right) S^{\theta\varphi},
\end{aligned}
\tag{7.37}
$$

and the equations of motion for the spin components take the following forms:

$$
\begin{aligned}
&\dot{S}^{r\varphi} + \frac{r\dot{r}}{r^2 + p_{\mathrm{n}}^2} S^{r\varphi} = 0, \\
&\dot{S}^{\theta\varphi} + \frac{2r\dot{r}}{r^2 + p_{\mathrm{n}}^2} S^{\theta\varphi} = -\frac{r\dot{\varphi}}{r^2 + p_{\mathrm{n}}^2} S^{r\theta}, \\
&\dot{S}^{r\theta} + \frac{r\dot{r}}{r^2 + p_{\mathrm{n}}^2} S^{r\theta} \\
&\quad = -\left[rV - (r^2 + p_{\mathrm{n}}^2) N\right] \dot{\varphi} S^{\theta\varphi} - \frac{p_{\mathrm{n}} V}{r^2 + p_{\mathrm{n}}^2} \dot{\varphi} S^{rt}
\end{aligned}
$$

$$\dot{S}^{rt} + \frac{N}{V}\dot{r}S^{rt}$$

$$= -\frac{2r}{p_{\rm n}V}\left[rV - (r^2 + p_{\rm n}^2)N\right]\dot{\varphi}S^{r\theta} - n\dot{\varphi}S^{r\theta}. \qquad (7.38)$$

The SUSY constraint $Q = 0$ with (7.37) gives

$$S^{r\varphi} = S^{rt} = 0,$$

$$\dot{r}S^{r\theta} = -V(r^2 + p_{\rm n}^2)\dot{\varphi}S^{\theta\varphi}. \qquad (7.39)$$

Case (i). The particle is moving along a fixed radius for which $\dot{\varphi} = 0$. The only nenule spin component $S^{\theta\varphi}$ is given by

$$S^{\theta\varphi} = \frac{C^{\theta\varphi}}{r^2 + p_{\rm n}^2}, \qquad (7.40)$$

$C^{\theta\varphi}$ being Grassmann constant and consequently, the gravitational red-shift is modied by the presence of spin-dependent part:

$$dt = \frac{d\tau}{V}\left(J^{(0)} - p_{\rm n}V\frac{C^{\theta\varphi}}{r^2 + p_{\rm n}^2}\right). \qquad (7.41)$$

The orbit of the particle also receives contribution from the spin and is described by

$$\frac{1}{V}\frac{dr}{dt} = \left\{ 1 + 2VE \left(J^{(0)} - p_{\mathrm{n}} V \frac{C^{\theta\varphi}}{r^2 + p_{\mathrm{n}}^2} \right)^{-2} \right\}^{\frac{1}{2}}. \qquad (7.42)$$

Case (ii). This possibility concerns motion for $\dot{\varphi} \neq 0$. The nonzero spin components $S^{r\theta}$, $S^{\theta\varphi}$ are linearly related as in (7.39), and they are given by

$$S^{r\theta} = \frac{C^{r\theta}}{\sqrt{V}}, \qquad S^{\theta\varphi} = -\frac{\dot{r}}{V(r^2 + p_{\mathrm{n}}^2)\dot{\varphi}} \frac{C^{r\theta}}{\sqrt{V}}, \qquad (7.43)$$

$C^{r\theta}$ being Grassmann constant.

Introducing two constants of motion:

$$L = (r^2 + p_{\mathrm{n}}^2)\dot{\varphi}, \qquad \Sigma = J^{(3)} - L = -2p_{\mathrm{n}}^2 F S^{\theta\varphi}, \qquad (7.44)$$

we obtain the time-dilation factor,

$$dt = \frac{d\tau}{V}\left[J^{(0)} + \frac{\Sigma}{2p_{\mathrm{n}}} \right], \qquad (7.45)$$

and for the orbit of the particle, the following equation:

$$\left(\frac{1}{r^2 + p_{\mathrm{n}}^2} \frac{dr}{d\varphi} \right)^2 = \left(2E - \frac{1}{r^2 + p_{\mathrm{n}}^2} \right) \frac{V}{L^2}$$
$$+ \frac{1}{4p_{\mathrm{n}}^2} \left[1 - \frac{1}{L} \left(2p_{\mathrm{n}} J^{(0)} + J^{(3)} \right) \right]^2. \qquad (7.46)$$

Thus the gravitational redshift formulae as well as the orbits of the pseudo-classical point particles receive contributions from the spin characterized by the Grassmann variables ψ^μ.

7.4 Concluding Remarks

The main concern of this study has been to investigate the motion of spinning point particles in the generalized NUT spacetime by pseudo-classical mechanics models in which spin degrees of freedom are characterized in terms of the Grassmann anticommuting spin variables ψ^μ. Particles' spin generalizes the usual Killing equations and Noether's theorem, which leads to obtain information about the solutions of the equations of motion of these particles in curved spacetime.

We investigate symmetries of the background spacetime and derive equations governing the motion of the spinning particle. We find a new supersymmetry to exist along with the four standard symmetries. The nongeneric fermionic symmetry is generated by Q_{f} in (7.25) and the appearance of it is closely related to the existence of Killing-Yano tensors, obtained in (7.23). The new conserved charge Z defined in (2.53) receives contribution from the spin-polarization tensor $S^{\mu\nu}$ (defined in (1.10)) and are given by (7.27) and (7.28). We note that although the Killing tensor in (7.26) is a constant of motion for a scalar (spinless) point particle, it receives contribution from spin in the case of spinning particles. The results of this study also demonstrates that the time dilation is not a purely geometric effect, it has a dynamical (i.e. spin) component too. This is illustrated in the first part of (7.36), (7.41) and (7.45).

The equations of motion for the pseudo-classical spinning point particles are complex enough and exact solutions are not illuminating. We have described the orbits for special cases of motion on a cone and on a plane. Equations (7.36), (7.42) and (7.46) display that the orbits of the spinning particles receive contribution from the spin of the particles. Having in mind the lack of a satisfactory quantum theory for gravitational interaction, our study is justified and not at all trivial.

Although the equations of motion of the pseudo-classical Dirac fermions are complex enough and exact solutions are not illuminating, we are able to present special solutions for the motion on a cone and on a plane. The supersymmetric constraint for physical fermions $Q = 0$ (1.48) plays an important role for the form of solutions.

The results of this chapter may be interesting in the study of fermion modes in gravitational instantons as well as in the long-range monopole dynamics. Our results reduce to the case of (i) the NUT-de Sitter spacetime for $p_{\rm n} = n$ (chapter 6), (ii) the peculiar [95] Taub-NUT [96, 142] spacetime for $p_{\rm n} = n$, $\Lambda = 0$ (chapter 5), (iii) the Reissner-Nordström-de Sitter spacetime for $p_{\rm n} = 0$ [44], (iv) the Schwarzschild-de Sitter spacetime for $p_{\rm n} = n$, $q = 0$ (chapter 3), (v) the Reissner-Nordström spacetime for $\Lambda = 0$, $p_{\rm n} = 0$ [43], and (vi) the Schwarzschild spacetime for $\Lambda = 0$, $p_{\rm n} = 0$, $q = 0$ [42]. Because of the presence of the cosmological constant, the result of this chapter may be interesting in view of the inflationary scenario of the early universe.

Discussion

The main concern of this thesis has been the investigation of the quantum objects, namely spin one half particles, in curved spaces by pseudo-classical mechanics model which is a world line supersymmetric extension of the ordinary relativistic point particle. The model involves, together with the usual spacetime coordinates, anticommuting Grassmann coordinates that take into account the spin degrees of freedom. Having in mind the lack of a satisfactory quantum theory for gravitational interaction, this study is interesting and not at all trivial.

For particles with internal degrees of freedom like spin, the usual Killing equations and Noether's theorem receive generalizations. These generalized equations then provide information about the solutions of the equations of motion for these particles in curved spacetime. The spacetime can have two types of symmetries: *generic* (chapter 1) and *nongeneric* (chapter 2) supersymmetries. Both kinds of symmetries are found for the Taub-NUT background spacetimes (chapters 4–7), but for the Schwarzschild type spacetimes there are only generic symmetries (chapter 3).

The results we obtain apply most directly to the formal aspects of the motion of fermions like electrons or, possibly, massive neutrinos (or photinos, gravitinos, etc.) in the external gravitational field. The formal aspects of motion include the proof of spin-orbit coupling and the corresponding fine-splitting, which result from dependence of the energy on the values and relative orientation of the orbital and spin angular momentum. This predicts that the time-dilation in a gravitational field, perihelion precession for bound state orbits, and the scattering of particles by gravitational fields are spin dependent. Thus there exists a gravitational analogue of the Stern-Gerlach type interaction well-known to appear in electromagnetic phenomena.

As stated in the introduction of this thesis, the equations of motion (1.11) and (1.13) remain valid when averaged inside a functional integral with the exponential of the action (1.3) in the integrand. That is, $S^{\mu\nu} = -i\psi^{\mu}\psi^{\nu}$ can be replaced by its quantum mechanical expectation value $\langle S^{\mu\nu}\rangle$. This permits to regard our results as a semi-classical approximation to the quantum Dirac theory. However, this approximation can only hold to first order in the spin, since $\langle S^{\mu\nu}\rangle^2 \neq \langle (S^{\mu\nu})^2\rangle$ in general.

It is obvious in the physical world that the effects of microscopic intrinsic spin of particles such as electrons in a gravitational field like that of a star can be completely neglected. In fact, the ratio Δ, defined in (3.37), for an electron orbiting the sun is of the order 10^{-17}. So, effects of particle spins most probably act a significant role only in strong gravitational interactions at short-distances, near the Planck-scale.

The spinning particle model used in this thesis work is a world line supersymmetric extension of the ordinary relativistic point particle and it is a theory that describes in a pseudo-classical way a Dirac fermion moving in an arbitrary spacetime. Since the Taub-NUT background spacetimes admit Killing-Yano type tensors, there exist new additional supersymmetries (chapters 4–7) which make possible a whole range of calculations, both classical and quantum mechanical, and can be applied to various physical processes in the background spacetime. The construction of the new supersymmetries in the pseudo-classical mechanics model can be carried over straightforwardly to the case of quantum mechanics. This is performed by the usual replacement of phase-space coordinates by operators and Poisson-Dirac brackets by anticommutators [1]. The supercharges in terms of these operators are replaced by Dirac-type operators [39]. In both cases, the correspondence principle expresses clearly the relations between these approaches and gives equivalent algebraic structures [73].

For the Dirac equation in curved spaces, it has been proved that the Killing-Yano tensors perform an essential role in the construction of new Dirac-type operators. The Dirac-type operators constructed from covariantly constant Killing-Yano tensors are equivalent with the standard Dirac operator [58, 148]. However, the non-covariantly constant Killing-Yano tensors generates non-standard Dirac operators which are not equivalent to the standard Dirac operator and are associated with the hidden symmetries of the space.

In view of the above considerations the study of this thesis is well motivated.

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